Chapter 10
Perturbative Introduction of Crossed-Channel Cut Singularities

Let us now consider the perturbative treatment of the crossed-channel cuts, that for simplicity are denoted generically as LHC, when unitarizing PWAs obtained from some EFT. We present four methods; the first one is based in the use of $\mathcal{N}(s)$ as introduced in Eq. (7.1). From this method we also derive another approach that is referred in the literature as the Inverse Amplitude Method (IAM). Other two approaches arise from the use of the N/D method where $\Delta(p^2)$ is calculated perturbative in the considered EFT, such that either the N/D IE is solved fully or in its first iterated form. This line of handling perturbatively the LHC contributions is discussed in Chaps. 11 and 12, in order.

Let us suppose that $T(s)$ is given by Eq. (7.2) in terms of $\mathcal{N}(s)$ and $g(s)$. In view of Eq. (7.10) a convenient choice for the subtraction constant in the unitarity loop function would be such that $g(s)$ is zero at some point along the near-threshold LHC. In this way, we might dismiss the dependence of $\mathcal{N}(s)$ along the physical region ($s \geq s_{\text{th}}$) on the iterated LHC contributions [multiplied by $g(s)$], at least for not too high s. For example, if one imposes that $g(0) = 0$ then, by taking the subtraction point at $s = 0$, the subtraction constant would be simply zero. Had we imposed that $g(s_0) = 0$, with s_0 along the LHC, then we would change the subtraction point to s_0, so that again $a(s_0) = 0$. In this case $g(s)$ would read

$$
\begin{aligned}
g(s) &= -\frac{s-s_0}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\rho(s')ds'}{(s'-s_0)(s'-s)} \\
&= -\frac{s}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\rho(s')ds'}{s'(s'-s)} + \frac{s_0}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\rho(s')ds'}{s'(s'-s_0)}\,.
\end{aligned}
\tag{10.1}
$$

This choice for the subtraction constant in order to weaken the influence of the iterated LHC for low s might be of relevance if one wished to provide a perturbative solution of Eq. (7.10) for $\mathcal{N}(s)$. Indeed, from Eq. (7.2) we have the geometric series

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$$T = \mathcal{N} - \mathcal{N} g \mathcal{N} - \mathcal{N} g \mathcal{N} g \mathcal{N} + \ldots \tag{10.2}$$

This expansion in powers of $\mathcal{N} g$ could be matched with a perturbative loop expansion of T and, in this way, $\mathcal{N}$ could be determined order by order [8, 48].

Let us give an explicit example based on ChPT applied in Ref. [48] to the massless $W_L W_L$ scattering (the subscript L stands for the longitudinal component of the W boson) by applying the equivalence theorem [49]. In this EFT the momentum expansion implies a loop expansion, so that the chiral dimension of a perturbative Feynman graph with L loops is $D = 2L + 2 + \sum_d N_d (d-2)$, with d the chiral dimension of a given monomial in the ChPT Lagrangian and N_d is the number of such vertices with dimension d [50]. The isoscalar scalar $W_L W_L$ scattering amplitude up to NLO or $\mathcal{O}(p^4)$ in ChPT is [48, 51]

$$T_2(s) = \frac{s}{v^2} \,, \tag{10.3}$$

$$\begin{aligned} T_4(s) = {} & \frac{3s^2}{2v^2(m_H^2 - s)} + \frac{m_H^4}{v^2 s}\left[\log\left(1 + \frac{s}{m_H^2}\right) - \frac{s}{m_H^2} + \frac{s^2}{2m_H^4}\right] \\ & - \frac{s^2}{1728\pi^2 v^4}\left[1673 - 297\sqrt{3}\,\pi + 108\log\frac{-s}{m_H^2} + 42\log\frac{s}{m_H^2}\right] , \end{aligned} \tag{10.4}$$

where $v = (\sqrt{2} G_F)^{-1/2} \simeq 1/4$ TeV is the analogous to f_π for the pion case, with G_F the Fermi coupling constant. The expression for $T_4(s)$ in Eq. (10.4) contains the exchange of a Standard Model Higgs boson of mass m_H. If we denoted by b the combination

$$\frac{11 v^2}{6 m_H^2} - \frac{1673 - 297\pi\sqrt{3}}{1728\pi^2} \to b \,, \tag{10.5}$$

then the expression in Eq. (10.4) becomes more general and it does not necessarily correspond to the exchange of a Standard Model Higgs boson, but to a general scenario of another underlying fundamental theory. The amplitude $T_4(s)$, up to $\mathcal{O}(p^4)$, becomes then

$$T_4(s) = b\frac{s^2}{v^4} - \frac{s^2}{1728\pi^2 v^4}\left[108\log\frac{-s}{m_H^2} + 42\log\frac{s}{m_H^2}\right] . \tag{10.6}$$

At NLO all of these alternative theories would give rise to $T_4(s)$ as written above in terms of v and b, although with the latter having different values. The scale m_H^2 is introduced above to refer to a high-energy scale in which bare resonance could appear. In the previous equation the first logarithm gives rise to the RHC and the last to the LHC.

In order to proceed with the unitarization of $T_2(s) + T_4(s)$ we employ the non-perturbative expression

$$T(s) = \frac{\mathcal{N}(s)}{1 + g(s)\mathcal{N}(s)} \,, \tag{10.7}$$

equivalent to Eq. (7.10). Next, we proceed with the chiral expansion of $\mathcal{N}(s)$ up to NLO as

$$\mathcal{N}(s) = \mathcal{N}_2(s) + \mathcal{N}_4(s) + \mathcal{O}(p^6) \tag{10.8}$$

with the subscript indicating the chiral order. Then, we match the chiral expansion of $T(s)$ in Eq. (10.7) by counting the loop function $g(s)$ as $\mathcal{O}(p^0)$, as it is clear from its loop expression in Eq. (8.5). This function in the present massless case reads

$$g(s) = \frac{1}{16\pi^2}\left(a + \log\frac{-s}{m_H^2}\right) . \tag{10.9}$$

Therefore we have,

$$\mathcal{N}_2(s) = T_2(s) \,, \tag{10.10}$$

$$\begin{aligned} \mathcal{N}_4(s) &= T_4(s) + T_2(s)^2 g(s) \\ &= \frac{s^2}{288\pi^2 v^4}\left(18(a + 16b\pi^2) - 7\log\frac{s}{m_H^2} .\right) . \end{aligned} \tag{10.11}$$

In this way we can employ Eq. (10.7) to calculate $T(s)$ by using $\mathcal{N} = \mathcal{N}_2 + \mathcal{N}_4$, the latter ones determined in Eqs. (10.10) and (10.11). In the limit in which $m_H \gg 4\pi v$, while keeping $|a|$ and $|b|$ of $\mathcal{O}(1)$ [b is around 0.1 for a heavy Standard Model Higgs boson of mass 1 TeV, cf. Eq. (10.5)], an isoscalar scalar resonance with vanishing mass and width is dynamically generated [48, 52, 53]. In the opposite limit, $m_H \ll 4\pi v$, we consider again the perturbative expression for $T_2(s) + T_4(s)$ given in Eqs. (10.3) and (10.4) corresponding to the exchange of a standard model Higgs. Further, we neglect the non-logarithmic terms divided by $4\pi v$ in comparison with those divided by the much smaller m_H. Additionally, near the bare pole, $s \simeq m_H^2$, the direct exchange of the resonance dominates over other contributions and, after these simplifications, we have now for $\mathcal{N}(s)$

$$\mathcal{N}(s) \approx \frac{3s^2}{2v^2(m_H^2 - s)} . \tag{10.12}$$

When inserted in Eq. (10.7) for calculating the unitarized the PWA $T(s)$ we obtain

$$T(s) \approx \frac{3m_H^2/(2v^2)}{m_H - \sqrt{s} - i\,\frac{3m_H^3}{64\pi v^2}} . \tag{10.13}$$

In this form, we end with a Breit–Wigner parameterization for the Higgs exchanged, with mass m_H and width

$$\Gamma_H = \frac{3m_H^3}{32\pi v^2} \,, \tag{10.14}$$

which coincides with the QFT expression for the width of the Higgs boson from the electroweak Lagrangian. This width is much smaller than m_H for $m_H \ll 4\pi v$.

In the literature there have been many other studies in which the LHC is included perturbatively and that could be understood by employing the basic point in the expansion given in Eq. (10.2).

Let us consider first the Inverse Amplitude Method, on which we briefly report. We come back again to Eq. (7.2) an express $\mathcal{N} = \mathcal{N}_2 + \mathcal{N}_4 + \mathcal{O}(p^6)$, with the former given by Eq. (10.10) and the latter by the first line Eq. (10.11), after matching with $T = T_2 + T_4 + \mathcal{O}(p^6)$ as explained above. Then,

$$T(s) = \left(\left[T_2(s) + T_4(s) + T_2(s)g(s)T_2(s) + \mathcal{O}(p^6)\right]^{-1} + g(s)\right)^{-1} \,. \tag{10.15}$$

We perform next the chiral expansion of the inverse matrix between the square brackets

$$\begin{aligned}
T(s) &= \left(T_2(s)^{-1}\left[I + T_4(s)T_2(s)^{-1} + T_2(s)g(s) + \mathcal{O}(p^4)\right]^{-1} + g(s)\right)^{-1} && (10.16)\\
&= \left(T_2(s)^{-1}\left[I - T_4(s)T_2(s)^{-1} - T_2(s)g(s) + \mathcal{O}(p^4)\right] + g(s)\right)^{-1} && (10.17)\\
&= \left[I - T_4(s)T_2(s)^{-1} + \mathcal{O}(p^4)\right]^{-1} T_2(s)\\
&= T_2(s)\left[T_2 - T_4 + \mathcal{O}(p^6)\right]^{-1} T_2(s) \,.
\end{aligned}$$

The last expression corresponds to the NLO IAM [32, 54–58]

$$T(s) = T_2(s)\,[T_2(s) - T_4(s)]^{-1}\,T_2(s) \,. \tag{10.18}$$

Despite it is based on a perturbative solution of Eq. (7.10), the IAM result is independent of the subtraction constant in $g(s)$. That this should be the case is clear if one considers that the IAM can also be recast as the expansion of the inverse of the PWA, $T(s)^{-1} = (T_2 + T_4)^{-1} = T_2^{-1}(T_2 - T_4 + \mathcal{O}(p^6))T_2^{-1}$, and then taking the inverse of this expansion.

There is an alternative derivation of the uncoupled IAM based on a DR for the inverse PWA $T^{-1}(s)$ [54, 55, 57]. Instead of taking directly $1/T(s)$ one consider the auxiliary function $G(s) = T_2(s)^2/T(s)$, whose imaginary part is, cf. Eq. (2.51),

$$\Im G = -T_2(s)^2\rho(s) \,. \tag{10.19}$$

We write down a three-times subtracted DR for $G(s)$ by applying the Sugawara–Kanazawa theorem discussed in Chap. 4, because $T_2(s)^2$ at most diverges like s^2 and $T(s) \to$constant for $s \to +\infty \pm i\varepsilon$ because of unitarity. It then follows that

$$G(s) = G(0) + G'(0)s + \frac{1}{2}G''(0)s^2 - \frac{s^3}{\pi}\int_{s_{\rm th}}^{\infty} ds' \frac{\rho(s')T_2(s')^2}{(s')^3(s'-s)} - LC(G) + PC(s)\,, \tag{10.20}$$

where $-LC(G)$ refers of the crossed-channel contributions in $G(s)$ and the pole contributions $PC(s)$ arise from possible zeroes of $T(s)$. We neglect this last contribution because the zeroes in the denominator are largely canceled by $T_2(s)^2$ when forming $G(s)$. There could be some slight mismatch between the zeroes of $T_2(s)$ and those of $T(s)$ which might give rise to a pathological behavior in narrow energy regions [32, 58, 59]. A modified version of the IAM formula was derived in Ref. [59] to cure this deficiency. Notice also that the expression for $T = 1/([N_2 + N_4]^{-1} + g)$, without the expansion of $[\mathcal{N}_2 + \mathcal{N}_4]^{-1}$, has no this pathology.

The subtraction constants $G(0)$, $G'(0)$ and $G''(0)$ are fixed by matching with the ChPT expansion of $G(s)$ up NLO,

$$G(s) = \frac{T_2(s)^2}{T_2(s) + T_4(s) + \mathcal{O}(p^6)} = T_2(s) - T_4(s) + \mathcal{O}(p^6)\ . \tag{10.21}$$

Therefore, by neglecting higher orders we can identify $G(0) = T_2(0) - T_4(0)$, $G'(0) = T_2'(0) - T_4'(0)$, and $G''(0) = T_2''(0) - T_4''(0)$. By the same token, $LC(G)$ is approximated from the crossed-channel cut contribution of $T_4(s)$, $LC(T_4)$. Furthermore, the dispersive integral in Eq. (10.20) is minus the one for the RHC contribution in $T_4(s)$, whose imaginary part along the RHC is $\Im T_4(s) = T_2(s)^2\rho(s)$, as required by perturbative unitarity. Thus, Eq. (10.20) becomes $G(s) = T_2(s) - T_4(s)$ and then $T(s)$ is given by Eq. (10.18). The IAM has also been extended to two-loop ChPT amplitudes in Ref. [60]. This method has been applied to meson–meson scattering [32, 57, 58, 61], quark-mass dependence of masses and decay constants [62], $W_L W_L$ scattering [52, 53], πN scattering [63], etc., among many other references.