Chapter 8
Nonlinear Localized Waves of Deformation in the Class of Metamaterials as Set as the Mass-in-mass Chain

Vladimir I. Erofeev, Daniil A. Kolesov & Alexey O. Malkhanov

Abstract A well-known mathematical model representing a chain of oscillators consisting of elastic elements and masses, each containing an internal oscillator and describing the class of acoustic metamaterials "mass-in-mass", is generalized by taking into account the nonlinearity of the external and (or) internal elastic elements. As a result of analysis of the long-wavelength approximation of the obtained system, it is shown that spatially localized nonlinear deformation waves (solitons) can be formed in a metamaterial, under dynamic influence on it. The dependencies connecting the parameters of a localized wave are determined: amplitude, velocity and width with inertial and elastic characteristics of the metamaterial.

Keywords: Mathematical modeling · Nonlinear waves · Metamaterial · Mass-in-mass chain · One-dimensional system

8.1 Introduction

The development of modern technologies is impossible without the creation of new promising materials with unusual properties. For example, defect-free carbon nanotubes are two orders of magnitude stronger and four times lighter than steel. Currently, a new class of substances with a complexly organized internal structure (microstructure) and possessing unique physicomechanical properties is called meta-

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materials. They first appeared in the field of optics and photonics (Cummer et al, 2016; Zhu and Zhang, 2018), but now they are increasingly found in other areas. For example, acoustic metamaterials (Zhang et al, 2009; Burov et al, 2011; Norris and Haberman, 2012; Deymier, 2013; Craster and Guenneau, 2013; di Cosmo and Laudato, 2018; Abali et al, 2017; Giorgio et al, 2017; Ming-Hui et al, 2009; Madeo et al, 2016; dell'Isola et al, 2015, 2016; El Sherbiny and Placidi, 2018) are widely used, in particular as sound and vibration absorbers as in Bobrovnitskii (2014, 2015); Bobrovnitskii et al (2016); Fedotovskii (2015, 2018); Bobrovnitskii and Tomilina (2018). Another example of materials with unusual properties are fullerites—solid structures formed based on fullerenes (Sidorov et al, 2005). Super- and ultra-hard fullerites are characterized by uniquely high velocities of longitudinal elastic waves and a wide diapason of these values ranging from 11 km/s to 26 km/s, depending on their structure, determined by the conditions of synthesis (Blank et al, 1998). The value of 26 km/s measured in one of the fullerite phases is a record—it is almost 20% more than the speed of longitudinal waves in graphite along atomic layers equal to 21.6 km/s (until recently this value was the highest for all known substances) and 40% more than the corresponding speed in diamond (18.6 km/s). The speeds of transverse waves in solid fullerite phases are also high (their values range from 7 km/s to 9.7 km/s), but they are still smaller than in diamond (11.6–12.8 km/s) which remain the highest among currently known substances.

Acoustic (or mechanical) metamaterials, being, in fact, not materials, but cellular periodic structures, in the long-wavelength range behave like continuous materials. The study of the features of dispersion, dissipation, and the appearance of nonlinearity of acoustic waves in metamaterials is of high interest (Altenbach et al, 2010; Dreyer et al, 2005; Agranovich et al, 2004; Berezovski et al, 2016; Engelbrecht et al, 2007; Madeo et al, 2015).

Guided by a mathematical analogy between acoustic and electromagnetic waves, many researchers have tried to construct continuous models of mechanical metamaterials. However, great success on this path was not achieved, since the mechanical analogs of actually existing materials with negative dielectric constant are deformable solids with negative mass, density or negative modulus of elasticity (Li and Chan, 2004; Fang et al, 2006; Ding et al, 2007; Cheng et al, 2008; Chan et al, 2006). And such materials do not exist in the reality.

It is obvious that an adequate description of the physicomechanical properties of metamaterials within the framework of the classical theory of elasticity is impossible. Recently, generalized micropolar theories of the Cosserat continuum type (Huang et al, 2009) have become widespread for modeling structurally inhomogeneous materials. However, these theories include a large number of material constants that require experimental determination and whose relationship with the structure of the material is not clear. This disadvantage is devoid of an alternative direction—structural modeling as in Altenbach et al (2011); Pavlov and Potapov (2008). In Pavlov (2010), a one-dimensional chain was considered containing identical masses m_1 connected by elastic elements (springs), having the same rigidity k_1, at the same time each mass inside itself contains another mass m_2 and one more elastic element—a spring with rigidity k_2 (see Fig. 8.1). Such a model, called the mass-in-mass chain, does not give the mentioned absurd results.

8.2 Mathematical Model

We generalize the model in Pavlov (2010) by taking into account the quadratic nonlinearity of the external and internal elastic elements. The potential energy of the unit cell of the mass-in-mass chain is written as:

$$W^{(j)} = \frac{1}{2}\left[k_1\left(u_1^{(j+1)} - u_1^{(j)}\right)^2 + k_2\left(u_2^{(j)} - u_1^{(j)}\right)^2 \right. \\ \left. +h_1\left(u_1^{(j+1)} - u_1^{(j)}\right)^3 + h_2\left(u_2^{(j)} - u_1^{(j)}\right)^2\right], \tag{8.1}$$

and its kinetic energy in the form:

$$T^{(j)} = \frac{1}{2}\left[m_1\left(\ddot{u}_1^{(j)}\right)^2 + m_2\left(\ddot{u}_2^{(j)}\right)^2\right]. \tag{8.2}$$

Let us suppose that $u_1(x)$ and $u_2(x)$ are continuous functions, which describe the displacements of all masses m_1 and m_2, respectively. Taking into account the expansion of displacements in a Taylor series up to the second term, we obtain

$$u_1^{(j+1)} = u_1(x+L) = u_1(x) + \frac{\partial u_1}{\partial x}L = u_1^{(j)} + \frac{\partial u_1}{\partial x}L. \tag{8.3}$$

The technique of expansion displacements in (8.3) was effectively applied by Kunin (1982) in the transformation of multimass discrete systems into a quasicontinuum.

The densities of the potential and kinetic energies for the equivalent continuum, obtained from (8.1) and (8.2), can be written in the form:

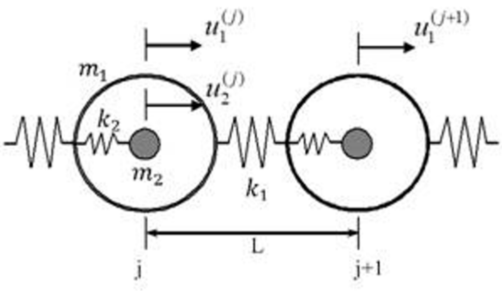

Fig. 8.1 Infinite mass-in-mass lattice structure

$$W=\frac{1}{2L}\left[k_1\left(\frac{\partial u_1}{\partial x}L\right)^2+k_2(u_2-u_1)^2+h_1\left(\frac{\partial u_1}{\partial x}L\right)^3+h_2(u_2-u_1)^2\right], \quad (8.4)$$

$$T=\frac{1}{2}\left[m_1(\ddot{u}_1)^2+m_2(\ddot{u}_2)^2\right]. \quad (8.5)$$

Let us construct from (8.4) and (8.5) the Lagrange function

$$\mathcal{L}=T-W=\mathcal{L}(\dot{u}_1,\dot{u}_2,u_{1x},u_1,u_2)$$

and take into account equations well known from analytical mechanics

$$\begin{cases}\dfrac{\partial}{\partial t}\left(\dfrac{d\mathcal{L}}{d\dot{u}_1}\right)+\dfrac{\partial}{\partial x}\left(\dfrac{\partial\mathcal{L}}{\partial u_{1x}}\right)-\dfrac{\partial\mathcal{L}}{\partial u_1}=0\\ \dfrac{\partial}{\partial t}\left(\dfrac{d\mathcal{L}}{d\dot{u}_2}\right)-\dfrac{\partial\mathcal{L}}{\partial u_2}=0\end{cases} \quad (8.6)$$

to get the system of equations in in displacements:

$$\begin{aligned}&\frac{m_1}{L}\ddot{u}_1-k_1L\frac{\partial^2u_1}{\partial x^2}-3h_1L^3\frac{\partial u_1}{\partial x}\frac{\partial^2u_1}{\partial x^2}-\frac{k_2}{L}(u_2-u_1)-\frac{3h_2}{2L}(u_2-u_1)^2=0,\\&\frac{m_1}{L}\ddot{u}_2-\frac{k_2}{L}(u_2-u_1)-\frac{3h_2}{2L}(u_2-u_1)^2=0.\end{aligned} \quad (8.7)$$

Further consider a particular case of system (8.6), where $h_1\neq 0,\ h_2=0$, i.e.:

$$\begin{gathered}\frac{m_1}{L}\ddot{u}_1-k_1L\frac{\partial^2u_1}{\partial x^2}-3h_1L^3\frac{\partial u_1}{\partial x}\frac{\partial^2u_1}{\partial x^2}-\frac{k_2}{L}(u_2-u_1)=0,\\ \frac{m_2}{L}\ddot{u}_2+\frac{k_2}{L}(u_2-u_1)=0\end{gathered} \quad (8.8)$$

The system (8.8) can be rewritten in the form of single equation:

$$\begin{gathered}\frac{\partial^2u_2}{\partial t^2}-\frac{k_1L^2}{m_1+m_2}\frac{\partial^2u_2}{\partial x^2}+\frac{m_1m_2}{k_2(m_1+m_2)}\frac{\partial^4u_2}{\partial t^4}-\frac{k_1L^2m_2}{k_2(m_1+m_2)}\frac{\partial^4u_2}{\partial x^2\partial t^2}\\ -\frac{3h_1L^4}{m_1+m_2}\left(\frac{\partial u_2}{\partial x}\frac{\partial^2u_2}{\partial x^2}+\frac{m_2}{k_2}\frac{\partial^2u_2}{\partial x^2}\frac{\partial^3u_2}{\partial t^2\partial x}+\frac{m_2}{k_2}\frac{\partial u_2}{\partial x}\frac{\partial^4u_2}{\partial t^2\partial x^2}\right.\\ \left.+\left(\frac{m_2}{k_2}\right)^2\frac{\partial^3u_2}{\partial t^2\partial x}\frac{\partial^4u_2}{\partial t^2\partial x^2}\right)=0.\end{gathered} \quad (8.9)$$

Let us introduce dimensionless variables—time, coordinate, and displacement:

$$\tau=\frac{t}{T},\quad y=\frac{x}{X},\quad u_2=u_0u. \quad (8.10)$$

The transformed equation (8.8) with the new variables (8.10) takes the form:

$$
\begin{gathered}
\frac{\partial^2 u}{\partial \tau^2}-\frac{k_1 L^2}{m_1+m_2}\frac{T^2}{X^2}\frac{\partial^2 u}{\partial y^2}+\frac{m_1 m_2}{k_2\left(m_1+m_2\right)}\frac{1}{T^2}\frac{\partial^4 u}{\partial \tau^4}\\
-\frac{k_1 L^2 m_2}{k_2\left(m_1+m_2\right)}\frac{1}{X^2}\frac{\partial^4 u}{\partial y^2 \partial \tau^2}-\frac{3 h_1 L^4}{m_1+m_2}\frac{T^2 u_0}{X^3}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{m_2}{k_2}\frac{1}{T^2}\frac{\partial^2 u}{\partial y^2}\frac{\partial^3 u}{\partial \tau^2 \partial y}\right.\\
\left.+\frac{m_2}{k_2}\frac{1}{T^2}\frac{\partial u}{\partial y}\frac{\partial^4 u}{\partial \tau^2 \partial y^2}++\left(\frac{m_2}{k_2}\frac{1}{T^2}\right)^2\frac{\partial^3 u}{\partial \tau^2 \partial y}\frac{\partial^4 u}{\partial \tau^2 \partial y^2}\right)=0.
\end{gathered}
\tag{8.11}
$$

We require that all the coefficients (8.11) are finite or small. We choose them so that among the nonlinear terms we can distinguish only one, the main item.

All the subsequent arguments are valid if two conditions are satisfied:

$$
\frac{k_1 L^2}{m_1+m_2}\frac{T^2}{X^2}=1 \text{ and } \frac{m_1 m_2}{k_2\left(m_1+m_2\right)}\frac{1}{T^2}=\varepsilon, \quad \varepsilon \ll 1. \tag{8.12}
$$

When these conditions are fulfilled, in equation (8.11) some of the terms can be discarded, since they have a larger order of smallness and do not have a significant effect on dynamic processes. Thus, equation (8.11) takes the form:

$$
\frac{\partial^2 u}{\partial \tau^2}-\frac{\partial^2 u}{\partial y^2}+\varepsilon\frac{\partial^2}{\partial \tau^2}\left[\frac{\partial^2 u}{\partial \tau^2}-\alpha\frac{\partial^2 u}{\partial y^2}\right]=\delta\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}, \tag{8.13}
$$

where

$$
\begin{gathered}
\frac{m_1 m_2}{k_2\left(m_1+m_2\right)}\frac{1}{T^2}=\varepsilon \ll 1,\\
\frac{3 h_1 L u_0 \sqrt{\varepsilon}\alpha}{k_1\sqrt{\frac{k_1}{k_2}\frac{m_2}{m_1}}}=\delta \ll 1,\ \alpha=\frac{m_1+m_2}{m_1}>1.
\end{gathered}
\tag{8.14}
$$

Returning to the original dimensional variables in equation (8.13), we obtain the simplified equation (8.8) in the form:

$$
\begin{gathered}
\frac{\partial^2 u_2}{\partial t^2}-\frac{k_1 L^2}{m_1+m_2}\frac{\partial^2 u_2}{\partial x^2}+\frac{m_1 m_2}{k_2\left(m_1+m_2\right)}\frac{\partial^4 u_2}{\partial t^4}-\frac{k_1 L^2 m_2}{k_2\left(m_1+m_2\right)}\frac{\partial^4 u_2}{\partial x^2 \partial t^2}\\
-\frac{3 h_1 L^4}{m_1+m_2}\frac{\partial u_2}{\partial x}\frac{\partial^2 u_2}{\partial x^2}=0.
\end{gathered}
\tag{8.15}
$$

8.3 Nonlinear Stationary Waves

We will seek the solution to this equation in the class of traveling stationary waves: $u_2=u_2(\xi)$, $\xi=x-Vt$, V – velocity of the stationary wave (unknown beforehand). With respect to deformation $\frac{du_2}{d\xi}=U$ the nonlinear partial differential equation (8.15) reduces to the anharmonic oscillator equation:

$$\frac{d^2U}{d\xi^2} + aU + bU^2 = 0,$$
$$a = \frac{k_2(m_1+m_2)\left(V^2 - c_1^2\right)}{m_1 m_2 \left(V^2 - c_2^2\right)},\ b = \frac{3h_1 L^4 k_2}{2V^2 m_1 m_2 \left(V^2 - c_2^2\right)}, \tag{8.16}$$
$$c_1^2 = \frac{k_1 L^2}{m_1 + m_2},\ c_2^2 = \frac{k_1 L^2}{m_1}.$$

Let us note that $c_2 > c_1$. Depending on the value of the velocity V, there are qualitatively different wave patterns, since equation (8.16) has different solutions in Erofeev et al (2002). Only the solutions which at infinity do not give a constant component for the strain wave U have physical meaning.

If the velocity of the stationary wave satisfies inequality: $c_2 > c_1 > V$, then equation (8.16) has a periodic solution expressed in terms of an elliptic sine:

$$U(\xi) = \frac{A}{3s^2}\left(1 + s^2 - \sqrt{1 - s^2 + s^4}\right) - A \cdot sn^2(Q\xi, s), \tag{8.17}$$

where $A = -\frac{3a}{2b}\frac{s^2}{\sqrt{1-s^2+s^4}}$ – amplitude of the stationary wave, s– elliptic function module, $Q = \sqrt{\frac{k_1}{4V^2 m_2 \sqrt{1-s^2+s^4}}}$ – nonlinear analog of the wave number.

It can be seen that $Q \sim \sqrt{k_1},\ Q \sim \frac{1}{\sqrt{m_1}}, Q \sim \frac{1}{V}$ when the other variables are fixed.

In the Fig. 8.2 the dependence $Q \sim \sqrt{k_1}$ is depicted: curve 1—the qualitative form of this dependence at a fixed velocity and internal mass of the element; curve 2—the trend of the behavior of the graph of the dependence with increasing mass and fixed value of velocity; curve 3—increase in velocity by the same order that the mass was increased in the previous case; curve 4—increase of both parameters. From the analysis of curves 2 and 3 and from the assumption that the mass, in comparison with the velocity, is a much more static parameter, we can conclude that the most significant effect on this dependence is exerted by speed.

In the Fig. 8.3 the dependence $Q \sim \frac{1}{\sqrt{m_1}}$ is shown: curve 1—a qualitative representation of this relationship at a fixed velocity and stiffness of the external spring of

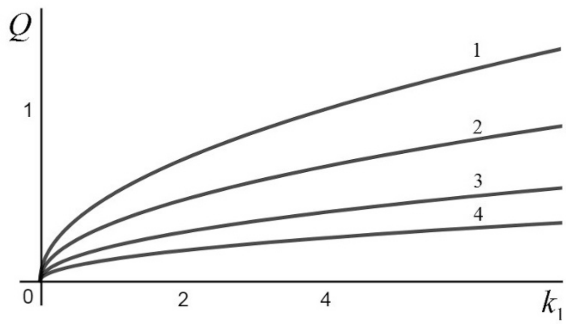

Fig. 8.2 The dependence of the wave number on the rigidity of the external elastic element of the system

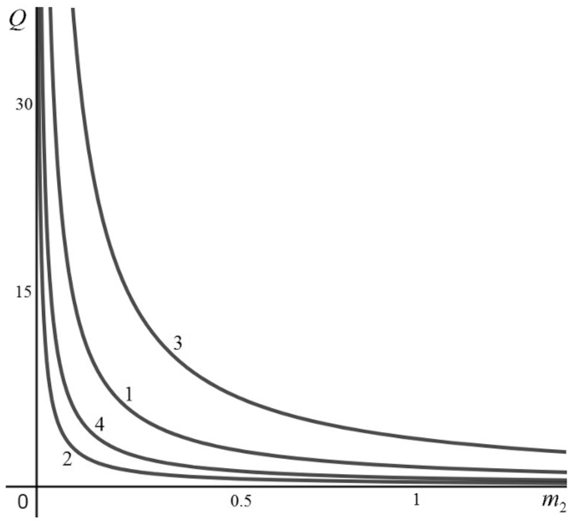

Fig. 8.3 The dependence of the wave number on the internal mass of the system element

the system element; curve 2—the trend of behavior of the graph of the dependence with increasing velocity and constant rigidity; curve 3—increase in rigidity by the same order, which was increased velocity in the previous case; line 4—increase of both parameters. From the analysis of the form of the curves 2 and 3 it can be seen that an increase in the parameters leads to a shift in the dependence curve in different directions. From the analysis of the curve 4 it follows that shows that the change in speed is "stronger."

In the Fig. 8.4 the dependence $Q \sim 1/V$ is depicted: curve 1—qualitative form of the relationship with fixed internal mass and stiffness of the external spring of the system element; curve 2—trend of behavior of the graph of the dependence with increasing mass and constant rigidity; curve 3—increase in rigidity by the same order that the mass was increased in the previous case. When both parameters are increased by the same order, the function graph coincides with curve 1. From the analysis of the curves it can be concluded that these parameters have the same influence on the indicated dependence.

The qualitative form of the periodic wave is shown in Fig. 8.5, where, through $K(s)$ the elliptic integral of the first kind is denoted.

If the velocity of the stationary wave satisfies the inequality $c_2 > V > c_1$, then equation (8.16) has an aperiodic solution, expressed in terms of the hyperbolic cosine:

$$U(\xi) = A_c \cosh^{-2}\left(\frac{\xi}{\Delta}\right). \tag{8.18}$$

This relation describes a solitary stationary wave (soliton) of deformation. Here $A_c = -\frac{3a}{2b}$ – the amplitude of soliton, $\Delta = \frac{2}{\sqrt{-a}}$ – the width of soliton. The analysis of the latter shows that $\Delta \sim V$, $\Delta \sim \frac{1}{\sqrt{k_2}}$, $\Delta \sim \sqrt{m_2}$.

In the Fig. 8.6 the dependence $\Delta \sim V$ is shown: curve 1 is a qualitative represen-

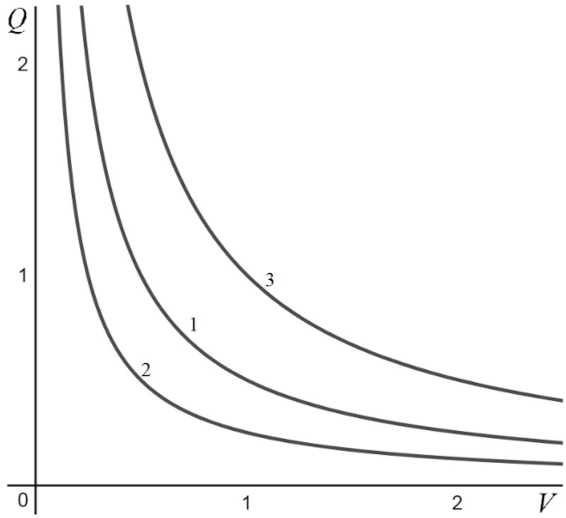

Fig. 8.4 The dependence of the wave number of a stationary wave on its speed

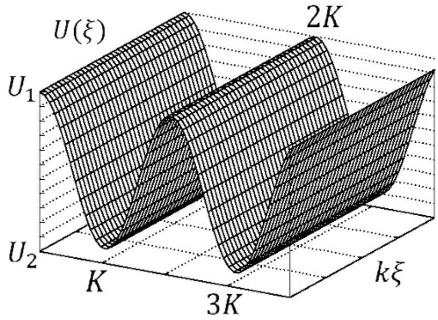

Fig. 8.5 Qualitative representation of the periodic wave

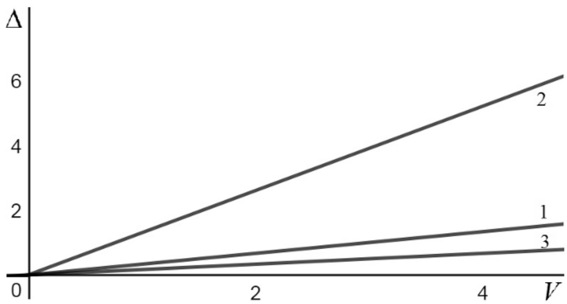

Fig. 8.6 The dependence of the soliton width on its speed

tation of this relationship for fixed internal spring stiffness and internal mass of the system element; curve 2—the trend of behavior of the graph of the dependence with increasing of the mass and constant rigidity; curve 3—increase in rigidity by the same order that the mass was increased in the previous case. When both parameters are increased by the same order, the function graph coincides with curve 1. From the analysis of the curves it can be concluded that these parameters have the same effect on the indicated dependence.

In the Fig. 8.7 the dependence $\Delta \sim 1/\sqrt{k_2}$ is depicted: curve 1—the qualitative form of this dependence at a fixed velocity and internal mass of the element of the system; curve 2—the trend of the behavior of the graph of the dependence with increasing velocity and constant mass; curve 3—increase in mass by the same order, which was increased velocity in the previous case; line 4—increase of both parameters. From the analysis of the curves 2 and 3 it can be seen that an increase in speed leads to a stronger shift up of the graph. Consequently, the change in velocity "stronger" affects the width of the soliton.

In the Fig. 8.8 the dependence $\Delta \sim \sqrt{m_2}$ is shown: curve 1—a qualitative representation of this relationship at a fixed velocity and stiffness of the internal spring of the element; curve 2—the trend of behavior of the graph of the dependence with increasing velocity and constant rigidity; curve 3—increase in rigidity by the same order, which was increased velocity in the previous case; curve 4—increase of both

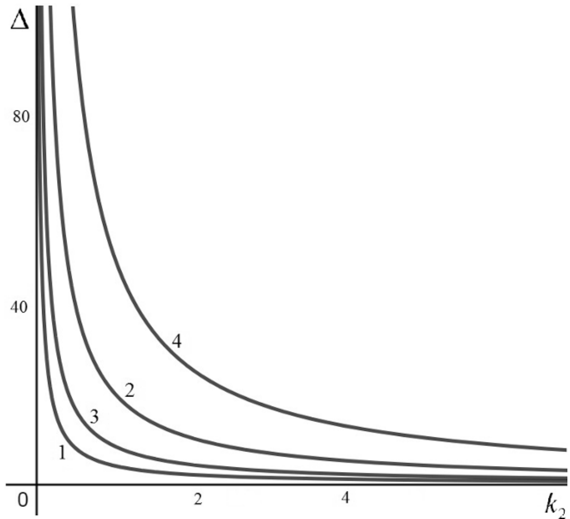

Fig. 8.7 The dependence of the soliton width on the rigidity of the internal elastic element of the system

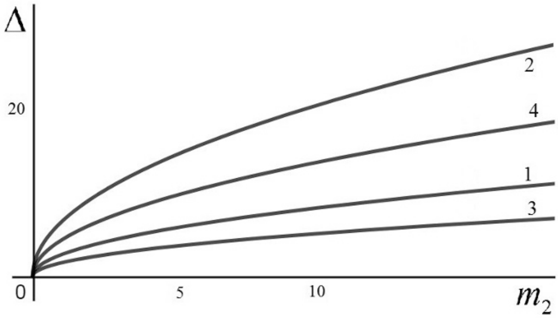

Fig. 8.8 The dependence of the soliton width on the internal mass of a system element

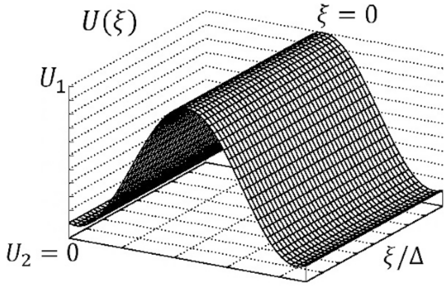

Fig. 8.9 The qualitative representation of a soliton

parameters. From the analysis of the form of curves 2 and 3 and from the assumption that the rigidity is a much more static parameter in comparison with the velocity, it can be concluded that the most significant effect on this dependence is exerted by speed.

The qualitative form of the soliton of deformations is shown in Fig. 8.9.

If the velocity of the stationary wave satisfies the inequality: $V > c_2 > c_1$, then the equation (8.16) has a periodic solution expressed in terms of an elliptic sine:

$$U(\xi) = \frac{A}{3s^2}\left(\sqrt{1-s^2+s^4} - 1 - s^2\right) + A \cdot sn^2(Q\xi, s), \tag{8.19}$$

where

$$A = -\frac{3a}{2b}\frac{s^2}{\sqrt{1-s^2+s^4}},\ Q^2 = \frac{a}{4\sqrt{1-s^2+s^4}}. \tag{8.20}$$

The dependencies shown in Figs. 8.2, 8.3, 8.4 remain valid for this case.

8.4 Conclusions

As a result of analysis of the long-wavelength approximation of the obtained system, it is shown that spatially localized nonlinear deformation waves (solitons) can be formed in a metamaterial, under dynamic influence on it. The dependencies connecting the parameters of a localized wave are determined: amplitude, velocity and width with inertial and elastic characteristics of the metamaterial.

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