Chapter 7
On Non-holonomic Boundary Conditions within the Nonlinear Cosserat Continuum

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Abstract Within the framework of the nonlinear micropolar elastic continuum we discuss non-holonomic kinematic boundary conditions. By non-holonomic boundary conditions we mean linear relations between virtual displacements and virtual rotations given on the boundary. Such boundary conditions can be used for modelling of complex material interactions in the vicinity of the boundaries and interfaces.

7.1 Introduction

The model of micropolar medium known also as Cosserat continuum was proposed by Cosserat brothers, see Cosserat and Cosserat (1909) and the contributions by Nowacki (1986) for infinitesimal deformations and by Eringen and Kafadar (1976); Eringen (1999); Eremeyev et al (2013); Altenbach and Eremeyev (2013); Eremeyev and Altenbach (2017) for finite deformations, where the further references can be found. The Cosserat model found various applications to description of such microstructured media as foams, granular media, composites, magnetic fluids, and thin-walled structures. Within the micropolar continuum the fields of translations and rotations are used as kinematical descriptors. In addition to stress tensor the couple stress tensor is also introduced in the theory which describes the rotational (moment-type) interactions in the medium.

Considering initial boundary-value problems of the micropolar mechanics one usually assumes kinematic or/and static boundary conditions expressed through

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translations and rotations or/and stress and couple stress vectors, respectively. These conditions play a role of principal and natural boundary conditions which follow from the stationarity of the corresponding functionals or from the principle of virtual work, see, e.g., Nowacki (1986); Eringen (1999); Pietraszkiewicz and Eremeyev (2009). Here we consider more general case of boundary conditions (BCs) when the latter cannot be derived from any functional, in general. For example, such type of boundary conditions one has in the case of nonconservative loading (Bolotin, 1963) or when some relations between linear and angular velocities are prescribed on a micropolar fluid surface (Migoun and Prokhorenko, 1984; Łukaszewicz, 1999).

The paper is organized as follows. First, in Section 7.2 we briefly recall basic equations of the micropolar continuum undergoing finite deformations. Considering the principle of virtual work in Section 7.3 we discuss the weak formulations of boundary conditions. In Section 7.4 we introduce non-holonomic boundary relations expressed as linear relations between virtual displacements and virtual rotations. Finally, we present few examples of non-holonomic boundary conditions.

7.2 Constitutive Relations

The deformation of a micropolar medium is described through kinematically independent fields of translations and rotations. So the kinematics of a micropolar continuum is described through the following vectorial fields:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{7.1}$$

$$\mathbf{d}_k = \mathbf{d}_k(\mathbf{X}), \quad k = 1 \ldots 3, \tag{7.2}$$

where $\mathbf{x}$ and $\mathbf{X}$ are positions vectors defined in current and reference placements, whereas $\mathbf{d}_k$ are unit orthogonal vectors called directors, see (Eringen and Kafadar, 1976; Eringen, 1999; Eremeyev et al, 2013) for details. Instead of (7.2) one can use the microrotation tensor defined as follows

$$\mathbf{Q} = \mathbf{D}_k \otimes \mathbf{d}_k, \tag{7.3}$$

where $\mathbf{D}_k$ are directors in a reference placement, $\otimes$ stands for the tensor (diadic) product, and Einstein's summation rule is utilized. Note that without loss of generality $\mathbf{Q}$ can be defined as a proper orthogonal tensor. To this end one have to chose the same orientation of triples $\mathbf{D}_k$ and $\mathbf{d}_k$. In what follows we use the direct (coordinate-free) tensor calculus presented in Lurie (1990); Simmonds (1994); Eremeyev et al (2018).

For an hyperelastic material there exists a strain energy density $\mathcal{W}$. We assume that $\mathcal{W}$ depends on $\mathbf{x}$, $\mathbf{Q}$ and their gradients

$$\mathcal{W} = \mathcal{W}(\mathbf{x}, \mathbf{F}, \mathbf{Q}, \nabla\mathbf{Q}), \tag{7.4}$$

where $\mathbf{F} = \nabla \mathbf{x}$ is the deformation gradient, ∇ is the gradient operator defined in the reference placement. For example, in the Cartesian coordinates X_k we have

$$\nabla = \mathbf{i}_k \frac{\partial}{\partial X_k},$$

where $\mathbf{i}_k$ are the Cartesian base vectors, $\mathbf{i}_k \cdot \mathbf{i}_m = \delta_{km}$, so $\mathbf{X} = X_k \mathbf{i}_k$, "$\cdot$" denotes scalar product, and δ_{mn} is the Kronecker symbol.

The principle of the material frame-indifference (Truesdell and Noll, 2004) says that $\mathcal{W}$ is invariant under changes

$$\mathbf{x} \to \mathbf{O} \cdot \mathbf{x} + \mathbf{a}, \quad \mathbf{d}_k \to \mathbf{O} \cdot \mathbf{d}_k \tag{7.5}$$

for any constant proper orthogonal tensor $\mathbf{O}$ and any constant vector $\mathbf{a}$. From (7.3) and (7.5) it follows that $\mathbf{Q}$ and $\nabla \mathbf{Q}$ change as follows

$$\mathbf{Q} \to \mathbf{D}_k \otimes (\mathbf{O} \cdot \mathbf{d}_k) = \mathbf{D}_k \otimes \mathbf{d}_k \cdot \mathbf{O}^T = \mathbf{Q} \cdot \mathbf{O}^T, \quad \nabla \mathbf{Q} \to \nabla \mathbf{Q} \cdot \mathbf{O}^T. \tag{7.6}$$

As a result of the invariance we get

$$\begin{aligned} \mathcal{W} =& \mathcal{W}(\mathbf{F}, \mathbf{Q}, \nabla \mathbf{Q}) \\ =& \mathcal{W}(\mathbf{F} \cdot \mathbf{O}^T, \mathbf{Q} \cdot \mathbf{O}^T, \nabla \mathbf{Q} \cdot \mathbf{O}^T), \end{aligned} \tag{7.7}$$

where the superscript T stands for the transpose tensors. Choosing $\mathbf{O} = \mathbf{Q}$ we have

$$\mathcal{W} = \mathcal{W}(\mathbf{F} \cdot \mathbf{Q}^T, \mathbf{I}, \nabla \mathbf{Q} \cdot \mathbf{Q}^T). \tag{7.8}$$

Hereinafter $\mathbf{I}$ is the unit tensor. This choice is possible as $\mathbf{O}$ can be any proper orthogonal tensor, so it can also coincide with $\mathbf{Q}$ given in any point. On the other hand Eq. (7.8) verifies the principle of material frame-indifference.

Note that $\frac{\partial \mathbf{Q}}{\partial X_k} \cdot \mathbf{Q}^T$ is a skew tensor. So it can be represented as

$$\frac{\partial \mathbf{Q}}{\partial X_k} \cdot \mathbf{Q}^T = -\mathbf{k}_k \times \mathbf{I} \tag{7.9}$$

through an axial vector $\mathbf{k}_k$. Here $\times$ stands for the cross product. Thus, the third-order tensor $\nabla \mathbf{Q} \cdot \mathbf{O}^T$ has the form

$$\nabla \mathbf{Q} \cdot \mathbf{O}^T = -\mathbf{K} \times \mathbf{I}, \quad \mathbf{K} = \mathbf{i}_k \otimes \mathbf{k}_k. \tag{7.10}$$

Note that in (7.10) we introduce the cross-product between two second-order tensors. For diads it was introduced by Gibbs, see (Wilson, 1901, p. 281), as follows

$$(\mathbf{a} \otimes \mathbf{b}) \times (\mathbf{c} \otimes \mathbf{d}) = \mathbf{a} \otimes (\mathbf{b} \times \mathbf{c}) \otimes \mathbf{d}$$

and can be easily extended for tensors of any order, see Eremeyev et al (2018).

Using Gibbsian cross operation $(\ldots)_\times$ introduced again by Gibbs (Wilson, 1901, p. 275), we get the formula

$$\mathbf{K} = \frac{1}{2}\mathbf{i}_k \otimes \left(\frac{\partial \mathbf{Q}}{\partial X_k} \cdot \mathbf{Q}^T \right)_\times . \tag{7.11}$$

Hereinafter for any second-order tensor $\mathbf{T} = T_{mn}\mathbf{i}_m \otimes \mathbf{i}_n$ the notation $\mathbf{T}_\times$ denotes the vectorial invariant of $\mathbf{T}$ defined as follows $\mathbf{T}_\times = T_{mn}\mathbf{i}_m \times \mathbf{i}_n$.

As a result, the strain energy density depends on two natural strain measures $\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T$ and $\mathbf{K}$, see, e.g., Pietraszkiewicz and Eremeyev (2009),

$$\mathcal{W} = \mathcal{W}(\mathbf{E}, \mathbf{K}). \tag{7.12}$$

Various examples of the micropolar constitutive equations can be found in the literature, see, e.g., Eringen (1999); Eremeyev and Pietraszkiewicz (2012, 2016).

7.3 Principle of Virtual Work

In order to formulate the virtual work principle we consider the first variation of the energy functional

$$\mathcal{E} = \int_V \mathcal{W} \mathrm{d}V,$$

where V is the micropolar body volume. Calculating $\delta\mathcal{E}$ we can find the consistent form of the work $\delta\mathcal{A}$ of external loads. We introduce first the variations of translations

$$\mathbf{u} = \delta \mathbf{x}. \tag{7.13}$$

In order to introduce the variation of rotations we consider $\delta\mathbf{d}_k$. As $\mathbf{d}_k$ are unit vectors, that is $\mathbf{d}_m \cdot \mathbf{d}_n = \delta_{mn}$, we have that $\delta\mathbf{d}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \delta\mathbf{d}_n = 0$, and

$$\delta\mathbf{d}_1 \cdot \mathbf{d}_1 = 0, \quad \delta\mathbf{d}_2 \cdot \mathbf{d}_2 = 0, \quad \delta\mathbf{d}_3 \cdot \mathbf{d}_3 = 0.$$

As a result, $\delta\mathbf{d}_k$, $k = 1, 2, 3$, can be represented through the same vector $\boldsymbol{\psi}$

$$\delta\mathbf{d}_k = \boldsymbol{\psi} \times \mathbf{d}_k. \tag{7.14}$$

From (7.14) it follows that

$$\delta\mathbf{Q} = -\mathbf{Q} \times \boldsymbol{\psi}.$$

Note that unlike $\mathbf{u}$, vector $\boldsymbol{\psi}$ does not coincide with a variation of any vector, in general.

Calculating $\delta\mathcal{W}$ we get

$$\delta\mathcal{W} = \frac{\partial \mathcal{W}}{\partial \mathbf{E}} : \delta\mathbf{E} + \frac{\partial \mathcal{W}}{\partial \mathbf{K}} : \delta\mathbf{K},$$

where ":" stands for the scalar product in the space of second-order tensors, for example, $\mathbf{T} : \mathbf{E} = \mathrm{tr}\,(\mathbf{T} \cdot \mathbf{E}^T)$, tr is the trace operator, and

$$\begin{aligned}\delta\mathbf{E} =& \delta(\mathbf{F}\cdot\mathbf{Q}^T) = \delta\mathbf{F}\cdot\mathbf{Q}^T + \mathbf{F}\cdot\delta\mathbf{Q}^T\\ =& \nabla\mathbf{u}\cdot\mathbf{Q}^T + \mathbf{F}\cdot(\boldsymbol{\psi}\times\mathbf{I})\cdot\mathbf{Q}^T, \qquad (7.15)\\ \delta\mathbf{K} =& \nabla\boldsymbol{\psi}\cdot\mathbf{Q}^T. \qquad (7.16)\end{aligned}$$

For the derivation here we used the relations

$$\delta\mathbf{Q} = \mathbf{D}_k\otimes\delta\mathbf{d}_k = \mathbf{D}_k\otimes\mathbf{d}_k\times\boldsymbol{\psi} = -\mathbf{Q}\times\boldsymbol{\psi}, \quad (\mathbf{Q}\times\boldsymbol{\psi})^T = -\boldsymbol{\psi}\times\mathbf{Q}^T,$$

see Eremeyev and Zubov (1994); Eremeyev et al (2013) for details. So we have

$$\delta\mathcal{W} = \left(\frac{\partial\mathcal{W}}{\partial\mathbf{E}}\cdot\mathbf{Q}\right):\nabla\mathbf{u} + \left(\frac{\partial\mathcal{W}}{\partial\mathbf{K}}\cdot\mathbf{Q}\right):\nabla\boldsymbol{\psi} + \left(\frac{\partial\mathcal{W}}{\partial\mathbf{E}}\cdot\mathbf{Q}\right):(\mathbf{F}\times\boldsymbol{\psi}). \quad (7.17)$$

Introducing the first Piola–Kirchhoff stress $\mathbf{T}$ and couple stress $\mathbf{M}$ tensors by the formulae

$$\mathbf{T} = \frac{\partial\mathcal{W}}{\partial\mathbf{E}}\cdot\mathbf{Q}, \quad \mathbf{M} = \frac{\partial\mathcal{W}}{\partial\mathbf{K}}\cdot\mathbf{Q}, \qquad (7.18)$$

we transform $\delta\mathcal{W}$ into the compact form

$$\delta\mathcal{W} = \mathbf{T}:\nabla\mathbf{u} + \mathbf{M}:\nabla\boldsymbol{\psi} + \mathbf{T}:(\mathbf{F}\times\boldsymbol{\psi}).$$

Calculating the first variation of the energy functional with the use of the integration by parts we get

$$\begin{aligned}\delta\mathcal{E} =& \int_V \delta\mathcal{W}\,\mathrm{d}V\\ =& -\int_V \left[(\nabla\cdot\mathbf{T})\cdot\mathbf{u} + \left(\nabla\cdot\mathbf{M} + (\mathbf{F}^T\cdot\mathbf{T})_\times\right)\cdot\boldsymbol{\psi}\right]\,\mathrm{d}V\\ &+ \int_{\partial V}(\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{u} + \mathbf{n}\cdot\mathbf{M}\cdot\boldsymbol{\psi})\,\mathrm{d}S. \qquad (7.19)\end{aligned}$$

Here $\mathbf{n}$ is the vector of outer unit normal to the boundary ∂V. The form of $\delta\mathcal{E}$ dictates the possible consistent expression of the external loadings work

$$\mathcal{A} = \int_V(\mathbf{f}\cdot\mathbf{u} + \mathbf{m}\cdot\boldsymbol{\psi})\,\mathrm{d}V + \int_{\partial V}(\boldsymbol{\phi}\cdot\mathbf{u} + \boldsymbol{\mu}\cdot\boldsymbol{\psi})\,\mathrm{d}S. \qquad (7.20)$$

In (7.20) $\mathbf{f}$ and $\boldsymbol{\phi}$ are external forces given in the volume and on its boundary, respectively, whereas $\mathbf{m}$ and $\boldsymbol{\mu}$ are external volumetric and surface couples (moments).

Finally, the virtual work principle takes the following form

$$\begin{aligned}\delta\mathcal{E} - \delta\mathcal{A} =& \int_V\left[-(\nabla\cdot\mathbf{T} - \mathbf{f})\cdot\mathbf{u} - \left(\nabla\cdot\mathbf{M} + (\mathbf{F}^T\cdot\mathbf{T})_\times - \mathbf{m}\right)\cdot\boldsymbol{\psi}\right]\,\mathrm{d}V\\ &+ \int_{\partial V}[(\mathbf{n}\cdot\mathbf{T} - \boldsymbol{\phi})\cdot\mathbf{u} + (\mathbf{n}\cdot\mathbf{M} - \boldsymbol{\mu})\cdot\boldsymbol{\psi}]\,\mathrm{d}S = 0. \qquad (7.21)\end{aligned}$$

Another method of derivation of (7.21) is presented in Pietraszkiewicz and Eremeyev (2009). Considering admissible variations, from (7.21) it follows the equilibrium equations and the corresponding natural boundary conditions.

For example, when the translations and rotations are both fixed on ∂V we have that

$$\mathbf{u} = \mathbf{0}, \quad \boldsymbol{\psi} = \mathbf{0} \quad \mathbf{X} \in \partial V \tag{7.22}$$

and the surface integral in (7.21) vanishes. So (7.22) play a role of incremental kinematic boundary conditions in the micropolar elasticity. Obviously, there is a straightforward correspondence between (7.22) and standard kinematic relations

$$\mathbf{x} = \mathbf{x}_0, \quad \mathbf{Q} = \mathbf{Q}_0 \quad \mathbf{X} \in \partial V,$$

where $\mathbf{x}_0$ and $\mathbf{Q}_0$ are given vector and tensor-valued functions, $\mathbf{Q}_0^T = \mathbf{Q}_0^{-1}$.

If $\mathbf{u}$ and $\boldsymbol{\psi}$ do not vanish on ∂V from (7.21) we have

$$\int_{\partial V} [(\mathbf{n} \cdot \mathbf{T} - \boldsymbol{\phi}) \cdot \mathbf{u} + (\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu}) \cdot \boldsymbol{\psi}] \, \mathrm{d}S = 0. \tag{7.23}$$

Equation (7.23) constitutes a weak form of a natural boundary conditions. In particular, if $\mathbf{u}$ and $\boldsymbol{\psi}$ are arbitrary, Eq. (7.23) results in the natural static boundary conditions

$$\mathbf{n} \cdot \mathbf{T} = \boldsymbol{\phi}, \quad \mathbf{n} \cdot \mathbf{M} = \boldsymbol{\mu}. \tag{7.24}$$

In what follows we consider a case intermediate between (7.22) and (7.24). In other word we will consider kinematic constraints that are relations between $\mathbf{u}$ and $\boldsymbol{\psi}$ given on ∂V or its part.

7.4 Non-holonomic Kinematic Boundary Conditions

In the analytical mechanics are known various incremental constraints on generalized variables. These constraints can be holonomic or non-holonomic, see, e.g., Lurie (2001). First, we formulate an incremental boundary condition as a linear relations between $\mathbf{u}$ and $\boldsymbol{\psi}$

$$\mathbf{L}_1 \cdot \mathbf{u} + \mathbf{L}_2 \cdot \boldsymbol{\psi} = \mathbf{0}, \tag{7.25}$$

where second-order tensors $\mathbf{L}_1$ and $\mathbf{L}_2$ depend on $\mathbf{x}$, $\mathbf{Q}$, and their spatial gradients, in general. Let us note that (7.25) does not correspond to any constrain written in terms of $\mathbf{x}$ and $\mathbf{Q}$, in general. So we call (7.25) *non-holonomic boundary conditions*. Such incremental constraints are known in the analytical mechanics, see, e.g., Lurie (2001). Such constraints can be applied using Lagrange multiplier technique or through the direct solving of (7.25) with respect to one on the variables. The conservatives conditions for micropolar solids and rigid bodies including action of external moments were discussed by Eremeyev and Zubov (1994); Zelenina and Zubov (2000).

For example, assuming that $\mathbf{L}_2$ is invertible from (7.25) we get that

$$\boldsymbol{\psi} = -\mathbf{L}_2^{-1} \cdot \mathbf{L}_1 \cdot \mathbf{u}.$$

Substituting this into (7.23) we have

$$\mathbf{n} \cdot \mathbf{T} - (\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu}) \cdot \mathbf{L}_2^{-1} \cdot \mathbf{L}_1 = \boldsymbol{\phi}. \tag{7.26}$$

Obviously, the using of the Lagrange multipliers technique gives the same result. Indeed, introducing a Lagrange multiplier $\boldsymbol{\lambda}$ we add to (7.23) the expression

$$\int_{\partial V} \boldsymbol{\lambda} \cdot (\mathbf{L}_1 \cdot \mathbf{u} + \mathbf{L}_2 \cdot \boldsymbol{\psi}) \, \mathrm{d}S = 0.$$

So we get

$$\int_{\partial V} [(\mathbf{n} \cdot \mathbf{T} - \boldsymbol{\phi}) \cdot \mathbf{u} + (\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu}) \cdot \boldsymbol{\psi} + \boldsymbol{\lambda} \cdot (\mathbf{L}_1 \cdot \mathbf{u} + \mathbf{L}_2 \cdot \boldsymbol{\psi})] \, \mathrm{d}S = 0. \tag{7.27}$$

From (7.27) it follows that

$$\mathbf{n} \cdot \mathbf{T} - \boldsymbol{\phi} + \boldsymbol{\lambda} \cdot \mathbf{L}_1 = \mathbf{0}, \tag{7.28}$$
$$\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu} + \boldsymbol{\lambda} \cdot \mathbf{L}_2 = \mathbf{0}. \tag{7.29}$$

Assuming again that $\mathbf{L}_2$ is invertible we exclude $\boldsymbol{\lambda}$ from (7.29)

$$\boldsymbol{\lambda} = -(\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu}) \cdot \mathbf{L}_2^{-1}. \tag{7.30}$$

Finally, substituting $\boldsymbol{\lambda}$ into (7.28) we get (7.26).

Let us consider particular cases of (7.25). Obviously, Eqs. (7.22) present the trivial case of (7.25). Indeed, (7.22) follows from (7.25) with $\mathbf{L}_1 = \mathbf{L}_2 = \mathbf{0}$. Another case is sliding with free rotations at the boundary, $\mathbf{n} \cdot \mathbf{u} = 0$, $\boldsymbol{\psi}$ has arbitrary values. This case corresponds to $\mathbf{L}_1 = \mathbf{n} \otimes \mathbf{n}$, $\mathbf{L}_2 = \mathbf{0}$. Eq. (7.23) results in the following static boundary conditions

$$(\mathbf{n} \cdot \mathbf{T} - \boldsymbol{\varphi}) \cdot \mathbf{A} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{M} = \boldsymbol{\mu},$$

where $\mathbf{A} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$.

Another assumption leading to (7.25) can be motivated as follows. Let us assume that the material particles behave as rolling stones in the vicinity of the body boundary. Assuming the same relation between $\mathbf{u}$ and $\boldsymbol{\psi}$ as for linear and angular velocities of a rigid body rolling on a surface we get

$$\mathbf{u} = r\mathbf{n} \times \boldsymbol{\psi}. \tag{7.31}$$

Here r plays a role of a characteristic length of a micropolar medium. For example, it is the distance between the mass center of a material particle and its boundary. Eq. (7.31) means that $\mathbf{L}_1 = \mathbf{I}$ and $\mathbf{L}_2 = r\mathbf{n} \times \mathbf{I}$. Note that here $\mathbf{L}_2$ is a singular tensor whereas $\mathbf{L}_1$ is invertible. This constraint results in the following natural boundary condition

$$\mathbf{n} \cdot \mathbf{M} - r\mathbf{n} \cdot \mathbf{T} \times \mathbf{n} = \boldsymbol{\mu} - r\boldsymbol{\phi} \times \mathbf{n}. \tag{7.32}$$

Let us consider more general case of non-holonomic surface constraints. We use the following linear relation

$$\mathbf{L}_1 \cdot \mathbf{u} + \mathbf{L}_2 \cdot \boldsymbol{\psi} + \mathbf{L}_3 : \nabla \mathbf{u} + \mathbf{L}_4 : \nabla \boldsymbol{\psi} = \mathbf{0}, \quad \mathbf{X} \in \partial V. \tag{7.33}$$

Here $\mathbf{L}_3$ and $\mathbf{L}_4$ are third-order tensors depending on $\mathbf{x}$, $\mathbf{Q}$ and their gradients. Eq. (7.33) is a system of first-order partial differential equations which first integral can be found through the characteristic technique, see, e.g., Arnold (2004). Instead we again use the Lagrange multiplier approach. Now instead of (7.23) we have

$$\begin{aligned}\int_{\partial V} &[(\mathbf{n} \cdot \mathbf{T} - \boldsymbol{\phi}) \cdot \mathbf{u} + (\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu}) \cdot \boldsymbol{\psi} \\ &+ \boldsymbol{\lambda} \cdot (\mathbf{L}_1 \cdot \mathbf{u} + \mathbf{L}_2 \cdot \boldsymbol{\psi} + \mathbf{L}_3 : \nabla \mathbf{u} + \mathbf{L}_4 : \nabla \boldsymbol{\psi})] \, \mathrm{d}S = 0.\end{aligned} \tag{7.34}$$

In order to transform (7.34) using the integration by parts we represent ∇ as a sum of the surface gradient and normal derivative

$$\nabla = \nabla_s + \mathbf{n} \frac{\partial}{\partial n},$$

where $\frac{\partial}{\partial n}$ is the derivative with respect to the coordinate normal to ∂V. Using the surface divergence theorem (Eremeyev et al, 2018) we apply the following integration by parts formula

$$\int_A \mathbf{Y} : \nabla_s \mathbf{y} \, \mathrm{d}S = \int_{\partial A} \boldsymbol{\nu} \cdot \mathbf{Y} \cdot \mathbf{y} \, \mathrm{d}s - \int_A [(\nabla_s \cdot \mathbf{Y}) \cdot \mathbf{y} + 2H \mathbf{n} \cdot \mathbf{Y} \cdot \mathbf{y}] \, \mathrm{d}S \tag{7.35}$$

for any fields $\mathbf{Y}$ and $\mathbf{y}$. Here $2H = -\nabla_s \cdot \mathbf{n}$ is the mean curvature of a surface A with the contour ∂A, and $\boldsymbol{\nu}$ is the normal to ∂A such that $\boldsymbol{\nu} \cdot \mathbf{n} = 0$, see Fig 7.1.

With (7.35) we have

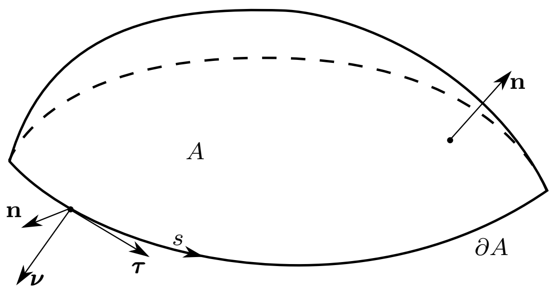

Fig. 7.1 For the surface divergence theorem: surface A with contour ∂A. The unit vectors $\mathbf{n}$, $\boldsymbol{\nu}$, and $\boldsymbol{\tau}$ are defined along ∂A. Here $\mathbf{n}$ is the unit vector normal to A, $\boldsymbol{\tau}$ is the unit vector tangent to ∂A, whereas $\boldsymbol{\nu}$ is the unit vector lying in the tangent plane to A and normal to ∂A.

$$\int_{\partial V} \boldsymbol{\lambda} \cdot \mathbf{L}_3 : \nabla \mathbf{u} \, \mathrm{d}S = \int_{\partial V} \boldsymbol{\lambda} \cdot \mathbf{L}_3 : \left(\mathbf{n} \frac{\partial \mathbf{u}}{\partial n} \right) \mathrm{d}S$$
$$- \int_{\partial V} \left[\nabla \cdot (\boldsymbol{\lambda}_s \cdot \mathbf{L}_3) + 2H \mathbf{n} \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_3) \right] \cdot \mathbf{u} \, \mathrm{d}S, \tag{7.36}$$
$$\int_{\partial V} \boldsymbol{\lambda} \cdot \mathbf{L}_4 : \nabla \boldsymbol{\psi} \, \mathrm{d}S = \int_{\partial V} \boldsymbol{\lambda} \cdot \mathbf{L}_4 : \left(\mathbf{n} \frac{\partial \boldsymbol{\psi}}{\partial n} \right) \mathrm{d}S$$
$$- \int_{\partial V} \left[\nabla_s \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_4) + 2H \mathbf{n} \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_4) \right] \cdot \boldsymbol{\psi} \, \mathrm{d}S. \tag{7.37}$$

Here we assumed that $\partial\partial V = \emptyset$.

With (7.36) and (7.37) Eq. (7.34) results in

$$(\boldsymbol{\lambda} \otimes \mathbf{n}) : \mathbf{L}_3 = \mathbf{0}, \tag{7.38}$$
$$(\boldsymbol{\lambda} \otimes \mathbf{n}) : \mathbf{L}_4 = \mathbf{0}, \tag{7.39}$$
$$\mathbf{n} \cdot \mathbf{T} - \boldsymbol{\phi} + \boldsymbol{\lambda} \cdot \mathbf{L}_1 = \nabla_s \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_3) + 2H \mathbf{n} \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_3), \tag{7.40}$$
$$\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu} + \boldsymbol{\lambda} \cdot \mathbf{L}_2 = \nabla_s \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_4) + 2H \mathbf{n} \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_4). \tag{7.41}$$

From (7.38) and (7.39) it follows that the curvature dependent terms in (7.40) and (7.41) are vanishing. So we get

$$\mathbf{n} \cdot \mathbf{T} - \boldsymbol{\phi} + \boldsymbol{\lambda} \cdot \mathbf{L}_1 = \nabla_s \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_3), \tag{7.42}$$
$$\mathbf{n} \cdot \mathbf{M} - \boldsymbol{\mu} + \boldsymbol{\lambda} \cdot \mathbf{L}_2 = \nabla_s \cdot (\boldsymbol{\lambda} \cdot \mathbf{L}_4). \tag{7.43}$$

For simplicity let us restrict ourselves by the case of $\mathbf{L}_3$ and $\mathbf{L}_4$ which have the following property

$$(\mathbf{a} \otimes \mathbf{n}) : \mathbf{L}_3 = \mathbf{0}, \quad (\mathbf{a} \otimes \mathbf{n}) : \mathbf{L}_4 = \mathbf{0} \quad \forall \, \mathbf{a}. \tag{7.44}$$

Then (7.38) and (7.39) vanish identically. The properties (7.44) means that the non-holonomic kinematic boundary constraint (7.34) takes the form

$$\mathbf{L}_1 \cdot \mathbf{u} + \mathbf{L}_2 \cdot \boldsymbol{\psi} + \mathbf{L}_3 : \nabla_s \mathbf{u} + \mathbf{L}_4 : \nabla_s \boldsymbol{\psi} = \mathbf{0}, \quad \mathbf{X} \in \partial V. \tag{7.45}$$

As an example of (7.33) or (7.45) let us recall the boundary conditions used in the micropolar hydrodynamics, see Migoun and Prokhorenko (1984); Łukaszewicz (1999), where the following relations between angular $\boldsymbol{\omega}$ and linear $\mathbf{v}$ velocity was discussed

$$\boldsymbol{\omega} = \frac{\alpha}{2} \nabla \times \mathbf{v}.$$

Here α is a material parameter, $0 \le \alpha \le 1$. Note that the constraint $\boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{v}$ is used in the theory of couple stresses (Cosserat continuum with constrained rotations or Cosserat pseudocontinuum), see Nowacki (1986).

Assuming similar relation between virtual rotations and translations we get

$$\boldsymbol{\psi} = \frac{\alpha}{2} \nabla \times \mathbf{u}, \tag{7.46}$$

which is a particular case of (7.33). Indeed, (7.46) follows from (7.33) with

$$\mathbf{L}_1 = \mathbf{0}, \quad \mathbf{L}_2 = \mathbf{I}, \quad \mathbf{L}_3 = \tfrac{1}{2}\alpha\mathbf{I}\times\mathbf{I}, \quad \mathbf{L}_4 = \mathbf{0}.$$

Here we used the following identity:

$$(\mathbf{I}\times\mathbf{I}) : \nabla\mathbf{u} = -\nabla\times\mathbf{u}.$$

From the physical point of view (7.46) means that the micro-rotations depends on macro-rotations on the boundary. In other words with (7.46) we model interactions between the medium and its boundary.

Thus, (7.38) and (7.39) result in one constraint

$$\begin{aligned}(\boldsymbol{\lambda}\otimes\mathbf{n}) : \mathbf{L}_3 =& \mathbf{n}\cdot(\boldsymbol{\lambda}\cdot\mathbf{L}_3) = \frac{\alpha}{2}\mathbf{n}\cdot(\boldsymbol{\lambda}\cdot\mathbf{I}\times\mathbf{I}) = \frac{\alpha}{2}\mathbf{n}\cdot(\boldsymbol{\lambda}\times\mathbf{I}) = \frac{\alpha}{2}\mathbf{n}\cdot(\mathbf{I}\times\boldsymbol{\lambda})\\ =&\frac{\alpha}{2}\mathbf{n}\times\boldsymbol{\lambda} = \mathbf{0},\end{aligned}$$

which means that $\boldsymbol{\lambda}$ is normal to ∂V: $\boldsymbol{\lambda} = \Lambda\mathbf{n}$.

Eq. (7.41) transforms into

$$\mathbf{n}\cdot\mathbf{M} - \boldsymbol{\mu} + \boldsymbol{\lambda} = \mathbf{0},$$

so one easily finds $\boldsymbol{\lambda}$ from it

$$\boldsymbol{\lambda} = \Lambda\mathbf{n}, \quad \Lambda = -(\mathbf{n}\cdot\mathbf{M} - \boldsymbol{\mu})\cdot\mathbf{n}. \tag{7.47}$$

With (7.47) and identities

$$\nabla_s\cdot(\boldsymbol{\lambda}\cdot\mathbf{L}_3) = \frac{\alpha}{2}\nabla_s\cdot(\boldsymbol{\lambda}\cdot\mathbf{I}\times\mathbf{I}) = \frac{\alpha}{2}\nabla_s\times\boldsymbol{\lambda},$$

$$\nabla_s\times(\Lambda\mathbf{n}) = \nabla_s\Lambda\times\mathbf{n},$$

we exclude $\boldsymbol{\lambda}$ from Eq. (7.40), which takes the following form

$$\mathbf{n}\cdot\mathbf{T} + \frac{\alpha}{2}\nabla_s\left[(\mathbf{n}\cdot\mathbf{M} - \boldsymbol{\mu})\cdot\mathbf{n}\right]\times\mathbf{n} = \boldsymbol{\phi}, \tag{7.48}$$

which plays a role of the natural boundary condition complementary to (7.46).

Let us note that (7.48) can be also derived without using of Lagrange multiplier technique. To this end one have to transform the variational equation

$$\int_{\partial V}\left[(\mathbf{n}\cdot\mathbf{T} - \boldsymbol{\phi})\cdot\mathbf{u} + \frac{\alpha}{2}\left(\mathbf{n}\cdot\mathbf{M} - \boldsymbol{\mu}\right)\cdot(\nabla\times\mathbf{u})\right]\,\mathrm{d}S = 0.$$

with integration by parts.

7.5 Conclusions

Within the nonlinear micropolar elasticity we introduced the non-holonomic kinematic boundary conditions. These conditions are formulated as linear relations between virtual translations and rotations. In other words we presented new incremental kinematic boundary conditions. The corresponding natural static boundary conditions are also derived. As for the derivation we used the principle of virtual work, the discussed results extend the class of possible boundary conditions also for inelastic micropolar materials such as considered by Altenbach and Eremeyev (2014). Let us note that, though the boundary condition for the translation field and its natural static counterpart is physically clear, for microrotation there is no general agreement on the vorticity of complex materials on the boundary and on the type of the corresponding boundary condition for the field of microrotation.

It is worth to underline that after Sedov (1965) and Germain (1973a,b) this variational approach became a powerful tool for modelling of media with microstructure, see also discussion by dell'Isola et al (2017); Eugster and dell'Isola (2017, 2018a,b). So in a similar way non-holonomic boundary conditions can be introduced for other generalized media, such as strain gradient elasticity. For the virtual work and the least action principles in strain gradient solids and fluids we refer to Auffray et al (2015); Abali et al (2015, 2017); Eremeyev and Altenbach (2014); Eremeyev (2016) and the reference therein. In particular, such boundary equations could be useful for modelling of the behaviour of complex fluids in the vicinity of a free surface and/or interface.

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