Chapter 33
How the Properties of Pantographic Elementary Lattices Determine the Properties of Pantographic Metamaterials

Emilio Turco

Abstract In this paper we describe a three-scales homogenization process which we use to determine a macroscopic model for pantographic metamaterials. The smallest scale refers to the length at which the considered deformable mechanical system can be modeled as a Cauchy's continuum. Of course, at this scale, its geometry is rather complex. The meso-scale refers to a length at which the system can be modeled as a Hencky-type discrete system constituted by masses interconnected by extensional and rotational springs. At macro-scale the model to be used is a generalized plate whose deformation energy depends on geodesic curvature. While the direct identification from the smallest scale to the macro-scale seems rather difficult, the identification from smallest scale to meso-scale can be successfully obtained. The geometrical properties, along with Young and Poisson coefficients of the used isotropic material, at the smallest scale determine the extensional and rotational stiffnesses to be used at the meso-scale. On the other hand, the Piola-type identification process allows us to determine the stiffnesses of the macroscopic generalized plate model, *via* a simple asymptotic expansion. We have observed that this process is valid in both cases when the smallest scale is of the order of microns and when the smallest scale is of the order of tenth of millimeters. Some experimental and numerical results supporting this statement are exhibited.

Keywords: Mechanics of (meta)materials · Lagrangian models · Micro-meso and meso-macro parameters identification

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33.1 Introduction

In technical literature there are not clearly established homogenization methods (however, see the attempts Franciosi et al, 2018; Karamooz et al, 2014; Karamooz and Kadkhodaei, 2015; Saeb et al, 2016; Abali et al, 2016) leading to higher gradient continua starting from mechanical systems showing complex geometrical microstructures and highly inhomogeneous stiffnesses fields. Therefore, we have decided to use a three-scales homogenization process building the macroscopic continuum model for pantographic metamaterials in a three-steps process. The study of this class of mechanical systems has a great importance in design of metamaterials, see, *e.g.*, , the review (Barchiesi et al, 2018c). Some identification results are however available, see, for instance, Barchiesi and Placidi (2017); Placidi et al (2017b); Yang et al (2018); Placidi et al (2015, 2017a); Placidi and El Dhaba (2017); Placidi et al (2017c); Barchiesi et al (2018b); Lekszycki et al (2018).

The analysis becomes more difficult when the considered pantographic specimens undergo damage and plasticity phenomena: the identification of macro-mechanical properties seems beyond actual state-of-the-art. However, the results found in Placidi et al (2018a,b) supply an important guidance for this more general identification procedure, as they give a possible target macro-model.

At the smallest scale the considered deformable body is, at least from the theoretical point of view, mechanically homogeneous but its geometry is involved, showing, see Fig. 33.1, large gaps and narrow material connections (pivots) in which deformation energy may be concentrate. At this length scale the standard three-dimensional Cauchy continuum model can be usefully introduced. Unfortunately due to the complex geometry and the expected deformation patterns, to use this modeling one needs heavy numerical codes, involving, for simple specimens, even several millions of degrees of freedom. Therefore, also with a view towards technological applications and having in mind some interesting optimization problems, there is a need to find so-called reduced order models.

However, a direct deduction algorithm, even using some simple formal perturbation methods, has not been yet clearly and satisfactorily developed. Therefore, it has been proposed to introduce a meso-scale where the deformable body is modeled by

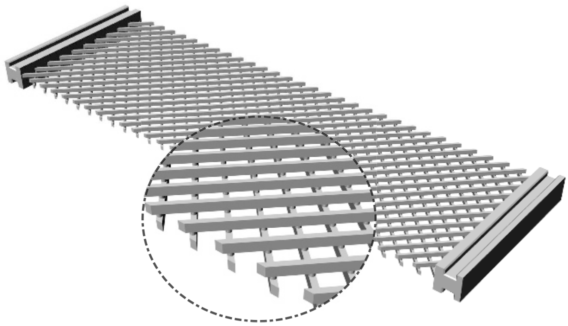

Fig. 33.1 Three-dimensional rendering of a pantographic lattice with a blow-up of a part.

means of a Hencky-type discrete Lagrangian system, see Turco et al (2016c) for the model and Turco et al (2016e,a,b) for the comparison, for some remarkable cases, between experiments and numerical simulations. This system is constituted by material points, in which one will concentrate masses when inertia has to be taken into account, interconnected by extensional and rotational springs, see Fig. 33.2. These springs will have deformation energies whose constitutive law can be linked with the geometrical and mechanical properties characterizing the system at the lowest scale. This identification is rather simple, at least from the theoretical point of view, and can be performed by using the methods already conjectured by Piola (dell'Isola et al, 2015, 2019) and developed by Hencky (1921) to obtain some estimates of the buckling load for *Elastica*.[1]

Finally, at the macro-scale the model which seems to us to be the most suitable is a generalized plate. In standard plate theories, the curvature is the only deformation measure due to the transverse displacement on the current configuration. Instead the true mechanical nature of pantographic sheets can be captured by a two-dimensional continuum only by introducing a deformation energy depending on so-called geodesic curvature (Steigmann and dell'Isola, 2015; Giorgio et al, 2016, 2015, 2017b, 2018) and on twisting of the material curves modeling the pantographic fibers.

Hencky-type discrete models, by using Piola's identification (see dell'Isola et al, 2016), based on the identification of finite differences with derivatives, immediately produces a macro-model which belongs to the just described class of generalized plates. Therefore even if the direct identification from the smallest scale to the macro-scale cannot be attained directly, the just described three steps process is rather successful, at least when limiting ourselves to numerical identifications. However the numerical tuning of constitutive parameters can be driven by some

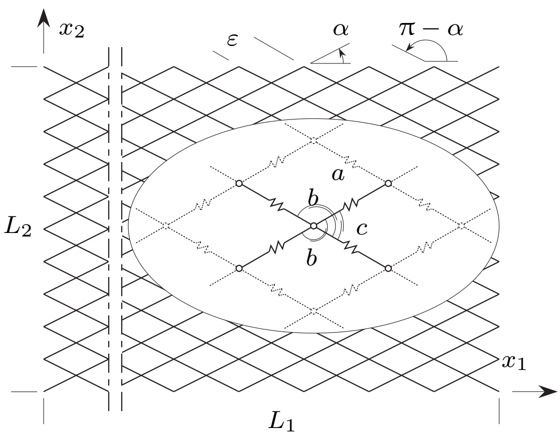

Fig. 33.2 Meso-mechanical model for the in-plane mechanical behavior of a pantographic lattice with non-orthogonal beams.

[1] The identification process is not so simple if we want to model the whole nonlinear mechanical behavior of the pantographic lattice by using, as we discuss later, only three parameters.

theoretical knowledge: indeed the main geometrical properties, together with the Young and Poisson coefficients of the Cauchy material used as a model at the smallest scale, can give an order of magnitude for the values of those extensional and rotational stiffnesses which are correct at the meso-scale.

The aforementioned general three-steps procedure is valid even if the lowest length scale has different values. Actually it has been experimentally proven that in both cases, when the smallest scale is of the order of microns and when the smallest scale is of the order of tenth of milli-meters, it can be applied successfully. Therefore the same numerical procedure for getting predictive models can be usefully applied in these two different cases simply varying a few constitutive parameters. We believe however that, when going down to lower length scales, some quantum effects may arise and in such cases the applicability of Cauchy models may be disputable.

In this work, after this brief introduction, we firstly describe in Sect. 33.2 the pantographic micro-structure starting from the presentation of the fundamental brick or unit. Successively, in Sect. 33.3 we propose a strategy to determine the triplets of stiffnesses which completely define the Lagrangian meso-mechanical model and we use, in Sect. 33.4, the obtained results to estimate the macro-mechanical parameters following the identification described in dell'Isola et al (2016). Some concluding remarks, in Sect. 33.5 close the paper along with some future challenges.

33.2 Description of Pantographic Units used to form Pantographic Micro-structures

The crucial feature of pantographic modules consists in the presence of pivots as interconnecting structural elements to link different, rectilinear or curvilinear, beams. Figure 33.3, for instance, is a shot of a pantographic lattice with orthogonal fibers inside a testing machine immediately before an elongation test. We remark that the obtained assembly of beams is, generally speaking, a lattice and not a truss. However, equivalent truss structures have been conceived having the same behavior as pantographic modules, see, *e.g.*, , Seppecher et al (2011); Turco et al (2017a); Khakalo and Niiranen (2018). The presence of pivots in a specifically designed geometrical pattern induce the existence of so-called *floppy modes*, *i.e.*, deformation modes whose associated deformation energy is vanishing or very small when compared

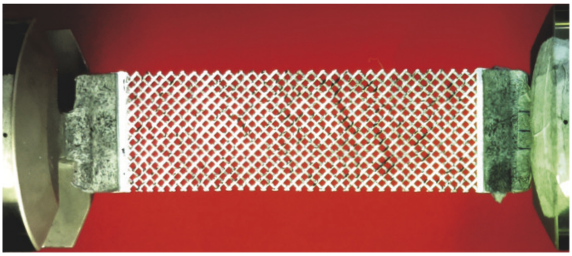

Fig. 33.3 Shot of a pantographic lattice with orthogonal fibers inside a testing machine immediately before an elongation test.

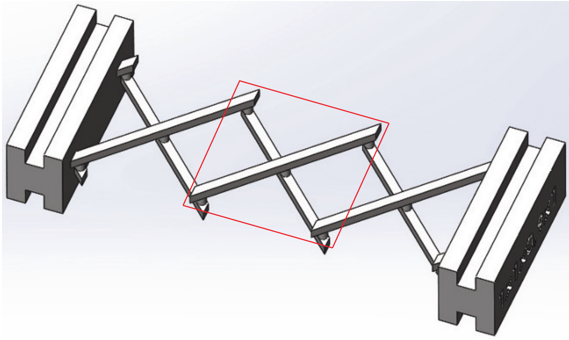

Fig. 33.4 Pantographic lattice formed by three units (the central unit is highlighted by means of a red box).

with the other deformation modes. The existence of floppy modes implies that at a macro-level the deformation energy exhibits some singularities and therefore some new mathematical tools were needed to start studying the well-posed problems for pantographic metamaterials, see, *e.g.*, , Boutin et al (2017).

Looking at Fig. 33.4 we distinguish three units each one formed by beams having rectangular cross-section and pivots, that is cylinders with circular cross-section. An exploded drawing of, for example, the central unit is reported in Fig. 33.5 highlighting the dimensions of the ℓ_b long beam with rectangular $b_b \times h_b$ cross-section and those of the circular cross-section cylinder (d_c and h_c are the diameter and the height, respectively).

Although in this work we refer to orthogonal families of rectilinear beam, there are examples, which show some interesting peculiarities, of non-orthogonal rectilinear beams, see Turco et al (2017b), and curvilinear beams, see Scerrato et al (2016); Giorgio et al (2016).

The composition of complex structures by using pantographic modules allows for the design of

1. pantographic beams: they can be obtained by a sequence of units which growths in one direction, see, *e.g.*, , Barchiesi et al (2018a); Birsan et al (2012);
2. pantographic sheets: they can be obtained by a sequence of units which growths in two directions, see, *e.g.*, , dell'Isola et al (2016); Eremeyev et al (2018); Khakalo et al (2018); Eremeyev and dell'Isola (2018) and the reviews Spagnuolo and Andreaus (2018); Laudato and Di Cosmo (2018); Golaszewski et al (2018);
3. pantographic blocks: they can be obtained by a sequence of lattices which growths in one direction, see, *e.g.*, , dell'Isola et al (2018).

33.3 How the unit Properties Determine the Meso-stiffnesses

Once the geometry is perfectly described we need the parameters which describe the mechanical behavior of the used material. Several specimens were built using 3D

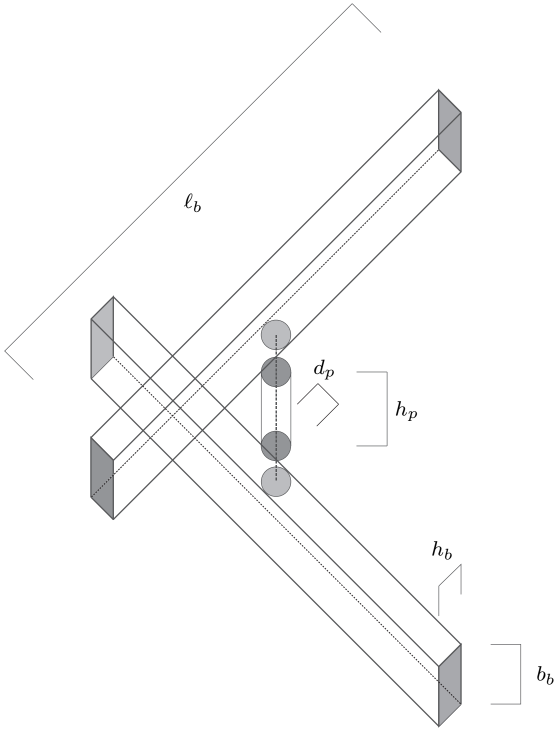

Fig. 33.5 Exploded drawing of a pantographic unit formed by two orthogonal beams with rectangular cross-section connected with a cylindrical pivot.

printing process both using as material the polyamide and aluminum or steel. Usual printing process are based on melting processes of pulverized material. This kind of process may alter the mechanical parameters of the used material. In addition, as we will discuss in the following, since it may introduce some hollows which surely could alter appreciably the mechanical parameters of the material. For the moment, we may start with the value of the Young modulus Y and the Poisson ratio ν given for the material used in the printing process. For example, the polyamide, following EN ISO 527, has $Y = 1700$ MPa and $\nu = 0.4$.

In Turco et al (2016c) a completely discrete Lagrangian model has been introduced to capture the in-plane mechanical response of pantographic lattice. It is based on three mechanical parameters for modeling the stretching, bending and shearing strain energy of the whole lattice[2]. The basic idea derives from the Hencky approach for modeling the bending strain energy. Roughly speaking, each beam is modeled by a series of rigid links and elastic joints (rotational springs of stiffness b). Increasing the number of the elastic joints, Hencky obtained estimates approaching to the bifurcation load of the *Elastica*. It is almost simple to prove, see Turco (2018); Turco et al (2018b), that using de Saint-Venant results for a beam under bending we might

[2] This idea has been enhanced, improving the bending energy by means of a h-refinement, in Turco et al (2018a,b).

estimate the parameter b form the geometry and the Young modulus. In practice, when the number of elastic joints is large enough[3], we have:

$$b = \frac{YJ}{\ell}\,, \quad \text{being} \quad J = \frac{b_b h_b^3}{12}\,. \tag{33.1}$$

In the same way we might estimate the spring stiffness a used to model the stretching strain energy. In formula:

$$a = \frac{YA}{\ell}\,, \quad \text{being} \quad A = b_b h_b\,. \tag{33.2}$$

The third parameter c is related to the torsional stiffness of the cylinder with circular cross-section or, briefly, of the pivot. Also in this case we can use the de Saint-Venant solution for the torsion to estimate c:

$$c = \frac{GI_p}{h_c}\,, \quad \text{being} \quad G = \frac{Y}{2(1+\nu)} \quad \text{and} \quad I_p = \frac{\pi d_p^4}{32}\,. \tag{33.3}$$

The aforementioned road to estimate the parameters is very simple, and therefore attractive, but it is based on hypotheses too far from the phenomenon which we want do describe. In particular:

1. de Saint-Venant results are accurate for beams and pivots long enough (respect to the dimensions of their cross-section); this is not verified for pantographic lattices where the ratios ℓ_b/h_b and h_p/d_p are approximatively, see again Fig. 33.5 for the meaning of the used symbols, 18.4 and 2.22, respectively;
2. the intrinsic porosity of the material obtained by 3D printing process reduces, often in a remarkable way, the Young modulus of the printed specimens; Fig. 33.6 shows a part of a beam using three different, and increasing, magnifications; it is almost clear the granular nature of the printed material, furthermore, micro X-ray computed tomography analysis show the presence of hollows of not negligible dimensions in the printed material;
3. the theoretical geometry is only an approximation of the true one, see again Fig. 33.6, however the presence of errors on the stiffness parameters gives stable results as is proved in Turco and Rizzi (2016);
4. in almost all the experiments pantographic lattices undergo large displacements, this is completely far from the de Saint-Venant results.

For all the aforementioned reasons the values suggested from the de Saint-Venant solutions can only be a starting point for an accurate enough estimation of the stiffness parameters of the discrete model. Refined values of these estimates can be derived comparing the results of numerical simulations[4] with those deriving from

[3] In Turco (2018) some quantitative results relative to the buckling load of an elastic beam are reported. Two and five rigid links estimate the buckling load with an error of 18.9% and 2.23%, respectively.

[4] Numerical simulations whose results are reported in Figs. 33.7 and 33.8 were performed by use an in-house made code which considers large displacements (but neglect material nonlinearities

Fig. 33.6 Blow-up sequence of a pantographic lattice part built by a 3D printing process using polyamide powder; the sequence clearly shows the granular nature of the specimen.

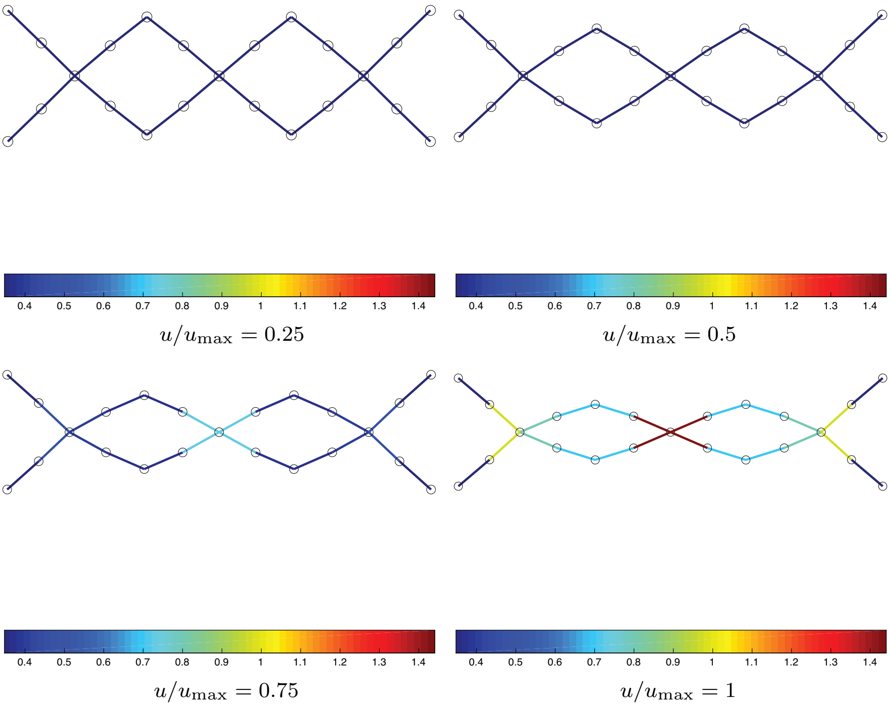

Fig. 33.7 Extensional test with controlled displacement u on the right side nodes: sequence of deformations for $u/u_{\max} = 0.25$, 0.5, 0.75 and 1 for a pantographic lattice formed by three units (colors show the strain energy level achieved on beams, small circles represent the nodes of the discrete model).

experiments following the methodologies reported in Turco (2017). In practice,

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and viscous effects) modeling the pantographic lattices as a set of rigid links and elastic joints to approximate the bending and shearing strain energy. The bending contributions are improved by adding an intermediate node between two close pivots (pivots and additional nodes are drawn in Fig. 33.7 using small circles). In addition, the model considers also the stretching energy, see Turco by choosing the triplet (a, b, c) which fits the experimental results, see Fig. 33.8, *i.e.*, the load-displacement curve and the deformations at some prefixed points of the curve, see Fig. 33.7 which reports the displacements for $u/u_{\max} = 0.25$, 0.5, 0.75 and 1 along with the strain energy level achieved on beams by means of colors for the pantograph depicted in Fig. 33.4. We remark that the nonlinearity of the force-displacement curve is only due to the large displacements attained both in the experiment and in its numerical simulation. Even if the viscoelastic effect is, in general, present in the polyamide, here we have not considered this effect in the numerical simulations since the experimental loading rate was designed to remove viscous phenomena.

In other words minimizing the discrepancy between the experiment and its numerical simulation in a least square fashion:[5]

$$\arg\min_{(a,b,c)} \left(\|\mathbf{n}(a,b,c) - \mathbf{m}\|^2 \right) , \tag{33.4}$$

being $\mathbf{n}$ and $\mathbf{m}$ the vector collecting the numerical and measured, in some experiment, response, respectively. We remark that the vector $\mathbf{m}$, and consequently $\mathbf{n}$, can collect different kind of information such as the measured force and displacements in a prefixed number of points of the structure both corresponding to an assigned displacement (if we consider a load-displacement curve obtained by assigning the displacement).

In addition, since experimental measurements surely include errors, for example those deriving from the instrumental precision, the least square formulation (33.4) can be made able to filter these errors by an additional term which imposes, by means a Lagrangian multiplier λ, a desired condition, in formula:[6]

$$\arg\min_{(a,b,c)} \left(\|\mathbf{n}(a,b,c) - \mathbf{m}\|^2 + \lambda^2 \|\mathbf{c}\|^2 \right) , \tag{33.5}$$

having used the constrain condition $\mathbf{c}$. In several cases to filter the errors in $\mathbf{m}$ it is enough to choose the solution which has the minimum norm corresponding to $\mathbf{c} = [a\, b\, c]^T$, see again Turco (2017).

Particularly useful for a fixed pantographic lattice is the plot of the load-displacement, such that reported in Fig. 33.9, which immediately suggests the values of the triplet (a, b, c) which best fit the experimental load-displacement curve.

et al (2018b) for more details. Numerical simulations try to reproduce the experimental tests whose results are displacement-controlled and the whole test reproduce quasi-static results.

[5] This method is also known as Levenberg-Marquardt algorithm.

[6] This method is also known as damped Levenberg-Marquardt algorithm.

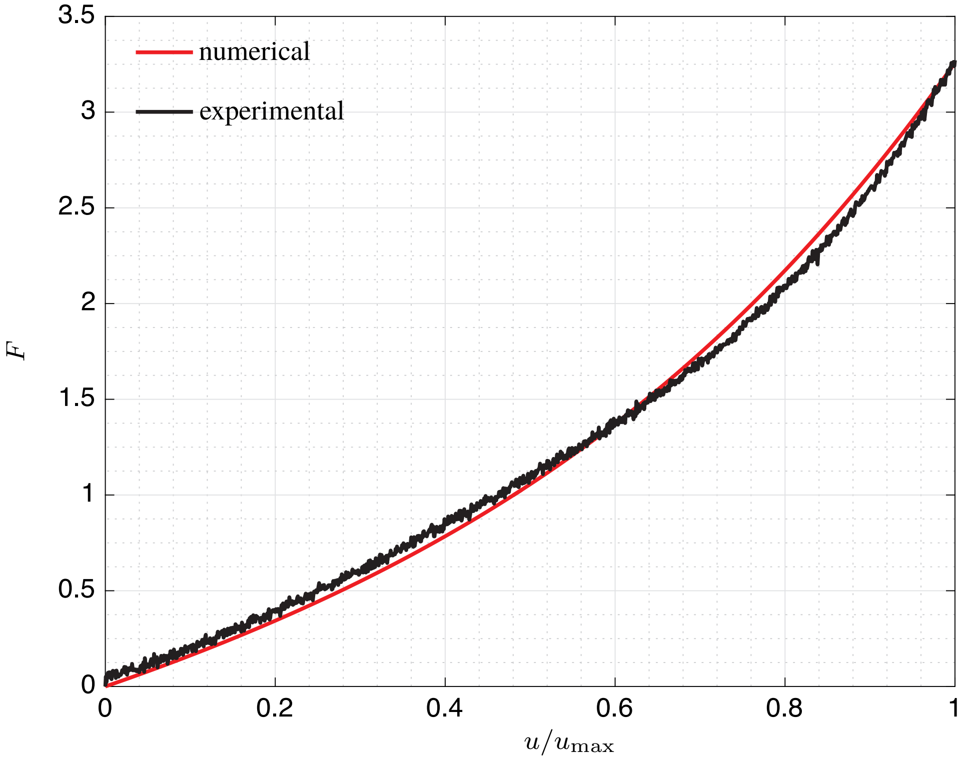

Fig. 33.8 Extensional test with controlled displacement u on the right side nodes: experimental, in black, and best fitting, in red, of the load-displacement curve for a pantographic lattice formed by three units.

33.4 Meso-macro Identification

The original work by Hencky deals with beams, see Hencky (1921). Hencky wanted to find a discrete Lagrangian system which approximates the Euler beam, when the number of used beams tends to infinity. The beams considered by Hencky are inextensible and the Euler bending stiffness is obtained by introducing suitable rotational springs. It is remarkable to note that already Piola (dell'Isola et al, 2014, 2019), although for different epistemological reasons, wanted to introduce a discrete Lagrangian system to approximate Euler continuum beam. Piola was motivated by the need of justifying physically the Euler's and Bernoulli's dependence of deformation energy on curvature. Therefore the correspondence between the Euler's continuum model and Hencky's discrete model has been established by Piola for justifying some modeling choices, while by Hencky for practical computing purposes.

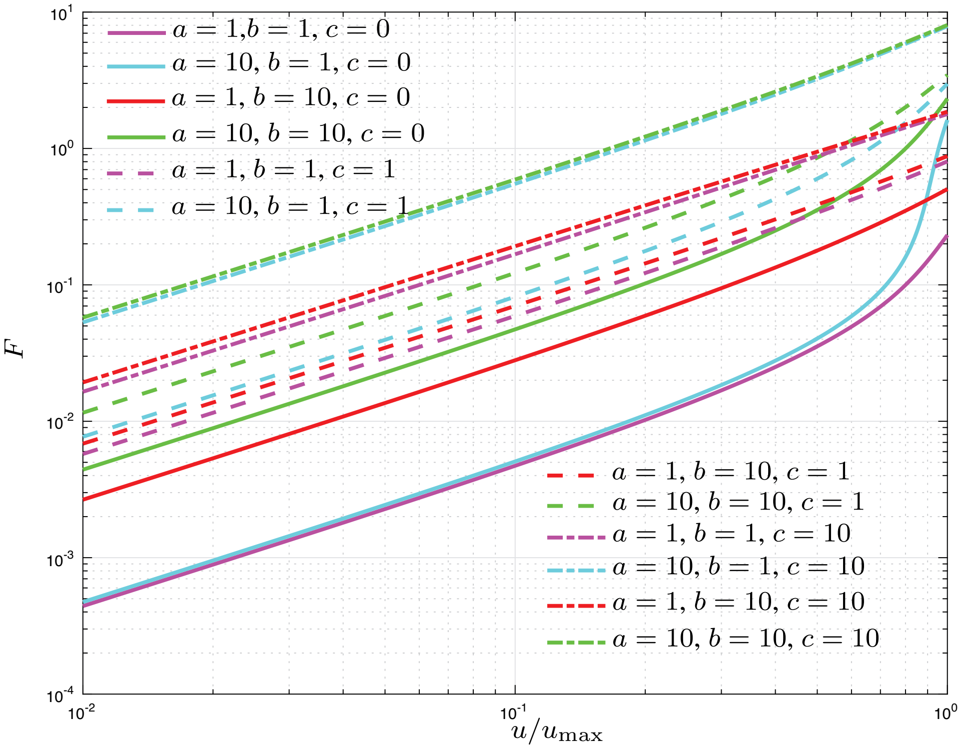

Fig. 33.9 Load-displacement curves useful for a quick estimate of the stiffness triplet (a, b, c).

The work of Piola and Hencky gave us the conceptual tool for finding, from the constitutive parameters of the meso-scale model the corresponding values of the constitutive parameters of the second gradient plate theory to be used at macro-level for pantographic sheets. The material coefficients for macro-generalised plate theory have to be determined in terms of the meso-springs stiffnesses. Here, it is crucial the procedure established in dell'Isola et al (2016). One has to remark that usually the identification procedures have been studied mainly in linear regimes (even if some remarkable results were obtained in Alibert and Della Corte (2015); Seppecher et al (2011); Braides et al (2018). The presented identification process allows for the treatment of problems in which large displacements occur.

The meso-stiffness parameters a, b and c identified by the process described in Sect. 33.3 can be used to identify the macro-stiffness parameters A, B and C to be used in an in-plane model such as well-described in dell'Isola et al (2016) or in an out-of-plane model such as presented in Giorgio et al (2018). Starting from Piola's ansatz, a straightforward procedure gives:

$$
\begin{aligned}
A &= a\,, \\
B &= b\,, \\
C &= \frac{c}{\varepsilon^2}\,,
\end{aligned}
\tag{33.6}
$$

being ε the distance between two successive elastic joints.[7] We remark that in dell'Isola et al (2016) the energy associated to the shearing part has a ρ exponent, *i.e.*, it is not quadratic in general, as that presented in Turco et al (2016c), therefore continuum and discrete models can be linked only in the case $\rho = 2$.

Equation (33.6) represents a formal statement which is assuring that the considered system, at macro-level, behaves as a second gradient material. It is a sufficient condition which, once verified at meso-level, assures that the macro-system must have a deformation energy depending on second gradient of displacement. In an intuitive way one usually says that: *to get a macro second gradient material one has to have high contrast in meso-stiffnesses*. In Equation (33.6) small letters characterize meso-stiffnesses of the Hencky-type springs, while the corresponding capital letters refer to macro elasticity coefficients, including second gradient ones. The characteristic length-scales at macro-level are given by ratio square roots $\sqrt{B/A}$ and $\sqrt{B/C}$. Therefore it is evident that such length-scales can be different from ε, when suitable choices for the values of meso-stiffnesses are made. In order to have large boundary layers where second gradients of displacement are concentrated, ratio square roots $\sqrt{b/a}$ and $\sqrt{b/c}$ must be very large. This statement substantiates the intuitive statement recalled few lines before.

33.5 Concluding Remarks and Future Challenges

The problem of designing novel metamaterials requires a modeling capability which includes the possibility to optimize structural parameters, see dell'Isola et al (2018). In the context of pantographic microstructures the optimization required involves the capacity to perform at high speed complex numerical calculations. Indeed the pantographic sheets are intended to supply a scientific concept suitable to obtain as technological output a metamaterial capable to undergo large deformations and displacements while remaining in elastic regimes.

Of course the existence of elongation floppy modes favors this kind of performances. Indeed the pantographic module has been conceived, see Alibert et al (2003) exactly to get an approximation of ideal pure second gradient materials exhibiting elongations with null or very small deformation energies. The initial motivation was purely scientific, however some demands from aeronautical engineering require the development of such kind of materials in the effort to build light composite structures for flying vehicles.

[7] The relation between macro- and micro-stiffnesses reported in Eq. (33.6) derive from the micro-macro identification based on the Piola's ansatz. A detailed explanation of the computations necessary to reach these results is reported in dell'Isola et al (2016).

The pure demand of existence of elongation floppy modes is not sufficient to calculate the most suitable microstructure for a given tailored purpose. Actually an optimization process is unavoidable. This process requires large computing burdens and, therefore, imposes the formulation of a modeling procedure which is the simplest possible. While it is undoubted that the modeling at the lowest scale is both more detailed and more complex if one wants to obtain predictive specifications, the macro continuum model, although of scientific interest, do requires a formulation of a discretized version to be implemented into a numerical code. It is therefore our opinion that the most suitable modeling procedure is what we have called meso-model, as we believe to have proven in the argumentations presented in this paper.

Finally, we list some open problems which we will tackle in the next future:

1. even if the proposed method seems to produce reliable results, we are strongly interested to any strategies able to correlate experimental measurements to the identification of stiffnesses, *e.g.*, , that reported in Placidi et al (2015);
2. it will be interesting to consider problems where the inertia forces are non negligible and therefore the hypothesis of quasi-static application of external loads or given displacements is not close enough to well-describe the underline problem; in these cases, following the guidelines reported in Giorgio et al (2017a); Engelbrecht and Berezovski (2015); Javili et al (2015), might be useful to verify if the proposed parameter identification furnishes results equivalent in accuracy to those obtainable in quasi-static loading, see, *e.g.*, , di Cosmo et al (2018); Abd-alla et al (2017); Berezovski et al (2016);
3. in this work we only considered the simplest law for describing the strain energy; however, the proposed strategy is open to consider more sophisticated models as in Braides et al (2018); Atai and Steigmann (1997); Challamel et al (2014); Placidi and Barchiesi (2018); Turco et al (2016d);
4. since the meso-mechanical model able to represent the behavior of pantographic structures can be considered as a Representative Elementary Volume (REV) for a continuum models, see, for example, the attempt Andreaus et al (2018), and that this has to be surely discretized, modern interpolation laws such as based on B-splines and NURBS, see Piegl and Tiller (1997); Cottrell et al (2009); Greco et al (2017); Greco and Cuomo (2013); Cuomo et al (2014); Balobanov and Niiranen (2018); Cazzani et al (2016), look like interesting;
5. discrete models are the fundamental brick to construct continuum models, see, *e.g.*, , Shirani et al (2018), able to treat plane or three-dimensional problems, see Turco (2018); Eugster et al (2014), or to consider problems involving foams using the suggestions reported in De Masi et al (2008, 2009); Grimmett (2016);
6. finally, it appears interesting and fruitful to consider as an alternative approach the peridynamic formulation, see *e.g.*, Diyaroglu et al (2015); Meo et al (2016).

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