Chapter 11
Eigenstresses in a Nonlinearly Elastic Sphere with Distributed Dislocations

Evgeniya V. Goloveshkina & Leonid M. Zubov

Abstract The problem of the eigenstresses due to distributed edge and screw dislocations in a hollow nonlinearly elastic sphere is considered. The dislocation density is given by an arbitrary spherically symmetric tensor field. For a general isotropic elastic material, the problem is reduced to a one-dimensional nonlinear boundary value problem. By replacing the unknown functions, the boundary value problem with nonlinear boundary conditions is transformed to a problem with linear ones. Numerical solutions are constructed for specific models of compressible and incompressible materials. The analysis of the influence of dislocations on a stress state of an elastic sphere at large deformations is carried out.

Keywords: Nonlinear elasticity · Dislocation density · Eigenstresses · Large deformations · Spherical symmetry · Rotation tensor

11.1 Introduction

A microstructure of a solid body largely determines the deformation, strength and other properties. Therefore, a study of the microstructure and its defects is necessary for analyzing the mechanical behavior of many crystalline bodies. There are many studies on this subject which emphasize such defects as dislocations (Bilby et al, 1955; Kondo, 1952; Kröner, 1960; Zubov, 1997; Derezin and Zubov, 2011, 1999). Dislocation models are applicable to the description of such phenomena as crystal growth, fatigue, failure, plastic flow, inelasticity, and also other defects of crystalline and nanostructured materials (Clayton, 2011; Clayton et al, 2006; Gutkin and Ovid'ko, 2004; Maugin, 2012; Zhbanova and Zubov, 2016). When there is a lot

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of dislocations in a bounded volume, it makes sense to go to their continuous distribution. In this case, the continuum dislocation theory is used. In the present paper, in the context of the continuum dislocation theory, the nonlinear eigenstress problem for a hollow elastic sphere is solved.

The general formulation of the nonlinear equilibrium problem for an elastic isotropic sphere with an arbitrary spherically symmetric dislocation distribution was given in Zubov (2014). In a number of special cases, exact spherically symmetric solutions of the nonlinear dislocation theory were found (Zubov, 2014; Zhbanova and Zubov, 2016; Goloveshkina and Zubov, 2018). In Zhbanova and Zubov (2016) within the framework of the harmonic (semi-linear) material model, the exact solution was found for any function characterizing the density of edge dislocations. In particular, the case of dislocations concentrated on a spherical surface inside a body was investigated. It was established that this surface was a surface of discontinuity of strains and stresses. In addition to the eigenstress problem, the problem for a hollow sphere under loading by external or internal hydrostatic pressure was solved in Zhbanova and Zubov (2016). In Zubov (2014) an analytical solution of nonlinear elasticity for a hollow sphere made of incompressible material with distributed screw dislocations of radial direction was obtained. In Goloveshkina and Zubov (2018), for a special distribution of screw and edge dislocations, a solution universal in the class of isotropic incompressible elastic bodies was found. With the help of the solution obtained, the eigenstresses in a solid elastic sphere and in an infinite space with a spherical cavity were determined. The interaction of dislocations with an external hydrostatic loading was also investigated. The dislocation distribution determining the spherically symmetric quasi-solid state of an elastic body characterized by zero stresses and a nonuniform elementary volumes rotation field was found.

In this paper, we investigate the general case of a spherically symmetric dislocation distribution. In this case, the exact solution can not be obtained analytically. Therefore, the nonlinear boundary value problem is solved numerically. In the eigenstress problem for an elastic sphere, we use a special technique that allows one to transform a boundary value problem with nonlinear boundary conditions into a problem with linear ones. This makes the numerical solving the boundary value problem for a nonlinear differential equation remarkably easy to perform. A numerical analysis is carried out for the semi-linear material model and the incompressible Bartenev–Khazanovich material model also known as the Varga model. The solution obtained describes the effect of distributed screw and edge dislocations on large spherically symmetric deformations of an elastic sphere.

11.2 Input Relations

We define the dislocation density as a second-rank tensor field α such that the total Burgers vector of dislocations crossing an arbitrary surface coincides with the flux of the tensor α through this surface (Nye, 1953; Vakulenko, 1991). The dislocation density tensor field must satisfy the solenoidality condition

$$\mathrm{div}\boldsymbol{\alpha} = 0\ . \tag{11.1}$$

Hereinafter, the divergence, rotor, and gradient operators (Lurie, 1990; Lebedev et al, 2010) are written in coordinates of the reference configuration. We introduce the deformation gradient (Lurie, 1990; Lebedev et al, 2010)

$$\mathbf{F} = \mathrm{grad}\mathbf{R}, \tag{11.2}$$

where $\mathbf{R} = X_k \mathbf{i}_k$ is the radius vector of a point of the elastic medium in the deformed configuration, X_k ($k = 1, 2, 3$) are Cartesian coordinates of the body in the final state, $\mathbf{i}_k$ are the fixed coordinate base vectors.

In the presence of dislocations in the body, the vector field $\mathbf{R}$ does not exist and the geometric relations (11.2) are replaced by the tensor incompatibility equation with respect to $\mathbf{F}$:

$$\mathrm{rot}\mathbf{F} = \boldsymbol{\alpha}, \tag{11.3}$$

and the tensor $\mathbf{F}$ is called the distortion tensor.

In the absence of mass forces, the equilibrium equations for an elastic medium (Lurie, 1990; Ogden, 1997) have the form

$$\mathrm{div}\mathbf{D} = 0, \tag{11.4}$$

where $\mathbf{D}$ is the asymmetric Piola stress tensor associated with the distortion tensor $\mathbf{F}$ by the constitutive equations of an elastic material (Lurie, 1990; Truesdell, 1977; Ogden, 1997)

$$\mathbf{D}(\mathbf{F}) = \mathrm{d}W(\mathbf{G})/\mathrm{d}\mathbf{F}\ , \qquad \mathbf{G} = \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}\ . \tag{11.5}$$

Here, W is the specific energy, $\mathbf{G}$ is the metric tensor (the Cauchy strain measure).

In the finite strain theory, along with the Piola stress tensor $\mathbf{D}$ we use the symmetric Cauchy tensor (Lurie, 1990; Ogden, 1997; Truesdell, 1977)

$$\mathbf{T} = (\det\mathbf{F})^{-1}\mathbf{F}^{\mathrm{T}}\cdot\mathbf{D} \tag{11.6}$$

and the symmetric Kirchhoff stress tensor also called the second Piola–Kirchhoff stress tensor

$$\mathbf{P} = \mathbf{D}\cdot\mathbf{F}^{-1}\ . \tag{11.7}$$

11.3 Spherically Symmetric State

We introduce the spherical coordinates r, φ, θ:

$$x_1 = r\cos\varphi\cos\theta, \qquad x_2 = r\sin\varphi\cos\theta, \qquad x_3 = r\sin\theta,$$

where x_s ($s = 1, 2, 3$) are the Cartesian coordinates of a sphere in the reference state. Then $\mathbf{e}_r$, $\mathbf{e}_\varphi$, and $\mathbf{e}_\theta$ are the unit vectors tangent to the coordinate lines, forming the basis.

The spherically symmetric dislocation distribution (Zubov, 2014) is represented by the dislocation density tensor

$$\boldsymbol{\alpha} = \alpha_1(r)\mathbf{g} + \alpha_2(r)\mathbf{d} + \alpha_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \ , \tag{11.8}$$
$$\mathbf{g} = \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta \ , \quad \mathbf{d} = \mathbf{e}_\varphi \otimes \mathbf{e}_\theta - \mathbf{e}_\theta \otimes \mathbf{e}_\varphi \ .$$

The first and last terms describe the distribution of screw dislocations, while the second one describes the distribution of edge dislocations. Note that the spherical symmetry of the tensor field (11.8) means that at all points of the sphere on a spherical surface $r = \mathrm{const}$, the components of the tensor field in the considered basis are equal. At the same time, the tensor itself is invariant under rotations about the vector $\mathbf{e}_r$, that is, for an arbitrary function $\chi(r)$, the following equality holds

$$\mathbf{Q} \cdot \boldsymbol{\alpha} \cdot \mathbf{Q}^{\mathrm{T}} = \boldsymbol{\alpha}, \quad \mathbf{Q} = \cos\chi(r)\mathbf{g} + \sin\chi(r)\mathbf{d} + \mathbf{e}_r \otimes \mathbf{e}_r \ .$$

By virtue of (11.8), the solenoidality condition (11.1) implies the equation determining the relation between the components α_1 and α_3 of the dislocation density tensor:

$$\alpha_1 = \alpha_3 + \frac{1}{2} r\alpha_3', \tag{11.9}$$

where $'$ denotes the derivative with respect to the radial coordinate. In the following, the scalar dislocation densities α_2 and α_3 are assumed to be the given functions of the radial coordinate r.

According to (Zubov, 2014), for an isotropic material the distortion tensor as well as the stress tensor are found in the form analogous to the dislocation density tensor:

$$\mathbf{F} = F_1(r)\mathbf{g} + F_2(r)\mathbf{d} + F_3(r)\mathbf{e}_r \otimes \mathbf{e}_r, \tag{11.10}$$
$$\mathbf{D} = D_1(r)\mathbf{g} + D_2(r)\mathbf{d} + D_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \ . \tag{11.11}$$

Taking into account (11.8) and (11.10), the incompatibility equation (11.3) is reduced to three scalar equations

$$(rF_2)' = r\alpha_1, \quad F_2 = \frac{r\alpha_3}{2}, \quad F_3 = (rF_1)' + r\alpha_2, \tag{11.12}$$

and the equilibrium equations (11.4) due to (11.11) reduce to a single equation

$$\frac{\mathrm{d}D_3}{\mathrm{d}r} + \frac{2(D_3 - D_1)}{r} = 0 \, . \tag{11.13}$$

If the prescribed hydrostatic pressure q_0 acts on the outer surface of the sphere $r = r_0$, and the pressure q_1 acts on the inner surface $r = r_1$, then the boundary conditions for the equation (11.13) will be

$$D_3(r_i) = -q_i \left[F_1^2(r_i) + \frac{1}{4}\alpha_3^2 r_i^2 \right], \quad i = 0, 1\,.$$

In the eigenstress problem, i. e. in the absence of external loads, the boundary conditions become

$$D_3(r_1) = 0, \quad D_3(r_0) = 0\,. \tag{11.14}$$

We compute the determinant of the distortion tensor:

$$\det\mathbf{F} = F_3 \left(F_1^2 + \frac{1}{4} r^2 \alpha_3^2 \right). \tag{11.15}$$

For physically realizable deformation, it is necessary that $\det\mathbf{F} > 0$. Therefore, from (11.15) it follows that $F_3 > 0$.

The polar decomposition of the distortion tensor has the form $\mathbf{F} = \mathbf{U} \cdot \mathbf{A}$, where the positive definite stretch tensor $\mathbf{U}$ and the proper orthogonal rotation tensor $\mathbf{A}$ in compliance with (11.10) are determined by the formulas

$$\mathbf{U} = \mathbf{G}^{1/2} = \sqrt{F_1^2 + \frac{1}{4} r^2 \alpha_3^2}\,\mathbf{g} + F_3 \mathbf{e}_r \otimes \mathbf{e}_r, \tag{11.16}$$

$$\mathbf{A} = \mathbf{U}^{-1} \cdot \mathbf{F} = \cos\psi(r)\mathbf{g} + \sin\psi(r)\mathbf{d} + \mathbf{e}_r \otimes \mathbf{e}_r\,. \tag{11.17}$$

Here,

$$\cos\psi = \frac{F_1}{\sqrt{F_1^2 + \frac{1}{4} r^2 \alpha_3^2}}, \quad \sin\psi = \frac{r\alpha_3}{2\sqrt{F_1^2 + \frac{1}{4} r^2 \alpha_3^2}}\,. \tag{11.18}$$

From the representation (11.17) one can see that the orthogonal tensor $\mathbf{A}$ describes a rotation through an angle ψ around the vector $\mathbf{e}_r$.

Given (11.10), we find the inverse distortion tensor $\mathbf{F}^{-1}$ and the metric tensor $\mathbf{G}$:

$$\mathbf{F}^{-1} = \left(F_1^2 + \frac{1}{4} r^2 \alpha_3^2 \right)^{-1} \left(F_1 \mathbf{g} - \frac{1}{2} r \alpha_3 F_2 \mathbf{d} \right) + F_3^{-1} \mathbf{e}_r \otimes \mathbf{e}_r, \tag{11.19}$$

$$\mathbf{G} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = \left(F_1^2 + \frac{r^2 \alpha_3^2}{4} \right) \mathbf{g} + F_3^2 \mathbf{e}_r \otimes \mathbf{e}_r\,. \tag{11.20}$$

The invariants of the tensor $\mathbf{G}$ for spherically symmetric deformation are expressed as follows:

$$\begin{aligned} I_1 &= \mathrm{tr}\mathbf{G} = 2\left(F_1^2 + \frac{1}{4} r^2 \alpha_3^2 \right) + F_3^2, \\ I_2 &= \frac{1}{2}\left(\mathrm{tr}^2\mathbf{G} - \mathrm{tr}\mathbf{G}^2 \right) = \left(F_1^2 + \frac{1}{4} r^2 \alpha_3^2 \right)^2 + 2F_3^2 \left(F_1^2 + \frac{1}{4} r^2 \alpha_3^2 \right), \\ I_3 &= \det\mathbf{G} = F_3^2 \left(F_1^2 + \frac{1}{4} r^2 \alpha_3^2 \right)^2. \end{aligned} \tag{11.21}$$

For an arbitrary isotropic elastic material, the constitutive equations are representable in the form (Lurie, 1990; Truesdell, 1977; Ogden, 1997)

$$\mathbf{D} = (\tau_1 + I_1\tau_2)\mathbf{F} - \tau_2\mathbf{G}\cdot\mathbf{F} + I_3\tau_3\mathbf{F}^{-\mathrm{T}}, \quad \tau_k = 2\frac{\partial W(I_1, I_2, I_3)}{\partial I_k}, \quad k = 1, 2, 3\,. \tag{11.22}$$

Here, τ_k are the material response functions dependent on the strain measure invariants.

Substituting (11.10) with account of (11.12) as well as (11.19) and (11.20) into the constitutive equations (11.22), we obtain the following representations of the Piola stress tensor components:

$$\begin{aligned} D_1 &= (\tau_1 + I_1\tau_2)F_1 - \tau_2 F_1\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right) + I_3\tau_3F_1\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right)^{-1}, \\ D_2 &= (\tau_1 + I_1\tau_2)\frac{r\alpha_3}{2} - \frac{1}{2}r\alpha_3\tau_2\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right) + \frac{1}{2}r\alpha_3 I_3\tau_3\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right)^{-1}, \\ D_3 &= (\tau_1 + I_1\tau_2)F_3 - \tau_2F_3^3 + I_3\tau_3F_3^{-1}\,. \end{aligned} \tag{11.23}$$

Since the stresses (11.23) are defined through the invariants (11.21) dependent on F_1 and F_3, then taking into account the relationship between the tensor $\mathbf{F}$ components (11.12), the stresses are expressed in terms of one distortion component F_1. Thus, for any isotropic material, the boundary value problem consists of the equilibrium equation (11.13), which is a second-order nonlinear ordinary differential equation with respect to the function $F_1(r)$, and the nonlinear boundary conditions (11.14).

As an example, we write this equation explicitly for a semi-linear (harmonic) material having the following constitutive equations (Lurie, 1990; Ogden, 1997; John, 1960):

$$\mathbf{D} = \frac{2\mu}{1-2\nu}\left(\nu\mathrm{tr}\mathbf{U} - 1 - \nu\right)\mathbf{A} + 2\mu\mathbf{F}, \tag{11.24}$$

where μ and ν are the material constants. In the small strain region, the semi-linear material follows Hooke's law with the shear modulus μ and the Poisson's ratio ν. The differential equation with respect to the distortion $F_1(r)$ for the material is written as follows:

$$\begin{aligned} F_1'' = {} & \frac{2(2-3\nu)}{(\nu-1)r}F_1' + \frac{(\alpha_2 + r\alpha_2')(1-\nu) + 2\alpha_2(1-2\nu)}{(\nu-1)r} \\ & + \frac{2\nu}{(\nu-1)r}\left(F_1^2 + \frac{r^2\alpha_3^2}{4}\right)^{-1/2}\left(F_1F_1' + \frac{r\alpha_3(\alpha_3 + r\alpha_3')}{4}\right) \\ & + \frac{2}{(\nu-1)r^2}\left[2\nu\sqrt{F_1^2 + \frac{r^2\alpha_3^2}{4}} + \nu(F_1 + rF_1' + r\alpha_2) - 1 - \nu\right] \\ & \times\left[1 - F_1\left(F_1^2 + \tfrac{r^2\alpha_3^2}{4}\right)^{-1/2}\right]. \end{aligned} \tag{11.25}$$

The constitutive equations in terms of the Kirchhoff stress tensor for any, including an anisotropic, elastic body have the form:

$$\mathbf{P} = 2\mathrm{d}W(\mathbf{G})/\mathrm{d}\mathbf{G} \,. \tag{11.26}$$

The semi-linear material belongs to models of an elastic medium, the specific energy of which is given as a function of the stretch tensor $\mathbf{U}$ and not the metric tensor $\mathbf{G}$. In this case, the symmetric Biot stress tensor is convenient to use:

$$\mathbf{S} = \mathrm{d}W/\mathrm{d}\mathbf{U} \,. \tag{11.27}$$

From (11.26) and (11.27), we obtain the formulas connecting the Biot stress tensor with the Kirchhoff and Piola stress tensors

$$\mathbf{S} = \frac{1}{2}(\mathbf{P}\cdot\mathbf{U} + \mathbf{U}\cdot\mathbf{P}) = \frac{1}{2}\left(\mathbf{D}\cdot\mathbf{A}^{\mathrm{T}} + \mathbf{A}\cdot\mathbf{D}^{\mathrm{T}}\right) \,. \tag{11.28}$$

If the material is isotropic, then the specific energy depends on three invariants of the stretch tensor, i. e. $W = W(J_1, J_2, J_3)$, where $J_1 = \mathrm{tr}\mathbf{U}$, $J_2 = \frac{1}{2}\left(\mathrm{tr}^2\mathbf{U} - \mathrm{tr}\mathbf{U}^2\right)$, $J_3 = \det\mathbf{U}$. Consequently, the tensor $\mathbf{S}$ can be rewritten as

$$\mathbf{S} = \left(\frac{\partial W}{\partial J_1} + J_1\frac{\partial W}{\partial J_2}\right)\mathbf{I} - \frac{\partial W}{\partial J_2}\mathbf{U} + J_3\frac{\partial W}{\partial J_3}\mathbf{U}^{-1} \,, \tag{11.29}$$

where $\mathbf{I}$ is the unit tensor.

In an isotropic body, the tensors $\mathbf{P}$ and $\mathbf{U}$ are coaxial and therefore commute: $\mathbf{P}\cdot\mathbf{U} = \mathbf{U}\cdot\mathbf{P}$. Then, $\mathbf{S} = \mathbf{P}\cdot\mathbf{U} = \mathbf{U}\cdot\mathbf{P}$ and $\mathbf{S} = \mathbf{D}\cdot\mathbf{A}^{\mathrm{T}}$.

Constitutive equations in terms of the Piola tensor for the material with the specific energy $W = W(J_1, J_2, J_3)$ due to (11.29) will be

$$\mathbf{D} = (\eta_1 + J_1\eta_2)\,\mathbf{A} - \eta_2\mathbf{F} + J_3\eta_3\mathbf{F}^{-\mathrm{T}}, \quad \eta_k = \frac{\partial W}{\partial J_k} \,. \tag{11.30}$$

In the spherically symmetric problem, the invariants J_1, J_2, J_3 are expressed in F_1, F_2, F_3 using formulas

$$J_1 = 2\sqrt{F_1^2 + \frac{1}{4}r^2\alpha_3^2} + F_3,$$
$$J_2 = F_1^2 + \frac{r^2\alpha_3^2}{4} + 2F_3\sqrt{F_1^2 + \frac{1}{4}r^2\alpha_3^2},$$
$$J_3 = F_3\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right) \,.$$

Taking into account (11.10), (11.19), and (11.17), the components of the Piola tensor are written in the form

$$D_1 = (\eta_1 + J_1\eta_2)\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right)^{-1/2} F_1 - \eta_2 F_1 + J_3\eta_3\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right)^{-1} F_1,$$

$$D_2 = \frac{1}{2}(\eta_1 + J_1\eta_2)\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right)^{-1/2} r\alpha_3 - \eta_2 F_2 + \frac{J_3\eta_3 r\alpha_3}{2}\left(F_1^2 + \frac{1}{4}r^2\alpha_3^2\right)^{-1}, \tag{11.31}$$

$$D_3 = \eta_1 + J_1\eta_2 - \eta_2 F_3 + J_3\eta_3 F_3^{-1}.$$

In the case of an incompressible material, the condition $\det\mathbf{F} = I_3 = J_3 = 1$ is satisfied. Therefore, for the material, the constitutive equations (11.22) are modified as follows:

$$\mathbf{D} = (\tau_1 + I_1\tau_2)\mathbf{F} - \tau_2\mathbf{G}\cdot\mathbf{F} - p\mathbf{F}^{-\mathrm{T}}, \tag{11.32}$$

where p is a pressure in an incompressible body not expressed in terms of strain. Given the incompressibility property, the constitutive equations (11.29) and (11.30) are reduced to

$$\mathbf{S} = (\eta_1 + J_1\eta_2)\,\mathbf{I} - \eta_2\mathbf{U} - p\mathbf{U}^{-1},$$
$$\mathbf{D} = (\eta_1 + J_1\eta_2)\,\mathbf{A} - \eta_2\mathbf{F} - p\mathbf{F}^{-\mathrm{T}}.$$

11.4 Transformation of the Boundary Value Problem

Since the boundary conditions (11.14) of the eigenstress problem are represented by the stress constraints, then, taking into account their expressions (11.31) in terms of the distortion, we finally obtain boundary conditions on the distortion. They represent a nonlinear relation with respect to the function F_1 and its derivative F_1'. To obtain a boundary value problem with linear boundary conditions, it is necessary to replace the unknown function. Instead of the equation with respect to $F_1(r)$, we derive a system of equations with respect to $D_2(r)$ and $D_3(r)$. For this, we need to solve the problem of inversion of the Piola stress tensor as a function of the distortion tensor: $\mathbf{D} = h(\mathbf{F})$, that is, find the tensor function H, inverse to the function h: $\mathbf{F} = H(\mathbf{D})$. The way to solve the problem for an isotropic material is indicated in Zubov (1976) and consists of the following. First, a more simple problem of inversion of the dependence $\mathbf{S} = l(\mathbf{U})$ between the symmetric tensors is solved, i. e. the function L such that $\mathbf{U} = L(\mathbf{S})$ is founded. Further we have

$$\mathbf{F} = \mathbf{U}\cdot\mathbf{A} = L(\mathbf{S})\cdot\mathbf{A} = L\left(\mathbf{D}\cdot\mathbf{A}^{\mathrm{T}}\right)\cdot\mathbf{A}.$$

The problem of constructing the function $\mathbf{F} = H(\mathbf{D})$ will be solved if we express the rotation tensor $\mathbf{A}$ in terms of the Piola stress tensor $\mathbf{D}$. This can be done by solving the equation with respect to $\mathbf{A}$, expressing the symmetry property of the Biot stress tensor

$$\mathbf{D}\cdot\mathbf{A}^{\mathrm{T}} = \mathbf{A}\cdot\mathbf{D}^{\mathrm{T}}. \tag{11.33}$$

In the spherically symmetric problem considered here, the last equation in view of (11.11) and (11.17) is equivalent to one scalar relation

$$D_1 \sin\psi = D_2 \cos\psi\,. \tag{11.34}$$

This equation has two solutions:

$$\cos\psi = \sqrt{\frac{D_1^2}{D_1^2 + D_2^2}}, \quad \sin\psi = \frac{D_2}{D_1}\sqrt{\frac{D_1^2}{D_1^2 + D_2^2}} \tag{11.35}$$

and

$$\cos\psi = -\sqrt{\frac{D_1^2}{D_1^2 + D_2^2}}, \quad \sin\psi = -\frac{D_2}{D_1}\sqrt{\frac{D_1^2}{D_1^2 + D_2^2}}\,. \tag{11.36}$$

If we assume that $-\pi \le \psi \le \pi$, then the first solution is described by the inequalities

$$-\frac{\pi}{2} \le \psi \le \frac{\pi}{2},$$

and the second by the inequalities

$$-\pi \le \psi \le -\frac{\pi}{2}, \quad \frac{\pi}{2} \le \psi \le \pi\,.$$

From the formula (11.18), it is clear that the first solution corresponds to the positive F_1, and the second to the negative. As shown in Zhbanova and Zubov (2016), F_1 is negative when the eversion deformation of a sphere occurs (Zubov and Moiseyenko, 1983), and positive in case of spherically symmetric deformation of a sphere without eversion. Consequently, the second solution corresponds to the eigenstress problem for the everted sphere with distributed dislocations. We note that in the absence of dislocations, in a sphere without eversion the stresses are identically equal to zero, while in an everted sphere the stresses are not zero due to eversion.

11.5 Problem for Semi-linear Material

Given the constitutive equations of the semi-linear material (11.24), the tensor $\mathbf{S}$ is written as

$$\mathbf{S} = \frac{2\mu\nu}{1-2\nu}\mathbf{I}\,\mathrm{tr}(\mathbf{U}-\mathbf{I}) + 2\mu(\mathbf{U}-\mathbf{I}). \tag{11.37}$$

We invert the expression (11.37):

$$\mathbf{U} = \mathbf{I} + \frac{1}{2\mu}\left(\mathbf{S} - \frac{\nu}{1+\nu}\mathbf{I}\,\mathrm{tr}\mathbf{S}\right)\,. \tag{11.38}$$

Let us find the distortion tensor $\mathbf{F}$ by the formula

$$\mathbf{F} = \mathbf{U} \cdot \mathbf{A}\,. \tag{11.39}$$

Taking into account (11.38) we have

$$\mathbf{F} = \mathbf{A} + \frac{1}{2\mu}\left[\mathbf{D} - \frac{\nu}{1+\nu}\mathbf{A}\mathrm{tr}\left(\mathbf{D}\cdot\mathbf{A}^{\mathrm{T}}\right)\right]. \tag{11.40}$$

Based on (11.11), (11.17), and (11.40), we finally obtain the tensor $\mathbf{F}$, expressed in terms of the Piola stresses:

$$\mathbf{F} = \frac{1}{2\mu}\left[(D_1 + K\cos\psi)\,\mathbf{g} + (D_2 + K\sin\psi)\,\mathbf{d} + (D_3 + K)\mathbf{e}_r \otimes \mathbf{e}_r\right], \tag{11.41}$$
$$K = \frac{2\mu(1+\nu) - \nu\left[2\left(D_1\cos\psi + D_2\sin\psi\right) + D_3\right]}{1+\nu},$$

where $\cos\psi$ and $\sin\psi$ are computed from the formulas (11.35) or (11.36).

The tensor $\mathbf{D}$ components are calculated numerically from the system of equations (11.9), (11.12), (11.13), and (11.41) with the boundary conditions (11.14). Knowing the Piola stresses, we find the distortion by the formulas (11.41).

11.6 Problem for Incompressible Material

We consider the problem for incompressible material using the Bartenev–Khazanovich model (Lurie, 1990) as an example. The equations of state of the material have the form:

$$\mathbf{D} = 2\mu\mathbf{A} - p\mathbf{F}^{-\mathrm{T}},$$

where p is the internal pressure in an incompressible body. Then the tensor $\mathbf{S}$ by virtue of (11.39) is written as

$$\mathbf{S} = 2\mu\mathbf{I} - p\mathbf{U}^{-1}\,. \tag{11.42}$$

Then, from the incompressibility condition $\det\mathbf{U} = 1$ we find

$$p = \sqrt[3]{\det\left(2\mu\mathbf{I} - \mathbf{S}\right)}\,. \tag{11.43}$$

Let us invert (11.42) by expressing $\mathbf{U}$ and substituting (11.43):

$$\mathbf{U} = \sqrt[3]{\det\left(2\mu\mathbf{I} - \mathbf{S}\right)}\left(2\mu\mathbf{I} - \mathbf{S}\right)^{-1}\,. \tag{11.44}$$

According to (11.17) and (11.44) we derive the tensor (11.39) in the form

$$\mathbf{F} = \frac{\sqrt[3]{2\mu - D_3}(B\cos\psi - C\sin\psi)}{A^{2/3}}\mathbf{g}$$
$$+ \frac{\sqrt[3]{2\mu - D_3}(B\sin\psi + C\cos\psi)}{A^{2/3}}\mathbf{d} + \frac{\sqrt[3]{A}}{(2\mu - D_3)^{2/3}}\mathbf{e}_r \otimes \mathbf{e}_r, \qquad (11.45)$$
$$A = 4\mu^2 + D_1^2 + D_2^2 - 4\mu(D_1\cos\psi + D_2\sin\psi),$$
$$B = 2\mu - D_1\cos\psi - D_2\sin\psi, \quad C = D_2\cos\psi - D_1\sin\psi,$$

with $\cos\psi$ and $\sin\psi$ computed from (11.35) or (11.36).

From the system of equations (11.9), (11.12), (11.13), and (11.45) with the boundary conditions (11.14), we find the numerical solution of the problem by calculating the stresses and strains.

11.7 Numerical Results

For numerical calculations within the framework of the semi-linear material, we choose the dislocation distribution

$$\alpha_1 = \frac{\gamma_0}{r}, \quad \alpha_2 = \frac{\beta_0}{r}, \quad \alpha_3 = \frac{2\gamma_0}{r},$$

and for the incompressible material

$$\alpha_3 = \frac{2\gamma_0}{r^2}, \quad \alpha_1 = \alpha_2 = 0,$$

where β_0 and γ_0 are some constants. The outer radius of the sphere is considered to be equal to one ($r_0 = 1$), which is equivalent to introducing a dimensionless radial coordinate. The following numerical results correspond to the value $r_1 = 0.5$. Similarly, assuming $\mu = 1$, we deal with the dimensionless stresses. For the dimensionless constant ν we take the value $\nu = 0.3$.

According to (11.34), the problem has two solutions since $\cos\psi$ can be positive (11.35) or negative (11.36). For the incompressible material, the numerical results are displayed in the case $\cos\psi > 0$ (Figs. 11.11–11.16), and for the semi-linear material, in the case $\cos\psi > 0$ (Figs. 11.1–11.5) as well as $\cos\psi < 0$ (Figs. 11.6–11.10).

It is established that for both material models, the stresses D_1 and D_2 in absolute value are maximal on the inner surface of the sphere, and D_3 on the surface close to the inner surface. For the incompressible material, for $\cos\psi > 0$ the maximum stress D_1 is an order of magnitude higher than the maximum stresses D_2 and D_3.

Different curves in each figure illustrate the influence of the dislocation intensity on a stress-strain state. Thus, for the semi-linear material, the stresses D_1 and D_3 decrease and become more uniformly distributed over the thickness of the sphere (Figs. 11.1, 11.3, 11.6, and 11.8), and the stress D_2 increases and its distribution becomes less uniform (Figs. 11.2 and 11.7). In addition, there are spherical sur-

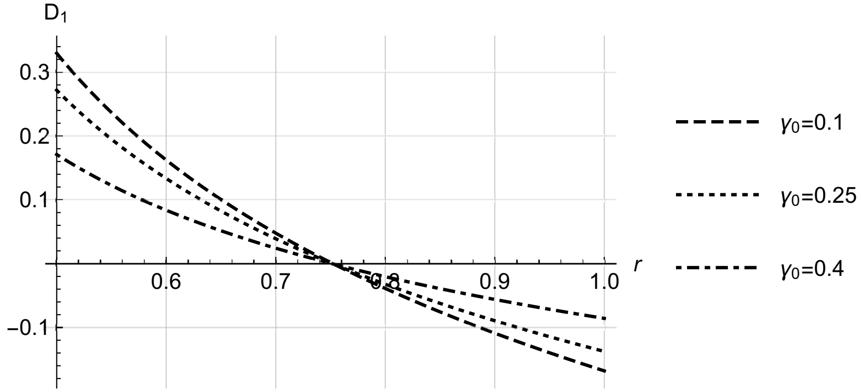

Fig. 11.1 Semi-linear material, $\cos\psi > 0$, $\beta_0 = 0.2$, stress D_1

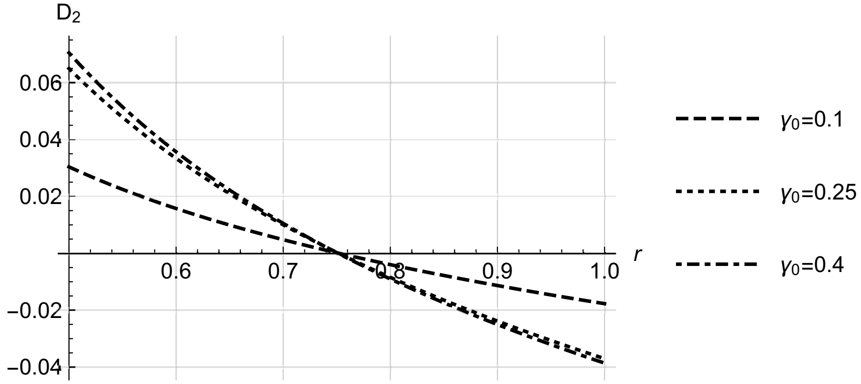

Fig. 11.2 Semi-linear material, $\cos\psi > 0$, $\beta_0 = 0.2$, stress D_2

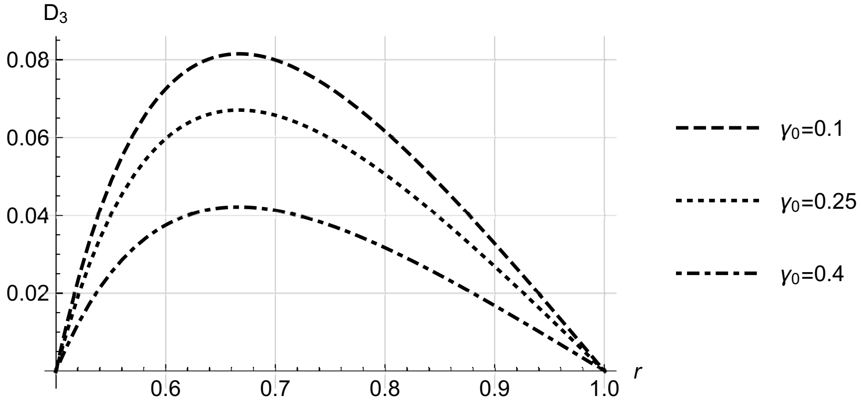

Fig. 11.3 Semi-linear material, $\cos\psi > 0$, $\beta_0 = 0.2$, stress D_3

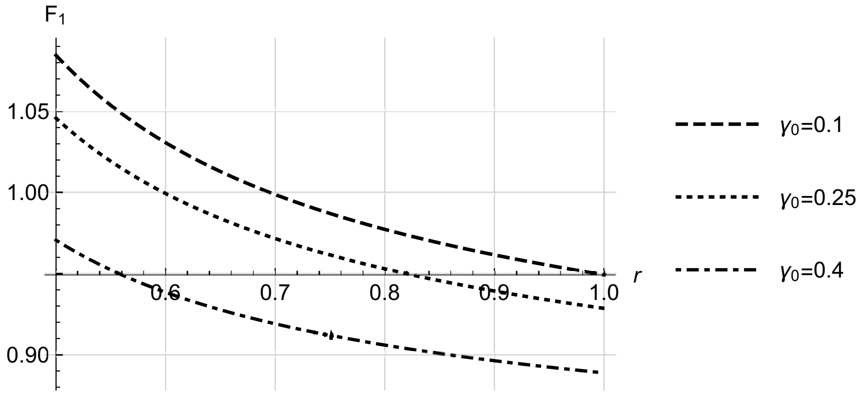

Fig. 11.4 Semi-linear material, $\cos\psi > 0$, $\beta_0 = 0.2$, distortion F_1

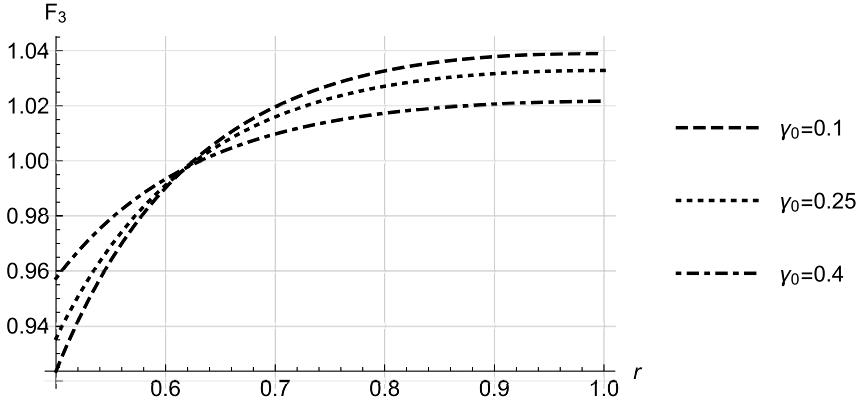

Fig. 11.5 Semi-linear material, $\cos\psi > 0$, $\beta_0 = 0.2$, distortion F_3

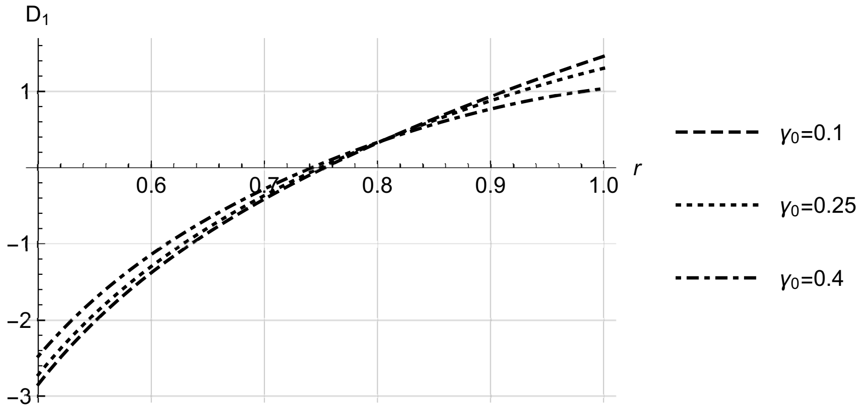

Fig. 11.6 Semi-linear material, $\cos\psi < 0$, $\beta_0 = 0.2$, stress D_1

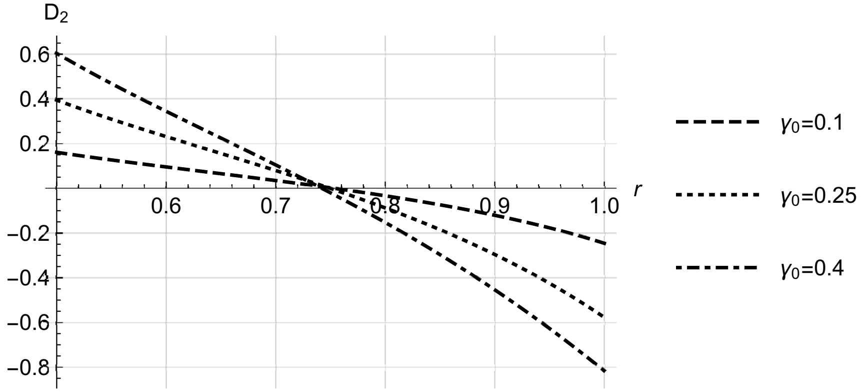

Fig. 11.7 Semi-linear material, $\cos\psi < 0$, $\beta_0 = 0.2$, stress D_2

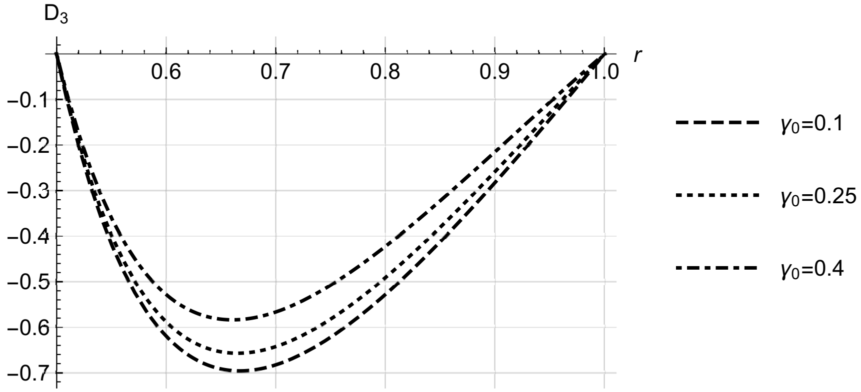

Fig. 11.8 Semi-linear material, $\cos\psi < 0$, $\beta_0 = 0.2$, stress D_3

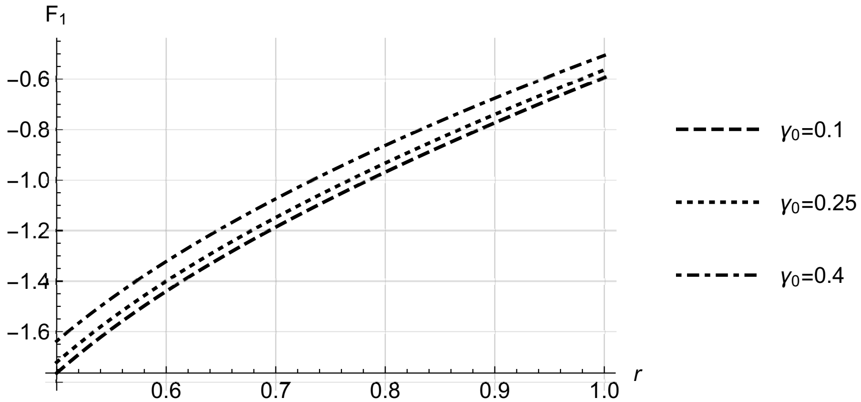

Fig. 11.9 Semi-linear material, $\cos\psi < 0$, $\beta_0 = 0.2$, distortion F_1

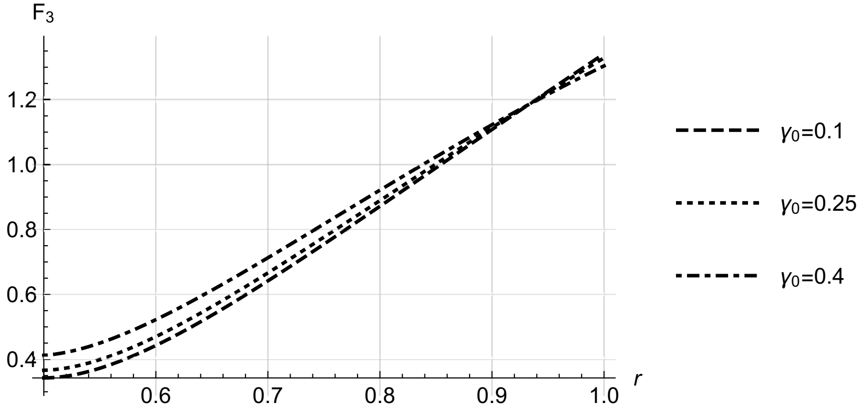

Fig. 11.10 Semi-linear material, $\cos\psi < 0$, $\beta_0 = 0.2$, distortion F_3

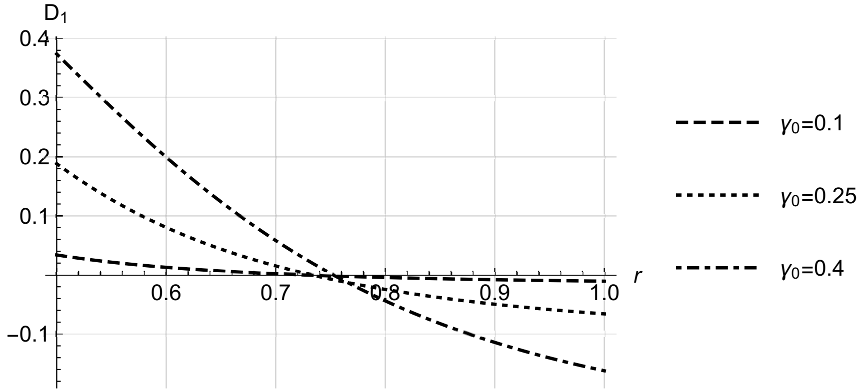

Fig. 11.11 Incompressible material, $\gamma_0 = 0.1$, stress D_1

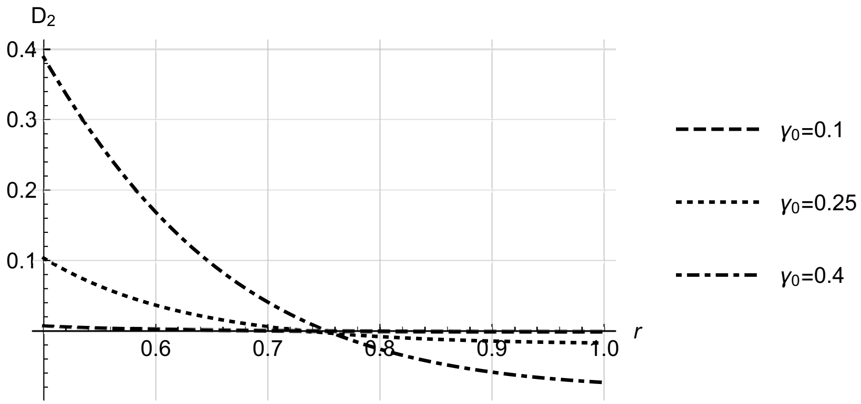

Fig. 11.12 Incompressible material, $\gamma_0 = 0.1$, stress D_2

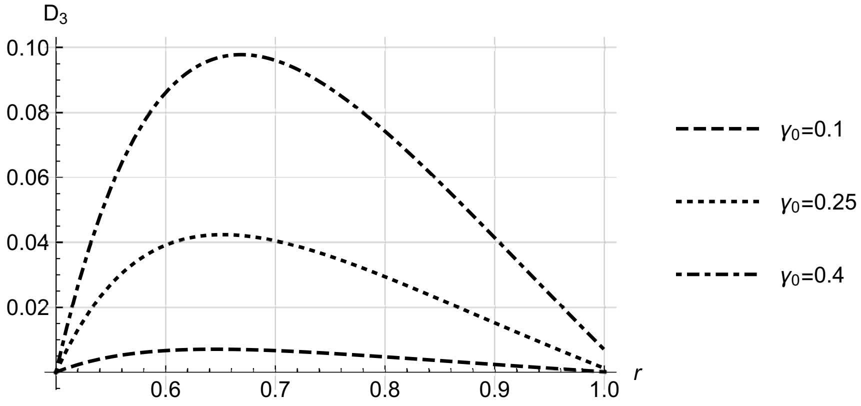

Fig. 11.13 Incompressible material, $\cos\psi > 0$, stress D_3

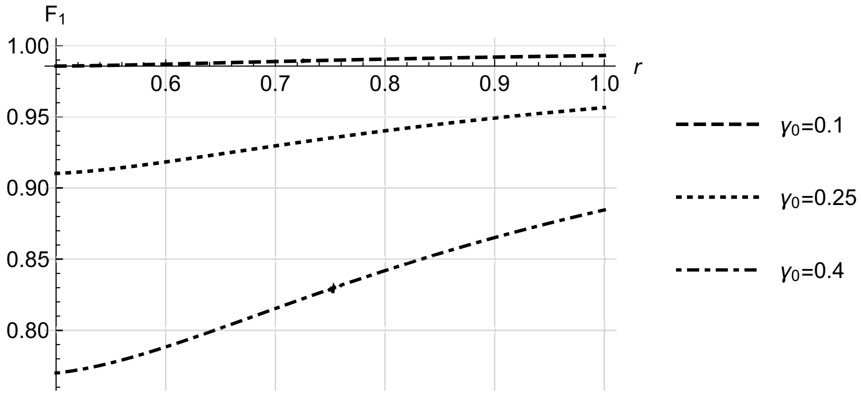

Fig. 11.14 Incompressible material, $\cos\psi > 0$, distortion F_1

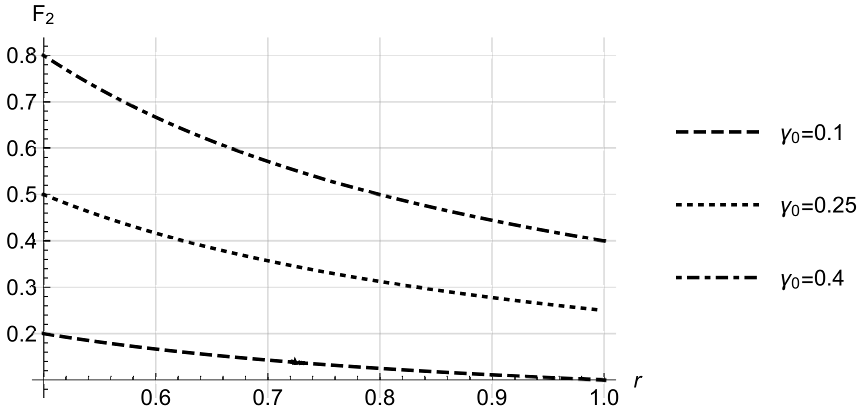

Fig. 11.15 Incompressible material, $\cos\psi > 0$, distortion F_2

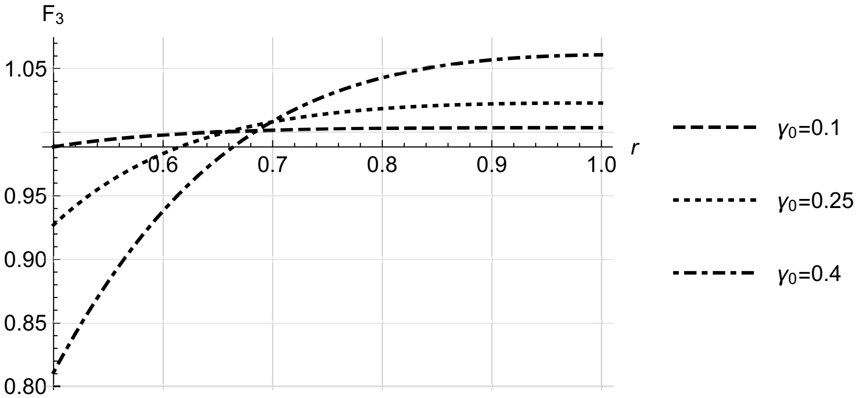

Fig. 11.16 Incompressible material, $\cos\psi > 0$, distortion F_3

faces on which the stress D_1 or D_2 does not depend on the dislocation density. For $\cos\psi < 0$, this kind of surface is located in the middle between the inner and outer surfaces of the sphere for D_2 (Fig. 11.7) and near the outer surface for D_1 (Fig. 11.6). In the case $\cos\psi > 0$, for D_1 and D_2 these surfaces coincide and are located in the middle (Figs. 11.1 and 11.2). Moreover, on said surfaces these stresses are zero. For the distortion F_3 there also exists a surface $r = r_*$ on which $F_3(r_*)$ does not depend on the dislocation density, with F_3 increasing before the surface $r = r_*$ and decreasing after it as moving from the inner surface of the sphere to the outer one (Figs. 11.5 and 11.10). In the sphere without eversion, the surface considered is near the inner surface, and in the everted sphere near the outside one. In both cases, the distortion F_2 due to (11.12) and (11.7) is a constant value. The distortion F_1 decreases in absolute value with increasing dislocation density (Figs. 11.4 and 11.9).

For the incompressible material, the stresses D_1 and D_2, which are approximately equal, also do not depend on the dislocation density on a certain spherical surface. Moreover, on this surface they vanish (Figs. 11.11 and 11.12). At different dislocation densities, on another spherical surface closer to the inner surface of the sphere, the distortion F_3 is about the same. With increasing the dislocations, when moving away from the inner surface of the sphere, F_3 decreases before the considered surface and increases after it (Fig. 11.16). The higher the dislocation density the higher the Piola stresses and the less uniformly the distribution of latter. With that the distortion F_1 decreases (Fig. 11.14), and F_2 increases (Fig. 11.15).

11.8 Conclusion

In the present paper, we have considered the problem of the nonlinear continuum dislocation theory for an elastic hollow sphere for an arbitrary spherically symmetric distribution of screw and edge dislocations. The system of solving equations consists of the equilibrium equations, the incompatibility equations, and the constitutive equations of the elastic medium. Using the properties of spherically symmetric tensor fields, for a general isotropic material we have reduced the original problem to a nonlinear boundary value problem for an ordinary second-order differential equation with respect to one component of the distortion tensor. This equation is obtained in two cases: the specific energy of the material is a function of the metric tensor invariants and a function of the stretch tensor invariants. The boundary conditions for a one-dimensional boundary value problem with respect to the distortion are nonlinear. To simplify the numerical solution of this problem, we have transformed it to a boundary value problem with the linear boundary conditions. The unknown functions of the radial coordinate in the transformed problem are the components of the Piola stress tensor.

We have established that the eigenstress problem for a hollow sphere always has two spherically symmetric solutions, one of which describes the equilibrium of an everted hollow sphere with dislocations.

For two specific models of an elastic medium: the compressible semi-linear material and the incompressible Bartenev–Khazanovich material, we have constructed a numerical solution of a one-dimensional boundary value problem, describing the eigenstresses due to given densities of screw and edge dislocations. Based on the obtained numerical results, we have analyzed the effect of the intensity of the dislocation distribution and its behavior on a stress state of the sphere at large deformations.

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