Chapter 10
Types of Physical Nonlinearity in the Theory of Constitutive Relations and the Generalized Poynting Effect

Dimitri V. Georgievskii

Abstract The certain class of constitutive relations are considered that connect the symmetric stress tensor and the symmetric strain tensor by means of isotropic potential tensor nonlinear functions in three-dimensional space. The various definitions of tensor nonlinearity are given as well as their equivalence is shown. From the perspective of mathematical theory about the tensor nonlinear functions, an interpretation of the Poynting effect is given, which is well known in experimental mechanics. It is demonstrated that such an effect is not necessarily the consequence of tensor nonlinearity in constitutive relations; instead, it is effected by the quadratic dependence on invariants in certain material functions. Therefore, in the physically linear case for a *small* strain, this dependence is absent. Concerning this "order of smallness," the Poynting effect is investigated and a possibility is discussed for simulating such an effect by means of the tensor linear constitutive relations.

Keywords: stress, strain, constitutive relation, material function, invariant, scalar potential, establishing experiment, the Poynting effect, tensor nonlinearity

10.1 Various Definitions of Tensor Nonlinearity and Their Equivalence

In the theory of constitutive relations for isotropic media, the considerable place belongs to the scleronomous models for which a connection of the strain tensor, $\tilde{\varepsilon}$, and the stress tensor, $\tilde{\sigma}$, are given in three-dimensional space by means of the isotropic tensor nonlinear function,

$$\tilde{\varepsilon} = B_0\tilde{\mathrm{I}} + B_1\tilde{\sigma} + B_2\tilde{\sigma}^2 \, , \tag{10.1}$$

Dimitri V. Georgievskii
Mechanical and Mathematical Department, Moscow State University, Moscow 119991, Russia,
e-mail: georgiev@mech.math.msu.su

B. E. Abali et al. (eds.), *New Achievements in Continuum Mechanics and Thermodynamics*, Advanced Structured Materials 108,
https://doi.org/10.1007/978-3-030-13307-8_10

where $\tilde{\mathrm{I}}$ is the identity tensor of the second rank, B_0, B_1 and B_2 are the material functions of three independence invariants:

$$I_{\sigma 1} = \mathrm{tr}\tilde{\sigma}, \quad I_{\sigma 2} = \sqrt{\mathrm{tr}(\tilde{\sigma}^2)}, \quad I_{\sigma 3} = \sqrt[3]{\mathrm{tr}(\tilde{\sigma}^3)}\,, \tag{10.2}$$

of the tensor $\tilde{\sigma}$. An extensive literature (see, for example, both the classic and recent works Rivlin, 1953; Rivlin and Ericksen, 1955; Il'yushin, 1963; Altenbach et al, 1995; Abali et al, 2013; Devendiran et al, 2017; Kulvait et al, 2017) is devoted to the problems of generality in continuum mechanics of the representation (10.1) and the inverse one

$$\tilde{\sigma} = A_0\tilde{\mathrm{I}} + A_1\tilde{\varepsilon} + A_2\tilde{\varepsilon}^2\,, \tag{10.3}$$

where A_0, A_1 and A_2 are the material functions of the invariants:

$$I_{\varepsilon 1} = \mathrm{tr}\tilde{\varepsilon}, \quad I_{\varepsilon 2} = \sqrt{\mathrm{tr}(\tilde{\varepsilon}^2)}, \quad I_{\varepsilon 3} = \sqrt[2]{\mathrm{tr}(\tilde{\varepsilon}^3)}\,, \tag{10.4}$$

where they can be expressed in B_0, B_1 and B_2.

If the medium has the scalar potential $w(I_{\sigma 1}, I_{\sigma 2}, I_{\sigma 3})$ such that $\tilde{\varepsilon} = \partial w/\partial\tilde{\sigma}$ then the following three potentiality conditions are fulfilled,

$$\frac{\partial B_0}{\partial I_{\sigma 2}} = I_{\sigma 2}\frac{\partial B_1}{\partial I_{\sigma 1}}, \quad \frac{\partial B_0}{\partial I_{\sigma 3}} = I_{\sigma 3}^2\frac{\partial B_2}{\partial I_{\sigma 1}}, \quad I_{\sigma 2}\frac{\partial B_1}{\partial I_{\sigma 3}} = I_{\sigma 3}^2\frac{\partial B_2}{\partial I_{\sigma 2}}\,, \tag{10.5}$$

that relate the material functions B_0, B_1 and B_2. The set of conditions (10.5) may be considered as the system of differential equations with respect to B_0, B_1 and B_2, which has the first integrals in the certain cases (Georgievskii, 2016b).

Tensor nonlinearity of the function (10.1) is stipulated by presence of the last term in the right part. If $B_2 \equiv 0$ then this function—just as the corresponding class of materials—is called quasilinear, i. e. linear in the tensorial sense; but possibly nonlinear in scalar sense. Among the latter the case of physical linearity corresponds that B_0 linearly depends on $I_{\sigma 1}$ and does not depend on $I_{\sigma 2}$ and $I_{\sigma 3}$, as B_1 is constant.

In this way, a difference from identical zero of the material function B_2 in (10.1) represents the natural definition of tensor nonlinearity. This definition is equivalent to the fact that the angle between deviators $\tilde{s} = \tilde{\sigma} - I_{\sigma 1}\tilde{\mathrm{I}}/3$ and $\tilde{e} = \tilde{\varepsilon} - I_{\varepsilon 1}\tilde{\mathrm{I}}/3$ is not equal to zero identically. Let us prove this.

We assume that $\tilde{\sigma}$ and $\tilde{\varepsilon}$ are not spherical tensors (according to (10.1) and (10.3) they are either spherical or nonspherical, simultaneously) such that $\tilde{s}$ and $\tilde{e}$ are not identically zero tensors of the second rank and the angle $\alpha = (\tilde{s}; \tilde{e})$ is defined. We calculate $\cos\alpha$:

$$\cos\alpha = \frac{\tilde{s} : \tilde{e}}{\sqrt{\tilde{s} : \tilde{s}}\sqrt{\tilde{e} : \tilde{e}}} \equiv \frac{\tilde{s} : \tilde{e}}{I_{s2}I_{e2}} \tag{10.6}$$

$$\tilde{s} : \tilde{s} = J, \quad \tilde{s} : \tilde{e} = JB_1 + KB_2, \quad \tilde{e} : \tilde{e} = JB_1^2 + 2KB_1B_2 + LB_2^2 \tag{10.7}$$

where J, K and L are the invariants of stress state depending on $I_{\sigma 1}$, $I_{\sigma 2}$ and $I_{\sigma 3}$ (10.2):

$$J = I_{\sigma 2}^2 - \frac{1}{3} I_{\sigma 1}^2, \quad K = I_{\sigma 3}^3 - \frac{1}{3} I_{\sigma 1} I_{\sigma 2}^2, \quad L = \frac{4}{3} I_{\sigma 1} I_{\sigma 3}^3 - I_{\sigma 1}^2 I_{\sigma 2}^2 + \frac{1}{6}(I_{\sigma 1}^4 + I_{\sigma 2}^4) \tag{10.8}$$

Since $\tilde{s} : \tilde{s} > 0$ and $\tilde{e} : \tilde{e} > 0$ then

$$J > 0, \quad JL - K^2 > 0 \tag{10.9}$$

Using the Hamilton – Cayley theorem, after calculations we write

$$\cos \alpha = \left(1 + \frac{(JL - K^2) B_2^2}{(JB_1 + KB_2)^2} \right)^{-1/2} \tag{10.10}$$

It should be noted that the material function B_0 is not present in the expression for α.

As is obvious from (10.10) that if $B_2 = 0$ and $B_1 > 0$ then $\tilde{s}$ and $\tilde{e}$ are co-directed, i. e. the unit directing tensors $\tilde{s}^0 = \tilde{s}/I_{s2}$ and $\tilde{e}^0 = \tilde{e}/I_{e2}$ are the same. The statement is also truly in reverse (here it is necessary to use both the inequalities (10.9). An equivalence of two definitions has been established.

A relative smallness of the tensor nonlinearity effect usually observable in experiments with deformable solids may be treated as a smallness of the angle α. The relation (10.10) results in the connection in linear approximation of the low values α and the dimensionless material function B_2:

$$\alpha = \sqrt{JL - K^2} \, \frac{B_2}{JB_1} + O(B_2^2) \tag{10.11}$$

If $\alpha \ll 1$ then tensor nonlinear effects of material behavior are said to have the second order of smallness. It is implied that the first order is inherent in the parameters of stress-strain state caused by presence in (10.1) of the material function B_1.

10.2 Establishing experiments to find the material functions B_0, B_1 and B_2

Let us pay attention to the establishing experiments to find the function B_0, B_1 and B_2 at any point $(I_{\sigma 1}, I_{\sigma_2}, I_{\sigma_3})$ in the domain of their definition (Georgievskii (2016a)). For this purpose it is proposed to use long hollow cylindrical specimens suitable to implement any combination of the following realizable stress states (the cylindrical coordinates r, θ and z associated with the specimen under consideration are used)

- uniaxial tension, $\sigma_{zz} = a = \text{const}$;
- torsion, $\sigma_{r\theta} = b = \text{const}$;
- longitudinal shear, $\sigma_{rz} = c = \text{const}$;
- uniform compression, $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = -d = \text{const}$.

In the above four cases, the other components of the stress tensor are assumed to be equal to zero. It is also assumed that in a certain range of the loads a, b, c and d the stress-strain relation is described by (10.1) and (10.3) with a sufficient accuracy.

Similar establishing experiments are proposed in Georgievskii et al (2012) for the case when $\tilde{\sigma}$ and $\tilde{\varepsilon}$ are deviators ($I_{\varepsilon 1} = 0$, $I_{\sigma 1} = 0$). For the material functions $\{A_0, A_1, A_2\}$ and $\{B_0, B_1, B_2\}$ the following additional relations are valid: $A_0 = -A_2 I_{\varepsilon 2}^2/3$ and $B_0 = -B_2 I_{\sigma 2}^2/3$. In the establishing experiments, hence, it is required to find two functions B_1 and B_2 dependent on $I_{\sigma 2}$ and $I_{\sigma 3}$. By virtue of incompressibility, in Georgievskii et al (2012) the tensor $\tilde{\varepsilon}$ is considered as a strain-rate tensor in a tensor nonlinear non-Newtonian viscous fluid. It is necessary to mention here the work Placidi et al (2015) devoted to the Gedanken experiments for the determination of two-dimensional linear second gradient elasticity coefficients as well as the work Placidi et al (2017) dealing with identification of two-dimensional pantographic structures.

For our original problem, we have

$$\begin{aligned}
\sigma_{rr} &= \sigma_{\theta\theta} = -d, \quad \sigma_{zz} = a - d, \quad \sigma_{r\theta} = b, \quad \sigma_{rz} = c, \quad \sigma_{\theta z} = 0, \\
(\sigma^2)_{rr} &= b^2 + c^2 + d^2, \quad (\sigma^2)_{\theta\theta} = b^2 + d^2, \quad (\sigma^2)_{zz} = c^2 + (a-d)^2, \\
(\sigma^2)_{r\theta} &= -2bd, \quad (\sigma^2)_{rz} = c(a - 2d), \quad (\sigma^2)_{\theta z} = bc, \\
I_{\sigma 1} &= a - 3d, \\
I_{\sigma 2}^2 &= a^2 + 2b^2 + 2c^2 + 3d^2 - 2ad, \\
I_{\sigma 3}^3 &= a(a^2 + 3c^2 + 3d^2) - 3d(a^2 + 2b^2 + 2c^2 + d^2)
\end{aligned} \tag{10.12}$$

Considering d as some parameter, from (10.12) we express a, b and c in terms of the invariants (10.2)

$$\begin{aligned}
a &= I_{\sigma 1} + 3d, \\
b^2 &= \frac{1}{2}(I_{\sigma 2}^2 - I_{\sigma 1}^2 - 4I_{\sigma 1}d - 6d^2) - c^2, \\
c^2 &= \frac{1}{3(I_{\sigma 1} + 3d)}(I_{\sigma 3}^3 - I_{\sigma 1}^3 - 9I_{\sigma 1}^2 d - 24 I_{\sigma 1} d^2 - 24d^3 + 3I_{\sigma 2}^2 d)
\end{aligned} \tag{10.13}$$

Using (10.1) and (10.3) we determine the strain components ε_{zz}, $\varepsilon_{r\theta}$ and ε_{rz}:

$$\begin{aligned}
\varepsilon_{zz} &= B_0 + B_1(a - d) + B_2[c^2 + (a-d)^2], \\
\varepsilon_{r\theta} &= B_1 b - 2B_2 bd, \\
\varepsilon_{rz} &= B_1 c + B_2 c(a - 2d)
\end{aligned} \tag{10.14}$$

The relations expressed by (10.14) can be considered as the system of equations to obtain B_0, B_1 and B_2 using the strain components ε_{zz}, $\varepsilon_{r\theta}$ and ε_{rz} measured experimentally. This system has the solution

$$\begin{aligned}
B_0 &= \varepsilon_{zz} + (d^2 - a^2 - c^2)\frac{\varepsilon_{rz}}{ac} + (c^2 - d^2 + ad)\frac{\varepsilon_{r\theta}}{ab}, \\
B_1 &= 2d\frac{\varepsilon_{rz}}{ac} + (a - 2d)\frac{\varepsilon_{r\theta}}{ab}, \\
B_2 &= \frac{\varepsilon_{rz}}{ac} - \frac{\varepsilon_{r\theta}}{ab}
\end{aligned} \tag{10.15}$$

which is unique if

$$a \neq 0, \quad b \neq 0, \quad c \neq 0 \tag{10.16}$$

Thus, in order to specify the material functions, B_0, B_1 and B_2, the experiments should follow the following steps:

1. The choice of the invariants $I_{\sigma 1}$, I_{σ_2} and I_{σ_3} as the arguments of the functions B_0, B_1 and B_2.
2. The calculation of a, b and c on the basis of (10.13) with a fixed value of d as a parameter.
3. The perfomance of experiments using a hollow cylindrical specimen with the stresses a, b, c, d and the measurements of the components ε_{zz}, $\varepsilon_{r\theta}$ and ε_{rz}.
4. The calculation of B_0, B_1 and B_2 on the basis of (10.15) at the chosen at step 1 point $(I_{\sigma 1}, I_{\sigma_2}, I_{\sigma_3})$ of three-dimensional space of arguments.

Although the parameter d is not mentioned in (10.16) and the formulas expressed by (10.13) and (10.15) are also valid for $d = 0$ the above discussion shows its importance. Changing this parameter, in the space $(I_{\sigma 1}, I_{\sigma_2}, I_{\sigma_3})$ we can enlarge the domain where the quantities a, b and c exist and where the denominator of (10.13) is not equal to zero.

10.3 The Generalized Poynting Effect

Returning to the notion "an order of smallness of tensor nonlinearity effects" discussed in Sect. 10.1, we should set the question which order of smallness is inherent in the Poynting effect. During the last century it attracted an attention in experimental mechanics of solids (Green, 1954; Lurie, 2005; Chen and Chen, 1991; dell'Isola et al, 1998; Akinola, 1999; Gavrilyachenko and Karyakin, 2000; Goldstein et al, 2015; Misra et al, 2018). Let us at once talk about the so-called generalized Poynting effect defined in the following way.

The stress tensor is supposed to have in some orthogonal coordinate system the only identically nonzero component $\sigma_{\alpha\beta} = \sigma_0(\mathbf{x})$, $\alpha \neq \beta$. The stress state of this type is characterized by the following invariants (10.2), (10.8)

$$I_{\sigma 1} = I_{\sigma 3} = 0, \quad I_{\sigma 2} = \sqrt{2}\,|\sigma_0|, \quad J = 2\sigma_0^2, \quad K = 0, \quad L = 2\sigma_0^4/3\,. \tag{10.17}$$

The domain of definition of the material functions B_0, B_1 and B_2 represents the positive real axis in three-dimensional invariant space. According to Eq. (10.1) the tensor $\tilde{\varepsilon}$ has the following nonzero components:

$$\begin{aligned} \varepsilon_{\alpha\alpha} &= \varepsilon_{\beta\beta} = B_0(0, I_{\sigma 2}, 0) + \sigma_0^2 B_2(0, I_{\sigma 2}, 0), \\ \varepsilon_{\gamma\gamma} &= B_0(0, I_{\sigma 2}, 0), \\ \varepsilon_{\alpha\beta} &= \sigma_0 B_1(0, I_{\sigma 2}, 0)\,. \end{aligned} \tag{10.18}$$

Difference from zero of the components $\varepsilon_{\alpha\alpha}$ and $\varepsilon_{\beta\beta}$ just makes up the essence of the generalized Poynting effect. We should not dwell here on the equations of equilibrium as well as on the compatibility equations which the stresses (10.17) and the strains (10.18) must comply with. These equations define more exactly a choice of the material functions B_0, B_1, B_2 and determine the stress $\sigma_0(\mathbf{x})$.

The particular case of the cylindrical coordinate system ($\alpha \to \theta$, $\beta \to z$, $\gamma \to r$) corresponds to the classical Poynting effect observable in experiments with twisted specimens and thoroughly described in the literature. To simulate this, in Goldstein et al (2015) they use initial or deformation anisotropy, in other works they choose various physical nonlinear models of isotropic solids. The formulae (10.18) show that one can attach the constitutive relations (10.1) to the second group.

The material function B_2 being the indication of tensor nonlinearity (as follows from Sect. 10.1 is contained only in the components $\varepsilon_{\alpha\alpha}$ and $\varepsilon_{\beta\beta}$ in (10.18). This fact means that

a) the effect of tensor nonlinearity in the stress-strain state (10.17), (10.18) appears only in difference of the component $\varepsilon_{\gamma\gamma}$ from two remaining diagonal components $\varepsilon_{\alpha\alpha}$ and $\varepsilon_{\beta\beta}$ which are equal to each other;
b) difference of $\varepsilon_{\beta\beta}$ (or $\varepsilon_{\alpha\alpha}$) from zero can be a consequence both tensor nonlinearity and dependence of the function B_0 on the quadratic invariant $I_{\sigma 2}$; this dependence may be realized among them in tensor linear materials when

$$B_0 = B_0(I_{\sigma 1}, I_{\sigma 2}), \quad B_1 = B_1(I_{\sigma 1}, I_{\sigma 2}), \quad B_2 \equiv 0 \tag{10.19}$$

c) the order of smallness of the component $\varepsilon_{\beta\beta}$ (or $\varepsilon_{\alpha\alpha}$), i.e. the generalized Poynting effect, determines by simultaneous smallness of the angle α which by virtue of (10.11) equal to

$$\alpha = \frac{|\sigma_0|}{\sqrt{3}}\,\frac{B_2}{B_1} + O(B_2^2) \tag{10.20}$$

and smallness of values of the dimensionless function $B_0(0, I_{\sigma 2}, 0)$ along the axis $I_{\sigma 2} > 0$.

Below we describe briefly a possibility of simulation of the generalized Poynting effect using the tensor linear constitutive relations (10.1) with the material functions (10.19). The second and the third potentiality conditions (10.5) are fulfilled identically while the first condition (10.5) connects the functions B_0 and B_1 as follows:

$$B_0 = -\frac{\nu}{E}\, I_{\sigma 1} + \frac{b_0}{E^2}\, I_{\sigma 2}^2, \quad B_1 = \frac{1+\nu}{E} + \frac{2b_0}{E^2}\, I_{\sigma 1} \tag{10.21}$$

$$\tilde{\varepsilon} = \frac{1}{E}\Big[\Big(-\nu I_{\sigma 1} + \frac{b_0}{E}\, I_{\sigma 2}^2\Big)\tilde{\mathrm{I}} + \Big(1 + \nu + \frac{2b_0}{E}\, I_{\sigma 1}\Big)\tilde{\sigma}\Big] \tag{10.22}$$

$$w(I_{\sigma 1}, I_{\sigma 2}) = -\frac{\nu}{2E}\, I_{\sigma 1}^2 + \frac{1+\nu}{2E}\, I_{\sigma 2}^2 + \frac{b_0}{E^2}\, I_{\sigma 1} I_{\sigma 2}^2 \tag{10.23}$$

Here E and ν are the material constants known as Young's modulus and Poisson's ratio, respectively; b_0 is the dimensionless material constant which characterizes a scalar nonlinearity of the constitutive relations (10.22). The potential (10.23) includes three constant and when $b_0 = 0$ it turns into the ordinary in linear elasticity potential of stress with respect to strains. Some other variants of a choice of the potential in conformity to the Poynting effect estimate, contain in Gavrilyachenko and Karyakin (2000).

By substituting (10.21) to (10.18) we receive

$$\varepsilon_{\alpha\alpha} = \varepsilon_{\beta\beta} = \varepsilon_{\gamma\gamma} = \frac{2b_0}{E^2}\,\sigma_0^2, \quad \varepsilon_{\alpha\beta} = \frac{1+\nu}{E}\,\sigma_0 \tag{10.24}$$

The value $\varepsilon_{\beta\beta}$ possesses more high order of smallness in comparison with $\varepsilon_{\alpha\beta}$ if $\varepsilon_{\beta\beta}/\varepsilon_{\alpha\beta} \ll 1$, i. e. $b_0\sigma_0 \ll E$. It is just the condition that the generalized Poynting effect within tensor linear connection of stresses and strains represents a phenomenon of the second order. In case of the classical Poynting effect ($\alpha \to \theta$, $\beta \to z$, $\gamma \to r$) the formulae (10.24) show that the relative extension ε_{zz} is in proportion to square of the strain $\varepsilon_{\theta z}$, which is conversely in proportion to the angle of twisting. This feature of the Poynting effect is often exploited in the literature.

It is necessary to mention here a so-called inverse Poynting effect as in Goldstein et al (2015) consisting in twisting of a specimen by action of one-dimensional stretching loading. Some off-diagonal components of the strain tensor are not equal to zero. It is obvious that this phenomenon can not be described by the relations (10.1) even by arbitrary form of tensor nonlinearity. However, one can simulate it using anisotropic models of continuum.

References

Abali BE, Müller WH, Georgievskii DV (2013) A discrete-mechanical approach for computation of three-dimensional flows. ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik 93(12):868–881

Akinola A (1999) An energy function for transversely-isotropic elastic material and the Poynting effect. Korean J Comput Appl Math 6(3):639 – 649

Altenbach H, Altenbach J, Zolochevsky A (1995) Erweiterte Deformationsmodelle und Versagenskriterien der Werkstoffmechanik. Deutscher Verlag für Grundstoffindustrie, Stuttgart

Chen M, Chen Z (1991) Second-order effect of an elastic circular shaft during torsion. Appl Math Mech 12(9):769 – 776

dell'Isola F, Ruta GC, Batra RC (1998) Generalized poynting effects in predeformed prismatic bars. Journal of Elasticity 50(2):181 – 196

Devendiran VK, Sandeep RK, Kannan K, Rajagopal KR (2017) A thermodynamically consistent constitutive equation for describing the response exhibited by several alloys and the study of a meaningful physical problem. International Journal of Solids and Structures 108(1):1 – 10

Gavrilyachenko TV, Karyakin MI (2000) Specific features of the nonlinearly elastic behavior of cylindrical compressible bodies in torsion. Journal of Applied Mechanics and Technical Physics 41(2):377–381

Georgievskii DV (2016a) Establishing experiments in tensor nonlinear theories of continuum mechanics. Moscow University Mechanics Bulletin 71(2):49–50

Georgievskii DV (2016b) Potentiality of isotropic nonlinear tensor functions relating two deviators. Mechanics of Solids 51(5):619–622

Georgievskii DV, Müller WH, Abali BE (2012) Establishing experiments to find material functions in tensor nonlinear constitutive relations. Bulletin of the Russian Academy of Sciences: Physics 76(12):1374–1377

Goldstein RV, Gorodtsov VA, Lisovenko DS (2015) Linear Poynting's effect at torsion and extension of curvilinearly anisotropic tubes. Doklady Physics 60(9):396–399

Green AE (1954) A note on second-order effects in the torsion of incompressible cylinders. Mathematical Proceedings of the Cambridge Philosophical Society 50(3):488–490

Il'yushin AA (1963) Plasticity. Foundations of the General Mathematical Theory (in Russ.). USSR Acad. Sci. Publ., Moscow

Kulvait V, Málek J, Rajagopal KR (2017) Modeling gum metal and other newly developed titanium alloys within a new class of constitutive relations for elastic bodies. Arch Mech 69(3):223 – 241

Lurie AI (2005) Theory of Elasticity. Foundations of Engineering Mechanics, Springer, Berlin, Heidelberg

Misra A, Lekszycki T, Giorgio I, Ganzosch G, Müller WH, dell'Isola F (2018) Pantographic metamaterials show atypical poynting effect reversal. Mechanics Reserch Communications 89:6–10

Placidi L, Andreaus U, Della Corte A, Lekszycki T (2015) Gedanken experiments for the determination of two-dimensional linear second gradient elasticity coefficients. ZAMP - Journal of Applied Mathematics and Physics / Zeitschrift für Angewandte Mathematik und Physik 66(6):3699–3725

Placidi L, Barchiesi E, Della Corte A (2017) Identification of two-dimensional pantographic structures with a linear d4 orthotropic second gradient elastic model accounting for external bulk double forces. Mathematical Modelling in Solid Mechanics Ser Advanced Structured Materials 69:211–232

Rivlin RS (1953) The solution of problems in second order elasticity theory. Journal of Rational Mechanics and Analysis 2:53–81

Rivlin RS, Ericksen JL (1955) Stress-deformation relations for isotropic materials. Journal of Rational Mechanics and Analysis 4:323–425