Kähler-Einstein Metrics via Moduli Continuity Method

Cristiano Spotti

If one figure is derived from another by a continuous change and the latter is as general as the former, then any property of the first figure can be asserted at once for the second figure.

Jean Victor Poncelet (1788–1867)
Traité des propriétés projectives des figures, 1822

Abstract We discuss some ideas behind a strategy that has been used to construct Kähler-Einstein metrics for *explicit families* of Fano varieties.

Keywords Kähler-Einstein metrics · Moduli spaces · Fano varieties

1 Introduction

A major problem in complex differential geometry consists in understanding which Fano manifolds admit Kähler-Einstein metrics. Recall that a n-dimensional complex manifold X is said to be Fano if it has positive first Chern class or, equivalently, its anticanonical bundle $K_X^{-1} = \bigwedge^n TX$ is ample. Geometrically, a Kähler-Einstein (KE) metric is simply an Einstein space for which parallel transport commutes with the underlying compatible complex rotation. It is a non-trivial fact that such metrics, necessarily with *positive* constant scalar curvature, are unique up to the natural symmetries (biholomorphisms and scalings). Thus KE metrics provide a way to canonically "geometrize" Fano manifolds. However, not all Fano manifolds admit such metrics, as the classical example of the blow-up of the projective plane in one point shows.

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Understanding exactly which Fano manifolds admit KE metrics has been the object of intense investigation in the last decades. These studies culminated in the recent solution of the so-called *Yau-Tian-Donaldson conjecture* in the Fano case ([8] for the "if" part, and [4] for the "only if"):

Theorem 1.1 *A smooth Fano manifold X admits a KE metric if and only if is K-polystable.*

Such result shows that the transcendental problem of finding a KE metric (a solution of a geometric PDE) is equivalent to the purely algebro-geometric property of K-stability. In short, checking K-stability amounts to compute the positivity of certain numerical invariant (Donaldson-Futaki invariant) on $\mathbb{C}^*$-equivariant degenerations of a Fano manifold X to a possibly singular variety X_0.

However, despite its fundamental theoretical importance, the above theorem is at present not very useful in constructing new KE metrics since, in general, it is still impossible to verify the K-stability property from its definition, due to the too many degenerations which a-priori need to be checked.

In this note, we would like to describe a different method for showing existence of KE metric on *explicitly given* Fano manifolds, which has been used in [31, 34] and it can possibly be applied in many other new situations.

Often Fano manifolds comes in *complex families* $\pi : \mathcal{X} \to \mathcal{H}$, with $\pi^{-1}(t) = X_t$ Fano variety. By varying the base parameter t we obtain in general non-biholomorphic (but still diffeomorphic) Fano manifolds. To keep an easy example in mind, consider the case of hypersurfaces of degree less than $n+2$ in $\mathbb{CP}^{n+1}$. By varying the coefficients of the defining polynomials, we obtain in general different Fano varieties. It is important to mention here that in low dimensions Fano manifolds are fully classified: each one is given as a member of some *explicit family*. In dimension two the classification is classical. In dimension three it is obtained thanks to works of Fano, Iskovskikh, Mori and Mukai [21].

It is then natural to ask the following question:

Question 1.2 *Which Fano varieties in a given* explicit family $\pi : \mathcal{X} \to \mathcal{H}$ *carry KE metrics?*

A natural strategy to analyze such situation consists in investigating the KE existence problem by studying variations of the parameters $t \in \mathcal{H}$. The idea of studying the KE problem by varying the complex structure is definitely not a new one. Indeed, it was used by Tian to solve the KE existence problem in dimension two [37]. He proved that, in each degree $d = c_1^2(X) < 5$ the KE condition is non-empty, open and closed within the subset of a natural parameter space $\mathcal{H}_d$ parameterizing *smooth* Fano surfaces of degree d.

Here we are going to explain an extension of such ideas. We do not focus on smooth Fanos only, but we also consider some degenerate singular limits. Moreover, we crucially make use of stability conditions (K-stability and classical GIT) in the study of how the KE condition varies in a family, by relating degenerations to concrete algebraic moduli constructions. We refer to this method as to the *moduli continuity method*. Such strategy was first used in [31] to study the so-called Gromov-Hausdorff (GH) moduli compactification of the space of smooth KE Fano surfaces (for degree $d = 4$ there is a previous work of T. Mabuchi and S. Mukai [28], which uses a slightly different approach). By applying this method, we do not only understand precisely which smoothable Fanos in a given family admit KE metrics, but we also provide a concrete description of the "abstract" GH moduli compactification, also known as K-moduli space (see [33] for a survey on such moduli spaces of algebraic varieties with their relation to special metrics). Moreover, this gives an explicit classification of the singularities of GH limits of certain Einstein manifolds, which is definitely interesting from a purely differential geometric point of view.

2 The *Moduli Continuity Method*

The moduli continuity method can be described as a strategy that can be used to answer Question 1.2. We now explain and comment the main steps, in a somehow idealized situation. Since our focus is on the main ideas, we refer to the literature for precise definitions and arguments.

Being a continuity method, it is not a surprise that it consists of three main parts: *non-emptiness*, *small variations* and *large variations*. We are going to describe very quickly the first two, while spending more time on the last one, since it has been the object of some very recent advances.

Non-emptiness The first step consists in finding within our family $\pi : \mathcal{X} \to \mathcal{H}$ a Fano variety X_{t_0} for which we can "easily" conclude that a KE metric exists on it. But where to look for such variety? As a general rule, we should look for a Fano which is *more symmetric* and apply some *existence criterion*, such as the G-invariant α-invariant, with $G \subseteq Aut(X)$ finite.

In principle, we could possibly even search among mildly singular Fano within the family. For example, the singular cubic surface $xyz = t^3$ is a finite quotient of the projective plane, and hence it has an obvious KE metric. Or we could look for a toric varieties since, in this case, the KE problem is fully understood [5]. Note that to actually apply the method starting from a singular variety, we would need to argue that some nearby smoothing is KE (see discussion in the next step).

Small variations The next step concerns how the KE condition varies for small perturbations of the complex parameters $t \in \mathcal{H}$. If the automorphism group $Aut(X_t)$ is discrete, a simple application of the implicit function theorem shows that all X_s sufficiently close to the KE manifold X_t admit KE metrics too. Actually, it can be proved that, in such case, the KE condition is Zariski open [11, 30].

In general, however, the automorphism group does not need to be discrete neither the KE condition open. Nevertheless, the situation is understood via a *local GIT picture* [7, 36]: we can look at the natural induced action of the reductive automorphism group on the space of infinitesimal deformations in order to understand which nearby Fano manifold remains KE. The prototypical situation is the case of the Mukai-Umemura Fano 3-fold and its deformations, first analyzed in [38]. We can then understand which are the Fanos near X_t in our family which remain KE.

The singular situation is more subtle, but it has been discussed at least for metric limits in [24, 35].

Large variations Let X_{t_i} a sequence of KE Fano manifolds in our family, and let $t_i \to t_\infty \in \mathcal{H}$. The question now is: does X_{t_∞} (even singular) need to be KE too?

In general, the answer is negative. However, in certain situations, we can give a positive answer. By the limit picture for KE Fanos of [12], eventually by passing to a subsequence, we can assume that X_{t_i} converge in a "refined" GH sense to a singular KE Fano variety X_∞. That is, they converge both in the metric GH sense and as complex cycles in a given uniform projective embedding. However, this abstract natural limit X_∞ a-priori does not need to be given by a variety within our original starting family $\pi : \mathcal{X} \to \mathcal{H}$.

To actually show that X_∞ is indeed a (special!) member of our family, we need three main steps:

1. Refined a-priori control on the singularities of GH (K-stable) limits.
2. Classifications of mildly singular Fanos.
3. Stability comparison argument.

The first point has seen recent advances, but we postpone its more careful discussion in the next section. For the moment, we could just say that a consequence of such analysis should give effective bounds (in terms of natural invariants of the general member of the family) of the so-called *Gorenstein index* of X_∞, that is, the minimal power to which we need to raise the $\mathbb{Q}$-Cartier anticanonical divisor $-K_{X_\infty}$ to find a genuine line bundle.

Let us suppose that a small a-priori bound on the index has been achieved. As I recalled at the beginning, in some situations, Fano manifolds have been classified. Thus, it becomes now important to extend the classification to the mildly singular Fanos as given by the first step. Usually, this ends up in showing an effective bound on the very-ampleness of the anticanonical bundle. A further information one could possibly use is that the limit X_∞ is $\mathbb{Q}$-Gorenstein smoothable. We think that this analysis should rise interesting problems for algebraic geometers.

In a lucky case, the extended classification may give that X_∞ is indeed biholomorphic to a member of our family, say X_{t_*} (see later for a discussion in the case this does not hold). But, is X_{t_*} biholomorphic to our starting flat limit X_{t_∞}?

It is here that the last step enters the game (and, also, it is here the reason why we have called such strategy a *moduli* continuity method). From Berman result [4] we know that $X_\infty \cong X_{t_*}$ is K-polystable. On the other hand, on our family there would usually be an equivariant action of a linear group G such that two varieties are abstractly isomorphic if and only if there is an element of the group carrying one to the other (just think for example to the natural action of $SL(n+2)$ on the space of projective Fano hypersurfaces). This can give rise to a *classical GIT problem*. It becomes now crucial to understand how K-stability relates to such classical GIT stability. In good situations (e.g., when one can check that the CM line bundle [32], whose weight is the Donaldson-Futaki invariant, is an equivariant positive multiple of a linearization considered in a classical GIT, the family is nice enough to avoid the Li-Xu pathology, etc. . .) we can infer that K-polystability implies GIT-polystability. Thus we can conclude that t_* is now in $\mathcal{H}^{ps}$, the GIT polystable locus. Moreover, a Luna's slice argument shows that $[X_{t_i}]$ converges to $[X_{t_*}]$ in the analytic topology of the explicit GIT quotient $\mathcal{H}//G$. Using the fact that $\mathcal{H}//G$ is *Hausdorff* we can now see that, if $t_\infty \in \mathcal{H}^{ps}$, $X_{t_\infty} \cong X_{t_*} \cong X_\infty$ carries a KE metric.

Finally, running an open-closed argument and using the density of smooth Fanos in our family, we can deduce that the natural injective (by uniqueness of the KE metric) continuous map $\phi : \overline{\mathcal{EM}}^{GH} \to \mathcal{H}//G$ we have constructed, is indeed surjective. Here $\overline{\mathcal{EM}}^{GH}$ is the *compactification* of the moduli spaces of KE Fanos manifolds in our family up to biholomorphic isometries equipped with the refined GH topology (also known as K-compactification). Hence ϕ is a homeomorphism by the standard compact-to-Hausdorff argument.

In conclusion, the problem of understanding which Fanos in our family is KE has been reduced to the study of a classical GIT quotient, that can be concretely analyzed via standard algebro-geometric techniques.

There are few points we would like to emphasize and comment on. It is worth noting that this approach requires to work necessarily with formation of singularities, even if one cares about the existence only of *smooth* KE Fanos. This is typical and not surprising in analysis (e.g., regularity theory, geometric flows, etc. . .). There is some "hard analysis" input also in this moduli continuity method approach: this is "hidden" in the "algebraic regularity" result [12], itself based on Cheeger-Colding regularity theory of limit spaces. After that, the argument becomes addressable with help from algebraic geometry. This is possible thanks to the presence of the underlying canonical algebraic structure, which make our KE case, in a certain sense, special among geometric PDEs.

As we mentioned in the introduction, as a non-trivial by-product of such moduli method we obtain a concrete description of all GH degenerations of the KE metrics in our family. A further interesting question to investigate is if such compactifications actually provide a compactification of a *connected component* of the full Einstein moduli space on the real underlying smooth manifold. That is, it would be interesting to see if there can be Einstein but *non-Kähler* deformations of a KE Fano manifold. As far as we are aware, this problem has not been solved yet, but it is very intriguing from a differential geometric viewpoint.

Finally, before discussing the crucial aspect of bounding a-priori the singularity types of GH limits, we should stress that the strategy described will require, in general, some adjustment in order to be applied: the main issue is that, in several cases, the abstract GH limits cannot always live in the family we started with! To deal with this problem, it would be needed to perform some birational modifications of our original family in order to accommodate such limits. Even for hypersurfaces in $\mathbb{CP}^{n+1}$ we cannot expect in general to reduce the KE problem just to the obvious GIT quotient: for example, for quartics 3-folds in $\mathbb{CP}^4$ it is clear that one should consider, at least, one blow-up of the family at the non-reduced double quadric, in order to accommodate the (KE) "hyperelliptic" Fanos which are given by taking double covers of a quadric. In any case, the moduli continuity method can be applied, with more work, also to such situations [31].

2.1 Refined A-Priori Control on the Singularities of GH Limits

By the general theory of Donaldson-Sun [12], we know that a GH limit X_∞ of smooth KE manifolds is a singular Fano variety with $\mathbb{Q}$-Cartier canonical divisor and Kamamata log terminal (klt) singularities, i.e., X_∞ is normal and for each log-resolution $r : \hat{X}_\infty \to X_\infty$, the canonical divisor $K_{\hat{X}_\infty} = r^* K_{X_\infty} + \sum_i a_i E_i$ with *discrepancies* $a_i > -1$. Moreover, X_∞ carries a weak KE metric in the sense of [14]. In this section we want to explain how to get further bounds on the singularities.

In complex two dimension, it was previously known by works of Anderson, Tian and many others, that GH limits are KE orbifolds, i.e., the singularities are locally of type $\mathbb{C}^2/\Gamma_p$, with $\Gamma_p \subseteq U(2)$ finite, acting freely on the sphere (precisely the klt condition in dimension two) and the metric is orbifold smooth. Thus, a natural invariant which measures the "sharpness" of a singularity is given by the order of the group at p. Since the KE metric satisfies the Bishop-Gromov monotonicity formula, we can relate the "local volume" (i.e., the order of the group) with the global volume (which is preserved in GH limits), thus obtaining some a-priori bounds on the order of the orbifold singularities. This was used for analyzing the two dimensional case [31].

In higher dimension the situation becomes more subtle. First of all, the expected general singularity won't be of quotient type (Schlessinger's rigidity). For example, this happens for the ordinary double point ODP singularity $\sum_i x_i^2 = 0$ in dimension bigger than or equal to three. Moreover, if we rescale the weak KE metric near a singularity $p \in X_\infty$, the metric tangent cone (a singular Calabi-Yau cone $C(Y_p)$) won't be in general locally biholomorphic (actually not even homeomorphic!) to the singularity germ $\mathcal{V} \subset X_\infty$ itself. Such local jump of the complex structure was first observed by Hein and Naber [19] (the isolated A_k-singularity in three dimension jumps to the flat splitting $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ as soon as $k \geq 3$), and the picture has been made general and precise in [13]. In any case, from a geometric measure theory viewpoint, it still makes sense to define the *density* as a measure of the sharpness:

$$\Theta_p := \lim_{r \to 0} \frac{Vol(B_p^{KE}(r))}{\omega_{2n} r^{2n}} = \frac{Vol(Y_p)}{Vol(S^{2n-1})} \leq 1.$$

As before, one could bound the densities using Bishop-Gromov estimates. However, by some experimental inspection, the estimate one gets this way is too weak to be of any use in higher dimension. In order to proceed further, we need new input from algebraic geometry.

Trying to find a purely algebro geometric construction of the metric tangent cones and inspired by previous work [29] in Sasakian geometry by Martelli, Sparks and Yau, Li introduced in [22] the following new invariant for a germ $(\mathcal{V}, p)$ of a klt singularity, the *infimum of the normalized volume of valuations*:

$$\hat{\mathrm{vol}}(\mathcal{V}, p) := \inf_{\nu \in Val_p} A^n(\nu)\,\mathrm{vol}(\nu) > 0,$$

where $A(\nu)$ is the log-discrepancy of a valuation and $\mathrm{vol}(\nu) := \limsup_{r\to 0} \frac{\mathrm{length}(\mathcal{O}_{V,p}/\{f|\nu(f)\geq r\})}{r^n/n!}$ its volume. It is proved in [6], that the infimum is actually a minimum.

This new invariant has to be considered as the algebro-geometric analogue of the metric density. The order of a germ of an holomorphic function at the singularity computed with respect to the distance induced by the KE metric induces a valuation ν_{KE}. In [20] it was proved that $A^n(\nu_{KE})\,\hat{\mathrm{vol}}(\nu_{KE}) \leq n^n \Theta_p$ and equality holds for quasi-regular tangent cones. More recently [23], the equality has been shown to hold in any situation, and moreover ν_{KE} is indeed the (unique among the so-called quasi-monomial valuations) minimizer. Hence for singularities in GH limits $\hat{\mathrm{vol}}(\mathcal{V}, p) = n^n \Theta_p$.

The next crucial ingredient is the following "algebro-geometric Bishop-Gromov" estimate, proved by Liu [25] as a generalization of Fujita's volume bound for K-semistable Fano manifolds [16]:

$$c_1^n(X) \leq \left(1 + \frac{1}{n}\right)^n \hat{\mathrm{vol}}(\mathcal{V}, p),$$

for any germ of singularity $(\mathcal{V}, p)$ in a n-dimensional K-semistable Fano variety X. This estimate is stronger than Bishop-Gromov estimate. Thus, provided that the volume of a sequence of KE Fano manifolds is large enough, we obtain good quantitative lower bounds for the volume densities at the singularities, since GH limits are K-polystable, as a consequence of Berman's result [4].

Moreover, it is very natural to expect that such densities satisfy certain gaps among their values (in analogy with minimal surfaces theory, in which, for example, the Willmore's conjecture can be interpreted as a gap for the density of certain minimal cone). Thus, since the ODP singularity is, in a rough sense, the simplest one, it is very natural to expect the following:

Conjecture 2.1 *ODP gap conjecture* [34]*:* $\Theta_p \left(= n^{-n}\hat{vol}(\mathcal{V}, p)\right) \leq 2\left(1 - \frac{1}{n}\right)^n$, *for any singularity* $p \in X^n$*, and the equality holds iff the singularity is an ODP (and the metric tangent cone is the ODP with its natural CY Stenzel's cone metric)*

This is clearly true thanks to the orbifold regularity of KE metrics for $n = 2$. Moreover, using classification results for three dimensional canonical singularities, it has been very recently proved in [26] that the value 16 is indeed the infimum of the normalized volume for all klt singularities (non-necessarily assumed to come from GH limit).

To state the next theorem, let us introduce the following quantity,

$$V(n) = \sup\{\Theta(C^n(Y)) \mid C^n(Y) \cong \mathbb{C}^k \times C(Y') \ncong \mathbb{C}^n,\ k \geq 0,\ Y'\text{smooth}\},$$

and recall that the *Fano index* is the maximal $r \in \mathbb{N}$ such that $K_X^{-1} = L^r$ for L an ample line bundle.

Theorem 2.2 ([34]) *Let X_∞ be a GH limit of n-dimensional smooth KE Fanos X_i of index r such that*

$$c_1^n(X_i) > \frac{(n+1)^n}{2} V(n).$$

Then X_∞ has Gorenstein canonical singularities *(i.e., the discrepancies are non-negative for any log resolution) and $K_{X_\infty}^{-1} = L^r$ for some line bunde L.*

It is clear that $2\left(1 - \frac{1}{n}\right)^n \leq V(n) \leq 1$. Thus, if $c_1^n(X) > (n+1)^n/2$ (that is, if the volume is bigger half of the volume of the projective space), the hypothesis holds. As we will see below, this condition is not empty. Moreover, since, as we have recalled, for $n = 3$ the volume gap holds true [23, 26], the above theorem implies (see [34]):

Corollary 2.3 *GH limits of KE Fano 3-folds of degree bigger than 20 are Gorenstein Fanos with canonical singularities and same Fano index. These include intersections of two quadrics, cubic hypersurfaces, and Fano 3-folds of rank one and degree 22 (deformations of the Mukai-Umemura manifold).*

More generally, one can obtain bounds on the Gorenstein index, which can also be very useful, as the two dimensional case shows.

In a nutshell the proof of Theorem 2.2 consists in:

1. Use the Liu's estimate of the volume to find a bound on the fundamental group of the possibly singular link of the metric tangent cone (by applying Colding-Naber convexity of its smooth locus [9]);
2. Bound the Cartier index on the cone via some covering trick;
3. Use the 2-steps construction of metric tangent cone in [13], to obtain the index bound on the original singular variety.

The definition of $V(n)$ is used in combination to Schlessinger's rigidity of quotient singularities to rule out certain situations.

As [26] suggest, part of the arguments can be made fully algebraic and more general by establishing that certain properties of the normalized volumes of valuations (mostly related to coverings) holds, thus avoiding to use more differential geometric techniques based on Cheeger-Colding theory.

3 Applications

The moduli continuity method has been applied in dimension two to fully study the GH compactification of KE surfaces in [31]. As a by-product we obtained an explicit classification of two dimensional KE Fano orbifolds with singularities of type $\mathbb{C}^2/\Gamma$, with $\Gamma \subseteq SU(2)$. In the proof we used, in combination with the bounds on the orbifold group, the classification of $\mathbb{Q}$-Gorenstein smoothable quotient singularities and classification results for certain smoothable Fano surfaces. We then constructed algebraic moduli spaces of Fano surfaces, which we showed to agree with the GH/K-moduli compactification via our strategy. While for degree 4 or 3 we could make use of classical GIT quotient, for degree 2 or 1 we performed certain birational modifications, resulting in a "gluing" of GIT quotients.

In the recent [34], we applied the above Theorem 2.2 and used the moduli strategy to show the following results.

Theorem 3.1 ([34]) *A possibly singular complete intersection of two quadrics* $X = Q_1 \cap Q_2$ *in* $\mathbb{P}^{n+2}$ *is KE (eq. is K-polystable) if and only if, up to reparametrization,* $Q_1 = 1$ *and* Q_2 *is diagonal with no more than* $(n+3)/2$ *equal eigenvalues and, if equality holds, then* $X \cong \{\sum_{i=0}^{(n+1)/2} x_i^2 = \sum_{(n+3)/2}^{n+2} x_i^2 = 0\}$. *In particular, all smooth intersections are KE, GH limits have at most bundles of ODP as singularities, and the GH compactification agrees with the GIT quotient* $Gr(2, Sym^2(\mathbb{C}^{n+3}))//SL(n+3)$ *obtained by associating to an intersection of two quadrics its pencil.*

Theorem 3.2 ([34]) *If ODP gap conjecture 2.1 holds for any* $k \leq n$ *(which does for* $n \leq 3$ [23, 26]*), then a possibly singular cubic* n*-fold admits a KE (eq. is K-polystable) if and only if it is GIT polystable for the* $SL(n+2)$ *action on* $Sym^3(\mathbb{C}^{n+2})$. *In particular, all smooth cubics are KE.*

Note that in dimension three GIT polystable cubics are fully classified [1]. For $n = 3$, Theorem 3.2 has been derived also in [26] as a consequence of their volume gap proof.

Thanks to the control of singularities provided by Theorem 2.2, the main point in the proof of Theorems 3.1 and 3.2 consists in showing that the GH limits embed naturally in the original family by applying Fujita's classification of singular del Pezzo varieties [15], and finally by using the continuity method strategy. The explicit form of Theorem 3.1 follows by the GIT analysis [3]. For $n = 3$, Theorem 3.1 is a bit special, since we hit the boundary of Theorem 2.2's inequality. Nevertheless, it can still be proven via rigidity arguments, avoiding to use the volume gap. We remark that the existence of KE on all smooth intersections of two quadrics was known before [2], but for cubics threefolds it was known only for special cases. The generic singularity in the above examples is an ODP. However, from a metric viewpoint, it is still unknown the full asymptotic to the Stenzel's CY cone metric [20].

Related to the above theorems, there are some interesting algebro geometric questions which deserve further investigation. In particular: are there other cases in higher dimension where Theorem 2.2 applies? Is indeed true that all GIT semistable cubics are normal in every dimension? What can we say about KE limits of Fano manifolds of Picard rank one and degree 22? Namely, do they embed as intersections of three sections of a tautological bundle on a Grassmannian similarly to the smooth case? Can one study explicitly the associated GIT problem?

Finally, another direction which is interesting to explore is the so-called *log case*, i.e., of Fano pairs (X, D) with $D = \sum_i (1-\beta_i)D_i$, $\beta_i \in (0, 1)$ admitting singular KE metrics with $2\pi\beta_i$ cone singularities at the generic points of D. This situation is not trivial, but well-understood, already in dimension one. It is known [27, 39] that the existence of a KE metric on log-$\mathbb{P}^1$s, i.e., $(\mathbb{P}^1, \sum_{i=1}^n (1-\beta_i)p_i)$ with $\beta_i \in (0, 1)$ and $d = 2 - \sum_i (1-\beta_i) > 0$, is equivalent to the Troyanov's condition $1-\beta_i < \sum_{j\neq i}(1-\beta_j)$ for $n \geq 3$, and $\beta_1 = \beta_2$ for $n = 2$. How the natural KE/K-compactification looks in this case? For fixed n, d and values of the cone angles, the only thing that can happen for limits is that *points collide*, since all the (marked) GH limits must still be given by a log-$\mathbb{P}^1$s by Gauss-Bonnet. To understand the possible limits one can use the moduli continuity method. It is in fact easy to see that the Troyanov's condition is equivalent (at least for rational angles) to GIT-stability for the rational polarization $\mathcal{L}_{\underline{\beta}} := \boxplus_{i=1}^n \mathcal{O}(1-\beta_i)$ on $(\mathbb{P}^1)^n$ (weighted points). Thus, by the continuity method, we have $\overline{\mathcal{EM}}^{GH}_{n,d,\underline{\beta}} \cong (\mathbb{P}^1)^n//_{\underline{\beta}} SL(2)$. Note that, by varying the cone angles, we obtain birational modifications of the moduli spaces.

In higher dimension, the situation is definitely more subtle, since also the divisor D_i may become quite singular in the limits. Even in dimension two, it is going to be essential to use the new advances related to valuations and the expected tangent cone description [10] to control the singularities of the divisors. Natural first steps to investigate are the KE-compactifications of moduli of $(\mathbb{P}^2, (1-\beta)D)$, with D degree $d \geq 3$ hypersurface and, for example, cubic surfaces with an hyperplane section [17]. At least for cone angles big enough, it is expected that the KE/K-compactification does agree with some GIT quotient naturally associated to the problem [18].

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