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Moduli of K-stable Varieties

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Moduli of K-stable Varieties

Springer

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Preface

This volume contains a collection of papers related to research presented at the INdAM Workshop “Moduli of K-stable varieties”, which was held in Rome, from 10 to 14 July 2017, at Sapienza Università di Roma.

The theme of the workshop, and hence also the theme of this book, was moduli theory. The basic goal of moduli theory is to form a parameter space for algebraic varieties. The centrality of this problem in geometry goes back at least to Riemann’s dissertation, and has been maintained in the twentieth century through the renowned work of Mumford and others. In modern algebraic geometry the three most important classes of varieties are Calabi-Yau varieties, canonically polarised varieties and Fano varieties. The main goal is to construct well-behaved projective moduli spaces for these classes of varieties. In the case of polarised Calabi-Yau manifolds, a quasi-projective moduli space parametrizing smooth varieties was formed in the 1980s and 1990s by Schumacher and Viehweg, but the structure of its compactifications remain unclear. The case of canonically polarised varieties was clarified greatly by Kollár-Shepherd-Barron in 1988, and a well-behaved projective moduli space has now been formed using many of the recent important advances in birational geometry.

The situation of Fano varieties is quite different, and was completely unclear until the last decade. The fundamental issue is that moduli spaces of Fano varieties are not automatically separated. With motivations more related to Kähler geometry rather than moduli theory, Tian and Donaldson introduced the notion of K-stability in order to understand the existence of Kähler-Einstein metrics on Fano varieties. Over time it became apparent that K-stability was also the right notion for forming moduli spaces: through work of Li-Wang-Xu, Spotti-Sun-Yao and Odaka, one can form a moduli space of smooth K-stable Fano varieties, and can even compactify this space by including certain singular K-stable Fano varieties at the boundary. Moreover, through work of Odaka, the moduli space of Calabi-Yau varieties and canonically polarised varieties can equally be seen as moduli spaces of K-stable varieties. Thus K-stability plays an important unifying role in moduli theory.

This volume contains two surveys, suitable for beginners in the field, on the moduli theory of K-stable varieties. The first is by Spotti, who details much of the analytic theory surrounding Kähler-Einstein metrics and how one can explicitly construct moduli spaces of K-stable Fano varieties using the theory of Gromov-Hausdorff convergence. The second is by Wang, who gives a detailed introduction to the general abstract construction of the moduli space of K-stable Fano varieties, which uses a combination of ideas from algebraic, differential and symplectic geometry.

The volume includes a further six research articles. On the more analytic side is work of Legendre and Sjöström Dyrefelt. Legendre studies the existence of special almost-Kähler metrics on almost complex manifolds, including a proof that the existence of such metrics is equivalent to a suitable notion of K-stability. Sjöström Dyrefelt demonstrates novel results in the theory of K-stability of arbitrary complex manifolds, proving the best known results in this direction.

On the algebraic side, the volume contains work of Ambro-Kollár, Codogni-Stoppa, Fujita and Odaka. Ambro-Kollár develop further the theory of semi-log canonical pairs, which are crucial in compactifying various moduli spaces. Codogni-Stoppa study the notion of equivariant K-stability, which is suitable for certain symmetric varieties. They show how one can use such results to reprove a link between canonical Kähler metrics and K-stability. Fujita also studies K-stability of highly symmetric varieties called toric varieties, from a different point of view to the traditional work. Odaka develops a new approach to compactifying moduli spaces of curves, which uses tropical geometry.

We gratefully thank the Istituto Nazionale di Alta Matematica "Francesco Severi" for providing funding and logistical support for our workshop.

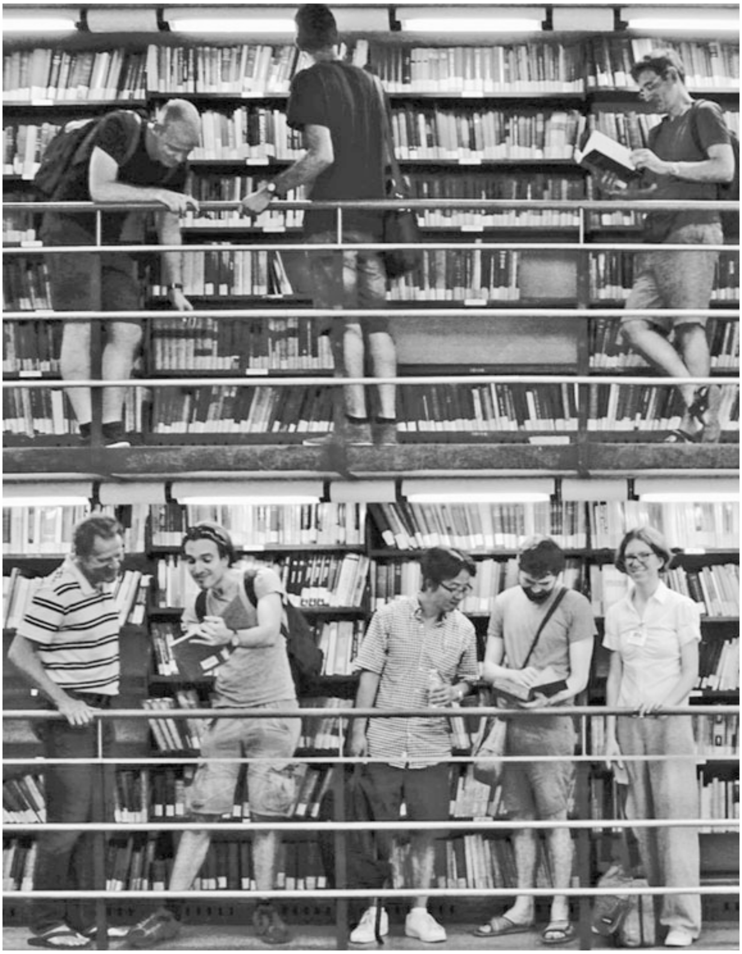
We also thank Jacopo Stoppa for organising the workshop with us.

Rome, Italy — Giulio Codogni
Cambridge, UK — Ruadhaí Dervan
Rome, Italy — Filippo Viviani
December 14, 2018

Conference Pictures

These pictures were taken in the library and courtyard of the Department of Mathematics “Guido Castelnuovo”, Sapienza University of Rome. The building for the School of Mathematics, inaugurated in 1935, was designed by Gio Ponti, who included a cutting-edge library with a reading room on four floors.

We thank the library staff for their hospitality and Andrea Fanelli for the pictures.



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Minimal Models of Semi-log-canonical Pairs

Florin Ambro and János Kollár

Abstract We compare the minimal model of a log canonical pair with the minimal model of its reduced boundary. These results are then used to study the existence of the minimal model of a semi-log-canonical pair using its normalization.

Keywords Minimal model · Semi-log-canonical · Adjunction · Flip

In birational geometry, it is frequently necessary to work not just with log canonical pairs (X, Δ), but with their non-normal variants, called *semi-log-canonical pairs.* Such pairs appear when one tries to compactify the moduli spaces of varieties and in inductive arguments.

Many properties of log canonical pairs have been generalized to the semi-log-canonical setting [1, 2, 8, 9, 18], but it was observed in [17] that log canonical rings of semi-log-canonical pairs are not always finitely generated and some flips of semi-log-canonical pairs do not exist. Note that, by contrast, abundance holds for a semi-log-canonical pair iff it holds for its normalization; this was proved in increasing generality in [7, 10–12, 15, 16, 22].

The aim of this note is to describe some conditions that guarantee the existence of minimal models for certain semi-log-canonical pairs. Our assumptions are rather restrictive, but they may be close to being optimal. The key is to understand an even simpler question involving log canonical pairs: *How does the boundary of a log canonical pair change under a flip?*

This is a very natural problem, that first appeared explicitly in Tsunoda's treatment of semi-stable flips [21], later in Shokurov's approach that reduces flips to special flips [16, 23] and in [13, Sec.4]; see also [5].

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We are thus led to the following general questions.

Question 1 Let $(X, D + \Delta)$ be an lc pair that is projective over a base scheme S with relatively ample divisor H, where all divisors in D appear with coefficient 1. Set $(X^0, D^0 + \Delta^0) := (X, D + \Delta)$ and for $i = 1, \dots, m$ let

$$\phi^i : (X^{i-1}, D^{i-1} + \Delta^{i-1}) \dashrightarrow (X^i, D^i + \Delta^i)$$

be the steps of the $(X, D + \Delta)$-MMP with scaling of H; see Definition 11. Let $\rho : \bar{D} \to D$ be the normalization. Do the restrictions

$$\phi^i_D := \phi^i|_{\bar{D}^{i-1}} : (\bar{D}^{i-1}, \mathrm{Diff}_{\bar{D}}\, \Delta^{i-1}) \dashrightarrow (\bar{D}^i, \mathrm{Diff}_{\bar{D}}\, \Delta^i)$$

form the steps of the MMP starting with $(\bar{D}^0, \mathrm{Diff}_{\bar{D}}\, \Delta^0) := (\bar{D}, \mathrm{Diff}_{\bar{D}}\, \Delta)$ and with scaling of $\rho^* H$?

Notation 2 We follow the terminology and notation of [18, 20].

From now on, whenever we write a divisor as $D + \Delta$, we assume that all irreducible components of D appear with coefficient 1 (Δ may also contain divisors with coefficient 1).

Let $\rho : \bar{D} \to D$ denote the normalization. The *different* of Δ on $\bar{D}$ is denoted by $\mathrm{Diff}_{\bar{D}}\, \Delta$. It is a $\mathbb{Q}$-divisor on $\bar{D}$ that satisfies a natural $\mathbb{Q}$-linear equivalence

$$K_{\bar{D}} + \mathrm{Diff}_{\bar{D}}\, \Delta \sim_{\mathbb{Q}} \rho^*\big(K_X + D + \Delta\big). \tag{1}$$

See [18, 4.2] for a precise definition and its main properties. In order to avoid secondary sub and superscripts, we usually write $\mathrm{Diff}_{\bar{D}}\, \Delta^i$ instead of the more precise $\mathrm{Diff}_{\bar{D}^i}\, \Delta^i$.

In the original definition, a step of the MMP corresponds to an extremal ray [6]. By (1), any contraction of an extremal ray on X induces the contraction of an extremal face on $\bar{D}$, but the face may well have dimension >1. In an MMP with scaling of an ample divisor, the steps correspond to certain contractions of extremal faces. The divisor H plays a very minor role in the sequel, but it makes it possible for us to tell exactly which MMP steps we get.

We see in Paragraph 21 that a positive answer to Question 1 can be used to answer the following problem on slc pairs.

Question 3 Let (X, Δ) be an slc pair that is projective over a base scheme S with normalization $\pi : (\bar{X}, \bar{D} + \bar{\Delta}) \to (X, \Delta)$, conductor $\bar{D} \subset \bar{X}$ and H an ample divisor on X. Set $(\bar{X}^0, \bar{D}^0 + \bar{\Delta}^0) := (\bar{X}, \bar{D} + \bar{\Delta})$ and for $i = 1, \dots, m$ let

$$\bar{\phi}^i : (\bar{X}^{i-1}, \bar{D}^{i-1} + \bar{\Delta}^{i-1}) \dashrightarrow (\bar{X}^i, \bar{D}^i + \bar{\Delta}^i)$$

be the steps of the $(\bar{X}, \bar{D} + \bar{\Delta})$-MMP with scaling of $\pi^* H$. Do we get

$$\phi^i : (X^{i-1}, \Delta^{i-1}) \dashrightarrow (X^i, \Delta^i)$$

that form the steps of the (X, Δ)-MMP with scaling of H and such that $(\bar{X}^i, \bar{D}^i + \bar{\Delta}^i)$ is the normalization of (X^i, Δ^i)?

Example 4 We give two types of examples showing that in Question 1 we usually do not get the steps of the $(\bar{D}, \mathrm{Diff}_{\bar{D}}\, \Delta)$-MMP.

(4.1) Start with a smooth variety X', a smooth divisor $D' \subset X'$ and another smooth divisor $C' \subset D'$. Assume that $K_{X'} + D'$ is ample. Set $X := B_{C'}X'$ with exceptional divisor E and let $D \subset X$ denote the birational transform of D'.

For any $1 \geq \epsilon > 0$, $(X, D + \epsilon E)$ is an lc pair whose canonical model is (X', D') and $(D', 0)$ is its own canonical model.

However, $(D, \epsilon\, \mathrm{Diff}_D\, E) \cong (D', \epsilon C')$ is different from $(D', 0)$.

Note further that $K_X + D$ is the pull-back of $K_{X'} + D'$, hence semiample and big. Thus the stable base locus of $K_X + D + \epsilon E$ is E. If $1 > \epsilon > 0$ then the only log canonical center of $(X, D + \epsilon E)$ is D and the other log centers are E and $E \cap D$; see Definition 6. Thus the stable base locus contains the log centers but not the log canonical center.

Here are some concrete examples.

(4.1.1) Let X' be a smooth surface, $D' \subset X'$ a smooth rational curve and $C' \subset D'$ a set of 3 points. Then $(D, \mathrm{Diff}_D\, E) \cong (D', C')$ has ample log canonical class but $(D', 0) \cong (\mathbb{P}^1, 0)$ has negative log canonical class.

(4.1.2) For $\dim X' \geq 3$ it can also happen that the $(D, \epsilon\, \mathrm{Diff}_D\, E)$-MMP tells us to contract C'. Take $X' = \mathbb{P}^3$ and let $D' \subset X'$ be a smooth surface of degree 5 that contains a line C'. Then the self-intersection of C' is -3, thus for $1 \geq \epsilon > \frac{1}{3}$ the first (and only) step of the $(D, \epsilon\, \mathrm{Diff}_D\, E)$-MMP is to contract C'.

(4.2) Let B be a smooth curve and $f : X \to B$ be a flat family of surfaces with quotient singularities and such that K_X is $\mathbb{Q}$-Cartier.

Let $g : X \to Z$ be a flipping contraction. (For concrete examples, see [20, 2.7] or the list in [19].) Thus there is a closed point $0 \in B$ such that g is an isomorphism over $B \setminus \{0\}$. Set $D := X_0$ and let $C \subset D$ denote the flipping curve. Our example is the pair (X, D). Here $\mathrm{Diff}_D\, 0 = 0$, hence we need to compare the MMP for (X, D) with the MMP for $(D, 0)$.

Over $0 \in B$ we have a birational contraction $g_0 : X_0 \to Z_0$ that contracts $C \subset X_0$ to a point. Moreover $(C \cdot K_{X_0}) = (C \cdot K_X) < 0$, thus Z_0 is again log terminal and the contraction $g_0 : X_0 \to Z_0$ is a step in the MMP for $X_0 = D$.

However, since $g : X \to Z$ a flipping contraction, the special fiber of the flip $g^+ : X^+ \to Z$ is another surface $X_0^+ \to Z_0$ with a new exceptional curve $C^+ \subset X_0^+$ such that $(C^+ \cdot K_{X_0^+}) = (C^+ \cdot K_{X^+}) > 0$. Thus X_0^+ is not the canonical model of X_0 and $X_0 \dashrightarrow X_0^+$ is not even a step of any minimal model program.

We can easily arrange that K_{X^+} is ample. In this case the stable base locus of K_X is the flipping curve $C \subset X_0 = D$. The only log canonical center of (X, D) is D which is not contained in the stable base locus of K_X.

It is easy to see that D must have at least 1 non-canonical singularity that is also contained in C. This gives a 0-dimensional log center of (X, D) that is contained in the stable base locus.

Example 5 Every counter example to Question 1, where D is normal, gives a counter example to Question 3 as follows.

Let $b \in B$ be a smooth, projective, pointed curve of genus ≥ 1. We can glue $(X, D+\Delta)$ to $(B \times D, \{b\} \times D + B \times \operatorname{Diff}_D \Delta)$ along D to get an slc pair (Y, Δ_Y) whose normalization is the disjoint union of $(X, D+\Delta)$ and $(B \times D, \{b\} \times D + B \times \operatorname{Diff}_D \Delta)$. On $(X, D+\Delta)$ we get the steps of the $(X, D+\Delta)$-MMP

$$\phi^i : (X^{i-1}, D^{i-1} + \Delta^{i-1}) \dashrightarrow (X^i, D^i + \Delta^i)$$

and these restrict to

$$\phi^i_D : (D^{i-1}, \operatorname{Diff}_D \Delta^{i-1}) \dashrightarrow (D^i, \operatorname{Diff}_D \Delta^i).$$

Let us denote the steps of the $(D, \operatorname{Diff}_D \Delta)$-MMP by

$$\psi_i : (D_{i-1}, \operatorname{Diff}_D \Delta_{i-1}) \dashrightarrow (D_i, \operatorname{Diff}_D \Delta_i).$$

Then the steps of the $(B \times D, \{b\} \times D + B \times \operatorname{Diff}_D \Delta)$-MMP are given by

$$\big(B \times D_{i-1}, \{b\} \times D_{i-1} + B \times \operatorname{Diff}_D \Delta_{i-1}\big) \dashrightarrow \big(B \times D_i, \{b\} \times D_i + B \times \operatorname{Diff}_D \Delta_i\big).$$

If $(D^i, \operatorname{Diff}_D \Delta^i) \not\cong (D_i, \operatorname{Diff}_D \Delta_i)$, then we can not glue the resulting pairs

$$(X^i, D^i + \Delta^i) \quad \text{and} \quad \big(B \times D_i, \{b\} \times D_i + B \times \operatorname{Diff}_D \Delta_i\big).$$

Thus the (Y, Δ_Y)-MMP does not exist.

We give positive answers to Questions 1 and 3 when the singularities of $(X, D+\Delta)$ (resp. of $(\bar{X}, \bar{D} + \bar{\Delta})$) are mild along the exceptional locus of ϕ (resp. of $\bar{\phi}$). We use discrepancies to make this assertion precise.

Definition 6 Let (X, Θ) be an lc pair. An irreducible subvariety $W \subset X$ is called a *log canonical center* (resp. a *log center*) of (X, Θ) if there is a divisor E over X such that $\operatorname{center}_X E = W$ and $a(E, X, \Theta) = -1$ (resp. $a(E, X, \Theta) < 0$).

Assume next that $\Theta = D + \Delta$ and let $\rho : \bar{D} \to D$ denote the normalization. By adjunction [18, 4.9], $W \subset \bar{D}$ is a log center of $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ iff $\rho(W)$ is a log center of $(X, D+\Delta)$. See [18, Chap.7] for more on log centers.

From now on we assume that the base scheme S is essentially of finite type over a field of characteristic 0. Our main result is the following.

Theorem 7 *Using the notation and assumptions of Question 1, assume in addition that the intersection of D with the exceptional locus of*

$$\Phi^m := \phi^m \circ \cdots \circ \phi^1 : X \dashrightarrow X^m$$

does not contain any log center of $(X, D+\Delta)$. *Then the maps*

$$\phi^i_{\bar{D}} : (\bar{D}^{i-1}, \operatorname{Diff}_{\bar{D}} \Delta^{i-1}) \dashrightarrow (\bar{D}^i, \operatorname{Diff}_{\bar{D}} \Delta^i)$$

form the steps of the MMP starting with $(\bar{D}^0, \operatorname{Diff}_{\bar{D}} \Delta^0) := (\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ *and with scaling of* $\rho^* H$.

Remark 8 As the Examples (4.1.1–4.1.2) show, we need to avoid all log centers, not just the log canonical centers.

It can happen that ϕ^i is an isomorphism along D^{i-1}. Thus the precise claim is that each $\phi^i_{\bar{D}}$ is either an isomorphism or an MMP step. (The literature is somewhat inconsistent. Usual definitions of MMP steps allow isomorphisms, but in many statements they are tacitly excluded.)

Theorem 9 *Using the notation and assumptions of Question 3, assume in addition that the intersection of* $\bar{D}$ *with the exceptional locus of*

$$\Phi^m_{\bar{X}} := \bar{\phi}^m \circ \cdots \circ \bar{\phi}^1 : \bar{X} \dashrightarrow \bar{X}^m$$

does not contain any log center of $(\bar{X}, \bar{D}+\bar{\Delta})$.

Then the first m steps of the (X, Δ)*-MMP with scaling of* H *exist*

$$\phi^i : (X^{i-1}, \Delta^{i-1}) \dashrightarrow (X^i, \Delta^i),$$

and $(\bar{X}^i, \bar{D}^i + \bar{\Delta}^i)$ *is the normalization of* (X^i, Δ^i).

Proof Let (X, Δ) be an slc pair with normalization $(\bar{X}, \bar{D}+\bar{\Delta}) \to (X, \Delta)$, where $\bar{D} \subset \bar{X}$ is the conductor. Let $\rho : \bar{D}^n \to \bar{D}$ denote its normalization.

The gluing theory of [18, Chap.5] says that there is a (regular) involution

$$\tau : (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}) \to (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}),$$

and X is obtained from $\bar{X}$ by identifying the equivalence classes of the relation generated by τ on $\bar{X}$.

Next let

$$\bar{\phi}^i : (\bar{X}^{i-1}, \bar{D}^{i-1} + \bar{\Delta}^{i-1}) \dashrightarrow (\bar{X}^i, \bar{D}^i + \bar{\Delta}^i)$$

be the steps of the $(\bar{X}, \bar{D}+\bar{\Delta})$-MMP with scaling of $\pi^* H$ and assume that Theorem 7 applies. Then

$$\bar{\phi}^i_D : \big((\bar{D}^{i-1})^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}^{i-1}\big) \dashrightarrow \big((\bar{D}^i)^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}^i\big)$$

are steps of the $\big(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}\big)$-MMP with scaling of $\rho^*\pi^*H$. Since both $\operatorname{Diff}_{\bar{D}^n} \bar{\Delta}$ and $\rho^*\pi^*H$ are τ-invariant, the τ-action descends to give (regular) involutions

$$\tau^i : \big((\bar{D}^i)^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}^i\big) \to \big((\bar{D}^i)^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}^i\big).$$

Let $Z^i \subset \bar{X}^i$ denote the intersection of D^i with the exceptional locus of

$$(\phi^i \circ \cdots \circ \phi^1)^{-1} : \bar{X}^i \dashrightarrow \bar{X}.$$

By our assumption, Z^i does not contain any of the log centers of $\big(\bar{X}^i, \bar{D}^i + \bar{\Delta}^i\big)$. Thus τ^i defines a finite equivalence relation on $\bar{X}^i$ by [18, 9.55]. Therefore the geometric quotient $\pi^i : \bar{X}^i \to X^i$ of $\bar{X}^i$ by the equivalence relation generated by τ^i exists by [18, 9.21]. Next [18, 5.38] shows that (X^i, Δ^i) is slc. By Lemma 12 the resulting rational map

$$\phi^i : (X^{i-1}, \Delta^{i-1}) \dashrightarrow (X^i, \Delta^i)$$

is an MMP step with scaling of H. □

Note that if X is a normal crossing variety [18, 1.7] then the log centers of $(X, 0)$ are exactly the log canonical centers of $(X, 0)$, which are also the strata of X, so the important distinction between log centers and log canonical centers is not visible in this case.

The normalization $\pi : (\bar{X}, \bar{D}) \to X$ is a normal crossing pair. It is conjectured that $(\bar{X}, \bar{D})$ has a minimal model. This is currently known if $K_{\bar{X}} + \bar{D}$ has non-negative Kodaira dimension (on every irreducible component) and the dimension is ≤ 5 [3].

If a minimal model $\phi : X \dashrightarrow X^{\min}$ exists, then its normalization $(\bar{X}^{\min}, \bar{D}^{\min})$ is a dlt pair whose canonical class is nef. The abundance conjecture predicts that its canonical class is semi-ample, but this is known only if the dimension is ≤ 4 [14]. However, if abundance holds for $(\bar{X}^{\min}, \bar{D}^{\min})$ then [12, Thm.1.4] implies that the canonical class of $X^{\min}$ is also semi-ample. In particular, the canonical ring of X is finitely generated. (Note that on $(\bar{X}^{\min}, \bar{D}^{\min})$ we always need the dlt case of the abundance conjecture, which is not even known in the general type case.)

Thus Theorem 9 implies the following. Conjecturally, the dimension restrictions should not be necessary.

Corollary 10 *Let X be a pure dimensional, projective, normal crossing variety. Assume that K_X has non-negative Kodaira dimension on every irreducible component of X and its stable base locus does not contain any stratum of X.*

(1) *If* $\dim X \leq 5$ *then X has a minimal model $\phi : X \dashrightarrow X^{\min}$, ϕ is a local isomorphism at all log canonical centers and $X^{\min}$ is semi-dlt* [18, 5.19].
(2) *If* $\dim X \leq 4$ *then the canonical ring of X is finitely generated.* □

Before we start the proof of Theorem 7, we need to define what a step of an MMP is.

Definition 11 (MMP steps) An *MMP step* is a diagram of S-schemes

$$\begin{array}{ccc} (X,\Theta) & \stackrel{\phi}{\dashrightarrow} & (X',\Theta') \\ f\searrow & & \swarrow f' \\ & Z & \end{array} \tag{2}$$

with the following properties.

(1) X and X' are pure dimensional,
(2) (X,Θ) and (X',Θ') are log canonical pairs,
(3) ϕ is birational,
(4) f, f' are projective and generically finite,
(5) $-(K_X+\Theta)$ is f-ample and $K_{X'}+\Theta'$ is f'-ample and
(6) $\Theta' = \phi_*\Theta$.

Equivalently, $-(K_X+\Theta)$ is f-ample and (X',Θ') is the canonical model of (X,Θ) [20, 3.50].

We frequently call $\phi : (X,\Theta) \dashrightarrow (X',\Theta')$ an MMP step if it sits in a diagram as in (2) for suitable Z. We prove in Lemma 14 that

(7) f' has no exceptional divisors. That is, if $E' \subset X'$ is a divisor then $f'|_{E'}$ is also generically finite.

Together with (3) this implies that ϕ is a rational contraction, that is, ϕ^{-1} has no exceptional divisors.

Our definition differs from traditional usage in 2 small ways. First, we do not assume that the relative Picard number of f is 1. Second, our Z is not uniquely determined by $\phi : (X,\Theta) \dashrightarrow (X',\Theta')$; if $Z \to Z_1$ is finite then we can replace Z by Z_1. The usual choice is to take the unique Z such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. However, the latter condition is not preserved when passing to the normalization of X or to a divisor in X. Thus allowing different choices of Z is convenient for us. We do not even assume that f and f' are dominant.

If H is a $\mathbb{Q}$-Cartier divisor on X then (2) is an MMP step *with scaling of* H if, in addition,

(8) H is f-ample, $-H' := -\phi_*H$ is f'-ample,
(9) $K_X+\Theta+cH$ is numerically f-trivial for some $c \in \mathbb{Q}$, (this implies that $K_{X'}+\Theta'+cH'$ is numerically f'-trivial) and
(10) $K_X+\Theta+cH$ has positive degree on every proper, irreducible curve $C \subset X$ that is not contracted by f and lies over a closed point of Z.

In practice we start with a pair $(X,\Theta+c'H)$ such that $K_X+\Theta+c'H$ is ample over S. We then decrease the value of c' until we reach $c \le c'$ such that $K_X+\Theta+cH$ is nef but not ample. If a multiple of $K_X+\Theta+cH$ is semiample, it gives us $f : X \to Z$; see [4] for details.

For slc pairs, one needs to pay extra attention to the non-normal locus, and there are various possible definitions. However, if ϕ is a local isomorphism at all codimension 1 singular points, then the above definition works without changes. This is the only case that we use in the sequel.

The following comparison result is clear from the definition.

Lemma 12 *Let* (X, Θ) *and* (X', Θ') *be pure dimensional slc pairs with normalizations* $\pi : (\bar{X}, \bar{D} + \bar{\Theta}) \to (X, \Theta)$ *and* $\pi' : (\bar{X}', \bar{D}' + \bar{\Theta}') \to (X', \Theta')$. *Then* (2) *is an MMP step iff*

$$\begin{array}{ccc} (\bar{X}, \bar{D} + \bar{\Theta}) & \stackrel{\bar{\phi}}{\dashrightarrow} & (\bar{X}', \bar{D}' + \bar{\Theta}') \\ & \bar{f} \searrow \quad \swarrow \bar{f}' & \\ & Z & \end{array} \tag{3}$$

is an MMP step, where $\bar{f} = f \circ \pi$ *and* $\bar{f}' = f' \circ \pi'$.

Furthermore, if H *is a* $\mathbb{Q}$*-Cartier divisor on* X *then* (2) *is an MMP step with scaling of* H *iff* (3) *is an MMP step with scaling of* $\pi^* H$. □

Next we consider a generalization of MMP steps.

Definition 13 A diagram as in (2) is called a *sub-MMP step* if

(1) the assumptions (11.2–11.5) hold,
(2) f' is allowed to have exceptional divisors and
(3) $\operatorname{coeff}_{G'} \Theta' \le \operatorname{coeff}_{G'} \Theta$ for every divisor $G' \subset X'$ that is not f'-exceptional. (By Lemma 14 this inequality then holds for all divisors over X.)

The following example is good to keep in mind. Let X be a smooth surface and $C \subset X$ a smooth, rational curve with self-intersection ≤ -3. Let $X \to X'$ denote the contraction of C.

Then $(X, C) \dashrightarrow (X, 0)$ and $(X', 0) \dashrightarrow (X, 0)$ are both sub-MMP step. Thus ϕ can be an isomorphism on the underlying varieties yet a non-trivial sub-MMP step.

The main reason for this definition is Lemma 16, but first we prove that the usual discrepancy inequalities (cf. [20, 3.38] or [18, 1.19 and 1.22]) also hold for sub-MMP steps.

Lemma 14 *Consider a sub-MMP step of lc pairs*

$$\begin{array}{ccc} (X, \Theta) & \stackrel{\phi}{\dashrightarrow} & (X', \Theta') \\ & f \searrow \quad \swarrow f' & \\ & Z & \end{array}$$

where f, f' are birational. Then $a(E, X', \Theta') \geq a(E, X, \Theta)$ for every divisor E over X. Furthermore, for every E, the following are equivalent.

(1) $a(E, X', \Theta') > a(E, X, \Theta)$.
(2) *ϕ is not a local isomorphism at the generic point of* $\operatorname{center}_X E$.
(3) *ϕ^{-1} is not a local isomorphism at the generic point of* $\operatorname{center}_{X'} E$.
(4) *Either f or f' has positive dimensional fiber over the generic point of* $\operatorname{center}_Z E$.

Proof Let Y be the normalization of the main component of the fiber product $X \times_Z X'$ with projections $X \stackrel{g}{\leftarrow} Y \stackrel{g'}{\rightarrow} X'$. Write

$$K_Y \sim_{\mathbb{Q}} g^*(K_X + \Theta) - F \quad \text{and} \quad K_Y \sim_{\mathbb{Q}} g'^*(K_{X'} + \Theta') - F' \tag{4}$$

where $g_* F = \Theta$ and $g'_* F' = \Theta'$. Thus

$$F' - F \sim_{\mathbb{Q}} g'^*(K_{X'} + \Theta') - g^*(K_X + \Theta) \quad \text{is } (f' \circ g')\text{-nef}. \tag{5}$$

Note that $(f' \circ g')_*(F - F') = f_*\Theta - f'_*\Theta'$ is effective by assumption (13.3). Therefore $F - F'$ is effective by [20, 3.39], proving the required inequality.

It is clear that (1) $\Rightarrow$ (2), (2) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (4). Thus assume (4).

By [20, 3.39] the support of $F - F'$ contains $\operatorname{Ex}(f' \circ g')$. Arguing similarly we get that it also contains $\operatorname{Ex}(f \circ g)$. Thus $a(E, X', \Theta') > a(E, X, \Theta)$ if either f or f' has positive dimensional fiber over the generic point of $\operatorname{center}_Z E$. □

Corollary 15 *A sub-MMP step $\phi : (X, \Theta) \dashrightarrow (X', \Theta')$ is an MMP step iff $a(G', X', \Theta') = a(G', X, \Theta)$ for every divisor $G' \subset X'$.*

Proof If ϕ is an MMP step then $\Theta' = \phi_*\Theta$, hence $a(G', X', \Theta') = a(G', X, \Theta)$ for every divisor $G' \subset X'$.

Conversely, if $G' \subset X'$ is an f'-exceptional divisor then $a(G', X', \Theta') > a(G', X, \Theta)$ by Lemma 14.1. Thus there are no f'-exceptional divisors and so $\Theta' = \phi_*\Theta$. □

Lemma 16 *Let $\phi : (X, \Theta) \dashrightarrow (X', \Theta')$ be an MMP step sitting in a diagram (2). Assume that (X, Θ) is lc, $\Theta = D + \Delta$ where D is reduced with normalization $\rho : \bar{D} \to D$ and none of the irreducible components of D is contracted by ϕ. Then the diagram*

$$\begin{array}{ccc} \left(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta\right) & \stackrel{\phi_D}{\dashrightarrow} & \left(\bar{D}', \operatorname{Diff}_{\bar{D}'} \Delta'\right) \\ & f_D \searrow \quad \swarrow f'_D & \\ & Z & \end{array} \tag{6}$$

is a sub-MMP step.

Proof Assumptions (11.2–11.4) are clear and (11.5) holds since

$$K_{\bar{D}} + \operatorname{Diff}_{\bar{D}} \Delta \sim_{\mathbb{Q}} \rho^*(K_X + D + \Delta).$$

It remains to show that (13.3) holds. More generally, we show that

$$a(E, \bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) \le a\big(E, \bar{D}', \operatorname{Diff}_{D'} \Delta'\big) \tag{7}$$

for every divisor E over $\bar{D}$.

We may assume that f, f' are birational. Let Y be the normalization of the main component of the fiber product $X \times_Z X'$ with projections $X \overset{g}{\leftarrow} Y \overset{g'}{\rightarrow} X'$. As in (4) write

$$g^*(K_X + D + \Delta) \sim_{\mathbb{Q}} g'^*\big(K_{X'} + D' + \Delta'\big) + F - F', \tag{8}$$

where $F - F'$ is effective by [20, 3.38] or by Lemma 14.

Let D_Y denote the normalization of the birational transform of D on Y. Restricting (8) to D_Y we get

$$(g|_{D_Y})^*(K_{\bar{D}} + \operatorname{Diff}_{\bar{D}} \Delta) \sim_{\mathbb{Q}} (g'|_{D_Y})^*\big(K_{\bar{D}'} + \operatorname{Diff}_{\bar{D}'} \Delta'\big) + F|_{D_Y} \tag{9}$$

and $F|_{D_Y}$ is also effective. □

Corollary 17 *Using the notation and assumptions of Lemma 16, let $p \in \bar{D}$ be a point. Then $\phi_D : \big(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta\big) \dashrightarrow \big(\bar{D}', \operatorname{Diff}_{\bar{D}'} \Delta'\big)$ is a local isomorphism at p iff $\phi : X \dashrightarrow X'$ is a local isomorphism at $\pi(p)$.*

Note that the claims about X and D are different. As in Example 4.1, it can happen that $\phi_D : \bar{D} \dashrightarrow \bar{D}'$ is an isomorphism but $\operatorname{Diff}_{\bar{D}'} \Delta' \neq (\phi_D)_* \operatorname{Diff}_{\bar{D}} \Delta$.

Proof If ϕ is a local isomorphism at $\pi(p)$ then clearly ϕ_D is a local isomorphism at p. Conversely, if $\phi_D : \bar{D} \dashrightarrow \bar{D}'$ is a local isomorphism at p then the maps $g_D : D_Y \to \bar{D}$ and $g'_D : D_Y \to \bar{D}'$ are isomorphic to each other near p. By (9)

$$g_D^* \operatorname{Diff}_{\bar{D}} \Delta - {g'_D}^* \operatorname{Diff}_{\bar{D}'} \Delta' = (g|_{D_Y})^*(F - F').$$

If ϕ is not a local isomorphism at $\pi(p)$ then $\operatorname{Supp}(F - F')$ contains p by [20, 3.38] or by Lemma 14, thus $\operatorname{Diff}_{\bar{D}} \Delta \neq \operatorname{Diff}_{\bar{D}'} \Delta'$ in every neighborhood of p. □

Proposition 18 *Using the notation of Lemma 16, assume in addition that $D \cap \operatorname{Ex}(\phi)$ does not contain any log center of $(X, D + \Delta)$. Then (6) is an MMP step.*

Proof Assume to the contrary that (6) is not an MMP step. Then, by Corollary 15, there is a divisor $G' \subset \bar{D}'$ such that

$$a(G', \bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) < a(G', \bar{D}', \operatorname{Diff}_{\bar{D}'} \Delta'). \tag{10}$$

Since $a(G', \bar{D}', \mathrm{Diff}_{\bar{D}'} \Delta') = -\operatorname{coeff}_{G'} \mathrm{Diff}_{\bar{D}'} \Delta' \le 0$, this implies that $\operatorname{center}_{\bar{D}} G'$ is a log center of $(\bar{D}, \mathrm{Diff}_{\bar{D}} \Delta)$. By adjunction [18, 4.8], $\operatorname{center}_X G'$ is also a log center of $(X, D + \Delta)$.

Finally (10) also shows that ϕ is not a local isomorphism at the generic point of $\operatorname{center}_X G'$. □

Note that Proposition 18 almost implies Theorem 7, except that it is not quite clear how to compare $\mathrm{Ex}(\Phi^m) \subset X$ with the $\mathrm{Ex}(\phi^i) \subset X^{i-1}$, and this would be needed in order to directly apply Proposition 18. The following variant of the concept of exceptional set gives a clearer picture.

Definition 19 (Divisorial exceptional set) Let $\phi : X \dashrightarrow X'$ be a birational map of schemes that are proper over S. The *divisorial exceptional set* of ϕ, denoted by $\mathrm{DEx}(\phi)$, is the set of all divisors E over X such that ϕ is not a local isomorphism at the generic point of $\operatorname{center}_X E$.

Thus the usual exceptional set $\mathrm{Ex}(\phi) \subset X$ is the union of the centers of the divisors in $\mathrm{DEx}(\phi)$. The advantage of divisorial exceptional sets is that we can compare them for different birational models.

Lemma 20 *Let $\phi_i : (X^{i-1}, \Delta^{i-1}) \dashrightarrow (X^i, \Delta^i)$ be a sequence of sub-MMP steps. Then*

(1) $\mathrm{DEx}(\phi^m \circ \cdots \circ \phi^1) = \{E : a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m)\}$ *and*
(2) $\mathrm{DEx}(\phi^m \circ \cdots \circ \phi^1) = \mathrm{DEx}(\phi^1) \cup \cdots \cup \mathrm{DEx}(\phi^m)$.

Proof The containments

$$\begin{aligned} &\mathrm{DEx}(\phi^m \circ \cdots \circ \phi^1) \supset \{E : a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m)\} \\ &\mathrm{DEx}(\phi^m \circ \cdots \circ \phi^1) \subset \mathrm{DEx}(\phi^1) \cup \cdots \cup \mathrm{DEx}(\phi^m) \end{aligned} \tag{11}$$

are clear. For a single MMP step $\phi : (X, \Delta) \dashrightarrow (X', \Delta')$, Lemma 14.1 shows that

$$\mathrm{DEx}(\phi) = \{E : a(E, X, \Delta) < a(E, X', \Delta')\}. \tag{12}$$

Combining with the inequalities $a(E, X^{i-1}, \Delta^{i-1}) \le a(E, X^i, \Delta^i)$ we obtain that $a(E, X^0, \Delta^0) \le a(E, X^m, \Delta^m)$ and

$$a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m) \Leftrightarrow E \in \mathrm{DEx}(\phi^1) \cup \cdots \cup \mathrm{DEx}(\phi^m).$$

This shows that

$$\{E : a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m)\} = \mathrm{DEx}(\phi^1) \cup \cdots \cup \mathrm{DEx}(\phi^m),$$

which completes the proof. □

21 (Proof of Theorem 7) By assumption none of the irreducible components of D is contained in $\mathrm{Ex}(\Phi^m)$, thus the maps ϕ^i_D are birational. They sit in diagrams

$$\begin{array}{ccc} (\bar{D}^{i-1}, \mathrm{Diff}_{\bar{D}}\,\Delta^{i-1}) & \overset{\phi^i_D}{\dashrightarrow} & (\bar{D}^i, \mathrm{Diff}_{\bar{D}}\,\Delta^i) \\ & f^i_D \searrow \quad \swarrow g^i_D & \\ & Z^i & \end{array} \tag{13}$$

that are sub-MMP steps by Lemma 16.

Assume to the contrary that ϕ^m_D is not an MMP step. Then, by Corollary 15, there is a divisor $G^m \subset \bar{D}^m$ such that

$$a(G^m, \bar{D}^{m-1}, \mathrm{Diff}_{\bar{D}}\,\Delta^{m-1}) < a(G^m, \bar{D}^m, \mathrm{Diff}_{\bar{D}}\,\Delta^m) \le 0. \tag{14}$$

Combining with the inequalities $a(G^m, \bar{D}^{i-1}, \mathrm{Diff}_{\bar{D}}\,\Delta^{i-1}) \le a(G^m, \bar{D}^i, \mathrm{Diff}_{\bar{D}}\,\Delta^i)$ of Lemma 14.1, we get that

$$a(G^m, \bar{D}, \mathrm{Diff}_{\bar{D}}\,\Delta) < a(G^m, D^m, \mathrm{Diff}_{\bar{D}}\,\Delta^m) \le 0.$$

Thus $\mathrm{center}_{\bar{D}}\, G^m$ is a log center of $(\bar{D}, \mathrm{Diff}_{\bar{D}}\,\Delta)$.

By (14) $G^m \in \mathrm{DEx}(\phi^m_D)$, hence by (20.2) $G^m \in \mathrm{DEx}(\phi^m_D \circ \cdots \circ \phi^1_D)$. Thus $\Phi^m = \phi^m \circ \cdots \circ \phi^1 : X \dashrightarrow X^m$ is also not an isomorphism at the generic point of $\mathrm{center}_D\, G^m \subset X$. This contradicts the assumptions of Theorem 7. □

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Torus Equivariant K-Stability

Giulio Codogni and Jacopo Stoppa

Abstract It is conjectured that to test the K-polystability of a polarised variety it is enough to consider test-configurations which are equivariant with respect to a torus in the automorphism group. We prove partial results towards this conjecture. We also show that it would give a new proof of the K-polystability of constant scalar curvature polarised manifolds.

Keywords Canonical Kaehler metrics · GIT · K-stability · Torus actions

1 Introduction

The Yau-Tian-Donaldson conjecture for Fano manifolds [7, 23, 25] predicts that a smooth Fano M admits a Kähler-Einstein metric if and only if it is K-polystable, a purely algebro-geometric condition expressed through the positivity of a certain limit of GIT weights (the Donaldson-Futaki weight or invariant). There are by now several proofs, in different degrees of generality (i.e. allowing M to have mild singularities, a boundary in the MMP sense, and/or slightly modifying the notion of K-stability), using different methods.

For an arbitrary polarised manifold (X, L) the most natural generalisation of a Kähler-Einstein metric is a constant scalar curvature Kähler (cscK) metric representing the first Chern class of L. If such a metric exists, (X, L) is called a cscK manifold.

A Kähler-Einstein metric, or more generally a cscK metric, if it exists, can always be taken invariant under the action of a compact group of automorphisms of M. From the GIT point of view, when the point whose stability we would

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like to investigate has a non-trivial reductive stabiliser H, the Hilbert-Mumford Criterion can be strengthened: it is enough to consider one-parameter subgroups which commute with H [10]. These facts suggest the following folklore conjecture (all the notions required in the rest of this introduction will be recalled in Sect. 3.)

Conjecture 1 *Let (X, L) be a polarised variety and let G be a reductive subgroup of $\mathrm{Aut}(X, L)$. Then (X, L) is K-polystable if and only if for every G-equivariant test-configuration the Donaldson-Futaki invariant is greater than or equal to zero, with equality if and only if the normalisation of the test-configuration is a product.*

An analytic proof in the case of Fano manifolds is given in [6], relying on an alternative approach to the Yau-Tian-Donaldson conjecture. An algebro-geometric proof in the Fano case and when G is a torus is given in [12].

Recall that a cscK manifold has reductive automorphism group, so K-polystable varieties are expected to have a reductive automorphism group as well; this problem is studied in [5]. Because of this it is natural to formulate Conjecture 1 just for reductive subgroups of $\mathrm{Aut}(X, L)$.

There is a general expectation that for the existence of a cscK metric one actually needs some enhancement of the original notion of K-stability. Quite a few different notions have been proposed. In this paper we focus on the generalisation of K-stability based on (possibly non-finitely generated) filtrations of the coordinate ring of (X, L) (see Definition 24). This notion has been proposed by G. Székelyhidi in [22], building on the work of D. Witt Nyström [24]; in [21], it is called $\hat{K}$-stability. In [22], it is shown that, given a cscK manifold (X, L), if the connected component of the identity of $\mathrm{Aut}(X, L)$ is equal to $\mathbb{C}^*$, then (X, L) is $\hat{K}$-stable. Importantly for us [22] also discusses a variant of $\hat{K}$-stability which replaces the Donaldson-Futaki invariant of a filtration with the asymptotic Chow weight Chow_∞, and proves that the $\hat{K}$-stability result remains true for this variant (the two notions coincide when dealing with classical test-configurations, corresponding to finitely generated filtrations).

Our main result is a step towards a proof of Conjecture 1 in the general case, or possibly of a variant of Conjecture 1 in the $\hat{K}$-stability setup.

Theorem 2 *Let (X, L) be a polarised variety. Fix a complex torus $T \subset \mathrm{Aut}(X, L)$ and let $(\mathcal{X}, \mathcal{L})$ be a test-configuration with Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$. Then we can associate to $(\mathcal{X}, \mathcal{L})$ a T-equivariant filtration χ of the coordinate ring of (X, L) whose asymptotic Chow weight satisfies $\mathrm{Chow}_\infty(\chi) \leq \mathrm{DF}(\mathcal{X}, \mathcal{L})$. If moreover χ is finitely generated, then it corresponds to a T-equivariant test-configuration which is a flat one-parameter limit of $(\mathcal{X}, \mathcal{L})$, and in particular has the same Donaldson-Futaki invariant and L^2 norm.*

Theorem 2 follows at once from Lemmas 29, 30 and Theorem 31, proved in Sect. 4. Theorem 31 shows that given a generalised test-configuration in the sense of [22], corresponding to a possibly non-finitely generated filtration χ, we can specialise it to a T-invariant filtration $\bar{\chi}$ with $\mathrm{Chow}_\infty(\bar{\chi}) \leq \mathrm{Chow}_\infty(\chi)$. In the "Appendix" we show that non-finitely generated filtrations can actually arise in Theorem 2.

In Sect. 5 we show that Conjecture 1 combined with ideas from [17, 19] naturally leads to a proof that cscK manifolds are K-polystable. K-polystability of cscK manifolds is proved in [2] using completely different methods.

Notation In this paper a polarised variety (X, L) is a complex projective variety X endowed with a very ample and projectively normal line bundle L. For the purposes of this paper one may always replace L with a positive tensor power, so these assumptions are not restrictive.

2 Some Results on Filtrations in Finite Dimensional GIT

In this section we discuss some preliminary notions in a finite dimensional GIT context.

Let V be a finite dimensional complex vector space. Pick an increasing filtration $F = \{F_i V\}_{i\in\mathbb{Z}}$ of V by complex subspaces (with index set $\mathbb{Z}$) and a $\mathbb{C}^*$-action λ on V.

Definition 3 The specialisation $\bar{F}$ of F via λ is the filtration given by

$$\bar{F}_i V = \lim_{\tau\to 0} \lambda(\tau) \cdot F_i V,$$

where the limit is taken in the appropriate Grassmannian.

Equivalently $\bar{F}_i V$ is the subspace spanned by the vectors $\bar{v}$ as v varies in $F_i V$, where $\bar{v}$ denotes the lowest weight term with respect to the action of λ. The filtration $\bar{F}$ is λ-equivariant by construction, that is each $\bar{F}_i V$ is preserved by λ.

Let G be a reductive group acting on V, and assume that the kernel of the action is a finite group.

Definition 4 Let γ be a one-parameter subgroup of G acting on V as above. The weight filtration of γ is the increasing filtration $F = \{F_i V\}_{i\in\mathbb{Z}}$ given by

$$F_i V = \bigoplus_{j\geq -i} V_j$$

where V_j is the weight j eigenspace for the action of γ.

Let $\mathcal{P}(\gamma)$ be the parabolic subgroup of G associated to the one-parameter subgroup γ. By definition this is the subgroup preserving the flag F.

Suppose that λ is an additional one-parameter subgroup of G. We wish to characterise the specialisation of the weight filtration F of γ via the action of λ. For this we recall that the intersection of parabolic subgroups $\mathcal{P}(\lambda)\cap\mathcal{P}(\gamma)$ contains a maximal torus $\mathcal{T}$ of G (see e.g. [3] Proposition 4.7). Moreover all maximal tori in a parabolic subgroup are conjugated by elements of the parabolic, hence there exists a one-parameter subgroup χ of $\mathcal{T}$ such that χ is conjugate to γ via an element in

$\mathcal{P}(\gamma)$, so that the weight filtration associated to χ is still F. Let

$$\bar{\gamma}(t) = \lim_{\tau \to 0} \lambda(\tau)\chi(t)\lambda(\tau)^{-1}.$$

This limit exists because χ lies in the parabolic $\mathcal{P}(\lambda)$, see [13] section 2.2.

Lemma 5 *Suppose that F is the weight filtration of γ. The specialisation $\bar{F}$ of F via λ coincides with the weight filtration of $\bar{\gamma}$. It follows in particular that $\bar{F}$ is induced by a one-parameter subgroup of G.*

Note that the filtration $\bar{F}$ is uniquely defined, but $\bar{\gamma}$ is not (for example, it depends on the choice of T).

Proof The key remark is that the weight j eigenspace of $\lambda(\tau)\chi(t)(\lambda(\tau))^{-1}$ is $\lambda(\tau) \cdot V_j$. Now for every $v \in V$ we have

$$\bar{\gamma}(t)(v) = \lim_{\tau \to 0} \lambda(\tau)\chi(t)(\lambda(\tau))^{-1}(v)$$

so v is a weight j eigenvector for $\bar{\gamma}$ if and only if v belongs to

$$\lim_{\tau \to 0} \lambda(\tau) \cdot V_j$$

where the limit is taken in the appropriate Grassmannian. □

Definition 6 The Hilbert-Mumford weight of a vector $v \in V$ with respect to the one-parameter subgroup γ is

$$\mathrm{HM}(v, \gamma) = \min_i \{v \in F_i V\}$$

where F is the weight filtration of γ.

This depends only on the weight filtration of γ and we will also denote it by $\mathrm{HM}(v, F)$ rather than $\mathrm{HM}(v, \gamma)$ if we wish to emphasise this fact. But notice that a general filtration of V will not come from a one-parameter subgroup of the fixed reductive group G.

Remark 7 With our sign convention $\mathrm{HM}(v, \gamma)$ is the weight of the induced action of γ on the fibre $\mathcal{O}_{\mathbb{P}(V)}(1)_{[v]_0}$ of the hyperplane line bundle on $\mathbb{P}(V)$ over $[v]_0 = \lim_{\tau \to 0} \lambda(\tau) \cdot [v]$. Thus for example the Hilbert-Mumford Criterion says that $[v]$ is GIT semistable if and only if $\mathrm{HM}(v, \gamma) \geq 0$ for all one-parameter subgroups γ.

Proposition 8 *Let λ be a one-parameter subgroup of the stabiliser of $[v] \in \mathbb{P}(V)$. The we have*

$$\mathrm{HM}(v, \bar{F}) \leq \mathrm{HM}(v, \gamma)$$

where $\bar{F}$ is the specialisation via λ of the weight filtration F of γ.

Recall that by Lemma 5 the filtration $\bar{F}$ is the weight filtration of a one-parameter subgroup of G.

Proof We only need to show that $v \in F_i V$ implies $v \in \bar{F}_i V$. This follows from the fact that v is an eigenvector of λ, so it is equal to its lowest weight term $\bar{v}$ with respect to the action of λ. □

It is easy to produce examples where the inequality of Proposition 8 is strict.

Example 9 We choose $G = SL(2, \mathbb{C})$ with its standard action on $V = \mathbb{C}^2$, and

$$v = e_2,\ \gamma(t) = \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix},\ \lambda(\tau) = \begin{pmatrix} \tau^h & 0 \\ \tau^h - \tau^{-h} & \tau^{-h} \end{pmatrix}$$

for fixed $h, k > 0$. Note that λ stabilises $[v] \in \mathbb{P}(V)$. One checks that γ is not contained in the parabolic $\mathcal{P}(\lambda)$. But conjugating γ with a suitable element in $\mathcal{P}(\gamma)$ gives

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^k & t^{-k} - t^k \\ 0 & t^{-k} \end{pmatrix} = \chi \in \mathcal{P}(\gamma) \cap \mathcal{P}(\lambda).$$

A straightforward computation gives

$$\lim_{\tau \to 0} \lambda(\tau)\chi(\lambda(\tau))^{-1} = \begin{pmatrix} t^{-k} & 0 \\ t^{-k} - t^k & t^k \end{pmatrix} = \bar{\gamma},$$

so we have

$$\mathrm{HM}(v, \bar{\gamma}) = -k < \mathrm{HM}(v, \gamma) = k.$$

It is important to realise that even if γ does not stabilise $[v] \in \mathbb{P}(V)$ its specialisation $\bar{\gamma}$ with respect to a λ in the stabiliser could well lie in the stabiliser (so abusing the K-stability terminology which will be recalled in the next section, in the present finite-dimensional setup and without imposing further restrictions, we can end up with a "product test-configuration").

Example 10 Let V, γ, λ be as in the previous example. We choose $v = e_1 + e_2$. Then $[v] \in \mathbb{P}V$ is stabilised by λ and by $\bar{\gamma}$, but not by γ. Note that in this case we have $\mathrm{HM}(v, \gamma) = \mathrm{HM}(v, \bar{\gamma}) = k$.

Let F, F' be filtrations of V with index set $\mathbb{Z}$. We say that F is included in F' if $F_i V \subset F'_i V$ holds for all i. The following observation follows immediately from the definition of the Hilbert-Mumford weight and will be useful in later applications.

Lemma 11 *Let F, F' be the weight filtrations of some one-parameter subgroups. If F is included in F' then we have*

$$\mathrm{HM}(v, F') \leq \mathrm{HM}(v, F)$$

for all v in V.

3 Filtrations, Test-Configurations, Approximations

Let (X, L) be a polarised variety. One of the main objects of study in this paper are test-configurations of (X, L). Let us briefly recall their definition.

Definition 12 Let $\mathbb{C}^*$ act in the standard way on $\mathbb{C}$. A test-configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) with exponent r is a $\mathbb{C}^*$-equivariant flat morphism $\pi : \mathcal{X} \to \mathbb{C}$, together with a π-ample line bundle $\mathcal{L}$ and a linearisation of the action of $\mathbb{C}^*$ on $\mathcal{L}$, such that the fibre over 1 is isomorphic to $(X, L^{\otimes r})$. We say that $(\mathcal{X}, \mathcal{L})$ is

- *very ample*, if $\mathcal{L}$ is π-very ample;
- a *product*, if it is isomorphic to $(X \times \mathbb{C}, L^{\otimes r} \boxtimes \mathcal{O}_{\mathbb{C}})$, where the action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ is induced by a one-parameter subgroup λ of $\mathrm{Aut}(X, L)$ by $\lambda(\tau) \cdot (x, t) = (\lambda(\tau) \cdot x, \tau t)$;
- *trivial*, if it is a product and, moreover, λ is trivial;
- *normal*, if the total space $\mathcal{X}$ is normal;
- *equivariant with respect to a subgroup $H \subset$* $\mathrm{Aut}(X, L)$, if the action of $\mathbb{C}^*$ can be extended to an action of $\mathbb{C}^* \times H$ such that the action of $\{1\} \times H$ is the natural action of H on $(X, L^{\otimes r})$;
- in the Fano case, a test-configuration is a *special degeneration* if $\mathcal{X}$ is normal, all the fibres are klt and a positive rational multiple of $\mathcal{L}$ equals $-K_{\mathcal{X}}$ (this notion is due to Tian [23], see also [11] Definition 1).

The normalisation of a test-configuration is the normalisation of $\mathcal{X}$ endowed with the natural induced line bundle and $\mathbb{C}^*$ action (or $\mathbb{C}^* \times H$ action). A test-configuration is a product if and only if the central fibre $\mathcal{X}_0$ is isomorphic to X: by standard theory in this case there is a trivialisation $\mathcal{X} \cong X \times \mathbb{C}$ and the $\mathbb{C}^*$-action on $\mathcal{X}$ corresponds to a $\mathbb{C}^*$-action on $X \times \mathbb{C}$ preserving $X \times \{0\}$, which must then be induced by a $\mathbb{C}^*$-action on X as above.

The following result summarises observations of Ross-Thomas [16] and Odaka [14].

Proposition 13 *For all sufficiently large r there is a bijective correspondence between increasing filtrations of $H^0(X, L^{\otimes r})^\vee$ (with index set $\mathbb{Z}$) and very ample test-configurations of exponent r. Such a test-configuration is a product if and only if the corresponding filtration is the weight filtration of a one-parameter subgroup of* $\mathrm{Aut}(X, L)$*, and it is equivariant with respect to a reductive subgroup $H \subset$* $\mathrm{Aut}(X, L)$ *if and only if the corresponding filtration is preserved by H.*

Proof An arbitrary increasing filtration of $H^0(X, L^{\otimes r})^\vee$ is induced by the weight filtration of a one-parameter subgroup of $GL(H^0(X, L^{\otimes r})^\vee)$, so we can associate to a filtration the (very ample) test-configuration induced by this one-parameter subgroup. If two one-parameter subgroups induce the same filtration then the corresponding test-configurations are isomorphic, see [14] Theorem 2.3 and its proof. Conversely, by [16, Proposition 3.7], for all sufficiently large r a very ample test-configuration of exponent r is always induced by a one-parameter subgroup of $GL(H^0(X, L^{\otimes r})^\vee)$, and this gives the filtration. The other claims are straightforward. □

One can act on a test-configuration $(\mathcal{X}, \mathcal{L})$ in two basic ways (see e.g. [8] section 2). Firstly we can pull-back $(\mathcal{X}, \mathcal{L})$ via a base-change $t \mapsto t^p$. The effect on the corresponding filtration is to multiply all the indices of the filtration by p. Equivalently the weights of the corresponding one-parameter subgroup are multiplied by p. Secondly we can rescale the linearisation of the action on $\mathcal{L}$ by a constant factor. The effect on the corresponding filtration is to shift all indices by some integer k. Equivalently we are composing the corresponding one-parameter subgroup with a one-parameter subgroup in the the center of $GL(H^0(X, L^{\otimes r})^\vee)$, which corresponds in turn to adding k to all the weights.

Combining the two operations above we can modify the weights to get a filtration with only positive indices, or alternatively to get a filtration induced by a one-parameter subgroup of $SL(H^0(X, L^{\otimes r})^\vee)$.

There is a more global correspondence between filtrations and test-configurations, which avoids fixing the exponent. We introduce the homogeneous coordinate ring

$$R = R(X, L) = \bigoplus_{k \geq 0} R_k = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k}).$$

We focus on filtrations of R of a special type.

Definition 14 We define a *filtration* χ of R to be sequence of vector subspaces

$$H^0(X, \mathcal{O}) = F_0 R \subset F_1 R \subset \cdots$$

which is

(i) *exhaustive*: for every k there exists a $j = j(k)$ such that $F_j R_k = H^0(X, L^{\otimes k})$,
(ii) *multiplicative*: $(F_i R_l)(F_j R_m) \subset F_{i+j} R_{l+m}$,
(iii) *homogeneous*: if f is in $F_i R$ then each homogeneous piece of f also lies in $F_i R$.

We denote by χ_k the filtration of $H^0(X, L^{\otimes k})$ induced by χ.

Note that when considering filtrations of R we restrict to those which only have non-negative indices; let us also notice that describing χ is equivalent to describe χ_k for every k. There are two basic algebraic objects attached to a filtration as above.

Definition 15 Let χ be a filtration. The corresponding *Rees algebra* is

$$\mathrm{Rees}(\chi) = \bigoplus_{i \geq 0} F_i R\, t^i$$

The graded modules are

$$\mathrm{gr}_i(H^0(X, L^{\otimes k})) = F_i(H^0(X, L^{\otimes k}))/F_{i-1}(H^0(X, L^{\otimes k}))$$

The graded algebra is

$$\mathrm{gr}(\chi) = \bigoplus_{k,i \geq 0} \mathrm{gr}_i(H^0(X, L^k))$$

The Rees algebra is a subalgebra of $R[t]$, and by the following elementary result, whose proof relies on the projective normality of L, it is possible to reconstruct χ from it.

Lemma 16 *Let A be a $\mathbb{C}$-subalgebra of $R[t]$. We define a filtration χ_A of R as follows*

$$F_i R = \{s \in R \mid t^i s \in A\}$$

The filtration χ_A satisfies the conditions of Definition 14 if and only if A satisfies the conditions

- *$A \cap R = H^0(X, \mathcal{O}_X)$;*
- *for every $s \in H^0(X, L)$ there exists an i such that $t^i s \in A$;*
- *if $t^i f$ is in A, then, for each of the homogenous component f_k of f, $t^i f_k$ is also in A.*

A filtration χ equals χ_A, where A is the Rees algebra of χ. There is an inclusion of filtrations $\chi_1 \subset \chi_2$ (i.e. an inclusion of filtered pieces) if and only if there is a corresponding inclusion of the Rees algebras $\mathrm{Rees}(\chi_1) \subset \mathrm{Rees}(\chi_2)$.

The following notion is crucial for us.

Definition 17 A filtration is called finitely generated if its Rees algebra is finitely generated.

Let us review the relation between finitely generated filtrations and test-configurations, as developed by Witt Nyström [24] and Székelyhidi [22] (see [4, Proposition 2.15] for a precise statement).

Let χ be a finitely generated filtration. The Rees algebra $\mathrm{Rees}(\chi)$ is a finitely generated flat $\mathbb{C}[t]$-module; this means that the associated relative Proj with its natural $\mathcal{O}(1)$ is a test-configuration $(\mathcal{X}, \mathcal{L})$. The central fibre is the Proj of the graded algebra $\mathrm{gr}(\chi)$; the $\mathbb{C}^*$-action on the central fibre is given by *minus* the i-grading of $\mathrm{gr}(\chi)$.

On the other hand let $(\mathcal{X}, \mathcal{L})$ be an exponent r test-configuration. Consider the filtration F of $H^0(X, L^{\otimes r})$ associated to it by Proposition 13. Up to base-change and scaling of the linearisation we can assume that all the weights are positive. Denote by N the length of this filtration. Let A be the $\mathbb{C}$-subalgebra of $R[t]$ generated by

$$H^0(X, L)t^N \oplus \bigoplus_{i=1}^{N} F_i H^0(X, L^{\otimes r})t^i$$

Then the filtration associated to A via Lemma 16 is the filtration of R induced by $(\mathcal{X}, \mathcal{L})$ (the second assumption in Lemma 16 holds because L is projectively normal, i.e. R is generated in degree 1).

Suppose that χ is a not necessarily finitely generated filtration. Following [22] Section 3.2 we can define finitely generated approximations $\chi^{(r)}$ as follows. Let F be the filtration induced by χ on $H^0(X, L^{\otimes r})$, this corresponds to an exponent r test-configuration $(\mathcal{X}, \mathcal{L})$, then $\chi^{(r)}$ is the finitely generated filtration corresponding to $(\mathcal{X}, \mathcal{L})$. Note that this construction also makes sense when χ is finitely generated and corresponds to $(\mathcal{X}, \mathcal{L})$, in which case $\chi^{(r)}$ corresponds to $(\mathcal{X}, \mathcal{L}^{\otimes r})$.

Definition 18 We introduce two "weight functions" attached to χ, given by

$$w_\chi(k) = w(k) = \sum_i (-i) \dim \mathrm{gr}_i(H^0(X, L^{\otimes k})),$$

respectively

$$d_\chi(k) = d(k) = \sum_i i^2 \dim \mathrm{gr}_i(H^0(X, L^{\otimes k})).$$

If χ is a finitely generated filtration (corresponding to a test-configuration $(\mathcal{X}, \mathcal{L})$) then by equivariant Riemann-Roch we have, for all sufficiently large k,

$$h(k) = h^0(X, L^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + \cdots$$
$$w(k) = b_0 k^{n+1} + b_1 k^n + \cdots$$
$$d(k) = c_0 k^{n+2} + c_1 k^{n+1} + \cdots$$

Definition 19 Let χ be a finitely generated filtration (which thus corresponds to a test-configuration). One defines the r-th Chow weight, Donaldson-Futaki weight (or invariant) and the L^2 norm as

$$\mathrm{Chow}_r(\chi) = \mathrm{Chow}_r(\mathcal{X}, \mathcal{L}) = r\frac{b_0}{a_0} - \frac{w(r)}{d(r)},$$

$$\mathrm{DF}(\chi) = \mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{a_1 b_0 - a_0 b_1}{a_0^2},$$

$$||\chi||_{L^2}^2 = ||(\mathcal{X}, \mathcal{L})||_{L^2} = c_0 - \frac{b_0^2}{a_0}.$$

Note that a straightforward computation shows that we have

$$\lim_{r\to\infty} \mathrm{Chow}_r(\mathcal{X}, \mathcal{L}^{\otimes r}) = \mathrm{DF}(\mathcal{X}, \mathcal{L}).$$

Definition 20 A polarised variety (X, L) is K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every test-configuration $(\mathcal{X}, \mathcal{L})$.

Given a subgroup H of $\mathrm{Aut}(X, L)$, we say that (X, L) is H-equivariantly K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every H-equivariant test-configuration $(\mathcal{X}, \mathcal{L})$.

Definition 21 A normal polarised variety (X, L) is K-polystable if for every test-configuration $(\mathcal{X}, \mathcal{L})$ with normal total space we have $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$, with equality if and only if $(\mathcal{X}, \mathcal{L})$ is a product.

Given a subgroup H of $\mathrm{Aut}(X, L)$, (X, L) is H-equivariantly K-polystable if for every H-equivariant test-configuration $(\mathcal{X}, \mathcal{L})$ with normal total space we have $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$, with equality if and only if $(\mathcal{X}, \mathcal{L})$ is a product.

Following [22] (Definition 3 and Equation (33)) we also define the following two invariants of a non-finitely generated filtration.

Definition 22 The Donaldson-Futaki and asymptotic Chow weights of a filtration χ are given by

$$\mathrm{DF}(\chi) = \liminf_{r\to\infty} \mathrm{DF}(\chi^{(r)}),$$

respectively

$$\mathrm{Chow}_\infty(\chi) = \liminf_{r\to\infty} \mathrm{Chow}_r(\chi^{(r)}).$$

Note that $\chi^{(r)}$ is an exponent r test configuration, so it is natural to consider its r-th Chow weight. Let us also emphasise that, when χ is finitely generated, both these invariants coincide with the classical Donaldson-Futaki weight, see [22, Section 3.2]. In general these two invariants differ, see [22, Example 4]; we do not know if there is an inequality relating them.

Definition 23 The L^2 norm of a filtration χ is given by

$$||\chi||_2 = \liminf_{r\to\infty} ||\chi^{(r)}|| .$$

In [22, Lemma 8] it is shown that the above liminf is actually a limit.

Definition 24 A polarised variety is $\hat{K}$-semistable if for any filtration χ of $R(X, L)$ we have

$$\mathrm{DF}(\chi) \geq 0.$$

It is $\hat{K}$-stable if the equality holds if and only if $||\chi||_2 = 0$. One can make parallel definitions replacing $\mathrm{DF}(\chi)$ with the asymptotic Chow weight $\mathrm{Chow}_\infty(\chi)$.

One easily checks that $\hat{K}$-semistability is equivalent to K-semistability. On the other hand $\hat{K}$-stability is (at least a priori) stronger than K-stability, and just as K-stability it implies that the automorphism group of (X, L) has no nontrivial one-parameter subgroups.

Székelyhidi [22] (Theorem 10 and Proposition 11) proves that if (X, L) is cscK with trivial automorphisms then it is $\hat{K}$-stable, including the variant notion using the Chow_∞ weight.

At present we do not know a good candidate for the notion of $\hat{K}$-polystability (i.e. allowing $\mathrm{Aut}(X, L)/\mathbb{C}^*$ to be non-finite, where by $\mathbb{C}^*$ we mean the central one parameter subgroup which acts as the identity on X and scales L).

4 Specialisation of a Test-Configuration

In the classical situation of a torus T acting on a projective variety one can specialise a point p to a fixed point $\bar{p}$ for the action of T: one picks a generic one-parameter subgroup λ of T and the specialisation is $\bar{p} = \lim_{\tau\to 0} \lambda(t) \cdot p$. This specialisation does depend on λ and when we need to emphasise this dependence we will denote it by $\bar{p}_\lambda$. In this section we first generalise this construction to test-configurations, and then prove some basic facts which imply our main result Theorem 2.

Definition 25 Let $(\mathcal{X}, \mathcal{L})$ be an exponent r test-configuration and F be the corresponding filtration of $H^0(X, L^{\otimes r})^\vee$ given by Proposition 13. Let T be a torus in $\mathrm{Aut}(X, L)$, and $\bar{F}$ the specialisation of F via a generic one-parameter subgroup λ of T. Then the specialisation $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ is the T-equivariant exponent r test-configuration corresponding to $\bar{F}$.

The specialisation depends on the choice of r and λ, but we will mostly suppress this in the notation.

We make a brief digression in order to discuss Definition 25. Recall that by Proposition 13 an exponent r test-configuration for (X, L) is obtained by embedding $\iota : X \hookrightarrow \mathbb{P}H^0(X, L^{\otimes r})^\vee$ with the complete linear system $|rL|$ and by taking the flat closure of $\iota(X)$ under the action of a one-parameter subgroup γ of $GL(H^0(X, L^{\otimes r})^\vee)$. The corresponding test-configuration $(\mathcal{X}, \mathcal{L})$ is a closed subscheme of $\mathbb{P}H^0(X, L^{\otimes r})^\vee \times \mathbb{C}$ (in fact it can be canonically completed to a closed subscheme of $\mathbb{P}H^0(X, L^{\otimes r})^\vee \times \mathbb{P}^1$ by gluing with the trivial family at infinity). If λ is a one-parameter subgroup of $\mathrm{Aut}(X, L)$ one could attempt to define the λ-specialisation of $(\mathcal{X}, \mathcal{L})$ by taking its flat closure as a closed subscheme of $\mathbb{P}H^0(X, L^{\otimes r})^\vee \times \mathbb{C}$ under the action of λ. We give a simple example showing that such a flat closure is not preserved by γ in general, so it is not a λ-equivariant test-configuration in a natural way. In fact we also show that in general the total space of the flat closure cannot support a test-configuration, and compute the corresponding specialisation $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ in the sense of Definition 25 in the example.

Example 26 Embed $\iota : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ via Veronese $[s_0 : s_1] \mapsto [s_0^2 : s_0 s_1 : s_1^2]$ and act with the one-parameter subgroup γ of $SL(3, \mathbb{C})$ given by $\mathrm{diag}(t^{-1}, t^2, t^{-1})$. This gives a test-configuration $(\mathcal{X}, \mathcal{L})$ of exponent 2 for $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ with total space $\mathcal{X} \subset \mathbb{P}^2 \times \mathbb{C}$ which is the variety $V(xz - t^6 y^2)$. Now choose

$$\lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau^h & 0 \\ 0 & \tau^{-h} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)).$$

The induced one-parameter subgroup in $SL(3, \mathbb{C})$, which we still denote by λ, is given by

$$\lambda = \begin{pmatrix} \tau^{2h} & 1 - \tau^{2h} & (\tau^{-h} - \tau^h)^2 \\ 0 & 1 & -2(1 - \tau^{-2h}) \\ 0 & 0 & \tau^{-2h} \end{pmatrix}.$$

One computes

$$\lambda(\tau) \cdot \mathcal{X} = V(\tau^{2h} x((\tau^{-h} - \tau^h)^2 x - 2(1 - \tau^{-2h}) y + \tau^{-2h} z) - t^6((1 - \tau^{2h}) x + y)^2).$$

Since $\lambda(\tau) \cdot \mathcal{X} \subset \mathbb{P}^2 \times \mathbb{C}$ is a familiy of divisors it is straightforward to take the flat limit at $\tau \to 0$. For $h > 0$ one finds

$$\lim_{\tau \to 0} \lambda(\tau) \cdot \mathcal{X} = V(x(x + 2y + z) - t^6 (x + y)^2) =: \bar{\mathcal{X}}. \tag{1}$$

The central fibre $V(x(x + 2y + z))$ is not preserved by γ, so the flat limit $\bar{\mathcal{X}}$ is not the total space of a test-configuration in a natural way. In this specific case, we can still find a non-canonical $\mathbb{C}^*$-action on $\bar{\mathcal{X}}$ which turns it into a λ-equivariant test-configuration. On the other hand, for $h < 0$, we find that the flat limit $\bar{\mathcal{X}}$ is given by the divisor

$$\lim_{\tau \to 0} \lambda(\tau) \cdot \mathcal{X} = V(x^2(t^6 - 1)).$$

This may be thought of as the product, thickened test-configuration $V(x^2)$ glued to six copies of $\mathbb{P}^2$, and clearly it cannot be the total space of a test-configuration for $\mathbb{P}^1$.

We can also consider the specialisation $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ in the sense of Definition 25. The conjugate one-parameter subgroup $\lambda(\tau)\gamma(t)(\lambda(\tau))^{-1}$ is given by

$$\begin{pmatrix} t^{-1} & -t^{-1}(-1+\tau^{2h})(-1+t^3) & -2t^{-1}(-1+\tau^{2h})^2(-1+t^3) \\ 0 & t^2 & 2t^{-1}(-1+\tau^{2h})(-1+t^3) \\ 0 & 0 & t^{-1} \end{pmatrix},$$

so γ lies in the parabolic $\mathcal{P}(\lambda)$ if and only if $h > 0$. In this case $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is obtained by acting on $V(xz - y^2)$ with $\bar{\gamma} = \lim_{\tau \to 0} \lambda(\tau)\gamma(t)(\lambda(\tau))^{-1}$. The resulting test-configuration is precisely (1). The central fibre $\bar{\mathcal{X}}_0 = V(x(2(x+y)+z))$ is preserved by $\bar{\gamma}$ and λ and we obtain a λ-equivariant test-configuration in a canonical way.

For $h < 0$ we have $\gamma \notin \mathcal{P}(\lambda)$ and we must first conjugate γ by some element $g \in \mathcal{P}(\gamma)$ to obtain $\chi \in \mathcal{P}(\lambda)$. A direct computation shows that one can choose

$$g = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \chi = \begin{pmatrix} t^{-1} & 0 & 0 \\ t^{-1} - t^2 & t^2 & t^{-1} - t^2 \\ 0 & 0 & t^{-1} \end{pmatrix}$$

yielding

$$\bar{\gamma} = \lim_{\tau \to 0} \lambda(\tau)\chi(t)(\lambda(\tau))^{-1} = \begin{pmatrix} t^2 & t^{-1} - t^2 & -t^{-1} + t^2 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}.$$

The corresponding test-configuration $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is given by

$$V(t^3 x(x + 2y + z) - (x + y)^2)$$

endowed with the action of $\bar{\gamma}$, which commutes with λ. Diagonalising $\bar{\gamma}$ (which is of course compatible with diagonalising λ) we see that $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is isomorphic to the test-configuration induced by $\mathrm{diag}(t^{-1}, t^{-1}, t^2)$ given by $V(t^3 xz - y^2)$.

Finally note that the test-configuration $(\mathcal{X}', \mathcal{L}')$ (isomorphic to $(\mathcal{X}, \mathcal{L})$) defined by χ is

$$V((x + y)(y + z) - t^3 y(x + 2y + z)).$$

Taking the flat closure of $(\mathcal{X}', \mathcal{L}')$ under the action of λ gives the one-parameter family of divisors of $\mathbb{P}^1 \times \mathbb{C}$ parametrised by τ

$$(x + y)^2 - t^3 x(x + 2y + z) + \tau^{-2h}(1 - t^3)(x + y)(x + 2y + z).$$

This is a flat one-parameter family taking $(\mathcal{X}', \mathcal{L}')$ to $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$.

We explain next an alternative approach to specialising test-configurations which is more global, i.e. independent of the exponent, and is based on filtrations of the homogeneous coordinate ring. Let χ be the filtration of $R = R(X, L)$ corresponding to $(\mathcal{X}, \mathcal{L})$, and T a torus in $\mathrm{Aut}(X, L)$.

Definition 27 Let $\lambda : \mathbb{C}^* \to T$ be a one-parameter subgroup. The specialisation $\bar{\chi}$ of χ with respect to λ is given by $\bar{\chi}_k = \lim_{\tau \to 0} \lambda(\tau) \cdot \chi_k$, where the limit is taken in the appropriate Grassmannian; the specialization depends on λ, but we suppress it from the notation. If the image of λ is generic in T (i.e. it avoids finitely many hyperplanes in the lattice of 1PS's of T), then $\bar{\chi}$ is T equivariant, and we call it a speicalization of χ with repsect to T.

It is straightforward to check that $\bar{\chi}$ is still a filtration of R in the sense of Definition 14. The limit filtration $\bar{\chi}$ can also be described as follows. Let $\mathrm{Rees}(\chi) \subset R$ be the Rees algebra of the finitely generated filtration χ. A one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{Aut}(X, L)$ acts on R and on $R[t]$ (trivially on t) and we may define a $\mathbb{C}[t]$-subalgebra $\mathrm{Rees}^{\lambda}(\chi) \subset R$ by

$$\mathrm{Rees}^{\lambda}(\chi) = \{\lim_{\tau \to 0} \lambda(\tau)(s) \ : \ s \in \mathrm{Rees}(\chi)\}.$$

Then $\bar{\chi}$ is precisely the filtration of R whose Rees algebra is $\mathrm{Rees}^{\lambda}(\chi)$, i.e.

$$\bar{F}_i R_k = \{s \in R_k \ : \ t^i s \in \mathrm{Rees}^{\lambda}(\chi)\}.$$

The crucial difficulty with this more global approach lies in the fact that the Rees algebra of $\bar{\chi}$ is not finitely generated in general. This is a well-known phenomenon in commutative algebra and an explicit example is given in the "Appendix".

Let $(\mathcal{X}, \mathcal{L})$ be a very ample test-configuration of exponent r. Given a generic one-parameter subgroup of $T \subset \mathrm{Aut}(X, L)$ we can perform two basic constructions. On the one hand we can specialise $(\mathcal{X}, \mathcal{L})$ to $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ in the sense of Definition 25. This specialisation corresponds to a finitely generated filtration η. The Veronese filtration $\eta^{(j)}$ corresponds to the Veronese test-configuration $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$ with exponent jr. On the other hand $(\mathcal{X}, \mathcal{L})$ corresponds to a finitely generated filtration χ of R via the construction described at the end of the previous section. We may specialise χ to $\bar{\chi}$ and consider a finitely generated approximation $\bar{\chi}^{(j)}$, corresponding to a test-configuration of exponent jr: by definition this is in fact $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$. Since $\bar{\chi}$ is not finitely generated (in general), the filtrations $\eta^{(j)}$, $\bar{\chi}^{(j)}$ will differ for infinitely many j, that is the test-configurations $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$ and $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$ differ for infinitely many j. However we can establish a simple comparison.

Proposition 28 *The filtration of $H^0(X, L^{\otimes jr})$ induced by $\bar{\chi}$ (or equivalently by $\bar{\chi}^{(j)}$ or $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$) is included in the filtration of the same vector space induced by $\eta^{(j)}$, i.e. by the filtration corresponding to $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$.*

Proof The result follows at once from the fact that the Rees algebra of $\bar{\chi}$ contains all the generators of the Rees algebra of η, by construction. □

Let us show that when $\bar{\chi}$ is finitely generated then $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is in fact a flat limit of $(\mathcal{X}, \mathcal{L})$ under a $\mathbb{C}^*$-action, and in particular the filtrations $\bar{\chi}^{(j)}$, $\eta^{(j)}$ coincide for all j, that is $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$ and $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$ coincide. In order to simplify the notation (without loss of generality) we assume in the following result that $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ has exponent 1 and χ is the corresponding finitely generated filtration.

Lemma 29 *Suppose that* $\mathrm{Rees}(\bar{\chi}) = \mathrm{Rees}^{\lambda}(\chi)$ *is a finitely generated* $\mathbb{C}[t]$-*subalgebra of* $R[t]$. *Then there exist an embedding* $\iota : \mathcal{X} \to \mathbb{P}^N \times \mathbb{C}$ *and a 1-parameter subgroup* $\widehat{\lambda}\colon \mathbb{C}^* \to GL(N+1, \mathbb{C})$ *such that*

- $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{L}^{\otimes r}$ *for some* $r \geq 1$,
- $\widehat{\lambda}$ *acting on* $\mathbb{P}^N$ *preserves* $\iota(\mathcal{X}_1) \cong X$ *and restricts to the induced action of* λ *on it,*
- *the 1-parameter flat family of subschemes of* $\mathbb{P}^N \times \mathbb{C}$ *induced by* $\widehat{\lambda}$ *(acting trivially on the second factor) has central fibre isomorphic to* $\bar{\mathcal{X}} := \mathrm{Proj}(\mathrm{Rees}(\bar{\chi}))$ *endowed with its natural Serre line bundle* $\mathcal{O}(r)$.

In particular it follows that the central fibre $(\bar{\mathcal{X}}_0, \mathcal{L}_0'^{\otimes r})$ *is a flat 1-parameter degeneration of the central fibre* $(\mathcal{X}_0, \mathcal{L}_0^{\otimes r})$ *(as closed subschemes of* $\mathbb{P}^N$*).*

Proof If $\mathrm{Rees}(\bar{\chi}) = \mathrm{Rees}^{\lambda}(\chi) \subset R[t]$ is a finitely generated $\mathbb{C}[t]$-subalgebra there exists a finite set of elements σ_i of $\mathrm{Rees}(\chi)$ such that the limits $\lim_{\tau\to 0} \lambda(\tau) \cdot \sigma_i$ generate $\mathrm{Rees}(\bar{\chi})$. Since $\lambda(\tau)$ is $\mathbb{C}[t]$-linear and we have $\lambda(\tau)\cdot(s_1+s_2) = \lambda(\tau)\cdot s_1 + \lambda(\tau)\cdot s_2$ and $\lambda(\tau)\cdot(s_1 s_2) = (\lambda(\tau)\cdot s_1)(\lambda(\tau)\cdot s_2)$ for all $s_1, s_2 \in R$, we can then choose our σ_i of the special form $\sigma_i = t^{p(i)} s_i$ where the s_i are homogeneous elements of R. Moreover, enlarging the collection of σ_i's, we can assume that the elements $t^{p(i)} s_i$, $i = 0, \ldots, N$ generate $\mathrm{Rees}(\chi)$. For a suitable $r \geq 1$ the monomials $\tilde{s}_j$ in our elements s_i of homogenous degree r generate the Veronese algebra $\tilde{R} = \bigoplus_{k\geqslant 0} R_{kr}$ (which is thus generated in degree 1) and so the corresponding elements $t^{p(j)}\tilde{s}_j$ generate the Veronese algebra $\bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr}$ and their limits $t^{p(j)} \lim_{\tau\to 0} \lambda(\tau)\cdot\tilde{s}_j$ generate the Veronese algebra $\bigoplus_{k\geqslant 0}(\bar{F}_{kr}\tilde{R})t^{kr}$.

With these assumptions we define a surjective morphism of $\mathbb{C}[t]$-algebras

$$\phi\colon \mathbb{C}[\xi_0, \ldots, \xi_N][t] \to \bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr}$$

by $\phi(t) = t$, $\phi(\xi_i) = t^{p(i)}\tilde{s}_i$. Suppose that the action of λ is given by $\lambda(\tau)\cdot\tilde{s}_i = \sum_j a_{ij}(\tau)\tilde{s}_j$. We define a one-parameter subgroup $\widehat{\lambda}\colon \mathbb{C}^* \to GL(\mathbb{C}_1[\xi_0, \ldots, \xi_N])$, acting on degree 1 elements by $\widehat{\lambda}(\tau)\cdot\xi_i = \sum_j a_{ij}(\tau)\xi_j$, and extend its action trivially on t. The morphism ϕ induces the required embedding

$$\iota\colon \mathcal{X} = \mathrm{Proj}_{\mathbb{C}[t]} \bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr} \to \mathrm{Proj}_{\mathbb{C}[t]} \mathbb{C}[\xi_0, \ldots, \xi_N][t],$$

which intertwines the actions of λ and $\widehat{\lambda}$. By construction the limit as $\tau \to 0$ of the flat family of closed subschemes of $\mathbb{P}^N \times \mathbb{C}$ given by

$$\widehat{\lambda}(\tau) \cdot \iota\big(\operatorname{Proj}_{\mathbb{C}[t]} \bigoplus_{k \geqslant 0} (F_{kr} \tilde{R}) t^{kr}\big)$$

is isomorphic to $\operatorname{Proj}_{\mathbb{C}[t]} \bigoplus_{k \geqslant 0} (\bar{F}_{kr} \tilde{R}) t^{kr}$ and so it gives a copy of $\bar{\mathcal{X}}$ embedded in $\mathbb{P}^N \times \mathbb{C}$ as a flat 1-parameter degeneration of $\mathcal{X}$.

To prove the statement on central fibres we look at the family of closed subschemes of $\mathbb{P}^N$ given by

$$\widehat{\lambda}(\tau) \cdot \iota(\mathcal{X}_0) = \widehat{\lambda}(\tau) \cdot \iota\big(\operatorname{Proj}_{\mathbb{C}[t]} \operatorname{gr} \bigoplus_{k \geqslant 0} (F_{kr} \tilde{R}) t^{kr}\big).$$

Taking the flat closure of this 1-parameter family we obtain a closed subscheme $\mathcal{Y}_0 \subset \mathbb{P}^N$ whose underlying reduced subscheme $\mathcal{Y}_0^{\mathrm{red}}$ is contained in $\bar{\mathcal{X}}_0 \subset \mathbb{P}^N$. By flatness the Hilbert function of $\mathcal{Y}_0$ is the same as that of the central fibre $(\mathcal{X}_0, \mathcal{L}_0^{\otimes r})$ and so the same as that of the general fibre $(X, L^{\otimes r})$. Similarly the Hilbert function of $\bar{\mathcal{X}}_0 \subset \mathbb{P}^N$ is the same as that of $(\bar{\mathcal{X}}_0, \bar{\mathcal{L}}_0^{\otimes r})$ and so the same as that of the general fibre $(X, L^{\otimes r})$. As we have $\mathcal{Y}_0^{\mathrm{red}} \subset \bar{\mathcal{X}}_0 \subset \mathbb{P}^N$ and $\bar{\mathcal{X}}_0, \mathcal{Y}_0 \subset \mathbb{P}^N$ have the same Hilbert functions we must actually have $\mathcal{Y}_0 = \bar{\mathcal{X}}_0$ as required. □

The following observation follows immediately from the definitions of the weight functions (Definitions 18 and 19) and of the specialisation $\bar{\chi}$ (Definition 27).

Lemma 30 *In the situation of Lemma 29 we have*

$$w_{(\bar{\mathcal{X}}, \bar{\mathcal{L}})}(k) = w_{(\mathcal{X}, \mathcal{L})}(k), \quad d_{(\bar{\mathcal{X}}, \bar{\mathcal{L}})}(k) = d_{(\mathcal{X}, \mathcal{L})}(k).$$

for all k. In particular we have

$$\mathrm{DF}(\bar{\mathcal{X}}, \bar{\mathcal{L}}) = \mathrm{DF}(\mathcal{X}, \mathcal{L}), \quad ||(\bar{\mathcal{X}}, \bar{\mathcal{L}})||_{L^2} = ||(\mathcal{X}, \mathcal{L})||_{L^2}.$$

Let us now consider the general case.

Theorem 31 *Let χ be a possibly non-finitely generated filtration, and let $\bar{\chi}$ be its specialisation with respect to a torus $T \subset \operatorname{Aut}(X, L)$ in the sense of Definition 27. Then we have*

$$\mathrm{Chow}_\infty(\bar{\chi}) \leq \mathrm{Chow}_\infty(\chi).$$

Proof We claim that the inequality $\mathrm{Chow}_r(\bar{\chi}^{(r)}) \leq \mathrm{Chow}_r(\chi^{(r)})$ holds for every r. By Definition 22 this will imply the Theorem.

Before proving the claim, let us recall the relation between the Chow weight and classical GIT, following [16, Section 3], [9, Section 7] and [22, Section 2]. Let

$V_r = H^0(X, L^{\otimes r})^\vee$, and denote by γ a 1PS of $GL(V_r)$ which induces the test configuration associated to $\chi^{(r)}$. The group $GL(V_r)$ acts on the appropriate Chow variety Z_r, and $X \subset \mathbb{P}(H^0(X, L^{\otimes r})^\vee)$ gives a point $[X] \in Z_r$. On Z_r we have the classical, ample Chow line bundle, giving a linearisation for the action of $GL(V_r)$. The r-th Chow weight of $\chi^{(r)}$ introduced in Definition 19 is the Hilbert-Mumford weight of the point $[X] \in Z_r$ under γ, computed with respect to a convenient rational rescaling of the ample Chow line bundle (with this normalisation the Chow line bundle becomes an ample $\mathbb{Q}$-line bundle, but this causes no difficulties).

The claim now follows from Proposition 8, i.e. the fact that Hilbert-Mumford weights decrease under specialisation. □

5 Application to cscK Polarised Manifolds

In this Section we show that Conjecture 1 combined with ideas from [17, 19] implies a new proof that cscK manifolds are K-polystable.

Theorem 32 *Let (X, L) be a cscK manifold and let T be a maximal torus in* $\mathrm{Aut}(X, L)$. *Then (X, L) is T-equivariantly K-polystable.*

More explicitly, Theorem 32 states that, given a normal T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$, we have

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$$

with equality if and only if $(\mathcal{X}, \mathcal{L})$ is a product.

Proof Let $(\mathcal{X}, \mathcal{L})$ be a normal T-equivariant test configuration. By a result of Donaldson [8] (X, L) is K-semistable, so it is enough to assume that $(\mathcal{X}, \mathcal{L})$ is not a product and to show that we cannot have $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$. We argue by contradiction assuming $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$.

Denote by α the $\mathbb{C}^*$ action on $(\mathcal{X}, \mathcal{L})$. Let β_i be an orthogonal basis of 1-parameter subgroups β_i of $\mathrm{Aut}(X, L)$ (see [20] for a discussion of the formal inner product on $\mathbb{C}^*$-actions). As $(\mathcal{X}, \mathcal{L})$ is T-equivariant, there are $\mathbb{C}^*$-actions $\tilde{\beta}_i$ on $(\mathcal{X}, \mathcal{L})$, preserving the fibres, commuting with each other and with α, and extending the action of β_i. Fixing i, the total space $(\mathcal{X}, \mathcal{L})$ endowed with the $\mathbb{C}^*$-action $\alpha \pm \tilde{\beta}_i$ is a test-configuration for (X, L), with Donaldson-Futaki invariant

$$\begin{aligned} \mathrm{DF}(\alpha \pm \tilde{\beta}_i) &= \mathrm{DF}(\alpha) \pm \mathrm{DF}(\tilde{\beta}_i) \\ &= \pm\, \mathrm{DF}(\tilde{\beta}_i) \end{aligned}$$

(the first equality follows since α, $\tilde{\beta}_i$ are commuting $\mathbb{C}^*$-actions on the same polarised scheme). Since we are assuming that (X, L) is cscK we know it is K-semistable and so we must have $\mathrm{DF}(\tilde{\beta}_i) = 0$ for all i. Let $(\mathcal{X}, \mathcal{L})_{\frac{\perp}{T}}$ denote the

L^2-orthogonal in the sense of [20], i.e. the test-configuration with total space $(\mathcal{X}, \mathcal{L})$ endowed with $\mathbb{C}^*$-action

$$\alpha - \sum_i \frac{\langle \alpha, \tilde{\beta}_i \rangle}{||\tilde{\beta}_i||^2} \tilde{\beta}_i.$$

Then we see that $\mathrm{DF}(\mathcal{X}, \mathcal{L})^{\perp}_{T} = 0$.

Since $\mathcal{X}$ is normal and not isomorphic to $X \times \mathbb{C}$, by [19] section 3 there exists a point $p \in (\mathcal{X}_1, \mathcal{L}_1)$ which is fixed by the maximal torus T, and such that denoting by $\overline{\alpha \cdot p}$ the closure of the orbit of p in $(\mathcal{X}, \mathcal{L})$ we have

$$\begin{aligned} \mathrm{DF}(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})^{\perp}_{T} &= \mathrm{DF}(\mathcal{X}, \mathcal{L})^{\perp}_{T} - C\epsilon^{n-1} + O(\epsilon^n) \\ &= -C\epsilon^{n-1} + O(\epsilon^n) \end{aligned} \tag{2}$$

for some constant $C > 0$. Here $(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})$ is the test-configuration for $(\mathrm{Bl}_p X, L - \epsilon E)$ (E, $\mathcal{E}$ denoting the exceptional divisors) induced by blowing up the orbit $\overline{\alpha \cdot p}$ in $\mathcal{X}$ with sufficiently small rational parameter $\epsilon > 0$. Since p is fixed by T there is a natural inclusion $T \subset \mathrm{Aut}(\mathrm{Bl}_p X, L - \epsilon E)$ and then $(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})^{\perp}_{T}$ denotes the L^2 orthogonal to T in the sense of [20].

As explained in [19] Theorem 2.4 a well-known result of Arezzo, Pacard and Singer [1] implies that the polarised manifold $(\mathrm{Bl}_p X, L - \epsilon E)$ admits an extremal metric in the sense of Calabi. The semistability result of [20] shows that we must have $\mathrm{DF}(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})^{\perp}_{T} \geq 0$. But this contradicts (2), so we must have in fact $\mathrm{DF}(\mathcal{X}, \mathcal{L}) > 0$ as claimed. □

Corollary 33 *If Conjecture 1 holds, then cscK manifolds are K-polystable.*

Proof Let (X, L) be a cscK manifold, and T a maximal torus in $\mathrm{Aut}(X, L)$. Theorem 32 implies that (X, L) is T-equivariantly K-polystable. Conjecture 1 then implies that (X, L) is K-polystable. □

Remark 34 The proof of the main result of [19] (Theorem 1.4) shows that if (X, L) is extremal and $T \subset \mathrm{Aut}(X, L)$ is a maximal torus then we have $\mathrm{DF}(\mathcal{X}, \mathcal{L})^{\perp}_{T} > 0$ for all T-equivariant test-configurations whose normalisation is not induced by a holomorphic vector field in T (or equivalently, which are not isomorphic to such a product outside a closed subscheme of codimension at least 2). If the assumption is dropped there are counterexamples. Note that Theorem 1.4 in [19] is mistakenly stated without this assumption. See [11] Remark 4 and the note [18] for further discussion.

Appendix

In this appendix we present an example of a test-configuration $(\mathcal{X}, \mathcal{L})$ with a 1-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{Aut}(X, L)$ such that the λ-equivariant filtration $\bar{\chi}$ of Definition 25 is not finitely generated. This is done by adapting a well-known example in the literature on canonical bases of subalgebras, due to Robbiano and Sweedler ([15] Example 1.20).

Consider the polynomial algebra $\mathbb{C}[t][x, y]$ over the ring $\mathbb{C}[t]$ and let A denote the $\mathbb{C}[t]$-subalgebra generated by

$$t(x + y), txy, txy^2, t^2y.$$

Then $A \subset R[t]$ is the Rees algebra of a homogeneous, multiplicative, pointwise left bounded finitely generated filtration χ of the homogeneous coordinate ring $R = \mathbb{C}[x, y]$ of the projective line $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. So $\mathrm{Proj}_{\mathbb{C}[t]} A$ endowed with its natural Serre bundle $\mathcal{O}(1)$ is a test-configuration for $\mathbb{P}^1$. Consider the 1-parameter subgroup $\lambda\colon \mathbb{C}^* \to SL(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)))$ acting by

$$\lambda(\tau) \cdot x = \tau^{-1}x, \quad \lambda(\tau) \cdot y = \tau y.$$

We let $\bar{\chi}$ be the limit of χ under the action of λ as in the proof of Proposition 27.

Proposition 35 *The limit filtration $\bar{\chi}$ is not finitely generated.*

Proof The 1-parameter subgroup λ induces a term ordering $>$ on the $\mathbb{C}[t]$-algebra $\mathbb{C}[t][x, y]$ which is compatible with the graded $\mathbb{C}[t]$-algebra structure and for which we have $x > y$. Let us denote the initial term of an element $\sigma \in \mathbb{C}[t][x, y]$ by $\mathrm{in}_> \sigma$. The Rees algebra $\mathrm{Rees}(\bar{\chi})$ coincides with the initial algebra of A defined by

$$\mathrm{in}_> A = \{\mathrm{in}_> \sigma \ : \ \sigma \in A\}.$$

We show that $\mathrm{in}_> A$ is not finitely generated. The proof follows closely the original argument in [15] Example 1.20.

Claim 1 The algebra A contains all the monomials of the form $t^{n-1}xy^n$ for $n \geq 3$, and does not contain elements which have a homogeneous component of the form t^kxy^n for $k < n-1$. In particular no element of A can have initial term of the form t^kxy^n for $k < n-1$. To check the first statement we observe that we have for $n \geq 3$

$$t^{n-1}xy^n = t(x + y)t^{n-2}xy^{n-1} - t(xy)t(t^{n-3}xy^{n-2})$$

and then argue by induction starting from the fact that A contains the monomials $t(x + y), txy, txy^2$. For the second statement it is enough to check that A does not contain t^kxy^n for $k < n-1$ (since A is a graded subalgebra). This is a simple check.

Claim 2 The algebra A does not contain elements which have a homogeneous component of the form $t^k y^j$ for $k \leq j$. In particular no element of A can have initial term of the form $t^k y^j$ for $k \leq j$. Since A is a graded subalgebra it is enough to show that $t^k y^j$ cannot belong to A if $k \leq j$. All the elements of A are of the form $f(t(x+y), txy, txy^2, t^2y)$ where $f(x_1, x_2, x_3, x_4)$ is a polynomial with coefficients in $\mathbb{C}[t]$. Assuming

$$f(t(x+y), txy, txy^2, t^2y) = t^k y^j$$

and setting $y = 0$ gives $f(tx, 0, 0, 0) = 0$. Similarly setting $x = 0$ gives $f(ty, 0, 0, t^2y) = t^k y^j$. If $k \leq j$ it follows that necessarily $k = j$ and $f(x_1, 0, 0, x_2) = x_1$. Comparing with $f(tx, 0, 0, 0)$ we find $tx = 0$, a contradiction.

Claim 3 $\text{in}_> A$ *is not finitely generated.* Assuming $\text{in}_> A$ is finitely generated we can find a finite set σ_i of elements of A such that $\text{in}_> \sigma_i$ generate $\text{in}_> A$. By finiteness we can choose $m \gg 1$ such that for all i we have $\text{in}_> \sigma_i \neq t^{m-1}xy^m$. On the other hand by Claim 1 we know that for all m we have $t^{m-1}xy^m \in \text{in}_> A$. By the definition of a term ordering we know thus that $t^{m-1}xy^m$ must be a product of powers of initial terms of the elements σ_i. As x appears linearly it follows that there must be two generators σ_i, σ_j with $\text{in}_> \sigma_i = t^p xy^r$, respectively $\text{in}_> \sigma_j = t^q y^s$ with $p+q = m-1, r+s = m$. By Claim 1 we must have $p \geq r-1$ and by Claim 2 we must have $q > s$. Hence $p+q > r+s-1 = m-1$ so $p+q \geq m$, a contradiction. □

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Notes on K-Semistability of Toric Polarized Varieties

Kento Fujita

Abstract We give a systematical construction of the blowup type test configuration, named the basic blowup type test configuration, for a toric polarized variety from a torus invariant prime divisor. If the barycenter of the associated polytope is not equal to the barycenter of its facets, then we can find a torus invariant prime divisor such that the Donaldson-Futaki invariant of the associated test configuration is negative.

Keywords K-stability · Toric varieties · Constant scalar curvature Kähler metrics

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1 Introduction

For the theory of toric varieties, we refer the readers to [4] and [8]. When we consider toric varieties, the base field $\Bbbk$ is assumed to be an algebraically closed field (of any characteristic). In this note, we systematically construct blowup type destabilizing test configurations for K-unstable toric polarized pairs. Let us recall some basic notions in order to state our result.

Notation 1.1 We set $N := \bigoplus_{i=1}^{d} \mathbb{Z}e_i$, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\{e_i^*\}_{1\leq i\leq d} \subset M$ be the dual basis of $\{e_i\}_{1\leq i\leq d}$, and let $\langle , \rangle\colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ be the natural pairing. Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope, let $\{Q_\lambda\}_{\lambda\in\Lambda}$ be the set of facets of P, let H_λ ($\lambda \in \Lambda$) be the linear span of Q_λ (i.e., the affine hyperplane in $M_{\mathbb{R}}$ with $Q_\lambda \subset H_\lambda$). Let dx (resp., dx_λ ($\lambda \in \Lambda$)) be the canonical Lebesgue measure on $M_{\mathbb{R}}$ (resp., on H_λ) such that $M_{\mathbb{R}}/M$ (resp., $H_\lambda/(H_\lambda \cap M)$) is of measure 1.

Definition 1.2 Take any $\lambda \in \Lambda$.

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(1) Let

$$\mathrm{vol}(P) := \int_P dx$$

be the *volume of* P. Similarly, let

$$\mathrm{vol}(Q_\lambda) := \int_{Q_\lambda} dx_\lambda$$

be the (relative) *volume of* Q_λ. Moreover, we set $\mathrm{vol}(\partial P) := \sum_{\lambda\in\Lambda} \mathrm{vol}(Q_\lambda)$.

(2) The *barycenter* $b(P) \in M_{\mathbb{R}}$ *of* P is defined by

$$\langle b(P), v\rangle = \frac{\int_P \langle x, v\rangle dx}{\mathrm{vol}(P)}$$

for any $v \in N_{\mathbb{R}}$. Similarly, the (relative) *barycenter* $b(Q_\lambda) \in M_{\mathbb{R}}$ of Q_λ is defined by

$$\langle b(Q_\lambda), v\rangle = \frac{\int_{Q_\lambda} \langle x, v\rangle dx_\lambda}{\mathrm{vol}(Q_\lambda)}$$

for any $v \in N_{\mathbb{R}}$, and the *barycenter* $b(\partial P) \in M_{\mathbb{R}}$ of ∂P is defined by

$$b(\partial P) := \frac{\sum_{\lambda\in\Lambda} \mathrm{vol}(Q_\lambda) b(Q_\lambda)}{\mathrm{vol}(\partial P)}.$$

Remark 1.3

(1) For any $\lambda \in \Lambda$, we have $b(Q_\lambda) \in H_\lambda$. Indeed, there exist $v_\lambda \in N_{\mathbb{R}}$ and $a_\lambda \in \mathbb{R}$ such that $H_\lambda \subset M_{\mathbb{R}}$ is defined by the equation

$$\{u \in M_{\mathbb{R}} \mid \langle u, v_\lambda\rangle = a_\lambda\}.$$

By the definition of $b(Q_\lambda)$, we have

$$\langle b(Q_\lambda), v_\lambda\rangle = \frac{\int_{Q_\lambda} a_\lambda dx_\lambda}{\mathrm{vol}(Q_\lambda)} = a_\lambda.$$

(2) From the definition of $b(\partial P)$, we have

$$\langle b(\partial P), v\rangle = \frac{\sum_{\lambda\in\Lambda} \int_{Q_\lambda} \langle x, v\rangle dx_\lambda}{\sum_{\lambda\in\Lambda} \mathrm{vol}(Q_\lambda)}$$

for any $v \in N_{\mathbb{R}}$.

We recall the following well-known result:

Theorem 1.4 (see [6]) *Let X be the projective toric variety and L be the ample Cartier divisor on X associated with $P \subset M_{\mathbb{R}}$. If $b(\partial P) \neq b(P)$, then (X, L) is K-unstable.*

The purpose of this note is to give an explicit blowup type destabilizing test configuration of (X, rL) (for $r \gg 1$) from some specific torus invariant prime divisor. More precisely, we will see the following result:

Theorem 1.5 *Let X be the projective toric variety and L be the ample Cartier divisor on X associated with $P \subset M_{\mathbb{R}}$. For any torus invariant prime divisor D on X, we can construct the basic blowup type test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of (X, rL) via D (see Definition 4.4) for any sufficiently divisible $r \in \mathbb{Z}_{>0}$. Moreover, if $b(\partial P) \neq b(P)$, then we can choose D with* $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$.

Thus, we give an alternative proof of Theorem 1.4. We emphasize that, when $b(P) \neq b(\partial P)$, the existence of destabilizing blowup type test configuration is well-known from Theorem 1.4. Our construction of the basic blowup type test configuration via a torus invariant prime divisor is similar to the construction in [7], very easy to construct, and very easy to compute the Donaldson-Futaki invariant. The computation of the Donaldson-Futaki invariant relies on the (weighted) Ehrhart theory.

2 On the Barycenters of Lattice Polytopes

Definition 2.1 For a bounded set $S \subset M_{\mathbb{R}}$, for $v \in N_{\mathbb{R}}$ and for $k \in \mathbb{Z}_{\geq 0}$, we set

$$L_S(k) := \#(kS \cap M),$$
$$f_S^v(k) := \sum_{u \in kS \cap M} \langle u, v \rangle.$$

From now on, we fix Notation 1.1.

Theorem 2.2

(1) (*see* [6, Proposition 4.1.3]) *$L_P(k)$ is a polynomial of degree d. Moreover, we have*

$$L_P(k) = \mathrm{vol}(P)k^d + \frac{1}{2}\mathrm{vol}(\partial P)k^{d-1} + O(k^{d-2}).$$

(2) *$f_P^v(k)$ is a polynomial of degree at most $d + 1$. Moreover, we have*

$$f_P^v(k) = \mathrm{vol}(P)\langle b(P), v \rangle k^{d+1} + \frac{1}{2}\mathrm{vol}(\partial P)\langle b(\partial P), v \rangle k^d + O(k^{d-1}).$$

Proof

(1) is well-known. $L_P(k)$ is called the *Ehrhart polynomial* of P. See [2, Lemma 3.19, Theorem 3.8 and Theorem 5.6].

(2) It is well-known that $f_P^v(k)$ is a polynomial of degree at most $d+1$ and

$$\lim_{k\to\infty}\frac{f_P^v(k)}{k^{d+1}} = \mathrm{vol}(P)\langle b(P), v\rangle.$$

See [1, Proposition 17] for example. Since

$$f_S^{av+a'v'}(k) = af_S^v(k) + a'f_S^{v'}(k)$$

holds for any $a, a' \in \mathbb{R}$ and for any $v, v' \in N_{\mathbb{R}}$, we may assume that $v = e_1$. Let us write

$$f_P^{e_1}(k) = f_{d+1}k^{d+1} + f_d k^d + O(k^{d-1}).$$

For a bounded set $S \subset M_{\mathbb{R}}$, let $\mathrm{cone}(S) \subset M_{\mathbb{R}} \oplus \mathbb{R}e_{d+1}^*$ be the cone spanned by $\{(s, 1) \mid s \in S\}$. Moreover, for any set $T \subset M_{\mathbb{R}} \oplus \mathbb{R}e_{d+1}^*$, we set the formal sum of monomials

$$\sigma_T(\vec{z}) := \sigma_T(z_1, \ldots, z_{d+1}) := \sum_{m\in T\cap(M\oplus\mathbb{Z}e_{d+1}^*)} \vec{z}^m,$$

where $\vec{z}^m := z_1^{m_1}\cdots z_{d+1}^{m_{d+1}}$ with $\langle m, e_i\rangle = m_i$. We note that

$$\frac{\partial\sigma_{\mathrm{cone}(S)}}{\partial z_1}(1, \ldots, 1, z_{d+1}) = \sum_{k\in\mathbb{Z}_{\geq 0}} f_S^{e_1}(k) z_{d+1}^k.$$

By Stanley's reciprocity (see [2, Theorem 4.3]), we have

$$\sigma_{\mathrm{cone}(P)}(z_1^{-1}, \ldots, z_{d+1}^{-1}) = (-1)^{d+1}\sigma_{\mathrm{Int}(\mathrm{cone}(P))}(z_1, \ldots, z_{d+1}).$$

By taking $\frac{\partial\bullet}{\partial z_1}(1, \ldots, 1, z_{d+1})$, we have

$$-\frac{\partial\sigma_{\mathrm{cone}(P)}}{\partial z_1}(1, \ldots, 1, z_{d+1}^{-1}) = (-1)^{d+1}\frac{\partial\sigma_{\mathrm{Int}(\mathrm{cone}(P))}}{\partial z_1}(1, \ldots, 1, z_{d+1}).$$

By the above equalities, together with [2, Exercise 4.5], we have

$$\sum_{k\geq 1} f_P^{e_1}(-k)z_{d+1}^k = -\sum_{k\leq 0} f_P^{e_1}(-k)z_{d+1}^k$$
$$= -\frac{\partial\sigma_{\mathrm{cone}(P)}}{\partial z_1}(1,\ldots,1,z_{d+1}^{-1}) = (-1)^{d+1}\frac{\partial\sigma_{\mathrm{Int}(\mathrm{cone}(P))}}{\partial z_1}(1,\ldots,1,z_{d+1})$$
$$= (-1)^{d+1}\sum_{k\geq 1} f_{\mathrm{Int}(P)}^{e_1}(k)z_{d+1}^k.$$

This implies that

$$f_P^{e_1}(-k) = (-1)^{d+1} f_{\mathrm{Int}(P)}^{e_1}(k)$$

for any $k \in \mathbb{Z}_{>0}$. Thus

$$f_{\partial P}^{e_1}(k) = f_P^{e_1}(k) - f_{\mathrm{Int}(P)}^{e_1}(k) = 2f_d k^d + O(k^{d-1})$$

holds. We already know that

$$f_{\partial P}^{e_1}(k) - \sum_{\lambda\in\Lambda} f_{Q_\lambda}^{e_1}(k) = O(k^{d-1}),$$

$f_{Q_\lambda}^{e_1}(k)$ is a polynomial of degree at most d for any $\lambda \in \Lambda$, and

$$\lim_{k\to\infty}\frac{f_{Q_\lambda}^{e_1}(k)}{k^d} = \mathrm{vol}(Q_\lambda)\langle b(Q_\lambda), e_1\rangle.$$

Therefore,

$$f_d = \frac{1}{2}\sum_{\lambda\in\Lambda}\mathrm{vol}(Q_\lambda)\langle b(Q_\lambda), e_1\rangle = \frac{1}{2}\mathrm{vol}(\partial P)\langle b(\partial P), e_1\rangle$$

holds. □

Definition 2.3 For a full-dimensional lattice polytope $P \subset M_{\mathbb{R}}$ and for $v \in N_{\mathbb{R}}$, we can write

$$L_P(k) = L_d k^d + L_{d-1}k^{d-1} + O(k^{d-2}),$$
$$f_P^v(k) = f_{d+1}k^{d+1} + f_d k^d + O(k^{d-1}).$$

We define

$$\mathrm{DF}(P, v) := L_{d-1}f_{d+1} - L_d f_d.$$

The following is trivial.

Lemma 2.4 *Let P, v be as in Definition 2.3.*

(1) *For any $r \in \mathbb{Z}_{>0}$, we have* $\mathrm{DF}(rP, v) = r^{2d}\,\mathrm{DF}(P, v)$.
(2) *For any $u \in M$, we have* $\mathrm{DF}(P + u, v) = \mathrm{DF}(P, v)$.
(3) *For any $a \in \mathbb{R}$, we have* $\mathrm{DF}(P, av) = a\,\mathrm{DF}(P, v)$.

Proof

(1) We have $L_{rP(k)} = L_P(rk)$ and $f^v_{rP}(k) = f^v_P(rk)$.
(2) We have $L_{P+u}(k) = L_P(k)$ and $f^v_{P+u}(k) = f^v_P(k) + k\langle u, v\rangle L_P(k)$.
(3) We have $f^{av}_P(k) = af^v_P(k)$. □

3 K-Semistability

We quickly recall the theory of K-stability. For detail, see [6] and [9] for example.

Definition 3.1 (see [6, 9–11] for example) Let X be a d-dimensional normal projective variety over $\Bbbk$ and L be an ample Cartier divisor on X.

(1) A coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{X\times\mathbb{A}^1_t}$ is said to be a *flag ideal* if $\mathcal{I}$ is of the form

$$\mathcal{I} = I_n + I_{n-1}t^1 + \cdots + I_1 t^{n-1} + (t^n),$$

where $\mathcal{O}_X \supset I_1 \supset \cdots \supset I_n$ are coherent ideal sheaves on X.
(2) Let $\mathcal{I}$ be a flag ideal, let $r \in \mathbb{Z}_{>0}$, let $\Pi\colon \mathcal{X} \to X \times \mathbb{A}^1$ be the blowup along $\mathcal{I}$, let $E \subset \mathcal{X}$ be the Cartier divisor defined by $\mathcal{O}_{\mathcal{X}}(-E) = \mathcal{I}\mathcal{O}_{\mathcal{X}}$, and we set $\mathcal{L} := \Pi^* p_1^* L^{\otimes r}(-E)$. Then $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is said to be a *blowup type test configuration of* (X, rL) if $\mathcal{L}$ is semiample over $\mathbb{A}^1$.
(3) Let $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ be a blowup type test configuration of (X, rL). For any $k \in \mathbb{Z}_{>0}$, we set

$$w(k) := -\dim\left(\frac{H^0(X \times \mathbb{A}^1, p_1^* L^{\otimes kr})}{H^0(X \times \mathbb{A}^1, p_1^* L^{\otimes kr}\mathcal{I}^k)}\right).$$

It is known that $w(k)$ is a polynomial of degree at most $d + 1$ for $k \gg 0$ (see [9, §3] for example). Set

$$w(k) = w_{d+1}k^{d+1} + w_d k^d + O(k^{d-1}),$$
$$h^0(X, L^{\otimes kr}) = L_d k^d + L_{d-1}k^{d-1} + O(k^{d-2}).$$

The *Donaldson-Futaki invariant* $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ of $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is defined by

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := L_{d-1}w_{d+1} - L_d w_d.$$

(4) (X, L) is said to be *K-unstable* if there exists a blowup type test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of (X, rL) for some $r \in \mathbb{Z}_{>0}$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$ holds. In this case, the test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is said to be a *blowup type destabilizing test configuration* of (X, rL). If (X, L) is not K-unstable, then we say that (X, L) is *K-semistable*.

Remark 3.2 It is obvious that (X, L) is K-semistable if and only if (X, rL) is K-semistable for some (equivalently, for any) $r \in \mathbb{Z}_{>0}$. Thus, we can define K-semistability for (X, L) with L an ample $\mathbb{Q}$-divisor.

4 Basic Blowup Type Test Configurations

Let X be the projective d-dimensional toric variety over $\Bbbk$ associated with a fan Σ in $N_{\mathbb{R}}$. Let $\{v_\lambda\}_{\lambda\in\Lambda}$ be the set of primitive generators of the set of one-dimensional cones in Σ. For any $\lambda \in \Lambda$, let $D_\lambda \subset X$ be the torus invariant prime divisor on X associated with a one-dimensional cone $\mathbb{R}_{\geq 0}v_\lambda \in \Sigma$.

Let $L := \sum_{\lambda\in\Lambda} d_\lambda D_\lambda$ be a torus invariant ample Cartier divisor on X ($d_\lambda \in \mathbb{Z}$), and let $P \subset M_{\mathbb{R}}$ be the corresponding lattice polytope, that is,

$$P := \{u \in M_{\mathbb{R}} \mid \langle u, v_\lambda\rangle \geq -d_\lambda \text{ for any } \lambda \in \Lambda\}.$$

Since L is ample, P is full-dimensional. Moreover, there is a one-to-one correspondence between the set of facets on P and Λ. For any $\lambda \in \Lambda$, let $Q_\lambda \subset P$ be the corresponding facet.

Lemma 4.1 *For any $k, j \in \mathbb{Z}_{\geq 0}$, we have*

$$H^0(X, L^{\otimes k}) = \bigoplus_{u\in kP\cap M} \Bbbk\chi^u.$$

Moreover, for any $\lambda \in \Lambda$, as a subset of $H^0(X, L^{\otimes k})$, we have

$$H^0(X, L^{\otimes k}(-jD_\lambda)) = \bigoplus_{\substack{u\in kP\cap M,\\ \langle u,v_\lambda\rangle\geq -kd_\lambda+j}} \Bbbk\chi^u.$$

Proof Well-known. See [8, p. 66 and p. 61] for example. □

Now, we construct a blowup type test configuration for a given torus invariant prime divisor. The construction is similar to the one in [7]. Fix $\lambda_0 \in \Lambda$ and set

$d_0 := d_{\lambda_0}$, $v_0 := v_{\lambda_0}$ and $D_0 := D_{\lambda_0}$ for simplicity. Fix $\tau \in \mathbb{Z}_{>0}$ such that

$$\{u \in M_{\mathbb{R}} \mid \langle u, v_0 \rangle \geq -d_0 + \tau\} = \emptyset$$

holds. Take $r \in \mathbb{Z}_{>0}$ sufficiently divisible such that $rP \subset M_{\mathbb{R}}$ is a normal lattice polytope (i.e., the graded $\Bbbk$-algebra

$$\bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(X, L^{\otimes kr})$$

is generated by $H^0(X, L^{\otimes r})$). By Lemma 4.1, the graded $\Bbbk$-algebra

$$\bigoplus_{k, j \in \mathbb{Z}_{\geq 0}} H^0(X, L^{\otimes kr}(-jD_0)) = \bigoplus_{\substack{k \in \mathbb{Z}_{\geq 0}, \\ j \in [0, kr\tau] \cap \mathbb{Z}}} H^0(X, L^{\otimes kr}(-jD_0))$$

is generated by

$$\bigoplus_{j \in [0, r\tau] \cap \mathbb{Z}} H^0(X, L^{\otimes r}(-jD_0)).$$

For any $j \in [0, r\tau] \cap \mathbb{Z}$, let us set the coherent ideal sheaf $I_j \subset \mathcal{O}_X$ as the image of the composition

$$H^0(X, L^{\otimes r}(-jD_0)) \otimes_{\Bbbk} L^{\otimes(-r)} \hookrightarrow H^0(X, L^{\otimes r}) \otimes_{\Bbbk} L^{\otimes(-r)} \xrightarrow{\text{ev}} \mathcal{O}_X,$$

where ev is the natural evaluation homomorphism. Let us consider the flag ideal

$$\mathcal{I} := I_{r\tau} + I_{r\tau-1}t^1 + \cdots + I_1 t^{r\tau-1} + (t^{r\tau}) \subset \mathcal{O}_{X \times \mathbb{A}^1_t}.$$

Note that, by construction, we have $I_j \subset \mathcal{O}_X(-jD_0)$ for any $j \in [0, r\tau] \cap \mathbb{Z}$ and $\mathcal{O}_X = I_0 \supset I_1 \supset \cdots \supset I_{r\tau} = 0$. For any $k \in \mathbb{Z}_{>0}$ and for any $j \in [0, kr\tau] \cap \mathbb{Z}$, we set

$$J_{(k,j)} := \sum_{\substack{j_1 + \cdots + j_k = j, \\ j_1, \ldots, j_k \in [0, r\tau] \cap \mathbb{Z}}} \prod_{l=1}^{k} I_{j_l}.$$

By construction, we have

$$\mathcal{I}^k = J_{(k,kr\tau)} + J_{(k,kr\tau-1)}t^1 + \cdots + J_{(k,1)}t^{kr\tau-1} + (t^{kr\tau})$$

for any $k \in \mathbb{Z}_{>0}$.

Lemma 4.2 (see [7, Lemma 3.1]) *For any $k \in \mathbb{Z}_{>0}$ and for any $j \in [0, kr\tau] \cap \mathbb{Z}$, the coherent ideal sheaf $J_{(k,j)}$ is equal to the image of the homomorphism*

$$H^0(X, L^{\otimes kr}(-jD_0)) \otimes_{\Bbbk} L^{\otimes(-kr)} \hookrightarrow H^0(X, L^{\otimes kr}) \otimes_{\Bbbk} L^{\otimes(-kr)} \xrightarrow{\text{ev}} \mathcal{O}_X.$$

In particular, we have

$$H^0(X, L^{\otimes kr}(-jD_0)) = H^0(X, L^{\otimes kr} \cdot J_{(k,j)})$$

as subspaces of $H^0(X, L^{\otimes kr})$.

Proof The proof is same as the proof of [7, Lemma 3.1]. □

Let $\Pi\colon \mathcal{X} \to X \times \mathbb{A}^1$ be the blowup along $\mathcal{I}$, let $E \subset \mathcal{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\mathcal{X}}(-E) = \mathcal{I}\mathcal{O}_{\mathcal{X}}$, and we set $\mathcal{L} := \Pi^* p_1^* L^{\otimes r}(-E)$.

Lemma 4.3 (see [7, Lemma 3.2]) *$\mathcal{L}$ is semiample over $\mathbb{A}^1$. Thus $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is a blowup type test configuration of (X, rL).*

Proof The proof is same as the proof of [7, Lemma 3.2]. □

Definition 4.4 (see [7, Definition 10]) We call the $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ the *basic blowup type test configuration of (X, rL) via D_0*.

The following proposition is important.

Proposition 4.5 *We have*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = r^{2d}\, \mathrm{DF}(P, v_0).$$

Proof Let

$$w(k) = w_{d+1}k^{d+1} + w_d k^d + O(k^{d-1}),$$
$$h^0(X, L^{\otimes kr}) = L_d k^d + L_{d-1}k^{d-1} + O(k^{d-2})$$

be as in Definition 3.1. Recall that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = L_{d-1}w_{d+1} - L_d w_d$. Note that $h^0(X, L^{\otimes kr})$ is the Ehrhart polynomial $L_{rP}(k)$ of rP. By the definition of $w(k)$ and by Lemma 4.2, we have

$$\begin{aligned} w(k) &= -kr\tau h^0(X, L^{\otimes kr}) + \sum_{j=1}^{kr\tau} h^0(X, L^{\otimes kr} \cdot J_{(k,j)}) \\ &= -kr\tau L_{rP}(k) + \sum_{j=1}^{kr\tau} h^0(X, L^{\otimes kr}(-jD_0)). \end{aligned}$$

Note that, by Lemma 4.1, we have

$$\begin{aligned}&\sum_{j=1}^{kr\tau} h^0(X, L^{\otimes kr}(-jD_0))\\&=\sum_{j=1}^{kr\tau}\#\{u\in krP\cap M \mid \langle u, v_0\rangle \geq -krd_0+j\}\\&=\sum_{u\in krP\cap M}(\langle u, v_0\rangle + krd_0) = krd_0 L_{rP}(k) + f_{rP}^{v_0}(k).\end{aligned}$$

Thus we get

$$w(k) = kr(d_0-\tau)L_{rP}(k) + f_{rP}^{v_0}(k).$$

This equality, together with Lemma 2.4, immediately implies that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \mathrm{DF}(rP, v_0) = r^{2d}\,\mathrm{DF}(P, v_0)$. □

By using Proposition 4.5, we can show the following theorem. Theorem 1.5 is a direct consequence of Theorem 4.6.

Theorem 4.6 *Assume that $b(P)\neq b(\partial P)$. Then we can find $\lambda\in\Lambda$ and $r\in\mathbb{Z}_{>0}$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$ holds, where $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is the basic blowup type test configuration of (X, rL) via D_λ.*

Proof By Theorem 2.2, for $v\in N_{\mathbb{R}}\setminus\{0\}$, the condition $\mathrm{DF}(P, v) < 0$ is equivalent to the condition

$$\langle b(P), v\rangle < \langle b(\partial P), v\rangle.$$

We know that $b(P)\in\mathrm{Int}(P)$. Let us set

$$t_0 := \min\{t\in\mathbb{R}_{\geq 0} \,|\, b(P)+t(b(P)-b(\partial P))\in\partial P\},$$

and we set

$$c := b(P)+t_0(b(P)-b(\partial P))\in\partial P.$$

Pick any facet $Q_\lambda\subset P$ with $c\in Q_\lambda$. By construction, we have $\langle c, v_\lambda\rangle = -d_\lambda$, and

$$\langle b(P)+t(b(P)-b(\partial P)), v_\lambda\rangle > -d_\lambda$$

for any $0\leq t < t_0$. Thus we have $\langle b(P), v_\lambda\rangle < \langle b(\partial P), v_\lambda\rangle$. Therefore the blowup type test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of (X, rL) given by X, L, λ, and r constructed as above satisfies that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$. □

5 Examples

We see several examples for the readers' convenience. We note that all the examples in the section are examples that have been considered before.

Example 5.1 (cf. [5, Example 4.4]) Set $d := 2$, $v_0 := -e_1 - e_2$, $v_1 := -e_1$, $v_2 := -e_2$, $v_3 := e_1$ and $v_4 := e_2$. Let Σ be the complete fan in $N_{\mathbb{R}}$ such that the set of one-dimensional cone is equal to the set

$$\{\mathbb{R}_{\geq 0}v_0, \ldots, \mathbb{R}_{\geq 0}v_4\}.$$

We know that the toric variety $X = X_\Sigma$ is the del Pezzo surface of degree 7. Let E_0, E_1, E_2 be the torus invariant curve on X corresponding with $\mathbb{R}_{\geq 0}v_0$, $\mathbb{R}_{\geq 0}v_1$, $\mathbb{R}_{\geq 0}v_2$, respectively. Then E_1, E_2 are the exceptional curves of the birational morphism $X \to \mathbb{P}^2$ and E_0 is the (-1)-curve on X apart from E_1, E_2.

Set $L := a_0E_0 + a_1E_1 + a_2E_2$. Then L is ample if and only if $a_1 + a_2 > a_0$, $a_0 > a_1$ and $a_0 > a_2$. From now on, we assume that L is ample. The associated lattice polytope $P \subset M_{\mathbb{R}}$ is the set of $x \in M_{\mathbb{R}}$ with $0 \leq \langle x, e_1\rangle \leq a_1$, $0 \leq \langle x, e_2\rangle \leq a_2$ and $\langle x, e_1\rangle + \langle x, e_2\rangle \leq a_0$. Thus we can show that

$$\begin{aligned}
\mathrm{vol}(P) &= \frac{1}{2}(2a_0a_1 + 2a_0a_2 - a_0^2 - a_1^2 - a_2^2),\\
\langle b(P), e_1\rangle &= \frac{-a_0^3 + 3a_0^2a_2 - 3a_0a_2^2 + a_2^3 + 3a_0a_1^2 - 2a_1^3}{3(2a_0a_1 + 2a_0a_2 - a_0^2 - a_1^2 - a_2^2)},\\
\langle b(P), e_2\rangle &= \frac{-a_0^3 + 3a_0^2a_1 - 3a_0a_1^2 + a_1^3 + 3a_0a_2^2 - 2a_2^3}{3(2a_0a_1 + 2a_0a_2 - a_0^2 - a_1^2 - a_2^2)},\\
\langle b(\partial P), e_1\rangle &= \frac{a_0a_1}{a_0 + a_1 + a_2},\\
\langle b(\partial P), e_2\rangle &= \frac{a_0a_2}{a_0 + a_1 + a_2}.
\end{aligned}$$

Assume that $b(P) = b(\partial P)$. Then, since $\langle b(\partial P), a_2e_1 - a_1e_2\rangle = 0$, we have $\langle b(P), a_2e_1 - a_1e_2\rangle = 0$. The condition is equivalent with the condition $(a_2 - a_1)(a_1 + a_2 - a_0)^3 = 0$. This implies that $a_1 = a_2$. Set $b := a_1/a_0 \in (1/2, 1)$. On the other hand, the condition $\langle b(P), e_1\rangle = \langle b(\partial P), e_1\rangle$ implies that $(2b - 1)(b - 1)(b^2 - b + 1) = 0$. This leads to a contradiction. Thus $b(P) \neq b(\partial P)$. Hence (X, L) is K-unstable for any ample L.

Example 5.2 (cf. [12, Proposition 5.2], [5, Example 4.5]) Set $d := 2$, $v_1 := -e_1$, $v_2 := -e_2$, $v_3 := e_1 + e_2$ and $v_i' := -v_i$ for $i = 1, 2, 3$. Let Σ be the complete fan in $N_{\mathbb{R}}$ such that the set of one-dimensional cones is equal to the set

$$\{\mathbb{R}_{\geq 0}v_i,\ \mathbb{R}_{\geq 0}v_i'\}_{i=1,2,3}.$$

We know that the toric variety $X = X_\Sigma$ is the del Pezzo surface of degree 6. Let E_1, E_2, E_3, E_1', E_2', E_3' be the torus invariant curve on X corresponding with $\mathbb{R}_{\geq 0}v_1$, $\mathbb{R}_{\geq 0}v_2$, $\mathbb{R}_{\geq 0}v_3$, $\mathbb{R}_{\geq 0}v_1'$, $\mathbb{R}_{\geq 0}v_2'$, $\mathbb{R}_{\geq 0}v_3'$, respectively. Then E_1, E_2, E_3 (resp., E_1', E_2', E_3') are mutually disjoint (-1)-curves.

Set $L := a_0(E_1' + E_2' + E_3') + a_1E_1 + a_2E_2 + a_3E_3$. Then L is ample if and only if $2a_0 > a_1$, $2a_0 > a_2$, $2a_0 > a_3$, $a_1 + a_2 > a_0$, $a_2 + a_3 > a_0$, and $a_3 + a_1 > a_0$. From now on, we assume that L is ample. The associated lattice polytope $P \subset M_\mathbb{R}$ is the set of $x \in M_\mathbb{R}$ with $-a_0 \leq \langle x, e_1\rangle \leq a_1$, $-a_0 \leq \langle x, e_2\rangle \leq a_2$ and $-a_3 \leq \langle x, e_1\rangle + \langle x, e_2\rangle \leq a_0$. Thus we can show that

$$\begin{aligned}
\mathrm{vol}(P) &= \frac{1}{2}(-3a_0^2 + 4a_0a_1 + 4a_0a_2 + 4a_0a_3 - a_1^2 - a_2^2 - a_3^2),\\
\langle b(P), e_1\rangle &= \frac{6a_0a_1^2 - 2a_1^3 + a_2^3 - 3a_0a_2^2 - 3a_0a_3^2 + a_3^3}{3(-3a_0^2 + 4a_0a_1 + 4a_0a_2 + 4a_0a_3 - a_1^2 - a_2^2 - a_3^2)},\\
\langle b(P), e_2\rangle &= \frac{6a_0a_2^2 - 2a_2^3 + a_1^3 - 3a_0a_1^2 - 3a_0a_3^2 + a_3^3}{3(-3a_0^2 + 4a_0a_1 + 4a_0a_2 + 4a_0a_3 - a_1^2 - a_2^2 - a_3^2)},\\
\langle b(\partial P), e_1\rangle &= \frac{a_0(2a_1 - a_2 - a_3)}{3a_0 + a_1 + a_2 + a_3},\\
\langle b(\partial P), e_2\rangle &= \frac{a_0(2a_2 - a_1 - a_3)}{3a_0 + a_1 + a_2 + a_3}.
\end{aligned}$$

Set $b_i := a_i/a_0$ for $i = 1, 2, 3$. We note that, for any $1 \leq i, j \leq 3$,

$$3 - 3b_i + b_i^2 + b_ib_j + b_j^2 = \frac{3}{4}(2 - b_i)^2 + \left(\frac{1}{2}b_i + b_j\right)^2 > 0$$

holds.

Assume that $b(P) = b(\partial P)$. The condition $\langle b(P), e_1\rangle = \langle b(\partial P), e_1\rangle$ and $\langle b(P), e_2\rangle = \langle b(\partial P), e_2\rangle$ is equivalent with the condition $b_1 + b_2 + b_3 = 3$, or

$$2b_1^3 + 6b_1 - 3b_1(b_2 + b_3) - 3(b_2 + b_3) + 6b_2b_3 - (b_2^3 + b_3^3) = 0 \tag{1}$$

and

$$2b_2^3 + 6b_2 - 3b_2(b_1 + b_3) - 3(b_1 + b_3) + 6b_1b_3 - (b_1^3 + b_3^3) = 0. \tag{2}$$

Assume that $b_1 + b_2 + b_3 \neq 3$. The Eq. (1) minus the Eq. (2) is equivalent to the condition $(b_1 - b_2)(b_1^2 + b_1b_2 + b_2^2 - 3b_3 + 3) = 0$. If $b_1^2 + b_1b_2 + b_2^2 - 3b_3 + 3 = 0$, then, from the Eq. (1), we have $(3 - 3b_1 + b_1^2 + b_1b_2 + b_2^2)(3 - 3b_2 + b_1^2 + b_1b_2 + b_2^2)(12 + 3b_1 + 3b_2 + b_1^2 + b_1b_2 + b_2^2) = 0$. This leads to a contradiction. Thus $b_1 = b_2$. From the Eq. (2), we have $(b_1 - b_3)(3 - 3b_1 + b_1^2 + b_1b_3 + b_3^2) = 0$. Thus we have $b_1 = b_2 = b_3$. Therefore, the condition $b(P) = b(\partial P)$ is equivalent to the condition $b_1+b_2+b_3=3$ or $b_1=b_2=b_3$. In particular, if (X, L) is K-semistable for L ample with $L=a_0(E_1'+E_2'+E_3')+a_1E_1+a_2E_2+a_3E_3$, then we have $a_1+a_2+a_3=3a_0$ or $a_1=a_2=a_3$.

Example 5.3 (cf. [3, Theorem C]) Set $r := d-1$ and assume that $r \geq 1$. Let $(m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^{\oplus r} \setminus \{(0, \ldots, 0)\}$. Set $e_0 := -(e_1+\cdots+e_r)$, $u_1 := -e_d$, $u_0 := -u_1+m_1e_1+\cdots+m_re_r$. Let Σ be the complete fan in $N_{\mathbb{R}}$ such that the set of full-dimensional cones is equal to the set

$$\{\mathbb{R}_{\geq 0}u_0+\mathbb{R}_{\geq 0}e_0+\cdots+\mathbb{R}_{\geq 0}e_{i-1}+\mathbb{R}_{\geq 0}e_{i+1}+\cdots+\mathbb{R}_{\geq 0}e_r\}_{0\leq i\leq r}$$
$$\cup\,\{\mathbb{R}_{\geq 0}u_1+\mathbb{R}_{\geq 0}e_0+\cdots+\mathbb{R}_{\geq 0}e_{i-1}+\mathbb{R}_{\geq 0}e_{i+1}+\cdots+\mathbb{R}_{\geq 0}e_r\}_{0\leq i\leq r}.$$

Then the toric variety $X=X_\Sigma$ is equal to

$$\mathbb{P}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(m_r)\right)$$

by [4, Example 7.3.5]. Let $D_{e_0}, D_{u_0} \subset X$ be the torus invariant prime divisor on X corresponding with $\mathbb{R}_{\geq 0}e_0$, $\mathbb{R}_{\geq 0}u_0$, respectively. Set $L := aD_{e_0}+bD_{u_0}$ for a, $b \in \mathbb{Z}$. We know that L is ample if and only if $a, b > 0$ holds. From now on, we assume that L is ample, i.e., $a, b \in \mathbb{Z}_{>0}$. Set $c := b/a \in \mathbb{Q}_{>0}$. The associated lattice polytope $P \subset M_{\mathbb{R}}$ is the set $x \in M_{\mathbb{R}}$ with

$$\begin{aligned}\langle x, e_1\rangle, \ldots, \langle x, e_r\rangle &\geq 0,\\ \sum_{i=1}^r\langle x, e_i\rangle &\leq a,\\ \langle x, e_d\rangle &\geq 0,\\ \langle x, e_d\rangle &\leq b+\sum_{i=1}^r m_i\langle x, e_i\rangle.\end{aligned}$$

Let $Q_{e_0}, \ldots, Q_{e_r}, Q_{u_0}, Q_{u_1} \subset P$ be the facet of P corresponding with $\mathbb{R}_{\geq 0}e_0, \ldots, \mathbb{R}_{\geq 0}e_r, \mathbb{R}_{\geq 0}u_0, \mathbb{R}_{\geq 0}u_1$, respectively. We can show that

$$\begin{aligned}\frac{\mathrm{vol}(P)}{a^{r+1}} &= \frac{1}{r!}c+\frac{1}{(r+1)!}\sum_{i=1}^r m_i,\\ \frac{\mathrm{vol}(Q_k)}{a^r} &= \frac{1}{(r-1)!}c+\frac{1}{r!}\sum_{i\in\{1,\ldots,r\}\setminus\{k\}} m_i \qquad (1\leq k\leq r),\\ \frac{\mathrm{vol}(Q_{e_0})}{a^r} &= \frac{1}{(r-1)!}c+\frac{1}{r!}\sum_{i=1}^r m_i,\\ \frac{\mathrm{vol}(Q_{u_j})}{a^r} &= \frac{1}{r!} \qquad (j\in\{0,\ 1\}),\end{aligned}$$

and

$$\frac{\mathrm{vol}(P)\langle b(P), -e_0\rangle}{a^{r+2}} = \frac{r}{(r+1)!}c + \frac{r+1}{(r+2)!}\sum_{i=1}^{r} m_i,$$

$$\frac{\mathrm{vol}(Q_{e_k})\langle b(Q_{e_k}), -e_0\rangle}{a^{r+1}} = \frac{r-1}{r!}c + \frac{r}{(r+1)!}\sum_{i\in\{1,\dots,r\}\setminus\{k\}} m_i \quad (1 \le k \le r),$$

$$\frac{\mathrm{vol}(Q_{e_0})\langle b(Q_{e_0}), -e_0\rangle}{a^{r+1}} = \frac{1}{(r-1)!}c + \frac{1}{r!}\sum_{i=1}^{r} m_i,$$

$$\frac{\mathrm{vol}(Q_{u_j})\langle b(Q_{u_j}), -e_0\rangle}{a^{r+1}} = \frac{r}{(r+1)!} \qquad (j \in \{0,\ 1\}).$$

Hence we have

$$\frac{\mathrm{vol}(\partial P)}{a^r} = \frac{r+1}{(r-1)!}c + \frac{1}{r!}\left(c + r\sum_{i=1}^{r} m_i\right),$$

$$\frac{\mathrm{vol}(\partial P)\langle b(\partial P), -e_0\rangle}{a^{r+1}} = \frac{r}{(r-1)!}c + \frac{1}{(r+1)!}\left(2r + (r^2+1)\sum_{i=1}^{r} m_i\right).$$

This implies that

$$\frac{1}{a^{2r+2}}\,\mathrm{vol}(P)\,\mathrm{vol}(\partial P)\langle b(P) - b(\partial P), e_0\rangle$$
$$= \frac{2}{(r+1)!(r+2)!}\left(\sum_{i=1}^{r} m_i\right)\left((r+1)c + \left(\sum_{i=1}^{r} m_i\right) - 1\right) > 0.$$

Therefore, we have $b(P) \neq b(\partial P)$. As a consequence, we have showed the following proposition.

Proposition 5.4 (cf. [3, Theorem C]) *Assume that* $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_r))$ *for some* $m_1, \dots, m_r \in \mathbb{Z}$*. If* X *is not isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^r$*, then* (X, L) *is K-unstable for any ample* L *on* X*.*

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A Note on Extremal Toric Almost Kähler Metrics

Eveline Legendre

Abstract An almost Kähler structure is *extremal* if the Hermitian scalar curvature is a Killing potential (Lejmi, Int J Math 21(12):1639–1662, 2010). When the almost complex structure is integrable it coincides with extremal Kähler metric in the sense of Calabi (Extremal Kähler metrics. II. In: Chavel I, Farkas HM (eds) Differential geometry and complex analysis. Springer, Berlin, 1985, pp 95–114). We observe that the existence of an extremal *toric* almost Kähler structure of involutive type implies uniform K-stability and we point out the existence of a formal solution of the Abreu equation for any angle along the invariant divisor. Applying the recent result of Chen and Cheng (On the constant scalar curvature Kähler metrics (III), General automorphism group. ArXiv1801.05907v1) and He (On Calabi's extremal metric and properness. arXiv:math.DG/1801.07636), we conclude that the existence of a compatible extremal toric almost Kähler structure of involutive type on a compact symplectic toric manifold is equivalent to its relative uniform K–stability (in a toric sense). As an application, using (Apostolov et al., Adv Math 227:2385–2424, 2011), we get the existence of an extremal toric Kähler metric in each Kähler class of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2))$.

Keywords Almost Kähler metrics · Toric geometry · Extremal Kähler metric

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1 Introduction

The objects and problems of toric Kähler geometry have been fruitfully translated in terms of convex affine geometry in the works of Abreu [1], Guillemin [22], Donaldson [16], Apostolov and al. [2] with important applications in the very hard and

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central problem of Calabi extremal Kähler metrics [6]. In particular, Donaldson used this theory to prove the celebrated Yau–Tian–Donaldson conjecture, [12, 33, 34], for toric surfaces with vanishing Futaki invariant in [12, 13, 15, 16]. There is a relative version of this conjecture due to Székelyhidi [31] which is more relevant in the presence of symmetries and for general extremal (non constant scalar curvature) Kähler metrics. This conjecture predicts that given a complex compact manifold (M^{2n}, J) with a Kähler class Ω and a maximal compact torus $T \subset \mathrm{Aut}(M, J)$, the existence of an invariant extremal Kähler metrics in Ω is equivalent to the "relative K–stability" of (M^{2n}, J, Ω) in a sense to be determined precisely but which would be related to an algebro-geometric notion of stability.

We recall briefly the toric counterpart of this theory, with more details in Sect. 2, as it was developed by Donaldson [12]. In the toric setting, (M^{2n}, J, Ω) is invariant by a compact torus $T = T^n$ and caracterized completely by a convex polytope P, open and relatively compact in $\mathfrak{t}^*$, the dual of the Lie algebra $\mathfrak{t}$ of T, together with an affine measure $\sigma \in \mathbf{M}(P)$ on the boundary of P. The K–stability (relative to T) is related to the positivity of a certain functional

$$\mathcal{L}_{(P,\sigma)}(f) = \int_{\partial P} f\sigma - \frac{1}{2}\int_P f \mathrm{A}_\sigma dx$$

on a set $\widetilde{\mathcal{C}}$ of convex functions f on P, see Definition 3.1. In this definition, $dx = dx_1 \wedge \cdots \wedge dx_n$ is a Lebesgue measure on $\mathfrak{t}^* \simeq \mathbb{R}^n$ and $\mathrm{A}_\sigma \in \mathrm{Aff}(\mathfrak{t}^*)$ is the *extremal affine function*, see Sect. 2.4. Following [12, 32], if there exists $\lambda > 0$ such that

$$\mathcal{L}_{(P,\sigma)}(f) \geq \lambda \int_{\partial P} f\sigma$$

for any "normalized" f in $\widetilde{\mathcal{C}}$ then (P, σ) is uniformly K–stable and K–stable if $\lambda = 0$ is the only possible choice , see Definition 3.1.

The K–stability or uniform K–stability only depends on P and σ and we define

$$\begin{aligned} \mathrm{uKs}(P) &= \{\sigma \in \mathbf{M}(\partial P) \mid (P, \sigma) \text{ is uniformly } K\text{–stable}\}, \\ \mathrm{Ks}(P) &= \{\sigma \in \mathbf{M}(\partial P) \mid (P, \sigma) \text{ is } K\text{–stable}\}. \end{aligned} \tag{1}$$

Of course we have $\mathrm{uKs}(P) \subset \mathrm{Ks}(P)$.

Compatible Kähler structures are essentially parametrized by a set of convex functions $\mathcal{S}(P, \sigma) \subset C^\infty(P)$, called symplectic potentials and satisfying some boundary condition, recalled in Sect. 2.2, depending on σ. Given $u \in \mathcal{S}(P, \sigma)$, the associated Kähler structure (g_u, J_u) is *extremal* in the sense of Calabi if it satisfies the following so-called *Abreu equation*

$$S(H^u) = -\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} \in \mathrm{Aff}(\mathfrak{t}^*) \tag{2}$$

where $H^u = (u^{ij}) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^{-1}$ is the inverse Hessian of u for a flat connection on $\mathfrak{t}^* \simeq \mathbb{R}^n$.

The relative version of the Yau–Tian–Donaldson conjecture for toric manifold is generalized following [12] by the prediction that, given a simple relatively compact polytope $P \subset \mathbb{R}^n$, one should have

$$\{\sigma \in \mathbf{M}(\partial P) \,|\, \exists u \in \mathcal{S}(P,\sigma) \text{ such that } S(H^u) \in \mathrm{Aff}(\mathfrak{t}^*)\} = \mathrm{Ks}(P). \tag{3}$$

Some experts think that the stability condition must be strengthened and one of the suggestion, see [9, 32], is to conjecture that

$$\{\sigma \in \mathbf{M}(\partial P) \,|\, \exists u \in \mathcal{S}(P,\sigma) \text{ such that } S(H^u) \in \mathrm{Aff}(\mathfrak{t}^*)\} = \mathrm{uKs}(P). \tag{4}$$

As we argue in Sect. 3.2, by combining Chen–Li–Sheng work [9] and the recent progress of Chen–Cheng [8] and He [24], with Donaldson [12] and Zhou–Zhu [35] results this conjecture is indeed true.

Theorem 1.1 *Given any compact convex labelled simple polytope (P,σ),*

$$\exists u \in \mathcal{S}(P,\sigma) \textit{ such that } S(H^u) \in \mathit{Aff}(\mathfrak{t}^*) \tag{5}$$

if and only if (P,σ) is uniformly K–stable (i.e. $\sigma \in$ uKs(P)).

In the constant scalar curvature case, that is when $\mathrm{A}_{(P,\sigma)}$ is a constant, this last statement is Theorem 1.8 of Chen–Cheng in [8] given that Donaldson showed in [12, Proposition 5.2.2] that uniform K-stability of (P,σ) is equivalent to the L^1–stability of Chen and Cheng. Theorem 1.1 above is an application of He's recent important result [24].

Remark 1.2 To pass from Theorem 1.1 to a positive resolution of the relative version of the Yau–Tian–Donaldson conjecture one would need to show that the uniform stability of a labelled polytope is equivalent to the stability with respect to toric degenerations, see Remark 3.2.

Observe that (2) is a non-linear 4-th order PDE problem on ϕ but only a linear second order PDE problem on H^ϕ. Denote $\mathcal{AK}(P,\sigma)$ the set of matrix-valued function $H : P \to Gl(\mathbb{R}^n)$ symmetric, positive definite and satisfying some boundary condition depending on σ detailed in Sect. 2.3. Then one can define a smooth toric *almost* Kähler structure (g_H, J_H) on (M,ω) as explained in [2, 28] and recalled in Sect. 2.3. Such an almost Kähler structure (g_H, J_H) is *extremal* in the sense of Lejmi if it satisfies the Abreu equation (6), that is

$$S(H) = -\sum_{i,j=1}^{n} \frac{\partial^2 H_{ij}}{\partial x_i \partial x_j} \in \mathrm{Aff}(\mathfrak{t}^*). \tag{6}$$

Lejmi studied the notion of extremal toric almost Kähler metrics in [28] and showed that a large and interesting part of them (the involutive type ones) is in one-to-one correspondence with $\mathcal{AK}(P, \sigma)$.

Chen–Li–Sheng proved that existence of a toric Calabi extremal Kähler metrics implies that the toric variety is uniformly K–stable, proving one side of the conjecture for toric manifolds [9]. In this note we observe and explain that their proof works equally well for extremal almost Kähler metrics and prove that

Proposition 1.3 *For any simple relatively compact $P \subset \mathbb{R}^n$, we have*

$$\{\sigma \in \mathbf{M}(\partial P) \,|\, \exists H \in \mathcal{AK}(P, \sigma) \text{ such that } S(H) \in \mathit{Aff}(\mathfrak{t}^*)\} \subset \ uKs(P). \tag{7}$$

In particular, if (M, J, g, ω) is a compact toric Kähler manifold such that (M, ω) admits a compatible extremal toric almost Kähler metrics of involutive type then $(M, J, [\omega])$ is uniformly K–stable[1] *with respect to toric degenerations.*

As a direct consequence of this last Proposition and Theorem 1.1 above we get

Corollary 1.4 *The existence of an extremal toric almost Kähler metric of involutive type compatible with ω implies the existence of a compatible extremal toric Kähler metric.*

Remark 1.5 It is unlikely that in general, for compact Kähler manifold of non-toric type, the existence of an extremal almost Kähler metric (M, J, ω) implies uniform K–stability of (M, J) or the existence of an extremal Kähler metric compatible with ω. However, as pointed out in [25], a certain notion of stability could generalize the conjecture and theory to almost Kähler metrics.

In [3], for any $k_2, k_1 > 0$ and any toric symplectic form ω on the total space of the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)) \to \mathbb{P}^1$, they construct explicit examples of extremal almost Kähler metrics compatible with ω. One can check directly that these metrics are of involutive type. As an application of Corollary 1.4 we get the following.

Corollary 1.6 *Each Kähler class of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2))$ admits a compatible extremal toric Kähler metric.*

The convex affine geometry point of view has been exploited successfully to provide a complete understanding of the situation, confirming the relative version of the Yau–Tian–Donaldson conjecture, when the moment polytope is a convex quadrilateral in [4, 5, 26, 30] (in particular for toric compact orbisurfaces with second betti number equal 2) including explicit solution or destabilizing test configuration whenever they exist. A key ingredient of the aforementioned papers is an explicit *formal* solution $H_{A,B} : P \to \mathrm{Sym}^2(\mathfrak{t}^*)$ depending on 2 polynomials A and B on one variable satisfying the boundary condition depending on σ and satisfying the second order PDE corresponding to the extremal equation of Calabi.

[1] Here uniform K-stability should be understand as defined above, see Remark 3.2.

One of the main observations of [4, 5, 26, 30] is that $H_{A,B}$ is positive definite if and only if the labelled polytope (P, σ) is K–stable and if and only if $H_{A,B}$ is the inverse Hessian of a symplectic potential.

A complete answer, like the one given for convex quadrilateral is certainly out of reach for convex polytope in general. However, we point out in this note that some parts of the strategy of [4, 5, 26, 30] may be extended in general thanks to the following observation.

Proposition 1.7 *For any simple labelled polytope (P, σ), there exists an infinite dimensional family of formal extremal solutions $H : P \to Sym^2(\mathfrak{t}^*)$ of Eq. (6) satisfying the boundary condition associated to σ. Whenever one of these solutions is positive definite on the interior of P, (P, σ) is uniformly K–stable.*

We discuss in Sect. 3.4 consequences of this last result and open problems in relation with the relative toric version of the Yau–Tian–Donaldson conjecture.

In the next section we gather facts, definition, key results and recall brief explanations on the topic of toric extremal (almost) Kähler metrics. Section 3 contains the proof of Propositions 1.3 and 1.7.

2 Labelled Polytope and Toric (Almost) Kähler Geometry

2.1 Rational Labelled Polytopes and Toric Symplectic Orbifolds

2.1.1 Notations

In the sequel a *polytope* P refers to an open, convex, polyhedral, *simple* and relatively compact subset of an affine space $\mathfrak{t}^* \simeq \mathbb{R}^n$. *Simple* means that each vertex is the intersection of exactly n facets (where n is the dimension of $\mathfrak{t}^*$). We order and denote the facets $F_1, \dots F_d \subset \overline{P}$. Choosing a non-zero inward normal vector $\vec{n}_s \in \mathfrak{t}$ to each facet F_s, we can write

$$P = \{x \in \mathfrak{t}^* \mid \ell_{\vec{n},s}(x) > 0,\ s = 1, \dots, d\}$$

where $\ell_{\vec{n},s}$ is the unique affine-linear function on $\mathfrak{t}^*$ such that $d\ell_{\vec{n},s} = \vec{n}_s$ and

$$F_s = \ell_{\vec{n},s}^{-1}(0) \cap \overline{P}.$$

Definition 2.1 Let $P \subset \mathfrak{t}^*$ be a polytope as above.

(a) A *labelling* for P is an ordered set of non-zero vectors $\vec{n} = (\vec{n}_1, \dots, \vec{n}_d) \in (\mathfrak{t})^d$ each $\vec{n}_s$ being normal to the facet F_s and inward to P. A *labelled polytope* is a pair $(P, \vec{n})$.

(b) A *rational labelled polytope* is a triple $(P, \vec{n}, \Lambda)$ where $(P, \vec{n})$ is a labelled polytope and $\Lambda \subset \mathfrak{t}$ is a lattice containing the labels $\vec{n}_1, \dots, \vec{n}_d$.

(c) A *Delzant polytope* is a pair (P, Λ) where $\Lambda \subset \mathfrak{t}$ is a lattice containing a set of labels $\vec{n} = (\vec{n}_1, \dots, \vec{n}_d)$ such that for each vertex $\{p\} = \cap_{s\in I_p} F_s$ the set $\{\vec{n}_s \mid s \in I_p\}$ is a $\mathbb{Z}$–basis of Λ.

We denote by $\mathbf{N}(P) := \{\vec{n} = (\vec{n}_1, \dots, \vec{n}_d) \in (\mathfrak{t})^d \mid (P, \vec{n})$ *labelled polytope*$\}$. Obviously $\mathbf{N}(P) \simeq \mathbb{R}^d_{>0}$. We will also be working on the dual space $\mathbf{M}(P)$ of measures σ on ∂P such that there exists a labelling $\vec{n} \in \mathbf{N}(P)$ satisfying

$$\vec{n}_s \wedge \sigma = -dx \qquad \text{on } F_s \tag{8}$$

where $dx = dx_1 \wedge \cdots \wedge dx_n$ is a fixed affine invariant volume form on $\mathfrak{t}^*$. Again $\mathbf{M}(P) \simeq \mathbb{R}^d_{>0}$ and $\sigma \in \mathbf{M}(P)$ is determined by its restriction to the facets of P. We write (formally) $\sigma = (\sigma_1, \dots, \sigma_d)$ where $\sigma_s = \sigma_{|F_s}$ is an affine invariant $(n-1)$–form on the hyperplane supporting F_s.

Remark 2.2 Fixing $dx = dx_1 \wedge \cdots \wedge dx_n$ once and for all, we get a bijection $\mathbf{N}(P) \simeq \mathbf{M}(P)$, $\vec{n} \mapsto \sigma_{\vec{n}}$ with inverse $\sigma \mapsto \vec{n}_\sigma$ given by the relation (8). In the following we use both notation (P, σ) or $(P, \vec{n})$ for the labelled polytope $(P, \vec{n}_\sigma)$.

2.1.2 Delzant–Lerman–Tolman Correspondence

Delzant showed that compact toric symplectic manifolds are in one to one correspondance with Delzant polytopes via the momentum map [11] and Lerman–Tolman [29] extended the correspondence to orbifolds by introducing rational labelled polytope. They are many ways to construct the corresponding (compact) toric symplectic orbifold $(M, \omega, T := \mathfrak{t}/\Lambda)$ from the data $(P, \vec{n}, \Lambda)$. We recall only the one we will use which, as far as we know, has been developed in [14, 18, 27].

(1) *Local toric charts:* Each vertex p of P is the intersection of n facets thus corresponds to a subset $I_p \subset \{1, \dots, d\}$ of n indices which in turn corresponds to a basis of $\mathfrak{t}$ namely $\{\vec{n}_s \mid s \in I_p\}$ that induces a sublattice $\Lambda_p = \text{span}_{\mathbb{Z}}\{\vec{n}_i \mid i \in I_p\}$ of Λ. Considering the torus $T_p = \mathfrak{t}/\Lambda_p$ we get a (non-compact) toric symplectic manifold

$$(M_p := \oplus_{s\in I_p} \mathbb{C}\vec{n}_s \simeq \mathbb{C}^n, \omega_{std}, T_p)$$

by identifying $T_p \simeq \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ via which T_p acts on $\mathbb{C}^n$. The momentum map $x_p : M_p \to \mathfrak{t}^*$ is given

$$x_p(z) = p + \frac{1}{2} \sum_{\in I_p} |z_s|^2 \alpha_s$$

where $\{\alpha_{\vec{n},i} \mid i \in I_p\} \subset \mathfrak{t}^*$ is the dual basis of $\{\vec{n}_i \mid i \in I_p\}$.

(2) *Gluing over $P \times T$:* Now using the exact sequence

$$\Lambda/\Lambda_p \hookrightarrow T_p \overset{\phi_p}{\twoheadrightarrow} T$$

where $T = \mathfrak{t}/\Lambda$ we get a way to glue equivariantly the (uniformizing) chart M_p over $P \times T$ seen as a toric symplectic manifold with momentum map x being the projection on the first factor, see [27] for more details.

In this construction, (M, ω) is obtained as the compactification of $(P \times T, dx \wedge d\theta)$. Here $dx \wedge d\theta$ is the canonical symplectic form of $P \times T$ coming from the one of the universal cover $P \times \mathfrak{t} \subset \mathfrak{t}^* \times \mathfrak{t}$. In particular, we get directly a set of action angle coordinates (x, θ) on the set where the action is free $\mathring{M} = P \times T = x^{-1}(P)$. These coordinates are usually constructed with the help of a Kähler metric [7] and one can prove that they are well defined up to an equivariant symplectomorphism.

2.2 Symplectic Potentials and Toric Kähler Metrics

Let (M, ω, T) be a compact toric symplectic orbifold associated with the rational labelled polytope $(P, \vec{n}, \Lambda)$. In particular $x : M \to \overline{P}$ is the momentum map. We fix a set of action angle coordinates (x, θ) on the set $\mathring{M}$ where the torus action is free. The next proposition gathers some now well-known facts establishing a correspondence between toric Kähler structures and symplectic potentials.

Proposition 2.3 ([1, 2, 13, 22]) *For any strictly convex function $u \in C^\infty(P)$,*

$$g_u = \sum_{i,j} u_{ij} dx_i \otimes dx_j + u^{ij} d\theta_i \otimes d\theta_j, \tag{9}$$

with $(u_{ij}) = \text{Hess}\, u$ and $(u^{ij}) = (u_{ij})^{-1}$, is a smooth Kähler structure on $P \times T$ compatible with the symplectic form $dx \wedge d\theta$. Conversely, any T–invariant compatible Kähler structure on $(P \times T, dx \wedge d\theta)$ is of this form.

Moreover, the metric g_u is the restriction of a smooth (in the orbifold sense) toric Kähler metric on (M, ω) if and only if

(1) *$u \in C^0(\overline{P})$ whose restriction to P or to any face's interior (except vertices), is smooth and strictly convex;*
(2) *$u - u_{\vec{n}}$ is the restriction of a smooth function defined on an open set containing $\overline{P}$ where*

$$u_{\vec{n}} = \frac{1}{2} \sum_{s=1}^{d} \ell_{\vec{n},s} \log \ell_{\vec{n},s} \tag{10}$$

is the so-called Guillemin potential.

The functions u satisfying the conditions of the previous Proposition are called *symplectic potentials* and we denote the set of such as $\mathcal{S}(P,\vec{n})$ or $\mathcal{S}(P,\sigma_{\vec{n}})$. In sum, the set of smooth compatible toric (orbifold) Kähler metrics on (M,ω,T) is in one-to-one correspondance with the quotient of $\mathcal{S}(P,\vec{n})$ by $\mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$, acting by addition. The correspondance is explicit and given by (9).

Remark 2.4 The Guillemin potential $u_{\vec{n}}$ lies in $\mathcal{S}(P,\vec{n})$ and corresponds to the Guillemin Kähler metric on the toric symplectic orbifold in the rational case.

The boundary conditions (1) and (2) of Proposition 2.3 appear when comparing the metrics g_u and $g_{u_{\vec{n}}}$ on the charts M_p as defined in Sect. 2.1.2.

Remark 2.5 Passing from symplectic to complex point of views is direct in toric geometry. Given $u\in\mathcal{S}(P,\sigma)$ the map $(x,\theta)\mapsto(\nabla u)_x+\sqrt{-1}\theta$ provides the complex coordinates as the coordinates on the universal covering of the big orbit $\mathring{M}\simeq(\mathbb{C}^*)^n$, see e.g. [14]. In these coordinates the Kähler potential of the Kähler form ω is the Legendre transform of u.

2.3 Toric Almost Kähler Metrics

An almost Kähler structure (g,J,ω) on M^{2n} has everything of a Kähler structure but the endomorphism $J\in\Gamma(\mathrm{End}(TM))$, is not necessarily integrable. That is, g is a Riemannian metric, ω is a symplectic form, and $J\in\Gamma(\mathrm{End}(TM))$ squares to minus the identity and they satisfy the following compatibility relation:

$$g(J\cdot,J\cdot)=g(\cdot,\cdot)\quad g(J\cdot,\cdot)=\omega(\cdot,\cdot).$$

A toric almost Kähler metric (g,J) is then an almost Kähler metric on a toric symplectic manifold/orbifold (M,ω,T) such that (g,J) is compatible with ω and g (equivalently J) is invariant by the torus T.

Let (M,ω,T) be a toric symplectic manifold with a momentum map $x:M\rightarrow\mathfrak{t}^*$ and moment polytope $\overline{P}=x(M)$ labelled by $\vec{n}\in\mathbf{N}(P)$. We use notation layed in Sect. 2.1.1 and fix a set of affine coordinates $x=(x_1,\dots,x_n)$ on $\mathfrak{t}^*$. In [28], the author proves among other things that T–invariant almost Kähler structures compatible with (M,ω) and such that the g–orthogonal distribution to the orbit is involutive (we call it toric almost Kähler structure of involutive type) are parametrized by symmetric bilinear forms

$$H:\overline{P}\rightarrow\mathrm{Sym}^2(\mathfrak{t}^*)\tag{11}$$

satisfying some conditions pointed out in [2] that we now recall.

(i) **Smoothness** H is the restriction on $\overline{P}$ of a smooth $\mathrm{Sym}^2(\mathfrak{t}^*)$–valued function defined on an open neighborhood of $\overline{P}$.

(ii) **Boundary condition** For any point y in interior of a codimension 1 face $F_s \subset \overline{P}$, we have

$$H_y(\vec{n}_s, \cdot) = 0 \tag{12}$$

$$dH_y(\vec{n}_s, \vec{n}_s) = 2\vec{n}_s. \tag{13}$$

(iii) **Positivity** For any point y in interior $\mathring{F}$ of a face $F \subset \overline{P}$, H is positive definite as $\mathrm{Sym}^2(T_y\mathring{F})$–valued function.

Proposition 2.6 ([2, 28]) *Let (M, ω, T) be a toric symplectic manifold and (g, J) be a compatible T–invariant almost Kähler metric of involutive type compatible with ω. Then the symmetric bilinear form defined for $a, b \in \mathfrak{t}$ and $x \in \overline{P}$ by $H_x(a, b) := g_p(X_a, X_b)$ for any $p \in M$ such that $x(p) = x$, satisfies the conditions* (i), (ii) *and* (iii). *Moreover, for any such symmetric bilinear form $H : \overline{P} \to Sym^2(\mathfrak{t}^*)$ satisfying conditions* (i), (ii) *and* (iii) *there is a unique compatible T–invariant almost Kähler metric (g_H, J_H) of involutive type satisfying $H_{x(p)}(a, b) = g^H_p(X_a, X_b)$ for any $p \in M$. With respect to action angle coordinates $(x, [\theta])$ on $\mathfrak{t}^* \times T \simeq \mathring{M}$, the metric g is given as*

$$g = \sum_{i,j} G_{ij} dx_i \otimes dx_j + H_{ij} d\theta_i \otimes d\theta_j, \tag{14}$$

where $G = (G_{ij}) = H^{-1}$.

Remark 2.7 Condition (12) implies that $H(u_s, \cdot) : \overline{P} \to \mathbb{R}$ vanishes on F_s and in particular is constant. Then for all $y \in \mathring{F}_s$, we have

$$(dH)_y(u_s, \cdot) \in \mathfrak{t}^* \otimes (T_y\mathring{F}_s)^0 = \mathfrak{t}^* \otimes \mathbb{R}u_s$$

where $(T_y\mathring{F}_s)^0 = \mathbb{R}u_s$ denotes the annihilator of $T_y\mathring{F}_s \subset T_y(\mathfrak{t}^*) = \mathfrak{t}^*$ in $\mathfrak{t}$. Therefore condition (13) is that the trace of $(dH)_y(u_s, \cdot)$ equals 2.

Fixing an affine invariant volume form $dx = dx_1 \wedge \cdots \wedge dx_n$, the labelling $\vec{n} \in \mathbf{N}(P)$ corresponds to a measure $\sigma \in \mathbf{M}(P)$ as defined in Sect. 2.1.1. Observe that the Boundary Condition above (i.e. condition (ii) namely (12), and (13)) implies that[2]

$$\sigma = \frac{1}{2} \sum_{i,j=1}^{n} (-1)^i H_{ij,j} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \tag{15}$$

[2]When a set of coordinates is fixed, we use the notation $f_{,i} = \frac{\partial}{\partial x_i} f$, $f_{,ij} = \frac{\partial^2}{\partial x_j \partial x_i} f$ ...

Assuming condition (12) holds condition (13) is equivalent to (15).

Thanks to Proposition 2.6 we can parametrize the space of compatible toric almost Kähler metrics of involutive type as

$$\mathcal{AK}(P,\sigma) := \{H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*) \mid H \text{ satisfies conditions (i), (ii) and (iii)}\}.$$

The inverse (u^{ij}) of the Hessian of symplectic potential $u \in \mathcal{S}(P,\vec{n})$ can be extended as a bilinear form $H^u \in \mathcal{AK}(P,\sigma)$. Observe also that for $H_0, H_1 \in \mathcal{AK}(P,\sigma)$ we have

$$H_t = tH_1 + (1-t)H_0 \in \mathcal{AK}(P,\sigma) \qquad \forall t \in [0,1].$$

The space $\mathcal{AK}(P,\sigma)$ is then a *convex* infinite dimensional set of metrics.

2.4 The Extremal Vector Field

Given a symplectic potential $u \in \mathcal{S}(P,\vec{n})$ the scalar curvature of the Kähler metric g_u is given by the pull back to M of the following expression, called the Abreu formula

$$S(H^u) = -\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} \tag{16}$$

as proved in [1] by direct computation. The function (16) extends as a smooth function on $\overline{P}$ because the boundary condition (2) of Proposition 2.3 implies that $(u^{ij}) \in \Gamma(P, \mathfrak{t}^* \otimes \mathfrak{t}^*)$ extends as a smooth bilinear form on $\overline{P}$, see [2]. It is shown in [28] that the suitable connection one should consider in case of almost extremal metrics is the Chern connection (which do not coincides with the Levi-Civita connection in the non-Kähler setting). It turns out that the formulas in the toric case coincide in the sense that for $H \in \mathcal{AK}(P,\sigma)$, the Hermitian scalar curvature is the pull-back of

$$S(H) := -\sum_{i,j=1}^{n} H_{ij,ij}.$$

Calabi's extremal Kähler metrics are caracterized by the condition that the Hamiltonian vector field of the scalar curvature is a Killing vector field [6] and extremal almost Kähler metric are defined with the same condition on the Hermitian scalar curvature [28]. Therefore, here, they correspond to the $H \in \mathcal{AK}(P,\sigma)$ such that

$$S(H) \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R}). \tag{17}$$

As observed by Donaldson in [16], picking an invariant volume form $dx = dx_1 \wedge \dots \wedge dx_n$ on $\mathfrak{t}^*$, the L^2–projection of $S(H^u)$ on $\mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ does not depend on the choice of $u \in \mathcal{S}(P, \vec{n})$. This fact holds for $H \in \mathcal{AK}(P, \sigma)$ and is the effect of a more general theory of invariant developed in [19, 20, 28] which in the toric case follows from integration by parts. Indeed, using the condition (ii) of definition of $\mathcal{AK}(P, \sigma)$ we have that for any $f \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ and $H \in \mathcal{AK}(P, \sigma)$

$$\int_P S(H) f dx = 2 \int_{\partial P} f \sigma_{\vec{n}}. \tag{18}$$

These computations do not require the existence of a lattice containing $\vec{n}_\sigma$, the labelling associated to $\sigma_{\vec{n}} \in \mathbf{M}(P)$ (see Sect. 2.1.1), or of a compact toric symplectic orbifold anywhere. Summing up these facts we get the following key result.

Proposition 2.8 ([16, 19, 20, 28]) *For any labelled polytope (P, σ), there exists a unique affine function $\mathrm{A}_{P,\sigma} \in \mathit{Aff}(\mathfrak{t}^*, \mathbb{R})$ such that*

$$\int_P \mathrm{A}_{P,\sigma} f dx = \int_P S(H) f dx = 2 \int_{\partial P} f \sigma \tag{19}$$

for any $f \in \mathit{Aff}(\mathfrak{t}^*, \mathbb{R})$ and any $H \in \mathcal{AK}(P, \sigma)$. Moreover, if there exists $H \in \mathcal{AK}(P, \sigma)$ such that the metric g^H is extremal almost Kähler in the sense of Calabi (and Lejmi) then

$$S(H) = \mathrm{A}_{P,\sigma}. \tag{20}$$

Remark 2.9 A direct corollary of the last Proposition is that the functional $\mathcal{L}_{(P,\sigma)}$ vanishes identically on affine-linear function.

Remark 2.10 The function $\mathrm{A}_{P,\sigma}$ depends linearly on $\sigma \in \mathbf{M}(P)$.

2.5 *Extremal Kähler Metrics Unicity and an Open Condition*

Uniqueness of extremal toric Kähler metric in a given class for a fixed torus is not an issue thanks to the proof of Guan in [21], using the convexity of the K–energy functional over geodesics. His proof works very well on symplectic potentials in $\mathcal{S}(P, \vec{n})$ as soon as $\overline{P}$ is compact using the works of [16], see e.g. [26, §2.2.1], because $\mathcal{S}(P, \vec{n})$ is a convex set with respect to smooth geodesics for the Mabuchi metric (which, here, are the affine lines $(1-t)u_0 + tu_1$) defined on the space of Kähler metrics [21]. Therefore, we get the following unicity result.

Proposition 2.11 *Let $(P, \vec{n})$ be a labelled polytope. If $u_0, u_1 \in \mathcal{S}(P, \vec{n})$ satisfy $S(u_0) = S(u_1) = \mathrm{A}_{P,\vec{n}}$ then $u_1 - u_0$ is the restriction to P of an affine linear function on $\mathfrak{t}^*$.*

Donaldson proved in [15] that the set of labelling $\vec{n} \in \mathbf{N}(P)$ for which the Abreu's equation has a solution is open in $\mathbf{N}(P)$, the d–dimensional open cone of labellings of P in $\mathfrak{t}^d$.

Proposition 2.12 (Donaldson [15]) *Let (P, σ) be a labelled polytope. Assume that there is a potential $u \in \mathcal{S}(P, \vec{n}_\sigma)$ satisfying the Abreu equation. Then there exists an open neighborhood $U \subset \mathbf{M}(P)$ of σ such that for each $\tilde{\sigma} \in U$ there exists a potential $\tilde{u} \in \mathcal{S}(P, \vec{n}_{\tilde{\sigma}})$ satisfying the Abreu equation.*

The statement in [15] is not exactly the one above but the proof works in this degree of generality. The argument is standard. The linearisation of $u \mapsto S(H^u)$ is an elliptic operator. To get around the lack of compacity of P, Donaldson argue that the system of charts associated to the vertices, see Sect. 2.1.2, provides the kind of compactification needed. This idea is developed with details in [27].

3 Uniform K–Stability and Extremal Almost Kähler Metrics

3.1 Uniform K–Stability and Chen–Li–Sheng Result

Consider the functional

$$\mathcal{L}_{(P,\sigma)}(f) = \int_{\partial P} f\sigma - \frac{1}{2}\int_P f \mathrm{A}_\sigma dx$$

which can be defined on various spaces of functions on $\overline{P}$, for example $C^0(\overline{P})$. From Proposition 2.8 we get that $\mathcal{L}_{(P,\sigma)}$ vanishes identically on the space of affine-linear functions.

Following [12], we define the set $\mathcal{C}_\infty$ of continuous convex function on $\overline{P}$ which are smooth on the interior, we have $\mathcal{S}(P, \sigma) \subset \mathcal{C}_\infty$ for all $\sigma \in \mathbf{M}(P)$. We fix $p_o \in P$, the set of a normalized functions is

$$\tilde{\mathcal{C}} := \{f \in \mathcal{C}_\infty \mid f(p) \geq f(p_o) = 0 \qquad \forall p \in P\}.$$

Note that the only affine-linear function in $\tilde{\mathcal{C}}$ is the trivial one.

Definition 3.1 A labelled polytope (P, σ) is *uniformly K–stable* if there exists $\lambda > 0$ such that

$$\mathcal{L}_{(P,\sigma)}(f) \geq \lambda \int_{\partial P} f\sigma$$

for any $f \in \tilde{\mathcal{C}}$.

Remark 3.2 Let $\mathcal{T}(P)$ be the set of continuous piecewise linear convex functions on $\overline{P}$, that is $f \in \mathcal{T}(P)$ if there are $f_1, \ldots, f_m \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ such that $f(x) =$

$\max\{f_1(x), \dots, f_m(x)\}$ for $x \in \overline{P}$. Given a lattice $\Lambda \subset \mathfrak{t}$, we define $\mathcal{T}(P, \Lambda) \subset \mathcal{T}(P)$, the set of continuous piecewise linear convex functions on P taking integral values on the dual lattice $\Lambda^* \subset \mathfrak{t}^*$. When (P, η, Λ) is rational Delzant and its vertices lie in the dual lattice $\Lambda^* \subset \mathfrak{t}^*$, the associated symplectic manifold (M, ω) is rational (that is $[\omega] \in H^2_{dR}(M, \mathbb{Q})$) and for any compatible toric complex structure J on M the Kähler manifold $(M, J, k[\omega])$ (for some k big enough) is polarized by a line bundle $L^k \to M$. In this situation, Donaldson presents in [12] a way to associate a test configuration $(\mathcal{X}_f, \mathcal{L}_f)$ over (M, L) to any function $f \in \mathcal{T}(P, \Lambda)$ such that the Donaldson–Futaki invariant of $(\mathcal{X}_f, \mathcal{L}_f)$ coincides, up to a positive multiplicative constant, with $\mathcal{L}_{(P,\sigma)}(f)$. These test configurations are called *toric degenerations* in [12] and [35]. The Yau–Tian–Donaldson conjecture predicts that if $\mathrm{A}_{P,\vec{n}}$ is a constant and there exists a solution $u \in \mathcal{S}(P, \vec{n})$ of the Abreu equation (2) then

$$\mathcal{L}_{(P,\sigma)}(f) \geq 0$$

for any $f \in \mathcal{T}(P, \Lambda)$ with equality if and only f is affine-linear.

Observe that the map $f \mapsto \int_{\partial P} f\sigma$ is a norm on $\tilde{\mathcal{C}}$. Therefore, Definition 3.1 coincides with the notion of uniform K–stability in the sense of Székelyhidi [32] but with a different norm and adapted to the toric situation. Moreover, this is the notion of stability in Definition 3.1 that Chen–Li–Sheng used in [9] to prove that

Theorem 3.3 ([9]) *If (P, σ) is a labelled polytope and that there exists a solution $u \in \mathcal{S}(P, \sigma)$ of the Abreu equation* (2) *then (P, σ) is uniformly K–stable.*

Proof of Proposition 1.3 Our Proposition 1.3 follows by observing that in the proof of the last Theorem, Chen–Li–Sheng only use the fact that the Hessian and inverse Hessian H^u of the solution $u \in \mathcal{S}(P, \sigma)$ are positive definite on the interior of P. One important step for their proof is to show that : *a labelled polytope (P, σ) is uniformly K–stable if and only if $\mathcal{L}_{(P,\sigma)}(f) \geq 0$ on some compactification $\mathcal{C}^K_*$ of* $\tilde{\mathcal{C}}$. But this is general and does not need any hypothesis on the existence of a solution of the Abreu equation. This latter hypothesis is only needed for Lemma 5.1 of [9]. The crucial observation is the following, if $H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*)$ satisfies Eq. (6), that is $S(H) = -\sum_{i,j=1}^n H_{ij,ij} = \mathrm{A}_{(P,\sigma)}$ then the boundary conditions (ii) of Sect. 2.3 implies that

$$\mathcal{L}_{(P,\sigma)}(f) = \int_P \langle H, \mathrm{Hess} f \rangle dx \tag{21}$$

whenever f is twice differentiable. Formula (21) goes back to [12].

Therefore, let H be a solution of equation (6), then for any interval $I \subset\subset P$ and sequence of convex functions $f_k \in \mathcal{C}_\infty \subset C^\infty(P)$ converging locally uniformly to f then we have, using (21) and weak convergence of Monge–Ampère measures, that

$$\mathcal{L}_{(P,\sigma)}(f_k) \geq \tau m_I(f)$$

where $m_I(f)$ is the Monge–Ampère measure induced by f on I and τ is a positive constant independant of k. This is the claim of Lemma 5.1 of [9] from which one can derive Proposition 1.3 using the same argument than [9] in the last paragraph of their section 5. □

3.2 Uniform K–Stability Implies the Existence of an Extremal Toric Kähler Metric

In this paragraph we will put together the work of Donaldson [12], He in [24] and Zhou–Zhu [35] to prove that

Proposition 3.4 *Let (P,σ) be a compact convex simple polytope. If $(P,\vec{n})$ is uniformly K–stable then there exists $u \in \mathcal{S}(P,\sigma)$ such that*

$$S(H^u) = \mathrm{A}_{(P,\sigma)}.$$

Given a compact group $K \subset \mathrm{Aut}_0(M,J)$ containing the extremal vector field (the Hamiltonian Killing version of it [20]) in its Lie algebra center and a fixed J–compatible K–invariant Kähler metric ω, one can define the *modified Mabuchi K–energy* as a functional $\mathcal{K}$ on the space of K–invariant Kähler potentials $\mathcal{H}_K := \{\phi \in \mathcal{C}^\infty(M)^K \mid \omega + dd^c\phi > 0\}$. This functional is important because it detects the K–invariant extremal Kähler metrics in $(M, J, [\omega])$. Let $K = K_0$ be a compact subgroup of $\mathrm{Aut}_0(M,J)$ whose complexified Lie algebra $\mathfrak{h}_0$ is the reduced part of $\mathfrak{h} := \mathrm{LieAut}_0(M,J)$. Denote G_0 the complexification of K_0 in $\mathrm{Aut}_0(M,J)$. An important ingredient in this theory is a certain distance d_{1,G_0} on $\mathcal{H}_K$ introduced by Darvas [10] and corresponding to the L^1–norm on $T_\phi\mathcal{H}_{K_0}$. That is for $\psi \in T_\phi\mathcal{H}_{K_0}$, the norm $\int_M |\psi|\omega_\phi^n$ allows to compute the length of curves and then $d_1(\phi_0,\phi_1)$ is the infimum of the length of the curves joining ϕ_0 and ϕ_1. Then $d_{1,G_0}(\phi_0,\phi_1) = \inf_{g\in G_0} d_1(\phi_0, g^*\phi_1)$.

Theorem 3.5 (Theorem 4 of He [24]) *There is a K_0–invariant extremal Kähler metrics in $(M, J, [\omega])$ if and only if the modified Mabuchi K–energy is bounded below on $\mathcal{H}_{K_0}$ and proper with respect to d_{1,G_0}.*

On a toric manifold, following Donaldson [12], it is more natural to define the K–energy on the space of symplectic potentials as follow. Let (P,σ) be a labelled compact simple polytope with extremal affine function $\mathrm{A}_{P,\vec{n}} \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$ and $u \in \mathcal{S}(P,\sigma)$, the modified Mabuchi K–energy (of the corresponding Kähler potential) is

$$\mathcal{F}_{(P,\sigma)}(u) = -\int_P \log\det(u_{ij})dx + \mathcal{L}_{(P,\sigma)}(u). \tag{22}$$

Indeed, direct calculation shows that the critical points of this functional on $\mathcal{S}(P, \vec{n})$ are the symplectic potentials satisfying

$$S(H^u) = \mathrm{A}_{(P,\sigma)}.$$

This allows us to translate He's Theorem (recalled in Theorem 3.5 above) in terms of (P, σ) only. As explained in [27], when it concerns T–invariant objects ($T \subset K_0$ in the toric case), analytic proofs e.g. estimates of Chen–Cheng [8], translate without problems using the smooth local complex charts (which do exist for any simple labeled polytope) and the compacity of $\overline{P}$. Then to prove Proposition 3.4 it is sufficient to show that $\mathcal{F}_{(P,\sigma)}$ is bounded below on $\widetilde{\mathcal{C}}$ and that it is proper with respect to d_{1,G_0}.

The first condition is given by Donaldson.

Lemma 3.6 (Lemma 3.2 of Donaldson [12]) *If* (P, σ) *is uniformly K–stable then* $\mathcal{F}_{(P,\sigma)}$ *is bounded below on* $\widetilde{\mathcal{C}}$.

We will derive the second using the following result.

Lemma 3.7 (Lemma 2.3 of Zhou–Zhu [35]) *If* (P, σ) *is uniformly K–stable then there exist real positive constants* C, D *such that*

$$\mathcal{F}_{(P,\sigma)}(u) \geq C \int_P u dx - D \tag{23}$$

for all $u \in \widetilde{\mathcal{C}}$.

Given two normalized symplectic potentials $u_0, u_1 \in \mathcal{S}(P, \sigma) \cap \widetilde{\mathcal{C}}_\infty$, we consider the curve $u_t = tu_1 + (1-t)u_0 \in \mathcal{S}(P, \sigma)$ and the curve given by its Legendre transform $\phi_t : \mathfrak{t} \to \mathbb{R}$ (which is a curve of Kähler potentials in the sense that $(\omega = dd^c\phi_t, J)$ is bihomorphically isometric to (ω, J_{u_t}) on $\mathring{M}$, see e.g. [1, 14, 27]).

Thanks to the normalization we have $\int_P u dx = \int_P |u| dx$ for $u \in \widetilde{\mathcal{C}}$ and $\dot{u_t}(x) = -\dot{\phi_t}((\nabla u_t)_x)$ thus

$$\begin{aligned}\int_P |u_0| dx + \int_P |u_1| dx &\geq \int_P |u_1 - u_0| dx = \int_0^1 \int_P |\dot{u_t}| dx\, dt \\ &= \int_0^1 \int_P |\dot{\phi_t}((\nabla u_t)_x)| dx\, dt = \int_0^1 \int_{\mathfrak{t}} |\dot{\phi_t}(y)| \det(D\nabla\phi_t)_y dy\, dt\end{aligned}$$

where the last equality uses the change of variables into complex coordinates, see Remark 2.5. This is used to get the expression

$$\int_0^1 \int_{\mathfrak{t}} |\dot{\phi_t}(y)| \det(D\nabla\phi_t)_y dy\, dt = \frac{1}{(2\pi)^n} \int_0^1 \int_M |\dot{\phi_t}| \omega_{\phi_t}^n\, dt.$$

Now, the right hand side of the last expression is the Darvas length [10] of the curve ϕ_t connecting two Kähler potentials $\psi_0 := \phi_0 - \phi$ and $\psi_1 := \phi_1 - \phi$ in $\mathcal{H}_{K_0}$, therefore

$$\frac{1}{(2\pi)^n}\int_0^1 \int_M |\dot{\phi}_t|\omega_{\phi_t}^n \, dt \geq d_1(\psi_0, \psi_1) \geq d_{1,G_0}(\psi_0, \psi_1).$$

Summing up, for any $u_1 \in \mathcal{S}(P,\sigma) \cap \widetilde{\mathcal{C}}_\infty$, we have that

$$\int_P |u_0| dx + \int_P u_1 dx \geq d_{1,G_0}(\psi_{u_0}, \psi_{u_1})$$

with ψ_u being the Kähler potential corresponding to the metric associated to u. In particular, fixing u_0 and substituting to u_1 a sequence $u_{1,k}$ such that $d_{1,G_0}(\phi_{u_0}, \phi_{u_{1,k}}) \to +\infty$ we get that $\int_P u_{1,k} dx \to +\infty$ which, using Zhou–Zhu properness Lemma 3.7, implies that

$$\mathcal{F}_{(P,\sigma)}(u_{1,k}) \to +\infty.$$

This, with Lemma 3.6 above, is enough to fulfill He's condition and get that there exists a torus invariant extremal Kähler metric. That is, it concludes the proof of Proposition 3.4 which, together with Theorem 3.3 of Chen–Li–Sheng [9] gives Theorem 1.1.

3.3 Extremal Almost Kähler Metrics

In this note we are interested in the $H \in \mathcal{AK}(P,\sigma)$ satisfying the Abreu equation (20). We will consider the following set of *formal solutions*

$$\mathcal{W}(\sigma) := \{H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*) \mid H \text{ satisfies conditions (i), (ii) and } S(H) = \mathrm{A}_{P,\sigma}\}$$

$$\mathcal{W} := \bigsqcup_{\sigma \in \mathbf{M}(P)} \mathcal{W}(\sigma).$$

The only thing a $\mathrm{Sym}^2(\mathfrak{t}^*)$–valued function $H \in \mathcal{W}$ misses to define an extremal toric almost Kähler metric in the sense of Lejmi is the positivity (that is condition (iii)). Therefore

$$\mathcal{W}^+(\sigma) := \mathcal{AK}(P,\sigma) \cap \mathcal{W}(\sigma)$$

parametrizes the space of extremal toric almost Kähler metrics of involutive type on $P \times \mathfrak{t}$ with boundary conditions imposed by the condition (ii) with respect to σ

(see (15)). Translated in our notation, Lejmi proved in [28], see also [12], that the set $\mathcal{W}^+(\sigma)$ is either empty or infinite dimensional.

Proposition 3.8 *Let P be a simple polytope. For any labelling $\sigma \in \mathbf{M}(P)$ the set $\mathcal{W}(\sigma)$ is not empty. Moreover, the set*

$$\{\sigma \in \mathbf{M}(P) \mid \mathcal{W}^+(\sigma) \neq \emptyset\}$$

is a non-empty open convex cone in $\mathbf{M}(P)$.

Proof First, note that the Abreu equation is linear on $\mathcal{W}$ and that the boundary condition data $\sigma \in \mathbf{M}(P)$ depends lineary on the $\mathrm{Sym}^2(\mathfrak{t}^*)$–valued function thanks to (15). Therefore, it is sufficient to find an open set $U \subset \mathbf{M}(P)$ of σ's such that $\mathcal{W}(\sigma)$ is not empty to prove the first assertion. Indeed, in this case U would contain a basis $\{\sigma_s\}_{s=1,\dots,d} \subset U$ and any $\tilde{\sigma} \in \mathbf{M}(P)$ is such $\sigma = \sum_{s=1}^d a_s\sigma_s$ with $a_s \in \mathbb{R}$. Picking any solution $H_s \in \mathcal{W}(\sigma_s)$ we have $\sum_{s=1}^d a_s H_s \in \mathcal{W}(\tilde{\sigma})$. According to [27] for each polytope there exists $\sigma_{KE} \in \mathbf{M}(P)$, unique up to dilatation, and a symplectic potential $u_{KE} \in \mathcal{S}(P, \vec{n}_{\sigma_{KE}})$ such that the metric $g_{u_{KE}}$ is Kähler–Einstein on $P \times \mathfrak{t}$ with respect to the natural symplectic structure on $\mathfrak{t}^* \times \mathfrak{t}$. In particular, $H^{u_{KE}}$ is a solution of Abreu's equation and thus $H^{u_{KE}} \in \mathcal{W}^+(\sigma_{KE})$. Thanks to Donaldson openness result, see Proposition 2.12 above, there exists an open set $U \subset \mathbf{M}(P)$ of σ's such that $\mathcal{W}^+(\sigma)$ is not empty. The second assertion follows the same argument with a special care for positive definite condition. □

Proposition 1.7 is a direct consequence of the last proposition.

3.4 The Space of Formal Solutions

Proposition 3.9 (Donaldson [12]) *Let (P, σ) be a labelled polytope. Assume the set $\mathcal{W}^+(\sigma)$ is non empty. Then the functional $N : \mathcal{W}^+(\sigma) \to \mathbb{R}$ defined by*

$$N(H) = \int_P \log(\det H)\, dx$$

is concave and the critical point, if it exists, is the inverse of a Hessian of a potential $u \in \mathcal{S}(P, \vec{n}_\sigma)$.

The union of the $\mathcal{W}^+(\sigma)$ is a convex cone

$$\mathcal{W}^+ := \bigsqcup_{\sigma \in \mathbf{M}(P)} \mathcal{W}^+(\sigma).$$

From the observation (15), the map $\mathbf{m} : \mathcal{W}^+ \to \mathbf{M}(P)$ taking $H \in \mathcal{W}^+$ to the measure $\mathbf{m}(H) = \sigma \in \mathbf{M}(P)$ is well-defined. The "fibers" of $\mathbf{m}$ are the $\mathcal{W}^+(\sigma)$. Proposition 1.3 implies that the image of the map $\mathbf{m}$ lies into $\mathrm{uKs}(P)$.

Note that $\mathcal{W}^+$ contains the inverse Hessians of the extremal Kähler potentials, that is the union over $\mathbf{M}(P)$ of $\mathcal{KW}^+(\sigma) := \{H^u \,|\, u \in \mathcal{S}(P, u_\sigma),\ H^u \in \mathcal{W}^+(\sigma)\}$. When non-empty, $\mathcal{KW}^+(\sigma)$ contains a unique point, the maximum of H on $\mathcal{W}^+(\sigma)$ thanks to Proposition 3.9. Since N is continuous on $\mathcal{W}^+$, $\mathcal{KW}^+ := \bigsqcup_{\sigma\in\mathbf{M}(P)} \mathcal{KW}^+(\sigma)$ is connected. The relative toric version of the Yau–Tian–Donaldson conjecture is then equivalent to

(i) $\mathcal{KW}^+$ meets each fiber $\mathcal{W}^+(\sigma)$,
(ii) $\mathbf{m}$ is onto.

The assertion (i) is that if there exists an extremal toric almost Kähler metric compatible with ω then there exists an extremal toric Kähler metric and assertion (ii) is that if (P, σ) is uniformly K–stable then there exists an extremal toric almost Kähler metric compatible with ω. This is Corollary 1.4.

4 Miscellaneous

4.1 The Normal and the Angle

Let $\vec{m} = (\vec{m}_1, \dots, \vec{m}_d)$ and $\vec{n} = (\vec{n}_1, \dots, \vec{n}_d)$ be two distinct sets of labels on the same polytope $P \subset \mathfrak{t}^*$ and assume that $(P, \vec{m}, \Lambda)$ is rational Delzant and thus associated to a compact toric symplectic manifold $(M, \omega, T = \mathfrak{t}/\Lambda)$ through the Delzant–Lerman–Tolman correspondance. For any $u \in \mathcal{S}(P, \vec{n})$ the metric g_u, see (9), defines a smooth Kähler metric on $P \times \mathfrak{t} \simeq \mathring{M} = x^{-1}(P)$ compatible with ω. However, since $u \notin \mathcal{S}(P, \vec{m})$ the metric g_u is not the restriction of a smooth metric on M. The behavior of g_u along the boundary of $\mathring{M}$ has been analysed in [27] and we recall the conclusion below.

Recall that $\vec{m}_s$ and $\vec{n}_s$ are inward to P and normal to the facet F_s. We denote $a_s > 0$ the real number such that

$$a_s\vec{n}_s = \vec{m}_s.$$

Note that the boundary condition of $\mathcal{S}(P, \vec{n})$ depends on the labelling via the Guillemin potential $u_{\vec{n}}$, see Remark 2.4. Also, all the potentials in $\mathcal{S}(P, \vec{n})$ have the same behavior along ∂P and for every $u \in \mathcal{S}(P, \vec{n})$, g_u differs from $g_{u_{\vec{n}}}$ only by the addition of a smooth tensor on $\overline{P} \times T \subset \mathfrak{t}^* \times T$. Therefore, without loss of generality, we pick $u_{\vec{n}} \in \mathcal{S}(P, \vec{n})$ to understand that behavior.

The metric $g_{u_{\vec{n}}}$ which is smooth on $\mathring{M} = P \times T = x^{-1}(P)$, has a

- singularity of cone angle type and angle $2a_s\pi$ along $x^{-1}(\mathring{F}_s)$, if $a_s < 1$;
- smooth extension on $x^{-1}(P \cup \mathring{F}_s)$, if $a_s = 1$;
- singularity caracterized by a large angle $2a_s\pi > 2\pi$ along $x^{-1}(\mathring{F}_s)$, if $a_s > 1$.

where, here, we have adopted the terminology in [17].

Proposition 4.1 ([27]) *Let (M,ω,T) be a toric compact symplectic manifold with labelled moment polytope $(P,\vec{m},\Lambda)$ and momentum map $x : M \to \mathfrak{t}^*$. For any labelling $\vec{n}$ of P, any potential $u \in \mathcal{S}(P,\vec{n})$ provides a Kähler metric g_u, defined via (9), smooth and compatible with ω on $\mathring{M} = x^{-1}(P)$ and with cone angle singularity $2\pi(\vec{n}_s/\vec{m}_s)$ transverse to the divisor $x^{-1}(\mathring{F}_s)$. Conversely, any compatible T–invariant Kähler metric smooth outside a divisor D and with cone angle singularity transverse to D is of this form.*

It is straighforward to extend the argument proving the last proposition to almost Kähler metric. Indeed we just compared the behaviour of the Hessian and inverse Hessian of $u_{\vec{n}}$ and $u_{\vec{m}}$. Therefore, any $H \in \mathcal{AK}(P,\sigma_{\vec{n}})$ defines an almost Kähler metrics on $\mathring{M}$ and with cone angle singularity $2\pi(\vec{n}_s/\vec{m}_s)$ transverse to the divisor $x^{-1}(\mathring{F}_s)$.

4.2 The Constant Scalar Curvature Case

In case $(P,\vec{n},\Lambda)$ is rational and associated to a compact toric symplectic orbifold (M,ω,T) via the Delzant–Lerman–Tolman correspondance and assuming we fix a compatible toric Kähler structure (g_u, J_u) (so that $u \in \mathcal{S}(P,\vec{n})$) then the classical Futaki invariant evaluated on the real holomorphic vector field J_uX_f induced by the affine linear function $f \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$ is defined in [19] to be

$$\mathrm{Fut}(M,[\omega])(f) := \int_M (S(H^u) - \overline{S}_{[\omega]})(x^* f)\omega^n/n! \tag{24}$$

where $\overline{S}_{[\omega]} = \int_M S(H^u)\omega^n / \int_M \omega^n$ is the normalized total scalar curvature. Now using (18) and the Fubini's Theorem of product integration, to express $\mathrm{Fut}(M,[\omega])$ in terms of $(P,\vec{n})$ and dx we see that $\overline{S}_{[\omega]} = 2\int_{\partial P}\sigma_u / \int_P dx$ and

$$\mathrm{Fut}(M,[\omega])(f) = \frac{2}{\int_P dx}\left(\int_{\partial P} f d\sigma_{\vec{n}} \int_P dx - \int_P f dx \int_{\partial P} d\sigma_{\vec{n}}\right).$$

This observation is a motivation to introduce the functional

$$\mathrm{Fut}(P,\vec{n})(f) := \int_{\partial P} f d\sigma_{\vec{n}} \int_P dx - \int_P f dx \int_{\partial P} d\sigma_{\vec{n}}, \tag{25}$$

which in the rational case, up to a multiplicative positive constant, is the classical Futaki invariant restricted to the complex Lie algebra $\mathfrak{t} \oplus J\mathfrak{t}$. Moreover, in the case the classical Futaki invariant vanishes, equivalently when A_σ is a constant (which is then $A_\sigma = 2\int_{\partial P}\sigma_u / \int_P dx)$ then

$$\mathrm{Fut}(P,\vec{n})(f) = \frac{2}{\int_P dx}\mathcal{L}_{(P,\sigma)}(f)$$

for any $f \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$.

Corollary 4.2 *Given any labelled polytope $(P,\vec{n})$, if there exists a symplectic potential $u \in \mathcal{S}(P,\vec{n})$ such that g_u has constant scalar curvature then $Fut(P,\vec{n})$ vanishes identically on $Aff(\mathfrak{t}^*,\mathbb{R})$.*

Let η and $\vec{n}$ be labellings for the same polytope P. Then, for each $s = 1,\dots,d$, η_s and $\vec{n}_s$ are inward to P and normal to the facet F_s and so there is a real number $a_s > 0$ such that

$$a_s\vec{n}_s = \eta_s.$$

When restricted on F_s, we have $d\sigma_{\vec{n}} = a_s d\sigma_\eta$. Therefore,

$$\mathrm{Fut}(P,\vec{n})(f) = \int_P dx \sum_s a_s \int_{F_s} f d\sigma_\eta - \int_P f dx \sum_s a_s \int_{F_s} d\sigma_\eta \tag{26}$$

and thus

$$\begin{aligned}\mathrm{Fut}(P,\vec{n})(f) =& \mathrm{Fut}(P,\eta)(f)\\ &- \int_P dx \sum_s (1-a_s)\int_{F_s} f d\sigma_\eta + \int_P f dx \sum_s (1-a_s)\int_{F_s} d\sigma_\eta.\end{aligned} \tag{27}$$

Note that, whenever (P,η,Λ) is rational Delzant and thus associated to a compact toric symplectic manifold $(M,\omega,T = \mathfrak{t}/\Lambda)$ through the Delzant–Lerman–Tolman correspondance, the last expression coincides, up to some multiplicative positive constant, with the *log Futaki invariant* (relative to the torus T) defined in [17]. Indeed, consider the case where $a_1 = \beta$ and $a_s = 1$ for $s = 2,\dots,d$ then we recover from (27) that

$$\mathrm{Fut}_{D,\beta}(\Xi_f,[\omega]) = \frac{2(2\pi)^n\mathrm{Fut}(P,\vec{n})(f)}{\int_M \omega^n} \tag{28}$$

where we follow the notation of [23] with $D = x^{-1}(F_1)$.

Observe from (26) that the vanishing of the Futaki invariant imposes linear conditions on the labelling normals.

Proposition 4.3 *Given a polytope $P \subset \mathfrak{t}^*$ of dimension n with d facets, there exists a $(d-n)$–dimensional cone $\mathbf{C}(P) \subset \mathfrak{t}^d$ of labelling $\vec{n} \in \mathbf{C}(P)$ such that $Fut(P, \vec{n})$ vanishes identically on $Aff(\mathfrak{t}^*, \mathbb{R})$.*

In [27] the last proposition follows non trivial consideratio, we give an elementary proof here.

Proof Put coordinates $x = (x_1, \ldots, x_n)$ on $\mathfrak{t}^*$ and translate P if necessary so that $\int_P x_i dx = 0$ for any $i = 1, \ldots, n$. The result follows if the linear map $\mathbb{R}^d \longrightarrow \mathbb{R}^n$ defined by

$$\mathbb{R}^d \ni a \mapsto \left(\sum_{s=1}^{d} a_s \left(\int_P x_i \, dx \int_{F_s} d\sigma_{\vec{n}} - \int_P dx \int_{F_s} x_i d\sigma_{\vec{n}} \right) \right)_{i=1,\ldots,n} \tag{29}$$

is onto and his kernel meets the positive quadrant of $\mathbb{R}^d$. With the suitable coordinate chosen the rhs of (29) is up to non-zero multiplicative constant

$$\left(\sum_{s=1}^{d} a_s \int_{F_s} x_1 d\sigma_{\vec{n}}, \ldots, \sum_{s=1}^{d} a_s \int_{F_s} x_n d\sigma_{\vec{n}} \right) \in \mathbb{R}^n.$$

This is onto by convexity of P, indeed, for any coordinates x_i there is a facet of P on which x_i is sign definite. Basic consideration on barycenter and the observation that $0 \in P$ imply that the kernel of the map (29) contains an element of the positive quadrant of $\mathbb{R}^d$. □

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Tropical Geometric Compactification of Moduli, I – M_g Case

Yuji Odaka

To the memory of Kentaro Nagao.

Abstract We compactify the classical moduli variety of compact Riemann surfaces by attaching moduli of (metrized) *graphs* as boundary. The compactifications do *not* admit the structure of varieties and patch together to form a big connected moduli space in which $\sqcup_g M_g$ is open dense.

The metrized graphs, which are often studied as "tropical curves", are obtained as Gromov-Hausdorff collapse by fixing diameters of the hyperbolic metrics of the Riemann surfaces. This phenomenon can be also seen as an archemidean analogue of the tropicalization of Berkovich analytification of M_g [1].

Keywords Moduli of curves · Compactification · Tropical geometry

1 Introduction

Let us recall that the moduli space of smooth projective curves admits a "canonical" modular compactification constructed in Deligne-Mumford [16] first as an algebraic stack $\overline{\mathcal{M}_g}^{\rm DM}$.[1] Later on, the moduli stack was proved to have a coarse projective variety which is normal and of dimension $3g-3$ [29, Especially, III], [19, 39].

The boundary of the compactification still parametrizes geometric objects which are certain nodal curves called "stable curves" characterized by the GIT stability [19, 39] or by the K-stability ([42, 4.1], also cf. [39, 40], [33, §7]). Hence the GIT

[1]Here we put the superscript "DM", often omitted in the literatures, to clearly distinguish from the compactifications we introduce in this paper.

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construction [38] applies [19, 39] while it also fits to more general moduli existence conjecture for K-(semi)stable polarized varieties ("K-moduli" cf., [43]).

In this paper, we introduce a pair of new compactifications of M_g which are *no longer varieties* but compact Hausdorff toplogical spaces. In the first compactification which we denote as $\overline{M_g}^{\mathrm{GH}}$, the boundaries parametrize the Gromov-Hausdorff limits of compact Riemann surfaces with rescaled Poincaré (i.e., Kähler-Einstein) metrics with diameter 1, which we identify as certain graphs (Theorem 2.4). Hence we would like to call the compactification $\overline{M_g}^{\mathrm{GH}}$ *Gromov-Hausdorff compactification*.

In the second compactifications of M_g, we further encode some non-negative integer weights on the vertices of the limit graphs. We call the metrized graphs with such weights, *weighted metrized graphs*. The class of our limits graph is very close to what has been studied as "(stable) tropical curves" in the literatures (e.g., [8, 10, 11, 35]). Our point is that we can construct a refined compactification of M_g than $\overline{M_g}^{\mathrm{GH}}$ by encoding the weights. The obtained compactifications will be called "*tropical geometric compactifications*". We chose the term because the boundaries coincides with the moduli spaces of such tropical curves, which are also studied in the literatures (e.g., [8, 10, 11, 35] again), while we also avoided the term "*tropical compactification*" already used by J. Tevelev whose context is very different, namely, the problem of compactifying subvarieties of a torus in a toric variety (cf., [52]).

Let us explain the backgrounds by discussing a broader picture for moduli spaces of more general varieties. There are two major backgrounds for this work, which we recall now:

(i) The current extensive approach to the Strominger-Yau-Zaslow mirror symmetry conjecture [51]. Indeed, conjectures of Gross-Wilson [22, §6], Todorov, Kontsevich-Soibelman [31] (cf., e.g., the survey on the Gross-Siebert program [21]) speculates certain families of Calabi-Yau varieties with its Ricci-flat Kähler metrics collapse to integral affine manifolds with singularities in the Gromov-Hausdorff sense, which are recently often regarded as some tropical version of Calabi-Yau varieties.

(ii) The algebraicity of *non-collapsed* Gromov-Hausdorff limits of Kähler- Einstein manifolds [17], its applications to moduli of Fano varieties [43, 47, 49], later followed by [34, 44, 50].

There is a similarity between the above two i.e. (i) and (ii) as the first i.e. (i) is in particular showing that the collapsed Gromov-Hausdorff limits of Kähler-Einstein manifolds are "tropical *algebraic*" objects while the second (ii) is showing that the non-collapsed limits of Kähler-Einstein (Fano) manifolds are *algebro-geometric* objects i.e., varieties.

For moduli spaces of Fano manifolds, which we discussed in (cf., [17, 34, 43, 44, 47, 50]), the two kinds of the compactifications

(α) the Gromov-Hausdorff metric compactification of the moduli space of Kähler-Einstein manifolds with the rescaled Kähler-Einstein metrics with fixed diameters (our $\overline{M_g}^{\mathrm{GH}}$ and $\overline{M_g}^{\mathrm{T}}$ to be introduced in this paper are on this side) which is closer to the spirit of (i) and

(β) algebro-geometric compactified moduli of K-stable varieties, e.g. $\overline{M_g}^{\mathrm{DM}}$ as in (ii)

essentially coincide because of the non-collapsing of the metrics. However they "look" completely different in the non-Fano case due to collapse of the Kähler-Einstein metrics as we show in the present series of papers. Indeed, the author believes that the Gromov-Hausdorff compactification while fixing the *volume* (rather than the diameter), if it exists in an appropriate sense, should be closer in spirit to (β). Nevertheless, as we observe in the case of M_g in this paper, we believe that the two series of compactifications (α) and (β) must be deeply connected in general.

In the present paper, first we start with the classification of all the possible Gromov-Hausdorff limits of the compact Riemann surfaces with Kähler-Einstein metrics of diameters 1. Then using the classification, we construct the compactifications and proceed to analyze their structures.

Our connection between classical algebro-geometric compactifications and tropical moduli spaces can be seen as an archimedean analogue of the *tropicalization* (*skeleton*) of non-archimedean analytification of the moduli varieties which is recently studied in [1]. We discuss this analogy towards the end of the Sect. 2.2.

Another interesting point of our compactifications $\overline{M_g}^{\mathrm{GH}}$, is that they naturally patch together to form a big (infinite dimensional) *conneted* moduli space in which M_g are open subsets for *all* g. We will call it *infinite join* and denotes it as $\overline{M_\infty}^{\mathrm{GH}}$.

It would be interesting to pursue this line of research for moduli varieties of other polarized varieties. For instance, the author conjectures that the moduli schemes of smooth canonical models, again with the rescaled Kähler-Einstein metrics of diameters 1, are also precompact for Gromov-Hausdorff distance and the corresponding collapses will be dual intersection complexes of KSBA semi-log-canonical models in certain generalized sense. Such speculation is inspired by the recent Kollár-Shepherd-Barron-Alexeev (KSBA) compactification (cf., e.g., the survey [30]) and the observation that it is a moduli scheme of K-stable varieties ([40, 41], also [4]).

Throughout this article, we work over the complex number field $\mathbb{C}$ unless otherwise stated.

Notes added, part Two years after our original preprint of this paper, Boucksom-Jonsson [7, §2] generalized the Morgan-Shalen compactification [37] which can be also further generalized to orbifolds in [45, Appendix]. It may be convenient to mention here that the compactification applied to M_g are *different* from our compactification. More precisely, although it can be set-theoritically identified with our $\overline{M_g}^{wT}$ but has *different* topology. See [45, Theorem 3.7] for the details.

Also, after that, we had other further developments with Yoshiki Oshima for the case of A_g and moduli of K-trivial varieties case (cf., [46]). In *loc.cit*, we put a focus on the moduli of K3 surfaces, after the works of [22, 23, 31].

2 Gromov-Hausdorff Compactification of M_g

2.1 Precompactness

For each compact Riemann surface of genus $g(\geq 2)$, we put *rescale* of the Kähler-Einstein metric with the *diameter* 1.[2] Recall that the Kähler-Einstein metric is nothing but the famous Poincaré metric in this case. The first point we should clarify is the precompactness of M_g with the associated Gromov-Hausdorff distance (for its definition we refer to e.g. [9, Chapter 7]) on it. We denote the Gromov-Hausdorff distance as d_{GH}. Recall that the precompactness of a subset of the space of all compact metric spaces means its closure with respect to the Gromov-Hausdorff topology is compact. During the process of degenerations i.e., going to boundary of M_g, the curvature tends to $-\infty$, so we can *not* apply the Gromov's precompactness theorem [20] in our situation. Instead we can apply the following theorem of Shioya [48] and the Gauss-Bonnet theorem to prove it.

Theorem 2.1 ([48, Theorem 1.1]) *For two fixed positive real numbers $D > 0$ and $c > 0$, consider the set $S(D, c)$ of closed 2-dimensional Riemannian manifolds (R, d) with*

(i) *the diameter* $\mathrm{diam}(d) < D$
(ii) *and the total absolute curvature* $\int_R |K_{(R,d)}| \mathrm{vol}(R) < c$ *where* $K_{(R,d)}$ *and* $\mathrm{vol}(R)$ *denotes the Gaussian curvature and the volume form with respect to the metric* d.

Then the set $S(D, c)$ is precompact with respect to the associated Gromov-Hausdorff distance.

By applying the above theorem, we get the following desired precompactness.

Corollary 2.2 (M_g, d_{GH}) *is precompact.*

First proof It directly follows from the Shioya's theorem above (2.1) since our total absolute curvature is constant due to the Gauss-Bonnet theorem. □

[2] Readers will find later that this specific constant 1 does not have any specific meaning as we only meant to fix it, so we can rather set it to be any fixed positive constant.

We include another proof of Corollary 2.2 in the next section, in which we also classify all the Gromov-Hausdorff limits.

2.2 Gromov-Hausdorff Collapse of Riemann Surfaces

Before stating a theorem, we precisely fix some graph theoretic terminology we use in this paper.

Definition 2.3 In the present paper, a *metrized (finite) graph* means a finite connected non-directed graph with finite positive lengths attached to all edges. It is not necessarily simple, i.e., loops and several edges with the same ends are allowed. A *contraction* of a finite graph is a graph which can be obtained from the original graph by contracting some of its edges.

The main result of this section is the following theorem, which implies the precompactness of M_g and also classify all the possible Gromov-Hausdorff limits of compact hyperbolic surfaces while fixing their diameters.

Theorem 2.4 *Let* $\{R_i\}_{i\in\mathbb{Z}_{>0}}$ *be an arbitrary sequence of compact Riemann surfaces of fixed genus* $g \geq 2$. *Suppose* $\{(R_i, \frac{d_{\mathrm{KE}}}{\mathrm{diam}(R_i)})\}_i$ *converges in the Gromov-Hausdorff sense. Here* d_{KE} *denotes the Poincaré metric*[3] *on each* R_i *and its diameter is* $\mathrm{diam}(R_i)$.

Then the limit is the metric space associated to either

(i) *a metrized graph of diameter* 1 *or*
(ii) *a compact Riemann surface of genus* g.

Assume furthermore that the sequence R_i *converges to* $[R_\infty] \in \overline{M}_g^{\mathrm{DM}}$ *(which can be always be achieved by passing to a subsequence since* $\overline{M}_g^{\mathrm{DM}}$ *is compact). Then if* $[R_\infty] \in M_g$ *we are in case (ii) and* R_i *converges in the Gromov-Hausdorff sense to the metric space underlying* R_∞*; if, on the other hand,* $[R_\infty] \notin M_g$ *then we are in case (i) and the* R_i *converges to the metric space underlying a metrized graph whose underlying graph is a contraction of the dual graph of* R_∞.

Conversely, any metrized graph with diameter 1 *whose underlying graph is a contraction of some (possibly* 0*) edges of the dual graph of a stable curve of genus* g*, can occur in this way (i).*

Proof We fix a reference compact Riemann surface S and regard the Teichmuller space T_g as the set of marked compact Riemann surfaces $[\phi\colon S \xrightarrow{\simeq} R]$ where we only care of the isotopy type of ϕ.

First we briefly recall the basic of the pair-of-pants decomposition of S, which we abbreviate as pants decomposition from now on for short, and later we will explain

[3] I.e., the hyperbolic metric which is also a Kähler-Einstein metric, hence the notation.

how to apply it.

$$S = \bigcup_{0 \le a \le g-2} P_a$$

with the associated simple closed boundary geodesics $s_1, \cdots, s_{3g-3}$. Then in turn it naturally induces the corresponding pants decompositions of R

$$R = \bigcup_{0 \le a \le g-2} P_a(R)$$

for all elements $[\phi \colon S \xrightarrow{\simeq} R]$ of T_g since we can take simple closed boundary geodesics in the corresponding homology classes. The associated simple closed boundary geodesics $\{s_j(R)\}_j$ of R gives the (real analytic) Fenchel-Nielsen coordinates on it

$$(l_1, \cdots, l_{3g-3}; \theta_1, \cdots, \theta_{3g-3}) \colon T_g \cong \mathbb{R}_{>0}^{3g-3} \times (\mathbb{R}/2\pi\mathbb{Z})^{3g-3},$$

where l_j is the length of s_j and θ_j is corresponding twist parameters (cf., [26]). Then the following well-known theorem is due to L. Bers.

Fact 2.5 ([5, Theorem 2 for the type $(g, 0)$ **case])** *Fix a positive integer* $g \ge 2$. *Then there is a uniform constant* C_g *such that for an arbitrary compact hyperbolic Riemann surface* R*, there is a pant decomposition whose corresponding lengths* l_j *of any dividing simple closed geodesic satisfy* $l_j < C_g$.

We now argue as follows. Suppose we are given a sequence $\{R_i\}_{i \in \mathbb{Z}_{>0}}$ of compact Riemann surfaces of the fixed genus $g \ge 2$, as in the statement of Theorem 2.4. We replace it by its certain subsequence, after several steps as follows. Firstly, due to the compactness of the Deligne-Mumford compactification $\overline{M_g}^{\mathrm{DM}}$, we can replace the sequence $\{R_i\}$ by subsequence, if necessary, to ensure the existence of a limit of $[R_i]$ inside $\overline{M_g}^{\mathrm{DM}}$. By applying the Bers' Theorem 2.5, for each i, we have a pants decomposition satisfying the assertion of Theorem 2.5, i.e., all the lengths of the corresponding simple closed geodesics are less than a uniform constant C_g. For each R_i, we fix such a pants decomposition from now on. On the other hand, note that for each pants decomposition there is a corresponding graph whose vertices are (pair of) pants while edges are common geodesics is 3-regular with $2g - 2$ vertices. We call this graph the combinatorial type of the pant decomposition. See for instance [24, around Definition 1.5] for the details. The number of edges of such a graph is $3g-3$ so obviously there is only a finite possibilities for such graphs. Hence, there is only a finite possibilities of combinatorial type of pants decomposition. Therefore, by passing to an appropriate subsequence of $\{R_i\}$ again, if necessary, we can and do assume the combinatorial type of the pants decompositions we took, which satisfies the condition $l_j < C_g$ of Fact 2.5, stays fixed. By the upper bound of l_j, by further passing to an appropriate subsequence of $\{R_i\}$ again, if necessary, we can and do assume moreover that $\lim_{i\to\infty} l_j(R_i) = L_j$ for some constants $L_j \in [0, C_g] \subset [0, \infty)$ for each j.

The simple geodesics $s_j(R_i)$ of R_i with $L_j = 0$ are representing the vanishing cycles, i.e., all the cycles that shrink to nodal singularities of the corresponding limit in the Deligne-Mumford compactification $\overline{M_g}^{\mathrm{DM}}$. We make the following claim, although the author believes this has been known to or expected by the experts.

Claim 2.6 *There is an index j with $L_j = 0$, if and only if the diameter of the non-rescaled hyperbolic metrics (i.e., with constant Gaussian curvature -1) tends to $+\infty$. This is also equivalent to that the limit of the sequence $[R_i]$ does not belong to M_g.*

Otherwise, passing to a subsequence, the Gromov-Hausdorff limit R_∞ of $\{R_i\}_i$ exists as a compact Riemann surface of the same genus g.

Proof of Claim 2.6 If all the L_j are non-zero, then the compactness of

$$\{(l_1, \cdots, l_{3g-3}; \theta_1, \cdots, \theta_{3g-3}) \mid L_i - \epsilon \le l_i \le C_g \text{ for } 1 \le \forall i \le 3g-3\} \subset T_g$$

for small enough positive real number ϵ straightforwardly implies that the corresponding points $[R_i] \in T_g$ converge inside T_g.

Now, we denote the space of all compact metric spaces with the Gromov-Hausdorff topology as CMet. Here, we recall the following standard fact well-known to experts.

Fact 2.7 (Gromov-Hausdorff continuity on M_g) *If we consider the map $\Phi\colon M_g \to \mathrm{CMet}$, sending $[R]$ to the underlying topological surface with the Poincaré metric. Also define $\Phi_1\colon M_g \to \mathrm{CMet}$ by sending $[R]$ to the underlying topological surface with the rescaled Poincaré metric with the diameter 1. Then these Φ and Φ_1 are both continuous with respect to the complex analytic topology on M_g.*

This is fairly standard but we write the arguments for convenience. Obviously, the continuity of Φ_1 follows from that of Φ because the diameters of the hyperbolic metrics vary continuously due to the continuity of Φ. In turn, the continuity of Φ follows, for instance, from the interpretation of the family as a family of the quotients of the upper half plane by continuously deforming Fuchsian subgroup of $PSL(2, \mathbb{R})$. (The isomorphic class of the Fuchsian group is not changed, as it is the isomorphic class of the fundamental group of genus g compact Riemann surface.) Or it also follows from the implicit function theorem applied to the constancy of the Gaussian curvature. Hence, in particular, the diameters of the (non-rescaled, original) Poincaré metrics of R_i are bounded and the Gromov-Hausdorff limit of R_i with the rescaled Poincaré metric is still a compact Riemann surface of genus g.

On the other hand, if $L_j = 0$ for at least one index j, then the famous collar theorem [28] applies and directly shows that for each i there is a cylinder (called "collar") inside R_i, including the closed geodesic l_j, whose diameter tends to $+\infty$. We end the proof of the Claim 2.6. □

From now on, we assume these equivalent conditions are satisfied i.e., $[R_\infty] \notin M_g$. Otherwise, the subsequence converges to a compact Riemann surface (i.e., "does not degenerate"), which corresponds to the case (ii) of Theorem 2.4. This is again because of the continuity of the surfaces with the rescaled Poincaré metrics parametrized by M_g with respect to the Gromov-Hausdorff topology.

Let us denote the diameter of the Poincaré (hyperbolic) metric $d_{\rm KE}$ of R_i as d_i. Then recall that what we are analysing is the metric behaviour of $(R_i, \frac{d_{\rm KE}}{d_i})$ and we wish to determine its Gromov-Hausdorff limit. For that, we analyze the behaviour of the pant $(P_a(R_i), \frac{d_{\rm KE}}{d_i})$ in this proof. We denote the three boundary geodesics of the pants as $s_b(P_a)(b = 1, 2, 3)$, or $s_b(R_i; P_a)(b = 1, 2, 3)$ for precision, which may partially be identified in the Riemann surface R_i i.e., e.g. $s_1(R_i; P_a) = s_2(R_i; P_a)$ can be possible. From now on, whenever the context is clear, we sometimes omit R_i and simply denote the pants of R_i as P_a, not $P_a(R_i)$ and its boundary geodesics $s_b(P_a)(b = 1, 2, 3)$ rather than $s_b(R_i; P_a)(b = 1, 2, 3)$.

Let us recall a standard fact in the Teichmuller theory (cf., [26, Chapter 3, §1.5, §2]) which claims that the pant $P_a(R_i)$ can be cut and separated into two isometric hyperbolic hexagons $Q_a(R_i)$ and $Q'_a(R_i)$ canonically by geodesics which connect different boundary geodesics of the pant $P_a(R_i)$. Let us also recall from [26, Chapter 3, §1.5, §2] that the interior part of the hyperbolic hexagons $Q_a(R_i)$, with its hyperbolic metric, can be regarded as an open subset of a unit disc with the hyperbolic metric $d_{\rm KE}$, in a unique way up to the isometry group of the disk i.e., $PGL(2, \mathbb{R})$. We denote the center of the unit disc as p.

Let us call the 3 boundaries of the hexagon which were originally part of the boundaries of the pant P_a as *boundary geodesics*. In any case, the important invariants are the lengths of the 3 boundary geodesics which are half of the lengths of the boundary geodesics $s_b(R_i; P_a)(b = 1, 2, 3)$ of the original pant P_a. Indeed, it is a well-known fact that biholomorphic type of Q_a (so also for P_a) is determined by the lengths of the three boundary geodesics (cf., e.g., [26]). We now study the Gromov-Hausdorff limit of the hyperbolic hexagon Q_a while fixing diameters. Then, recall from the Claim 2.6, it follows that $d_{\rm KE}(p, s_b(R_i; P_a)) \to +\infty$ for $i \to \infty$ if and only if the corresponding boundary geodesic $s_b(R_i; P_a)$ shrinks i.e., $\text{length}(s_b(R_i; P_a)) \to 0$ for $i \to \infty$.

To each P_a, we associate a tree Γ_a, just as a combinatorial graph, with

- the vertex set $V(\Gamma_a) := \{v_a\} \sqcup \{w_b \mid s_b(R_i; P_a) \text{ shrinks}\}$ and
- the edge set $E(\Gamma_a) := \{\overline{v_a w_b} \mid s_b(R_i; P_a) \text{ shrinks}\}$.

Denote the diameter of the hyperbolic hexagon $Q_a(R_i)$ with respect to Poincaré metric as $d_i(a)$. (Recall that the diameter of whole R_i is d_i.) We analyze the asymptotic behaviour of $(R_i, \frac{d_{\rm KE}}{d_i})$ by further "decomposing" into that of $Q_a(R_i)$ as above.

First we fix a constant $0 < \epsilon \ll 1$ so that the sequence of the half pant $\{Q_a(R_i)\}_i$ satisfies that the disk $D(p, (1-\epsilon))$ with center p and radius $(1-\epsilon)$ contains all non-shrinking boundary geodesics of $Q_a(R_i)$. Then thinking of the distance between each point in $(Q_a(R_i) \cap D(p, (1 - \epsilon))$ and p, we straightforwardly obtain that the diameter of $\{(Q_a(R_i) \cap D(p, (1-\epsilon)), d_{\rm KE})\}_i$ is bounded above by C_ϵ. On the other hand, the diameters of the collar neighborhoods of shrinking boundary geodesics tends to $+\infty$ by the collar theorem [28]. Hence, we have that

Claim 2.8 (Diverging hyperbolic hexagon) *$d_i(a) \to \infty$ for $i \to \infty$ if and only if there is an index b with* $\text{length}(s_b(R_i; P_a)) \to 0$ *for $i \to \infty$.*

Claim 2.9 (Limit of hyperbolic hexagon, I) *If we consider the sequence $(Q_a(R_i), \frac{d_{\text{KE}}}{d_i(a)})$ for $i = 1, 2, \cdots$, it has the Gromov-Hausdorff limit as a metrized tree Γ_a in the case when* $\text{length}(s_b(R_i; P_a)) \to 0$ *for some b when $i \to \infty$. Otherwise its Gromov-Hausdorff limit is still some hyperbolic hexagon.*

The last sentence of Claim 2.9 holds because, for any b, $\text{length}(s_b(R_i; P_a))$ converges to a positive real number from our assumption when $i \to +\infty$ and $d_i(a)$ are bounded above, converging to a positive real number as well. Thus, the Gromov-Hausdorff limit of $(Q_a(R_i), d_{\text{KE}})$ can be taken simply as the Hausdorff limit inside the unit disk which implies the desired claim.

Next, we compare the diameters of each hyperbolic hexagon $Q_a(R_i)$ and the whole Riemann surface R_i.

Claim 2.10 (Diameters comparison)

(i) *For any i there is at least one $Q_a(R_i)$ (or equivalently, its index a) such that*

$$d_i \leq 12(g-1)d_i(a). \tag{1}$$

(ii) *Suppose that an index a satisfies that $d_i(a) \to \infty$ when $i \to \infty$. Then, for any a and large enough i, we have*

$$\frac{d_i(a)}{2} \leq d_i. \tag{2}$$

Proof of Claim 2.10 The second assertion (ii) easily follows from the definition. Indeed, it can be proven as follows. First we can assume $d_i(a)$ is the length of a geodesic $\gamma\colon [0,1] \to R_i$ connecting two points $\gamma(0), \gamma(1)$ in the union of the boundary geodesics. Then its midpoint $\gamma(\frac{1}{2})$ and one of the endpoints, say $\gamma(1)$, of the geodesic has the same distance in whole R_i i.e., after gluing the boundary geodesics. Hence (ii) follows.

Our first assertion (i) is proved as follows. Take a shortest geodesic $\delta\colon [0,1] \to R_i$ connecting two points in R_i with $\text{length}(\delta) = \text{diam}(R_i)$. An elementary observation shows that the maximum number of the connected components of $\text{Im}(\delta) \cap Q_a(R_i)$ is at most 3 so that we have

$$\text{length}(\text{Im}(\delta) \cap Q_a(R_i)) \leq 3d_i(a) \tag{3}$$

$$\text{length}(\text{Im}(\delta) \cap Q'_a(R_i)) \leq 3d_i(a). \tag{4}$$

Indeed, if we write

$$I_a := \{t \in [0,1] \mid \delta(t) \in Q_a\} = [\alpha_1, \alpha_2] \sqcup \cdots \sqcup [\alpha_{2m-1}, \alpha_{2m}],$$

with $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \alpha_{2m}$, then note that $\delta(\alpha_2)$ and $\delta(\alpha_{2m-1})$ are connected by a geodesic of length at most $d_i(a)$, by the definition of $d_i(a)$. Since δ is taken to be a shortest geodesic, $\sum_{1\leq k<m} \text{length}(\delta([\alpha_{2k-1}, \alpha_{2k}])) \leq d_i(a)$ which gives our desired estimate (3), and also (4) similarly. Hence, by summing up, we obtain $d_i \leq 6\sum_a d_i(a)$. Since $\#\{a\} = 2(g-1)$, we obtain the desired inequality (2). □

From the Claims 2.8 and 2.10 (i),(ii) we have that $d_i \to \infty$ if and only if there is some P_a with $d_i(a) \to \infty$. Also it follows from the Claim 2.10, if P_a satisfies that for some b $\text{length}(s_b(R_i; P_a)) \to 0$ for $i \to \infty$, by further passing to a subsequence we can assume that R_i satisfies that

$$\frac{d_i(a)}{2} \leq d_i \leq 12(g-1)d_i(a),$$

for a fixed a, say $a = 1$. On the other hand,

$$\frac{d_i(a)}{2} \leq d_i$$

holds for any a. Hence, combining Claims 2.9 and 2.10, we have that

Claim 2.11 (Limit of hyperbolic hexagon, II) *Under our assumption that $[R_\infty] \notin M_g$, if we consider the sequence $(Q_a(R_i), \frac{d_{\mathrm{KE}}}{d_i})$ for $i \to \infty$, it converges in the Gromov-Hausdorff sense to either a metrized tree Γ or a point.*

The convergence to the point occurs exactly when $\dfrac{d_i}{d_i(a)} \to +\infty$ for $i \to +\infty$. From the above Claim 2.11, it follows that the global Gromov-Hausdorff limit of $(R_i, \frac{d_{\mathrm{KE}}}{d_i})$ is a metrized graph which is obtained by gluing all Γ_a at w_b's whose corresponding boundary geodesics s_j are the same in the whole Riemann surface R_i. The resulting graph is either the dual graph of the corresponding stable curve R_∞ or a graph obtained from the dual graph after contracting several edges to points. (We simply call such procedure a *contraction* of a graph in this paper.)

Now let us move on to the proof of the converse direction (the last paragraph of the statements of Theorem 2.4). That is, starting from an arbitrary finite metrized graph Γ of diameter 1 which satisfies the assumption of the last paragraph of Theorem 2.4, we wish to prove there is a sequence of compact Riemann surfaces $R_i (i = 1, 2, \cdots)$ of genus g such that Γ is the Gromov-Hausdorff limit of $(R_i, \frac{d_{\mathrm{KE}}}{d_i})$ i.e., the rescaled Poincaré metrics of diameter 1.

We fix an arbitrary stable curve R whose dual graph contracts to the underlying graph of Γ. Such R exists due to our assumption on Γ. Then take a smooth point in each of the irreducible components of R and denote them by p_i. Here the index i corresponds to each irreducible component. We take a semi-universal deformation of R as $\{R_{\vec{t}}\}_{\vec{t}\in U}$ with an open neighborhood $U' \subset \mathbb{C}^{3g-3}$ of $\vec{0}$, satisfying $R_{\vec{0}} = R$ and take $p_{i,\vec{t}}$ of $R_{\vec{t}}$ with $p_{i,\vec{0}} = p_i$ which is continuous with respect to $\vec{t}$. From here, we use a smaller open neighborhood of $\vec{0}$ denoted by $U \subset U'$ with $\bar{U} \subset U'$. Note that there is a discriminant locus $D \subset U$ such that $\vec{t} \notin D$ if and only if $R_{\vec{t}}$ is smooth.

We fix a uniform pants decomposition of $R_{\vec{t}}$ so that the nodes x_k of R are shrunk dividing geodesics s_k of the decomposition. For each node x_k of R connecting the irreducible components including p_i and p_j, there is a corresponding shortest geodesic $\gamma_{k,\vec{t}}$ connecting $p_{i,\vec{t}}$ and $p_{j,\vec{t}}$ if $R_{\vec{t}}$ is smooth which intersects with s_k.

Recall that there is a standard submersive holomorphic map $\phi = \{\phi_k\}_k : U \to \prod_k \mathrm{Kur}(x_k)$, where $\mathrm{Kur}(x_k)$ stands for the Kuranishi space underlying a semi-universal deformation of the node singularity x_k, and ϕ_k is induced by restricting the deformation of R to a neighborhood of each node x_k. In this case, $\mathrm{Kur}(x_k)$ can be regarded as an open neighborhood of 0 in $\mathbb{C}$ and the discriminant locus D is the divisor $\cup_k \phi_k^{-1}(0)$. For the proof of the fact that ϕ is submersive, i.e., its differential $d\phi$ is surjective, see [16, Proposition 1.5]. Since the distance of p_i, p_j for $i \neq j$ in R with respect to the hyperbolic metric is $+\infty$ (i.e., not defined as a real number), for a sequence $\{\vec{t}_m\}_{m=1,2,\cdots} \subset U \setminus D$,

$$\mathrm{length}(\gamma_{k,\vec{t}_m}; R_{\vec{t}_m}) \to +\infty$$

for $m \to \infty$ if and only if $\phi_k(\vec{t}_m) \to \vec{0}$.

On the side of Γ, for each node x_k of R, also an edge γ_k of Γ corresponds, which may be possibly contracted to a point. If it is contracted, we regard it as an edge of length 0.

From the above discussions with the surjectivity of ϕ, for large enough positive integers $m \gg 1$, there is $\vec{t}_m \in U \setminus D$

$$\mathrm{length}(\gamma_{k,\vec{t}_m}; R_{\vec{t}_m}) = m \cdot \mathrm{length}(\gamma_k; \Gamma) \text{ if } \gamma_k \text{ is not contracted in } \Gamma \tag{5}$$

$$\mathrm{length}(\gamma_{k,\vec{t}_m}; R_{\vec{t}_m}) = \sqrt{m} \text{ if } \gamma_k \text{ is contracted in } \Gamma. \tag{6}$$

Then, the above taken sequence of smooth compact Riemann surfaces $\{R_{\vec{t}_m}\}_m$ with the rescaled Poincaré metric converges to a metrized graph and from (5) and (6), the limit metrized graph coincides with Γ. We complete the proof of the last paragraph of Theorem 2.4. □

Remark 2.12 A while after the first version of this paper, we essentially gave another (logically independent) more moduli-theoritic proof of Theorem 2.4 in the sequel [45] by using [54]. Precisely speaking, Theorem 2.4 follows from [45, §3.2.1, Theorem 3.7 and its proof] which depends on [54].

Remark 2.13 In the simpler case of $g = 1$, i.e., elliptic curves case, we also have a similar phenomenon as discussed in the introduction of [22]. It can be regarded as the easiest prototypical example of the sequel paper [45] on the moduli spaces of principally polarized abelian varieties and also well-known to the experts of the Strominger-Yau-Zaslow mirror symmetry conjetures. Thus we give only brief description as an introduction to our sequels [45, 46].

Suppose there is a sequence of elliptic curve $\{\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau_i)\}_i$ where τ_i belongs to the standard fundamental domain W of the upper half plane $\mathbb{H}$ modulo the modular group SL(2, $\mathbb{Z}$), that is

$$W := \{\tau \in \mathbb{H} \mid |\mathrm{Re}(\tau)| \leq 1, |\tau| \geq 1\}.$$

If $\mathrm{Im}(\tau_i)$ does *not* diverge, then after passing to a subsequence, they converge in the Gromov-Hausdorff sense to an elliptic curve. If $\mathrm{Im}(\tau_i)$ diverges, then the Gromov-Hausdorff limit of a subsequence of $\left\{\left(R_i, \dfrac{d_{\mathrm{KE}}}{\mathrm{diam}(d_{\mathrm{KE}})}\right)\right\}_{i=1,2,\cdots}$ is $S^1(\frac{1}{2\pi})$, the circle of radius $\frac{1}{2\pi}$. On the other hand, for a family of elliptic curves over the punctured disk, the compactified Néron model after suitable base change is well-known to be n-gon with some $n \in \mathbb{Z}_{>0}$. Thus their dual graphs are topologically S^1, which is homeomorphic to the Gromov-Hausdorff limit discussed above.

Remark 2.14 For the case of curves with punctures (marked points), i.e., elements of $M_{g,n}$ with $n \geq 1$, as the natural hyperbolic metric has hyperbolic cusp singularities of *infinite* diameters around the punctures, we have not been able to make a suitable formulation to study Gromov-Hausdorff collapses.

Professor Y-G.Oh kindly pointed out to me that a different but similar kind of "graph-like thin" metrics also appear as "*(general) minimal area metric*" studied by Zwiebach and Wolf-Zwiebach (cf., e.g., [53, 55]) for constructing closed string field theory. The metrics are expected to be isometric to *flat* semi-infinite cylinders around the punctures. The graph structure is regarded as a version of Feynman diagrams there.

Remark 2.15 Our Theorem 2.4 suggests that the conjectures of Gross-Wilson [22, §6], Kontsevich-Soibelman [31] and Gross-Siebert (cf., [21]) on the correspondence of Gromov-Hausdorff limit and dual complex of degenerating *Calabi-Yau manifolds* may well have an analogue in *negative* Ricci curvature Kähler-Einstein case, i.e., those projective manifolds with ample canonical classes.

Let us trace again the proof of our Theorem 2.4 to see some analogy with the tropicalization of the Berkovich analytification [1]. The one page arguments below does *not* contain any substantially concrete results and rather we mean to give a re-interpretation of our Theorem 2.4 and compare with [1]. In our theorem 2.4, starting with an arbitrary sequence of compact hyperbolic surfaces, we took a nice subsequence which converges to a stable curve in the Deligne-Mumford compactification and also converging in the Gromov-Hausdorff sense (while fixing the diameter). Let us call such sequence of compact hyperbolic surfaces of genus $g(\geq 2)$ "strongly convergent sequence". We denote the set of such strongly convergent sequences of compact hyperbolic Riemann surfaces as[4] $\mathcal{SM}_g$.

[4]Here, $\mathcal{S}$ stands for a sequence.

Definition 2.16 For the positive integer $g \geq 2$, let S_g be the set of the underlying metric spaces of the metrized graphs which appear as the Gromov-Hausdorff limits of sequences of compact Riemann surfaces of genus $g(\geq 2)$, and associate Gromov-Hausdorff distance structure on it.

Note that S_g is also compact by Theorem 2.4 and the simple fact that S_g is closed under the Gromov-Hausdorff convergence.

Then what we have constructed in the proof of Theorem 2.4 is the following two kinds of limiting maps

$$r : \mathcal{SM}_g \to \overline{M_g}^{\mathrm{DM}} \tag{7}$$

which maps $\{R_i\}$ to the limit (Deligne-Mumford) stable curve and

$$t : \mathcal{SM}_g \to S_g \tag{8}$$

which maps $\{R_i\}$ to the Gromov-Hausdorff limit. Furthermore, we proved that r and t are compatible in the sense that the underlying graph of $t(\{R_i\})$ is a contraction of the dual graph of the limit stable curve $r(\{R_i\})$.

On the other hand, in the recent paper [1] by Abramovich-Caporaso-Payne, the following is proved.

> Fix an algebraically closed base field k with trivial valuation. If we consider the Berkovich analytification $\overline{M_g}^{an}$ [2] of the Deligne-Mumford compactification $\overline{M_g}$, then the deformation retraction to the Berkovich skeleton [3] is the "tropicalization" map towards the moduli of tropical curves of genus g.

Note that the Berkovich analytification parametrises stable curves over valuation fields which contains k (with trivial valuation) and it can be regarded as (a subspace of) this as an "algebro-geometric" analogue of the set of strongly convergent sequence of compact Riemann surfaces $\mathcal{SM}_g$. From this viewpoint, their tropicalization (deformation retract) is an analogue of our map t. The analogue of r in the Berkovich geometric setting [1] is the reduction map $\overline{M_g}^{an} \to \overline{M_g}^{\mathrm{DM}}$.

2.3 The Construction of $\overline{M_g}^{\mathrm{GH}}$

We define our *Gromov-Hausdorff compactification* of the moduli space of curves, first set-theoretically as

$$\overline{M_g}^{\mathrm{GH}} := M_g \sqcup S_g.$$

Recall that we have defined S_g in Definition 2.16 as the moduli space of the underlying metric spaces of the metrized graphs which appear as the Gromov-Hausdorff limits of sequences of compact Riemann surfaces of genus $g(\geq 2)$. Then we put a topology on it, whose open basis consists of the following two kinds of subsets:

(i) open subsets of M_g (with respect to the complex analytic topology) and
(ii) the metrics balls with centers are in S_g.

What we mean by the metric ball, with its center $[G] \in S_g \subset \overline{M_g}^{\mathrm{GH}}$ (G is a metrized graph) and radius $r \in \mathbb{R}_{>0}$, is simply defined as

$$B([G], r) := \{[C] \in \overline{M_g}^{\mathrm{GH}} \mid d_{\mathrm{GH}}([C], [G]) < r\}.$$

The obtained topological space $\overline{M_g}^{\mathrm{GH}}$ is compact due to our Theorem 2.4. It also satisfies the Hausdorff separation axiom simply because the Gromov-Hausdorff limit as compact metric space is unique as general theory (cf., [9]).

The readers may wonder why we do not simply use the notion of the metric completion above. However, note that the complex conjugate $\iota \in \mathrm{Aut}(\mathbb{C}/\mathbb{R})$ reverses the natural orientation of the corresponding Riemann surface, which does not change it metric space structure. A subtle technical point here is that $\overline{M_g}^{\mathrm{GH}}$ is not exactly the metric completion with respect to the Gromov-Hausdorff topology, of the set of compact Riemann surfaces of genus g by regarding the Riemann surfaces just as metric spaces. That is because it would discard the complex structures and ignore the effect of ι above (cf., e.g., [47, 49]).

Recall that S_g is defined as the moduli spaces of the underlying metric spaces of our limit metrized graphs as in Theorem 2.4. For each finite (metrized) graph Γ, let us denote the number of 1- valent vertices by $v_1(\Gamma)$ and denote the first betti number of Γ by $b_1(\Gamma)$. Then, more specifically and concretely, S_g can be described as follows.

Proposition 2.17 *The metric spaces parametrized by S_g can be characterized by a purely topological condition that the underlying topological spaces of the metrized graphs satisfy $v_1(\Gamma) + b_1(\Gamma) \leq g$.*

Note there is a subtle distinction between the metrized graph and the underlying metric space, which is simply a 1-dimensional CW complex with a metric. The reason is that the underlying metric space does *not* see the 2-valent vertices. It is also not enough to consider metrized graphs without 2-valent vertices since a circle can not be obtained in that way.

Proof of Proposition 2.17 From Theorem 2.4, we only need to specify the class of dual graphs of stable curves with genus g.

A stable curve C of genus g whose irreducible decomposition is $\cup_i C_i$ with dual graph Γ satisfies

$$g = \sum_i g(C_i^{\nu}) + b_1(\Gamma), \tag{9}$$

where ν denotes the normalization and b_1 denotes the first Betti number. From the stability condition, for each component C_i which corresponds to a 1-valent vertex of Γ, $g(C_i^{\nu}) \geq 1$. This is essentially the only numerical stability condition. Thus we have $g = \sum_i g(C_i^{\nu}) + b_1(\Gamma) \geq v_1(\Gamma) + b_1(\Gamma)$. Tracing back the above discussion, it is also easy to see that it is a sufficient condition as well. □

Remark 2.18 One remark, which the author hopes to be useful, is that in the above characterisation of metrized graphs which are parametrised in S_g, rather than putting the "diameter 1" condition, it may be easier to impose that "the sum of lengths of edges is 1" when we try to concretely describe the structure of our compactifications. Note that these two moduli spaces are naturally homeomorphic, simply by rescaling the metrics on our metrized graphs.

3 Related Moduli Spaces and Comparison

In this section, we further study $\overline{M_g}^{\mathrm{GH}}$ somewhat indirectly by comparing with other moduli spaces in literatures, and also construct some variants of $\overline{M_g}^{\mathrm{GH}}$ on the way, including what we call tropical geometric compactifications and denote by $\overline{M_g}^{\mathrm{T}}$.

3.1 Comparison with Tropical Moduli Spaces

Recently Brannetti-Melo-Viviani [8] constructed moduli space M_g^{tr} of the weighted metrized graphs, i.e., $(\Gamma, w\colon V(\Gamma) \to \mathbb{Z}_{\geq 0})$ of where Γ is a metrized graph such that

- 1-valent vertex v must have $w(v) \geq 2$,
- 2-valent vertex v must have $w(v) \geq 1$,
- $b_1(\Gamma) + \sum_{v \in V(\Gamma)} w(v) = g$,

with a natural topology (as well as some finer "stacky fan" structure) on it. Caporaso [10] introduced its log versions $M_{g,n}^{trop}$. See [8, 10] for the details. The moduli space M_g^{tr} is similar to our boundary S_g but there is an essential difference which is the presence of the weight function w above that morally encodes *genus* of each component of the limit Deligne-Mumford stable curves.

Similarly to what is done in [8, 10, 15], the combinatorial type of the underlying graph of a metrized graph gives a stratification on S_g such that each strata is a finite quotient of a simplex. A basic property of our moduli space S_g is the following.

Proposition 3.1 *The function $S_g \ni [\Gamma] \mapsto v_1(\Gamma) + b_1(\Gamma)$ is a lower semicontinuous function on S_g with respect to the Gromov-Hausdorff topology which has been previously considered.*

Proof The assersion follows easily from Theorem 2.4 combined with the precompactness (Corollary 2.2) but let us give a more straightforward combinatorial proof.

It is enough to see that if we contract one edge e, the function $v_1 + b_1$ does not increase. If the edge e is a loop, then the process decreases b_1 by 1 and v_1 increases at most 1. If the edge e is not a loop, then the contraction does not change the homotopy type of the graph so that it keeps b_1 unchanged, and v_1 does not increase (it may decrease by 1 or 2). □

Note that through the modular interpretations, there is a sequence of *canonical* closed embeddings

$$S_g \hookrightarrow S_{g+1} \hookrightarrow \cdots , \tag{10}$$

while other compactifications of moduli of curves and the moduli of *weighted* tropical curves by [8, 10, 11] do *not* have this chain of canonical inclusions.

Inside the moduli space M_g^{tr} of (weighted) tropical curves in the sense of [8], let us consider the closed locus S_g^{wt} which parametrizes those with the diameter 1 ("*wt*" of S_g^{wt} stands for weights.)

Proposition 3.2 *We have natural morphisms as follows.*

$$\partial M_g^{tr} := M_g^{tr} \setminus \{a \text{ point with weight } g\} \cong S_g^{wt} \times \mathbb{R}_{>0} \twoheadrightarrow S_g^{wt} \twoheadrightarrow S_g. \tag{11}$$

S_g has a finite stratification which satisfies that each strata is a finite group quotient of an open simplex and S_g is "purely" $(3g-4)$ dimensional for each $g (\geq 2)$ in the sense that, if we denote the union of $(3g-4)$-dimensional strata as $S_g^{oo} \subset S_g$, then it is an open dense subset. In addition, the last morphism of (11) is a proper map such that each fiber over S_g^{oo} is finite.

Proof A tropical curve in the sense of [8] has finite non-zero diameter unless it is a point, so that we get the first isomorphism. Secondly, starting from a tropical curve which is not topologically a point, just by forgetting the weights and the 2-valent vertices, we get the underlying metric space of a metrized graph. It defines the last morphism $S_g^{wt} \twoheadrightarrow S_g$, which we denote as r. It follows straightforward from the topology on S_g^{wt} in [8] that this morphism is continuous and this is surjective by Proposition 2.17. From the compactness of S_g^{wt} and S_g, it follows automatically that the morphism is proper. Note that for any point p in S_g^{wt} which has 2-valent vertices, $r^{-1}(r(p))$ is non-finite. It is because that for each metric space X corresponding to a point in S_g, if it is underlying metric space of certain weighted tropical curve (weighted metrized graph) Γ parametrized in S_g^{wt}, once we know the locations of vertices in X, there are only finite choices of Γ which corresponds to the decomposition of $g - b_1(X)$ into non-negative integer weights attached to the vertices.

It is easy to see that S_g has a natural finite stratification by the homeomorphic class of the underlying graphs. Each strata can be seen as the moduli of metrized graphs with the same underlying graph, with the sum of the length of edges are

1 by rescaling the metrics. Hence it is homeomorphic to the quotient of an open simplex with respect to a linear action of a finite group, which is the automorphism group of each graph. Next we proceed to the proof of the fact that S_g is purely $3g - 4$-dimensional as in the statement of Proposition 3.2. Indeed, for any given (underlying metric space of) a metrized graph Γ of the diameter 1 which satisfies $v_1(\Gamma) + b_1(\Gamma) < g$, by attaching small circles or short edges and rescaling, the corresponding point $[\Gamma] \in S_g$ can be easily perturbed to a point inside the strata with $v_1 + b_1 = g$. The strata can be easily checked to have dimension $3g - 4$, as $3g - 3$ is the number of edges inside Γ following elementary graph theory. This fact is also well known in the algebro-geometric field of study of the so-called Mumford curves. Thus, the union S_g^{oo} of $(3g-4)$-dimensional cells form open dense subset. For each $p = [\Gamma] \in S_g^{oo}$, the r-fiber $r^{-1}(r(p)) = \{p\}$ since for a point $[\Gamma']$ in the fiber, the vertices of the graph Γ' are nothing but the non-smooth points of $r(\Gamma') = r(\Gamma)$ as an underlying topological space and furthermore Γ' does not have any positive weights on the vertices because of the formula (9). We complete the proof of Proposition 3.2. □

3.2 Construction of $\overline{M_g}^{\mathrm{T}}$

It is possible to modify our construction of $\overline{M_g}^{\mathrm{GH}}$ to make more compatibility with the above "*weighted* tropical moduli spaces" of [8, 10, 11]. That is, for a collapsing sequence of genus g compact Riemann surfaces as in Theorem 2.4, we can encode the information of the genera of the irreducible components of the limiting stable curves on the limiting graph. More precisely speaking, first we consider the set

$$\overline{M_g}^{\mathrm{T}} := M_g \sqcup S_g^{wt},$$

on which we put a topology as follows. A subset C of $\overline{M_g}^{\mathrm{T}}$ is *closed* if and only if

- $C \cap S_g^{wt}$ is closed in S_g^{wt} and
- any Gromov-Hausdorff collapsed graphs of compact Riemann surfaces which are in $C \cap M_g$, attached with the genera of components of the normalization of the limit stable curve in [16] sense, which we suppose to exist, is actually in $C \cap S_g^{wt}$.

The compactness, the Hausdorff property of $\overline{M_g}^{\mathrm{T}}$, and the fact that M_g is open and dense inside $\overline{M_g}^{\mathrm{T}}$ all follow straightforwardly from our Theorem 2.4 and its proof. We would like to call this compactification $\overline{M_g}^{\mathrm{T}}$ of M_g as the *tropical geometric compactification* of M_g.

From the construction we have a natural continuous surjective map

$$\overline{M_g}^{\mathrm{T}} \twoheadrightarrow \overline{M_g}^{\mathrm{GH}},$$

which restricts to the identity map on the open subset M_g.

3.3 Finite Join $\overline{M_{\leq g}}^{\mathrm{GH}}$ and Infinite Join $\overline{M_{\infty}}^{\mathrm{GH}}$

An interesting phenomenon is that, as the following definitions show, our Gromov-Hausdorff compactification $\overline{M_g}^{\mathrm{GH}}$ naturally patches together for different g thanks to the sequence of the canonical inclusions (10) of S_g.

Definition 3.3 The *finite joins* of our Gromov-Hausdorff compactifications are defined inductively as topological spaces

$$\overline{M_{\leq 0}}^{\mathrm{GH}} := \overline{M_0}^{\mathrm{GH}} = \{ \text{ Riemann sphere } \mathbb{CP}^1\} \text{ (singleton)},$$

$$\overline{M_{\leq 1}}^{\mathrm{GH}} := \overline{M_1}^{\mathrm{GH}} := M_1 \sqcup \{S^1(\frac{1}{2\pi})\}(= \overline{A_1}^{\mathrm{T}} \text{ in the next section })$$

(one point compactification)

and for $g \geq 2$ as

$$\overline{M_{\leq g}}^{\mathrm{GH}} := (\overline{M_{\leq (g-1)}}^{\mathrm{GH}} \cup \overline{M_g}^{\mathrm{GH}})/\sim,$$

where the equivalence relation $\sim$ is simply the identification of the closed subset $S_{g-1} \subset S_g$ and another closed subset $S_{g-1} \subset \overline{M_{\leq (g-1)}}^{\mathrm{GH}}$. From the definition, we have natural inclusion relations

$$\cdots \overline{M_{\leq (g-1)}}^{\mathrm{GH}} \subset \overline{M_{\leq g}}^{\mathrm{GH}} \cdots .$$

Then we set

$$\overline{M_{\infty}}^{\mathrm{GH}} := \varinjlim_{g} \overline{M_{\leq g}}^{\mathrm{GH}} = \cup_g \overline{M_{\leq g}}^{\mathrm{GH}},$$

and call it *infinite join* of our Gromov-Hausdorff compactifications.

The boundary of $\overline{M_\infty}^{\mathrm{GH}}$ by which we mean the natural subset $\cup_g(\partial\overline{M_g}^{\mathrm{GH}} = S_g)$, should be regarded as a tropical version of the space[5] "M_∞" introduced and studied recently by Ji-Jost [27].

Also note that $\overline{M_\infty}^{\mathrm{GH}}$ is connected and all our Gromov-Hausdorff compactification $\overline{M_g}^{\mathrm{GH}}$ is inside this infinite join.

3.4 Comparison with the Outer Spaces

There is a classical theory of the *outer space* X_n by Culler-Vogtman [15], which is an analogue of the Teichmuller space for metrized graphs. There, the analogous discrete group to the mapping class group is the outer automorphism group $\mathrm{Out}(F_n)$ of the free group F_n with rank n. From now on, we use g instead of their n to unify our notation.

Recall that the quotient $X_g/\mathrm{Out}(F_g)$ parametrizes graphs Γ with $b_1(\Gamma) = g$ with $v_1(\Gamma) = 0$.

We introduce another moduli space of graphs as a subset of S_g (with the induced topology) as

$$S_g^o := \{\Gamma \in S_g \mid v_1(\Gamma) + b_1(\Gamma) = g\} \subset S_g.$$

It is simply the complement of $S_{g-1} \subset S_g$ by the definition. The following proposition essentially goes back to [13].

Proposition 3.4 *There is a canonical cellular open embedding $X_g/\mathrm{Out}(F_g) \hookrightarrow S_g^o (\subset S^g)$. The image of $X_g/\mathrm{Out}(F_g)$ is open dense in S_g (thus so is S_g^o).*

Proof First of all, it follows from the lower semicontinuity of the first Betti number of metrized graphs $b_1(\Gamma)$ that $X_g/\mathrm{Out}(F_g)$ is an open subset of S_g^o. For each $\Gamma \in S_g^o$ with $v_1(\Gamma) + b_1(\Gamma) = g$ and $0 < \epsilon \ll 1$, we define graph(s) $\phi_\epsilon(\Gamma)$ as follows. For each leave vw where v is a 1-valent vertix, we put a small loop of length $\epsilon l(vw)$. Doing the same for all edges and rescale the metric on whole graph to make its diameter 1, we get a metrized graph which we denote as $\phi_\epsilon(\Gamma)$. This construction naturally defines a perturbation of elements of S_g^o to those of $X_g/\mathrm{Out}(F_g)$. The fact that all of these are unions of relative interiors of the cells with respect to that CW complex structure follow straightforward from the definitions.

We also need to prove S_g^o is dense inside S_g. We provide an elementary proof for convenience. Let us analyze the neighborhood of $\Gamma \in S_{g-1} \subset S_g$. Starting from any such Γ with a point $p \in \Gamma$, we can similarly consider Γ's deformation $\psi_t(\Gamma) \in X_g/\mathrm{Out}(F_g)$ for $t > 0$, for example, as follows. Set $v_1(\Gamma)+b_1(\Gamma) = g-d$. Taking a point p, we define $\psi_t(\Gamma)$ as a union of Γ and a bouquet i.e., the union of d

[5] They call it "universal moduli spaces".

length t loops which passes through p. Thus in particular $X_g/\mathrm{Out}(F_g)$ is open and dense in S_g and hence so is S_g^o as well. □

Notes added, part We end this section with the following notes added, about the relation with [32] which was kindly taught by its author L.Lang in June of 2015. I appreciate him for informing it.

Remark 3.5 L. Lang defined "tropical convergence" of compact Riemann surfaces to metrized graphs as the convergence of the ratios of the lengths of shrinking geodesics, which represent vanishing cycles, in his [32, Definition 1.1]. As also written in [32, v2, §1.3], that notion of convergence is *not* equivalent to ours, i.e. Gromov-Hausdorff convergence of hyperbolic metrics. See more details on the original paper [32]. The author also gives more detailed arguments in [45, §3].

4 Investigating Topology

We would like to make the first step of investigation of the topology of our compactifications and their boundaries.

First, we recall the fact that the moduli space of smooth projective curves has vanishing higher homology groups, proved by J. Harer [25]. His proof shows the existence of a deformation retract via the cell complex structure of the Teichmuller space (the so called "arc complex").

Theorem 4.1 ([25, Theorem 4.1]) *For $g \geq 2$ and $i > 4g-5$, we have*

$$H_i(M_g;\mathbb{Q}) = 0 \text{ and } H^i(M_g;\mathbb{Q}) = 0.$$

So combined with the Poincaré-Lefschetz duality for orbifold, we get that for $i \leq 2g-2$

$$H_c^i(M_g;\mathbb{Q}) = 0 \text{ and } H_i^{\mathrm{BM}}(M_g;\mathbb{Q}) = 0,$$

where H_c^i denotes the cohomology group with compact supports and H_i^{BM} denotes the Borel-Moore homology group.

The above Theorem 4.1 has the following consequence.

Corollary 4.2 *For $i < 2g-2$, we have*

$$H^i(\overline{M_g}^{\mathrm{T}};\mathbb{Q}) = H^i(S_g;\mathbb{Q}),$$

$$H_i(\overline{M_g}^{\mathrm{T}};\mathbb{Q}) = H_i(S_g;\mathbb{Q}).$$

Proof It follows simply from the exact sequences of compactly supported cohomology groups or the Borel-Moore homology groups. □

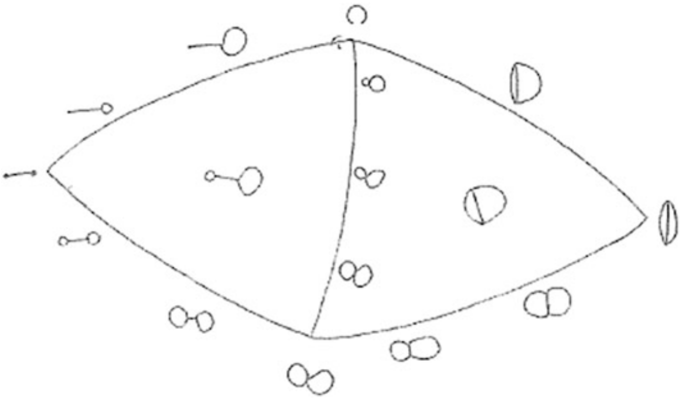

Fig. 1 The boundary S_2 of $\overline{M_2}^T$

Thus the study of homology and cohomology of our Gromov-Hausdorff compactification is reduced to that of the boundary for a specific range of degrees. Motivated by it, let us study the topology[6] of our boundary S_g. First, we sketch the following cases of small g.

Example 4.3 S_1 is just a point which stands for the circle of length 1. S_2 is a two 2-simplices (triangles) patched together along one of their edges for each. In one side of the 2-simplex, the inner points parametrize a union of two circles and a segment connecting them. The other side of the 2-simplex, the inner points parametrize a union of circle with a segment connecting two points in the circle. We refer to the picture below, where the parametrized metrized graphs are pictured around each stratum (Fig. 1).

Note that obviously S_1 and S_2 are both contractable.

Since the open dense locus S_g^o of S_g is a rational classifying space of $\mathrm{Out}(F_g)$ as known to [15], it has in general highly nontrivial topology. Indeed its cohomology is those of $\mathrm{Out}(F_g)$ (cf., e.g., [18] for non-vanishing cohomology for $g = 5$ case), we expect interesting topological structure on S_g for large g.

We define

$$S_\infty := \varinjlim S_g = \cup_g S_g,$$

the injective limit with respect to the canonical embeddings $S_{g-1} \hookrightarrow S_g \hookrightarrow S_{g+1} \cdots$ (cf., (10)). After a kind suggestion of the referee, the author learnt that our

[6] A while after the appearance of the first version of this paper as arXiv:1406.7772, Chan-Galatius-Payne [12] appears which systematically studies the topology of the moduli of weighted metrized graphs *with* $n(>0)$*-marked points* i.e. the "log version" of S_g^{wT}.

S_∞ can be informally (but not logically) seen as a tropical analogue of the infinite union of the classical moduli spaces studied in [14, 27].

While we expect that each S_g has highly nontrivial topologies in general, we observe the following.

Theorem 4.4 *The topological space S_∞ is contractible. In particular, for any $k \geq 0$, $\varinjlim_g H_k(S_g; \mathbb{Q}) = 0$.*

Proof Consider the cone of S_g, i.e., $CS_g := (S_g \times [0,1])/(S_g \times \{1\})$. It is enough to construct a series of continuous maps $\{\phi_g : CS_g \to S_\infty\}_{g \geq 2}$ which satisfies

- (i) ϕ_g maps $(S_g \times \{1\})$ to a point as $\phi_g(S_g \times \{1\}) = \{$the unit interval $[0,1]$ (as a metrized graph)$\}$,
- (ii) $\phi_g|_{S_g \times \{0\}} = \mathrm{id}|_{S_g}$,
- (iii) and $\phi_{g+1}|_{CS_g} = \phi_g$.

Indeed, from the third condition, they glue together to form a continuous map

$$\phi_\infty : CS_\infty \to S_\infty$$

and this gives a deformation retract of S_∞ into a point of S_∞ which corresponds to the unit interval $[0,1]$ again as a metrized graph.

We construct the map ϕ_g by the following three steps.

Step 1 (Adding vertices) First we construct $\phi_g|_{S_g \times [0,\frac{1}{3}]}$. For any $(\Gamma, t) \in S_g \times [0, \frac{1}{3}]$, suppose the set of vertices of Γ is $V(\Gamma) = \{p_1, \cdots, p_m\}$ and the set of edges is $E(\Gamma) = \{e_1, \cdots, e_n\}$. We define a new metrized graph $\psi_g(\Gamma, t)$ for $t \in (0, \frac{1}{3}]$ by setting the vertices set as $\{p_1, \cdots, p_m\} \sqcup \{p'_1, \cdots, p'_m\}$ and define the set of edges and their lengths as follows. The set of edges of $\psi_g(\Gamma, t)$ is $E(\Gamma) \sqcup \{\overline{p_i p'_i} \mid 1 \leq i \leq m\}$. We call an edge in $E(\Gamma) \subset E(\psi_g(\Gamma, t))$ as *old edge* in this proof, while the edges of the form $\overline{p_i p'_i}$ will be called *new edges*. We put their length $l(\overline{p_i p'_i}) = t$ while we keep the length of old edges as the same as Γ. Then we rescale the length of all edges of $\psi_g(\Gamma, t) (0 < t \leq \frac{1}{3})$ to make the diameter 1 and denote the obtained metrized graph as $\phi_g(\Gamma, t)$. Note that the image of $\phi_g|_{S_g \times [0,\frac{1}{3}]}$ is a priori *not* inside S_g. Indeed, while the metrized graphs parametrized in S_g are characterized by $v_1 + b_1$ by Proposition 2.17, we have that

$$v_1(\phi_g(\Gamma, t)) = \# V(\Gamma),$$

which is bigger than $v_1(\Gamma)$ if and only if Γ is not homeomorphic to the closed interval. This $\phi_g|_{S_g \times [0,\frac{1}{3}]}$ is continuous from the construction.

Step 2 (Contraction of old edges) Our next step is the construction of $\phi_g|_{S_g\times[\frac{1}{3},\frac{2}{3}]}$. Roughly speaking, in this step of t increasing from $\frac{1}{3}$ to $\frac{2}{3}$, we gradually contract the old edges i.e., those which belong to $E(\Gamma)$. We make this rigorous as follows.

First, as in Step 1, we construct $\psi_g(\Gamma,t)$ for $t\in[\frac{1}{3},\frac{2}{3}]$ by setting its vertices set and edges set as

$$V(\psi_g(\Gamma,t)) := V(\phi_g(\Gamma,\frac{1}{3}))$$
$$= \{v_1,\cdots,v_m\}\sqcup\{w_1,\cdots,w_m\} \text{ for } t\in\Big[\frac{1}{3},\frac{2}{3}\Big),$$
$$V(\psi_g(\Gamma,t)) := \{v\}\sqcup\{w_1,\cdots,w_m\} \text{ for } t=\frac{2}{3},$$
$$E(\psi_g(\Gamma,t)) := E(\phi_g(\Gamma,\frac{1}{3}))$$
$$= E(\Gamma)\sqcup\{\overline{v_iw_i}\mid 1\le i\le n\} \text{ for } t\in\Big[\frac{1}{3},\frac{2}{3}\Big),$$
$$E(\psi_g(\Gamma,t)) := \{\overline{vw_i}\mid 1\le i\le n\} \text{ for } t=\frac{2}{3}.$$

Then we put the metrics on the edges of $\phi_g(\Gamma,t)$ as follows.[7]

$$\text{length}(\overline{v_iw_i};\phi_g(\Gamma,t)) := \frac{1}{3},$$
$$\text{length}(\overline{v_iv_j};\phi_g(\Gamma,t)) := (2-3t)\text{length}(\overline{v_iv_j};\Gamma).$$

The above construction of $\psi_g(\Gamma,t)$ realizes shrink of old edges in $\phi_g(\Gamma,\frac{1}{3})$. Then finally we define the metrized graph $\phi_g(\Gamma,t)$ as rescale of $\psi_g(\Gamma,t)$ with the diameter 1.

From the construction, the continuity of $\psi_g|_{S_g\times[\frac{1}{3},\frac{2}{3}]}$ and $\phi_g|_{S_g\times[\frac{1}{3},\frac{2}{3}]}$ are obvious. The limit graph $\phi_g|_{t=\frac{2}{3}}$ is a metrized tree whose edges all share a common vertex so that its shape looks like "$*$". Precisely speaking, it is a metrized graph graphs whose

- vertices set is $\{v\}\sqcup\{w_i\mid 1\le i\le m\}$ and
- edges set is $\{\overline{vw_i}\mid 1\le i\le m\}$.

Let us call this type of tree "$*$-type" with $n(=\#E(\Gamma))$ leaves.

Step 3 (Deforming to the unit interval) The final step is the construction of $\phi_g|_{S_g\times[\frac{2}{3},1]}$. The moduli space of $*$-type trees Γ (as we defined and discussed above in Step 2) with n leaves of diameter 1, with the Gromov-Hausdorff topology, is

[7]The notation of the following is that the length of edge l in a graph G is denoted as length(l,G).

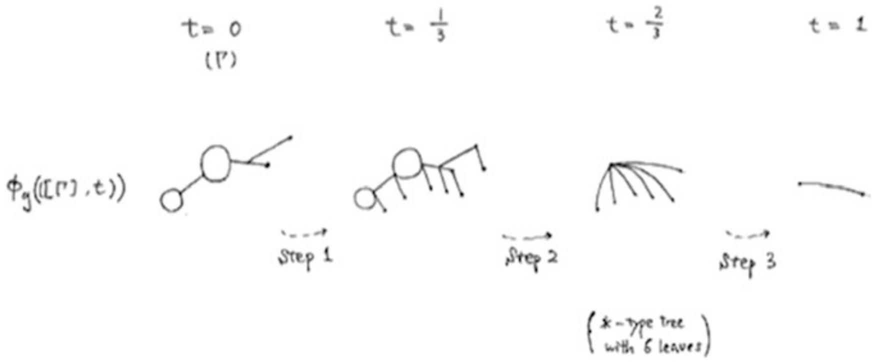

Fig. 2 Picture proof of Theorem 4.4

homeomorphic to the moduli space of those whose sum of lengths of edges is 1, simply by rescaling. And the latter is the simplex

$$\Delta_n := \{(x_1, \cdots, x_n) \mid 0 \leq x_1 \leq x_2 \leq \cdots x_n \leq 1, \sum_{i=1}^{n} x_i = 1\}.$$

The contractability of the simplex above ensures, or we can directly see that there is a deformation retract of each $\Gamma \in \Delta_n$ to the interval [0, 1]. This gives $\phi_g|_{S_g \times [\frac{2}{3}, 1]}$.

The desired properties (i), (ii), (iii) are all straightforward from the construction. We complete the proof of Theorem 4.4. To help understanding for the readers, we summarize our three Steps below as an example picture (Fig. 2). □

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A Partial Comparison of Stability Notions in Kähler Geometry

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Abstract In this follow up work to Dyrefelt (J Geom Anal, 2017. https://doi.org/10.1007/s12220-017-9942-9), Dervan and Ross (Math Res Lett 24, 2017), Dervan (Math Ann, 2017. https://doi.org/10.1007/s00208-017-1592-5), and Sjöström Dyrefelt (Int Math Res Not 2018. https://doi.org/10.1093/imrn/rny094) we introduce and study a notion of geodesic stability restricted to rays with prescribed singularity types. A number of notions of interest fit into this framework, in particular algebraic- and transcendental K-polystability, equivariant K-polystability, and the geodesic K-polystability notion introduced by the author in Sjöström Dyrefelt (Int Math Res Not 2018. https://doi.org/10.1093/imrn/rny094). We provide a partial comparison of the above notions, and show equivalence of some of these notions provided that the underlying manifold satisfies a certain 'weak cscK' condition. As an application this proves K-polystability of a new family of cscK manifolds with irrational polarization.

Keywords K-stability · Geodesic stability · Constant scalar curvature · Kähler metrics · Yau-Tian-Donaldson conjecture

1 Introduction

An important open problem in Kähler geometry is the Yau-Tian-Donaldson (YTD) conjecture, which predicts that existence of canonical metrics (in the sense of Calabi [12]) is equivalent to a suitable stability notion in algebraic geometry. In the case of Fano manifolds $(X, -K_X)$ equipped with the anticanonical polarization the conjecture was proven with respect to the classical algebraic notion of K-stability with roots in geometric invariant theory [19–21]. For polarized manifolds (X, L), or even completely arbitrary Kähler manifolds (X, ω), finding the precise stability notion that makes the conjecture hold is then a central part of the problem. Indeed,

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it is then debated which ones of a rapidly growing number of proposed stability notions (transcendental/algebraic/equivariant/filtration/uniform K-stability) should be the most relevant to the conjecture, and the relationship between these a priori differing notions is largely unexplored. In this follow up work to [31, 32, 43, 44] we aim to address this knowledge gap, by means of comparing some of the central stability notions in Kähler geometry to each other.

We will focus on the case of constant scalar curvature Kähler (cscK) metrics and related algebro-geometric stability notions. In particular we will investigate possible natural comparisons with the geodesic stability notion used in the recent proof of the properness conjecture, due to Chen-Cheng [17, 18]. In this context, a notion of special interest to us is the notion of *geodesic K-polystability*, which was introduced in [44]. This is a new notion that means that $(X, [\omega])$ is K-semistable, and moreover, the Donaldson-Futaki invariant vanishes precisely for the test configurations whose "associated geodesic ray" is induced by a holomorphic vector field (in a sense made precise in the aforementioned paper [44]). As such, it can be interpreted as a weaker version of the geodesic stability notion used by Chen-Cheng [17], which is in turn known to be equivalent to existence of cscK metrics [16–18]. This is particularly interesting in order to better understand the relationship between geodesic stability and the algebraic notions of K-polystability. Ultimately, such a comparison is precisely what is required to prove or disprove the YTD conjecture.

In view of the classical correspondence between geodesic rays and test configurations, see e.g. [1, 3, 7, 9, 23, 32, 40, 42–44] and references therein, there are a number of reasons to believe that geodesic K-polystability is a natural stability notion. First of all, it was proven in [7, 44] that constant scalar curvature Kähler (cscK) manifolds are geodesically K-polystable, thus proving one direction of a natural YTD conjecture in this setting. Moreover, if $\mathrm{Aut}_0(X) = \emptyset$ and the underlying class is cscK, then geodesic K-polystability is equivalent to the usual K-polystability notion [44]. It was also checked by R. Dervan in an appendix to [44] that geodesic K-polystability implies equivariant K-polystability (as introduced in his paper [31]), generalizing a notion introduced in [34, 46], which is conjectured to be equivalent to the cscK condition. When $\mathrm{Aut}_0(X) \neq \emptyset$ or the underlying Kähler class is not cscK, the relationship to the full non equivariant K-polystability notion however remains an open problem (of importance to understanding the YTD conjecture).

To study the above stability notions we introduce the terminology of *stability loci* in the Kähler cone: Denote by *K-polystable locus* the set of Kähler classes α in the Kähler cone of X such that (X, α) is K-polystable, and use similar terminology for other stability notions. Likewise, we say that the *cscK locus* is the set of all Kähler classes α on X which contain a cscK metric. In particular, the YTD conjecture then translates to the statement that the cscK locus coincides with the K-polystable locus. This way stability may be considered not as a question on a single given polarization, but as a question about characterizing a certain subset of the Kähler cone. This is sometimes a useful point of view, as we shall see in this note. We may in particular ask the following broad but central questions: How can we compare the cscK locus and the various stability loci? What stability loci coincide in the Kähler cone of X (i.e. which stability notions are equivalent)? In this note we will give some partial answers to the second part of this question, and set up the framework for continuing to study such problems in future work.

1.1 On Comparing Geodesic Stability and K-Stability

The main results of this paper are partial results towards comparing geodesic stability in the sense of Chen-Cheng [18] with classical K-stability notions, as well as (transcendental) K-polystability of $(X, [\omega])$ in the sense of [44]. The status of the comparison problem for stability notions in Kähler geometry is as follows: For arbitrary compact Kähler manifolds (X, ω) (such that the associated Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$ is possibly irrational) we have inclusions

$$\text{cscK locus} \subseteq \text{Geodesically K-polystable locus},$$

and it is straightforward to see that K-polystable locus $\subseteq$ Geodesically K-polystable locus. It is however open whether the cscK locus is included in the K-polystable locus, and it is unknown what is the precise relationship between the geodesically K-polystable locus and the K-polystable locus (especially if the underlying class does not admit a cscK metric). These are questions that concern the relationship between test configurations and their associated geodesic rays in the space of Kähler metrics. Indeed, the problem here posed is equivalent to asking if a test configuration is a product (in the sense that $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$) precisely if its associated geodesic ray is induced by a holomorphic vector field on X (Definition 2.3). In this paper we observe that it is enough to show K-polystability is equivalent to geodesic K-polystability for any given privileged polarization $(X, [\omega])$, and the above equivalence will automatically extend to the whole Kähler cone of X. As a first main result, we prove the following:

Main Theorem 1 *Let (X, ω) be a compact Kähler manifold and suppose that the K-polystable locus $\neq \emptyset$. Then $(X, [\omega])$ is K-polystable if and only if it is geodesically K-polystable.*

In particular, this gives a partial answer to the question of comparing K-polystability and geodesic K-polystability. In light of [44, Theorem 1.1] we also have the following first result of K-polystability for cscK manifolds that are not necessarily polarized and are allowed to admit holomorphic vector fields:

Corollary 1.1 *Let (X, ω) be a cscK manifold with K-polystable locus $\neq \emptyset$. Then $(X, [\omega])$ is K-polystable.*

Note that his proves one direction of the YTD conjecture for a new family of compact Kähler manifolds (X, ω) with irrational polarization, i.e. when $[\omega] \in H^{1,1}(X, \mathbb{R})$ is an arbitrary Kähler class on X not necessarily in the rational lattice $H^2(X, \mathbb{Q})$. As part of the proof we in particular obtain the following result, which sheds additional light on the connection between geodesic rays and test configurations, extending results of [7].

Theorem 1.2 *Let* (X, ω) *be a compact Kähler manifold and suppose that the K-polystable locus* $\neq \emptyset$*. Suppose that* $(\mathcal{X}, \mathcal{A})$ *is a test configuration for* $(X, [\omega])$*. Then the following are equivalent:*

- $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$
- *The associated geodesic ray is induced by a holomorphic vector field on* X (Definition 2.3).

Main Theorem 1 and Theorem 1.2 together strengthen the expectation that the notions of K-polystability and geodesic K-polystability are equivalent in general. Combined with [44, Theorem 1.1], Theorem 1.2 moreover reduces the statement "cscK manifolds are K-polystable", which is an important problem still open for arbitrary Kähler manifolds, to understanding whether the K-polystable locus is non-empty.

As a natural family of examples we may consider compact Kähler manifolds that we shall refer to as *weakly cscK*, i.e. such that the cscK locus $\neq \emptyset$ in the Kähler cone $\mathcal{C}_X$ of X. Indeed, polarized weakly cscK manifolds (X, L) satisfy the hypothesis that the K-polystable locus $\neq \emptyset$, since cscK then implies K-polystability (see [3, 7]). Moreover, there are many interesting concrete examples of weakly cscK manifolds; in particular any Kähler-Einstein Fano manifold is weakly cscK. We have the following immediate corollary of Theorem 1 for weakly cscK polarized manifolds:

Theorem 1.3 *Let* (X, L) *be a polarized weakly cscK manifold. Then*

(1) (X, L) *is K-polystable if and only if it is geodesically K-polystable.*
(2) (X, L) *is equivariantly geodesically K-polystable if and only if it is equivariantly K-polystable.*

The stability notions referred to in (1) are the classical (algebraic) K-polystability and (algebraic) geodesic K-polystability for polarized manifolds, see Sect. 3.3 for precise definitions. In (2) we say that $(X, [\omega])$ is *equivariantly geodesically K-polystable* if and only if it is geodesically K-polystable with respect to equivariant test configuratons (see [31, 44] for the Kähler case). Hence, this extends result of [7] from the case of polarized cscK manifolds to weakly cscK polarized manifolds. Note further that the result (2) holds also for arbitrary compact Kähler manifolds, using the formalism for [32, 43]. The result (1) can be checked in this setting at least if the automorphism group is discrete, but it is not known in general. This remains an open question.

We also record the following comparison of stability notions, which holds even for non-polarized Kähler manifolds (X, ω) (for the compatibility notion see Sect. 3.4 and references therein):

Theorem 1.4 *Suppose that* (X, ω) *is a weakly cscK Kähler manifold with* $\mathrm{Aut}_0(X)$ *discrete. Then* $(X, [\omega])$ *is uniformly K-stable if and only if* $(X, [\omega])$ *is coercive* *with respect to the set of subgeodesic rays compatible with a relatively Kähler test configuration for* $(X, [\omega])$*. Likewise,* $(X, [\omega])$ *is K-stable if and only if* $(X, [\omega])$ *is geodesically stable with respect to the set of subgeodesic rays compatible with a relatively Kähler test configuration for* $(X, [\omega])$.

1.2 Equivalence of Notions of Product Configuration

Another corollary of the techniques of this paper concerns the equivalence of various notions of product configurations occurring in the literature. This is interesting in its own right, since it addresses the question of equivalence of several commonly seen (and a priori different) candidate notions of K-polystability. Indeed, these notions have in commmon that they ask that the so called Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{A})$ is non-negative for all test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α), with equality if and only if $(\mathcal{X}, \mathcal{A})$ is a "product", in a suitable sense. Addressing a question asked in the author's thesis [45], the following result proves that several commonly seen notions of product configuration are in fact equivalent:

Theorem 1.5 *Suppose that* (X, L) *is a polarized weakly cscK manifold. Let* $(\mathcal{X}, \mathcal{L})$ *be a relatively Kähler test configuration for* (X, L)*, with associated geodesic ray* $(\varphi_t)_{t\geq 0}$*. Then the following are equivalent:*

(1) $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$
(2) $\mathcal{X}_{\pi^{-1}(\Delta_r)} \simeq X \times \Delta_r$ *for each* $r > 0$*, where* $\Delta_r := \{z \in \mathbb{C} \mid |z| < r\}$.
(3) $\mathcal{X}_0 \simeq X$
(4) *The associated geodesic ray* $(\varphi_t)_{t\geq 0}$ *is induced by a holomorphic vector field* V *on* X.

As before, the point is that this holds even if the underlying Kähler class $c_1(L)$ does not admit any cscK metrics, as long as (X, L) is weakly cscK. In fact, note that if we want the K-polystable locus to contain the properness locus, then there is no choice but to define products as objects whose associated subgeodesic rays satisfy

$$\inf_{g\in G} \mathrm{J}(g.\varphi_t) = 0,$$

where $G := \mathrm{Aut}_0(X)$ is the connected component of the automorphism group of X, the action $g.\varphi$ on potentials is defined as in Sect. 2.1, and

$$\mathrm{J}(\varphi) := \int_X \varphi\omega^n - \frac{1}{n+1}\sum_{k=0}^{n}\int_X \varphi\omega^k \wedge \omega_\varphi^{n-k}.$$

This is why geodesic K-polystability is such a natural notion, because it is "the most obvious" notion satisfying the above requirement.

The techniques used to prove the above results rely on understanding paths of test configurations when changing the underlying class, and in particular the existence of such paths that preserve the associated geodesic ray.

1.3 Idea of the Proofs: Special Paths of Test Configurations and an Injectivity Lemma

Let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two Kähler classes on X. In order to study how K-polystability notions vary as we vary the underlying class, one has to understand how to relate test configurations for (X, α) to test configurations of (X, β). A first straightforward observation is the following (part (1) on convex combinations of test configurations should be compared to e.g. [35] in the setting of polarized manifolds, part (2) is a direct consequence of the intersection theoretic point of view due to [4, 32, 39, 43, 50] and part (3) is a direct consequence of [44]):

Theorem 1.6 *Let $\alpha, \beta \in \mathcal{C}_X$ and set $\alpha_s := (1-s)\alpha + s\beta$, for $s \in [0, 1]$. Suppose that $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$ is a relatively Kähler test configuration for (X, α) and (X, β) respectively. Then*

(1) *$(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$ is a relatively Kähler test configuration for (X, α_s).*
(2) *The maps $[0, 1] \ni s \mapsto \mathrm{DF}(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$ and $[0, 1] \ni s \mapsto \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$ are continuous.*
(3) *Suppose that $\rho_A(t)$ and $\rho_B(t)$ are the uniquely associated geodesic rays respectively, and write $\rho_s(t) := (1-s)\rho_A(t) + s\rho_B(t)$ If $\alpha_s = [\omega_s]$, then*

$$\mathrm{DF}(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B}) = \lim_{t \to +\infty} t^{-1} \mathrm{M}_{\omega_t}(\rho_s(t)) - ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n).$$

In practice, it is however not a given to know something about the set of test configurations for a different polarization than the one considered. A key question then becomes: *How can one relate the test configurations for (X, α) to the test configurations for (X, β)?* In this direction, we prove the following extended version of the injectivity lemma [44, Theorem 1.8] (now allowing for a change of the underlying class):

Main Theorem 2 *Let $\alpha := [\omega]$ and $\beta := [\theta]$ be Kähler classes on X. Suppose that there is a subgeodesic ray $\rho(t) \in \mathrm{PSH}(X, \omega) \cap \mathrm{PSH}(X, \theta)$ which is compatible with two relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ for (X, α) and (X, β) respectively. Then the canonical $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \to \mathcal{Y} \setminus \mathcal{Y}_0$ extends to an isomorphism $\mathcal{X} \to \mathcal{Y}$.*

A slightly more precise result relating test configurations of (X, α) to those of (X, β) is given below:

Main Theorem 3 *Let $\alpha \in \mathcal{C}_X$ and suppose that $(\mathcal{X}, \mathcal{A})$ is a relatively Kähler smooth and dominating test configuration for (X, α). Then, for each $\beta \in \mathcal{C}_X$ there* *is a* $\lambda > 0$ *such that* $\lambda\beta > \alpha$ *and a relatively Kähler test configuration* $(\mathcal{Y}, \mathcal{B})$ *for* $(X, \lambda\beta)$ *such that*

(1) $\mathcal{Y} = \mathcal{X}$,
(2) *The test configurations* $(\mathcal{X}, \mathcal{A}) \sim (\mathcal{Y}, \mathcal{B})$*, i.e. there is a geodesic ray* $\rho(t)$ *compatible with both.*

In particular, if $\alpha = [\omega]$ *and* $\lambda\beta = [\theta]$*, then we have*

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}) = \lim_{t\to+\infty} t^{-1}\mathrm{M}_{\omega}(\rho(t)) - ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n)$$

and

$$\mathrm{DF}(\mathcal{X}, \mathcal{B}) = \lim_{t\to+\infty} t^{-1}\mathrm{M}_{\theta}(\rho(t)) - ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{B}^n).$$

This result has a number of straightforward applications, in particular to proving Theorem 1.2.

1.4 Applications to the Topology of the K-Semistable and Uniformly K-Stable Loci

The techniques of this paper also yield some basic properties of the K-semistable and uniformly K-stable loci, where uniform K-stability is defined with respect to the norm J^{NA}, i.e. $(X, [\omega])$ is uniformly K-stable if there is a $\delta > 0$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq \delta\mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$ for all relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for $(X, [\omega])$ (see Sect. 5.3 for further details). In particular, the techniques of variation of the underlying class in the Kähler cone (Theorem 1.6) immediately yield the following characterization of the K-semistable locus of the Kähler cone (cf. [35, Theorem G] for an analogous result in the projective setting).

Theorem 1.7 *The K-semistable locus is closed in Euclidean topology in the Kähler cone of* X.

Recall moreover that the cscK locus is open relative to the Futaki vanishing locus (see [11]), i.e. the cscK locus can be written as $U \cap \mathcal{C}_F$, where U is an open set in the Kähler cone $\mathcal{C}_X$). As a consequence of this, the K-semistable and cscK loci can only coincide whenever they both equal $\emptyset$ or $\mathcal{C}_X$. The fact that these stability loci are in general not equal has been known by means of counterexamples (see e.g. [48] and [36, Corollary 1.2]), but this yields a complementary perspective on this question. Since the K-polystable locus is moreover expected to be open relative to the Futaki vanishing locus, we expect in the same way that the set of strictly K-semistable Kähler classes form the complement of an open set inside a closed set in the Kähler cone. As before, it is known by example (see [36, 48]) that strictly K-semistable classes exist, but this would give some additional information (for example we would expect strictly K-semistable to exist in abundance, except exceptional cases.)

In case the automorphism group is discrete, we may also give a similar characterization of the uniformly K-stable locus. To do this we associate to each Kähler class $\alpha \in \mathcal{C}_X$ its 'stability threshold', that is

$$\Delta(\alpha) := \sup\{\delta > 0 \,|\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq \delta \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})\} > -\infty,$$

where the supremum is taken over all relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α). Moreover, introduce the level sets

$$\mathcal{U}_\delta := \{\alpha \in \mathcal{C}_X \mid \Delta(\alpha) \geq \delta\}.$$

We then make the following observation:

Theorem 1.8 *The uniformly K-stable locus can be written as a union*

$$\mathcal{U} := \bigcup_{\delta > 0} \mathcal{U}_\delta,$$

where each set $\mathcal{U}_\delta$ is closed in the Euclidean topology in the Kähler cone.

Note that the K-semistable locus equals $\mathcal{U}_0$, so Theorem 1.7 is a special case of Theorem 1.8.

1.5 Organization of the Paper

The goal of Sects. 2, 3, and 4 is to rigorously clarify how to view various K-stability notions as special cases of the classical geodesic stability notion (which is now known to be equivalent to existence of cscK metrics, due to recent progress of Chen-Cheng [17, 18]). In order to do this, some standard preliminary notions are recalled in Sect. 2. The slightly non-standard geodesic stability notion used in this paper is discussed in Sect. 2.3.2. In Sect. 3 the definitions of a wide variety of (transcendental) K-polystability notions are given. In Sect. 4 we compare these notions, and give proofs of other main results. Section 5 contains three applications of the methods used in our proof, in particular to weakly cscK manifolds and basic topological properties of the K-semistable and uniformly K-stable loci. The formalism for test configurations used in this paper is based on the notions introduced in [31, 32, 43, 44].

2 Variants of the Geodesic Stability Notion in Kähler Geometry

Throughout this paper, let (X, ω) be a compact Kähler manifold. Let $n := \dim_{\mathbb{C}}(X)$ be the complex dimension of X. Write

$$V := \int_X \omega^n := (\alpha^n)_X$$

for the Kähler volume of X.

2.1 The Space of Kähler Metrics and Geodesics

Consider the space

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(X) \mid \omega_\varphi := \omega + dd^c\varphi > 0\}, \quad (dd^c := \frac{i}{2\pi}\partial\bar{\partial})$$

of smooth Kähler potentials. In a landmark paper by Mabuchi [38] it was shown that $\mathcal{H}_\omega$ is a Riemannian symmetric space (of infinite dimension, with $T_\varphi\mathcal{H} \simeq C^\infty(X)$). Following Darvas [25, 26, 28, 29] and others we however privilege the point of view of considering $\mathcal{H}$ as a path metric space endowed with a certain *Finsler metric* d_1. To introduce it, let $d_1 : \mathcal{H}_\omega \times \mathcal{H}_\omega \to \mathbb{R}_+$ be the path length pseudometric associated to the weak Finsler metric on $\mathcal{H}_\omega$ defined by

$$||\xi||_\varphi := V^{-1} \int_X |\xi|\omega_\varphi^n, \quad \xi \in T_\varphi\mathcal{H}_\omega = \mathcal{C}^\infty(X).$$

More explicitly, if $[0, 1] \ni t \mapsto \phi_t$ is a smooth path in X, then let

$$l_1(\phi_t) := \int_0^1 ||\dot{\phi}_t||_{\phi_t} dt$$

be its length, and set

$$d_1(\varphi, \psi) = \inf\left\{l_1(\phi_t), \quad (\phi_t)_{0\le t\le 1} \subset \mathcal{H}_\omega, \ \phi_0 = \varphi, \ \phi_1 = \psi\right\},$$

where the infimum is taken over smooth paths $t \mapsto \phi_t$ as above. It can then be seen that $(\mathcal{H}_\omega, d_1)$ is a metric space which is *not* complete (see [28, Theorem 2] and the survey article [27] for details and background). The completion $\mathcal{E}$ of $(\mathcal{H}_\omega, d_1)$ was described by Darvas [26]. For the purpose of discussing energy functionals and geodesic stability we will in particular consider the subspace $\mathcal{E}^1 \subset \mathcal{E}$ of ω-psh functions of finite L^1-energy, i.e. the subspace of all $\varphi \in \mathcal{E}$ such that

$$\int_X |\varphi| \omega_\varphi^n < +\infty.$$

2.1.1 Group Actions

Let $G := \mathrm{Aut}_0(X)$ be the connected component of the complex Lie group of biholomorphisms of (X, J), whose Lie algebra consists of real vector fields V satisfying $\mathcal{L}_V J = 0$. For each $g \in G$ we then have $[g^*\omega] = [\omega]$ (this follows from Moser's trick in symplectic geometry, see e.g. [13, Chapter III.7]). The group G thus acts naturally on the space $\mathcal{K} := \{\omega_\varphi := \omega + dd^c\varphi : \varphi \in \mathcal{C}^\infty(X), \omega_\varphi > 0\}$ of Kähler metrics on X, so that $g \cdot \xi := g^*\xi,\ g \in G,\ \xi \in \mathcal{K}$. The space $\mathcal{K}$ is moreover in one-to-one correspondence with the space $\mathcal{H}_0 := \mathcal{H} \cap E^{-1}(0)$ of normalized Kähler potentials. Following [29, Section 5.2] the group G therefore also acts on $\mathcal{H}_0$, so that $g \cdot \varphi$ is the unique element in $\mathcal{H}_0$ satisfying $g \cdot \omega_\varphi = \omega_{g \cdot \varphi}$. As in [29, Lemma 5.8]) one may moreover show that

$$g \cdot \varphi = g \cdot 0 + \varphi \circ g. \tag{1}$$

By the dd^c-lemma the function $g \cdot 0$ is smooth, hence bounded, on X.

2.1.2 Geodesics in the Space of Kähler Metrics

There is also a natural notion of geodesic (and subgeodesic) rays in $\mathcal{H}$. To define it, suppose that $I \subseteq (0, +\infty)$ is an open interval. Let $I \ni t \mapsto \varphi_t$ be any curve of functions on X. Then $(\varphi_t)_{t \in I}$ can be identified with an S^1-invariant function Φ on $X \times \Delta_I$, where $\varphi_t(x) = \Phi(x, e^{-t+is})$, and

$$\Delta_I := \{\tau \in \mathbb{C} \mid -\log|\tau| \in I\}.$$

We will be mainly interested in the case $I = (0, +\infty)$, when Δ_I is the punctured unit disc in the complex plane. Let $p_1 : X \times \Delta_I \to X$ the first projection. We then say that a collection $(\varphi_t)_{t \in I}$ of locally bounded Kähler potentials on X is a *subgeodesic ray* if $\Phi \in \mathrm{PSH}(X \times \Delta_I, p_1^*\omega)$, i.e. $p_1^*\omega + dd^c\Phi \geq 0$ in the weak sense of currents. Moreover, it is said to be a *geodesic ray* if it is subgeodesic and maximal with respect to this property, or equivalently, if the S^1-invariant associated function Φ satisfies the following homogeneous complex Monge-Ampère equation

$$(\pi_1^*\omega + dd^c\Phi)^{n+1} = 0,$$

on $\pi^{-1}(\bar{\Delta})$ seen as a manifold with boundary. We refer to e.g. [5, 33], and references therein, for details on this notion. For the delicate question of regularity of geodesic rays in this setting, see [24].

2.2 The K-Energy Functional and cscK Metrics

Let (X, ω) be a compact Kähler manifold and let

$$\mathrm{Ric}(\omega) := -dd^c \log \omega^n$$

be the associated Ricci curvature form (where $dd^c := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$ is normalized so that $[\mathrm{Ric}(\omega)] = c_1(X)$). We say that ω is a *cscK metric* if it satisfies the cscK equation

$$\mathcal{S}(\omega) = \bar{\mathcal{S}}, \tag{2}$$

where

$$\mathcal{S}(\omega) := \mathrm{tr}_\omega \mathrm{Ric}(\omega) := n\frac{\mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}$$

is the scalar curvature of ω and $\bar{\mathcal{S}}$ is the mean scalar curvature, given by

$$\bar{\mathcal{S}} := V^{-1}\int_X S(\omega)\,\omega^n = n\frac{\int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\int_X \omega^n} := n\frac{(c_1(X)\cdot\alpha^{n-1})_X}{(\alpha^n)_X}. \tag{3}$$

As observed by Mabuchi in [37] the cscK metrics can be characterized by variational methods, as the minima of a certain functional called the *Mabuchi K-energy functional*. It is the unique functional $M : \mathrm{H}_\omega \to \mathbb{R}$ satisfying $\mathrm{M}(0) = 0$ and

$$\frac{d}{dt}\mathrm{M}(\varphi_t) = -V^{-1}\int_X \dot{\varphi}_t(\mathcal{S}(\omega_{\varphi_t}) - \bar{\mathcal{S}})\,\omega^n_{\varphi_t}.$$

for any smooth path $(\varphi_t)_{t\geq 0}$ in the $\mathcal{H}$. Note that part of the assertion of Mabuchi was that such a functional exists, and whenever they exist, the minimizers of this functional are precisely the cscK potentials $\varphi \in \mathcal{H}_\omega$, i.e. the corresponding Kähler form $\omega_\varphi := \omega + dd^c\varphi$ satisfies the cscK equation (2).

The K-energy can moreover be extended to the setting of locally bounded ω-psh functions on X, i.e. to a functional $\mathrm{M} : \mathrm{PSH}(X, \omega) \cap L^\infty(X) \to \mathbb{R} \cup \{+\infty\}$. Similarily, there is an extension to the space $\mathcal{E}^1$ of locally finite energy potentials. To see this, recall that the K-energy functional can be written explicity using the so called Chen-Tian formula as the sum

$$\mathrm{M} = \mathrm{M}_{pp} + \mathrm{M}_{ent}$$

of a pluripotential and an entropy part. Here

$$\mathrm{M}_{ent}(\varphi) := V^{-1} \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right) \omega_\varphi^n \in [0, +\infty)$$

and $\mathrm{M}_{pp}(\varphi)$ is a linear combination of terms of the form

$$\int_X \varphi \omega^k \wedge \omega_\varphi^{n-k}$$

and

$$\int_X \varphi \omega^k \wedge \omega_\varphi^{n-k-1} \wedge \mathrm{Ric}(\omega).$$

The pluripotential terms can be made sense of due to [5, 26]. The entropy term in the formula for M can always be made sense of as a lower semicontinous functional $\mathrm{M}_{ent} : \mathcal{E}^1 \to [0, +\infty]$, defined as the relative entropy of the probability measures ω_φ^n / V and ω^n / V (see [8, 17] and references therein).

2.3 Holomorphic Vector Fields, the ¥-Invariant and Geodesic Stability

In order to define the notion of geodesic stability, we first introduce the notation for holomorphic vector fields that we will use: Suppose that (X, ω) is a compact Kähler manifold and denote by $J : TX \to TX$ the associated complex structure. A real vector field on X is a section of the real tangent bundle TX of X. It is said to be *real holomorphic* if its flow preserves the complex structure, i.e. it has vanishing Lie derivative $L_V J = 0$. A holomorphic vector field on a *compact* manifold is automatically $\mathbb{C}$-complete, and its flow ϕ_t is an action of $(\mathbb{C}, +)$ on X by holomorphic automorphisms. Conversely, one may associate to every additive action $\phi : \mathbb{C} \times X \to X$ by holomorphic automorphisms on X the vector field

$$V_\phi(x) := \frac{d}{dt}\phi(t, x)_{|t=0},$$

called the infinitesimal generator of X. The vector field V_ϕ is holomorphic and $\mathbb{C}$-complete on X, with the flow ϕ.

Definition 2.1 A real holomorphic vector field V on X is said to be *Hamiltonian* if it admits a *Hamiltonian potential* $h_\omega^V \in C^\infty(X, \mathbb{R})$ such that the contraction

$$i_V(\omega) := V \rfloor \omega = \sqrt{-1}\bar{\partial} h_\omega^V.$$

Remark 2.2 Equivalently, a real holomorphic vector field admits a Hamiltonian potential if and only if it has a zero somewhere, see LeBrun-Simanca [11].

Note further that the Hamiltonian potential is unique up to constants, so to relieve this ambiguity we impose the normalization

$$\int_X h_\omega^V \omega^n = 0.$$

For the purpose of comparing with the situation for polarized manifolds (X, L) it is interesting to recall that Hamiltonian vector fields are precisely those that lift to line bundles, see [34, Lemma 12]. A real holomorphic Hamiltonian vector field is automatically a *Killing field*, since $L_V J = L_V \omega = 0$ implies that also $L_V g = 0$ for the Riemannian metric associated to the Kähler form ω.

2.3.1 Geodesic Rays Induced by Holomorphic Vector Fields

For future use we recall also the notion of geodesic rays arising from holomorphic vector fields on X: In order to explain this notion, recall that the connected component of the Lie group $G := \mathrm{Aut}_0(X)$ of automorphisms of X act on $\mathcal{K}$ by pullback $g.\omega := g^*\omega$, and induces a corresponding action on $\mathcal{H}_0$ via the identification $\mathcal{H}_0 \simeq \mathcal{K}$, as described in (1). If V is a real holomorphic Hamiltonian vector field on X, then $\exp(tJV)$ is an element of the Lie group G for each $t \in [0, +\infty)$. If we set

$$\omega_t := \exp(tJV)^*\omega, \quad t \in [0, +\infty)$$

then $(\omega_t)_{t\geq 0}$ is a geodesic ray in $\mathcal{K}$, see [38]. The corresponding geodesic ray in $\mathcal{H}_0$ is denoted by $\varphi_t := \exp(tJV).\varphi_0$, where $\varphi_0 = 0$ such that $\omega_{\varphi_0} = \omega$.

Definition 2.3 A geodesic ray is said to be *induced by the holomorphic vector field* if it is of the form $\varphi_t = \exp(tJV).\varphi_0$ for some real holomorphic Hamiltonian vector field V on X,

2.3.2 The ¥-Invariant and Geodesic Stability

In order to state the definition of geodesic stability that we will use, let $[0, +\infty) \ni t \mapsto \rho(t) := \varphi_t$ be a given locally finite energy geodesic ray in $\mathcal{E}_0^1 := \mathcal{E}^1 \cap E^{-1}(0)$. Following [17, 18, 23] we consider the following numerical invariant

$$¥(\rho(t)) := \lim_{t\to+\infty} t^{-1}\mathrm{M}(\rho(t)),$$

associated to the given geodesic ray $\rho(t)$. This quantity is well-defined by convexity of the K-energy functional, see [2, 22] and also [6] for convexity of the extension of the K-energy to finite energy spaces. Recall also that two geodesic rays (φ_t) and (ξ_t) are said to be *parallel* if

$$d_{1,G}(\varphi_t, \xi_t) := \inf_{g,h \in G} d_1(g.\varphi_t, h.\xi_t) < C$$

for some constant $C > 0$ independent of t.

Definition 2.4 The pair $(X, [\omega])$ is said to be geodesically stable if and only if $¥(\rho(t)) \geq 0$ for every unit speed geodesic ray $\rho(t)$, with equality precisely when $\rho(t)$ is induced by a holomorphic vector field on X.

Note that in the paper [18] of Chen-Cheng geodesic stability was defined with respect to rays parallel to geodesics induced by holomorphic vector fields, which we do not do here. However, our definition turns out to be equivalent to geodesic stability with respect to rays induced by holomorphic vector fields. Indeed, we have the following:

Proposition 2.5 (cf. [44, Proposition 4.10]) *Suppose that* (X, ω) *is a cscK manifold. Let* $[0, +\infty) \ni t \mapsto \rho(t)$ *be a unit speed geodesic ray in* $\mathcal{E}^1_0$*. Then the following are equivalent:*

(1) *The ray* $\rho(t)$ *is of finite* $d_{1,G}$*-length, i.e.* $d_{1,G}(\varphi_t, \varphi_0) < C$ *for some constant* $C > 0$ *independent of* t
(2) *The ray* $\rho(t)$ *is parallel to a ray induced by a holomorphic vector field*
(3) *The ray* $\rho(t)$ *is itself induced by a holomorphic vector field*

Proof The implication (1) and (3) is precisely the statement of [44, Proposition 4.10]. The implication (3) $\Rightarrow$ (2) is immediate, since any ray is parallel to itself. Finally, if $\rho(t)$ is parallel to a ray ξ_t^V induced by a holomorphic vector field (and with $\xi_0^V = \rho(0)$), then

$$d_{1,G}(\rho(t), \rho(0)) \leq d_{1,G}(\rho(t), \xi_t^V) + d_{1,G}(\xi_t^V, \rho(0)) < C$$

by the triangle inequality. Indeed, the first term is bounded by C by the assumption that $\rho(t)$ and ξ_t^V are parallel. Moreover

$$d_{1,G}(\xi_t^V, \rho(0)) = \inf_{g,h \in G} d_1(g.\xi_t^V, h.\rho(0)) = d_1(\rho(0), \rho(0)) = 0,$$

since $xi_t^V = \exp(tJV).\rho(0)$ for some real holomorphic Hamiltonian vector field V on X (so in particular, $exp(tJV) \in G$). Hence (2) $\Rightarrow$ (1). Putting this together, we conclude that (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2), thus completing the proof. □

As a corollary we have the following:

Corollary 2.6 *Geodesic stability (Definition 2.4) is equivalent to geodesic stability in the sense of Chen-Cheng* [18].

Proof Geodesic stability in our sense clearly implies geodesic stability in the sense of Chen-Cheng [18] (any ray is in particular parallel to itself). The latter geodesic stability condition was moreover proven in [18] to be equivalent to existence of cscK metrics. On the other hand, suppose that (X, ω) is a cscK manifold. Then it follows from Proposition 2.5 that a geodesic ray is parallel to a ray induced by a holomorphic vector field, if and only if it is itself induced by a holomorphic vector field. In other words, geodesic stability in the sense of Chen-Cheng implies the geodesic stability notion of Definition 2.4. Putting this together, the considered geodesic stability notions must be equivalent. □

2.4 A Weak Geodesic Stability Notion

In order to later compare geodesic stability to K-stability notions (see Sects. 3.5 and 4) it is also natural to introduce a slightly more flexible terminology. In this direction, we give the following definition, which emphasizes possible differences in the vanishing condition for the ¥-invariant:

Definition 2.7 (**(S, S_0)-geodesic stability**) Let $S_0 \subset S$ be subsets of the set of locally finite energy geodesic rays in $\mathcal{E}^1_0$. The pair $(X, [\omega])$ is then (S, S_0)-geodesically stable if and only if $¥(\rho(t)) \geq 0$ for every unit speed geodesic ray $\rho(t) \in S$, with equality precisely when $\rho(t) \in S_0$.

If S is taken to be the full set of unit speed locally finite energy geodesic rays in $\mathcal{E}^1_0$, and S_0 is as any of the conditions (1)–(3) in Proposition 2.5, then (S, S_0)-geodesic stability of $(X, [\omega])$ is equivalent to geodesic stability of $(X, [\omega])$ (in the sense of Chen-Cheng, alternatively Definition 2.4). We next recall the definitions of various stability notions in algebraic geometry, and show that they fit into the framework of the above notion of (S, S_0)-geodesic stability.

3 Notions of K-Polystability in Kähler Geometry

In this section we recall the general formalism of transcendental K-stability for Kähler manifolds, first introduced in [32, 43], and describe how various stability notions in algebraic geometry can be naturally defined from this point of view.

3.1 Preliminaries on Test Configurations

We first recall the concept of test configurations for X, following [43]. As a reference for this section we use [43, Section 3].

Definition 3.1 A test configuration for X consists of

- a normal complex space $\mathcal{X}$, compact and Kähler, with a flat morphism $\pi : \mathcal{X} \to \mathbb{P}^1$
- a $\mathbb{C}^*$-action ρ on $\mathcal{X}$ lifting the canonical action on $\mathbb{P}^1$
- a $\mathbb{C}^*$-equivariant isomorphism

$$\mathcal{X} \setminus \pi^{-1}(0) \simeq X \times (\mathbb{P}^1 \setminus \{0\}) \tag{4}$$

Remark 3.2 Note that since π is flat the central fiber $\mathcal{X}_0 := \pi^{-1}(0)$ is a Cartier divisor, so $\mathcal{X} \setminus \mathcal{X}_0$ is dense in $\mathcal{X}$ in Zariski topology.

The *trivial* test configuration for X is given by $(\mathcal{X} := X \times \mathbb{P}^1, \lambda_{\text{triv}}, p_2)$, where $p_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection on the 2nd factor, and $\lambda_{\text{triv}} : \mathbb{C}^* \times \mathcal{X} \to \mathcal{X}$, $(\tau, (x, z)) \mapsto (x, \tau z)$ is the $\mathbb{C}^*$-action that acts trivially on the first factor. If we instead let $\sigma : \mathbb{C}^* \times X \to X$ be any $\mathbb{C}^*$-action on X, then we obtain an induced test configuration as above with $\lambda(\tau, (x, z)) := (\sigma(\tau, x), \tau z)$ (by also taking the compactification so that the fiber at inifinity is trivial). Such test configurations are called *product* test configurations of (X, α). In either case, we identify X with $X \times \{1\}$ and the canonical equivariant isomorphism (4) is then explicitly induced by the isomorphisms $X \cong X \times \{1\} \to X \times \{\tau\}$ given by $x \mapsto \lambda(\tau, (x, 1)) =: \lambda(\tau) \cdot x$. Note moreover that if V is any real holomorphic Hamiltonian vector field on X, then it may or may not generate a $\mathbb{C}^*$-action, and only if it does there is a clear way to associate a product test configuration to it (as described above). This is a subtle key issue.

We further define the notion of test configuration for (X, α), where $\alpha \in H^{1,1}(X, \mathbb{R})$ is any Kähler class on X: In order to do so, we first recall that the notions of Kähler forms and plurisubharmonic functions can be defined on complex spaces, see [44, 45] for details on this in the present context. If (X, L) is a polarized manifold, then a test configuration for (X, L) is given by a $\mathbb{C}^*$-equivariant flat family $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$, see e.g. [9] and references therein for details and background on this classical definition first used in this form in [34]. More generally, we will work with the formalism for arbitrary Kähler manifolds, of which the above can be considered a special case. A test configuration for the polarized pair (X, α) is then defined as follows:

Definition 3.3 A test configuration for (X, α) is a pair $(\mathcal{X}, \mathcal{A})$ where $\mathcal{X}$ is a test configuration for X, and $\mathcal{A} \in H^{1,1}_{BC}(X, \mathbb{R})^{\mathbb{C}^*}$ is a $\mathbb{C}^*$-invariant $(1, 1)-$Bott-Chern cohomology class whose image under (4) is $p_1^*\alpha$.

We give a few remarks and examples on how to compare cohomological test configurations with *algebraic test configurations* $(\mathcal{X}, \mathcal{L})$ *for a polarized manifold* (X, L).

(1) If (X, L) is any compact Kähler manifold endowed with an ample line bundle L (so X is projective), and $(\mathcal{X}, \mathcal{L})$ is a test configuration for (X, L) in the usual algebraic sense, cf. e.g. [34], then $(\mathcal{X}, c_1(\mathcal{L}))$ is a cohomological test configuration for $(X, c_1(L))$. This observation is useful, since many examples of algebraic test configurations $(\mathcal{X}, \mathcal{L})$ for polarized manifolds (X, L) are known, see e.g. [47, 49] and references therein.
(2) There are more cohomological test configurations for $(X, c_1(L))$ than there are algebraic test configurations for (X, L) (take for instance $(\mathcal{X}, \mathcal{A})$ with $\mathcal{A}$ a transcendental class as in the above definition), but in some cases the ensuing stability notions can nonetheless be seen to be equivalent (see [43, Section 3]).

3.2 Intersection Theoretic Numerical Invariants

Following [43] we recall the following intersection theoretic definition of the classical Donaldson-Futaki invariant:

Definition 3.4 ([32, 43]) To any cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) we may associate its *Donaldson-Futaki invariant* $\mathrm{DF}(\mathcal{X}, \mathcal{A})$ and its *non-Archimedean Mabuchi functional* $\mathrm{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$, first introduced in [9]. They are given respectively by the following intersection numbers

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}) := \frac{\bar{\mathcal{S}}}{n+1} V^{-1}(\mathcal{A}^{n+1})_{\hat{\mathcal{X}}} + V^{-1}(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}^n)_{\hat{\mathcal{X}}}$$

and

$$\mathrm{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A}) := \mathrm{DF}(\mathcal{X}, \mathcal{A}) + ((\mathcal{X}_{0,\mathrm{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\hat{\mathcal{X}}}$$

computed on any smooth and dominating model $\tilde{\mathcal{X}}$ of $\mathcal{X}$ (due to the projection formula it does not matter which one). Note that $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq \mathrm{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$ with equality precisely when $\mathcal{X}_0$ is reduced.

In case $\mathcal{X}$ is smooth, $K_{\mathcal{X}/\mathbb{P}^1} := K_{\mathcal{X}} - \pi^* K_{\mathbb{P}^1}$ denotes the relative canonical class taken with respect to the flat morphism $\pi : \mathcal{X} \to \mathbb{P}^1$. In the general case of a normal (possibly singular) test configuration $\mathcal{X}$ for X, we however need to give meaning to the intersection number $K_{\mathcal{X}} \cdot \mathcal{A}_1 \cdot \dots \cdot \mathcal{A}_n$, for $\mathcal{A}_i \in H^{1,1}_{\mathrm{BC}}(\mathcal{X}, \mathbb{R})$. To do this, suppose that $\tilde{\mathcal{X}}$ is a smooth model for $\mathcal{X}$, with $\pi' : \tilde{\mathcal{X}} \to \mathcal{X}$ the associated morphism. Since $\tilde{\mathcal{X}}$ is smooth the canonical class $K_{\tilde{\mathcal{X}}} := \omega_{\tilde{\mathcal{X}}}$ is a line bundle. Now consider $\omega_{\mathcal{X}} := \mathcal{O}(K_{\mathcal{X}}) := (\pi'_* \omega_{\tilde{\mathcal{X}}})^{**}$, i.e. the "reflexive extension" of $\omega_{\tilde{\mathcal{X}}}$, which is a rank 1 reflexive sheaf on $\mathcal{X}$. We then set

$$(\omega_{\mathcal{X}} \cdot \mathcal{A}_1 \cdot \dots \cdot \mathcal{A}_n) := (K_{\tilde{\mathcal{X}}} \cdot \pi'^* \mathcal{A}_1 \cdot \dots \cdot \pi'^* \mathcal{A}_n). \tag{5}$$

Using the projection formula (or an argument of the type [32, Lemma 2.15] in the Kähler category) it is straightforward to see that the above intersection number is independent of the choice of model/resolution $\pi' : \tilde{\mathcal{X}} \to \mathcal{X}$. In particular this holds for the Donaldson-Futaki invariant DF and the non-archimedean Mabuchi functional M^{NA}.

3.3 Product Test Configurations and Several Definitions of K-Polystability

A number of natural variants of K-polystability for Kähler manifolds are given as follows:

Definition 3.5 In analogy with the usual definition for polarized manifolds, and following [43, Section 3], we say that

- (X, α) is *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ for all normal and relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α).
- (X, α) is *K-polystable* if it is K-semistable, and in addition $\mathrm{DF}(\mathcal{X}, \mathcal{A}) = 0$ if and only if $\mathcal{X}$ is a product, where the latter means that one of the following conditions hold:

  (1) $\mathcal{X}_{|\pi^{-1}(\mathbb{C})}$
  (2) $\mathcal{X}_0 := \pi^{-1}(0) \simeq X$
  (3) $\mathcal{X}_{|\pi^{-1}(\Delta_r)} = X \times \Delta_r$, where $r \in (0, +\infty)$ and $\Delta_r := \{z \in \mathbb{C} \mid |z| \leq r\}$.

  We will refer to these conditions as strong, weak and r-K-polystability respectively.

Note that demanding that $\mathcal{X}$ is $\mathbb{C}^*$-equivariantly isomorphic to $X \times \mathbb{P}^1$ is not enough: For instance, there are (algebraic) product test configurations $(X, L) \times \mathbb{C}$ whose Donaldson-Futaki invariant vanishes, but whose compactifications over $\mathbb{P}^1$ (and thus their corresponding cohomological test configuration $(\bar{\mathcal{X}}, c_1(\bar{\mathcal{L}}))$) is *not* a product. See e.g. [9, Example 2.8]. Hence the definition (1) is the strongest notion of product that makes sense to consider in the context of K-polystability.

When it is necessary to make the distinction, we will refer to the above stability notions as *cohomological*. In the same vein, we refer to the analogous stability notions for polarized manifolds (see e.g. [3, 9, 34]) as *algebraic*. It is an interesting topic to compare cohomological and algebraic stability notions to eachother.

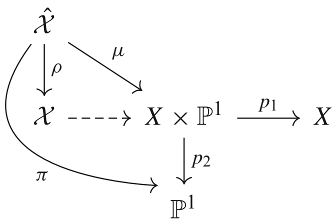
Regarding the cohomological notions, it was proven in [43, Theorem A] that cscK manifolds are always K-semistable. Moreover, if (X, L) is a polarized manifold, then (X, L) is K-semistable in the usual algebraic sense iff $(X, c_1(L))$ are (cohomologically) K-semistable [43, Proposition 3.14]. In other words, the algebraic and the cohomological notions of K-semistability are equivalent. It is an open question whether the same holds for K-polystability, but at least one of the implications always holds: if (X, L) is a polarized manifold such that $(X, c_1(L))$

is cohomologically K-polystable, then (X, L) is algebraically K-polystable (cf. [44, Proposition 2.22]). This holds regardless of the notion (1)–(3) of product that one uses. In particular, the above notions of K-polystability generalizes the usual notion for polarized manifolds considered in [3, 7].

3.4 Test Configurations Embedded in the Space of Subgeodesic Rays

We here briefly recall a key notion from the papers [43, 44], making precise the relationship between subgeodesic rays and test configurations, The goal is to view test configurations as "embedded" in the space of subgeodesic rays on X, in a sense made precise below. This allows in particular to compare the ¥ and the Donaldson-Futaki invariants, and more generally, to interpret certain K-polystability notions as weak versions of geodesic stability, by restricting the set of rays along which one tests the ¥-invariant.

In order to recall the definition of subgeodesic rays *compatible* with a given test configuration, we suppose that $(\mathcal{X}, \mathcal{A})$ is a (possibly singular) relatively Kähler test configuration for (X, α). By taking the normalization of the graph of $\mathcal{X} \dashrightarrow X \times \mathbb{P}^1$ and resolving singularities, we can always find a smooth model $\hat{\mathcal{X}}$ for $\mathcal{X}$, i.e. a $\mathbb{C}^*$-equivariant bimeromorphic morphism $\rho : \hat{\mathcal{X}} \to \mathcal{X}$, where $\hat{\mathcal{X}}$ is smooth and dominates the product $X \times \mathbb{P}^1$. This yields the following situation:



Now let (φ_t) be a locally bounded subgeodesic ray on X, with Φ the S^1-invariant function on $X \times \bar{\Delta}$ associated to the given ray $(\varphi_t)_{t \geq 0}$, such that $\varphi_t(x) = \Phi(x, e^{-t+is})$ for each $t \in [0, +\infty)$. By [43, Proposition 3.10] we then have

$$\rho^* \mathcal{A} = \mu^* p_1^* \alpha + [D],$$

where $D = \sum_{j=1}^n a_i D_i$ is a divisor on $\hat{\mathcal{X}}$ supported on the central fiber $\hat{\mathcal{X}}_0$. We can further decompose the current of integration of D as $\delta_D = \theta_D + dd^c \psi_D$, where θ_D is any smooth S^1-invariant $(1, 1)$-form on $\hat{\mathcal{X}}$. Locally, we then have

$$\psi_D = \sum_j a_j \log |f_j| \mod \mathcal{C}^\infty,$$

where the f_j are local defining equations for the irreducible components D_j respectively. Note that the choice of ψ_D is uniquely determined modulo a smooth function on $\hat{\mathcal{X}}$, so in particular it determines a unique singularity type. A locally bounded subgeodesic ray $(\varphi_t)_{t\geq 0}$ on X is then said to be L^∞-compatible with $(\mathcal{X}, \mathcal{A})$ if $\Psi := \Phi \circ \mu + \psi_D$ extends to a locally bounded $\rho^*\Omega$-psh function on $\hat{\mathcal{X}}$. Similarily, a smooth curve $(\varphi_t)_{t\geq 0}$ is C^∞-compatible with $(\mathcal{X}, \mathcal{A})$ if $\Psi := \Phi \circ \mu + \psi_D$ extends smoothly across $\hat{\mathcal{X}}_0$. In particular, an important point is that the singularity type of $\Phi \circ \mu$ is determined by the Green function ψ_D.

Example 3.6 We give two examples:

(1) As a central example, let Ω be a smooth S^1-invariant $(1, 1)$-form such that $[\Omega] = \mathcal{A}$. For $\tau \in (0, 1]$ we denote by Ω_τ the restriction of Ω to the fiber $\mathcal{X}_\tau$. As noted in [32], Ω_τ and Ω_1 are cohomologous, so we may define a family of functions $(\varphi_\tau)_{\tau\in(0,1]}$ on X by the relation $\lambda(\tau)^*\Omega_\tau - \Omega_1 = dd^c\varphi_\tau$. We can in turn define a $(\psi_t)_{t\in[0,+\infty)}$ on X defined by the relation $\psi_t := \varphi_{e^{-t}}$. It is smooth and C^∞-compatible with $(\mathcal{X}, \mathcal{A})$, but not in general a subgeodesic ray (although it is still a useful tool in many cases, see e.g. [43, Section 4]).

(2) Moreover, there is a well known construction that yields a unique (up to certain choices) geodesic ray associated to a given test configuration, obtained by solving a certain homogeneous complex Monge-Ampère equation on $\hat{\mathcal{X}}$. We refer to [43, Section 4] for details on the construction. This geodesic ray is then L^∞-compatible but in general not C^∞-compatible with $(\mathcal{X}, \mathcal{A})$ (this relates to the intricate question of regularity of such geodesics, see e.g. [24]).

The main results of [44] consist of an injectivity lemma as well as results on asymptotics of energy functionals along compatible subgeodesic rays. They can be summarized by considering the assignment

$$\mathrm{R} : (\mathcal{X}, \mathcal{A}) \mapsto [(\varphi_t)^{(\mathcal{X},\mathcal{A})}],$$

that maps any relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) to the set of subgeodesic rays compatible with $(\mathcal{X}, \mathcal{A})$. This map satisfies the following two key properties:

Theorem 3.7 ([44, Theorem 1.5 and Theorem 1.8]) *Suppose that (X, ω) is a compact Kähler manifold. Let $(\mathcal{X}, \mathcal{A})$ be a relatively Kähler test configuration for (X, α), and denote by $(\varphi_t)_{t\geq 0}$ any compatible subgeodesic ray in* $\mathrm{R}(\mathcal{X}, \mathcal{A})$. *Then*

(1) (Asymptotics of the K-energy)

$$\lim_{t\to+\infty} t^{-1}\mathrm{M}(\varphi_t) = \mathrm{DF}(\mathcal{X}, \mathcal{A}) + ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n).$$

(2) (Injectivity) *Let $(\mathcal{Y}, \mathcal{B})$ be another relatively Kähler test configuration for (X, α). Suppose that*

$$\mathrm{R}(\mathcal{X}, \mathcal{A}) \cap \mathrm{R}(\mathcal{Y}, \mathcal{B}) \neq \emptyset.$$

Then the canonical $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \to \mathcal{Y} \setminus \mathcal{Y}_0$ extends to an isomorphism $\mathcal{X} \to \mathcal{Y}$.

By relaxing the cscK assumption, we will improve on this result in Sect. 4 below.

3.5 Interpretation of K-Polystability as (S, S_0)-Geodesic Stability

It is useful to take a point of view which promotes that K-stability is really something tested along geodesic rays, and we discuss how various K-stability notions can be viewed rather explicitly as special cases of the geodesic stability notion used in a recent series of remarkable papers by Chen-Cheng [16–18]. The aim is thus to help clarifying the precise relationship between the abundance of stability notions available in the literature today. In order to do this, suppose that $\rho(t)$ is a locally finite energy unit speed geodesic ray in the space $\mathcal{E}^1_0 := \mathcal{E}^1 \cap E^{-1}(0)$. First recall the invariant

$$¥(\rho(t)) := \lim_{t\to+\infty} t^{-1}\mathrm{M}(\rho(t))$$

introduced in [16–18] (here M is the Mabuchi K-energy functional). If the geodesic ray $\rho(t)$ is "compatible" with a test configuration $(\mathcal{X}, \mathcal{A})$ for $(X, [\omega])$, in a sense made precise in [43, 44], then the ¥-invariant essentially coincides with the Donaldson-Futaki invariant (up to an explicit error term that vanishes when the total space of the test configuration is reduced). More precisely, we have

$$¥(\rho(t)) = \mathrm{DF}(\mathcal{X}, \mathcal{A}) + ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n),$$

by Theorem 3.7. With reference to the definition of (S, S_0)-geodesic stability introduced in Sect. 2.4, recall that when S is taken to be the set of *all* unit speed geodesic rays in $\mathcal{E}^1_0$, and S_0 is the set of all geodesic rays induced by holomorphic vector fields on X, then (S, S_0)-geodesic stability turns out to be equivalent to the geodesic stability notion used in [17, 18]. When it comes to geodesic K-polystability, we have an analogous interpretation as follows:

Theorem 3.8 *The pair $(X, [\omega])$ is geodesically K-polystable if and only if it is (S, S_0)-geodesically stable, where S is the set of subgeodesic rays compatible with a relatively Kähler test configuration for $(X, [\omega])$ and S_0 is the set of geodesic rays induced by some holomorphic vector field on X.*

Remark 3.9 A similar result applies also to other stability notions, such as *slope stability*, introduced in [41]. More precisely, one then chooses S as the set of all subgeodesic rays compatible with test configurations given by the deformation to the normal cone construction (see e.g. [4, Example 5.3]). Moreover, all the alternative K-polystability notions discussed in Sect. 3.3 are also of this form, by varying the set S_0 in the obvious way (the set of subgeodesic rays compatible with product test configurations, in the various senses respectively).

The above discussion fits well with the well known connection between test configurations and geodesic rays, as well as geodesic stability and the Yau-Tian-Donaldson conjecture. The notation introduced may also serve as a convenient common framework for all these different stability notions in Kähler geometry.

4 Stability Loci and Proof of Main Results

4.1 Stability Loci in the Kähler Cone

Let $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ be the open cone of Kähler classes on X. The classical point of view on the question of existence of canonical metrics is to exploit a variational approach, when it is natural to characterize existence of constant scalar curvature Kähler metrics *in a given Kähler class* on a given compact Kähler manifold. Thanks to the introduction of stability notions for pairs (X, α) of a given Kähler manifold and a Kähler class $\alpha \in \mathcal{C}_X$, we may however ask the following question: Given a compact Kähler manifold (X, ω), can we characterize the subsets of $\mathcal{C}_X$ consisting of the Kähler classes α for which the pair (X, α) is K-polystable/geodesically K-stable/cscK. The same question can of course be asked for any stability condition (K-semistability, slope stability etc). This slight change in point of view is sometimes useful, as we show below. In particular, note that the Yau-Tian-Donaldson conjecture can be reformulated as the statement that the cscK locus (alternatively, the geodesically stable locus) equals to K-polystable locus (in a suitable sense).

From the work of Berman-Darvas-Lu [7] and Chen-Cheng [18] we have equality of the cscK locus and the geodesically stable locus, and from the work [31, 32, 43, 44] we have inclusions

$$\text{cscK locus} \subseteq \text{geodesically K-polystable locus} \subseteq \text{K-semistable locus},$$

and in [44, Appendix] it was proven that the

$$\text{geodesically K-polystable locus} \subseteq \text{equivariantly K-polystable locus}.$$

The various K-polystability notions discussed in Sect. 3.3 have similar inclusions, but it is an open question whether equality holds (we will show in Theorem 4.12 below that this is indeed the case). When it comes to one of the main questions of this paper, namely comparing K-polystability with geodesic K-polystability, we know that there is an inclusion

$$\text{K-polystable locus} \subseteq \text{geodesically K-polystable locus}.$$

However, it is quite possible that for certain unquantized (i.e. unpolarized) compact Kähler manifolds the K-polystable locus is in fact empty (using the stronger transcendental stability notion, see Sect. 3.3. More precisely, the following questions are of particular interest to us here:

Question 4.1 *Do we have an inclusion cscK locus $\subseteq$ K-polystable locus? Do the K-polystable and geodesically K-polystable loci coincide in general?*

By the above discussion, an affirmative answer to the second question implies an affirmative answer also to the first one. In the sections that follow we will develop the tools to state and prove some partial results in this direction.

4.2 Convex Combinations and Changing the Underlying Kähler Class

We first focus on relating test configurations to eachother in the case when we change also the underlying Kähler class. More precisely, we set out to compare test configurations for (X, α) and (X, β), where $\alpha, \beta \in \mathcal{C}_X$ are different Kähler classes on X, which in turn yields a proof of Theorem 1.6. As a first observation, we note the following result on the ¥-invariant under convex combinations of rays along convex combinations of the underlying Kähler classes:

Proposition 4.2 *Let $\alpha, \beta \in \mathcal{C}_X$ and set $\alpha_s := (1-k)\alpha + s\beta$, for $s \in [0, 1]$. Suppose that $\rho_\alpha(t)$ and $\rho_\beta(t)$ are smooth subgeodesic rays with respect to (X, α) and (X, β) respectively. Then $\rho_s(t) := (1-s)\rho_\alpha(t) + s\rho_\beta(t)$ are subgeodesic rays with respect to (X, α_s), and the map*

$$[0, 1] \ni s \mapsto \yen(\rho_s(t))$$

is continuous.

Proof This follows immediately from the definitions. Indeed, fix Kähler forms ω_0, ω_1 on X such that $\alpha := [\omega_0]$ and $\beta := [\omega_1]$. In turn, let $\omega_s := (1-s)\omega_0 + s\omega_1$, such that $\alpha_s = [\omega_s]$. By hypothesis we then have

$$\omega_0 + dd^c\rho_\alpha(t) \geq 0$$

and

$$\omega_1 + dd^c\rho_\beta(t) \geq 0.$$

Therefore also

$$\omega_s + dd^c\rho_s(t) = (1-s)(\omega_0 + dd^c\rho_\alpha(t)) + s(\omega_1 + dd^c\rho_\beta(t)) \geq 0,$$

i.e. $\rho_s(t)$ is a subgeodesic ray. Finally, the map

$$[0, 1] \ni s \mapsto \yen(\rho_s(t))$$

is clearly a polynomial in s, hence continuous. □

As a particular case we obtain the following result on convex combinations of test configurations, where parts (1)–(2) should be compared with an observation in A. Isopoussu's thesis [35] (where only the setting of polarized manifolds was considered):

Theorem 4.3 *Let $\alpha, \beta \in \mathcal{C}_X$ and set $\alpha_s := (1-s)\alpha + s\beta$, for $s \in [0, 1]$. Suppose that $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$ is a relatively Kähler test configuration for (X, α) and (X, β) respectively. Then*

(1) *$(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$ is a relatively Kähler test configuration for (X, α_s).*
(2) *The maps*

$$[0, 1] \ni s \mapsto \mathrm{DF}(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$$

and

$$[0, 1] \ni s \mapsto \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, (1-s)\mathcal{A} + k\mathcal{B})$$

are continuous.
(3) *Suppose that $\rho_A(t)$ and $\rho_B(t)$ are subgeodesic rays C^∞-compatible with $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$ respectively, and write $\rho_s(t) := (1-s)\rho_A(t) + k\rho_B(t)$ If $\alpha_s = [\omega_s]$, then*

$$\mathrm{DF}(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B}) = \lim_{t\to+\infty} t^{-1}\mathrm{M}_{\omega_t}(\rho_s(t)) - ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n).$$

Proof The first assertion follows from the basic fact that the set of relatively Kähler classes on $\mathcal{X}$ is convex. In order to see that $(\mathcal{X}, (1-s)\mathcal{A}+s\mathcal{B})$ is a test configuration for (X, α_s) we may pass to a resolution $\rho : \hat{\mathcal{X}} \to \mathcal{X}$. Then $\rho^*\mathcal{A} = \mu^* p_1^*\alpha + [D]$ and $\rho^*\mathcal{B} = \mu^* p_1^*\alpha + [E]$. Hence

$$\rho^* ((1-s)\mathcal{A} + s\mathcal{B}) = \mu^* p_1^*\alpha_s + (1-s)[D] + s[E],$$

and the conclusion (1) follows. The assertion (2) follows immediately from the definition of DF and J^{NA} as intersection numbers (it is straightforward to see that $[0, 1] \ni s \mapsto \mathrm{DF}(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$ and $[0, 1] \ni s \mapsto \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$ are polynomials in k of degree at most $n + 1$, thus continuous). Finally, in order to prove (3) it suffices (by [44, Theorem 1.5]) to show that $\rho_s(t)$ is C^∞-compatible with $(\mathcal{X}, (1-s)\mathcal{A} + s\mathcal{B})$. This is also immediate. To see it, let Φ_0 and Φ_1 denote the S^1-invariant functions on $X \times \bar{\Delta}$ associated to $\rho_\alpha(t)$ and $\rho_\beta(t)$ respectively (so that $\rho_\alpha(t) = \Phi_0(x, e^{-t+iv})$ and $\rho_\beta(t) = \Phi_1(x, e^{-t+iv})$ as before). If we let μ, $[E]$, $[D]$ be as above, then

$$\mu^*((1-s)\Phi_0 + s\Phi_1) + (1-s)[D] + s[E] = (1-s)(\mu^*\Phi_0 + [D]) + s(\mu^*\Phi_1 + [E])$$

extends smoothly across $\mathcal{X}_0$ (since both terms do so, by hypothesis). This concludes the proof. □

An interesting point is to emphasize that these proofs become very simple once we take the point of view chosen above.

4.3 The Set of Product Configurations

It is a subtle but important point to understand how to properly define the concept of a product test configuration. A suggestion in [43] was that a test configuration should be called a product (or "geodesic product") if and only if it is compatible with a geodesic ray induced by a holomorphic vector field. Of course, if $(\mathcal{X}, \mathcal{A})$ is a product in the traditional sense (i.e. the total space is isomorphic to $X \times \mathbb{C}$ away from the fiber at infinity), then it is compatible with a ray of this form. The more difficult part is to establish the converse, in which case of only partial results are known: First, if we restrict to the case of polarized manifold (X, L) and their usual algebraic test configurations $(\mathcal{X}, \mathcal{L})$, then it was proven in [7] that this holds whenever the underlying class is cscK. Secondly, assuming existence of a cscK metric, the same holds for the more general transcendental test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α), provided that the test configuration is taken to be *equivariant* (see [31, 44]).

The goal of this section is to explain that the hypothesis that the underlying class is cscK can be weakened. Indeed, we will show that it is enough to assume that there exists a cscK metric in some (possibly different) Kähler class on X, i.e. the cscK locus $\neq \emptyset$. In order to establish this result, the following lemma constitutes the key step:

Lemma 4.4 *Let $\alpha, \beta \in \mathcal{C}_X$ such that also $\beta - \alpha \in \mathcal{C}_X$. Suppose that $(\mathcal{X}, \mathcal{A})$ is a relatively Kähler test configuration for (X, α), with associated geodesic ray $\rho(t)$. Then there exists a relatively Kähler test configuration $(\mathcal{X}, \mathcal{B})$ for (X, β), with the same total space $\mathcal{X}$, and which is C^∞-compatible with the same geodesic ray $\rho(t)$.*

Proof By resolution of indeterminacy there is a smooth and dominating test configuration $\rho : \hat{\mathcal{X}} \to \mathcal{X}$ for X such that

$$\begin{array}{ccccc} \hat{\mathcal{X}} & & & & \\ \downarrow{\scriptstyle \rho} & \overset{\mu}{\searrow} & & & \\ \mathcal{X} & \dashrightarrow & X \times \mathbb{P}^1 & \xrightarrow{p_1} & X \end{array}$$

Then it follows from [43, Proposition 3.10] that $\rho^*\mathcal{A}_\alpha = \mu^* p_1^*\alpha + [D]$ for some $\mathbb{R}$-divisor D on $\hat{\mathcal{X}}$ supported on $\hat{\mathcal{X}}_0$. Now set $\mathcal{A}_\beta := \mathcal{A} + \eta$, where η is a (1, 1)-cohomology class on $\mathcal{X}$ which satisfies

$$\rho^*\mathcal{A}_\beta - \rho^*\mathcal{A}_\alpha = \mu^* p_1^*(\beta - \alpha).$$

Since $\beta - \alpha \in \mathcal{C}_X$, it follows that $\mu^* p_1^*(\beta - \alpha)$ is nef. Therefore $\rho^*\eta$ is nef on $\hat{\mathcal{X}}$, so also η is nef on $\mathcal{X}$, and $\mathcal{A}_\beta = \mathcal{A} + \eta$ is relatively Kähler (as a sum of a relatively Kähler and relatively nef classes). Hence we have a cohomological test configuration $(\mathcal{X}, \mathcal{A}_\beta)$ for (X, β). Moreover, this new test configuration satisfies $\rho^*\mathcal{A}_\beta = \mu^* p_1^*\beta + [D]$, with the same μ and $[D]$ as before. As a consequence, also $(\mathcal{X}, \mathcal{A}_\beta)$ is C^∞-compatible with the geodesic ray $\rho(t)$, which is what we wanted to prove. □

In particular we then obtain the following corollary, where $\mathrm{Fut}_\alpha(X, V)$ denotes the classical Futaki invariant of the vector field V on X, and $\mathcal{C}_F \subseteq \mathcal{C}_X$ denotes the set of all Kähler classes α for which $\mathrm{Fut}_\alpha(X, \cdot)$ vanishes identically:

Proposition 4.5 *Suppose that $(\mathcal{X}, \mathcal{A})$ is a relatively Kähler test configuration for (X, α) whose associated geodesic ray $\rho(t)$ is induced by a holomorphic vector field V on X. Then for each $\beta \in \mathcal{C}_X$ there is a relatively Kähler test configuration $(\mathcal{X}, \mathcal{A}_\beta)$ for (X, β), with the same total space $\mathcal{X}$, such that*

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}_\beta) = \mathrm{Fut}_\beta(X, V).$$

In particular, if $\beta \in \mathcal{C}_F$, then $\mathrm{DF}(\mathcal{X}, \mathcal{A}_\beta) = 0$.

Proof Pick $\lambda > 0$ such that $\lambda\beta - \alpha \in \mathcal{C}_X$. By Lemma 4.4 there exists a relatively Kähler test configuration $(\mathcal{X}, \mathcal{A}_{\lambda\beta})$ for $(X, \lambda\beta)$ that is compatible with a ray $\rho(t)$ induced by a holomorphic vector field, i.e. of the form $\rho(t) := \exp(tJV).\rho(0)$ for some real holomorphic Hamiltonian vector field V on X. By [44, Theorem 3.10] we then have

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}_{\lambda\beta}) = \lim_{t\to+\infty} \frac{d}{dt}\mathrm{M}(\rho(t)) = \mathrm{Fut}_{\lambda\beta}(X, V),$$

which vanishes if $\beta \in \mathcal{C}_F$ Finally, set $\mathcal{A}_\beta := \lambda^{-1}\mathcal{A}_{\lambda\beta}$. Then $(\mathcal{X}, \mathcal{A}_\beta)$ is a relatively Kähler test configuration for (X, β) and

$$\bar{S}_{\lambda\beta} = \lambda^{-1}\bar{S}_\beta$$

$$V_{\lambda\beta} = \lambda^n V_\beta.$$

One then checks that

$$\mathrm{DF}_{\lambda\beta}(X, \lambda\mathcal{A}_\beta) = \frac{\bar{S}_{\lambda\beta}}{(n+1)V_{\lambda\beta}}(\lambda\mathcal{A}_\beta)^{n+1} + \frac{1}{V_{\lambda\beta}}(K_{\mathcal{X}/\mathbb{P}^1} \cdot (\lambda\mathcal{A}_\beta)^n) =$$

$$= \frac{\bar{S}_\beta}{(n+1)V_\beta}(\mathcal{A}_\beta)^{n+1} + \frac{1}{V_\beta}(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}_\beta^n) = \mathrm{DF}_\beta(X, \mathcal{A}_\beta),$$

which in turn equals $\mathrm{Fut}_\beta(X, V)$. This completes the proof. □

This result has several immediate and key applications below.

4.4 Proof of Theorems 3 and 1

As a first application of Lemma 4.4 and Proposition 4.5, we prove the following (the main new point being that we allow changing the underlying Kähler class):

Theorem 4.6 *Let $\alpha \in \mathcal{C}_X$ and suppose that $(\mathcal{X}, \mathcal{A})$ is a relatively Kähler smooth and dominating test configuration for (X, α). Then, for each $\beta \in \mathcal{C}_X$ there is a $\lambda > 0$ such that $\lambda\beta > \alpha$, and a relatively Kähler test configuration $(\mathcal{Y}, \mathcal{B})$ for $(X, \lambda\beta)$ such that*

(1) $\mathcal{Y} = \mathcal{X}$,
(2) *The test configurations $(\mathcal{X}, \mathcal{A}) \sim (\mathcal{Y}, \mathcal{B})$, i.e. there is a subgeodesic ray $\rho(t)$ that is C^∞-compatible with both.*

In particular, if $\alpha = [\omega]$ and $\lambda\beta = [\theta]$, then we have

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}) = \lim_{t\to+\infty} t^{-1}\mathrm{M}_\omega(\rho(t)) - ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n)$$

and

$$\mathrm{DF}(\mathcal{X}, \mathcal{B}) = \lim_{t\to+\infty} t^{-1}\mathrm{M}_\theta(\rho(t)) - ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{B}^n).$$

Proof The statements (1) and (2) are simply Lemma 4.4. The last statement regarding the asymptotics of the K-energy is precisely [44, Theorem 1.5]. □

The first main point of the above discussion is that we may now deduce the following main result:

Theorem 4.7 (cf. Theorem 1.2) *Let (X, ω) be a compact Kähler manifold and suppose that the K-polystable locus $\neq \emptyset$. Suppose that $(\mathcal{X}, \mathcal{A})$ is a test configuration for $(X, [\omega])$. Then the following are equivalent:*

- $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$
- *The associated geodesic ray is induced by a holomorphic vector field on X.*

Proof Since the K-polystable locus $\subseteq \mathcal{C}_F$ we may without loss of generality assume that $\alpha := [\omega] \in \mathcal{C}_F$. Now suppose for contradiction that the K-polystable locus is *strictly* contained in the geodesically K-polystable locus. Then there is a relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ which is a geodesic product (i.e. C^∞-compatible with a subgeodesic ray induced by a real holomorphic Hamiltonian vector field V on X), but not a product configuration (in the sense that $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X$). Assuming that $\alpha \in \mathcal{C}_F$ we then have

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}) = \mathrm{Fut}_\alpha(X, V) = 0.$$

Moreover, the K-polystable locus is non-empty, so we may pick $\beta \in \mathcal{C}_X$ in such a way so that (X, β) is K-polystable. By Proposition 4.5 there is then a test configuration $(\mathcal{X}, \mathcal{B})$ for (X, β), with the same total space $\mathcal{X}$, such that $\mathrm{DF}(\mathcal{X}, \mathcal{B}) = 0$ (indeed $(\mathcal{X}, \mathcal{B})$ is a geodesic product and $\beta \in \mathcal{C}_F$ because (X, β) is K-polystable). Since $(\mathcal{X}, \mathcal{B})$ is a relatively Kähler *non*-product configuration, this contradicts that (X, β) is K-polystable. Hence, if the K-polystable locus is non-empty then it must coincide with the geodesically K-polystable locus. In particular, the conditions (1) and (2) are equivalent. This finishes the proof. □

In particular, the above proof gives a partial answer to the question of comparing the K-polystability and geodesic K-polystability notions:

Corollary 4.8 *Let (X, ω) be a compact Kähler manifold and suppose that the K-polystable locus $\neq \emptyset$. Then the K-polystable locus equals the geodesically K-polystable locus.*

As a next key point, the above results are independent of whether we consider K-polystability with respect to $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$ or $\mathcal{X}_0 \simeq X$, as explained below.

4.5 Equivalence of Notions of Product Configuration

We now discuss the equivalence of various notions of product configurations and their corresponding K-polystability notions. For the purpose of this discussion, consider the following list of reasonable variants of the usual algebraic notion of product configuration:

Definition 4.9 We say that (X, α) is

(1) strongly K-polystable if it is K-polystable with respect to product configurations in the sense that $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$.
(2) weakly K-polystable if it is K-polystable with respect to product configurations in the sense that $\mathcal{X}_0 \simeq X$.
(3) r-K-polystable if it is K-polystable with respect to product configurations in the sense that $\mathcal{X}_{\pi^{-1}(\Delta_r)} \simeq X \times \Delta_r$, for any $r > 0$.

Remark 4.10 In the case of polarized manifolds (X, L) the strong K-polystability condition is rather that $\mathcal{X} \simeq X \times \mathbb{C}$, since then test configurations are usually defined over $\mathbb{C}$ rather than directly over $\mathbb{P}^1$.

The strong and weak K-polystability notions are both used frequently in the literature surrounding the YTD conjecture, see e.g. [8] and references therein. The goal is now to seize the opportunity to address the question of whether or not these conditions (1)–(3) are in fact equivalent. As preparation, we first check the following simple claim, suggested by the terminology:

Proposition 4.11 *If $r > r'$ then strong K-polystability $\Rightarrow$ r-K-polystability $\Rightarrow$ r'-K-polystability $\Rightarrow$ weak K-polystability*

Proof Suppose that there is a Kähler class $\alpha \in \mathcal{C}_X$ which is a strongly K-polystable but not weakly K-polystable. Then there is a test configuration satisfying $\mathcal{X}_0 \simeq X$, $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \not\simeq X \times \mathbb{C}$ and $\mathrm{DF}(\mathcal{X}, \mathcal{A}) > 0$. But this is a contradiction. The same argument goes through if $r > r'$, since then $\mathcal{X}_{\pi^{-1}(\Delta_r)} \simeq X \times \Delta_r$ implies that $\mathcal{X}_{\pi^{-1}(\Delta_{r'})} \simeq X \times \Delta_{r'}$. □

We now address the question of whether these a priori differing K-polystability notions, used by various authors in the literature, are in fact equivalent. Conveniently, it turns out that this is the case, thus clarifying the relationship between various results regarding the respective notions of K-polystability:

Theorem 4.12 *Suppose that (X, ω) is a compact Kähler manifold with non-empty strong K-polystability locus. Then the following notions are equivalent:*

(1) *Strong K-polystability*
(2) *r-K-polystability for any $r \in (0, +\infty)$*
(3) *Weak K-polystability*
(4) *Geodesic K-polystability*
(5) *S-geodesic stability with respect to the set of all geodesic rays compatible with relatively Kähler test configurations for (X, α)*

Proof Let S be a subset of all relatively Kähler test configurations for X. Suppose for contradiction that there is an $\alpha \in \mathcal{C}_X$ such that (X, α) is weakly K-polystable but not strongly K-polystable. Then there is, by definition, a test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) which is relatively Kähler, and satisfies $\mathcal{X}_0 \simeq X$, $\mathrm{DF}(\mathcal{X}, \mathcal{A}) = 0$, but $\mathrm{X}_{\pi^{-1}(\mathbb{C})} \not\simeq X \times \mathbb{C}$. Now pick β strongly K-polystable. Then $(\mathcal{X}, \mathcal{A} + \mu^* p_1^*(\beta - \alpha))$ is a relatively Kähler test configuration for (X, β), with the same total space as $(\mathcal{X}, \mathcal{A})$. But by Proposition 4.11 the pair (X, β) is, in particular, weakly K-polystable, and $\mathrm{X}_0 \simeq X$. Hence, by definition, $\mathrm{DF}(\mathcal{X}, \mathcal{A} + \mu^* p_1^*(\beta - \alpha)) = 0$. Finally, since (X, β) is also strongly K-polystable, we then have $\mathrm{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$). Conversely, it is clear that if $\mathrm{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$) then also $\mathcal{X}_0 \simeq X$. This finishes the proof of the equivalence (1) $\Leftrightarrow$ (3).

The exact same proof applies to any situation when we compare K-polystability notions with respect to notions of product where one notion implies the other, and both satisfy the requirement that $\mathrm{DF}(\mathcal{X}, \mathcal{A}) = 0$ for products. This way we prove that (1) $\Leftrightarrow$ (2). Finally, the equivalence (1) $\Leftrightarrow$ (4) is Theorem 1.2. □

Remark 4.13 If we were to consider K-polystability with respect to products in the sense that $\mathcal{X} \simeq X \times \mathbb{P}^1$ (for transcendental test configurations as in [24, 31, 32, 43, 44]), then the corresponding K-polystable locus would not contain the cscK locus in general. In fact, whenever $\mathrm{Aut}_0(X) \neq \emptyset$ the K-polystable locus would always be empty, so $\mathcal{X}_{\pi^{-1}(\mathbb{C})} \simeq X \times \mathbb{C}$ yields the strongest notion of product configuration of this type that is worth considering.

5 Weakly cscK Manifolds and Applications

5.1 The Special Case of Weakly cscK Manifolds

In view of the above main results, it is interesting to study situations when some of the above mentioned stability loci are non-empty (and for which we will then be able to establish that certain stability notions must be equivalent). A natural candidate for such manifolds are those compact Kähler manifolds (X, ω) that admit a cscK metric *in some possibly different Kähler class* $\alpha \neq [\omega] \in \mathcal{C}_X$. We will refer to such manifolds as weakly cscK.

Definition 5.1 We say that a compact Kähler manifold is *weakly cscK* if the associated cscK locus $\neq \emptyset$.

Note that a manifold can be weakly cscK without being cscK. Examples of this phenomenon can in particular be obtained by any Kähler-Einstein manifolds which also admits K-unstable polarizations. Concretely, it was shown through a study of *slope stability*, in [41, Example 5.30], that e.g. $\mathbb{P}^2$ blown up in 8 points in generic position satisfies this condition (see also [30] and [14, 15] for a more explicit treatment of this and other Del Pezzo surface examples). The idea is then to use the techniques of changing the underlying Kähler class, to reduce the study of arbitrary polarizations to the case when the underlying Kähler class admits a cscK metric. Some noteworthy corollaries of Theorems 1.2 and 1 follow:

Theorem 5.2 *Let (X, L) be a polarized weakly cscK manifold. Then the following holds:*

(1) *(X, L) is K-polystable if and only if it is geodesically K-polystable.*
(2) *(X, L) is equivariantly geodesically K-polystable if and only if it is equivariantly K-polystable*

For arbitrary compact Kähler manifolds (X, ω) the result (1) is known to hold if the automorphism group is discrete, see [44], and the second point (2) holds in general. In particular, we record the following result related to Remark 3.9:

Theorem 5.3 *Suppose that X is a weakly cscK Kähler manifold with* $\mathrm{Aut}_0(X)$ *discrete. Then (X, α) is uniformly K-stable if and only if (X, α) is coercive with respect to the set of subgeodesic rays compatible with a relatively Kähler test configuration for (X, α). Likewise, (X, α) is K-stable if and only if (X, α) is geodesically stable with respect to the set of subgeodesic rays compatible with a relatively Kähler test configuration for (X, α).*

Proof (Proof of Theorem 5.2) This is an immediate consequence of Theorem 1. Indeed, under the stated hypotheses the K-polystable locus is non-empty, by results of [3], so (X, L) is K-polystable if and only if it is geodesically K-polystable, even $c_1(L)$ does not itself admit a cscK metric. Finally, K-polystability trivially implies equivariant K-polystability, so also the equivariantly K-polystable locus is non-empty. In the same way as above, this proves (2). □

A reformulation of the above Theorem 5.2 is that, on weakly cscK polarized manifolds, a ray compatible with a test configuration is induced by a holomorphic vector field precisely if the test configuration is a product:

Theorem 5.4 *Suppose that (X, L) is a polarized weakly cscK manifold. Let $(\mathcal{X}, \mathcal{L})$ be a relatively Kähler test configuration for (X, L) with compatible subgeodesic ray $(\varphi_t) \in \mathrm{R}(\mathcal{X}, \mathcal{L})$. Then $\mathcal{X} \simeq X \times \mathbb{C}$ if and only if $(\varphi_t)_{t \geq 0}$ is induced by a holomorphic vector field on X.*

This extends a result of [3] from the case of cscK manifolds, to the larger class of weakly cscK manifolds.

5.2 An Extended Injectivity Lemma

Finally, it is worth noting that the above techniques can be used to extend the injectivity lemma (see Theorem 3.7, part (2)) from the setting of a fixed underlying Kähler class, to the setting of different underlying Kähler classes $\alpha, \beta \in \mathcal{C}_X$. Such injectivity type results were in [44] a key tool in proving equivariant K-polystability, geodesic K-polystability, and K-polystability whenever the automorphism group is discrete. It is also of independent interest.

In order to state the result, recall the assignment $\mathrm{R} : (\mathcal{X}, \mathcal{A}) \mapsto \left[(\varphi_t)^{(\mathcal{X}, \mathcal{A})}\right]$ from Sect. 3.4. We then have the following:

Theorem 5.5 *Suppose that $\alpha := [\omega]$ and $\beta := [\theta]$ are Kähler classes on X and let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be relatively Kähler test configurations for (X, α) and (X, β) respectively. Suppose that*

$$\mathrm{R}(\mathcal{X}, \mathcal{A}) \cap \mathrm{R}(\mathcal{Y}, \mathcal{B}) \neq \emptyset.$$

Then the canonical $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \to \mathcal{Y} \setminus \mathcal{Y}_0$ extends to an isomorphism $\mathcal{X} \to \mathcal{Y}$.

Remark 5.6 The hypothesis $\mathrm{R}(\mathcal{X}, \mathcal{A}) \cap \mathrm{R}(\mathcal{Y}, \mathcal{B}) \neq \emptyset$ here means that there is a subgeodesic ray $\rho(t) \in \mathrm{PSH}(X, \omega) \cap \mathrm{PSH}(X, \theta)$ which is compatible with two relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ for (X, α) and (X, β) respectively.

Proof of Theorem 5.5 The idea of the proof is to extend [44, Theorem 1.8] using the key Lemma 4.4 in order to control the change of the underlying Kähler class. Indeed, first fix Kähler forms ω_α and ω_β such that $[\omega_\alpha] = \alpha$ and $[\omega_\beta] = \beta$. By hypothesis $\mathrm{R}(\mathcal{X}, \mathcal{A}) \cap \mathrm{R}(\mathcal{Y}, \mathcal{B}) \neq \emptyset$ there is a subgeodesic ray $\rho(t) \in \mathrm{PSH}(X, \omega_\alpha) \cap \mathrm{PSH}(X, \omega_\beta)$ which is compatible with two relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ for (X, α) and (X, β) respectively. Now pick $(\mathcal{X}, \mathcal{A}_\beta)$ as in Lemma 4.4. Then $(\mathcal{X}, \mathcal{A}_\beta)$ and $(\mathcal{Y}, \mathcal{B})$ are relatively Kähler test configurations for (X, β), both compatible with the same subgeodesic ray $\rho(t)$. By applying the injectivity lemma [44, Theorem 1.8] we then finally see that the canonical $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \to \mathcal{Y} \setminus \mathcal{Y}_0$ extends to an isomorphism $\mathcal{X} \to \mathcal{Y}$. This is what we wanted to prove. □

5.3 Topology of the K-Semistable and Uniformly K-Stable Loci

The techniques of variation of the underlying class in the Kähler cone immediately yield some basic information on the structure and topology of the K-semistable and uniformly K-stable loci in the Kähler cone. Here (X, α) is said to be uniformly K-stable if there is a $\delta > 0$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq \delta \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$ for all relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) that dominate $X \times \mathbb{P}^1$ via a morphism $\mu : \mathcal{X} \to X \times \mathbb{P}^1$ (testing for these is enough by [45]). For such test configurations the norm $\mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$ is defined as the intersection number

$$\mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A}) := (\mu^* p_1^* \alpha \cdot \mathcal{A}) - \frac{(\mathcal{A}^{n+1})}{n+1}.$$

computed on $\mathcal{X}$ (as before $p_1 : X \times \mathbb{P}^1 \to X$ denotes the first projection). We refer to [32, 43–45] for details.

The following first result should be compared to [35, Theorem G]:

Theorem 5.7 *The K-semistable locus is closed in Euclidean topology in the Kähler cone of X.*

Proof By [43, Proposition 3.12] it suffices to test K-semistability for relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) that are smooth and dominating, i.e. there is a morphism $\mu : \mathcal{X} \to X \times \mathbb{P}^1$ such that $p_1 \circ \mu = \pi : \mathcal{X} \to \mathbb{P}^1$. Hence, we may fix any given relatively Kähler smooth and dominating test configuration $\mathcal{X}$ for X. By [43, Proposition 3.10] we moreover have

$$\mathcal{A} = \mu^* p_1^* \alpha + [D]$$

for some $\mathbb{R}$-divisor D on $\mathcal{X}$ supported on the central fiber $\mathcal{X}_0$. Since $\mathcal{A}$ is Kähler, there is an open neighbourhood $U_\alpha \subset \mathcal{C}_X$ of α such that $\mathcal{A}_\beta := \mu^* p_1^* \beta + [D] \in \mathcal{C}_X$ for every $\beta \in U_\alpha$. In view of the intersection theoretic interpretation of the Donaldson-Futaki invariant, note that the map

$$U_\alpha \ni \beta \mapsto \mathrm{DF}(\mathcal{X}, \mathcal{A}_\beta)$$

is continuous. As a consequence, suppose that $\alpha \notin$ K-semistable locus. Then there exists a smooth and dominating test configuration $(\mathcal{X}, \mathcal{A}_\alpha)$ as above such that $\mathrm{DF}(\mathcal{X}, \mathcal{A}_\alpha) < 0$. But by continuity there exists an open neighbourhood $V_\alpha \subset U_\alpha$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{A}_\beta) < 0$ for each $\beta \in V_\alpha$. In other words, if $\alpha \notin$ K-semistable locus, then there is an open neighbourhood satisfying $V_\alpha \subset \mathcal{C}_X \backslash$K-semistable locus. Hence the K-semistable locus is open in the Kähler cone of X, which is what we wanted to prove. □

Due to the fact that the cscK locus is open (see [11]) a consequence of this is that K-semistability is not equivalent to existence of cscK metrics. This has been known previously by means of counterexamples (see e.g. [36, 48]). Nonetheless, this yields a complementary perspective on this question. From this, we we also record the following corollary of independent interest:

Corollary 5.8 *The inclusion cscK locus $\subset$ K-semistable locus is strict whenever the K-semistable locus $\neq \emptyset, \mathcal{C}_X$.*

This also yields concrete examples of manifolds with "many" strictly semistable (i.e. K-semistable but not K-stable) Kähler classes:

Example 5.9 (Strictly semistable examples) Consider the Del Pezzo surface $X = \mathrm{Bl}_{p_1,\dots,p_8}\mathbb{P}^2$ to be the blowup of $\mathbb{P}^2$ in 8 points $p_1, \dots, p_8$ in general position. First of all, it is well known that X is Kähler-Einstein, so $(X, -K_X)$ is K-stable by [48]. In other words, the K-stable locus, thus also the K-semistable locus, is non-empty. On the other hand, it was shown in [41] that X admits K-unstable polarizations, so the K-semistable locus is $\neq \mathcal{C}_X$. Since both the K-stable locus and the K-semistable locus are $\neq \emptyset, \mathcal{C}_X$, whereas the former is open and the latter is closed in the Euclidean topology in $\mathcal{C}_X$, it follows that the strictly K-semistable locus is non-empty, i.e. the set K-semistable locus \ K-stable locus $\neq \emptyset$. This gives a new method of answering the question of existence of strictly K-semistable classes.

5.3.1 The Uniformly K-Stable Locus

Now suppose that (X, ω) is a compact Kähler manifold with discrete automorphism group, i.e $\mathrm{Aut}_0(X) = \emptyset$. Then similar arguments can also be made for the uniformly K-stable locus in the Kähler cone of X. To see this, we associate to each Kähler class $\alpha \in \mathcal{C}_X$ the finite real number

$$\Delta(\alpha) := \sup\{\delta > 0 \,|\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq \delta \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})\}$$

where the condition above should hold for all relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α). Moreover, introduce the sets

$$\mathcal{U}_\delta := \{\alpha \in \mathcal{C}_X \mid \Delta(\alpha) \geq \delta\}$$

We then make the following observation:

Theorem 5.10 *The uniformly K-stable locus can be written as a union*

$$\mathcal{U} := \bigcup_{\delta > 0} \mathcal{U}_\delta,$$

where each $\mathcal{U}_\delta$ is closed in the Euclidean topology in the Kähler cone.

Proof The proof is analogous as the one in Theorem 5.7, but applied to $\mathrm{DF} - \delta \mathrm{J}^{\mathrm{NA}}$ instead. For the convenience of the reader we give the argument: By [45, Proposition 3.2.20] it suffices to test uniform K-stability for relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) that are smooth and dominating, i.e. there is a morphism $\mu : \mathcal{X} \to X \times \mathbb{P}^1$ such that $p_1 \circ \mu = \pi : \mathcal{X} \to \mathbb{P}^1$. Hence, we may fix any given relatively Kähler smooth and dominating test configuration $\mathcal{X}$ for X. As before, by [43, Proposition 3.10] we moreover have $\mathcal{A} = \mu^* p_1^* \alpha + [D]$ for some $\mathbb{R}$-divisor D on $\mathcal{X}$ supported on the central fiber $\mathcal{X}_0$. Since $\mathcal{A}$ is Kähler, there is an open neighbourhood $U_\alpha \subset \mathcal{C}_X$ of α such that $\mathcal{A}_\beta := \mu^* p_1^* \beta + [D] \in \mathcal{C}_X$ for every $\beta \in U_\alpha$. In view of the intersection theoretic interpretation of both the Donaldson-Futaki invariant and the non-Archimedean J-functional, note that the map

$$U_\alpha \ni \beta \mapsto \mathrm{DF}(\mathcal{X}, \mathcal{A}_\beta) - \delta \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A}_\beta)$$

is continuous for each $\delta \in \mathbb{R}$. As a consequence, fix a $\delta \in \mathbb{R}$ and suppose that $\alpha \notin \mathcal{U}_\delta$. Then, by definition, there exists a smooth and dominating test configuration $(\mathcal{X}, \mathcal{A}_\alpha)$ as above such that $\mathrm{DF}(\mathcal{X}, \mathcal{A}_\alpha) < \delta \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A}_\beta)$. But by continuity there exists an open neighbourhood $V_\alpha \subset U_\alpha$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{A}_\beta) < \delta \mathrm{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{A}_\beta$ for each $\beta \in V_\alpha$. In other words, if $\alpha \notin \mathcal{U}_\delta$, then there is an open neighbourhood satisfying $V_\alpha \subset \mathcal{C}_X \setminus \mathcal{U}_\delta$. Hence for each $\delta \in \mathbb{R}$, the set $\mathcal{U}_\delta$ is open in the Kähler cone of X. Finally, it is clear that the uniformly K-stable locus can be written

$$\mathcal{U} := \bigcup_{\delta > 0} \mathcal{U}_\delta,$$

completing the proof. □

Remark 5.11 The K-semistable locus equals

$$\mathrm{Kss} = \bigcup_{\delta \geq 0} \mathcal{U}_\delta \ (= \mathcal{U}_0).$$

so Theorem 5.7 is a special case of Theorem 5.10.

We finally note the following reformulation of Theorem 5.10:

Corollary 5.12 *The stability threshold*

$$\mathcal{C}_X \ni \alpha \mapsto \Delta(\alpha)$$

is upper semicontinous.

We expect that it is also lower semicontinuous, but leave this question for future work.

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Kähler-Einstein Metrics via Moduli Continuity Method

Cristiano Spotti

If one figure is derived from another by a continuous change and the latter is as general as the former, then any property of the first figure can be asserted at once for the second figure.

Jean Victor Poncelet (1788–1867)
Traité des propriétés projectives des figures, 1822

Abstract We discuss some ideas behind a strategy that has been used to construct Kähler-Einstein metrics for *explicit families* of Fano varieties.

Keywords Kähler-Einstein metrics · Moduli spaces · Fano varieties

1 Introduction

A major problem in complex differential geometry consists in understanding which Fano manifolds admit Kähler-Einstein metrics. Recall that a n-dimensional complex manifold X is said to be Fano if it has positive first Chern class or, equivalently, its anticanonical bundle $K_X^{-1} = \bigwedge^n TX$ is ample. Geometrically, a Kähler-Einstein (KE) metric is simply an Einstein space for which parallel transport commutes with the underlying compatible complex rotation. It is a non-trivial fact that such metrics, necessarily with *positive* constant scalar curvature, are unique up to the natural symmetries (biholomorphisms and scalings). Thus KE metrics provide a way to canonically "geometrize" Fano manifolds. However, not all Fano manifolds admit such metrics, as the classical example of the blow-up of the projective plane in one point shows.

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Understanding exactly which Fano manifolds admit KE metrics has been the object of intense investigation in the last decades. These studies culminated in the recent solution of the so-called *Yau-Tian-Donaldson conjecture* in the Fano case ([8] for the "if" part, and [4] for the "only if"):

Theorem 1.1 *A smooth Fano manifold X admits a KE metric if and only if is K-polystable.*

Such result shows that the transcendental problem of finding a KE metric (a solution of a geometric PDE) is equivalent to the purely algebro-geometric property of K-stability. In short, checking K-stability amounts to compute the positivity of certain numerical invariant (Donaldson-Futaki invariant) on $\mathbb{C}^*$-equivariant degenerations of a Fano manifold X to a possibly singular variety X_0.

However, despite its fundamental theoretical importance, the above theorem is at present not very useful in constructing new KE metrics since, in general, it is still impossible to verify the K-stability property from its definition, due to the too many degenerations which a-priori need to be checked.

In this note, we would like to describe a different method for showing existence of KE metric on *explicitly given* Fano manifolds, which has been used in [31, 34] and it can possibly be applied in many other new situations.

Often Fano manifolds comes in *complex families* $\pi : \mathcal{X} \to \mathcal{H}$, with $\pi^{-1}(t) = X_t$ Fano variety. By varying the base parameter t we obtain in general non-biholomorphic (but still diffeomorphic) Fano manifolds. To keep an easy example in mind, consider the case of hypersurfaces of degree less than $n+2$ in $\mathbb{CP}^{n+1}$. By varying the coefficients of the defining polynomials, we obtain in general different Fano varieties. It is important to mention here that in low dimensions Fano manifolds are fully classified: each one is given as a member of some *explicit family*. In dimension two the classification is classical. In dimension three it is obtained thanks to works of Fano, Iskovskikh, Mori and Mukai [21].

It is then natural to ask the following question:

Question 1.2 *Which Fano varieties in a given* explicit family $\pi : \mathcal{X} \to \mathcal{H}$ *carry KE metrics?*

A natural strategy to analyze such situation consists in investigating the KE existence problem by studying variations of the parameters $t \in \mathcal{H}$. The idea of studying the KE problem by varying the complex structure is definitely not a new one. Indeed, it was used by Tian to solve the KE existence problem in dimension two [37]. He proved that, in each degree $d = c_1^2(X) < 5$ the KE condition is non-empty, open and closed within the subset of a natural parameter space $\mathcal{H}_d$ parameterizing *smooth* Fano surfaces of degree d.

Here we are going to explain an extension of such ideas. We do not focus on smooth Fanos only, but we also consider some degenerate singular limits. Moreover, we crucially make use of stability conditions (K-stability and classical GIT) in the study of how the KE condition varies in a family, by relating degenerations to concrete algebraic moduli constructions. We refer to this method as to the *moduli continuity method*. Such strategy was first used in [31] to study the so-called Gromov-Hausdorff (GH) moduli compactification of the space of smooth KE Fano surfaces (for degree $d = 4$ there is a previous work of T. Mabuchi and S. Mukai [28], which uses a slightly different approach). By applying this method, we do not only understand precisely which smoothable Fanos in a given family admit KE metrics, but we also provide a concrete description of the "abstract" GH moduli compactification, also known as K-moduli space (see [33] for a survey on such moduli spaces of algebraic varieties with their relation to special metrics). Moreover, this gives an explicit classification of the singularities of GH limits of certain Einstein manifolds, which is definitely interesting from a purely differential geometric point of view.

2 The *Moduli Continuity Method*

The moduli continuity method can be described as a strategy that can be used to answer Question 1.2. We now explain and comment the main steps, in a somehow idealized situation. Since our focus is on the main ideas, we refer to the literature for precise definitions and arguments.

Being a continuity method, it is not a surprise that it consists of three main parts: *non-emptiness*, *small variations* and *large variations*. We are going to describe very quickly the first two, while spending more time on the last one, since it has been the object of some very recent advances.

Non-emptiness The first step consists in finding within our family $\pi : \mathcal{X} \to \mathcal{H}$ a Fano variety X_{t_0} for which we can "easily" conclude that a KE metric exists on it. But where to look for such variety? As a general rule, we should look for a Fano which is *more symmetric* and apply some *existence criterion*, such as the G-invariant α-invariant, with $G \subseteq Aut(X)$ finite.

In principle, we could possibly even search among mildly singular Fano within the family. For example, the singular cubic surface $xyz = t^3$ is a finite quotient of the projective plane, and hence it has an obvious KE metric. Or we could look for a toric varieties since, in this case, the KE problem is fully understood [5]. Note that to actually apply the method starting from a singular variety, we would need to argue that some nearby smoothing is KE (see discussion in the next step).

Small variations The next step concerns how the KE condition varies for small perturbations of the complex parameters $t \in \mathcal{H}$. If the automorphism group $Aut(X_t)$ is discrete, a simple application of the implicit function theorem shows that all X_s sufficiently close to the KE manifold X_t admit KE metrics too. Actually, it can be proved that, in such case, the KE condition is Zariski open [11, 30].

In general, however, the automorphism group does not need to be discrete neither the KE condition open. Nevertheless, the situation is understood via a *local GIT picture* [7, 36]: we can look at the natural induced action of the reductive automorphism group on the space of infinitesimal deformations in order to understand which nearby Fano manifold remains KE. The prototypical situation is the case of the Mukai-Umemura Fano 3-fold and its deformations, first analyzed in [38]. We can then understand which are the Fanos near X_t in our family which remain KE.

The singular situation is more subtle, but it has been discussed at least for metric limits in [24, 35].

Large variations Let X_{t_i} a sequence of KE Fano manifolds in our family, and let $t_i \to t_\infty \in \mathcal{H}$. The question now is: does X_{t_∞} (even singular) need to be KE too?

In general, the answer is negative. However, in certain situations, we can give a positive answer. By the limit picture for KE Fanos of [12], eventually by passing to a subsequence, we can assume that X_{t_i} converge in a "refined" GH sense to a singular KE Fano variety X_∞. That is, they converge both in the metric GH sense and as complex cycles in a given uniform projective embedding. However, this abstract natural limit X_∞ a-priori does not need to be given by a variety within our original starting family $\pi : \mathcal{X} \to \mathcal{H}$.

To actually show that X_∞ is indeed a (special!) member of our family, we need three main steps:

1. Refined a-priori control on the singularities of GH (K-stable) limits.
2. Classifications of mildly singular Fanos.
3. Stability comparison argument.

The first point has seen recent advances, but we postpone its more careful discussion in the next section. For the moment, we could just say that a consequence of such analysis should give effective bounds (in terms of natural invariants of the general member of the family) of the so-called *Gorenstein index* of X_∞, that is, the minimal power to which we need to raise the $\mathbb{Q}$-Cartier anticanonical divisor $-K_{X_\infty}$ to find a genuine line bundle.

Let us suppose that a small a-priori bound on the index has been achieved. As I recalled at the beginning, in some situations, Fano manifolds have been classified. Thus, it becomes now important to extend the classification to the mildly singular Fanos as given by the first step. Usually, this ends up in showing an effective bound on the very-ampleness of the anticanonical bundle. A further information one could possibly use is that the limit X_∞ is $\mathbb{Q}$-Gorenstein smoothable. We think that this analysis should rise interesting problems for algebraic geometers.

In a lucky case, the extended classification may give that X_∞ is indeed biholomorphic to a member of our family, say X_{t_*} (see later for a discussion in the case this does not hold). But, is X_{t_*} biholomorphic to our starting flat limit X_{t_∞}?

It is here that the last step enters the game (and, also, it is here the reason why we have called such strategy a *moduli* continuity method). From Berman result [4] we know that $X_\infty \cong X_{t_*}$ is K-polystable. On the other hand, on our family there would usually be an equivariant action of a linear group G such that two varieties are abstractly isomorphic if and only if there is an element of the group carrying one to the other (just think for example to the natural action of $SL(n+2)$ on the space of projective Fano hypersurfaces). This can give rise to a *classical GIT problem*. It becomes now crucial to understand how K-stability relates to such classical GIT stability. In good situations (e.g., when one can check that the CM line bundle [32], whose weight is the Donaldson-Futaki invariant, is an equivariant positive multiple of a linearization considered in a classical GIT, the family is nice enough to avoid the Li-Xu pathology, etc. . .) we can infer that K-polystability implies GIT-polystability. Thus we can conclude that t_* is now in $\mathcal{H}^{ps}$, the GIT polystable locus. Moreover, a Luna's slice argument shows that $[X_{t_i}]$ converges to $[X_{t_*}]$ in the analytic topology of the explicit GIT quotient $\mathcal{H}//G$. Using the fact that $\mathcal{H}//G$ is *Hausdorff* we can now see that, if $t_\infty \in \mathcal{H}^{ps}$, $X_{t_\infty} \cong X_{t_*} \cong X_\infty$ carries a KE metric.

Finally, running an open-closed argument and using the density of smooth Fanos in our family, we can deduce that the natural injective (by uniqueness of the KE metric) continuous map $\phi : \overline{\mathcal{EM}}^{GH} \to \mathcal{H}//G$ we have constructed, is indeed surjective. Here $\overline{\mathcal{EM}}^{GH}$ is the *compactification* of the moduli spaces of KE Fanos manifolds in our family up to biholomorphic isometries equipped with the refined GH topology (also known as K-compactification). Hence ϕ is a homeomorphism by the standard compact-to-Hausdorff argument.

In conclusion, the problem of understanding which Fanos in our family is KE has been reduced to the study of a classical GIT quotient, that can be concretely analyzed via standard algebro-geometric techniques.

There are few points we would like to emphasize and comment on. It is worth noting that this approach requires to work necessarily with formation of singularities, even if one cares about the existence only of *smooth* KE Fanos. This is typical and not surprising in analysis (e.g., regularity theory, geometric flows, etc. . .). There is some "hard analysis" input also in this moduli continuity method approach: this is "hidden" in the "algebraic regularity" result [12], itself based on Cheeger-Colding regularity theory of limit spaces. After that, the argument becomes addressable with help from algebraic geometry. This is possible thanks to the presence of the underlying canonical algebraic structure, which make our KE case, in a certain sense, special among geometric PDEs.

As we mentioned in the introduction, as a non-trivial by-product of such moduli method we obtain a concrete description of all GH degenerations of the KE metrics in our family. A further interesting question to investigate is if such compactifications actually provide a compactification of a *connected component* of the full Einstein moduli space on the real underlying smooth manifold. That is, it would be interesting to see if there can be Einstein but *non-Kähler* deformations of a KE Fano manifold. As far as we are aware, this problem has not been solved yet, but it is very intriguing from a differential geometric viewpoint.

Finally, before discussing the crucial aspect of bounding a-priori the singularity types of GH limits, we should stress that the strategy described will require, in general, some adjustment in order to be applied: the main issue is that, in several cases, the abstract GH limits cannot always live in the family we started with! To deal with this problem, it would be needed to perform some birational modifications of our original family in order to accommodate such limits. Even for hypersurfaces in $\mathbb{CP}^{n+1}$ we cannot expect in general to reduce the KE problem just to the obvious GIT quotient: for example, for quartics 3-folds in $\mathbb{CP}^4$ it is clear that one should consider, at least, one blow-up of the family at the non-reduced double quadric, in order to accommodate the (KE) "hyperelliptic" Fanos which are given by taking double covers of a quadric. In any case, the moduli continuity method can be applied, with more work, also to such situations [31].

2.1 Refined A-Priori Control on the Singularities of GH Limits

By the general theory of Donaldson-Sun [12], we know that a GH limit X_∞ of smooth KE manifolds is a singular Fano variety with $\mathbb{Q}$-Cartier canonical divisor and Kamamata log terminal (klt) singularities, i.e., X_∞ is normal and for each log-resolution $r : \hat{X}_\infty \to X_\infty$, the canonical divisor $K_{\hat{X}_\infty} = r^* K_{X_\infty} + \sum_i a_i E_i$ with *discrepancies* $a_i > -1$. Moreover, X_∞ carries a weak KE metric in the sense of [14]. In this section we want to explain how to get further bounds on the singularities.

In complex two dimension, it was previously known by works of Anderson, Tian and many others, that GH limits are KE orbifolds, i.e., the singularities are locally of type $\mathbb{C}^2/\Gamma_p$, with $\Gamma_p \subseteq U(2)$ finite, acting freely on the sphere (precisely the klt condition in dimension two) and the metric is orbifold smooth. Thus, a natural invariant which measures the "sharpness" of a singularity is given by the order of the group at p. Since the KE metric satisfies the Bishop-Gromov monotonicity formula, we can relate the "local volume" (i.e., the order of the group) with the global volume (which is preserved in GH limits), thus obtaining some a-priori bounds on the order of the orbifold singularities. This was used for analyzing the two dimensional case [31].

In higher dimension the situation becomes more subtle. First of all, the expected general singularity won't be of quotient type (Schlessinger's rigidity). For example, this happens for the ordinary double point ODP singularity $\sum_i x_i^2 = 0$ in dimension bigger than or equal to three. Moreover, if we rescale the weak KE metric near a singularity $p \in X_\infty$, the metric tangent cone (a singular Calabi-Yau cone $C(Y_p)$) won't be in general locally biholomorphic (actually not even homeomorphic!) to the singularity germ $\mathcal{V} \subset X_\infty$ itself. Such local jump of the complex structure was first observed by Hein and Naber [19] (the isolated A_k-singularity in three dimension jumps to the flat splitting $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ as soon as $k \geq 3$), and the picture has been made general and precise in [13]. In any case, from a geometric measure theory viewpoint, it still makes sense to define the *density* as a measure of the sharpness:

$$\Theta_p := \lim_{r \to 0} \frac{Vol(B_p^{KE}(r))}{\omega_{2n} r^{2n}} = \frac{Vol(Y_p)}{Vol(S^{2n-1})} \leq 1.$$

As before, one could bound the densities using Bishop-Gromov estimates. However, by some experimental inspection, the estimate one gets this way is too weak to be of any use in higher dimension. In order to proceed further, we need new input from algebraic geometry.

Trying to find a purely algebro geometric construction of the metric tangent cones and inspired by previous work [29] in Sasakian geometry by Martelli, Sparks and Yau, Li introduced in [22] the following new invariant for a germ $(\mathcal{V}, p)$ of a klt singularity, the *infimum of the normalized volume of valuations*:

$$\hat{\mathrm{vol}}(\mathcal{V}, p) := \inf_{\nu \in Val_p} A^n(\nu)\,\mathrm{vol}(\nu) > 0,$$

where $A(\nu)$ is the log-discrepancy of a valuation and $\mathrm{vol}(\nu) := \limsup_{r\to 0} \frac{\mathrm{length}(\mathcal{O}_{\mathcal{V},p}/\{f|\nu(f)\geq r\})}{r^n/n!}$ its volume. It is proved in [6], that the infimum is actually a minimum.

This new invariant has to be considered as the algebro-geometric analogue of the metric density. The order of a germ of an holomorphic function at the singularity computed with respect to the distance induced by the KE metric induces a valuation ν_{KE}. In [20] it was proved that $A^n(\nu_{KE})\,\hat{\mathrm{vol}}(\nu_{KE}) \leq n^n \Theta_p$ and equality holds for quasi-regular tangent cones. More recently [23], the equality has been shown to hold in any situation, and moreover ν_{KE} is indeed the (unique among the so-called quasi-monomial valuations) minimizer. Hence for singularities in GH limits $\hat{\mathrm{vol}}(\mathcal{V}, p) = n^n \Theta_p$.

The next crucial ingredient is the following "algebro-geometric Bishop-Gromov" estimate, proved by Liu [25] as a generalization of Fujita's volume bound for K-semistable Fano manifolds [16]:

$$c_1^n(X) \leq \left(1 + \frac{1}{n}\right)^n \hat{\mathrm{vol}}(\mathcal{V}, p),$$

for any germ of singularity $(\mathcal{V}, p)$ in a n-dimensional K-semistable Fano variety X. This estimate is stronger than Bishop-Gromov estimate. Thus, provided that the volume of a sequence of KE Fano manifolds is large enough, we obtain good quantitative lower bounds for the volume densities at the singularities, since GH limits are K-polystable, as a consequence of Berman's result [4].

Moreover, it is very natural to expect that such densities satisfy certain gaps among their values (in analogy with minimal surfaces theory, in which, for example, the Willmore's conjecture can be interpreted as a gap for the density of certain minimal cone). Thus, since the ODP singularity is, in a rough sense, the simplest one, it is very natural to expect the following:

Conjecture 2.1 *ODP gap conjecture* [34]: $\Theta_p \,(= n^{-n}\hat{vol}(\mathcal{V}, p)) \leq 2\left(1 - \frac{1}{n}\right)^n$, *for any singularity $p \in X^n$, and the equality holds iff the singularity is an ODP (and the metric tangent cone is the ODP with its natural CY Stenzel's cone metric)*

This is clearly true thanks to the orbifold regularity of KE metrics for $n = 2$. Moreover, using classification results for three dimensional canonical singularities, it has been very recently proved in [26] that the value 16 is indeed the infimum of the normalized volume for all klt singularities (non-necessarily assumed to come from GH limit).

To state the next theorem, let us introduce the following quantity,

$$V(n) = \sup\{\Theta(C^n(Y)) \mid C^n(Y) \cong \mathbb{C}^k \times C(Y') \ncong \mathbb{C}^n,\ k \geq 0,\ Y' \text{smooth}\},$$

and recall that the *Fano index* is the maximal $r \in \mathbb{N}$ such that $K_X^{-1} = L^r$ for L an ample line bundle.

Theorem 2.2 ([34]) *Let X_∞ be a GH limit of n-dimensional smooth KE Fanos X_i of index r such that*

$$c_1^n(X_i) > \frac{(n+1)^n}{2} V(n).$$

Then X_∞ has Gorenstein canonical singularities *(i.e., the discrepancies are non-negative for any log resolution) and $K_{X_\infty}^{-1} = L^r$ for some line bunde L.*

It is clear that $2\left(1 - \frac{1}{n}\right)^n \leq V(n) \leq 1$. Thus, if $c_1^n(X) > (n+1)^n/2$ (that is, if the volume is bigger half of the volume of the projective space), the hypothesis holds. As we will see below, this condition is not empty. Moreover, since, as we have recalled, for $n = 3$ the volume gap holds true [23, 26], the above theorem implies (see [34]):

Corollary 2.3 *GH limits of KE Fano 3-folds of degree bigger than 20 are Gorenstein Fanos with canonical singularities and same Fano index. These include intersections of two quadrics, cubic hypersurfaces, and Fano 3-folds of rank one and degree 22 (deformations of the Mukai-Umemura manifold).*

More generally, one can obtain bounds on the Gorenstein index, which can also be very useful, as the two dimensional case shows.

In a nutshell the proof of Theorem 2.2 consists in:

1. Use the Liu's estimate of the volume to find a bound on the fundamental group of the possibly singular link of the metric tangent cone (by applying Colding-Naber convexity of its smooth locus [9]);
2. Bound the Cartier index on the cone via some covering trick;
3. Use the 2-steps construction of metric tangent cone in [13], to obtain the index bound on the original singular variety.

The definition of $V(n)$ is used in combination to Schlessinger's rigidity of quotient singularities to rule out certain situations.

As [26] suggest, part of the arguments can be made fully algebraic and more general by establishing that certain properties of the normalized volumes of valuations (mostly related to coverings) holds, thus avoiding to use more differential geometric techniques based on Cheeger-Colding theory.

3 Applications

The moduli continuity method has been applied in dimension two to fully study the GH compactification of KE surfaces in [31]. As a by-product we obtained an explicit classification of two dimensional KE Fano orbifolds with singularities of type $\mathbb{C}^2/\Gamma$, with $\Gamma \subseteq SU(2)$. In the proof we used, in combination with the bounds on the orbifold group, the classification of $\mathbb{Q}$-Gorenstein smoothable quotient singularities and classification results for certain smoothable Fano surfaces. We then constructed algebraic moduli spaces of Fano surfaces, which we showed to agree with the GH/K-moduli compactification via our strategy. While for degree 4 or 3 we could make use of classical GIT quotient, for degree 2 or 1 we performed certain birational modifications, resulting in a "gluing" of GIT quotients.

In the recent [34], we applied the above Theorem 2.2 and used the moduli strategy to show the following results.

Theorem 3.1 ([34]) *A possibly singular complete intersection of two quadrics* $X = Q_1 \cap Q_2$ *in* $\mathbb{P}^{n+2}$ *is KE (eq. is K-polystable) if and only if, up to reparametrization,* $Q_1 = 1$ *and* Q_2 *is diagonal with no more than* $(n+3)/2$ *equal eigenvalues and, if equality holds, then* $X \cong \{\sum_{i=0}^{(n+1)/2} x_i^2 = \sum_{(n+3)/2}^{n+2} x_i^2 = 0\}$. *In particular, all smooth intersections are KE, GH limits have at most bundles of ODP as singularities, and the GH compactification agrees with the GIT quotient* $Gr(2, Sym^2(\mathbb{C}^{n+3}))//SL(n+3)$ *obtained by associating to an intersection of two quadrics its pencil.*

Theorem 3.2 ([34]) *If ODP gap conjecture 2.1 holds for any* $k \leq n$ *(which does for* $n \leq 3$ [23, 26]*), then a possibly singular cubic n-fold admits a KE (eq. is K-polystable) if and only if it is GIT polystable for the* $SL(n+2)$ *action on* $Sym^3(\mathbb{C}^{n+2})$. *In particular, all smooth cubics are KE.*

Note that in dimension three GIT polystable cubics are fully classified [1]. For $n = 3$, Theorem 3.2 has been derived also in [26] as a consequence of their volume gap proof.

Thanks to the control of singularities provided by Theorem 2.2, the main point in the proof of Theorems 3.1 and 3.2 consists in showing that the GH limits embed naturally in the original family by applying Fujita's classification of singular del Pezzo varieties [15], and finally by using the continuity method strategy. The explicit form of Theorem 3.1 follows by the GIT analysis [3]. For $n = 3$, Theorem 3.1 is a bit special, since we hit the boundary of Theorem 2.2's inequality. Nevertheless, it can still be proven via rigidity arguments, avoiding to use the volume gap. We remark that the existence of KE on all smooth intersections of two quadrics was known before [2], but for cubics threefolds it was known only for special cases. The generic singularity in the above examples is an ODP. However, from a metric viewpoint, it is still unknown the full asymptotic to the Stenzel's CY cone metric [20].

Related to the above theorems, there are some interesting algebro geometric questions which deserve further investigation. In particular: are there other cases in higher dimension where Theorem 2.2 applies? Is indeed true that all GIT semistable cubics are normal in every dimension? What can we say about KE limits of Fano manifolds of Picard rank one and degree 22? Namely, do they embed as intersections of three sections of a tautological bundle on a Grassmannian similarly to the smooth case? Can one study explicitly the associated GIT problem?

Finally, another direction which is interesting to explore is the so-called *log case*, i.e., of Fano pairs (X, D) with $D = \sum_i (1-\beta_i)D_i$, $\beta_i \in (0,1)$ admitting singular KE metrics with $2\pi\beta_i$ cone singularities at the generic points of D. This situation is not trivial, but well-understood, already in dimension one. It is known [27, 39] that the existence of a KE metric on log-$\mathbb{P}^1$s, i.e., $(\mathbb{P}^1, \sum_{i=1}^n (1-\beta_i)p_i)$ with $\beta_i \in (0,1)$ and $d = 2 - \sum_i (1-\beta_i) > 0$, is equivalent to the Troyanov's condition $1-\beta_i < \sum_{j\neq i}(1-\beta_j)$ for $n \geq 3$, and $\beta_1 = \beta_2$ for $n=2$. How the natural KE/K-compactification looks in this case? For fixed n, d and values of the cone angles, the only thing that can happen for limits is that *points collide*, since all the (marked) GH limits must still be given by a log-$\mathbb{P}^1$s by Gauss-Bonnet. To understand the possible limits one can use the moduli continuity method. It is in fact easy to see that the Troyanov's condition is equivalent (at least for rational angles) to GIT-stability for the rational polarization $\mathcal{L}_{\underline{\beta}} := \boxplus_{i=1}^n \mathcal{O}(1-\beta_i)$ on $(\mathbb{P}^1)^n$ (weighted points). Thus, by the continuity method, we have $\overline{\mathcal{EM}}^{GH}_{n,d,\underline{\beta}} \cong (\mathbb{P}^1)^n //_{\underline{\beta}} SL(2)$. Note that, by varying the cone angles, we obtain birational modifications of the moduli spaces.

In higher dimension, the situation is definitely more subtle, since also the divisor D_i may become quite singular in the limits. Even in dimension two, it is going to be essential to use the new advances related to valuations and the expected tangent cone description [10] to control the singularities of the divisors. Natural first steps to investigate are the KE-compactifications of moduli of $(\mathbb{P}^2, (1-\beta)D)$, with D degree $d \geq 3$ hypersurface and, for example, cubic surfaces with an hyperplane section [17]. At least for cone angles big enough, it is expected that the KE/K-compactification does agree with some GIT quotient naturally associated to the problem [18].

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GIT Stability, K-Stability and the Moduli Space of Fano Varieties

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Abstract This is a slightly extended version of the lecture notes of a mini-course in the workshop of *Moduli of K-stable Varieties* given by the author, in which the main construction of the proper moduli space of $\mathbb{Q}$-Gorenstein smoothable K-semistable Fano varieties in Li et al. (On proper moduli space of smoothable Kähler-Einstein Fano varieties, ArXiv:1411.0761 v3, 2014) is outlined.

Keywords Geometric invariant theory (GIT) · Symplectic quotient · K-stability · Kähler-Einstein metric · $\mathbb{Q}$-Fano variety

1 Introduction

Constructing moduli spaces for polarized algebraic varieties is a fundamental problem in algebraic geometry. One of the main motivation of Geometric Invariant Theory (GIT) invented by Mumford [37] is exactly for this purpose. In particular, it has been successfully applied to construct the moduli space of canonically polarized varieties in dimension one. For the two dimensional case, in [16] David Gieseker successfully applied GIT to prove the existence of a quasi-projective moduli space for minimal surfaces of general type. The advantage of GIT is that it produce a projective coarse moduli, however, quite often it is difficult to characterize the singular varieties appearing in the GIT compactification even for dimension two (cf. [43]). However, if one wants to have a geometrically *natural* compactification for these moduli spaces, the GIT method in its classical form fails to produce that (cf. [49]). Thanks to the recent breakthrough coming from the theory of the Minimal Model Program (MMP) (see [3] etc.), there is a canonical projective moduli space parameterizing KSBA-stable varieties, named after Kollár-Shepher-Barron-Alexeev (cf. [19, 23]).

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As for Fano varieties, the story is much subtler due to the appearance of the *continuous* automorphism group and the difficulties that MMP becomes less canonical in the Fano situation. Fortunately, the recent breakthrough in the Kähler-Einstein problem, namely the solution to the Yau-Tian-Donaldson Conjecture [4–6, 48] in a sense play an alternative role to [3] in canonically polarized case. In [39], the authors make the first progress by constructing a proper moduli spaces for K-semistable Del Pezzo surfaces. They have taken the advantage of the existence of explicit GIT moduli spaces constructed by algebraic geometers in dimension two, which is hard to generalize to higher dimensions.

In this note, we will outline our construction of moduli space of $\mathbb{Q}$-Gorenstein smoothable K-semistable Fano varieties of any dimension in [33]. The new feature of our construction is that it is a hybrid of classical GIT and the Gromov-Hausdorff compactification of the moduli of Kähler-Einstein manifolds. To put in another word, we successfully endow a *proper* topological moduli space with an algebraic structure. This process can be regarded as an generalization of the classical fact *GIT=symplectic quotient*, the difficulty here is that our situation is neither a classical GIT nor a symplectic quotient but luckily each aspect supplies exactly what is missing in its counterpart.

Before we close the introduction, let us summarize the organization of the paper. In Sect. 2, we recall some basic facts on classical GIT; in Sect. 3, we introduce the definition of (log) K-stability and the master space Z^* that will be used for our construction of the moduli space; in Sect. 4, we sketch the main ideas of proving the separatedness and Zariski openness of K-semistability; in Sect. 5, we present the main construction; in Sect. 6, we state our results on projectivity of the moduli space constructed in Sect. 5. Finally, in the last section we propose some problems for future study.

2 GIT and Symplectic Quotient

In this section, we recall some basics of GIT and symplectic quotient which serve as the classical way of constructing proper algebraic moduli space for algebro-geometric objects. Let us start with a reductive algebraic group G acting on a projective variety $(Z, \mathcal{O}_Z(1))$ polarized by a *very ample*[1] line bundle $\mathcal{O}_Z(1)$ such that it is G*-linearized*, i.e. there is a G-action on the total space of $\mathcal{O}_Z(1)$ covering the G-action of Z such that $\forall g \in G$, the isomorphism $g : \mathcal{O}_Z(1)|_z \to \mathcal{O}_Z(1)|_{g\cdot z}$ is *linear*. Now let us fix a maximal subgroup $K < G$ together with a K-invariant Hermitian metric $(\mathcal{O}_Z(1), h) \to (Z, \omega)$ with *positive* curvature form ω. It is known that (cf. [37]) $\mathcal{O}_Z(1)$ being linearized is equivalent to the existence of a

[1]We assume $\mathcal{O}_Z(1)$ being *very* ample only to simplify our notation.

K-equivariant *moment map*

$$\mu_K : Z \longrightarrow \mathfrak{k}^* \quad \text{satisfying} \quad d\langle \mu(z), \xi \rangle_{\mathfrak{k}} = \omega(\cdot, \sigma_z(\xi)), \ \forall \xi \in \mathfrak{k} = \mathrm{Lie}(K). \tag{1}$$

where $\sigma_z : \mathfrak{k} \longrightarrow T_z Z$ denotes the infinitesimal action and $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is the Ad_K-invariant inner product on $\mathfrak{k}$, with respect to which we identify $\mathfrak{k}$ and its dual $\mathfrak{k}^*$ canonically.[2]

Definition 2.1 (Hilbert-Mumford (cf. Ch. 2 of [37])) Let $z \in Z$ and $\lambda : \mathbb{C}^\times \to G$ be a *one parameter subgroup* (1-PS) of G, we define

$$w_z(\lambda) := \text{ weight of } \mathbb{C}^\times \curvearrowright \mathcal{O}_Z(1)\big|_{z_0} \text{ with } z_0 := \lim_{t \to 0} \lambda(t) \cdot z.$$

We say z is *semistable* with respect to λ if $w_z(\lambda) \geq 0$, and z is *semistable* if it is semistable with respect to any 1-PS of G; z is *polystable* if it is semistable with $w_z(\lambda) = 0$ if and only if $z_0 = \lim_{t \to 0} \lambda(t) \cdot z \in O_z := G \cdot z$ and is *stable* if z is polystable and its stabilizer $G_z < G$ is finite.

Theorem 2.2 (Kempf-Ness and Kirwan (cf. Ch. 8 of [37])) *$z \in Z$ is semistable (resp. polystable) if and only if $\inf_{y \in O_z} |\mu(y)| = 0$ (resp. $\min_{y \in O_z} |\mu(y)| = 0$) or equivalently $\overline{O_z} \cap \mu^{-1}(0) \neq \varnothing$ (resp. $O_z \cap \mu^{-1}(0) \neq \varnothing$). Furthermore, we have a homeomorphism (with respect to the complex analytic topolgy)*

$$\begin{array}{ccc} Z^{\mathrm{ss}} /\!\!/ G := \mathrm{Proj} \bigoplus_{k \geq 1} H^0(Z, \mathcal{O}_Z(k))^G & \longrightarrow & \mu^{-1}(0) /\!\!/ K \\ \overline{O_z} \cap Z^{\mathrm{ss}} & \longmapsto & K \cdot z^{\min} \end{array} \tag{2}$$

where $Z^{\mathrm{ss}} \subset Z$ denotes the Zariski open subset of semistable points in Z and $O_{z^{\min}} \subset \overline{O_z} \cap Z^{\mathrm{ss}}$ is the unique closed orbit satisfying $|\mu(K \cdot z^{\min})| = 0$.

Example 2.3 Consider the $G = \mathbb{C}^\times$-action on $(Z, \mathcal{O}_Z(1)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ given by $t \cdot [x, y, z] = [tx, t^{-1}y, z]$. Then $\mathcal{O}_Z(1)$ is G-linearized and the moment map for $K = \mathrm{U}(1) < G = \mathbb{C}^\times$ is given by:

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\ \mu\ } & \mathfrak{k} \cong \sqrt{-1}\mathbb{R} \\ z := [z_0, z_1, z_2] & \longmapsto & \dfrac{\sqrt{-1}(|z_0|^2 - |z_1|^2)}{2\pi |z|^2} \end{array}, \tag{3}$$

[2]From now on, we will abuse our notation by regarding the moment map μ_K to be $\mathfrak{k}$ valued.

and we have $Z^{\mathrm{ss}} = \mathbb{P}^2 \setminus \{[1,0,0],[0,1,0]\}$ and $[0,0,1] \in Z^{\mathrm{ss}}$ is the only *strictly polystable* point. Furthermore, we have homeomorphism:

$$\mathbb{P}^1 \cong \frac{\mathbb{P}^2 \setminus \{[1,0,0],[0,1,0]\}}{\mathbb{C}^\times} \longrightarrow \{|z_0| = |z_1|\}/\mathrm{U}(1).$$

Suppose $z_0 \in Z^{\mathrm{ps}} \subset Z^{\mathrm{ss}}$, the subset of *polystable* points, then we may reduce the GIT problem from group G to a smaller $G_{z_0} < G$, which is also *reductive* by Matsushima's theorem [38]. To see that, let us consider the G-equivariant embedding (as $\mathcal{O}_Z(1)$ is very ample)

$$\begin{array}{ccc} \mathcal{O}_Z(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^M}(1) \\ \downarrow & & \downarrow \\ Z & \xhookrightarrow{H^0(\mathcal{O}_Z(1))} & \mathbb{P}^M \end{array} . \tag{4}$$

Since G_{z_0} is reductive, there is a maximal compact subgroup $K < G$ satisfying $G_{z_0} = (K_{z_0})^{\mathbb{C}} := (G_{z_0} \cap K)^{\mathbb{C}}$. Let $\mathfrak{k}_{z_0} = \mathrm{Lie}(K_{z_0})$ be the Lie algebra and we fix a bi-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on $\mathfrak{k}$ and let $\mathfrak{k}_{z_0}^\perp \subset \mathfrak{k}$ be its orthogonal complement with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$. Then the infinitesimal action $\sigma_{z_0} : \mathfrak{g} \longrightarrow T_{z_0}\mathbb{P}^M$ is G_{z_0}-*equivariant* in the sense that

$$\sigma_{z_0}(\mathrm{Ad}_g \xi) = g \cdot \sigma_{z_0}(\xi) \ \text{ for all } g \in G_{z_0} ,$$

and there is a G_{z_0}-invariant linear subspace $z_0 \in \mathbb{P}W := \mathbb{P}(W' \oplus \mathbb{C}\hat{z}_0) \subset \mathbb{P}^M$ such that

$$\mathbb{P}^M = \mathbb{P}(W' \oplus (\mathfrak{k}_{z_0}^\perp)^{\mathbb{C}}) = \mathbb{P}(W \oplus \mathbb{C}\hat{z}_0 \oplus (\mathfrak{k}_{z_0}^\perp)^{\mathbb{C}}) \text{ with } (\mathfrak{k}_{z_0}^\perp)^{\mathbb{C}} := \mathfrak{k}_{z_0}^\perp \otimes \mathbb{C}, \tag{5}$$

where $0 \neq \hat{z}_0 \in \mathbb{C}^{M+1}$ is a lift of $z_0 \in \mathbb{P}^M$ and $\mathbb{C}^{M+1} = W' \oplus \mathbb{C}\hat{z}_0 \oplus (\mathfrak{k}_{z_0}^\perp)^{\mathbb{C}}$ is a decomposition as G_{z_0}-module. Now consider the *multiplication* morphism

$$\begin{array}{ccc} G \times \mathbb{P}W & \xrightarrow{\ \phi\ } & G \cdot \mathbb{P}W \subset \mathbb{P}^M \\ (g, w) & \longmapsto & g \cdot w \end{array} \tag{6}$$

then for $\xi \in \mathfrak{g}_{z_0}$ and $\delta w \in T_{z_0}\mathbb{P}W$ we have

$$d\phi|_{(e,z_0)}(\xi, \delta w) = \sigma_{z_0}(\xi) + \delta w \in T_{z_0}\mathbb{P}^N \cong (\mathfrak{k}_{z_0}^\perp)^{\mathbb{C}} \oplus T_{z_0}\mathbb{P}W$$

where $e \in G$ is the identity, and as a consequence $\ker d\phi|_{(e,z_0)} = \mathfrak{g}_{z_0}$. Now let us define an open set

$$U_0 := \left\{ w \in \mathbb{P}W \,\middle|\, \mathrm{rk}\left(q \circ d\phi|_{\{1\}\times\mathbb{P}W} : \mathfrak{g} \times T\mathbb{P}W \to (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W \right) = \dim \mathfrak{g}_{z_0}^{\perp} \right\} \subset \mathbb{P}W$$

with $q : T\mathbb{P}^N|_{\mathbb{P}W} \to (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W$ being the quotient morphism between vector bundles over $\mathbb{P}W$. Then it follows from the Implicit Function Theorem that:

Lemma 2.4 *$U_0 \subset \mathbb{P}W$ is a G_{z_0}-invariant Zariski open set and $\forall z \in U_0$ we have* $\ker(\sigma_z) \subset \mathfrak{g}_{z_0}$.

Notice that one only needs G_{z_0} being *reductive* to obtain Lemma 2.4 above, and hence a purely infinitesimal statement. But if we evoke the *polystability* of z_0, a *global* property of z_0, then we obtain a *stronger* and *global* consequence. To state it let us introduce

$$G \times_{G_{z_0}} U_W := G \times U_W / \sim \ \text{ with } (g, w) \sim (gh, h^{-1}w) \ \ \forall h \in G_{z_0},$$

then the morphism ϕ (cf. (11)) descends to the quotient $G \times_{G_{z_0}} \mathbb{P}W$ which by abusing of notation is still denoted by:

$$\begin{array}{ccc} G \times_{G_{z_0}} \mathbb{P}W & \xrightarrow{\ \phi\ } & G \cdot \mathbb{P}W \subset \mathbb{P}^M \\ (g, w) & \longmapsto & g \cdot w \end{array} \tag{7}$$

Theorem 2.5 (Luna's slice Theorem (cf. Theorem 1.12 of [44] and Theorem 5.3 of [10])) *Let G be a reductive algebraic group acting on $\mathbb{P}^M$ via a representation $\rho : G \to \mathrm{SL}(M+1)$ as above, hence $\mathcal{O}_{\mathbb{P}^M}(1)$ is G-linearized. Suppose $z_0 \in \mathbb{P}^M$ is G-polystable then we have*

(1) *G_{z_0} is reductive.*
(2) *There is a G_{z_0}-invariant Zariski open set $z_0 \in U_W \subset U_0 \subset \mathbb{P}W$ as above such that the morphism*

$$\begin{array}{ccc} G \times_{G_{z_0}} U_W & \xrightarrow{\ \phi\ } & G \cdot U_W \subset \mathbb{P}^M \\ (g, w) & \longmapsto & g \cdot w \end{array} \tag{8}$$

is strongly étale (cf. [10, Definition 4.14]). In particular, we have $G_w < G_{z_0}$ for any $w \in U_W$. and after a possible shrinking of U_W, we may assume that ϕ is an isomorphism.

We remark that although K-stability is a not GIT stability and we do not have Luna's slice Theorem a priori, we will establish all the consequences of Theorem 2.5 via a complete different mean. These consequences allow us to construct local affine chart for our moduli space.

Example 2.6 Applying Theorem 2.5 to the Example 2.3, we obtain:

(1) If $z_0 = [1, 1, 0]$ then $G_{z_0} = \{\pm 1\}$ and $U_W = \mathbb{P}W \setminus [0, 0, 1] \subset \mathbb{P}W = \{x - y = 0\} \subset \mathbb{P}^2$ such that

$$\begin{array}{ccc} \mathbb{C}^\times \times_{\{\pm 1\}} U_W & \xrightarrow{\ \phi\ } & \mathbb{P}^2 \setminus \{[0,1,0],[1,0,0]\} \subset \mathbb{P}^2 ; \\ (g, w) & \longmapsto & g \cdot w \end{array} \tag{9}$$

is an isomorphism to $\mathrm{Im}\phi = \mathbb{C}^\times \cdot U_W$.

(2) If $z_0 = [0, 0, 1]$ then $G_{z_0} = G$ and $U_W = \mathbb{P}W \setminus \{[1, 0, 0], [0, 1, 0]\} \subset \mathbb{P}W = \mathbb{P}^2$ such that

$$\begin{array}{ccc} \mathbb{C}^\times \times_{\mathbb{C}^\times} U_W & \xrightarrow{\ \phi\ } & \mathbb{P}^2 \setminus \{[0,1,0],[1,0,0]\} \subset \mathbb{P}^2 . \\ (g, w) & \longmapsto & g \cdot w \end{array} \tag{10}$$

is an isomorphism $\mathrm{Im}\phi = \mathbb{C}^\times \cdot U_W$.

Unfortunately, it is not known that K-stability can be fitted into a classical GIT problem (cf. [13, 40]). In particular the *ampleness* of the CM-line bundle first introduced by Tian [47] (cf. also [40]) over the Hilbert scheme is lacking. So instead we develop an alternative approach, to achieve that let us introduce the following:

Assumption 2.7 (Properness) There is a *closed* K-invariant subset

$$\Sigma \hookrightarrow \mathbb{P}^M$$

satisfying:

(1) $\forall z \in \mathbb{P}^M$, $(G \cdot z) \cap \Sigma$ consists of *at most one* K-orbit. Σ is *continuous* in the sense that for any sequence of $\{z_i\}_{i=1}^\infty \subset \mathbb{P}^M$ satisfying $(G \cdot z_i) \cap \Sigma \neq \varnothing$ and $\lim_{i \to \infty} z_i = z_\infty \in \Sigma$, we have

$$\lim_{i \to \infty} \mathrm{dist}_{\mathbb{P}^M}((G \cdot z_i) \cap \Sigma, K \cdot z_\infty) = 0.$$

(2) $G_z = (G_z \cap K)^{\mathbb{C}}$ for all $z \in \Sigma$.

Now ϕ is G_{z_0}-*invariant* with respect to the action $h \cdot (g, w) = (gh^{-1}, h \cdot w)$, hence it descends to a K-invariant map, which by abuse of notation it is still denoted by

$$\begin{array}{ccc} G \times_{G_{z_0}} \mathbb{P}W & \xrightarrow{\ \phi\ } & G \cdot \mathbb{P}W \subset \mathbb{P}^M . \\ (g, w) & \longmapsto & g \cdot w \end{array} \tag{11}$$

Moreover, it is a *bi-holomorphism* (see the proof of [44, Theorem 1.12]) from a K-invariant tubular neighborhood

$$N_\epsilon(K\cdot z_0) := \left\{(g\exp\sqrt{-1}\xi, w)\in G\times_{G_{z_0}} V \,\middle|\, g\in K,\ \xi\in\mathfrak{k}_{<\epsilon}\right\} \text{ with } \mathfrak{k}_{<\epsilon} := \{\xi\in\mathfrak{k}\mid |\xi|<\epsilon\} \tag{12}$$

of the orbit $K\cdot z_0 \cong K/K_{z_0}$ onto $\phi(U_\epsilon) = K\cdot\exp\mathfrak{k}_{<\epsilon}\cdot V$ for $0<\epsilon\ll 1$, where $z_0\in V\subset\mathbb{P}W$ is a K-invariant *analytic* open neighborhood.

Now suppose $\tilde g = g\cdot\exp\sqrt{-1}\xi$ satisfies $g\in K$ and $\xi\in\mathfrak{k}$ with $|\xi|<\epsilon$ such that $\tilde g\cdot w = w\in V$ then:

$$\phi(g\cdot\exp\sqrt{-1}\xi, w) = \phi(\tilde g, w) = \tilde g\cdot w = w = \phi(e,w) \text{ and } (\tilde g, w)\in N_\epsilon(K\cdot z_0).$$

These together with the fact that $\phi|_{U_\epsilon}$ is bi-holomorphic imply that

$$(\tilde g, w) \overset{G_{z_0}}{\sim} (e,w)\in G\times\mathbb{P}W$$

i.e. there is a $h\in G_{z_0}$ such that $(\tilde g h^{-1}, hw) = (e,w)$, hence $\tilde g = h\in G_{z_0}\cap G_w$. To summarize, we obtain the following:

Lemma 2.8 *Let $w\in V\subset\mathbb{P}W$ (defined in (12)) and suppose $\tilde g\in G_w$ is of the form $\tilde g = g\cdot\exp\xi$ with $g\in K$ and $\xi\in\mathfrak{g}$ satisfies $|\xi|<\epsilon$. Then $\tilde g\in G_{z_0}$.*

Theorem 2.9 *Let K be a compact Lie group acting on $\mathbb{P}^M$ via a representation $K\to \mathrm{U}(M+1)$ and $G=K^{\mathbb{C}}$ be its complexification. Let $z_0\in\mathbb{P}^M$ with its stabilizer G_{z_0} satisfying $G_{z_0} = (G_{z_0}\cap K)^{\mathbb{C}}$ and $z_0\in\Sigma\subset\mathbb{P}^M$ satisfying* Condition 2.7. *Then there is an G_{z_0}-invariant Zariski open neighborhood $z_0\in U^{\mathrm{sp}}\subset\mathbb{P}W$ such that for $\forall w\in U^{\mathrm{sp}}\cap G\cdot\Sigma$ we have $G_w < G_{z_0}$.*

Proof First, it suffices for us to prove our statement for an *analytic* neighborhood, then by the constructibility we can pass it to a Zariski open neighborhood.

Suppose Assumption 2.7 holds, then the continuity of the slice Σ implies that there is a sufficiently small analytic K_{z_0}-invariant neighborhood $z_0\in\tilde V\subset V\subset\mathbb{P}W$ such that for any $w\in\tilde V\cap(G\cdot\Sigma)$, there is a $\xi\in(\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}}$ satisfying $|\xi|<\delta<\epsilon$ and $z\in\Sigma$ such that $w=\exp\xi\cdot z$. In particular, $\exp\xi\cdot K_z\cdot\exp(-\xi)\subset G_w$ is a maximal compact subgroup of G_w. Since $K_z<K$ is compact we have

$$\begin{aligned}\exp\xi\cdot K_z\cdot\exp(-\xi) &= \{h\cdot\exp(\mathrm{Ad}_{h^{-1}}\xi)\cdot\exp(-\xi)\mid h\in K_z\}\\ &\subset\{g\cdot\exp\sqrt{-1}\zeta\,|\,\zeta\in\mathfrak{g},\ |\zeta|<\epsilon \text{ and } g\in K\}\,.\end{aligned}$$

By Lemma 2.8, we must have $\exp(-\xi)\cdot K_z\cdot\exp\xi\subset G_{z_0}$. Hence

$$G_{z_0}\supset\Big(\exp(-\xi)\cdot K_z\cdot\exp\xi\Big)^{\mathbb{C}} = G_w,$$

since G_{z_0} is reductive. Finally, one notices that the set

$$\{w \in \mathbb{P}W \mid G_w < G_{z_0}\} \supset G_{z_0} \cdot \tilde{V}$$

is G_{z_0}-invariant and constructible. This allows us to choose a G_{z_0}-invariant Zariski open subset $U^{\mathrm{sp}} \supset G_{z_0} \cdot \tilde{V}$, and our proof is completed. □

Assumption 2.10 (Stabilizer Preserving) There is a G_{z_0}-invariant Zariski open neighborhood of $z_0 \in U^{\mathrm{sp}} \subset \mathbb{P}W$ such that $G_w < G_{z_0}$ for all $w \in U^{\mathrm{sp}}$.

Example 2.11 Notice that Assumption 2.10 does not hold in general, even in the situation that a 1-PS $\alpha(t)$ degenerating $\lim_{t\to 0} \alpha(t) \cdot z = z_0$, we cannot conclude that $G_{z_t} < G_{z_0}$. Consider the SL(2)-action on $\mathbb{P}(\mathrm{Sym}^{\otimes 3}\mathbb{C}^2)$. The 1-PS

$$\alpha(t) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} t + 1/t & -t + 1/t \\ -t + 1/t & t + 1/t \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

degenerates $p(X, Y) = (X - Y)(X - \zeta Y)(X - \zeta^2 Y)$ to $p_0(X, Y) = \frac{3}{4}(X - Y)(X + Y)^2 \in \mathbb{P}(\mathrm{Sym}^{\otimes 3}\mathbb{C}^2)$. Then $\mathbb{Z}/3\mathbb{Z} \cong \mathrm{SL}(2)_{p_t} \not\subset \mathrm{SL}(2)_{p_0} = \langle 1 \rangle$, and the map

$$\mathrm{SL}(2) \times_{\mathbb{G}_m} \mathbb{P}W \longrightarrow \mathrm{SL}(2) \cdot \mathbb{P}W$$

is *not* finite.

Twisting the linearization of G_{z_0} on $\mathcal{O}_{\mathbb{P}M}(1)|_{\mathbb{P}W}$ by the inverse of the character corresponding to the action $G_{z_0} \curvearrowright \mathcal{O}_{\mathbb{P}M}(1)|_{z_0}$, we obtain that $z_0 \in \mathbb{P}W$ is GIT-polystable with respect to the new G_{z_0}-linearization on $\mathcal{O}_{\mathbb{P}M}(1)|_{\mathbb{P}W}$. Let $U_W^{\mathrm{ss}} \subset \mathbb{P}W$ denote the GIT-semistable points with respect to this linearization and

$$\pi_W : \mathbb{P}W \supset U_W^{\mathrm{ss}} \longrightarrow \mathcal{M} := \mathbb{P}W /\!\!/ G_{z_0} \text{ with } \pi_W(z_0) = 0 \in \mathcal{M} \tag{13}$$

denote the GIT quotient map. Let $0 \in B_{\mathcal{M}}(0, r) \subset \mathcal{M}$ be the *open* ball of radius r with respect to a prefixed continuous metric. Then for each $r > 0$, we introduce

Definition 2.12 Let $U_{W,r}^{\mathrm{ss}}$ be the *connected component* of

$$(G \cdot \pi_W^{-1}(B(0, r))) \cap \mathbb{P}W \subset U^{\mathrm{ss}}$$

containing z_0. In particular, $U_{W,r}^{\mathrm{ss}}$ is G_{z_0}*-invariant.*

Remark 2.13 Under the Assumption 2.10, $U_{W,r}^{\mathrm{ss}}$ is actually G_{z_0}-invariant. For simplicity, let us first assume that G_{z_0} is finite, we notice the G_{z_0} acts on $G \cdot z \cap U_{W,r}^{\mathrm{ss}}$ then our claim follows from the following

$$G \cdot z \cap U_{W,r}^{\mathrm{ss}} = \mathrm{mult}_{z_0}(\overline{G \cdot z_0}, \mathbb{P}W) = |G_{z_0}| \;\text{ for } 0 < r \ll 1.$$

The argument for general G_{z_0} is similar.

Let $[\cdot] : G \to G/G_{z_0}$ denote the quotient map. We say a sequence $\{h_i\} \subset G$ is *bounded* in G/G_{z_0} if and only if $\{\psi^{-1}([h_i])\}$ is contained in a *bounded* subset of $K \times_{K_{z_0}} (\sqrt{-1}\mathfrak{k}_{z_0}^{\perp})$, where ψ is the Cartan decomposition (cf. [44, equation (1.8)])

$$\begin{array}{rccc} \psi : K \times_{K_{z_0}} (\sqrt{-1}\mathfrak{k}_{z_0}^{\perp}) & \longrightarrow & G/G_{z_0} & , \\ (g, \sqrt{-1}\xi) & \longmapsto & (g \cdot \exp \sqrt{-1}\xi) \cdot G_{z_0} & \end{array} \tag{14}$$

which is a K-equivariant *diffeomophism.*

Assumption 2.14 (Finite Distance) An analytic *open neighborhood* of $z_0 \in U^{\mathrm{fd}} \subset \mathbb{P}W$ is of *finite distance* if there is a *bounded* (in the sense above) set $G_{U^{\mathrm{fd}}} \Subset G/G_{z_0}$ depending only on U^{fd} and z_0 such that for any pair $(z, g) \in U^{\mathrm{fd}} \times G$ satisfying $g \cdot z \in U^{\mathrm{fd}}$, there is an $h \in G,\ [h] \in G_{U^{\mathrm{fd}}} \Subset G/G_{z_0}$ such that $g \cdot z = h \cdot z$, where $[\cdot] : G \to G/G_{z_0}$ is the quotient map. It follows from the definition that U^{fd} is G_{z_0}*-invariant.*

Lemma 2.15 *Suppose both Assumptions 2.10 and 2.14 are satisfied. Then there is a positive* $\epsilon > 0$ *such that for any* $0 < r < \epsilon$, $U^{\mathrm{ss}}_{W,r}$ *satisfies the following: for any sequence* $\{(g_i, y_i)\} \in G \times_{G_{z_0}} U^{\mathrm{ss}}_{W,r}$ *satisfying* $z_i = g_i \cdot y_i \to z_\infty \in G \cdot U^{\mathrm{ss}}_{W,r}$, *as* $i \to \infty$, *after passing to a subsequence, there is a*

$$(g_\infty, y_\infty) \in \overline{\{(g_i, y_i)\}_i} \subset G \times_{G_{z_0}} U^{\mathrm{ss}}_{W,r} \text{ such that } g_\infty \cdot y_\infty = z_\infty.$$

In particular, the map $\phi|_{G \times_{G_{z_0}} U^{\mathrm{ss}}_{W,r}} : G \times_{G_{z_0}} U^{\mathrm{ss}}_{W,r} \to G \cdot U^{\mathrm{ss}}_{W,r}$ *is a finite morphism.*

Proof First, we notice that after translating z_∞ by a $g \in G$ if necessary, we may assume that $z_\infty \in U^{\mathrm{ss}}_{W,r}$. Since we can always pass to a subsequence, we may and will assume $y_i \xrightarrow{i \to \infty} z_\infty \in U^{\mathrm{ss}}_{W,r}$ after a possible decreasing of r as $\overline{U^{\mathrm{ss}}_{W,r}} \subset \mathbb{P}W$ is compact by Definition 2.12.

By Assumption 2.14, we may choose $0 < r \ll 1$ such that $U^{\mathrm{ss}}_{W,r} \subset U^{\mathrm{fd}}$ then there is a sequence $\{h_i\} \subset G$, with $\{[h_i]\}$ being bounded in G/G_{z_0} and satisfying $g_i \cdot y_i = h_i \cdot y_i$, hence $h_i^{-1} \cdot g_i \in G_{y_i},\ \forall i$. Now by Assumption 2.10, we have

$$h_i^{-1} \cdot g_i \in G_{y_i} < G_{z_0},\ \forall i,$$

from which we conclude that $\{[g_i]\}$ is *bounded* in G/G_{z_0} and hence the set $\{(g_i, y_i)\} \subset G \times_{G_{z_0}} U^{\mathrm{ss}}_{W,r}$ is precompact. Thus the morphism $\phi|_{G \times_{G_{z_0}} U^{\mathrm{ss}}_{W,r}} : G \times_{G_{z_0}} U^{\mathrm{ss}}_{W,r} \to G \cdot U^{\mathrm{ss}}_{W,r}$ is a proper and étale morphism hence finite. □

3 K-Stability and Properties of Moduli Space

In this section, let us first recall the definition of K-stability. Let X be a Fano manifold then for $r \gg 1$,[3] we have the embedding

$$(X, \mathcal{O}_X(-rK_X)) \subset (\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \text{ with } N+1 = \dim H^0(X, \mathcal{O}_X(-rK_X)).$$

Definition 3.1 Let X be a normal projective variety with an effective $\mathbb{Q}$-divisor D. Suppose that

- (X, D) admits at worst *Kawamata log terminal* (klt) singularities (see [21, 2.34]);
- $-(K_X + D)$ is an ample $\mathbb{Q}$-Cartier divisor.

Then we call (X, D) a *log* $\mathbb{Q}$*-Fano pair* (resp. $\mathbb{Q}$*-Fano variety* if $D = 0$).

Definition 3.2 Let $(X; \mathcal{O}_X(-rK_X))$ be an n-dimensional $\mathbb{Q}$-Fano variety and $D \in |-mK_X|$ be an effective prime divisor so that $(X, D/m)$ is a log $\mathbb{Q}$-Fano pair. A *log test configuration* of $(X, D/m; \mathcal{O}_X(-rK_X))$ consists of

(1) A projective flat morphism $\pi : (\mathcal{X}, \mathcal{D}; \mathcal{L}) \to \mathbb{A}^1$;
(2) A $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, such that π is $\mathbb{G}_m$-equivariant with respect to the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ via multiplication;
(3) $\mathcal{L}$ is relative ample and we have a $\mathbb{G}_m$-equivariant isomorphism

$$(\mathcal{X}^\circ, \mathcal{D}^\circ; \mathcal{L}|_{\mathcal{X}^\circ}) \cong (X \times \mathbb{G}_m, D \times \mathbb{G}_m; \pi_X^* \mathcal{O}_X(-rK_X)) \tag{15}$$

where $(\mathcal{X}^\circ, \mathcal{D}^\circ) = (\mathcal{X}, \mathcal{D}) \times_{\mathbb{A}^1} \mathbb{G}_m$ and $\pi_X : X \times \mathbb{G}_m \to X$.

A log test configuration is called a *product* test configuration if $(\mathcal{X}, \mathcal{D}; \mathcal{L}) \cong (X \times \mathbb{A}^1, D \times \mathbb{A}^1; \pi_X^* \mathcal{O}_X(-rK_X))$ where $\pi_X : X \times \mathbb{A}^1 \to X$, and a *trivial* test configuration if $\pi : (\mathcal{X}, \mathcal{D}; \mathcal{L}) \to \mathbb{A}^1$ is a product test configuration with $\mathbb{G}_m$ acting trivially on X.

To proceed, let χ denote the Hilbert polynomial and we introduce $a_i, \tilde{a}_i, b_i, \tilde{b}_i \in \mathbb{Q}$ via the following expansions.

- $\chi(X, L^{\otimes k}) := \dim H^0(X, L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$;
- $\chi(D, (L|_D)^{\otimes k}) := \dim H^0(D, L^k|_D) = \tilde{a}_0 k^{n-1} + O(k^{n-2})$;
- $w(k) :=$ weight of $\mathbb{G}_m$-action on $\wedge^{\mathrm{top}} H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0}) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$;
- $\tilde{w}(k) :=$ weight of $\mathbb{G}_m$-action on $\wedge^{\mathrm{top}} H^0(\mathcal{D}_0, \mathcal{L}^{\otimes k}|_{\mathcal{D}_0}) = \tilde{b}_0 k^n + O(k^{n-1})$.

[3]Note, r will be chosen sufficiently divisible (whose existence is guaranteed by [33, Lemma 8.3]) so that all $\mathbb{Q}$-Gorentstein smoothable K-semistable Fano varieties are embedded in $\mathbb{P}^N$ (cf. Definition of Z in (17) and Sect. 5).

Now we are ready to state the algebro-geometric criterion for the existence of conical Kähler-Einstein metric on a log Fano manifold (X, D) with cone angle $2\pi(1-(1-\beta)/m)$ along the divisor $D \in |-mK_X|$.

Definition 3.3 For a $\mathbb{Q}$-Fano variety X with $D \in |-mK_X|$ and a real number $\beta \in [0, 1]$, we define the *log generalized Futaki invariant with the angle* β as following:

$$\mathrm{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = \mathrm{DF}(\mathcal{X}; \mathcal{L}) + (1-\beta) \cdot \mathrm{CH}(\mathcal{X}, \mathcal{D}; \mathcal{L})$$

with

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) := \frac{a_1 b_0 - a_0 b_1}{a_0^2} \text{ and } \mathrm{CH}(\mathcal{X}, \mathcal{D}; \mathcal{L}) := \frac{1}{m} \cdot \frac{a_0 \tilde{b}_0 - b_0 \tilde{a}_0}{2a_0^2} \text{ (cf. [32, Definition 3.3]) }.$$

Then

$$\mathrm{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}^{\otimes r}) = \mathrm{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) .$$

We say $(X, D; L)$ is called *β-K-semistable* if $\mathrm{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, and *β-K-polystable* (resp. β-K-stable) if it is β-K-semistable with $\mathrm{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ is a product test configuration (resp. trivial test configuration).

Thanks to the linear dependence of $\mathrm{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L})$ on β, we obtain that $(X, D; L)$ is β-K-semistable for any $\beta \in (\beta_1, \beta_2]$ if $(X, D; L)$ is both β_1-K-semistable and β_2-K-polystable with $\beta_1 < \beta_2$.

Definition 3.4 Let

$$(\mathbb{H}^{\chi;N} := \mathrm{Hilb}_\chi(\mathbb{P}^N), \mathcal{O}_{\mathbb{H}}(1)) \xhookrightarrow{\text{Plücker}} (\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \tag{16}$$

denote the Hilbert scheme of closed subschemes of $\mathbb{P}^N$ with Hilbert polynomial χ and *Plücker* denote the Plücker embedding. For a closed subscheme $X \subset \mathbb{P}^N$ with Hilbert polynomial $\chi\left(X, \mathcal{O}_{\mathbb{P}^N}(k)\big|_X\right) = \chi(k)$, let $\mathrm{Hilb}(X) \in \mathbb{H}^{\chi;N}$ denote its *Hilbert* point, and

$$Z := \left\{ \mathrm{Hilb}(Y) \;\middle|\; \begin{array}{l} Y \subset \mathbb{P}^N \text{ be a smooth Fano manifold with } N = \dim H^0(Y, \mathcal{O}_Y(-rK_Y)), \\ \mathcal{O}_{\mathbb{P}^N}(1)\big|_Y \cong \mathcal{O}_Y(-rK_Y) \text{ and } \chi\left(Y, \mathcal{O}_{\mathbb{P}^N}(k)\big|_Y\right) = \chi(k). \end{array} \right\} \subset \mathbb{H}^{\chi;N} . \tag{17}$$

By the boundedness of smooth Fano manifolds with fixed dimension (see [22]), we may choose $r \gg 1$ such that Z includes all such Fano manifolds.

Now let $\overline{Z} \subset \mathbb{H}^{\chi;N}$ be the closure of $Z \subset \mathbb{H}^{\chi;N}$ and $\overline{Z}^\circ$ be the *open* subset of $\overline{Z}$ that parameterizes $\mathbb{Q}$-Fano subvarieties $X \subset \mathbb{P}^N$ such that $\mathcal{O}_Y(-rK_X) \sim \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Then one can introduce the CM line bundle $\lambda_{\mathrm{CM}} \to \overline{Z}^\circ$

(cf. Definition 6.1) and apply the GIT machinery introduced in Sect. 2 to the $\mathrm{SL}(N+1)$-invariant subscheme $\lambda_{\mathrm{CM}} \to \overline{Z}^{\circ}$. Then for any test configuration (cf. Definition 3.2) of a point $\mathrm{Hilb}(X) \in \overline{Z}^{\circ}$ coming from a 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}(N+1)$, the Donaldson-Futaki invariant DF is precisely the weight $w_{\mathrm{Hilb}(X)}(\lambda)$ with respect to the line bundle λ_{CM}. But this interpretation does not help us in the sense that $\lambda_{\mathrm{CM}} \to \overline{Z}^{\circ}$ is *not* known to be ample (cf. [13] and [50]), in particular, the traditional GIT machinery *does not* apply.

4 Separatedness and Zariski Openness of the K-Semistable Locus

Inspired by the recent book by Kollár's [20, §1.21 of Section 1.1] on the construction of KSBA moduli space and the classical GIT machinery, in order to obtain well behaved moduli spaces for a *reasonable* class of varieties **V**, it is necessary for us to first establish the following properties:

(i) *Boundedness*. The class of schemes **V** is called bounded if there is a flat morphism of schemes of *finite type* $u : U \to T$ such that every scheme in **V** occurs as a fiber of $U \to T$.

(ii) *Properness.*

(a) *Valuative criterion of properness.* Let B be a smooth curve, $B^{\circ} \subset B$ an open set and $\pi_{B^{\circ}} : \mathcal{X}^{\circ} = \mathcal{X} \times_B B^{\circ} \to B^{\circ}$ a proper, flat family whose fibers are in **V**. Then there is a finite surjection $p : A \to B$ such that there is an extension

$$\begin{array}{ccccc} \mathcal{X}^{\circ} \times_B C & =: & \mathcal{W}^{\circ} & \hookrightarrow & \mathcal{W} \\ \downarrow & & \downarrow{\scriptstyle \pi_{C^{\circ}}} & & \downarrow{\scriptstyle \pi_C} \\ B^{\circ} \times_B C & =: & C^{\circ} & \hookrightarrow & C \end{array} \qquad (18)$$

where $\pi_A : W \to A$ is also a proper, flat family whose fibers are in **V**.

(b) *Separatedness.* Suppose there are two flat families of schemes in **V** over a smooth curve B satisfying:

$$\begin{array}{ccc} \mathcal{X} & \overset{f}{\dashrightarrow} & \mathcal{X}' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & B & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}^{\circ} & \xrightarrow{f|_{\mathcal{X}^{\circ}} = \cong} & (\mathcal{X}')^{\circ} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & B^{\circ} & \end{array} \qquad (19)$$

Then f extends to an isomorphism over B.

(iii) *Zariski openness*. That is, for any flat family of varieties $\mathcal{X} \to S$, there is a *open* subscheme $T \subset S$ such that $t \in T \iff \mathcal{X}_t \in \mathbf{V}$.

In our situation the right class $\mathbf{V}$ of varieties are the $\mathbb{Q}$-Fano varieties that are *$\mathbb{Q}$-Gorenstein smoothable and K-semistable*. It is quite different from the KSBA situation in the sense that the automorphism for X might be *continuous*, thus the closed points of the moduli space are represented by *S-equivalent* (instead of isomorphic) class of K-semistable varieties. And the condition (ii-b) above should modified accordingly:

(ii-b)' *Separatedness*. If there are two extensions in (19), then $\mathcal{X}_0$ and $\mathcal{X}'_0$ are *S-equivalent*, i.e. there are two test configurations:

$$\begin{array}{ccc} (\mathcal{X}_0; \mathcal{O}_{\mathcal{X}_0}(-rK_{\mathcal{X}_0})) \times \mathbb{G}_m & \hookrightarrow & (\mathcal{Y}; \mathcal{O}_{\mathcal{Y}}(-rK_{\mathcal{Y}})) \\ \downarrow \pi_{\mathbb{G}_m} & & \downarrow \pi_{\mathbb{A}} \\ \mathbb{G}_m & \hookrightarrow & \mathbb{A}^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} (\mathcal{X}'_0; \mathcal{O}_{\mathcal{X}'_0}(-rK_{\mathcal{X}'_0})) \times \mathbb{G}_m & \hookrightarrow & (\mathcal{Y}'; \mathcal{O}_{\mathcal{Y}'}(-rK_{\mathcal{Y}'})) \\ \downarrow \pi'_{\mathbb{G}_m} & & \downarrow \pi'_{\mathbb{A}} \\ \mathbb{G}_m & \hookrightarrow & \mathbb{A}^1 \end{array}, \tag{20}$$

such that $\mathcal{Y}_0 \cong \mathcal{Y}'_0$ is a *K-polystable* $\mathbb{Q}$-Fano variety.

This makes our construction of the moduli space much subtler, and it partially explains why we have to evoke Alper's theory in Sect. 5.

In [12], Donaldson and Sun successfully established Property (i) and (ii-a). So the first goal of [33], whose proof will be sketched below, is to establish the remaining Property (ii-b) and (iii).

Theorem 4.1 *Let $\mathcal{X} \to C$ be a flat family of projective varieties over a pointed smooth curve $(C, 0)$ with $0 \in C$. Suppose*

(1) *$K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier and $-K_{\mathcal{X}/C}$ is relatively ample over C;*
(2) *for any $t \in C^\circ := C \setminus \{0\}$, $\mathcal{X}_t$ is smooth and $\mathcal{X}_0$ is klt;*
(3) *$\mathcal{X}_0$ is K-polystable.*

Then

(i) *there is a Zariski open neighborhood U of $0 \in C$ on which $\mathcal{X}_t$ is K-semistable for all $t \in U$, and K-stable if we assume further that $\mathcal{X}_0$ has a discrete automorphism group;*
(ii) *for any other flat projective family $\mathcal{X}' \to C$ satisfying (1)–(3) as above and*

$$\mathcal{X}' \times_C C^\circ \cong \mathcal{X} \times_C C^\circ,$$

we can conclude $\mathcal{X}'_0 \cong \mathcal{X}_0$;
(iii) *$\mathcal{X}_0$ admits a weak Kähler-Einstein metric $\omega_{\mathrm{KE}}(\mathcal{X}_0)$. If one assume further that $\mathcal{X}_t$ is K-polystable for all $t \in C^\circ$ then $(\mathcal{X}_0, \omega_{\mathrm{KE}}(\mathcal{X}_0))$ is the Gromov-Hausdorff limit of $\{(\mathcal{X}_t, \omega_{\mathrm{KE}}(\mathcal{X}_t)\}_{t \in C^\circ}$ as $t \to 0$, when the latter is endowed with a Kähler-Einstein metric $\omega_{\mathrm{KE}}(\mathcal{X}_t)$ (which is unique up to $\mathrm{Aut}(\mathcal{X}_t)$) for each $t \in C^\circ$.*

Our approach is a continuity method very similar to the one proposed by Donaldson in [9]. Indeed, by throwing in an *auxiliary divisor* $\mathcal{D} \in |-mK_{\mathcal{X}}|$, we consider the following log extension of Theorem 4.1.

Theorem 4.2 *For a fixed* $\beta \in [0, 1]$*, let* $\mathcal{X} \to C$ *be a flat family over a pointed smooth curve* $(C, 0)$ *with a relative codimension 1 cycle* $\mathcal{D}$ *over* C*. Suppose*

(1) $-K_{\mathcal{X}/C}$ *is ample and* $\mathcal{D} \sim_C -mK_{\mathcal{X}/C}$ *for some positive integer* $m > 1$*;*
(2) *for any* $t \in C^\circ := C \setminus \{0\}$*,* $\mathcal{X}_t$ *and* $\mathcal{D}_t$ *are smooth,* $(\mathcal{X}_0, \frac{1}{m}\mathcal{D}_0)$ *is klt;*
(3) $(\mathcal{X}_0, \mathcal{D}_0)$ *is* β*-K-polystable. (cf. Definition 3.3).*

Then

(i) *there is a Zariski neighborhood* U *of* $0 \in C$*, on which* $(\mathcal{X}_t, \mathcal{D}_t)$ *is* β*-K-semistable (in fact* β*-K-polystable if* $\beta < 1$*) for all* $t \in U$*;*
(ii) *for any other flat projective family* $(\mathcal{X}', \mathcal{D}') \to C$ *with a relative codimension 1 cycle* $\mathcal{D}'$ *satisfying* (1)–(3) *as above and*

$$(\mathcal{X}', \mathcal{D}') \times_C C^\circ \cong (\mathcal{X}, \mathcal{D}) \times_C C^\circ,$$

we can conclude $(\mathcal{X}'_0, \mathcal{D}'_0) \cong (\mathcal{X}_0, \mathcal{D}_0)$*;*
(iii) $(\mathcal{X}_0, \mathcal{D}_0)$ *admits a conical weak Kähler-Einstein metric with cone angle* $2\pi(1-(1-\beta)/m)$ *along* $\mathcal{D}_0$*, which is the Gromov-Hausdorff limit of* $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ *endowed with the conical Kähler-Einstein metric with cone angle* $2\pi(1-(1-\beta_i)/m)$ *along* $\mathcal{D}_{t_i} \subset \mathcal{X}_{t_i}$ *for any sequence* $t_i \to 0$ *and* $\beta_i \nearrow \beta$*.*

The proof of Theorem 4.2 is of algebro-geometric nature, but it is heavily based on the analytic input from the recent breakthrough made in [4–6] and [48]. To save the space, we will simply outline the ideas of the proof and ignore the technical details. It is established via the following steps[4]:

(1) Using the fact that the set of log canonical thresholds satisfies ascending chain condition (ACC) (see [17]) one can show that there is a constant $\beta_0 > 0$ depending only on the dimension n such that for $0 < \beta \le \beta_0$ there is only at most one possible extension of $(\mathcal{X}, \mathcal{D}) \times_C C^\circ \to C^\circ$ with at worst klt singularities.
(2) Fix ϵ, such that $0 < \epsilon < \beta_0$. We define a set $\mathbf{B} \subset [\epsilon, 1]$ for which the conclusions of Theorem 4.2 hold for the angles belonging to the set $\mathbf{B}$. Step one implies $\mathbf{B} \supset [\epsilon, \beta_0]$.
(3) Now to prove Theorem 4.1, let us first *assume* that all the nearby (in the analytic topology[5]) fibers $\mathcal{X}_t$ are K-semistable. Then it suffices to show that $\mathbf{B}$ is open and closed in $[\epsilon, 1)$. This based on two crucial facts. The first one says that if we consider a reductive group G-acting on $\mathbb{P}^M$ via a linear representation

[4] To avoid lengthy context of technicality, here we include a slight modification of what is already included in the introduction section of [33].

[5] which we know in the end that it also holds in the Zariski topology.

$G \to \mathrm{SL}(M+1)$ then the G-orbit structure near any p with *reductive stabilizer* $G_p < G$ admits a holomorphic fiberation over a neighborhood of a GIT polystable point thanks to the infinitesimal action of $\mathfrak{g}_p^{\perp} = \mathrm{Lie}(G_p) \subset \mathfrak{g} = \mathrm{Lie}(G)$ (cf. (5) in Sect. 2). In particular, it guarantees that there are no nearby distinct G-orbits of K-polystable points on the limiting orbit, that is, one has the local uniqueness. (cf. [33, Lemma 3.1] for the precise statement). Second, by using a crucial *Intermediate Value Theorem* type of results [33, Lemma 6.9]), we prove that if there is a different limit, which a priori could be far away from the given central fiber in the parametrizing Hilbert scheme, then we can indeed always find *another* limit which either specializes to $(\mathcal{X}_0, \mathcal{D}_0)$ in a test configuration or becomes the central fiber of a test configuration of $(\mathcal{X}_0, \mathcal{D}_0)$, violating the K-stability assumption. Similarly, this argument can also be applied to study the case when $\beta \nearrow 1$.

(4) To finish the proof, we need to verify the assumption that all the nearby fibers $\mathcal{X}_t$ are *K-semistable*. For this, one needs two observations. First, it follows from the work of [5, 6] and [48] that to check K-semistability of $\mathcal{X}_t,\ t \neq 0$, it suffices to test for all one-parameter-group (1-PS) degenerations in a fixed $\mathbb{P}^N$. Second, it follows from the classical GIT that *K-semistable threshold* (kst)[6] is a constructible function. So what remains to be shown is that it is also lower semi-continuous (also observed in [46]), which is a consequence of the upper semi-continuity of the dimension of the automorphism groups and the continuity method deployed in the proof of Theorem 4.2.

We remark that in [46] a slightly weaker result (under an additional assumption that $\mathrm{Aut}(\mathcal{X}_0)$ is finite) has been obtained along this line using a more analytical approach.

Finally, by an appropriate modification of the argument in the above we obtained in [33] the following analogy of classical GIT.

Theorem 4.3 *Suppose X is a $\mathbb{Q}$-Gorenstein smoothable K-semistable Fano manifold, then there is a test configuration $\mathcal{X} \to \mathbb{C}$ that degenerates X to a K-polystable X_0. Moreover, X_0 is uniquely determined by X.*

5 The Moduli Space Exists as a Proper Scheme

With all the necessary properties in hand (cf. Sect. 4), we are ready to outline our main construction in [33].

[6]For a $\mathbb{Q}$-Fano variety X together with a $\mathbb{Q}$-Cartier divisor $D \in |-K_X|$, we define the *K-semistable threshold* for the log pair (X, D) as following:

$$\mathrm{kst}(X, D) := \sup\{\beta \in (0, 1] \mid (X, D) \text{ is } \beta\text{-K-semistable}\}. \tag{21}$$

In particular, it is positive by [33, Theorem 5.2].

5.1 Set Up

First, we first fix our notation. Let X be a $\mathbb{Q}$-Fano variety and we fix an $r \gg 1$ sufficiently divisible such that $\mathcal{O}_X(-rK_X)$ is a *very ample* line bundle.

Definition 5.1 Let $h^{\otimes r}$ be a *continuous* Hermitian metric on $\mathcal{O}_X(-rK_X)$ and Ω_h be the corresponding volume form on $X^{\text{reg}} \subset X$, the smooth part of X. We say X admits a *Kähler-Einstein* metric, if the curvature form ω_h for the metric h satisfies

$$\omega_h^n = \Omega_h \text{ on } X^{\text{reg}}. \tag{22}$$

Let h_{KE} denote a solution to (22) and $\{s_i\}_{i=0}^N$ be an orthonormal basis of $H^0(\mathcal{O}_X(-rK_X))$ with respect to the inner product

$$|s|^2 := \int_X |s|^2_{h_{\text{KE}}} \Omega_{h_{\text{KE}}}. \tag{23}$$

The embedding given by

$$(X, \mathcal{O}_X(-rK_X)) \xrightarrow{\{s_i\}_{i=0}^N} (\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$$

is called a Tian's embedding, which is *unique* up to a $\mathrm{U}(N+1)$-translation.

Now let

$$\begin{array}{ccc} X \subset \mathcal{X} & \xrightarrow{\ i\ } & \mathbb{P}^N \times S \\ \pi \downarrow & & \downarrow \\ 0 \in S & \longrightarrow & S \end{array} \tag{24}$$

be a flat family of Kähler-Einstein $\mathbb{Q}$-Fano varieties (e.g. X is K-stable in Theorem 4.1) and $\{s_i\}$ be the local basis of the vector bundle $\pi_*\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/Z}) = \pi_*(\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}})$ induced from the coordinate sections of $\mathcal{O}_{\mathbb{P}^N}(1)$ over a open neighborhood $0 \in \{|z| < \epsilon\} \subset Z$ such that $\{s_i(0)\}$ induces Tian's embedding for X. We define a matrix

$$A_{\text{KE}}(z) := [(s_i, s_j)_{\text{KE}}(z)] \in \sqrt{-1}\mathfrak{u}(N+1)$$

with

$$(s_i, s_j)_{\text{KE}}(z) = \int_{\mathcal{X}_z} \langle s_i(z), s_j(z) \rangle_{h_{\text{KE}}^{\otimes r}} \Omega_{\text{KE}},$$

then we obtain a family of Tian's embeddings

$$T : \mathcal{X} \longrightarrow \mathbb{P}^N. \tag{25}$$

given by $\{A_{\mathrm{KE}}^{-1/2}(z) \circ s_j(z)\}_{j=0}^N$. The map T extends *continuously* to $\mathcal{X}_0 = X$ thanks to Theorem 4.1 and the continuity of the metric $h_{\mathrm{KE}}(z)$ at $0 \in S$. In particular, we have

$$A_{\mathrm{KE}}(z) = I_{N+1} + O(|z|). \tag{26}$$

Now let $\overline{Z} \subset \mathbb{H}^{\chi;N}$ be the closure of $Z \subset \mathbb{H}^{\chi;N}$ (defined in (17)) and Z° be the *open* set of $\overline{Z}$ that parameterizes the *K-semistable* $\mathbb{Q}$-Fano subvarieties Y satisfying $\mathcal{O}_Y(-rK_Y) \sim \mathcal{O}_{\mathbb{P}^N}(1)|_Y$. Notice that the Gromov-Hausdroff limits of Fano Kähler-Einstein manifolds are automatically in Z°, so are the $\mathbb{Q}$-Gorenstein smoothable K-polystable $\mathbb{Q}$-Fano varieties thanks to the seminal work of [4–6] and [48]. By [33, Lemma 8.3], the $\mathbb{Q}$-Gorenstein smoothable K-semistable $\mathbb{Q}$-Fano varieties of dimension n with a fixed volume form a bounded family. This in particular allows us to prefix a sufficiently divisible $r \gg 1$ such that $Z^\circ \subset \mathbb{H}^{\chi,N}$ contains *all* $\mathbb{Q}$-Gorenstein smoothable K-semistable $\mathbb{Q}$-Fano varieties Hilbert polynomial χ. Let Z^* to be the semi-normalization of Z°_{red} which is the reduction of Z°, the purpose of introducing Z^* is to guarantee that the *scheme structure* does not depend on $r \gg 1$ in the end.

Then we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{X}^* & \xrightarrow{i} & \mathbb{P}^N \times Z^* & \longrightarrow & \mathbb{P}^N \times Z^\circ_{\mathrm{red}} \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\ Z^* & \longrightarrow & Z^* & \longrightarrow & Z^\circ_{\mathrm{red}} \end{array} \tag{27}$$

where $\mathcal{X}^*$ is the universal family over Z^*. Notice that Z^* inherit a canonical $\mathrm{SL}(N+1)$-action from $\mathbb{H}^{\chi;N}$.

Let $(Z^*)^{\mathrm{kps}} \subset Z^*$ denote the locus of K-polystable points in Z^*, which is a constructible set thanks to the work in [33, Section 7 and 9]. By Theorem 4.3, we have $\overline{\mathrm{SL}(N+1)\cdot z} \cap (Z^*)^{\mathrm{kps}} \neq \varnothing$ for every point $z \in Z^*$. By Theorem 4.1 and [12, Theorem 1.2], the set of Hilbert points of the universal family $\mathcal{X}$ of Kähler-Einstein $\mathbb{Q}$-Fano varieties obtained via Tian's embedding induces a *proper* $\mathrm{U}(N+1)$-invariant slice

$$\begin{array}{ccccc} \Sigma_{\mathrm{KE}} := \{\mathrm{Hilb}(\mathcal{X}_z, \omega_{\mathrm{KE}}(\mathcal{X}_z)) \mid z \in (Z^*)^{\mathrm{kps}}\} & \hookrightarrow & \mathbb{H}^{\chi;N} & \xhookrightarrow{\text{Plücker}} & \mathbb{P}^M \\ & \searrow & & & \downarrow \\ & & & & \mathbb{P}^M/\mathrm{U}(N+1) \end{array} \tag{28}$$

where $\mathrm{Hilb}(\mathcal{X}_z, \omega_{\mathrm{KE}}(\mathcal{X}_z))$ denotes the Hilbert point of $\mathcal{X}_z$ corresponding to the Tian's embedding (unique up to $\mathrm{U}(N+1)$-translation) of $\mathcal{X}_z \subset \mathbb{P}^N$ using the Kähler-Einstein metric $\omega_{\mathrm{KE}}(\mathcal{X}_z)$. In particular, Σ_{KE} is *compact* and satisfied Assumption 2.7. And our goal is to endow an algebraic structure on the proper *topological* moduli space $\Sigma_{\mathrm{KE}}/\mathrm{U}(N+1)$. To be more precise, our goal is to prove the following:

Theorem 5.2 *For $N \gg 0$, let Z^* be the semi-normalization of the locus inside $\mathrm{Hilb}_\chi(\mathbb{P}^N)$ parametrizing all $\mathbb{Q}$-Gorenstein smoothable K-semistable Fano varieties in $\mathbb{P}^N$ with fixed Hilbert polynomial χ. Then the algebraic stack $[Z^*/\mathrm{SL}(N+1)]$ admits a proper semi-normal scheme $\mathcal{KF}_N$ as its good moduli space. Furthermore, for sufficiently large N, $\mathcal{KF}_N$ does not depend on N.*

To achieve that, our naive idea is to build a system of affine charts for each point in $\Sigma_{\mathrm{KE}}/\mathrm{U}(N+1)$ and then verify that they can be glued together and form a proper scheme. The framework of this process has been established by Alper in [2] and [1], which we recall first.

5.2 Alper's Framework

In this subsection, let us recall the theory developed [2] and [1].

Definition 5.3 (Section 4.1 in [2]) Let $\mathcal{X}$ be an Artin stack and M be an algebraic space. We say a morphism $\phi : \mathcal{Z} \to M$ is a *good moduli space* if

(1) The push-forward functor on quasi-coherent sheaves is exact;
(2) The induced morphism on sheaves $\mathcal{O}_M \to \phi_* \mathcal{O}_{\mathcal{Z}}$ is an isomorphism

The notion of *good moduli space* is introduced to extend the traditional GIT quotient since we have the following:

Example 5.4 (Theorem 13.6 in [2]) Let G be a reductive algebraic group acting on a polarized pair $(Z, \mathcal{O}_Z(1))$, i.e. $\mathcal{O}_Z(1)$ is G-linearized as in Sect. 2. Then the morphism from the Artin stack $\mathcal{Z} := [Z/G] \to Z /\!\!/ G$ is a *good moduli* space in the sense of Definition 5.3. Notice that GIT quotients are *good* quotients in the sense of [10, Definition 2.12].

Definition 5.5 Let $\mathcal{Z}$ be an algebraic stack of finite type over $\mathbb{C}$, and let $z \in \mathcal{Z}(\mathbb{C})$ be a closed point with reductive stabilizer G_z. We say $f_z : \mathcal{V}_z \to \mathcal{Z}$ is a *local quotient presentation around* z if

(1) $\mathcal{V}_z = [\mathrm{Spec}\, A/G_z]$, with A being a finite type $\mathbb{C}$-algebra.
(2) f_z is étale and affine.
(3) There exists a point $v \in \mathcal{V}_z$ such that $f_z(v) = z$ and f_z induces isomorphism $G_v \cong G_z$.

We say $\mathcal{Z}$ *admits a local quotient presentation* if there exists a local quotient presentation around every closed point $z \in \mathcal{Z}$.

Then we have the following

Theorem 5.6 (Theorem 10.3 in [2] and Theorem 4.1 in [1]) *Let $\mathcal{Z}$ be an algebraic stack of finite type over $\mathbb{C}$, Suppose that*

(1) *for every closed point $z \in \mathcal{Z}$, there is a local quotient presentation $f_z : \mathcal{V}_z \to \mathcal{Z}$ around z such that*

  (a) *f_z is stabilizer preserving at closed points of $\mathcal{V}_z$, i.e. for any $v \in \mathcal{V}_z(\mathbb{C})$, $\mathrm{Aut}_{\mathcal{V}_z(\mathbb{C})}(v) \to \mathrm{Aut}_{\mathcal{Z}(\mathbb{C})}(f(v))$ is an isomorphism.*

  (b) *f_z sends closed points to closed points.*

(2) *For any $\mathbb{C}$-point $z \in \mathcal{Z}$, the closed substack $\overline{\{z\}}$ admits a good moduli spaces (cf. [2, Section 1.2]).*
Then $\mathcal{Z}$ admits a good moduli space M. If we assume further that

(3) *$\mathcal{Z}$ admits a line bundle $\mathcal{L} \to \mathcal{Z}$ such that for any closed point $z \in \mathcal{Z}(\mathbb{C})$, the stabilizer $G_z < G$ acts on $\mathcal{L}|_z$ trivially.*

Then $\mathcal{L}$ descends to a line bundle L on M.

Remark 5.7 We may regard that the local quotient presentations correspond to the collection of local charts covering our moduli space, and the stabilizer preserving condition as a gluing condition for those charts. The general local condition for descending the line bundle to a good quotient, already appeared in the work [11, Theorem 2.3]. In particular, in order for $\mathcal{L}$ to descend it is *crucial* to establish the assumption (2), (3) in the Theorem 5.6. This was firstly achieved for the moduli space of Fano Kähler-Einstein varieties in [33, Section 8].

5.3 Existence of Good Moduli for $\mathbb{C}$-Points in Z^*

In this subsection, we explain how to establish the assumptions needed to apply Theorem 5.6.

Let us fix a K-polystable $\mathbb{Q}$-Fano variety X so that $\mathrm{Hilb}(X) \in Z^*$, then it admits a weak Kähler-Einstein metric by Theorem 4.1 from which we deduce that $\mathrm{Aut}(X) \subset \mathrm{SL}(N+1)$ is reductive. By abusing the notation, let $\mathrm{Hilb}(X)$ denote the Hilbert point for the Tian's embedding of $X \subset \mathbb{P}^N$ after we fix a basis of $H^0(\mathcal{O}_X(-rK_X))$. Let $\mathbb{H}^{\chi;N} \subset \mathbb{P}^M$ be the Plücker's embedding which is clearly $\mathrm{SL}(N+1)$-equivariant. Then by [8, Proposition 1] or the proof of [33, Lemma 3.1], there is an $\mathrm{Aut}(X)$-invariant linear subspace $W' \subset \mathbb{C}^{M+1}$ such that

$$\mathbb{C}^{M+1} = W \oplus \mathfrak{aut}(X)^{\perp} := W' \oplus \mathbb{C}\cdot\hat{z}_0 \oplus \mathfrak{aut}(X)^{\perp} \text{ with } \mathfrak{aut}(X)^{\perp} \oplus \mathfrak{aut}(X) = \mathfrak{sl}(N+1)$$

is a decomposition as $\mathrm{Aut}(X)$-invariant subspaces, where $0 \neq \hat{z}_0 \in \mathbb{C}^{M+1}$ is a lift of $z_0 := \mathrm{Hilb}(X) \in \mathbb{P}W \subset \mathbb{P}^M$ and we have

$$\mathbb{P}^M = \mathbb{P}(W \oplus \mathfrak{aut}(X)^\perp) = \mathbb{P}(W' \oplus \mathbb{C}z_0 \oplus \mathfrak{aut}(X)^\perp). \tag{29}$$

In particular, this induces a representation $\rho : \mathrm{Aut}(X) \to \mathrm{SL}(W)$. On the other hand, $\mathrm{Hilb}(X)$ is fixed by $\mathrm{Aut}(X)$. We let $\rho_X : \mathrm{Aut}(X) \to \mathbb{G}_m$ denote the character corresponding to the linearization of $\mathrm{Aut}(X)$ on $\mathcal{O}_{\mathbb{H}^{\chi;N}}(1)|_{\mathrm{Hilb}(X)}$ induced from the embedding $\mathrm{Aut}(X) \subset \mathrm{SL}(N+1)$. Then we can introduce the following

Definition 5.8 A point $z \in \mathbb{P}W$ is *GIT-polystable (resp. GIT-semistable)* if z is *polystable(resp. semistable)* with respect the linearization $\rho \otimes \rho_X^{-1}$ on $\mathcal{O}_{\mathbb{P}W}(1) \to \mathbb{P}W$ in the GIT sense.

To establish the assumption (1a) of Theorem 5.6, we have the following:

Theorem 5.9 (Theorem 8.5 [33]) *There is an* $\mathrm{Aut}(X)$*-invariant linear subspace* $\mathbb{P}W \subset \mathbb{H}^{\chi;N}$ *and a Zariski open neighborhood* $\mathrm{Hilb}(X) \in U_W \subset \mathbb{P}W \times_{\mathbb{H}^{\chi;N}} Z^*$ *such that for any* $\mathrm{Hilb}(Y) \in U_W$*, Y is K-polystable if and only if* $\mathrm{Hilb}(Y)$ *is GIT-polystable with respect to* $\mathrm{Aut}(X)$*-action on* $\mathbb{P}W \times_{\mathbb{H}^{\chi;N}} Z^*$.

Moreover, for all GIT-polystable $\mathrm{Hilb}(Y) \in U_W$*, we have* $\mathrm{Aut}(Y) < \mathrm{Aut}(X)$*, i.e. the local GIT presentation induced from the multiplication morphism* (11) *in Sect.* 2*:*

$$\begin{array}{ccc} U_W /\!\!/ \mathrm{Aut}(X) & \longrightarrow & Z^*/\mathrm{SL}(N+1),^{7} \\ {[w]}_{\mathrm{Aut}(X)} & \longmapsto & {[w]}_{\mathrm{SL}(N+1)} \end{array} \tag{30}$$

is stabilizer preserving in the sense of Theorem 5.6*.*

Proof To give a outline of the proof, let

$$\begin{array}{ccc} \Delta : Z^* & \longrightarrow & \mathbb{H}^{\chi;N} \times Z^* \\ z & \longmapsto & (z, z)\ . \end{array} \tag{31}$$

be the diagonal morphism, we define $O_{Z^*} := \mathrm{SL}(N+1) \cdot \Delta(Z^*) \subset \mathbb{H}^{\chi;N} \times Z^*$ where $\mathrm{SL}(N+1)$ acts *trivially* on Z^* and acts on $\mathbb{H}^{\chi;N}$ via the action induced from $\mathbb{P}^N$. This allows us to construct the family of limiting orbits space associated to the family (27) as follows:

$$\begin{array}{ccccc} \overline{BO}_z \subset & \overline{BO}_{Z^*} & \xrightarrow{\ i\ } & \mathbb{H}^{\chi;N} \times Z^* \\ \downarrow & \downarrow & & \downarrow \pi_{Z^*} \\ z \ \in & Z^* & = & Z^* \end{array} \tag{32}$$

[7] Where $[\cdot]_{\mathrm{Aut}(X)}$ and $[\cdot]_{\mathrm{SL}(N+1)}$ denote the equivalent classes of the categorical quotients of U_W and Z^* by $\mathrm{Aut}(X)$ and $\mathrm{SL}(N+1)$ respectively.

with $\overline{BO}_{Z^*} \subset \mathbb{H}^{\chi;N} \times Z^*$ be the closure of O_{Z^*} and $\overline{BO}_z$ is the union of limiting *broken orbits*. Now we claim that there is a *unique* closed K-polystable orbit inside $\overline{BO}_z$. To see this, one only needs to notice that for any $z \in Z^*$, we can always find a smooth curve $f : C \to Z^*$ that passes through z and the image $f(C)$ meets the dense open locus inside of Z^* corresponding to *smooth K-polystable Fano manifolds* with the *maximal* dimension of its $\mathrm{SL}(N+1)$-orbit space. Then our claim follows by applying Theorem 4.1 to the pull back family over C.

Now since in each $\overline{BO}_z$ there is a *unique closed orbit* in Z^*, on the other hand, as $\mathrm{Hilb}(X) \in \mathbb{P}W \times_{\mathbb{H}^{\chi;N}} Z^*$ is GIT-polystable there is an $\mathrm{Aut}(X)$-invariant Zariski open neighborhood U_W of $\mathrm{Hilb}(X) \in \mathbb{P}W \times_{\mathbb{H}^{\chi;N}} Z^*$ inside the GIT-semistable locus such that the intersection of $\overline{BO}_z$ with U_W is contains a *unique closed orbit*, i.e. the GIT-polystable orbit. This implies that these two classes of *closed orbits* must agree with each other. This finishes the proof of the first statement.

Finally to establish the last statement of Theorem 5.9, we only need to notice that the slice Σ_{KE} defined in (28) satisfies the Assumption 2.7 thanks to the following fact in [6, Theorem 4]

Lemma 5.10 *Let X be a $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano variety admitting weak Kähler-Einstein metric. Then* $\mathrm{Aut}(X) = (\mathrm{Isom}(X))^{\mathbb{C}}$. *In particular,* $\mathrm{Aut}(X) = (\mathrm{Aut}(X) \cap \mathrm{U}(N+1))^{\mathbb{C}}$.

So we are able to construct an analytic open set $U_W \subset \mathbb{P}W \times_{\mathbb{H}^{\chi;N}} Z^*$ that is stabilizer preserving. To obtain the Zariski openness, one only needs to observe the fact that

$$\mathrm{Aut}(Z^*) := \{(z, G_z) \in Z^* \times \mathrm{SL}(N+1) \mid G_z < \mathrm{SL}(N+1) \text{ is the stabilizer of } z \text{ in } \mathrm{SL}(N+1)\}$$

is a constructible set. Hence our proof is completed. □

Finally to prove Theorem 5.2, we need to establish the assumption (2) of Theorem 5.6, that is, for each $\mathbb{C}$-point $[z] \in [Z^*/\mathrm{SL}(N+1)]$, $\overline{\{z\}}$ has a good moduli space. But this will follow if we can show that the fibers of the quotient are *affine*. Notice being affine (or equivalently z_0 being GIT polystable) is part of the assumption in Theorem 2.5. But K-stability is not GIT stability *globally*. So instead of obtaining Assumption 2.10 as a consequence of Theorem 2.5, we establish Assumption 2.10 first. This is the major difference between the classical GIT and K-stability.

Let $z = \mathrm{Hilb}(Y) \in U_W$ specializing to a K-polystable $z_0 = \mathrm{Hilb}(X) \in U_W \subset \mathbb{H}^{\chi;N}$ via a 1-PS $\lambda(t) : \mathbb{G}_m \to \mathrm{Aut}(X) < \mathrm{SL}(N+1)$. Let $(\mathcal{Y} = \mathcal{X}|_C, X) \to (C = \overline{\lambda(t) \cdot z}, z_0) \subset U_W$ be the restriction of the universal family $\mathcal{X} \to Z^*$ to the pointed curve (C, z_0) and also we prefix a basis $\{s_i\} \subset \mathcal{O}_{\mathcal{Y}}(-rK_{\mathcal{Y}/C})$.

Lemma 5.11 *Under the notation introduced above, we have* $\mathrm{Aut}(Y) < \mathrm{Aut}(X)$ *for* $z := \mathrm{Hilb}(Y)$. *As a direct consequence, after a possible shrinking of the Zariski open neighborhood $z_0 \in U_W \subset \mathbb{P}W \times_{\mathbb{H}^{\chi;N}} Z^*$, we have*

$$\mathrm{SL}(N+1)_z < \mathrm{Aut}(X), \ \forall z \in U_W$$

where $\mathrm{SL}(N+1)_z$ *is the stabilizer of* z *inside* $\mathrm{SL}(N+1)$*. In particular, Assumption 2.10 holds.*

Proof The idea of proving the statement is to introduce an auxillary divisor $\mathcal{D} \in |-mK_{\mathcal{X}}|$, and extend our argument in the proof of Theorem 5.9 to the log $\mathbb{Q}$-Fano varieties. For details, please see the proof of [33, Lemma 8.10]. □

Next in order to apply Lemma 2.15, we now establish Assumption 2.14. Let us fix $G = \mathrm{SL}(N+1)$ and $G_{z_0} = \mathrm{Aut}(X)$.

Lemma 5.12 *Let* $z_0 \in U^{\mathrm{ss}}_{W,r} \subset \mathbb{P}W$ *be defined as in Definition 2.12 and*

$$U_{Z^*,r} := U^{\mathrm{ss}}_{W,r} \times_{\mathbb{H}^{\chi;N}} Z^*.$$

Then for $0 < r$ *sufficient small, we have* $U_{Z^*,r} \subset U^{\mathrm{fd}}$*, i.e. Assumption 2.14 is satisfied for* $U_{Z^*,r}$*.*

Proof In order to better illustrate the idea without getting into messy technicalities, we will prove a simple case that z_0 is K-stable, hence $G_{z_0} < \infty$. As we have seen Sect. 5.1, there is a *proper* $\mathrm{U}(N+1)$-invariant slice $z_0 \in \Sigma_{\mathrm{KE}} \subset \mathbb{H}^{\chi;N}$ obtained via Tian's embedding. By the continuity of Σ_{KE} and transversality of the $\mathfrak{g}^{\perp}_{z_0}$-action on U_0, for some $0 < r'' < r' \ll 1$ and $0 < \epsilon \ll 1$ we have

$$B_{Z^*}(z_0, r'') \subset U^{\mathrm{ss}}_{W,r'} \cap \exp \mathfrak{g}^{\perp}_{z_0,<\epsilon} \cdot \Sigma, \tag{33}$$

where $\mathfrak{g}^{\perp}_{z_0,<\epsilon} := \{\xi \in \mathfrak{g}^{\perp}_{z_0} \mid |\xi| < \epsilon\}$ and $B_{Z^*}(z_0, r'')$ denotes the ball of radius ϵ centered at $z_0 \in Z^*$ with respect to a prefixed continuous metric on Z^*. Moreover, by choosing a small r if necessary, we may assume $\mathcal{X}_z$ is K-stable for all $z \in B_{Z^*}(z_0, r'')$.

To see the lemma, let $\{s_i\}$ be the local basis of $\pi_*(\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}})$ corresponding to the coordinate sections of $\mathbb{P}^N$ such that the induced embedding of $X = \mathcal{X}_{z_0} \subset \mathbb{P}^N$ gives rise to $\mathrm{Hilb}(X)$. Now let us equip the line bundle $\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/Z^{*,\mathrm{kps}}}) \cong \mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}}$ with a Hermitian metric which gives rise to the *unique* Kähler-Einstein metric when restricted to each $\mathcal{X}_z$ with $z \in B_{Z^*}(z_0, r'')$, and we can introduce the matrix $A_{\mathrm{KE}}(z)$

As a consequence, for any pair $(z, g) \in B_{Z^*}(z_0, r'') \times G$ satisfying $g \cdot z \in B_{Z^*}(z_0, r'')$, there are $h', h'' \in G$ such that under the quotient map

$$[\cdot] : G \to G/G_{z_0},$$

$h' \cdot z, h'' \cdot g \cdot z \in \Sigma$ and $[h'], [h''] \in G/G_{z_0}$ are perturbations of $[1] \in G/G_{z_0}$ thanks to (26). Since both $h \cdot z$ and $h' \cdot g \cdot z$ are the Hilbert points of Tian's embedding of the *same* $\mathbb{Q}$-Fano variety, we know that $u := h'^{-1} \cdot h'' \cdot g \in \mathrm{U}(N+1)$. This implies that $g \cdot z = h \cdot z$ with $h = h''^{-1} \cdot h' \cdot u$ and that $[h]$ is *uniformly bounded* in G/G_{z_0} with a bound depending only on $B_{Z^*}(z_0, r'')$ and z_0. Since whether or not z lies in U^{fd} is independent of the G_{z_0}-translation, we conclude that Assumption 2.14 holds for K-polystable points lies in $U_{Z^*,r} \subset G_{z_0} \cdot B_{Z^*}(z_0, r'')$ for some $0 < r < r''$. □

Proof of Theorem 5.2 By Sect. 5.2, proving our statement boils down to establishing the following: for any closed point $[z_0] \in [Z^*/\mathrm{SL}(N+1)]$ there is an *affine* neighbourhood $z_0 \in U_W \subset \mathbb{P}W$ determined in Theorem 5.9 such that

(1) The morphism $[U_W/G_{z_0}] \to [Z^*/G]$ is *affine and strongly étale* (i.e. stabilizer preserving and sending closed point to closed point), and
(2) For any $z \in Z^*$ specializing to z_0 under G-action, the closure of the substack $[z]$ inside $[Z^*/G]$, $\overline{\{[z]\}} \subset [Z^*/G]$ admits a good moduli space.

Here we fix $G = \mathrm{SL}(N+1)$ and $G_{z_0} = \mathrm{Aut}(X)$.

We have shown the morphism is strongly étale in Theorem 5.9. Next we confirm the affineness. Since $Z^* \to [Z^*/\mathrm{SL}(N+1)]$ is faithfully flat, it suffices to show that

$$\phi\colon G \times_{G_{z_0}} U_W \to Z^*$$

is affine. Since ϕ is quasi-finite and Z^* is separated, it suffices to choose U_W such that $G \times_{G_{z_0}} U_W$ is affine. Let $U_W \subset Z^* \cap \mathbb{P}(W)$ be a G_{z_0}-invariant affine open set. Then we know $G \times_{G_{z_0}} U_W$ is *affine* since it is a quotient of the affine scheme $G \times U_W$ by the free action of the reductive group G_{z_0}. Furthermore, we have an isomorphism

$$(G \times_{G_{z_0}} U_W) /\!\!/ G \cong U_W /\!\!/ G_{z_0}.$$

Now we establish the second condition. Since we have already established the *uniqueness* of minimal orbit contained in $\overline{BO}_{z_0}$ stated after diagram (32), all we need is the affineness of $G \cdot \pi_W^{-1}(0)$ as it implies that for any $z \in Z^*$ satisfying $\overline{G \cdot z} \ni z_0$ the closure of $[z] \in [Z^*/G]$ is a closed substack of $[G \cdot \pi_W^{-1}(0)/G]$, which can be written as the form $[\mathrm{Spec}(A)/G]$ for some affine scheme $\mathrm{Spec}(A)$, hence $\overline{[z]}$ admits a good moduli space.

To obtain the affineness, one notices that Theorem 5.9 and Corollary 5.11 guarantee the Assumption 2.10, also we have already established Assumption 2.14 by Lemma 5.12. Thus the morphism

$$\phi|_{G \times_{G_{z_0}} U_r} : G \times_{G_{z_0}} U_{Z^*,r} \to G \cdot U_{Z^*,r} \tag{34}$$

is a *finite* morphism for $0 < r \ll 1$ by Lemma 2.15. By choosing $0 < r$ even smaller, we may conclude that $\phi\Big|_{G \times_{G_{z_0}} U_{Z^*,r}}$ is an *analytic isomorphism*, since $\phi|_{G \cdot z_0}$ is an isomorphism and immersion near $G \cdot z_0$. Now we restrict ϕ to the fiber over $[z_0] \in [Z^*/G]$, we have a finite morphism

$$G \times_{G_{z_0}} \pi_W^{-1}(0) \longrightarrow G \cdot \pi_W^{-1}(0)\,.$$

Since $G \times_{G_{z_0}} \pi_W^{-1}(0)$ is a fiber of a GIT quotient morphism, we conclude that $G \cdot \pi_W^{-1}(0)$ is affine.

As a consequence, the étale chart $\phi/G : (G \times_{G_{z_0}} U_W)/\!\!/G \to G \cdot U_W/G$ is actually a *finite* morphism, which implies $G \cdot U_W/G$ is affine. This gives an *affine* neighborhood of $[z_0] \in \mathcal{KF}_N$. This proves that the algebraic space $\mathcal{KF}_N$ is actually a *scheme*. Finally to prove the last statement of Theorem 5.2, we observe that the boundedness of $\mathbb{Q}$-Gorenstein smoothable K-semistable $\mathbb{Q}$-Fano varieties (cf. [33, Lemma 8.3]) implies that the closed points of $\mathcal{KF}_N$ stabilize. However, since $\mathcal{KF}_N$ is semi-normal, we indeed know that they are isomorphic (see [18, 7.2]). □

5.4 Moment Map and Σ_{KE}

In this subsection, we give a moment map interpretation of the slice Σ_{KE}. Let $(L, h) \to (X, \omega)$ be a symplectic manifold together with a Hermitian Line bundle (L, h) whose curvature form is given by $\omega \in H^2(X, \mathbb{Z})$. Let

$$\mathcal{J}(X, \omega) = \{J \in \mathrm{End}(TX) \mid J^2 = -\mathrm{id},\ \omega(J\cdot, J\cdot) = \omega(\cdot,\cdot), N_J = 0 \text{ and } \omega(\cdot, J\cdot) > 0\}$$

denote the space of *integrable* complex structures compatible with the symplectic form ω, where N_J is the Nijenhuis tensor. Let $\Gamma(X, L)$ denote the space of *smooth* sections of $L \to X$. Let us consider the incidence variety

$$Z = \{(s_0, \cdots, s_N; J) \mid \bar{\partial}_J s_i = 0, \forall i\} \subset \overbrace{\Gamma(X, L) \times \cdots \times \Gamma(X, L)}^{N+1} \times \mathcal{J}(X, \omega).$$

which is a ∞-dimensional Kähler manifold invariant under the diagonal action of Hamiltonian group $\mathrm{Ham}(X, \omega)$. On the other hand, the natural $\mathrm{SU}(N+1)$-action on $\overbrace{\Gamma(X, L) \times \cdots \times \Gamma(X, L)}^{N+1}$ commutes with the action of $\mathrm{Ham}(X, \omega)$. Both actions are Hamiltonian actions with moment map given by

$$\begin{array}{rcl} \pi : \overbrace{\Gamma(X, L) \times \cdots \times \Gamma(X, L)}^{N} \times \mathcal{J}(X, \omega) & \longrightarrow & \mathfrak{su}(N+1) \times \wedge_0^{2n} X \\ (s_0, \cdots, s_N; J) & \longmapsto & (\mu_{\mathfrak{su}}(\{s_i\}), \mu_{\mathcal{J}}(J) := (\mathrm{Ric}(\omega, J) - \omega) \wedge \omega^{n-1}) \end{array} ; \tag{35}$$

with

$$\mu_{\mathfrak{su}}(\{s_i\}) = \int_X (\langle s_i, s_j \rangle_h - \delta_{ij})\omega^n.$$

Then we have

$$\Sigma_{\mathrm{KE}}/\mathrm{SU}(N+1) = \frac{(\mu_{\mathfrak{su}}, \mu_{\mathcal{J}})^{-1}(0,0)}{\mathrm{SU}(N+1) \times \mathrm{Ham}(X, \omega)}$$

when restricted to the open locus

$$U = \{J \in \mathcal{J}(X, \omega) \mid (X, \omega, J) \text{ is K-stable } \} \subset \mathcal{J}(X, \omega).$$

One should notice difference between our setting and the one in [7] is that the symplectic form we use on $\mathcal{J}(X, \omega)$ is *not* the same as the one used in [7], which is the restriction of the Fubini-Study form of $\mathbb{G}(N_r + 1, \Gamma(X, L))$ obtained from the embedding

$$\begin{array}{ccc} \mathcal{J}(X,\omega) & \longrightarrow & \mathbb{G}(N_r+1, \Gamma(X,L)) \\ J & \longmapsto & H^0_J(X,L) = \operatorname{span}\{s \mid \bar{\partial}_J s = 0\} \subset \Gamma(X,L) \end{array} \quad \text{for } r \gg 1.$$

We hope to elaborate these points of view in a future work.

6 Toward the Projectivity of $\overline{\mathcal{KF}}$

In this section, we address the projectivity of the moduli space $\mathcal{KF}$ constructed in Theorem 5.2. First, let us recall the natural line bundle over Z^* introduced by Tian.

Definition 6.1 ([47]) Let $\pi : \mathcal{X} \to S$ be a flat family of $\mathbb{Q}$-Fano varieties such that $mK_{\mathcal{X}/S}$ is *Cartier* for some integer m. We define the CM $\mathbb{Q}$-line bundle $\lambda_{\mathrm{CM}} = \lambda_{\mathrm{CM}}(S)$ on S as the determinant line bundle associated to the push-forward of a virtual $\mathbb{Q}$-line bundle (in the sense of Grothendieck):

$$\frac{1}{2^{n+1}m^{n+1}} \det\left[\pi_!\left(-(K_{\mathcal{X}/S}^{-m} - K_{\mathcal{X}/S}^{m})^{n+1}\right)\right]. \tag{36}$$

Remark 6.2 In the following if it's clear from the context we will just write line bundle instead of $\mathbb{Q}$-line bundle for convenience. Equivalently, we can define the CM-line bundle using Knudsen-Mumford expansion (see [41], [42]):

$$\det\left(\pi_*\left(K_{\mathcal{X}/S}^{-mr}\right)\right) = -\lambda_{\mathrm{CM}}\frac{(mr)^{n+1}}{(n+1)!} + O(r^n).$$

By the Grothendieck-Riemann-Roch theorem, the first Chern class of $\lambda_{\mathrm{CM}}(S)$ is given by the formula:

$$\begin{aligned} c_1(\lambda_{\mathrm{CM}}) &= \frac{1}{2^{n+1}m^{n+1}}\pi_*\left[\mathrm{Ch}\left(-(K_{\mathcal{X}/S}^{-m} - K_{\mathcal{X}/S}^{m})^{n+1}\right)\mathrm{Td}(\mathcal{X}/S)\right]_{(2)} \\ &= \pi_*\left(-c_1(K_{\mathcal{X}/S}^{-1})^{n+1}\right). \end{aligned} \tag{37}$$

Theorem 6.3 (Theorem 1.1 in [34]) *The* CM *line bundle* $\lambda_{\mathrm{CM}} \to Z^*$ *descends to a* $\mathbb{Q}$*-line bundle* Λ_{CM} *on the proper moduli space* $\overline{\mathcal{KF}}$*. There is a canonically defined continuous Hermitian metric* h_{DP} *on* Λ_{CM} *whose curvature form is a positive current* ω_{WP} *on* $\overline{\mathcal{KF}}$ *which extends the canonical Weil-Petersson form* $\omega^{\circ}_{\mathrm{WP}}$ *on* $\mathcal{KF}^{\circ} \subset \overline{\mathcal{KF}}$*, the open set of Fano manifolds with discrete automorphism.*

Let us sketch the main idea behind the proof of Theorem 6.3:

(1) By applying the theory of Deligne pairings, for any smooth variety S together with a flat family of Kähler-Einstein Fano varieties $\mathcal{X} \to S$ containing an open dense subset $S^{\circ} \subset S$ such that all fibers of $\mathcal{X}|_{S^{\circ}} \to S^{\circ}$ are Kähler-Einstein Fano manifolds, one can construct a Hermitian metric h_{DP} on the CM line bundle $\lambda_{\mathrm{CM}} \to S$ whose restriction to S° is the classical Weil-Petersson form. This is based on the work of [14, Theorem 7.9].

(2) Theorem 4.1 and the partial-C^0 estimate established in [12] together with an extension of continuity results in [25] (cf. [34, Section 7]) allow us to show that this metric is indeed *continuous* and its curvature form can be extended to a *positive* current on S.

(3) Finally, to descend $\lambda_{\mathrm{CM}} \to Z^*$ to $\overline{\mathcal{KF}}$, two assumptions of Theorem 5.6 need to be satisfied. First, the Assumption (3) of Theorem 5.6 which is an easy consequence of the vanishing of Donaldson-Futaki invariants for $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics. Second, which is much more *serious* is the Assumption (1) of Definition 5.3 (or Assumption (2) of Theorem 5.6). This is the consequence of *goodness* of our moduli space whose proof occupies the major part of Sect. 5. In conclusion, CM line bundle λ_{CM} together with the metric h_{DP} can be descended to an Hermitian line bundle $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ on $\overline{\mathcal{KF}}$. Moreover the curvature form of h_{DP} is exactly the Weil-Petersson current we want.

Finally, we close this section by indicating two important consequences of Theorem 6.3: (1).$\Lambda_{\mathrm{CM}} \to \overline{\mathcal{KF}}$ is nef and big; (2). $\mathcal{KF}^{\circ}$ is quasi-projective (cf. [34, Theorem 1.2] for details).

7 Problems

The construction sketched above produces a merely abstract existence result, which is far from being explicit, except for the work of [39] in dimension two. Recent progress made by Liu-Xu [31] and Spotti-Sun [45] based on the work of Fujita, Li, Liu and Li-Xu (cf. [15, 26–30] and [35, 36] etc.) produces quite a few explicit examples. One interesting feature of those explicit constructions is that they all admit a GIT construction. So it is natural to have a good understanding relation between GIT and K-stability for the Fano varieties. On the other hand, by exhibiting explicit examples in [49] it is shown that classical asymptotic GIT stability fails to produce proper algebraic moduli space for *canonically polarized* varieties. It remains unclear if asymptotic GIT stability also fails for Fano case. To understand this question, the first example we want to understand is

Problem 7.1 Whether or not asymptotic GIT stability can be used to produces proper moduli spaces for Del Pezzo surfaces.

Some progresses are made in [24], but a complete answer to the above question is still lacking since there is still quite limited methods of checking asymptotic stability.

For the moduli space of sheaves over surfaces, we know that there is a *dominant* map from the Gieseker compactification to Uhlenbeck's compactification, which is the vector bundle analogue of K-stable compactification thanks to the work of Jun Li [28]. So the Uhlenbeck compactification obtained from the Hermitian-Einstein metric is *smaller*. It was asked in [50] if such philosophy remains to be true for the moduli space of Fano varieties, that is, in some sense the Kähler-Einstein compactification is smaller than *any* GIT compactification. The explicit construction mentioned in the above examples and [39, Theorem 3.4] seems to suggest that Kähler-Einstein compactification is indeed *minimal* in the following sense: Suppose a GIT quotient $Z /\!\!/ G = M$ is the moduli space of Fano varieties with a fixed Hilbert polynomial χ, suppose further that the general members admit Kähler-Einstein metric and that the master space Z has *minimal* Picard rank, e.g. one. Then M is very *likely* the Kähler-Einstein compactification.

Problem 7.2 Can we make a more precise statement on the *minimality* of the moduli space of K-semistable Fano varieties?

Finally, we finish this note by addressing the following well-expected though challenging question rooted from Sect. 6.

Problem 7.3 ([39] and [34]) The descending of CM line bundle $\Lambda_{\mathrm{CM}} \to \overline{\mathcal{KF}}$ is ample and hence $\overline{\mathcal{KF}}$ is projective.

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