Introduction to Classical and Quantum Markov Semigroups

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Abstract We provide a self-contained and fast-paced introduction to the theories of operator semigroups, Markov semigroups and quantum dynamical semigroups. The level is appropriate for well-motivated graduate students who have a background in analysis or probability theory, with the focus on the characterisation of infinitesimal generators for various classes of semigroups. The theorems of Hille–Yosida, Hille–Yosida–Ray, Lumer–Phillips and Gorini–Kossakowski–Sudarshan–Lindblad are all proved, with the necessary technical prerequisites explained in full. Exercises are provided throughout.

1 Introduction

These notes are an extension of a series of lectures given at the Winter School on Dynamical Methods in Open Quantum Systems held at Georg-August-Universität Göttingen during November 2016. These lectures were aimed at graduate students with a background in analysis or probability theory. The aim has been to make the notes self-contained but brief, so that they are widely accessible. Exercises are provided throughout.

We begin with the basics of the theory of operator semigroups on Banach spaces, and develop this up to the Hille–Yosida and Lumer–Phillips theorems; these provide characterisations for the generators of strongly continuous semigroups and strongly continuous contraction semigroups, respectively. As those with a background in probability theory may not be comfortable with all of the necessary material from functional analysis, this is covered rapidly at the start. The reader can find much more on these topics in Davies's book [9].

After these fundamentals, we recall some key ideas from probability theory. The correspondence between time-homogeneous Markov processes and Markov

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semigroups is explained, and we explore the concepts of Feller semigroups and Lévy processes. We conclude with the Hille–Yosida–Ray theorem, which characterises generators of Feller semigroups via the positive maximum principle. Applebaum [3, Chapter 3] provides another view of much of this material, as do Liggett [20, Chapter 3] and Rogers and Williams [26, Chapter III].

The final part of these notes addresses the theory of quantum Markov semigroups, and builds to the characterisation of the generators of uniformly continuous conservative semigroups, and the Gorini–Kossakowski–Sudarshan–Lindblad form. En route, we establish Stinespring dilation and Kraus decomposition for linear maps defined on unital C^* algebras and von Neumann algebras, respectively, which are important results in the theories of open quantum systems and quantum information. The lecture notes of Alicki and Lendi [2] provide a useful complement, and those of Fagnola [14] study quantum Markov semigroups from the fruitful perspective of quantum probability. There is much scope, and demand, for further developments in this subject.

1.1 Acknowledgements

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1.2 Conventions

The notation "$P := Q$" means that the quantity P is defined to equal Q.

The sets of natural numbers, non-negative integers, non-negative real numbers, real numbers and complex numbers are denoted $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}$ and $\mathbb{C}$, respectively; the square root of -1 is denoted i. Note that we follow the Anglophone rather than Francophone convention, in that 0 is both non-negative and non-positive but is neither positive nor negative.

The indicator function of the set A is denoted 1_A, with the domain determined by context. If $f : A \to B$ and $C \subseteq A$, then $f|_C : C \to B$, the restriction of f to C, takes the same value at any point in C as f does.

Inner products on complex vector spaces are taken to be linear on the right and conjugate linear on the left. Given our final destination, we work with complex vector spaces and complex-valued functions by default.

2 Operator Semigroups

2.1 Functional-Analytic Preliminaries

Throughout their development, there has been a fruitful interplay between abstract functional analysis and the theory of operator semigroups. Here we give a rapid introduction to some of the basic ideas of the former. We cover a little more material that will be used in the sequel, but the reader will find it useful for their further studies in semigroup theory.

Definition 2.1 In these notes, a *normed vector space* V is a vector space with complex scalar field, equipped with a *norm* $\|\cdot\| : V \to \mathbb{R}_+$ which is

(i) subadditive: $\|u+v\| \leqslant \|u\| + \|v\|$ for all $u, v \in V$;
(ii) homogeneous: $\|\lambda v\| = |\lambda|\,\|v\|$ for all $v \in V$ and $\lambda \in \mathbb{C}$; and
(iii) faithful: $\|v\| = 0$ if and only if $v = 0$, for all $v \in V$.

The normed vector space V is *complete* if, whenever $(v_n)_{n\in\mathbb{N}} \subseteq V$ is a Cauchy sequence, there exists $v_\infty \in V$ such that $v_n \to v_\infty$ as $n \to \infty$. A complete normed vector space is called a *Banach space*. Thus Banach spaces are those normed vector spaces in which every Cauchy sequence is convergent.

[Recall that a sequence $(v_n)_{n\in\mathbb{N}} \subseteq V$ is *Cauchy* if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|v_m - v_n\| < \varepsilon$ for all $m, n \geqslant N$.]

Exercise 2.2 (Banach's Criterion) Let $\|\cdot\|$ be a norm on the complex vector space V. Prove that V is complete for this norm if and only if every absolutely convergent series in V is convergent.

[Given $(v_n)_{n\in\mathbb{N}} \subseteq V$, the series $\sum_{n=1}^\infty v_n$ is said to be *convergent* precisely when the sequence of partial sums $(\sum_{j=1}^n v_j)_{n\in\mathbb{N}}$ is convergent, and *absolutely convergent* when $(\sum_{j=1}^n \|v_j\|)_{n\in\mathbb{N}}$ is convergent.]

Example 2.3 If $n \in \mathbb{N}$, then the finite-dimensional vector space $\mathbb{C}^n$ is a Banach space for any of the ℓ^p norms, where $p \in [1,\infty]$ and

$$\|(v_1,\ldots,v_n)\|_p := \begin{cases} \left(\sum_{j=1}^n |v_j|^p\right)^{1/p} & \text{if } p < \infty, \\ \max\{|v_j| : j = 1,\ldots,n\} & \text{if } p = \infty. \end{cases}$$

These norms are all *equivalent*: for all $p, q \in [1,\infty]$ there exists $C_{p,q} > 1$ such that

$$C_{p,q}^{-1}\|v\|_q \leqslant \|v\|_p \leqslant C_{p,q}\|v\|_q \qquad \text{for all } v \in \mathbb{C}^n.$$

Example 2.4 For all $p \in [1,\infty]$, let the *sequence space*

$$\ell^p := \{v = (v_n)_{n\in\mathbb{Z}_+} \subseteq \mathbb{C} : \|v\|_p < \infty\},$$

where

$$\|v\|_p := \begin{cases} \left(\sum_{n=0}^{\infty} |v_n|^p \right)^{1/p} & \text{if } p \in [1, \infty), \\ \sup\{|v_n| : n \in \mathbb{Z}_+\} & \text{if } p = \infty, \end{cases}$$

and the vector-space operations are defined coordinate-wise: if $u, v \in \ell^p$ and $\lambda \in \mathbb{C}$, then

$$(u + v)_n := u_n + v_n \quad \text{and} \quad (\lambda v)_n := \lambda v_n \qquad \text{for all } n \in \mathbb{Z}_+.$$

These are Banach spaces, with $\ell^p \subseteq \ell^q$ if $p, q \in [1, \infty]$ are such that $p \leqslant q$.

If $p \in [1, \infty)$, then $\ell^p \subseteq c_0 \subseteq \ell^\infty$, where

$$c_0 := \{v = (v_n)_{n \in \mathbb{Z}_+} \subseteq \mathbb{C} : \lim_{n \to \infty} v_n = 0\}$$

is itself a Banach space for the norm $\| \cdot \|_\infty$.

Example 2.5 An *inner product* on the complex vector space V is a form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}; \ (u, v) \mapsto \langle u, v \rangle$$

which is

(i) linear in the second argument: the map $V \to \mathbb{C}; v \mapsto \langle u, v \rangle$ is linear for all $u \in V$;

(ii) Hermitian: $\overline{\langle u, v \rangle} = \langle v, u \rangle$ for all $u, v \in V$; and

(iii) positive definite: $\langle v, v \rangle \geqslant 0$ for all $v \in V$, with equality if and only if $v = 0$.

Any inner product determines a norm on V, by setting $\|v\| := \langle v, v \rangle^{1/2}$ for all $v \in V$. Furthermore, the inner product can be recovered from the norm by *polarisation*: if $q : V \times V \to \mathbb{C}$ is a sesquilinear form on V, so is conjugate linear in the first argument and linear in the second, then

$$q(u, v) = \sum_{j=0}^{3} \mathrm{i}^{-j} q(u + \mathrm{i} v, u + \mathrm{i} v) \qquad \text{for all } u, v \in V.$$

A Banach space with norm which comes from an inner product is a *Hilbert space*. For example, the sequence space ℓ^2 is a sequence space, since setting

$$\langle u, v \rangle := \sum_{n=0}^{\infty} \overline{u_n} v_n \qquad \text{for all } u, v \in \ell^2$$

defines an inner product on ℓ^2 such that $\langle v, v\rangle = \|v\|^2$ for all $v \in \ell^2$. In any Hilbert space H, the *Cauchy–Schwarz inequality* holds:

$$|\langle u, v\rangle| \leqslant \|u\|\,\|v\| \qquad \text{for all } u, v \in \mathsf{H}.$$

It may be shown that a Banach space V is a Hilbert space if and only if the norm satisfies the *parallelogram law*:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \qquad \text{for all } u, v \in V.$$

Exercise 2.6 Let H be a Hilbert space. Given any set $S \subseteq H$, prove that its *orthogonal complement*

$$S^\perp := \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in S\}$$

is a closed linear subspace of H. Prove further that $L \subseteq H$ is a closed linear subspace of H if and only if $L = (L^\perp)^\perp$.

Example 2.7 Let $C(K)$ denote the complex vector space of complex-valued functions on the compact Hausdorff space K, with vector-space operations defined pointwise: if $x \in K$ then

$$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) := \lambda f(x)$$

for all $f, g \in C(K)$ and $\lambda \in \mathbb{C}$. The *supremum norm*

$$\|\cdot\| : f \mapsto \|f\|_\infty := \sup\{f(x) : |x| \in K\}$$

makes $C(K)$ a Banach space. [Completeness is the undergraduate-level fact that uniform convergence preserves continuity.]

Example 2.8 Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, so that $\mu : \mathcal{F} \to [0, \infty]$ is a measure and there exists a countable cover of Ω with elements in $\mathcal{F}$ of finite measure.

For all $p \in [1, \infty]$, the *L^p space*

$$L^p(\Omega, \mathcal{F}, \mu) := \{f : \Omega \to \mathbb{C} \mid \|f\|_p < \infty\}$$

is a Banach space when equipped with the *L^p norm*

$$\|f\|_p := \begin{cases} \left(\int_\Omega |f(x)|^p \,\mu(\mathrm{d}x)\right)^{1/p} & \text{if } p \in [1, \infty), \\ \inf\big\{\sup\{\,|f(x)| : x \in \Omega \setminus V\} : V \subseteq \Omega \text{ is a null set}\big\} & \text{if } p = \infty, \end{cases}$$

and where functions are identified if they differ on a null set. [Note that if $f \in L^p(\Omega, \mathcal{F}, \mu)$ then $\|f\|_p = 0$ if and only if $f = 0$ on a null set.]

The space $L^2(\Omega, \mathcal{F}, \mu)$ is a Hilbert space, with inner product such that

$$\langle f, g \rangle := \int_\Omega \overline{f(x)} g(x)\, \mu(\mathrm{d}x) \qquad \text{for all } f, g \in L^2(\Omega, \mathcal{F}, \mu).$$

If $p, q, r \in [1, \infty]$ are such that $p^{-1} + q^{-1} = r^{-1}$, where $\infty^{-1} := 0$, then

$$\|fg\|_r \leqslant \|f\|_p \, \|g\|_q \qquad \text{for all } f \in L^p(\Omega, \mathcal{F}, \mu) \text{ and } g \in L^q(\Omega, \mathcal{F}, \mu); \tag{2.1}$$

this is *Hölder's inequality*. The subadditivity of the L^p norm, known as *Minkowski's inequality*, may be deduced from Hölder's inequality. When $r = 1$ and $p = q = 2$, Hölder's inequality is known as the *Cauchy–Bunyakovsky–Schwarz inequality*.

Example 2.9 Let $d \geqslant 1$. The space $C_c^\infty(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ with compact support is a subspace of $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$, and is dense for $p \in [1, \infty)$, when $\mathbb{R}^d$ is equipped with Lebesgue measure.

Given a *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, let $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and

$$D^\alpha f := \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_d}}{\partial x_d} f \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^d).$$

Note that $D^\alpha f \in C_c^\infty(\mathbb{R}^d)$ for all $f \in C_c^\infty(\mathbb{R}^d)$ and $\alpha \in \mathbb{Z}_+^d$.

Let $f \in L^p(\mathbb{R}^d)$, where $p \in [1, \infty]$, and note that $fg \in L^1(\mathbb{R}^d)$ for all $g \in C_c^\infty(\mathbb{R}^d)$, by Hölder's inequality. If there exists $F \in L^p(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} f(x) D^\alpha g(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^d} F(x) g(x)\, \mathrm{d}x \qquad \text{for all } g \in C_c^\infty(\mathbb{R}^d)$$

then F is the *weak derivative* of f, and we write $F = D^\alpha f$. [It is a straightforward exercise to verify that the weak derivative is unique, and that this agrees with the previous definition if $f \in C_c^\infty(\mathbb{R}^d)$.]

Given $p \in [1, \infty)$ and $k \in \mathbb{Z}_+$, the *Sobolev space*

$$W^{k,p}(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) : D^\alpha f \in L^p(\mathbb{R}^d) \text{ whenever } |\alpha| \leqslant k\}$$

is a Banach space when equipped with the norm

$$f \mapsto \|f\| := \Big(\sum_{|\alpha| \leqslant k} \|D^\alpha f\|_p^p \Big)^{1/p}$$

and contains $C_c^\infty(\mathbb{R}^d)$ as a dense subspace.

The Sobolev space $W^{k,2}(\mathbb{R}^d)$ is usually abbreviated to $H^k(\mathbb{R}^d)$ and is a Hilbert space, with inner product such that

$$\langle f, g \rangle := \sum_{|\alpha| \leqslant k} \langle D^\alpha f, D^\alpha g \rangle \qquad \text{for all } f, g \in H^k(\mathbb{R}^d).$$

Exercise 2.10 Prove that the normed vector space $W^{k,p}(\mathbb{R}^d)$, as defined in Example 2.9, is complete.

Example 2.11 Let U and V be normed vector spaces. A linear operator $T : U \to V$ is *bounded* if

$$\|T\| := \{\|Tu\| : u \in U\} < \infty.$$

If T is bounded, then $\|Tu\| \leqslant \|T\| \, \|u\|$ for all $u \in U$, and $\|T\|$ is the smallest constant with this property.

The set of all such linear operators is denoted by $B(U; V)$, or $B(U)$ if U and V are equal.

This set is a normed vector space, with *operator norm* $T \mapsto \|T\|$ and algebraic operations defined pointwise, so that

$$(S + T)u = Su + Tu \quad \text{and} \quad (\lambda T)u := \lambda T u$$

for all $S, T \in B(U; V)$, $\lambda \in C$ and $U \in U$. Furthermore, the space $B(U; V)$ is a Banach space whenever V is.

Exercise 2.12 Prove the claims in Example 2.11.

Exercise 2.13 Let V be a normed vector space. Prove that the norm on $B(V)$ is *submultiplicative*: if $S, T \in B(V)$, then $ST : v \mapsto S(Tv) \in B(V)$, with $\|ST\| \leqslant \|S\| \, \|T\|$.

Exercise 2.14 Let U and V be normed vector spaces and let $T : U \to V$ be a linear operator. Prove that T is bounded if and only if it is continuous when U and V are equipped with their norm topologies.

Example 2.15 Given any normed space V, its *topological dual* or *dual space* is the Banach space $V^* := B(V; \mathbb{C})$. An element of V^* is called a *linear functional* or simply a *functional*.

If $p, q \in (1, \infty)$ are *conjugate indices*, so that such that $p^{-1} + q^{-1} = 1$, then $(\ell^p)^*$ is naturally isomorphic to ℓ^q via the *dual pairing*

$$[u, v] := \sum_{n=0}^{\infty} u_n v_n \qquad \text{for all } u \in \ell^p \text{ and } v \in \ell^q.$$

Hölder's inequality shows that $u \mapsto [u, v]$ is an element of $(\ell^p)^*$ for any $v \in \ell^q$; proving that every functional arises this way is an exercise. Furthermore, the same pairing gives an isomorphism between $(\ell^1)^*$ and ℓ^∞. [The dual of ℓ^∞ is much larger than ℓ^1; it is isomorphic to the space $M(\beta\mathbb{N})$ of regular complex Borel measures on the Stone–Čech compactification of the natural numbers.]

Similarly, for conjugate indices $p, q \in (1, \infty)$, the dual of $L^p(\Omega, \mathcal{F}, \mu)$ is identified with $L^q(\Omega, \mathcal{F}, \mu)$, and the dual of $L^1(\Omega, \mathcal{F}, \mu)$ with $L^\infty(\Omega, \mathcal{F}, \mu)$, via the pairing

$$[f, g] := \int_\Omega f(x) g(x)\, \mu(\mathrm{d}x).$$

In particular, ℓ^2 and $L^2(\Omega, \mathcal{F}, \mu)$ are conjugate-linearly isomorphic to their dual spaces. This is a general fact about Hilbert spaces, known as the *Riesz–Fréchet theorem*: if H is a Hilbert space, then

$$H^* = \big\{ \langle u| : u \in H \big\}, \qquad \text{where } \langle u|v := \langle u, v\rangle \quad \text{for all } v \in H.$$

If K is a compact Hausdorff space, then the dual of $C(K)$ is naturally isomorphic to the space $M(K)$ of regular complex Borel measures on K, with dual pairing

$$[f, \mu] := \int_K f(x)\, \mu(\mathrm{d}x) \qquad \text{for all } f \in C(K) \text{ and } \mu \in M(K).$$

The Hahn–Banach theorem [25, Corollary 2 to Theorem III.6] implies that the dual space separates points: if $v \in V$, then there exists $\phi \in V^*$ such that $\|\phi\| = 1$ and $\phi(v) = \|v\|$.

Example 2.16 Duality makes an appearance at the level of operators. If U and V are normed spaces and $T \in B(U; V)$, then there exists a unique *dual operator* $T' \in B(V^*; U^*)$ such that

$$(T'\psi)(v) = \psi(Tu) \qquad \text{for all } u \in U \text{ and } \psi \in V^*.$$

The map $T \mapsto T'$ from $B(U; V)$ to $B(V^*; U^*)$ is linear and reverses the order of products: if $S \in B(U; V)$ and $T \in B(V; W)$, then $(TS)' = S'T'$.

If H and K are Hilbert spaces, and we identify each of these with its dual via the Riesz–Fréchet theorem, then the operator dual to $T \in B(\mathsf{H}; \mathsf{K})$ is identified with the *adjoint operator* $T^* \in B(\mathsf{K}; \mathsf{H})$, since

$$\big(T'\langle v|\big)u = \langle v, Tu\rangle_{\mathsf{K}} = \langle T^*v, u\rangle_{\mathsf{H}} = \langle T^*v|u \qquad \text{for all } u \in \mathsf{H} \text{ and } v \in \mathsf{K}.$$

2.2 Semigroups on Banach Spaces

Definition 2.17 A family of operators $T = (T_t)_{t\in\mathbb{R}_+} \subseteq B(V)$ is a *one-parameter semigroup* on V, or a *semigroup* for short, if

(i) $T_0 = I$ the identity operator and (ii) $T_s T_t = T_{s+t}$ for all $s, t \in \mathbb{R}_+$.

The semigroup T is *strongly continuous* if

$$\lim_{t\to 0+} \|T_t v - v\| = 0 \qquad \text{for all } v \in V,$$

and is *uniformly continuous* if

$$\lim_{t\to 0+} \|T_t - I\| = 0.$$

Exercise 2.18 Prove that a uniformly continuous semigroup is strongly continuous. [The converse is false: see Exercise 2.29.]

Theorem 2.19 *Let T be a strongly continuous semigroup on the Banach space V. There exist constants $M \geqslant 1$ and $a \in \mathbb{R}$ such that $\|T_t\| \leqslant Me^{at}$ for all $t \in \mathbb{R}_+$.*

Proof See [9, Theorem 6.2.1]. □

Remark 2.20 The semigroup T of Theorem 2.19 is said to be of *type* (M, a). A semigroup of type $(1, 0)$ is also called a *contraction semigroup*.

By replacing T_t with $e^{-at}T_t$, one can often reduce to the case of semigroups with uniformly bounded norm. However, it is not always possible to go further and reduce to contraction semigroups; see [9, Example 6.2.3 and Theorem 6.3.8].

Exercise 2.21 Prove that a strongly continuous semigroup is strongly continuous at every point: if $t \geqslant 0$, then $\lim_{h\to 0} \|T_{t+h}x - T_t x\| = 0$. Prove further that the same is true if "strongly" is replaced by "uniformly".

Exercise 2.22 Given any $A \in B(V)$, let $\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

(i) Prove that this series is convergent, so that $\exp(A) \in B(V)$. Prove further that $\|\exp(A)\| \leqslant \exp \|A\|$.

(ii) Prove that if $B \in B(V)$ commutes with A, so that that $AB = BA$, then $\exp(A)$ and $\exp(B)$ also commute, with $\exp(A)\exp(B) = \exp(A + B)$. [Hint: consider the derivatives of

$$t \mapsto \exp(tA)\exp(-tA) \quad \text{and} \quad t \mapsto \exp(tA)\exp(tB)\exp\bigl(-t(A+B)\bigr).]$$

(iii) Prove that setting $T_t := \exp(tA)$ for all $t \in \mathbb{R}_+$ produces a uniformly continuous one-parameter semigroup T.

The converse of Exercise 2.22(iii) is true, and we state it as a theorem.

Theorem 2.23 *If T is a uniformly continuous one-parameter semigroup, then there exists an operator $A \in B(V)$ such that $T_t = \exp(tA)$ for all $t \in \mathbb{R}_+$.*

Proof By continuity at the origin, there exists $t_0 > 0$ such that

$$\|T_s - I\| < 1/2 \qquad \text{for all } s \in [0, t_0].$$

Then

$$\Big\| t_0^{-1} \int_0^{t_0} T_s \,\mathrm{d}s - I \Big\| = t_0^{-1} \Big\| \int_0^{t_0} T_s - I \,\mathrm{d}s \Big\| \leqslant 1/2 < 1.$$

Thus $X := t_0^{-1} \int_0^{t_0} T_s \,\mathrm{d}s \in B(V)$ is invertible, because the Neumann series

$$\sum_{n=0}^{\infty} (I - X)^n = I + (I - X) + (I - X)^2 + \dots$$

is absolutely convergent, so convergent, by Banach's criterion. Furthermore,

$$\begin{aligned} h^{-1}(T_h - I) \int_0^{t_0} T_s \,\mathrm{d}s &= h^{-1} \int_0^{t_0} T_{s+h} - T_s \,\mathrm{d}s \\ &= h^{-1} \int_h^{t_0+h} T_s \,\mathrm{d}s - h^{-1} \int_0^{t_0} T_s \,\mathrm{d}s \\ &= h^{-1} \int_{t_0}^{t_0+h} T_s \,\mathrm{d}s - h^{-1} \int_0^{h} T_s \,\mathrm{d}s \\ &\to T_{t_0} - I \end{aligned}$$

as $h \to 0+$. Hence

$$A := \lim_{h \to 0+} h^{-1}(T_h - I) = (T_{t_0} - I)(t_0 X)^{-1}.$$

Moreover, for any $t \in [0, t_0]$,

$$\begin{aligned} T_{t_0} = I + A \int_0^t T_{t_1} \,\mathrm{d}t_1 &= I + A\Big(tI + \int_0^t \int_0^{t_1} T_{t_2} \,\mathrm{d}t_2 \,\mathrm{d}t_1\Big) \\ &= I + tA + \frac{t^2}{2} A^2 + \dots \end{aligned}$$

$$+ A^n \int_0^t \ldots \int_0^{t_n} T_{t_{n+1}} \, \mathrm{d}t_{n+1} \ldots \mathrm{d}t_1$$

$$\to \sum_{n \geqslant 0} \frac{1}{n!}(tA)^n = \exp(tA)$$

as $n \to \infty$, since

$$\left\| A^n \int_0^t \ldots \int_0^{t_n} T_{t_{n+1}} \, \mathrm{d}t_{n+1} \ldots \mathrm{d}t_1 \right\| \leqslant \frac{3t^{n+1} \|A\|^n}{2(n+1)!}.$$

This working shows that $T_t = \exp(tA)$ for any $t \in [0, t_0]$, so for all $t \in \mathbb{R}_+$, by the semigroup property: there exists $n \in \mathbb{Z}_+$ and $s \in [0, t_0)$ such that $t = nt_0 + s$, and

$$T_t = T_{t_0}^n T_s = \exp(nt_0 A + sA) = \exp(tA).$$

□

Remark 2.24 The integrals in the previous proof are *Bochner integrals*; they are an extension of the Lebesgue integral to functions which take values in a Banach space. We will only be concerned with continuous functions, so do not need to concern ourselves with notions of measurability. All the standard theorems carry over from the Lebesgue to the Bochner setting, such as the inequality $\| \int f(t) \, \mathrm{d}t \| \leqslant \int \|f(t)\| \, \mathrm{d}t$, and if T is a bounded operator then $T \int f(t) \, \mathrm{d}t = \int T f(t) \, \mathrm{d}t$.

Definition 2.25 If T is a uniformly continuous semigroup, then the operator $A \in B(V)$ such that $T_t = \exp(tA)$ for all $t \in \mathbb{R}_+$ is the *generator* of the semigroup.

Exercise 2.26 Prove that the generator of a uniformly continuous one-parameter semigroup T is unique. [Hint: consider the limit of $t^{-1}(T_t - I)$ as $t \to 0+$.]

Example 2.27 Given $t \in \mathbb{R}_+$ and $f \in V := L^p(\mathbb{R}_+)$, where $p \in [1, \infty)$, let

$$(T_t f)(x) := f(x + t) \qquad \text{for all } x \in \mathbb{R}_+.$$

Then $T_t \in B(V)$, with $\|T_t\| = 1$, and $T = (T_t)_{t \in \mathbb{R}_+}$ is a one-parameter semigroup. If f is continuous and has compact support, then an application of the Dominated Convergence Theorem gives that $T_t f \to f$ as $t \to 0+$; since such functions are dense in V, it follows that T is strongly continuous.

Exercise 2.28 Prove the assertions in Example 2.27. Prove also that if $f \in V = L^p(\mathbb{R}_+)$ is absolutely continuous, with $f' \in V$ such that

$$f(x) = f(0) + \int_0^x f'(y) \, \mathrm{d}y \qquad \text{for all } x \in \mathbb{R}_+,$$

then

$$\lim_{t \to 0+} t^{-1}(T_t f - f) = f',$$

where the limit exists in V. [Hint: show that

$$\|t^{-1}(T_t f - f) - f'\|_p^p = t^{-1} \int_0^t \|T_y f' - f'\|_p^p \, \mathrm{d}y$$

and then use the strong continuity of T at the origin.]

Exercise 2.29 Prove that the semigroup of Example 2.27 is not uniformly continuous. [Hint: let $f_n = \lambda_n 1_{[n^{-1}, 2n^{-1}]}$, where the positive constant λ_n is chosen to make f_n a unit vector in V, and consider $\|T_t f_n - f_n\|$ for $n > t^{-1}$.]

2.3 Beyond Uniform Continuity

As shown above, uniformly continuous one-parameter semigroups are in one-to-one correspondence with bounded linear operators. To move beyond this situation, we need to introduce linear operators which are only partially defined on the ambient Banach space V.

Definition 2.30 An *unbounded operator* in V is a linear transformation A defined on a linear subspace $V_0 \subseteq V$, its *domain*; we write $\operatorname{dom} A = V_0$.

An *extension* of A is an unbounded operator B in V such that $\operatorname{dom} A \subseteq \operatorname{dom} B$ and the restriction $B|_{\operatorname{dom} A} = A$. In this case, we write $A \subseteq B$.

An unbounded operator A in V is *densely defined* if $\operatorname{dom} A$ is dense in V for the norm topology.

Definition 2.31 Given operators A and B, let $A + B$ and AB be defined by setting

$$\operatorname{dom}(A + B) := \operatorname{dom} A \cap \operatorname{dom} B, \quad (A + B)v := Av + Bv$$

and

$$\operatorname{dom} AB := \{v \in \operatorname{dom} A : Av \in \operatorname{dom} B\}, \quad (AB)v := A(Bv).$$

Note that neither $A + B$ nor AB need be densely defined, even if both A and B are.

Definition 2.32 Let T be a strongly continuous one-parameter semigroup on V. Its *generator* A is an unbounded operator with domain

$$\operatorname{dom} A := \left\{v \in V : \lim_{t \to 0+} t^{-1}(T_t v - v) \text{ exists in } V\right\}$$

and action

$$Av := \frac{\mathrm{d}}{\mathrm{d}t} T_t v\Big|_{t=0} := \lim_{t\to 0+} t^{-1}(T_t v - v) \qquad \text{for all } v \in \operatorname{dom} A.$$

It is readily verified that A is an unbounded operator.

Exercise 2.33 Prove that if $v \in V$ and $t \in \mathbb{R}_+$ then

$$\int_0^t T_s v \,\mathrm{d}s \in \operatorname{dom} A \qquad \text{and} \qquad (T_t - I)v = A \int_0^t T_s v \,\mathrm{d}s.$$

Deduce that $\operatorname{dom} A$ is dense in V. [Hint: begin by imitating the proof of Theorem 2.23.]

Lemma 2.34 *Let the strongly continuous semigroup T have generator A. If $v \in \operatorname{dom} A$ and $t \in \mathbb{R}_+$, then $T_t v \in \operatorname{dom} A$ and $T_t A v = A T_t v$; thus, $T_t(\operatorname{dom} A) \subseteq \operatorname{dom} A$. Furthermore,*

$$(T_t - I)v = \int_0^t T_s A v \,\mathrm{d}s = \int_0^t A T_s v \,\mathrm{d}s.$$

Proof First, note that

$$h^{-1}(T_h - I)T_t v = T_t h^{-1}(T_h - I)v \to T_t A v \quad \text{as } h \to 0+,$$

by the boundedness of T_t, so $T_t v \in \operatorname{dom} A$ and $A T_t v = T_t A v$, as claimed. For the second part, let

$$F : \mathbb{R}_+ \to V;\ t \mapsto (T_t - I)v - \int_0^t T_s A v \,\mathrm{d}s.$$

Note that F is continuous and $F(0) = 0$; furthermore, if $t > 0$, then

$$h^{-1}(F(t+h) - F(t)) = T_t h^{-1}(T_h - I)v - h^{-1}\int_0^h T_{s+t} A v \,\mathrm{d}s \to T_t A v - T_t A v = 0$$

as $h \to 0+$, whence $F \equiv 0$. □

Definition 2.35 An operator A in V is *closed* if, whenever $(v_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom} A$ is such that $v_n \to v \in V$ and $A v_n \to u \in V$, it follows that $v \in \operatorname{dom} A$ and $Av = u$. Note that a bounded operator is automatically closed.

The operator A is *closable* if it has a closed extension, in which case the *closure* $\overline{A}$ is the smallest closed extension of A, where the ordering of operators is given in Definition 2.30.

Exercise 2.36 Prove that the *graph*

$$\mathcal{G}(A) := \{(v, Av) : v \in \operatorname{dom} A\}$$

of an unbounded operator A in V is a normed vector space for the product norm

$$\| \cdot \| : (v, Av)\| \mapsto \|v\| + \|Av\|.$$

Prove further that A is closed if and only if $\mathcal{G}(A)$ is a Banach space, and that A is closable if and only if the closure of its graph in $V \oplus V$ is the graph of some operator. Finally, prove that if A is closable then $\mathcal{G}(\overline{A})$ is the intersection of the graphs of all closed extensions of A.

Exercise 2.37 Let A be the generator of the strongly continuous one-parameter semigroup T. Use Lemma 2.34 and Theorem 2.19 to show that A is closed.

Proof Suppose $(v_n)_{n \in \mathbb{N}} \subseteq \operatorname{dom} A$ is such that $v_n \to v$ and $Av_n \to u$. Let $t > 0$ and note that

$$T_t v_n - v_n = \int_0^t T_s A v_n \, \mathrm{d}s \qquad \text{for all } n \geqslant 1.$$

Furthermore,

$$\left\| \int_0^t T_s A v_n \, \mathrm{d}s - \int_0^t T_s u \, \mathrm{d}s \right\| \leqslant \int_0^t M e^{as} \|Av_n - u\| \, \mathrm{d}s \leqslant M t e^{\max\{a,0\}t} \|Av_n - u\| \to 0$$

as $n \to \infty$, so

$$T_t v - v = \int_0^t T_s u \, \mathrm{d}s.$$

Dividing both sides by t and letting $t \to 0+$ gives that $v \in \operatorname{dom} A$ and $Av = u$, as required. □

Definition 2.38 Let H be Hilbert space. If A is a densely defined operator in H, then the *adjoint* A^* is defined by setting

$$\operatorname{dom} A^* := \{u \in \mathsf{H} : \text{there exists } v \in \mathsf{H} \text{ such that } \langle u, Aw \rangle = \langle v, w \rangle \text{ for all } w \in \operatorname{dom} A\}$$

and

$$A^* u = v, \quad \text{where } v \text{ is as in the definition of } \operatorname{dom} A^*.$$

When A is bounded, this agrees with the earlier definition. If A is not densely defined, then there may be no unique choice for v, so this definition cannot immediately be extended further.

It is readily verified that the adjoint A^* is always closed: if $(u_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom} A^*$ is such that $u_n \to u \in \mathsf{H}$ and $A^*u_n \to v \in \mathsf{H}$ then

$$\langle u, Aw\rangle = \lim_{n\to\infty} \langle u_n, Aw\rangle = \lim_{n\to\infty} \langle A^*u_n, w\rangle = \lim_{n\to\infty} \langle v, w\rangle \qquad \text{for all } w \in \operatorname{dom} A,$$

so $x \in \operatorname{dom} A^*$ and $A^*u = v$.

Exercise 2.39 Prove that a densely defined operator A is closable if and only if its adjoint A^* is densely defined, in which case $\overline{A} = (A^*)^*$ and $\overline{A}^* = A^*$.

Definition 2.40 A densely defined operator A in a Hilbert space is *self-adjoint* if and only if $A^* = A$. This is stronger than the condition that

$$\langle u, Av\rangle = \langle Au, v\rangle \qquad \text{for all } u, v \in \operatorname{dom} A,$$

which is merely the condition that $A \subseteq A^*$. An operator satisfying this inclusion is called *symmetric*.

Exercise 2.41 Let A be a densely defined operator in the Hilbert space H. Prove that A is self-adjoint if and only if A is symmetric and such that both $A + \mathrm{i}I$ and $A - \mathrm{i}I$ are surjective, so that

$$\{Av + \mathrm{i}v : v \in \operatorname{dom} A\} = \{Av - \mathrm{i}v : v \in \operatorname{dom} A\} = \mathsf{H}.$$

Proof Suppose first that A is symmetric and the range conditions hold. Let $u, v \in \mathsf{H}$ be such that

$$\langle u, Aw\rangle = \langle v, w\rangle \qquad \text{for all } w \in \operatorname{dom} A,$$

so that $u \in \operatorname{dom} A^*$ and $A^*u = v$. We wish to prove that $u \in \operatorname{dom} A$ and $Au = v$.

Let $x, y \in \operatorname{dom} A$ be such that $(A - \mathrm{i}I)x = v - \mathrm{i}u$ and $(A + \mathrm{i}I)y = u - x$. Then

$$\begin{aligned}\langle u, u - x\rangle &= \langle u, (A + \mathrm{i}I)y\rangle = \langle v - \mathrm{i}u, y\rangle \\ &= \langle (A - \mathrm{i}I)x, y\rangle = \langle x, (A + \mathrm{i}I)y\rangle = \langle x, u - x\rangle,\end{aligned}$$

where the penultimate equality holds because A is symmetric and $x, y \in \operatorname{dom} A$. It follows that $\|u - x\|^2 = 0$, so $u = x \in \operatorname{dom} A$ and $Au = Ax = v - \mathrm{i}u + \mathrm{i}x = v$.

Now suppose that A is self-adjoint, and note that it suffices to prove that $A + \mathrm{i}I$ is surjective, since $-A$ is self-adjoint whenever A is.

Note first that

$$\|(A + \mathrm{i}I)v\|^2 = \|Av\|^2 + \|v\|^2 \qquad \text{for all } v \in \operatorname{dom} A, \tag{2.2}$$

which implies that $\operatorname{ran}(A + \mathrm{i}I)$ is closed: if the sequence $(v_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom} A$ is such that $\big((A+\mathrm{i}I)v_n\big)_{n\in\mathbb{N}}$ is convergent, then both $(v_n)_{n\in\mathbb{N}}$ and $(Av_n)_{n\in\mathbb{N}}$ are Cauchy, so convergent, with $v_n \to v \in \mathsf{H}$ and $Av_n \to u \in \mathsf{H}$. Since A is closed, it follows that $v \in \operatorname{dom} A$ and $Av = u$, from which we see that $(A+\mathrm{i}I)v_n \to u+\mathrm{i}v = (A+\mathrm{i}I)v$.

It is also follows from (2.2), with A replaced by $-A$, that $A - \mathrm{i}I$ is injective. As

$$\begin{aligned} u \in \ker(A - \mathrm{i}I) &\iff \langle (A - \mathrm{i}I)u, v\rangle = 0 \qquad \text{for all } v \in \operatorname{dom} A \\ &\iff \langle u, (A + \mathrm{i}I)v\rangle = 0 \qquad \text{for all } v \in \operatorname{dom} A = \operatorname{dom} A^* \\ &\iff u \in \operatorname{ran}(A + \mathrm{i}I)^{\perp}, \end{aligned}$$

so

$$\operatorname{ran}(A + \mathrm{i}I) = (\operatorname{ran}(A + \mathrm{i}I)^{\perp})^{\perp} = \ker(A - \mathrm{i}I)^{\perp} = \{0\}^{\perp} = \mathsf{H}.$$

□

Definition 2.42 Let A be an unbounded operator in V. Its *spectrum* is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ has no inverse in } B(V)\}$$

and its *resolvent* is the map

$$\mathbb{C} \setminus \sigma(A) \to B(V);\ \lambda \mapsto (\lambda I - A)^{-1}.$$

In other words, $\lambda \in \mathbb{C}$ is not in the spectrum of A if and only if there exists a bounded operator $B \in B(V)$ such that $B(\lambda I - A) = I_{\operatorname{dom} A}$ and $(\lambda I - A)B = I_V$; in particular, the operator $\lambda I - A$ is a bijection from $\operatorname{dom} A$ onto V.

Remark 2.43 If the operator $T : V \to V$ is bounded, then its spectrum $\sigma(T)$ is contained in the closed disc $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|T\|\}$ [22, Lemma 1.2.4].

Exercise 2.44 Let A be an unbounded operator in V and suppose $\lambda \in \mathbb{C}$ is such that $\lambda I - A$ is a bijection from $\operatorname{dom} A$ onto V. Prove that $(\lambda I - A)^{-1}$ is bounded if and only if A is closed. [Thus algebraic invertibility of $\lambda I - A$ is equivalent to its topological invertibility if and only if A is closed.]

The following theorem shows that the resolvent of a semigroup generator may be thought of as the Laplace transform of the semigroup.

Theorem 2.45 *Let A be the generator of a one-parameter semigroup T of type (M, a) on V. Then $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant a\}$. Furthermore, if $\operatorname{Re}\lambda > a$, then*

$$(\lambda I - A)^{-1}v = \int_0^\infty e^{-\lambda t} T_t v \,\mathrm{d}t \qquad \textit{for all } v \in V \tag{2.3}$$

and $\|(\lambda I - A)^{-1}\| \leqslant M(\operatorname{Re}\lambda - a)^{-1}$.

Proof Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > a$ and note first that

$$R : V \mapsto V;\ v \mapsto \int_0^\infty e^{-\lambda t} T_t v \,\mathrm{d}t$$

is a bounded linear operator, with $\|R\| \leqslant M(\operatorname{Re}\lambda - a)^{-1}$.

If $v \in V$ and $u = Rv$, then

$$T_t u = \int_0^\infty e^{-\lambda s} T_{s+t} v \,\mathrm{d}s = \int_t^\infty e^{-\lambda(r-t)} T_u v \,\mathrm{d}r = e^{\lambda t} \int_t^\infty e^{-\lambda u} T_r v \,\mathrm{d}r,$$

and therefore, if $t > 0$,

$$\begin{aligned} t^{-1}(T_t - I)u &= t^{-1} e^{\lambda t} \int_t^\infty e^{-\lambda s} T_s v \,\mathrm{d}s - t^{-1} \int_0^\infty e^{-\lambda s} T_s v \,\mathrm{d}s \\ &= -t^{-1} e^{\lambda t} \int_0^t e^{-\lambda s} T_s v \,\mathrm{d}s + t^{-1}(e^{\lambda t} - 1) \int_0^\infty e^{-\lambda s} T_s v \,\mathrm{d}s \\ &\to -v + \lambda u \qquad \text{as } t \to 0+. \end{aligned}$$

Thus $u \in \operatorname{dom} A$ and $(\lambda I - A)u = v$. It follows that $\operatorname{ran} R \subseteq \operatorname{dom} A$ and $(\lambda I - A)R = I_V$.

However, since $(T_t - I)R = R(T_t - I)$ and R is bounded, the same working shows that

$$RAu = -u + \lambda Ru \iff R(\lambda I - A)u = u \qquad \text{for all } u \in \operatorname{dom} A.$$

Thus $R(\lambda I - A) = I_{\operatorname{dom} A}$ and $R = (\lambda I - A)^{-1}$, as claimed. □

The Laplace-transform formula of Theorem 2.45 allows one to recover a semigroup from its resolvent.

Theorem 2.46 *Let A be the generator of a one-parameter semigroup T of type (M, a) on V, and let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > a$. Then*

$$(\lambda I - A)^{-n} v = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-\lambda t} T_t v \,\mathrm{d}t \qquad \textit{for all } n \in \mathbb{N} \textit{ and } v \in V,$$

and

$$\begin{aligned} T_t v &= \lim_{n\to\infty} (I - n^{-1} t A)^{-n} v \\ &= \lim_{n\to\infty} (n/t)^n \big((n/t)I - A\big)^{-n} v \qquad \textit{for all } t > 0 \textit{ and } v \in V. \end{aligned}$$

Proof The first claim follows by induction, with Theorem 2.45 giving the case $n = 1$.

As noted by Hille and Phillips [16, Theorem 11.6.6], the second follows from the Post–Widder inversion formula for the Laplace transform. For all $n \in \mathbb{N}$, let

$$f_n : \mathbb{R}_+ \to \mathbb{R}_+;\ t \mapsto \frac{n^n}{(n-1)!} t^n e^{-nt},$$

and note that f_n is strictly increasing on $[0, 1]$ and strictly decreasing on $[1, \infty)$, and its integral $\int_0^\infty f_n(t)\,\mathrm{d}t = 1$; this last fact may be proved by induction. If n is sufficiently large, then a short calculation shows that

$$(n/t)^n \big((n/t)I - A\big)^{-n} v = (1 - n^{-1})^{-n} \int_0^\infty f_{n-1}(r) e^{-r} T_{tr} v \,\mathrm{d}r.$$

The result follows by splitting the integral into three parts. Fix $\varepsilon \in (0, 1)$ and note first that $f_n(r) \leqslant n e^n r^n e^{-nr}$ for all $r \in \mathbb{R}_+$, with the latter function strictly increasing on $[0, 1]$, so

$$\Big\| \int_0^{1-\varepsilon} f_n(r) e^{-r} T_{tr} v \,\mathrm{d}r \Big\| \leqslant n(1-\varepsilon)^{n+1} e^{n\varepsilon} M \max\{1, e^{at(1-\varepsilon)}\} \|v\| \to 0 \quad \text{as } n \to \infty.$$

Similarly, if $b := \varepsilon/(1+\varepsilon)$, then $f_n(r) e^{bnr} \leqslant n e^n (1+\varepsilon)^n e^{(b-1)n(1+\varepsilon)} \leqslant n(1+\varepsilon)^n$ for all $r \geqslant 1 + \varepsilon$, and so

$$\begin{aligned} \Big\| \int_{1+\varepsilon}^\infty f_n(r) e^{-r} T_{tr} v \,\mathrm{d}r \Big\| &\leqslant M \|v\| n(1+\varepsilon)^n \int_{1+\varepsilon}^\infty e^{(a-bn)r} \,\mathrm{d}r \\ &\leqslant M \|v\| \frac{n}{bn - a} (1+\varepsilon)^n e^{(a-bn)(1+\varepsilon)} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

since $b(1+\varepsilon) = \varepsilon$ and $(1+\varepsilon)e^{-\varepsilon} < 1$. A standard approximation argument now completes the proof. □

We have now obtained enough necessary conditions on the generator of a strongly continuous semigroup for them to be sufficient as well.

Theorem 2.47 (Feller–Miyadera–Phillips) *A closed, densely defined operator A in V is the generator of a strongly continuous semigroup of type (M, a) if and only if*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leqslant a\}$$

and

$$\|(\lambda I - A)^{-m}\| \leqslant M(\lambda - a)^{-m} \qquad \text{for all } \lambda > a \text{ and } m \in \mathbb{N}. \tag{2.4}$$

Proof Let A be the generator of a strongly continuous semigroup T of type (M, a). The spectral condition is a consequence of Theorem 2.45, and the norm inequality follows from Theorem 2.46.

For the converse, let the operator A be closed, densely defined, such that (2.4) holds and having spectrum not containing (a, ∞). Setting $A_\lambda := \lambda A(\lambda I - A)^{-1}$, note that $\{A_\lambda : \lambda \in (a, \infty)\}$ is a commuting family of bounded operators such that $A_\lambda v \to Av$ as $\lambda \to \infty$, for all $v \in \operatorname{dom} A$; see Exercise 2.48 for more details.

With $T_t^\lambda := \exp(tA_\lambda)$, the inequalities (2.4) imply $\|T_t^\lambda\| \leqslant M \exp\bigl(a\lambda t/(\lambda - a)\bigr)$ for all $\lambda > a$ and $t \in \mathbb{R}_+$, so $\limsup_{\lambda\to\infty} \|T_t^\lambda\| \leqslant Me^{at}$. Since

$$(T_t^\lambda - T_t^\mu)v = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\bigl(T_s^\lambda T_{t-s}^\mu v\bigr)\,\mathrm{d}s = \int_0^t T_s^\lambda T_{t-s}^\mu (A_\lambda - A_\mu)v\,\mathrm{d}s,$$

if $\lambda, \mu > 2a_+ = 2\max\{a, 0\}$ and $v \in \operatorname{dom} A$ then

$$\|(T_t^\lambda - T_t^\mu)v\| \leqslant tM^2 e^{2a_+ t}\|(A_\lambda - A_\mu)v\| \to 0 \qquad \text{as } \lambda, \mu \to \infty,$$

locally uniformly in t. An approximation argument shows that $T_t u = \lim_{\lambda\to\infty} T_t^\lambda u$ exists for all $t \in \mathbb{R}_+$ and $u \in V$, and that $T = (T_t)_{t\in\mathbb{R}_+}$ is a strongly continuous one-parameter semigroup of type (M, a).

To see that the generator of T is A, note that the previous working and Lemma 2.34 imply that

$$T_t v - v = \lim_{\lambda\to\infty} T_t^\lambda v - v = \lim_{\lambda\to\infty} \int_0^t T_s^\lambda A_\lambda v\,\mathrm{d}s = \int_0^t T_s A v\,\mathrm{d}s \qquad \text{for all } v \in \operatorname{dom} A;$$

dividing by t and letting $t \to 0$ shows that the generator B of T is an extension of A. Note that (a, ∞) is not in the spectrum of B, by Theorem 2.45; it is a simple exercise to show that $(\lambda I - A)^{-1} = (\lambda I - B)^{-1}$ for $\lambda > a$, and since the ranges of these operators are the domain of A and B, the result follows. □

Exercise 2.48 Let A be an unbounded operator in V, with spectrum not containing (a, ∞) and such that $\|(\lambda I - A)^{-1}\| \leqslant M(\lambda - a)^{-1}$ for all $\lambda > a$, where M and a are constants. Prove that

$$A_\lambda := \lambda A(\lambda I - A)^{-1} = \lambda^2(\lambda I - A)^{-1} - \lambda I$$

commutes with A_μ for all $\lambda, \mu > a$. Prove also that

$$\lim_{\lambda\to\infty} \lambda(\lambda I - A)^{-1}u = u \qquad \text{for all } u \in V,$$

by showing this first for the case $u \in \operatorname{dom} A$. Deduce that $A_\lambda v \to Av$ when $v \in \operatorname{dom} A$.

For contraction semigroups, we have the following refinement of Theorem 2.47.

Theorem 2.49 (Hille–Yosida) *Let A be a closed, densely defined linear operator in the Banach space V. The following are equivalent.*

(i) *A is the generator of a strongly continuous contraction semigroup.*

(ii) $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$ *and*

$$\|(\lambda I - A)^{-1}\| \leqslant (\operatorname{Re}\lambda)^{-1} \qquad \textit{whenever } \operatorname{Re}\lambda > 0.$$

(iii) $\sigma(A) \cap (0, \infty)$ *is empty and*

$$\|(\lambda I - A)^{-1}\| \leqslant \lambda^{-1} \qquad \textit{whenever } \lambda > 0.$$

Proof Note that (i) implies (ii), by Theorem 2.45, and (ii) implies (iii) trivially. That (iii) implies (i) follows from the extension of Theorem 2.47 noted in its proof. □

In practice, verifying the norm conditions in Theorems 2.47 and 2.49 may prove to be challenging. The next section introduces the concept of operator dissipativity, which is often more tractable.

2.4 The Lumer–Phillips Theorem

Throughout this subsection, V denotes a Banach space and V^* its topological dual.

Definition 2.50 For all $v \in V$, let

$$TF(v) := \{\phi \in V^* : \phi(v) = \|v\|^2 = \|\phi\|^2\}$$

be the set of *normalised tangent functionals* to v. The Hahn–Banach theorem [25, Theorem III.6] implies that $TF(v)$ is non-empty for all $v \in V$.

Exercise 2.51 Prove that if H is a Hilbert space then $TF(v) = \{\langle v|\}$ for all $v \in H$, where the Dirac functional $\langle v|$ is such that $\langle v|u := \langle v, u\rangle$ for all $u \in H$. [Recall the Riesz–Fréchet theorem from Example 2.15.]

Exercise 2.52 Prove that if $f \in V = C(K)$ and $x_0 \in K$ is such that $|f(x_0)| = \|f\|$ then setting $\phi(g) := \overline{f(x_0)}g(x_0)$ for all $g \in V$ defines a normalised tangent functional for f. Deduce that $TF(f)$ may contain more than one element.

Definition 2.53 An unbounded operator A in V is *dissipative* if and only if there exists $\phi \in TF(v)$ such that $\operatorname{Re}\phi(Av) \leqslant 0$, for all $v \in \operatorname{dom} A$. [Note that it suffices to check this condition for unit vectors only.]

Exercise 2.54 Prove that an operator A in the Hilbert space H is dissipative if and only if $\|(I + A)v\| \leqslant \|(I - A)v\|$ for all $v \in \operatorname{dom} A$.

Exercise 2.55 Suppose T is a contraction semigroup with generator A. Prove that A is dissipative.

Proof If $v \in \operatorname{dom} A$ and $\phi \in TF(v)$, then

$$\operatorname{Re}\phi(Av) = \lim_{t\to 0+} t^{-1}\operatorname{Re}\phi(T_t v - v) \leqslant \lim_{t\to 0+} t^{-1}\|\phi\|\,\|v\| - \|v\|^2 = 0,$$

so A is dissipative. □

We now seek to find a converse to the result of the preceding exercise.

Lemma 2.56 *The unbounded operator A in V is dissipative if and only if*

$$\|(\lambda I - A)v\| \geqslant \lambda\|v\| \qquad \text{for all } \lambda > 0 \text{ and } v \in \operatorname{dom} A. \tag{2.5}$$

If A is dissipative and $\lambda I - A$ is surjective for some $\lambda > 0$, then $\lambda \notin \sigma(A)$ and $\|(\lambda I - A)^{-1}\| \leqslant \lambda^{-1}$.

Proof Suppose first that (2.5) holds, let $v \in \operatorname{dom} A$ be a unit vector and, for all $\lambda > 0$, choose $\phi_\lambda \in TF\big((\lambda I - A)v\big)$. Then $\phi_\lambda \neq 0$, so $\psi_\lambda = \|\phi_\lambda\|^{-1}\phi_\lambda$ is well defined, and

$$\lambda \leqslant \|(\lambda I - A)v\| = \psi_\lambda(\lambda v - Av) = \lambda\operatorname{Re}\psi_\lambda(v) - \operatorname{Re}\psi_\lambda(Av).$$

Since $\operatorname{Re}\psi_\lambda(v) \leqslant 1$ and $-\operatorname{Re}\psi_\lambda(Av) \leqslant \|Av\|$, it follows that

$$\operatorname{Re}\psi_\lambda(Av) \leqslant 0 \qquad \text{and} \qquad \operatorname{Re}\psi_\lambda(v) \geqslant 1 - \lambda^{-1}\|Av\|.$$

The Banach–Alaoglu theorem [25, Theorem IV.21] implies that the unit ball of V^* is weak* compact, so the net $(\psi_\lambda)_{\lambda>0}$ has a weak*-convergent subnet with limit in the unit ball. Hence there exists $\psi \in V^*$ such that

$$\|\psi\| \leqslant 1, \quad \operatorname{Re}\psi(Av) \leqslant 0 \quad \text{and} \quad \operatorname{Re}\psi(v) \geqslant 1.$$

In particular,

$$|\psi(v)| \leqslant \|\psi\| \leqslant 1 \leqslant \operatorname{Re}\psi(v) \leqslant |\psi(v)|,$$

so $\psi \in TF(v)$ and A is dissipative.

Conversely, if $\lambda > 0$, $v \in \operatorname{dom} A$ and $\phi \in TF(v)$ is such that $\operatorname{Re}\phi(Av) \leqslant 0$ then

$$\|v\|\,\|(\lambda I - A)v\| \geqslant |\phi\big((\lambda I - A)v\big)| = |\lambda\|v\|^2 - \phi(Av)| \geqslant \lambda\|v\|^2.$$

Thus (2.5) holds, and $\lambda I - A$ is injective.

If $\lambda I - A$ is also surjective, then (2.5) gives that $\|u\| \geqslant \lambda\|(\lambda I - A)^{-1}u\|$ for all $u \in V$, whence the final claim. □

Exercise 2.57 Let A be dissipative. Prove that $\lambda I - A$ is surjective for some $\lambda > 0$ if and only if $\lambda I - A$ is surjective for all $\lambda > 0$. [Hint: for a suitable choice of λ and λ_0, consider the series $R_\lambda := \sum_{n=0}^\infty (\lambda - \lambda_0)^n (\lambda_0 I - A)^{-(n+1)}$.]

Proof Suppose that $\lambda_0 > 0$ is such that $\lambda_0 I - A$ is surjective. It follows from Lemma 2.56 that $\|(\lambda_0 I - A)^{-1}\| \leqslant \lambda_0^{-1}$. The series

$$R_\lambda = \sum_{n=0}^\infty (\lambda_0 - \lambda)^n (\lambda_0 I - A)^{-(n+1)}$$

is norm convergent for all $\lambda \in (0, 2\lambda_0)$; if we can show that $R_\lambda = (\lambda I - A)^{-1}$, then the result follows.

If $C \in B(V)$ is such that $\|C\| < 1$ then $I - C$ is invertible, with $(I - C)^{-1} = \sum_{n=0}^\infty C^n$. Hence if $C = (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}$, then

$$R_\lambda = (\lambda_0 I - A)^{-1}(I - C)^{-1} = (I - C)^{-1}(\lambda_0 I - A)^{-1},$$

so $\operatorname{ran} R_\lambda \subseteq \operatorname{dom}(\lambda_0 I - A) = \operatorname{dom}(\lambda I - A)$,

$$(\lambda I - A)R_\lambda = \big((\lambda - \lambda_0)I + (\lambda_0 I - A)\big)R_\lambda = \big((\lambda - \lambda_0)(\lambda_0 I - A)^{-1} + I\big)(I - C)^{-1} = I_V$$

and

$$\begin{aligned} R_\lambda(\lambda I - A) &= R_\lambda\big((\lambda - \lambda_0)I + (\lambda_0 I - A)\big) \\ &= (I - C)^{-1}\big((\lambda - \lambda_0)(\lambda_0 I - A)^{-1} + I\big) = I_{\operatorname{dom} A}. \end{aligned}$$

□

Theorem 2.58 (Lumer–Phillips) *A closed, densely defined operator A generates a strongly continuous contraction semigroup if and only if A is dissipative and $\lambda I - A$ is surjective for some $\lambda > 0$.*

Proof One implication follows from Exercise 2.57, Lemma 2.56 and Theorem 2.49. The other implication follows from Theorem 2.49 and Exercise 2.55. □

Example 2.59 Let $V = L^2[0, 1]$, and let $Af := g$, where

$$\operatorname{dom} A := \Big\{ f \in V : \text{there exists } g \in V \text{ such that } f(t) = \int_0^t g(s)\,\mathrm{d}s \text{ for all } t \in [0, 1] \Big\}.$$

Thus $f \in \operatorname{dom} A$ if and only if $f(0) = 0$ and f is absolutely continuous on $[0, 1]$, with square-integrable derivative, and then $Af = f'$ almost everywhere. For such f, note that

$$\operatorname{Re}\langle f, Af\rangle = \operatorname{Re}\int_0^1 \overline{f(t)}\, f'(t)\,\mathrm{d}t = \frac{1}{2}\int_0^t \big(\overline{f}f\big)'(t)\,\mathrm{d}t = \frac{1}{2}|f(1)|^2 \geqslant 0,$$

so $-A$ is a dissipative operator, but A is not.

Let $g \in V$ and $\lambda > 0$; we wish to find $f \in \operatorname{dom} A$ such that

$$(\lambda I + A)f = g \iff \lambda f + f' = g \iff f = \int (g - \lambda f).$$

We proceed by iterating this relation: given $h \in \{f, g\}$, let $h_0 := h$ and, for all $n \in \mathbb{Z}_+$, let $h_{n+1} \in V$ be such $h_{n+1}(t) = \int_0^t h_n(s)\,\mathrm{d}s$ for all $t \in [0, 1]$. Then

$$f = g_1 - \lambda \int f = g_1 - \lambda \int\int (g - \lambda f) = \cdots = \sum_{j=0}^{n-1} (-\lambda)^j g_{j+1} + (-\lambda)^n f_n$$

for all $n \in \mathbb{N}$. The series $\sum_{j=0}^{\infty} (-\lambda)^j g_{j+1}$ is uniformly convergent on $[0, 1]$, so defines a function $F \in \operatorname{dom} A$, whereas $(-\lambda)^n f_n \to 0$ as $n \to \infty$. Thus

$$(\lambda I + A)F = -\sum_{j=0}^{\infty} (-\lambda)^{j+1} g_{j+1} + \sum_{j=0}^{\infty} (-\lambda)^j g_j = g_0 = g,$$

so $\lambda I + A$ is surjective. By the Lumer–Phillips theorem, the operator $-A$ generates a contraction semigroup.

Exercise 2.60 Fill in the details at the end of Example 2.59. [Hint: with the notation of the example, show that if $h \in \{f, g\}$ then $|h_n(t)|^2 \leqslant t^n \|h\|_2^2/n!$ for all $n \in \mathbb{N}$.]

Remark 2.61 We can explain informally why the operator A defined in Example 2.59 does not generate a semigroup, and why $-A$ does. Recall that each element of a semigroup leaves the domain of the generator invariant, by Lemma 2.34, and A would generate a left-translation semigroup, which does not preserve the boundary condition $f(0) = 0$. Moreover, $-A$ generates the right-translation semigroup, and this does preserve the boundary condition.

If we let A_0 be the restriction of A to the domain

$$\operatorname{dom} A_0 := \{f \in \operatorname{dom} A : f(1) = 0\},$$

so adding a further boundary condition, then both A_0 and $-A_0$ are dissipative, but neither generates a semigroup. We cannot solve the equation $(\lambda I \pm A_0)f = g$ for all g when subject to the constraint that $f \in \operatorname{dom} A_0$. [Take $g \in L^2[0, 1]$ such that $g(t) = t$ for all $t \in [0, 1]$, construct F as in Example 2.59 and note that $F(1) \neq 0$.]

Example 2.62 Recall the weak derivatives D^α and Sobolev spaces $H^k(\mathbb{R}^d)$ defined in Example 2.9, and let $2e_j \in \mathbb{Z}_+^d$ be the multi-index with 2 in the jth coordinate and 0 elsewhere. The *Laplacian*

$$\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} = \sum_{j=1}^{d} D^{2e_j}$$

is a densely defined operator in $L^2(\mathbb{R}^d)$ with domain $\operatorname{dom}\Delta := H^2(\mathbb{R}^d)$. It may be shown that

$$\langle \Delta f, g\rangle_{L^2(\mathbb{R}^d)} = -\langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d)} \qquad \text{for all } f, g \in H^2(\mathbb{R}^d), \tag{2.6}$$

where

$$\nabla := (D^{e_1}, \ldots, D^{e_d}) : f \mapsto \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\Big);$$

consequently, the Laplacian Δ is dissipative. One way to establish (2.6) is to use the Fourier transform. Fourier-theoretic results can also be used to prove that $\lambda I - \Delta$ is surjective for all $\lambda > 0$, essentially because the map $x \mapsto 1/(\lambda + |x|^2)$ is bounded on $\mathbb{R}^d$. Thus the Laplacian generates a contraction semigroup.

Exercise 2.63 Let A be a densely defined operator on the Hilbert space H. Prove that if A is symmetric, so that

$$\langle u, Av\rangle = \langle Au, v\rangle \quad \text{for all } u, v \in \operatorname{dom} A,$$

then $\mathrm{i}A$ is dissipative. Deduce that if H is self-adjoint then $\mathrm{i}H$ and $-\mathrm{i}H$ are the generators of contraction semigroups.

Prove, further, that if $T = (T_t)_{t\in\mathbb{R}_+}$ has generator $\mathrm{i}H$, with H self-adjoint, then T_t is unitary, so that $T_t^* T_t = I = T_t T_t^*$, for all $t \in \mathbb{R}_+$.

Proof The first part is an immediate consequence of Theorem 2.58, the Lumer–Phillips theorem, together with Exercise 2.41.

For the next part, fix $u, v \in \operatorname{dom} H$ and $t \in \mathbb{R}_+$. If $h > 0$ then

$$\begin{aligned} h^{-1}\langle u, (T_{t+h}^* T_{t+h} - T_t^* T_t)v\rangle &= \langle T_{t+h}u, h^{-1}(T_h - I)T_t v\rangle \\ &\quad + \langle h^{-1}(T_h - I)T_t u, T_t v\rangle \\ &\to \langle T_t u, \mathrm{i}HT_t v\rangle + \langle \mathrm{i}HT_t u, T_t v\rangle = 0 \end{aligned}$$

as $h \to 0+$, since $T_t u, T_t v \in \operatorname{dom} H$ and T is strongly continuous. A real-valued function on $\mathbb{R}_+$ is constant if it is continuous and its right derivative is identically zero, so this working shows that $T_t^* T_t = I$.

Now let $S = (S_t)_{t \in \mathbb{R}_+}$ be the strongly continuous semigroup with generator $-\mathrm{i}H$. The previous working shows that $S_t^* S_t = I$ for all $t \in \mathbb{R}_+$, so it suffices to let $t > 0$ and prove that $S_t = T_t^*$. To see this, let $u, v \in \operatorname{dom} H$ and consider the function

$$F : [0, t] \to \mathbb{C};\ s \mapsto \langle u, T_{t-s}^* S_s v \rangle.$$

Working as above, it is straightforward to show that $F' \equiv 0$ on $(0, t)$, so $F(0) = F(t)$ and the result follows. □

Exercise 2.64 Suppose U is a strongly continuous one-parameter semigroup on the Hilbert space H, with U_t unitary, so that $U_t^* U_t = I = U_t U_t^*$, for all $t \in \mathbb{R}_+$. Let A be the generator of U.

Prove that $U^* = (U_t^*)_{t \in \mathbb{R}_+}$ is also a strongly continuous one-parameter semigroup, with generator $-A$. Deduce that $H := \mathrm{i}A$ is self-adjoint.

Proof The semigroup property for U^* is immediate, and strong continuity holds because

$$\|(U_t^* - I)v\|^2 = \langle (I - U_t)v, v \rangle - \langle v, (U_t - I)v \rangle \leqslant 2\|(U_t - I)v\|\,\|v\| \to 0$$

as $t \to 0+$, for any $v \in \mathsf{H}$.

Next, denote the generator of U^* by B, and let $v \in \operatorname{dom} A$. Then

$$t^{-1}(U_t^* - I)v = -U_t^* t^{-1}(U_t - I)v \to -Av \qquad \text{as } t \to 0+,$$

so $-A \subseteq B$. Since $(U^*)^* = U$, applying this argument with U replaced by U^* gives the reverse inclusion. Thus U^* has generator $B = -A$, as claimed.

Finally, let $H = \mathrm{i}A$ and suppose first that $u, v \in \operatorname{dom} H = \operatorname{dom} A$. Then

$$\langle -\mathrm{i}Hu, v \rangle = \lim_{t \to 0+} \langle t^{-1}(U_t - I)u, v \rangle = \lim_{t \to 0+} \langle u, t^{-1}(U_t^* - I)v \rangle = \langle u, \mathrm{i}Hv \rangle,$$

so $H \subseteq H^*$. For the reverse inclusion, note that

$$U_t^* v = v + \int_0^t U_s^* A^* v \,\mathrm{d}s \qquad \text{for all } v \in \operatorname{dom} A^*,$$

by Lemma 2.34 applied to U and properties of the adjoint. Thus $A^* \subseteq -A$, the generator of U^*, and therefore $H^* = -\mathrm{i}A^* \subseteq \mathrm{i}A = H$. □

Remark 2.65 Exercises 2.63 and 2.64 lead to Stone's theorem, which gives a one-to-one correspondence between self-adjoint operators and strongly continuous one-parameter groups of unitary operators. This result has significant consequences for the mathematical foundations of quantum theory; see [25, Section VIII.4].

3 Classical Markov Semigroups

Throughout this section, the triple $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space, so that $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure on the σ-algebra $\mathcal{F}$ of subsets of Ω, and E will denote a topological space, with $\mathcal{E}$ its Borel σ-algebra, generated by the open subsets.

An *E-valued random variable* is a $\mathcal{F}$-$\mathcal{E}$-measurable mapping $X : \Omega \to E$. If X is an E-valued random variable, then $\sigma(X)$ is the smallest sub-σ-algebra $\mathcal{F}_0$ of $\mathcal{F}$ such that X is $\mathcal{F}_0$-$\mathcal{E}$ measurable. More generally, if $(X_i)_{i\in I}$ is an indexed set of E-valued random variables, then $\sigma(X_i : i \in I)$ is the smallest sub-σ-algebra $\mathcal{F}_0$ of $\mathcal{F}$ such that X_i is $\mathcal{F}_0$-$\mathcal{E}$ measurable for all $i \in I$.

3.1 Markov Processes

Definition 3.1 Given a real-valued random variable X which is integrable, so that

$$\mathbb{E}\big[|X|\big] := \int_\Omega |X(\omega)|\,\mathbb{P}(\mathrm{d}\omega) < \infty,$$

and a sub-σ-algebra $\mathcal{F}_0$ of $\mathcal{F}$, the *conditional expectation* $\mathbb{E}[X|\mathcal{F}_0]$ is a real-valued random variable Y which is $\mathcal{F}_0$-$\mathcal{E}$ measurable and such that

$$\mathbb{E}[1_A X] = \mathbb{E}[1_A Y] \qquad \text{for all } A \in \mathcal{F}_0.$$

The choice of Y is determined *almost surely*: if Y and Z are both versions of the conditional expectation $\mathbb{E}[X|\mathcal{F}_0]$, then $\mathbb{P}(Y \neq Z) = 0$. The existence of Y is guaranteed by the Radon–Nikodým theorem.

The fact that $\mathbb{E}[X|\mathcal{F}_0]$ is determined almost surely can be recast as saying that $\mathbb{E}[\cdot|\mathcal{F}_0]$ is a linear operator from $L^1(\Omega, \mathcal{F}, \mathbb{P})$ to $L^1(\Omega, \mathcal{F}_0, \mathbb{P}|_{\mathcal{F}_0})$. In fact, the map $X \mapsto \mathbb{E}[X|\mathcal{F}_0]$ is a contraction from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ onto $L^p(\Omega, \mathcal{F}_0, \mathbb{P}|_{\mathcal{F}_0})$, for all $p \in [1, \infty]$.

Remark 3.2 Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Informally, we can think of $Y := \mathbb{E}[X|\mathcal{F}_0]$ as the best guess for X given the information in $\mathcal{F}_0$. In other words, the conditional expectation Y of X with respect to $\mathcal{F}_0$ is the essentially unique choice of $\mathcal{F}_0$-measurable random variable Z which minimises the least-squares distance $\|Z - X\|_2$.

Definition 3.3 Given a topological space E, let the Banach space

$$B_b(E) := \{f : E \to \mathbb{C} \mid f \text{ is Borel measurable and bounded}\},$$

with vector-space operations defined pointwise and supremum norm

$$\|f\| := \sup\{|f(x)| : x \in E\}.$$

Exercise 3.4 Verify that $B_b(E)$ is a Banach space. Show further that the norm $\|\cdot\|$ is submultiplicative, where multiplication of functions is defined pointwise, so that $B_b(E)$ is a *Banach algebra*. Show also that the Banach algebra $B_b(E)$ is *unital*: the multiplicative unit 1_E is such that $\|1_E\| = 1$. Show finally that the C^* *identity* holds:

$$\|f\|^2 = \|\overline{f} f\| \qquad \text{for all } f \in B_b(E),$$

where the isometric involution $f \mapsto \overline{f}$ is such that $\overline{f}(x) := \overline{f(x)}$ for all $x \in E$.

Definition 3.5 (Provisional) A *Markov process* with *state space* E is a collection of E-valued random variables $X = (X_t)_{t\in\mathbb{R}_+}$ on a common probability space such that, given any $f \in B_b(E)$,

$$\mathbb{E}[f(X_t) \mid \sigma(X_r : 0 \leqslant r \leqslant s)] = \mathbb{E}[f(X_t) \mid \sigma(X_s)]$$

for all $s, t \in \mathbb{R}_+$ such that $s \leqslant t$.

A Markov process is *time homogeneous* if, given any $f \in B_b(E)$,

$$\mathbb{E}[f(X_t) \mid X_s = x] = \mathbb{E}[f(X_{t-s}) \mid X_0 = x] \tag{3.1}$$

for all $s, t \in \mathbb{R}_+$ such that $s \leqslant t$ and $x \in E$.

Definition 3.5 is well motivated by Remark 3.2, but it is somewhat unsatisfactory; for example, what should be the proper meaning of (3.1)? To improve upon it, we introduce the following notion.

Definition 3.6 A *transition kernel* on $(E, \mathcal{E})$ is a map $p : E \times \mathcal{E} \to [0, 1]$ such that

(i) the map $x \mapsto p(x, A)$ is Borel measurable for all $A \in \mathcal{E}$ and
(ii) the map $A \mapsto p(x, A)$ is a probability measure for all $x \in E$.

We interpret $p(x, A)$ as the probability that the transition ends in A, given that it started at x.

Exercise 3.7 If p and q are transition kernels on $(E, \mathcal{E})$, then the *convolution* $p * q$ is defined by setting

$$(p * q)(x, A) := \int_E p(x, \mathrm{d}y) q(y, A) \qquad \text{for all } x \in E \text{ and } A \in \mathcal{E}.$$

Prove that $p * q$ is a transition kernel. Prove also that convolution is associative: if p, q and r are transition kernels then $(p * q) * r = p * (q * r)$.

Definition 3.8 A triangular collection $\{p_{s,t} : s, t \in \mathbb{R}_+,\ s \leqslant t\}$ of transition kernels is *consistent* if $p_{s,t} * p_{t,u} = p_{s,u}$ for all $s, t, u \in \mathbb{R}_+$ with $s \leqslant t \leqslant u$; that is,

$$p_{s,u}(x, A) = \int_E p_{s,t}(x, \mathrm{d}y) p_{t,u}(y, A) \qquad \text{for all } x \in E \text{ and } A \in \mathcal{E}. \tag{3.2}$$

Equation (3.2) is the *Chapman–Kolmogorov equation*. We interpret $p_{s,t}(x, A)$ as the probability of moving from x at time s to somewhere in A at time t.

Similarly, a one-parameter collection $\{p_t : t \in \mathbb{R}_+\}$ of transition kernels is *consistent* if $p_s \star p_t = p_{s+t}$ for all $s, t \in \mathbb{R}_+$. In this case, the Chapman–Kolmogorov equation becomes

$$p_{s+t}(x, A) = \int_E p_s(x, \mathrm{d}y) p_t(y, A) \qquad \text{for all } x \in E \text{ and } A \in \mathcal{E}. \tag{3.3}$$

We interpret $p_t(x, A)$ as the probability of moving from x into A in t units of time.

Definition 3.9 A family of E-valued random variables $X = (X_t)_{t \in \mathbb{R}_+}$ on a common probability space is a *Markov process* if there exists a consistent triangular collection of transition kernels such that

$$\mathbb{E}[1_A(X_t) \mid \sigma(X_r : 0 \leqslant r \leqslant s)] = p_{s,t}(X_s, A) \quad \text{almost surely}$$

for all $A \in \mathcal{E}$ and $s, t \in \mathbb{R}_+$ such that $s \leqslant t$.

The family X is a *time-homogeneous Markov process* if there exists a consistent one-parameter collection of transition kernels such that

$$\mathbb{E}[1_A(X_t) \mid \sigma(X_r : 0 \leqslant r \leqslant s)] = p_{t-s}(X_s, A) \quad \text{almost surely}$$

for all $A \in \mathcal{E}$ and $s, t \in \mathbb{R}_+$ such that $s \leqslant t$.

The connection between time-homogeneous Markov processes and semigroups is provided by the following definition and theorem.

Definition 3.10 A *Markov semigroup* is a contraction semigroup T on $B_b(E)$ such that, for all $t \in \mathbb{R}_+$, the bounded linear operator T_t is *positive*: whenever $f \in B_b(E)$ is such that $f \geqslant 0$, that is, $f(x) \in \mathbb{R}_+$ for all $x \in E$, then $T_t f \geqslant 0$. [Note that we impose no condition with respect to continuity at the origin.]

If T_t preserves the unit, that is, $T_t 1_E = 1_E$ for all $t \in \mathbb{R}_+$, then the Markov semigroup T is *conservative*.

Remark 3.11 Positive linear maps preserve order: if T is such a map and $f \leqslant g$, in the sense that $f(x) \leqslant g(x)$ for all $x \in E$, then $Tf \leqslant Tg$. The image of a real-valued function h under a positive linear map is real valued, since if h takes real values, then $h = h^+ - h^-$, where $h^+ : x \mapsto \max\{h(x), 0\}$ and $h^- := x \mapsto \max\{-h(x), 0\}$. Consequently, positive linear maps also commute with the conjugation, in the sense that $T\overline{f} = \overline{Tf}$.

Exercise 3.12 Suppose the mapping $T : B_b(E) \to B_b(E)$ is linear and positive. Show that $|Tf|^2 \leqslant T|f|^2\, T1_E$ for all $f \in B_b(E)$, and deduce that T is bounded, with norm $\|T\| \leqslant \|T1_E\|$.

Proof If $f \in B_b(E)$, $x \in E$ and $\lambda \in \mathbb{R}$, then

$$0 \leqslant T\big(|f - \lambda(Tf)(x)|^2\big)(x) = \lambda^2 (T1_E)(x)\,|(Tf)(x)|^2 - 2\lambda|(Tf)(x)|^2 + (T|f|^2)(x).$$

Inspecting the discriminant of this polynomial in λ gives the first claim, and the second follows because.

$$|(T_t f)(x)|^2 \leqslant (T|f|^2)(x)\,(T1_E)(x) \leqslant \|f\|^2 (T1_E)^2(x) \leqslant \|f\|^2 \|T1_E\|^2.$$

□

Theorem 3.13 *Let $p = \{p_t : t \in \mathbb{R}_+\}$ be a family of transition kernels. Setting*

$$(T_t f)(x) := \int_E p_t(x, \mathrm{d}y) f(y) \qquad \textit{for all } f \in B_b(E) \textit{ and } x \in E$$

defines a bounded linear operator on $B_b(E)$ which is positive, contractive and unit preserving. Furthermore, the family $T = (T_t)_{t \in \mathbb{R}_+}$ is a Markov semigroup if and only if p is consistent.

Proof If $f \in B_b(E)$, $x \in E$ and $s, t \in \mathbb{R}_+$, then the Chapman–Kolmogorov equation (3.3) implies that

$$\begin{aligned}(T_{s+t} f)(x) = \int_E p_{s+t}(x, \mathrm{d}z) f(z) &= \int_E \int_E p_s(x, \mathrm{d}y) p_t(y, \mathrm{d}z) f(z) \\ &= \int_E p_s(x, \mathrm{d}y)(T_t f)(y) \\ &= \big(T_s(T_t f)\big)(x).\end{aligned}$$

Verifying the remaining claims is left as an exercise. □

If we have more structure on the semigroup T, then it is possible to provide a converse to Theorem 3.13. This will be sketched in the following section.

3.2 *Feller Semigroups*

Definition 3.14 Let the topological space E be locally compact. Then

$$C_0(E) := \{f : E \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\} \subseteq B_b(E)$$

is a Banach space when equipped with pointwise vector-space operations and the supremum norm. [A function $f : E \to \mathbb{C}$ *vanishes at infinity* if, for all $\varepsilon > 0$, there exists a compact set $K \subseteq E$ such that $|f(x)| < \varepsilon$ for all $x \in E \setminus K$.]

Exercise 3.15 Prove that $C_0(E)$ lies inside $B_b(E)$ and is indeed a Banach space. Prove that the multiplicative unit 1_E is an element of $C_0(E)$ if and only if E is compact.

Definition 3.16 A Markov semigroup T is *Feller* if the following conditions hold:

(i) $T_t\big(C_0(E)\big) \subseteq C_0(E)$ for all $t \in \mathbb{R}_+$ and

(ii) $\lim_{t\to 0+} \|T_t f - f\| = 0$ for all $f \in C_0(E)$.

Remark 3.17 If a time-homogeneous Markov process X has Feller semigroup T, then

$$\mathbb{E}\big[f(X_{t+h}) - f(X_t) \mid \sigma(X_t)\big] = (T_h f - f)(X_t) = h\,(Af)(X_t) + o(h),$$

so the generator A describes the change in X over an infinitesimal time interval.

Definition 3.18 An $\mathbb{R}^d$-valued stochastic process $X = (X_t)_{t\in\mathbb{R}_+}$ is a *Lévy process* if and only if X

(i) has independent increments, so that $X_t - X_s$ is independent of the past σ-algebra $\sigma(X_r : 0 \leqslant r \leqslant s)$ for all $s, t \in \mathbb{R}_+$ with $s \leqslant t$,

(ii) has stationary increments, so that $X_t - X_s$ has the same distribution as $X_{t-s} - X_0$, for all $s, t \in \mathbb{R}_+$ with $s \leqslant t$ and

(iii) is continuous in probability at the origin, so $\lim_{t\to 0+} \mathbb{P}\big(|X_t - X_0| \geqslant \varepsilon\big) = 0$ for all $\varepsilon > 0$.

Remark 3.19 Lévy processes are well behaved; they have cádlág modifications, and such a modification is a semimartingale, for example.

Exercise 3.20 Prove that if X is a stochastic process with independent and stationary increments, and with cádlág paths, then X is continuous at the origin in probability.

Theorem 3.21 *Every Lévy process gives rise to a conservative Feller semigroup.*

Proof (Sketch Proof) For all $t \in \mathbb{R}_+$, define a transition kernel p_t by setting

$$p_t(x, A) := \mathbb{E}[1_A(X_t - X_0 + x)] \qquad \text{for all } x \in \mathbb{R}^d \text{ and Borel } A \subseteq \mathbb{R}^d.$$

If $s \in \mathbb{R}_+$, then

$$p_t(x, A) = \mathbb{E}[1_A(X_{s+t} - X_s + x)] = \mathbb{E}[1_A(X_{s+t} - X_s + x) \mid \mathcal{F}_s], \tag{3.4}$$

where $\mathcal{F}_s := \sigma(X_r : 0 \leqslant r \leqslant s)$; the first equality holds by stationarity and the second by independence. In particular,

$$p_t(X_s, A) = \mathbb{E}[1_A(X_{s+t}) \mid \mathcal{F}_s],$$

so X is a Markov process with transition kernels $\{p_t : t \in \mathbb{R}_+\}$ if these are consistent. For consistency, we use Theorem 3.13; let T be defined as there and note that

$$(T_t f)(x) = \int_E p_t(x, \mathrm{d}y) f(y) = \mathbb{E}[f(X_t - X_0 + x)]. \tag{3.5}$$

From the previous working, it follows that

$$(T_t f)(x) = \mathbb{E}[f(X_{s+t} - X_s + x) \mid \mathcal{F}_s],$$

and replacing x with the $\mathcal{F}_s$-measurable random variable $X_s - X_0 + x$ gives that

$$(T_{s+t} f)(x) = \mathbb{E}[f(X_{s+t} - X_0 + x)] = \mathbb{E}[(T_t f)(X_s - X_0 + x)] = \big(T_s(T_t f)\big)(x),$$

as required. Equation (3.5) also shows that T is conservative.

If $f \in C_0(\mathbb{R}^d)$, then $x \mapsto f(X_t - X_0 + x) \in C_0(\mathbb{R}^d)$ almost surely, and therefore the Dominated Convergence Theorem gives that $T_t f \in C_0(\mathbb{R}^d)$.

For continuity, let $\varepsilon > 0$ and note that $f \in C_0(\mathbb{R}^d)$ is uniformly continuous, so there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Hence

$$\begin{aligned}
\|T_t f - f\| &\leqslant \sup_{x \in \mathbb{R}^d} \mathbb{E}\big[|f(X_t - X_0 + x) - f(x)|\big] \\
&= \sup_{x \in \mathbb{R}^d} \Big(\mathbb{E}\big[1_{|X_t - X_0| < \delta} |f(X_t - X_0 + x) - f(x)|\big] \\
&\qquad + \mathbb{E}\big[1_{|X_t - X_0| \geqslant \delta} |f(X_t - X_0 + x) - f(x)|\big]\Big) \\
&\leqslant \varepsilon + 2\|f\| \, \mathbb{P}\big(|X_t - X_0| \geqslant \delta\big) \\
&\to \varepsilon \qquad \text{as } t \to 0+.
\end{aligned}$$

□

Theorem 3.22 *Let T be a conservative Feller semigroup. If the state space E is metrisable, then there exists a time-homogeneous Markov process which gives rise to T.*

Proof (Sketch Proof) For all $t \in (0, \infty)$, let

$$p_t(x, A) := (T_t 1_A)(x) \quad \text{for all } x \in E \text{ and } A \in \mathcal{E}.$$

Then p_t is readily verified to be a transition kernel.

Let μ be a probability measure on E. If $t_n \geqslant \cdots \geqslant t_1 \geqslant 0$ and $A_1, \ldots A_n \in \mathcal{E}$, then

$$p_{t_1,\ldots,t_n}(A_1 \times \cdots \times A_n) = \int_E \mu(\mathrm{d}x_0) \int_{A_1} p_{t_1}(x_0, \mathrm{d}x_1) \ldots \int_{A_n} p_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n).$$

By the Chapman–Kolmogorov equation (3.3), these finite-dimensional distributions form a projective family. The Daniell–Kolmogorov extension theorem now yields a probability measure on the product space

$$\Omega := E^{\mathbb{R}_+} = \{\omega = (\omega_t)_{t \in \mathbb{R}_+} : \omega_t \in E \text{ for all } t \in \mathbb{R}_+\}$$

such the coordinate projections $X_t : \Omega \to E;\ \omega \mapsto \omega_t$ form a time-homogeneous Markov process X with associated semigroup T. □

Example 3.23 (Uniform Motion) If $E = \mathbb{R}$ and $X_t = X_0 + t$ for all $t \in \mathbb{R}_+$, then

$$(T_t f)(x) = f(x+t) = \int_{\mathbb{R}} p_t(x, \mathrm{d}y) f(y) \qquad \text{for all } f \in C_0(\mathbb{R}) \text{ and } x \in \mathbb{R},$$

where the transition kernel $p_t : (x, A) \mapsto \delta_{x+t}(A)$. It follows that X gives rise to a Feller semigroup with generator A such that $Af = f'$ whenever $f \in \operatorname{dom} A$.

Example 3.24 (Brownian Motion) If $E = \mathbb{R}$ and X is a standard Brownian motion, then Itô's formula gives that

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}s \qquad \text{for all } f \in C^2(\mathbb{R}).$$

It follows that the Lévy process X has a Feller semigroup with the generator A such that $Af = \frac{1}{2} f''$ for all $f \in C^2(\mathbb{R}) \cap \operatorname{dom} A$. [Informally,

$$t^{-1}\big(\mathbb{E}[f(X_t) \mid X_0 = x] - f(x)\big) = \frac{1}{2t}\int_0^t \mathbb{E}[f''(X_s) | X_0 = x]\,\mathrm{d}s \to \frac{1}{2} f''(x)$$

as $t \to 0+$.]

Example 3.25 (Poisson Process) If $E = \mathbb{R}$ and X is a homogeneous Poisson process with unit intensity and unit jumps, then

$$\mathbb{E}[f(X_t) | X_0 = x] = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} f(x+n) \qquad \text{for all } t \in \mathbb{R}_+.$$

Hence the Lévy process X has a Feller semigroup with the bounded generator A such that $(Af)(x) = f(x+1) - f(x)$ for all $x \in \mathbb{R}$ and $f \in C_0(\mathbb{R})$. [To see this, note that

$$\frac{(T_t f - f)(x)}{t} = \frac{e^{-t} - 1}{t} f(x) + e^{-t} f(x+1) + O(t) \qquad \text{as } t \to 0+,$$

uniformly for all $x \in \mathbb{R}$.]

The following exercise and theorem show that it is possible to move from the non-conservative to the conservative setting, and from a locally compact state space to a compact one.

Exercise 3.26 Let $\mathcal{T}$ be a locally compact topology on E and let ∞ denote a point not in E. Prove that $\widehat{E} := E \cup \{\infty\}$ is compact when equipped with the topology

$$\widehat{\mathcal{T}} := \mathcal{T} \cup \big\{(E \setminus K) \cup \{\infty\} : K \in \mathcal{T} \text{ is compact}\big\},$$

and that $\widehat{\mathcal{T}}$ is Hausdorff if and only if $\mathcal{T}$ is. [This is the *Alexandrov one-point compactification*.] Prove further that $C_0(E)$ has co-dimension one in $C(\widehat{E})$.

Theorem 3.27 *Let T be a Feller semigroup with locally compact state space E. If*

$$\widehat{T}_t f := f(\infty) + T_t\big(f|_E - f(\infty)\big) \qquad \textit{for all } t \in \mathbb{R}_+ \textit{ and } f \in B_b(\widehat{E}),$$

then $\widehat{T} = \big(\widehat{T}_t\big)_{t \in \mathbb{R}_+}$ is a conservative Feller semigroup with compact state space $\widehat{E}$.

Proof Fix $t \in \mathbb{R}_+$. The hardest step is to prove that $\widehat{T}_t$ is positive, that is, if $\lambda \in \mathbb{R}_+$ and $g \in B_b(E)$ are such that $\lambda + g(x) \geqslant 0$ for all $x \in E$, then $\lambda + (T_t g)(x) \geqslant 0$ for all $x \in E$. Note that g is real valued, and T_t maps real-valued functions to real-valued functions, by positivity. Let the function $g^- := x \mapsto \max\{-g(x), 0\}$ and note that $\lambda \geqslant g^-(x)$ for all $x \in E$. Hence

$$(T_t g^-)(x) \leqslant \|T_t g^-\| \leqslant \|g^-\| \leqslant \lambda$$

and $(T_t g)(x) \geqslant (-T_t g^-)(x) \geqslant -\lambda$, as required.

It is immediate that $\widehat{T}_t$ preserves the unit, so $\widehat{T}_t$ is contractive, by Exercise 3.12. The remaining claims are straightforward to verify. □

3.3 The Hille–Yosida–Ray Theorem

As noted above, it can be difficult to show that the hypotheses of the Hille–Yosida theorem, Theorem 2.49, hold. The Lumer–Phillips theorem gives an alternative for contraction semigroups, via the notion of dissipativity. Here, we will show that the additional structure available for Feller semigroups gives another possible approach.

Throughout this subsection, E denotes a locally compact Hausdorff space. Here, a *Feller semigroup* on $C_0(E)$ means a strongly continuous contraction semigroup on $C_0(E)$ composed of positive operators. This is the restriction to $C_0(E)$ of the Feller semigroups considered above.

Let

$$C_0(E;\mathbb{R}) := \{f : E \to \mathbb{R} \mid f \in C_0(E)\}$$

denote the real subspace of $C_0(E)$ containing those functions which take only real values.

Definition 3.28 A linear operator A in $C_0(E)$ is *real* if and only if

(i) $\overline{f} \in \operatorname{dom} A$ whenever $f \in \operatorname{dom} A$, so that the domain of A is closed under conjugation, and

(ii) $\overline{Af} = A\overline{f}$ for all $f \in \operatorname{dom} A$, so that A commutes with the conjugation.

Exercise 3.29 Show that (i) and (ii) are equivalent to

(i) $f + \mathrm{i}g \in \operatorname{dom} A$ implies $f, g \in \operatorname{dom} A$ whenever $f, g \in C_0(E;\mathbb{R})$, and

(ii) $A\big(\operatorname{dom} A \cap C_0(E;\mathbb{R})\big) \subseteq C_0(E;\mathbb{R})$,

respectively.

Exercise 3.30 Prove that T is real whenever T is positive.

Prove further that if $T = (T_t)_{t\in\mathbb{R}_+}$ is a Feller semigroup on $C_0(E)$ and T_t is real for all $t \in \mathbb{R}_+$ then the generator A of T is real.

Proof The first claim is an immediate consequence of Remark 3.11.

For the second, suppose A is the generator of the Feller semigroup T on $C_0(E)$, with each T_t real, and let $f \in \operatorname{dom} A$. Then, since conjugation is isometric, if $t > 0$, then

$$\|t^{-1}(T_t f - t) - Af\| = \|t^{-1}(\overline{T_t f} - \overline{f}) - \overline{Af}\| = \|t^{-1}(T_t\overline{f} - \overline{f}) - \overline{Af}\|,$$

and so $\overline{f} \in \operatorname{dom} A$, with $A\overline{f} = \overline{Af}$. The result follows. □

Definition 3.31 A linear operator A in $C_0(E)$ satisfies the *positive maximum principle* if, whenever $f \in \operatorname{dom} A \cap C_0(E;\mathbb{R})$ and $x_0 \in E$ are such that $f(x_0) = \|f\|$, it holds that $(Af)(x_0) \leqslant 0$.

Theorem 3.32 (Hille–Yosida–Ray) *A closed, densely defined operator A in $C_0(E)$ is the generator of a Feller semigroup on $C_0(E)$ if and only if A is real and satisfies the positive maximum principle, and $\lambda I - A$ is surjective for some $\lambda > 0$*

Proof Suppose first that A generates a Feller semigroup on $C_0(E)$. By the Lumer–Phillips theorem, Theorem 2.58, and Exercise 3.30, it suffices to prove that A satisfies the positive maximum principle. For this, let $f \in \operatorname{dom} A \cap C_0(E;\mathbb{R})$ and

$x_0 \in E$ be such that $f(x_0) = \|f\|$. Setting $f^+ := x \mapsto \max\{f(x), 0\}$, we see that

$$(T_t f)(x_0) \leqslant (T_t f^+)(x_0) \leqslant \|T_t f^+\| \leqslant \|f^+\| = f(x_0).$$

Thus

$$(Af)(x_0) = \lim_{t \to 0+} \frac{(T_t f - f)(x_0)}{t} \leqslant 0.$$

Conversely, suppose A is real and satisfies the positive maximum principle. Given any $f \in \operatorname{dom} A$, there exist $x_0 \in E$ and $\theta \in \mathbb{R}$ such that $e^{i\theta} f(x_0) = \|f\|$. The real-valued function $g := \operatorname{Re} e^{i\theta} f \in \operatorname{dom} A$, since A is real, and $\|f\| = g(x_0) \leqslant \|g\| \leqslant \|f\|$, so $\operatorname{Re}(Ae^{i\theta} f)(x_0) = (Ag)(x_0) \leqslant 0$, by the positive maximum principle. If $\lambda > 0$, then

$$\begin{aligned}\|(\lambda I - A)f\| = \|(\lambda I - A)e^{i\theta} f\| &\geqslant |\lambda e^{i\theta} f(x_0) - (Ae^{i\theta} f)(x_0)| \\ &\geqslant \operatorname{Re} \lambda e^{i\theta} f(x_0) - \operatorname{Re}(Ae^{i\theta} f)(x_0) \geqslant \lambda \|f\|,\end{aligned}$$

so A is dissipative, by Lemma 2.56, and $\lambda I - A$ is injective. In particular, T is a strongly continuous contraction semigroup, by the Lumer–Phillips theorem.

To prove that each T_t is positive, let $\lambda > 0$ be such that $\lambda I - A$ is surjective, so invertible, let $f \in C_0(E)$ be non-negative, and consider $g = (\lambda I - A)^{-1} f \in C_0(E)$. Either g does not attain its infimum, in which case $g \geqslant 0$ because g vanishes at infinity, or there exists $x_0 \in E$ such that $g(x_0) = \inf\{g(x) : x \in E\}$. Then

$$\lambda g - Ag = (\lambda I - A)g = f \iff \lambda g - f = Ag,$$

so $\lambda g(x_0) - f(x_0) = (Ag)(x_0) \geqslant 0$, by the positive maximum principle applied to $-g$. Thus if $x \in E$, then

$$\lambda g(x) \geqslant \lambda g(x_0) \geqslant f(x_0) \geqslant 0,$$

which shows that $\lambda(\lambda I - A)^{-1}$ is positive and therefore so is $(\lambda I - A)^{-1}$. Finally, Theorem 2.46 gives that

$$\begin{aligned} T_t f &= \lim_{n \to \infty} (I - tn^{-1} A)^{-n} f \\ &= \lim_{n \to \infty} (t^{-1} n)^n (t^{-1} n I - A)^{-n} f \qquad \text{for all } f \in C_0(E), \end{aligned} \tag{3.6}$$

so each T_t is positive also. □

Exercise 3.33 Prove that if the operator A is real then its resolvent $(\lambda I - A)^{-1}$ is real for all $\lambda \in \mathbb{R} \setminus \sigma(A)$. Deduce with the help of Theorem 2.46 that the Feller semigroup T is real if its generator A is.

Proof Suppose A is real and $\lambda \in \mathbb{R} \setminus \sigma(A)$. If $f \in C_0(E)$, then $f = (\lambda I - A)g$ for some $g \in C_0(E)$, and

$$\overline{f} = \overline{(\lambda I - A)g} = \lambda \overline{g} - \overline{Ag} = (\lambda I - A)\overline{g}.$$

Hence

$$\overline{(\lambda I - A)^{-1} f} = \overline{g} = (\lambda I - A)^{-1} \overline{f},$$

as required. Since conjugation is isometric, the deduction is immediate. □

Example 3.34 Let the linear operator A be defined by setting

$$\operatorname{dom} A := \{ f \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) : f'' \in C_0(\mathbb{R}) \} \quad \text{and} \quad Af = \frac{1}{2} f''.$$

It is a familiar result from elementary calculus that A satisfies the positive maximum principle

Remark 3.35 Courrège has classified the linear operators in $C_0(\mathbb{R}^d)$ with domains containing $C_c^\infty(\mathbb{R}^d)$ which satisfy the positive maximum principle. See [3, §3.5.1] and references therein.

4 Quantum Feller Semigroups

To move beyond the classical, we need to replace the commutative domain $C_0(E)$ with the correct non-commutative generalisation. This is what we introduce in the following section.

4.1 C Algebras*

Definition 4.1 A *Banach algebra* is a complex Banach space and simultaneously a complex associative algebra: it has an associative multiplication compatible with the vector-space operators and the norm, which is submultiplicative. If the Banach algebra is *unital*, so that it has a multiplicative identity 1, called its *unit*, then we require the norm $\|1\|$ to be 1.

An *involution* on a Banach algebra is an isometric conjugate-linear map which reverses products and is self-inverse.

A Banach algebra with involution A is a C^* *algebra* if and only if the C^* *identity* holds:

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } a \in \mathsf{A}.$$

Remark 4.2 The C^* identity connects the algebraic and analytic structures in a very rigid way. For example, there exists at most one norm for which an associative algebra is a C^* algebra, and $*$-homomorphisms between C^* algebras are automatically contractive [30, Proposition I.5.2].

Theorem 4.3 (Gelfand) *Every commutative C^* algebra is isometrically isomorphic to $C_0(E)$, where E is a locally compact Hausdorff space. The algebra is unital if and only if E is compact, in which case $C_0(E) = C(E)$.*

Theorem 4.4 (Gelfand–Naimark) *Any C^* algebra is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra of $B(\mathsf{H})$ for some Hilbert space H, a so-called* concrete C^* algebra*.*

Remark 4.5 Let A be a C^* algebra. Given any $n \in \mathbb{N}$, let $M_n(\mathsf{A})$ be the complex algebra of $n \times n$ matrices with entries in A, equipped with the usual algebraic operations. By the Gelfand–Naimark theorem, we may assume that $\mathsf{A} \subseteq B(\mathsf{H})$ for some Hilbert space H, and so $M_n(\mathsf{A}) \subseteq B(\mathsf{H}^n)$, where matrices of operators act in the usual manner on column vectors with entries in H. We equip $M_n(\mathsf{A})$ with the restriction of the operator norm on $B(\mathsf{H}^n)$, and then $M_n(\mathsf{A})$ becomes a C^* algebra.

Remark 4.5 is the root of the theory of operator spaces [10, 24].

Definition 4.6 A unital concrete C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$ is a *von Neumann algebra* if and only if any of the following equivalent conditions hold.

(i) Closure in the strong operator topology: if the net $(a_i) \subseteq \mathsf{A}$ and $a \in B(\mathsf{H})$ are such that $a_i v \to av$ for all $v \in \mathsf{H}$, then $a \in \mathsf{A}$.
(ii) Closure in the weak operator topology: if the net $(a_i) \subseteq \mathsf{A}$ and $a \in B(\mathsf{H})$ are such that $\langle v, a_i v\rangle \to \langle v, av\rangle$ for all $v \in \mathsf{H}$, then $a \in \mathsf{A}$.
(iii) Equality with its bicommutant: letting

$$S' := \{a \in \mathsf{A} : ab = ba \text{ for all } b \in S\}$$

denote the commutant of $S \subseteq \mathsf{A}$, then $\mathsf{A}'' := (\mathsf{A}')' = \mathsf{A}$ [von Neumann].
(iv) Existence of a predual: there exists a Banach space A_* with $(\mathsf{A}_*)^* = \mathsf{A}$ [Sakai].

Sakai's characterisation (iv) prompts consideration of the predual of $B(\mathsf{H})$. The predual A_* is naturally a subspace of A^*, and a bounded linear functional ϕ on $B(\mathsf{H})$ is an element of $B(\mathsf{H})_*$ if and only it is *σ-weakly continuous*: there exist square-summable sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty \subseteq \mathsf{H}$ such that

$$\sum_{n=1}^\infty \big(\|u_n\|^2 + \|v_n\|^2\big) < \infty \quad \text{and} \quad \phi(T) = \sum_{n=1}^\infty \langle u_n, T v_n\rangle \qquad \text{for all } T \in B(\mathsf{H}). \tag{4.1}$$

This yields a fifth characterisation of von Neumann algebras.

(v) Closure in the σ-weak topology: if the net $(a_i) \subseteq \mathsf{A}$ and $a \in B(\mathsf{H})$ are such that $\phi(a_i) \to \phi(a)$ for all $\phi \in B(\mathsf{H})_*$, then $a \in \mathsf{A}$.

The predual A_* consists of all those bounded linear functionals on A which are continuous in the σ-weak topology; equivalently, they are the restriction to A of elements of $B(\mathsf{H})_*$ as described in (4.1).

Example 4.7 Recall from Example 2.15 that $L^\infty(\Omega, \mathcal{F}, \mu) \cong \big(L^1(\Omega, \mathcal{F}, \mu)\big)^*$, and so every L^∞ space is a commutative von Neumann algebra. Furthermore, every commutative von Neumann algebra is isometrically $*$-isomorphic to $L^\infty(\Omega, \mathcal{F}, \mu)$ for some locally compact Hausdorff space Ω and positive Radon measure μ; see [30, Theorem III.1.18].

4.2 Positivity

Definition 4.8 In a C^* algebra A we have the notion of *positivity*: we write $a \geqslant 0$ if and only if there exists $b \in \mathsf{A}$ such that $a = b^* b$. The set of positive elements in A is denoted by A_+, is closed in the norm topology and is a *cone*: it is closed under addition and multiplication by non-negative scalars. Note that a positive element is self-adjoint.

This notion of positivity agrees with that encountered previously.

Lemma 4.9 *Let $T \in B(\mathsf{H})$ be such that $\langle v, Tv \rangle \geqslant 0$ for all $v \in \mathsf{H}$. There exists a unique operator $S \in B(\mathsf{H})$ such that $\langle v, Sv \rangle \geqslant 0$ for all $v \in \mathsf{H}$, and $S^2 = T$. Furthermore, S is the limit of a sequence of polynomials in T with no constant term.*

Proof This may be established with the assistance of the Maclaurin series for the function $z \mapsto (1 - z)^{1/2}$. See [25, Theorem VI.9] for the details. □

Corollary 4.10 *If $a \in \mathsf{A}_+$, then there exists a unique element $a^{1/2} \in \mathsf{A}_+$, the* square root *of a, such that $(a^{1/2})^2 = a$. The square root $a^{1/2}$ lies in the closed linear subspace of A spanned by the set of monomials $\{a^n : n \in \mathbb{N}\}$.*

Proof This is a straightforward exercise. □

Exercise 4.11 Prove that $f \in C_0(E)_+$ if and only if $f(x) \geqslant 0$ for all $x \in E$. Prove also that if the C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$, where H is a Hilbert space, then $a \in \mathsf{A}_+$ if and only if $\langle v, av \rangle \geqslant 0$ for all $v \in \mathsf{H}$. [The existence of square roots is crucial for both parts.]

Proposition 4.12 *Let A by a C^* algebra. Then any element $a \in \mathsf{A}$ may be written in the form $(a_1 - a_2) + \mathrm{i}(a_3 - a_4)$, where $a_1, \ldots, a_4 \in \mathsf{A}_+$.*

Proof The self-adjoint elements $\operatorname{Re} a := (a + a^*)/2$ and $\operatorname{Im} a := (a - a^*)/(2\mathrm{i})$ are such that $a = \operatorname{Re} a + \mathrm{i} \operatorname{Im} a$. Thus it suffices to show that any self-adjoint element of A is the difference of two positive elements.

Let $a \in \mathsf{A}$ be self-adjoint and let A_0 be the closed linear subspace of A spanned by the set of monomials $\{a^n : n \in \mathbb{N}\}$. As A_0 is a commutative C^* algebra, Theorem 4.3 gives an isometric $*$-isomorphism $j : \mathsf{A}_0 \to C_0(E)$, where E is a locally compact Hausdorff space. Then $f := j(a)$ is real valued, so

$$f^+ := x \mapsto \max\{f(x), 0\} \qquad \text{and} \qquad f^- := x \mapsto \max\{-f(x), 0\}$$

are well-defined elements of $C_0(E)_+$ such that $f = f^+ - f^-$. Hence $a = a^+ - a^-$, where $a^+ := j^{-1}(f^+)$ and $a^- := j^{-1}(f^-)$ are positive, as desired. □

Remark 4.13 The proof of Proposition 4.12 shows that if $a \in \mathsf{A}$ is self-adjoint, then there exist $a^+, a^- \in \mathsf{A}_+$ such that $a = a^+ - a^-$ and $a^+ a^- = 0$.

Definition 4.14 The positive cone provides a partial order on the set of self-adjoint elements of A. Given elements $a, b \in \mathsf{A}$, we write $a \leqslant b$ if and only if $a = a^*$, $b = b^*$ and $b - a \in \mathsf{A}_+$.

This order respects the norm.

Proposition 4.15 *Let $a, b \in \mathsf{A}_+$ be such that $a \leqslant b$. Then $\|a\| \leqslant \|b\|$.*

Proof Suppose without loss of generality that $\mathsf{A} \subseteq B(\mathsf{H})$. Then $a \leqslant b \leqslant \|b\| I$, by transitivity, Exercise 4.11 and the Cauchy–Schwarz inequality. If A_0 denotes the unital commutative C^* algebra generated by the set of monomials $\{a^n : n \in \mathbb{Z}_+\}$, then Theorem 4.3 gives an isometric $*$-isomorphism $j : \mathsf{A}_0 \to C(E)$, where E is a compact Hausdorff space. Hence

$$0 \leqslant j(\|b\| I - a)(x) = \|b\| - j(a)(x) \qquad \text{for all } x \in E,$$

so $0 \leqslant j(a)(x) \leqslant \|b\|$ for all such x and $\|a\| = \|j(a)\|_\infty \leqslant \|b\|$, as claimed. □

Exercise 4.16 Prove that if $a \in \mathsf{A}_+$ and $n \in \mathbb{Z}_+$, then $\|a^n\| = \|a\|^n$. [Hint: work as in the proof of Proposition 4.15.]

Definition 4.17 A linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between C^* algebras is *positive* if and only if $\Phi(\mathsf{A}_+) \subseteq \mathsf{B}_+$.

Note that any algebra $*$-homomorphism is positive; this fact has been utilised in the proof of Proposition 4.15.

Corollary 4.18 *Let $\Phi : \mathsf{A} \to \mathsf{B}$ be a positive linear map between C^* algebras. Then*

(i) *the map Φ commutes with the involution, so that $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathsf{A}$, and*

(ii) *the map Φ is bounded.*

Proof Part (i) is an exercise.

For (ii), it suffices to prove that Φ is bounded on A_+; suppose otherwise for contradiction. For all $n \in \mathbb{N}$, let $a_n \in \mathsf{A}_n$ be such that $\|a_n\| = 1$ and $\|\Phi(a_n)\| > 3^n$. If $a := \sum_{n\geqslant 1} 2^{-n} a_n \in \mathsf{A}_+$, then $a \geqslant 2^{-n} a_n$ for all $n \in \mathbb{N}$. Hence $\Phi(a) \geqslant 2^{-n}\phi(a_n)$ and $\|\phi(a)\| \geqslant 2^{-n}\|\Phi(a_n)\| > (3/2)^n$, by Proposition 4.15, which is a contradiction for sufficiently large n. □

We will now begin to investigate the generators of positive semigroups, following in the footsteps of Evans and Hanche-Olsen [12].

Theorem 4.19 *Let $T = (T_t)_{t\in\mathbb{R}_+}$ be a uniformly continuous one-parameter semigroup on the C^* algebra A. If T_t is positive for all $t \in \mathbb{R}_+$, then the semigroup generator $\mathcal{L}$ is bounded and $*$-preserving.*

Proof The boundedness of $\mathcal{L}$ follows immediately from Theorem 2.23, and if $a \in \mathsf{A}$, then

$$\mathcal{L}(a)^* = \lim_{t\to 0+} t^{-1}(T_t(a) - a)^* = \lim_{t\to 0+} t^{-1}(T_t(a^*) - a^*) = \mathcal{L}(a^*),$$

by continuity of the involution and the fact that positive maps are $*$-preserving. □

The following result is a variation on [12, Theorem 2]. The proof exploits an idea of Fagnola [14, Proof of Proposition 3.10].

Theorem 4.20 *Let $\mathcal{L}$ be a $*$-preserving bounded linear map on the C^* algebra A. The following are equivalent.*

(i) *If $a, b \in \mathsf{A}_+$ are such that $ab = 0$, then $a\mathcal{L}(b)a \geqslant 0$.*

(ii) *$(\lambda I - \mathcal{L})^{-1}$ is positive for all sufficiently large $\lambda > 0$.*

(iii) *$T_t = \exp(t\mathcal{L})$ is positive for all $t \in \mathbb{R}_+$.*

Proof Suppose (i) holds; we will show that $(\lambda I - \mathcal{L})^{-1}$ is positive if $\lambda > \|\mathcal{L}\|$. It suffices to take $a \in \mathsf{A}$ such that $(\lambda I - \mathcal{L})(a)$ is positive, and prove that $a \in \mathsf{A}_+$. Note that a is self- adjoint, so Remark 4.13 gives b and $c \in \mathsf{A}_+$ with $a = b - c$ and $bc = 0$. Thus (ii) holds if $c = 0$.

The condition $bc = 0$ implies that $b^{1/2}c = 0$, so (i) gives that $c\mathcal{L}(b)c \geqslant 0$. Hence

$$0 \leqslant c^*\big(\lambda a - \mathcal{L}(a)\big)c = \lambda c(b-c)c - c\mathcal{L}(b)c + c\mathcal{L}(c)c \leqslant -\lambda c^3 + c\mathcal{L}(c)c,$$

and therefore $0 \leqslant \lambda c^3 \leqslant c\mathcal{L}(c)c$. It follows that $\lambda\|c\|^3 = \lambda\|c^3\| \leqslant \|\mathcal{L}\|\,\|c\|^3$, which holds only when $c = 0$, as required.

That (ii) and (iii) are equivalent is a consequence of Theorems 2.45 and 2.46. To see that (iii) implies (i), note that if $a, b \in \mathsf{A}_+$ are such that $ab = 0$, then

$$0 \leqslant t^{-1}aT_t(b)a = t^{-1}a\big(b + t\mathcal{L}(b) + O(t)\big)a = a\mathcal{L}(b)a + O(t) \to a\mathcal{L}(b)a$$

as $t \to 0+$. □

In the quantum world, we can go beyond positivity to find a stronger notion, complete positivity, which is of great importance to the theories of open quantum systems and quantum information.

4.3 Complete Positivity

Recall from Remark 4.5 that matrix algebras over C^* algebras are also C^* algebras.

Definition 4.21 Let $n \in \mathbb{N}$. A linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between C^* algebras is *n-positive* if and only if the ampliation

$$\Phi^{(n)} : M_n(\mathsf{A}) \to M_n(\mathsf{B});\ (a_{ij})_{i,j=1}^n \mapsto \big(\Phi(a_{ij})\big)_{i,j=1}^n$$

is positive. If Φ is n-positive for all $n \in \mathbb{N}$, then Φ is *completely positive*.

Remark 4.22 Choi [6] produced examples of maps which are n-positive but not $n+1$-positive.

Exercise 4.23 Let $n \in \mathbb{N}$ and let $T = (T_t)_{t \in \mathbb{R}_+}$ be a one-parameter semigroup on the C^* algebra A. Prove that $T^{(n)} = (T_t^{(n)})_{t \in \mathbb{R}_+}$ is a one-parameter semigroup on $M_n(\mathsf{A})$, Prove further that if T is uniformly continuous, with generator $\mathcal{L}$, then $T^{(n)}$ is also uniformly continuous, with generator $\mathcal{L}^{(n)}$.

Proposition 4.24 (Paschke [23]) *Let $A = (a_{ij})_{i,j=1}^n \in M_n(\mathsf{A})$, where A is a C^* algebra. The following are equivalent.*

(i) *The matrix $A \in M_n(\mathsf{A})_+$.*
(ii) *The matrix A may be written as the sum of at most n matrices of the form $(b_i^* b_j)_{i,j=1}^n$, where $b_1, \ldots, b_n \in \mathsf{A}$.*
(iii) *The sum $\sum_{i,j=1}^n c_i^* a_{ij} c_j \in A_+$ for any $c_1, \ldots, c_n \in \mathsf{A}$.*

Proof To see that (iii) implies (i), we use the fact that any C^* algebra has a faithful representation which is a direct sum of cyclic representations [30, Theorem III.2.4]. Thus we may assume without loss of generality that $\mathsf{A} \subseteq B(\mathsf{H})$ and there exists a unit vector $u \in \mathsf{H}$ such that $\{au : a \in \mathsf{H}\}$ is dense in H.

Given this and Exercise 4.11, let $c_1, \ldots, c_n \in \mathsf{A}$. Then (iii) implies that

$$0 \leqslant \sum_{i,j=1}^n \langle u, c_i^* a_{ij} c_j u \rangle_{\mathsf{H}} = \langle v, Av \rangle_{\mathsf{H}^n},$$

where $v = (c_1 u, \ldots, c_n u)^T \in \mathsf{H}^n$. Vectors of this form are dense in H^n as $c_1, \ldots, c_n$ vary over A, so the result follows by another application of Exercise 4.11.

The other implications are straightforward to verify. □

Exercise 4.25 Let $n \in \mathbb{N}$. Use Proposition 4.24 to prove that a linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between C^* algebras is n-positive if and only if

$$\sum_{i,j=1}^{n} b_i^* \Phi(a_i^* a_j) b_j \geqslant 0$$

for all $a_1, \ldots, a_n \in \mathsf{A}$ and $b_1, \ldots, b_n \in \mathsf{B}$. Deduce that any $*$-homomorphism between C^* algebras is completely positive, as is any map of the form

$$B(\mathsf{K}) \to B(\mathsf{H});\ a \mapsto T^* a T, \qquad \text{where } T \in B(\mathsf{H}; \mathsf{K}).$$

Theorem 4.26 *A positive linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between C^* algebras is completely positive if A is commutative or B is commutative.*

Proof The first result is due to Stinespring [29] and the second to Arveson [4]. We will prove the latter.

We may suppose that $\mathsf{B} = C_0(E)$, where E is a locally compact Hausdorff space, by Theorem 4.3. If $a_1, \ldots, a_n \in \mathsf{A}$, $b_1, \ldots, b_n \in B$ and $x \in E$, then

$$\Big(\sum_{i,j=1}^{n} b_i^* \Phi(a_i^* a_j) b_j \Big)(x) = \sum_{i,j=1}^{n} \overline{b_i(x)} \Phi(a_i^* a_j)(x) b_j(x) = \Phi\big(c(x)^* c(x)\big)(x) \geqslant 0,$$

where $c(x) := \sum_{i=1}^{n} b_i(x) a_i \in A$. Exercises 4.11 and 4.25 give the result. □

Definition 4.27 A map $\Phi : \mathsf{A} \to \mathsf{B}$ between unital algebras is *unital* if $\Phi(1_{\mathsf{A}}) = 1_{\mathsf{B}}$, where 1_{A} and 1_{B} are the multiplicative units of A and B, respectively.

Theorem 4.28 (Kadison) *A 2-positive unital linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between unital C^* algebras is such that*

$$\Phi(a)^* \Phi(a) \leqslant \Phi(a^* a) \qquad \textit{for all } a \in \mathsf{A}. \tag{4.2}$$

Proof Note first that if $a \in \mathsf{A}$ then

$$A := \begin{bmatrix} 1 & a \\ a^* & a^* a \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \geqslant 0,$$

so

$$0 \leqslant \Phi^{(2)}(A) = \begin{bmatrix} 1 & \Phi(a) \\ \Phi(a)^* & \Phi(a^* a) \end{bmatrix}.$$

Suppose without loss of generality that $\mathsf{B} \subseteq B(\mathsf{H})$ for some Hilbert space H, and note that, by Exercise 4.11, if $u \in \mathsf{H}$ and

$$v := \begin{bmatrix} -\Phi(a)u \\ u \end{bmatrix} \in \mathsf{H}^2 \quad \text{then} \quad 0 \leqslant \langle v, \Phi^{(2)}(A)v \rangle = \langle u, \big(\Phi(a^*a) - \Phi(a)^*\Phi(a)\big)u \rangle.$$

As u is arbitrary, the claim follows. □

Remark 4.29 The inequality (4.2) is known as the *Kadison–Schwarz inequality*.

Exercise 4.30 Show that the inequality (4.2) holds if Φ is required only to be positive as long as a is *normal*, so that $a^*a = aa^*$. [Hint: use Theorem 4.26.]

4.4 Stinespring's Dilation Theorem

Exercise 4.25 gives two classes of completely positive maps. The following result makes clear that these are, in a sense, exhaustive.

Theorem 4.31 (Stinespring [29]) *Let $\Phi : \mathsf{A} \to B(\mathsf{H})$ be a linear map, where A is a unital C^* algebra and H is a Hilbert space. Then Φ is completely positive if and only if there exists a Hilbert space K, a unital $*$-homomorphism $\pi : \mathsf{A} \to B(\mathsf{K})$ and a bounded operator $T : \mathsf{H} \to \mathsf{K}$ such that*

$$\Phi(a) = T^*\pi(a)T \qquad (a \in \mathsf{A}).$$

Proof One direction is immediate. For the other, let $\mathsf{K}_0 := \mathsf{A} \underline{\otimes} \mathsf{H}$ be the algebraic tensor product of A with H, considered as complex vector spaces. Define a sesquilinear form on K_0 such that

$$\langle a \otimes u, b \otimes v \rangle = \langle u, \Phi(a^*b)v \rangle_{\mathsf{H}} \qquad \text{for all } a, b \in \mathsf{A} \text{ and } u, v \in H.$$

It is an exercise to check that this form is positive semidefinite, using the assumption that Φ is completely positive, and that the kernel

$$\mathsf{K}_{00} := \{x \in \mathsf{K}_0 : \langle x, x \rangle = 0\}$$

is a vector subspace of K_0. Let K be the completion of $\mathsf{K}_0/\mathsf{K}_{00} = \big\{[x] : x \in \mathsf{K}_0\big\}$.

If

$$\pi(a)[b \otimes v] := [ab \otimes v] \qquad \text{for all } a, b \in \mathsf{A} \text{ and } v \in \mathsf{H},$$

then $\pi(a)$ extends by linearity and continuity to an element of $B(\mathsf{K})$, denoted in the same manner. Furthermore, the map $a \mapsto \pi(a)$ is a unital $*$-homomorphism from A to $B(\mathsf{K})$.

To conclude, let $T \in B(\mathsf{H};\mathsf{K})$ be defined by setting $Tv = [1 \otimes v]$ for all $v \in \mathsf{H}$. It is a final exercise to verify that $\Phi(a) = T^*\pi(a)T$, as required. □

The following result extends the Kadison–Schwarz inequality, Theorem 4.28.

Corollary 4.32 *If* $\Phi : \mathsf{A} \to B(\mathsf{H})$ *is unital and completely positive then*

$$\sum_{i,j=1}^{n} \langle v_i, \big(\Phi(a_i^* a_j) - \Phi(a_i)^*\Phi(a_j)\big) v_j \rangle \geqslant 0$$

for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathsf{A}$ *and* $v_1, \ldots, v_n \in \mathsf{H}$.

Proof Let π and T be as in Theorem 4.31. Then $\|T\|^2 = \|T^*\pi(1_{\mathsf{A}})T\| = \|\Phi(1_{\mathsf{A}})\| = 1$ and

$$\begin{aligned}\sum_{i,j=1}^{n} \langle v_i, \Phi(a_i^* a_j) v_j \rangle = \sum_{i,j=1}^{n} \langle T v_i, \pi(a_i^* a_j) T v_j \rangle &= \Big\| \sum_{i=1}^{n} \pi(a_i) T v_i \Big\|^2 \\ &\geqslant \Big\| T^* \sum_{i=1}^{n} \pi(a_i) T v_i \Big\|^2 \\ &= \Big\| \sum_{i=1}^{n} \Phi(a_i) v_i \Big\|^2 \\ &= \sum_{i,j=1}^{n} \langle v_i, \Phi(a_i)^* \Phi(a_j) v_j \rangle.\end{aligned}$$

□

Definition 4.33 A triple (K, π, T) as in Theorem 4.31 is a *Stinespring dilation* of Φ. Such a dilation is *minimal* if

$$\mathsf{K} = \overline{\text{lin}}\{\pi(a)Tv : a \in \mathsf{A},\ v \in \mathsf{H}\}.$$

Proposition 4.34 *Any unital completely positive map* $\Phi : \mathsf{A} \to B(\mathsf{H})$ *has a minimal Stinespring dilation.*

Proof One may take (K, π, T) as in Theorem 4.31 and restrict to the smallest closed subspace of K containing $\{\pi(a)Tv : a \in \mathsf{A},\ v \in \mathsf{H}\}$. □

Exercise 4.35 Prove that the minimal Stinespring dilation is unique in an appropriate sense.

Definition 4.36 Let $(a_i) \subseteq \mathsf{A}$ be a net in the von Neumann algebra $\mathsf{A} \subseteq B(\mathsf{H})$. We write $a_i \searrow 0$ if $a_i \geqslant a_j \geqslant 0$ whenever $i \geqslant j$ and $\langle v, a_i v \rangle \to 0$ for all $v \in \mathsf{H}$.

[It follows from Vigier's theorem [22, Theorem 4.1.1.] that the decreasing net (a_i) converges in the strong operator topology to some element $a \in \mathsf{A}_+$.]

A linear map $\Phi : \mathsf{A} \to B(\mathsf{K})$ is *normal* if $a_i \searrow 0$ implies that $\langle v, \Phi(a_i)v\rangle \to 0$ for all $v \in \mathsf{K}$.

Proposition 4.37 *Let A be a von Neumann algebra. If the linear map $\Phi : \mathsf{A} \to B(\mathsf{H})$ is completely positive and normal, then the unital $*$-homomorphism π of Theorem 4.31 may be chosen to be normal also.*

Proof Let (K, π, T) be a minimal Stinespring dilation for Φ. If $v \in \mathsf{H}$, $a \in \mathsf{A}$ and the net $(a_i) \subseteq \mathsf{A}_+$ is such that $a_i \searrow 0$, then

$$\langle \pi(a)Tv, \pi(a_i)\pi(a)Tv\rangle = \langle v, T^*\pi(a^*a_ia)Tv\rangle = \langle v, \Phi(a^*a_ia)v\rangle \to 0,$$

since $a^*a_ia \searrow 0$. It now follows by polarisation and minimality that $\pi(a_i) \searrow 0$, as required. □

Proposition 4.38 *A linear map $\Phi : \mathsf{A} \to B(\mathsf{H})$ is normal if and only if it is σ-weakly continuous.*

Proof It suffices to prove that if $(b_i) \subseteq B(\mathsf{K})$ is a norm-bounded net then $b_i \to 0$ in the σ-weak topology if and only if $\langle v, b_i v\rangle \to 0$ for all $v \in \mathsf{K}$. Furthermore, by polarisation, we need only consider σ-weakly continuous functionals of the form

$$\phi : B(\mathsf{K}) \to \mathbb{C};\ a \mapsto \sum_{n=1}^{\infty} \langle x_n, ax_n\rangle, \qquad \text{where } \sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

The result now follows by a standard truncation argument. □

4.5 Semigroup Generators

We will now introduce the class of quantum Feller semigroups, and proceed toward a classification of the semigroup generators for a uniformly continuous subclass. As above, we will first establish some necessary conditions that hold in greater generality.

Definition 4.39 A *quantum Feller semigroup* $T = (T_t)_{t\in\mathbb{R}_+}$ on a C^* algebra A is a strongly continuous contraction semigroup such that each T_t is completely positive.

If A is unital, with unit 1, and $T_t 1 = 1$ for all $t \in \mathbb{R}_+$ then T is *conservative*.

Exercise 4.40 Let T be a quantum Feller semigroup on a unital C^* algebra. Prove that T is conservative if and only if $1 \in \operatorname{dom} \mathcal{L}$, with $\mathcal{L}(1) = 0$. [Hint: Theorem 2.46 may be useful.]

To begin the characterisation of the generators of these semigroups, we introduce a concept due to Evans [11].

Proposition 4.41 *Let* $\Phi : \mathsf{A} \to B(\mathsf{H})$ *be a linear map on the unital concrete* C^* *algebra* $\mathsf{A} \subseteq B(\mathsf{H})$*. The following are equivalent.*

(i) *If* $n \in \mathbb{N}$ *and* $a \in M_n(\mathsf{A})$*, then*

$$\Phi^{(n)}(a^* a) + a^* \Phi^{(n)}(1) a - \Phi^{(n)}(a^*) a - a^* \Phi^{(n)}(a) \in M_n\big(B(\mathsf{H})\big)_+.$$

(ii) *If* $n \in \mathbb{N}$ *and* $a_1, \ldots, a_n \in \mathsf{A}$*, then*

$$\big(\Phi(a_i^* a_j) + a_i^* \Phi(1) a_j - \Phi(a_i^*) a_j - a_i^* \Phi(a_j)\big)_{i,j=1}^n \in M_n\big(B(\mathsf{H})\big)_+.$$

(iii) *If* $n \in \mathbb{N}$*,* $a_1, \ldots, a_n \in \mathsf{A}$ *and* $v_1, \ldots, v_n \in \mathsf{H}$ *are such that* $\sum_{i=1}^n a_i v_i = 0$*, then*

$$\sum_{i,j=1}^n \langle v_i, \Phi(a_i^* a_j) v_j \rangle \geqslant 0.$$

(iv) *If* $n \in \mathbb{N}$*,* $a_1, \ldots, a_n \in \mathsf{A}$ *and* $b_1, \ldots, b_n \in B(\mathsf{H})$ *are such that* $\sum_{i=1}^n a_i b_i = 0$*, then*

$$\sum_{i,j=1}^n b_i^* \Phi(a_i^* a_j) b_j \geqslant 0.$$

Proof Given $a_1, \ldots, a_n \in \mathsf{A}$, let $a = (a_{ij}) \in M_n(\mathsf{A})$ be such that $a_{1j} = a_j$ and $a_{ij} = 0$ otherwise. Then

$$\begin{aligned}&\big(\Phi^{(n)}(a^* a) + a^* \Phi^{(n)}(1) a - \Phi^{(n)}(a^*) a - a^* \Phi^{(n)}(a)\big)_{ij} \\ &\quad = \Phi(a_i^* a_j) + a_i \Phi(1) a_j - \Phi(a_i^*) a_j - a_i^* \Phi(a_j)\end{aligned}$$

for all $i, j = 1, \ldots, n$, so (i) implies (ii).

Conversely, let $a = (a_{ij}) \in M_n(\mathsf{A})$. Applying (ii) to $a_{k1}, \ldots, a_{kn}$ and then summing over k gives that

$$\begin{aligned} 0 &\leqslant \sum_{k=1}^n \big[\Phi(a_{ki}^* a_{kj}) + a_{ki}^* \Phi(1) a_{kj} - \Phi(a_{ki}^*) a_{kj} - a_{ki}^* \Phi(a_{ki}^*)\big]_{i,j=1}^n \\ &= \Phi^{(n)}(a^* a) - a^* \Phi^{(n)}(1) a - \Phi^{(n)}(a^*) a - a^* \Phi^{(n)}(a).\end{aligned}$$

Thus (ii) implies (i).

The implication from (ii) to (iii) is clear, as is that from (iii) to (iv). For the final part, let $a_1, \ldots, a_n \in \mathsf{A}$ and $b_1, \ldots, b_n \in B(\mathsf{H})$, let $a_0 = 1$ and $b_0 = -\sum_{i=1}^n a_i b_i$, and note that $\sum_{i=0}^{n} a_i b_i = 0$. Hence (iv) gives that

$$0 \leqslant \sum_{i,j=0}^{n} b_i^* \Phi(a_i^* a_j) b_j = \sum_{i,j=1}^{n} b_i^* \big(\Phi(a_i^* a_j) + a_i^* \Phi(1) a_j - a_i^* \Phi(a_j) - \Phi(a_i^*) a_j\big) b_j.$$

Thus (ii) now follows from the first part of Exercise 4.25. □

Definition 4.42 A linear map $\Phi : \mathsf{A} \to B(\mathsf{H})$ on the unital C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$ is *conditionally completely positive* if and only if any of the equivalent conditions in Proposition 4.41 hold.

Exercise 4.43 Prove that the set of conditionally completely positive maps from A to $B(\mathsf{H})$ is a cone, that is, closed under addition and multiplication by non-negative scalars. Prove also that this cone contains all completely positive maps and scalar multiples of the identity map. Finally, prove that the cone is closed under pointwise weak-operator convergence: the net $\Phi_i \to \Phi$ if and only if $\langle v, \Phi_i(a)v\rangle \to \langle v, \Phi(a)v\rangle$ for all $a \in \mathsf{A}$ and $v \in \mathsf{H}$.

Exercise 4.44 Let A be as in Definition 4.42. A linear map $\delta : \mathsf{A} \to B(\mathsf{H})$ is a *derivation* if and only if

$$\delta(ab) = a\delta(b) + \delta(a)b \qquad \text{for all } a, b \in \mathsf{A}.$$

Prove that a derivation is conditionally completely positive. Prove also that the map

$$\mathsf{A} \to B(\mathsf{H});\ a \mapsto G^* a + aG$$

is conditionally completely positive and normal for all $G \in B(\mathsf{H})$.

Theorem 4.45 *Let T be a uniformly continuous quantum Feller semigroup on the unital C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$. The semigroup generator $\mathcal{L}$ is bounded, $*$-preserving and conditionally completely positive.*

Proof The first two claims follow immediate from Theorem 4.19. For conditional complete positivity, let $a_1, \ldots, a_n \in \mathsf{A}$ and $v_1, \ldots, v_n \in \mathsf{H}$. By Corollary 4.32, if $t > 0$, then

$$t^{-1} \sum_{i,j=1}^{n} \langle v_i, \big(T_t(a_i^* a_j) - T_t(a_i)^* T_t(a_j)\big) v_j\rangle \geqslant 0.$$

Letting $t \to 0+$ gives that

$$\sum_{i,j=1}^{n} \langle v_i, \big(\mathcal{L}(a_i^* a_j) - \mathcal{L}(a_i)^* a_j - a_i^* \mathcal{L}(a_j)\big) v_j\rangle \geqslant 0,$$

and if $\sum_{i=1}^{n} a_i v_i = 0$ then the second and third terms vanish. □

Exercise 4.46 Use Exercise 4.43 to provide an alternative proof that $\mathcal{L}$ in Theorem 4.45 is conditionally completely positive.

The following result is [11, Theorem 2.9] of Evans, who credits Lindblad [21].

Theorem 4.47 (Lindblad, Evans) *Let $\mathcal{L}$ be a $*$-preserving bounded linear map on the unital C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$. The following are equivalent.*

(i) *$\mathcal{L}$ is conditionally completely positive.*
(ii) *$(\lambda I - \mathcal{L})^{-1}$ is completely positive for all sufficiently large $\lambda > 0$.*
(iii) *$T_t = \exp(t\mathcal{L})$ is completely positive for all $t \in \mathbb{R}_+$.*

Proof The equivalence of (ii) and (iii) is given by Theorems 2.45 and 2.46, together with Exercise 4.23. The solution to Exercise 4.46 gives that (iii) implies (i); to complete the proof, it suffices to show that (i) implies (iii).

Suppose first that $\mathcal{L}(1) \leqslant 0$. Then $\mathcal{L}^{(n)}(1) \leqslant 0$ for all $n \in \mathbb{N}$, so if $a \in M_n(\mathsf{A})$ then

$$\mathcal{L}^{(n)}(a^*a) \geqslant a^*\mathcal{L}^{(n)}(a^*)a + a^*\mathcal{L}^{(n)}(a).$$

Thus if $b, c \in M_n(\mathsf{A})_+$ are such that $bc = 0$ then $b^{1/2}c = 0$ and

$$c\mathcal{L}^{(n)}(b)c \geqslant c\mathcal{L}^{(n)}(b^{1/2})b^{1/2}c + cb^{1/2}\mathcal{L}^{(n)}(b)c = 0.$$

Theorem 4.20 now gives that $T_t^{(n)} = \exp(t\mathcal{L}^{(n)})$ is positive for all $t \in \mathbb{R}_+$, so (iii) holds.

Finally, if $\mathcal{L}(1) > 0$, then the conditionally completely positive map

$$\mathcal{L}' : \mathsf{A} \to B(\mathsf{H});\ a \mapsto \mathcal{L}(a) - \|\mathcal{L}(1)\|a$$

is such that $\mathcal{L}'(1) \leqslant 0$, since $0 \leqslant \mathcal{L}(1) \leqslant \|\mathcal{L}(1)\|I$. It follows that $T'_t = \exp(t\mathcal{L}')$ is completely positive for all $t \in \mathbb{R}_+$, and therefore so is $T_t = \exp\big(\|\mathcal{L}(1)\|t\big)T'_t$. □

Remark 4.48 Since completely positive unital linear maps between unital C^* algebras are automatically contractive, by Theorem 4.31 and the fact that $*$-homomorphisms between C^* algebras are contractive, the previous result characterises the generators of uniformly continuous conservative quantum Feller semigroups.

4.6 The Gorini–Kossakowski–Sudarshan–Lindblad Theorem

In order to provide a more explicit description of the generators of quantum Feller semigroups, we will establish some results of Lindblad and Christensen, and of Kraus. The Kraus decomposition is a key tool in quantum information theory.

Theorem 4.49 (Lindblad, Christensen) *Let $\mathcal{L}$ be a $*$-preserving bounded linear map on the von Neumann algebra A. Then $\mathcal{L}$ is conditionally completely positive and normal if and only if there exists a completely positive, normal map $\Psi : \mathsf{A} \to \mathsf{A}$ and an element $g \in \mathsf{A}$ such that*

$$\mathcal{L}(a) = \Psi(a) + g^* a + ag \qquad \textit{for all } a \in \mathsf{A}.$$

Proof The second part of Exercise 4.44 shows that $\mathcal{L}$ is conditionally completely positive or normal if and only if Ψ has the same property.

Given this, it remains to prove that if $\mathcal{L}$ is conditionally completely positive, then there exists $g \in \mathsf{A}$ such that $a \mapsto \mathcal{L}(a) - g^* a - ag$ is completely positive. We will show this under the assumption that $\mathsf{A} = B(\mathsf{H})$; see [14, Proof of Theorem 3.14]. The general case [7] requires considerably more work.

Given $u, v \in \mathsf{H}$, let the Dirac dyad

$$|u\rangle\langle v| : \mathsf{H} \to \mathsf{H};\ w \mapsto \langle v, w\rangle u.$$

Fix a unit vector $u \in \mathsf{H}$, and let $G \in B(\mathsf{H})$ be such that

$$G^* : \mathsf{H} \to \mathsf{H};\ v \mapsto \mathcal{L}\big(|v\rangle\langle u|\big)u - \frac{1}{2}\langle u, \mathcal{L}\big(|u\rangle\langle u|\big)u\rangle v.$$

Given $a_1, \ldots a_n \in \mathsf{A}$ and $v_1, \ldots, v_n \in \mathsf{H}$, let $v_0 = u$ and $a_0 = -\sum_{i=1}^n |a_i v_i\rangle\langle u|$, so that $\sum_{i=0}^n a_i v_i = 0$. The conditional complete positivity of $\mathcal{L}$ implies that

$$\begin{aligned}
0 &\leqslant \sum_{i,j=1}^n \Big(\langle v_i, \mathcal{L}(a_i^* a_j) v_j\rangle - \langle v_i, \mathcal{L}\big(a_i^* |a_j v_j\rangle\langle u|\big)u\rangle - \langle u, \mathcal{L}\big(|u\rangle\langle a_i v_i| a_j\big) v_j\rangle \\
&\qquad + \langle u, \mathcal{L}\big(|u\rangle\langle a_i v_i||a_j v_j\rangle\langle u|\big)u\rangle\Big) \\
&= \sum_{i,j=1}^n \langle v_i, \mathcal{L}(a_i^* a_j) v_j\rangle - \langle v_i, \mathcal{L}\big(|a_i^* a_j v_j\rangle\langle u|\big)u\rangle - \langle u, \mathcal{L}\big(|u\rangle\langle a_j^* a_i v_i|\big) v_j\rangle \\
&\qquad + \langle u, \mathcal{L}\big(|u\rangle\langle u|\big)u\rangle\langle a_i v_i, a_j v_j\rangle \\
&= \sum_{i,j=1}^n \langle v_i, \big(\mathcal{L}(a_i^* a_j) - G^* a_i^* a_j - a_i^* a_j G\big) v_j\rangle.
\end{aligned}$$

The result follows. □

Remark 4.50 If A is required only to be a C^* algebra, then Christensen and Evans [7] showed that Theorem 4.49 remains true if $\mathcal{L}$ and Ψ no longer required to be normal, but then g and the range of Ψ must be taken to lie in the σ-weak closure of A.

Theorem 4.51 (Kraus [18]) *Suppose* $\mathsf{A} \subseteq B(\mathsf{H})$ *is a von Neumann algebra. A linear map* $\Psi : \mathsf{A} \to B(\mathsf{K})$ *is normal and completely positive if and only if there exists a family of operators* $(L_i)_{i \in \mathbb{I}} \subseteq B(\mathsf{K}; \mathsf{H})$ *such that*

$$\Psi(a) = \sum_{i \in \mathbb{I}} L_i^* a L_i \qquad \textit{for all } a \in \mathsf{A},$$

with convergence in the strong operator topology. The cardinality of the index set $\mathbb{I}$ *may be taken to be no larger than* $\dim \mathsf{K}$.

Proof If Ψ has this form, then it is completely positive and normal. The first claim is readily verified; for the second, let $a_j \searrow 0$, fix j_0 and note that $\langle u, a_j u \rangle \leqslant \langle u, a_{j_0} u \rangle$ for all $u \in \mathsf{H}$ and $j \geqslant j_0$. Fix $\varepsilon > 0$ and $v \in \mathsf{K}$, choose a finite set $\mathbb{I}_0 \subseteq \mathbb{I}$ such that the sum $\sum_{i \in \mathbb{I}_0} \langle L_i v, a_{j_0} L_i v \rangle > \langle v, \Psi(a_{j_0}) v \rangle - \varepsilon$, and note that

$$\langle v, \Psi(a_j) v \rangle \leqslant \sum_{i \in \mathbb{I}_0} \langle L_i v, a_j L_i v \rangle + \sum_{i \in \mathbb{I} \setminus \mathbb{I}_0} \langle L_i v, a_{j_0} L_i v \rangle < \sum_{i \in \mathbb{I}_0} \langle L_i v, a_j L_i v \rangle + \varepsilon.$$

This shows that Ψ is normal, as required.

For the converse, Theorem 4.31 shows it suffices to prove that if $\pi : \mathsf{A} \to B(\mathsf{K})$ is a normal unital $*$-homomorphism, then π can be written as in the statement of the theorem.

Let $(e_i)_{i \in \mathbb{I}}$ be an orthonormal basis for H, and consider the net $(I_{\mathsf{H}} - \sum_{i \in \mathbb{I}_0} |e_i\rangle\langle e_i|)$, where the index $\mathbb{I}_0$ runs over all finite subsets of $\mathbb{I}$, ordered by inclusion. Since π is normal and unital, we have that $I_{\mathsf{K}} = \sum_{i \in \mathbb{I}} \pi\big(|e_i\rangle\langle e_i|\big)$ in the weak-operator sense; thus, there exists some $i_0 \in \mathbb{I}$ such that $P := \pi\big(|e_{i_0}\rangle\langle e_{i_0}|\big)$ is a non-zero orthogonal projection.

Let $u \in \mathsf{K}$ be a unit vector such that $Pu = u$, let $a \in \mathsf{A}$, and note that

$$\|\pi(a)u\|^2 = \langle Pu, \pi(a^* a) Pu \rangle = \langle u, \pi\big(|e_{i_0}\rangle\langle e_{i_0}| a^* a |e_{i_0}\rangle\langle e_{i_0}|\big) u \rangle = \|a e_{i_0}\|^2.$$

Hence there exists a partial isometry $L_0 : \mathsf{K} \to \mathsf{H}$ with initial space K_0, the norm closure of $\{\pi(a)u : a \in \mathsf{A}\}$, and final space H_0, the norm closure of $\{a e_0 : a \in \mathsf{A}\}$, and such that $L_0 \pi(a) u = a e_0$ for all $a \in \mathsf{A}$. Note that K_0 is invariant under the action of $\pi(a)$, for all $a \in \mathsf{A}$, so

$$\pi(a)\pi(b)u = P_0 \pi(ab) u = L_0^* L_0 \pi(ab) u = L_0^* a b e_0 = L_0^* a L_0 \pi(b) u \qquad \text{for all } b \in \mathsf{A}.$$

Thus $\pi(a)|_{\mathsf{K}_0} = L_0^* a L_0|_{\mathsf{K}_0}$, and since $L_0(\mathsf{K}_0^\perp) = \{0\}$, it follows that $\pi(a) P_0 = L_0^* a L_0$ for all $a \in \mathsf{A}$, where $P_0 := L_0^* L_0$ is the orthogonal projection onto the initial space K_0.

Repeating this argument, but on $\mathsf{K}_0^\perp$, there exists a partial isometry $L_1 : \mathsf{K} \to \mathsf{H}$ with initial projection P_1 such that $P_0 P_1 = 0$ and $\pi(a) P_1 = L_1^* a L_1$ for all $a \in \mathsf{A}$. An application of Zorn's lemma now gives the result. □

Remark 4.52 With Ψ and $(L_i)_{i\in\mathbb{I}}$ as in Theorem 4.51, we may write

$$\Psi(a) = L^*(a \otimes I_{\mathsf{K}_\mathbb{I}})L \qquad \text{for all } a \in \mathsf{A},$$

where $\mathsf{K}_\mathbb{I}$ is the Hilbert space with orthonormal basis $(e_i)_{i\in\mathbb{I}}$ and $L \in B(\mathsf{K}; \mathsf{H} \otimes \mathsf{K}_\mathbb{I})$ is such that

$$Lv = \sum_{i\in\mathbb{I}} L_i v \otimes e_i \qquad \text{for all } v \in \mathsf{K}.$$

Exercise 4.53 Use Theorem 4.51 and the second part of Theorem 4.26 to show that every positive normal linear functional on the von Neumann algebra A has the form

$$a \mapsto \sum_{n=1}^{\infty} \langle x_n, a x_n \rangle, \qquad \text{where } \sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

[Every bounded linear functional is the linear combination of four positive ones [22, Theorem 3.3.10], and Grothendieck [15] observed that each of these may be taken to be normal if the original is [17, Theorem 7.4.7]. Hence every normal linear functional is of the form used to define the σ-weak topology in Definition 4.6.]

Lemma 4.54 *Let T be a uniformly continuous semigroup on a von Neumann algebra with generator $\mathcal{L}$. Then $\mathcal{L}$ is normal if and only if T_t is normal for all $t \in \mathbb{R}_+$.*

Proof This holds because the limit of a norm-convergent sequence of normal maps is normal. To see this, let $\Phi_n, \Phi : \mathsf{A} \to B(\mathsf{H})$ be such that $\|\Phi_n - \Phi\| \to 0$, let the net $(a_i) \subseteq \mathsf{A}_+$ be such that $a_i \searrow 0$, and let $v \in \mathsf{H}$. Fix i_0 and note that $\|a_i\| \leqslant \|a_{i_0}\|$ whenever $i \geqslant i_0$, so

$$|\langle v, \Phi(a_i)v\rangle| \leqslant \|v\|^2 \|a_{i_0}\| \, \|\Phi_n - \Phi\| + |\langle v, \Phi(a_i)v\rangle| \qquad \text{for all } i \geqslant i_0.$$

The claim follows. □

Theorem 4.55 (Gorini–Kossakowski–Sudarshan, Lindblad) *Let $\mathsf{A} \subseteq B(\mathsf{H})$ be a von Neumann algebra. A bounded linear map $\mathcal{L} \in B(\mathsf{A})$ is the generator of a uniformly continuous conservative quantum Feller semigroup composed of normal maps if and only if*

$$\mathcal{L}(a) = -\mathrm{i}[h, a] - \frac{1}{2}\big(L^*La - 2L^*(a \otimes I)L + aL^*L\big) \qquad \textit{for all } a \in \mathsf{A},$$

where $h = h^* \in \mathsf{A}$ and $L \in B(\mathsf{H}; \mathsf{H} \otimes \mathsf{K})$ for some Hilbert space K.

Proof If $\mathcal{L}$ has this form, then it is straightforward to verify that the semigroup it generates is as claimed.

Conversely, suppose $\mathcal{L}$ is the generator of a semigroup as in the statement of the theorem. Then Theorem 4.47 gives that $\mathcal{L}$ is conditionally completely positive and $\mathcal{L}(1) = 0$. Moreover, $\mathcal{L}$ is normal, by the preceding lemma, and so Theorem 4.49 gives that

$$\mathcal{L}(a) = \Psi(a) + g^* a + ag \qquad \text{for all } a \in \mathsf{A},$$

where $\Psi : \mathsf{A} \to \mathsf{A}$ is completely positive and normal, and $g \in \mathsf{A}$. Taking $a = 1$ in this equation shows that $g^* + g = -\Psi(1)$, so $g = -\frac{1}{2}\Psi(1) + \mathrm{i}h$ for some self-adjoint element $h \in \mathsf{A}$. The result now follows by Theorem 4.51. □

The story of the previous theorem is very well told in [8]. Going beyond the case of bounded generators is the subject of much interest. See the survey [28] for some recent developments.

4.7 Quantum Markov Processes

We will conclude by giving a very brief indication of how a quantum process may be defined.

Remark 4.56 Let E be a compact Hausdorff space. If X is an E-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then

$$j_X : \mathsf{A} \to \mathsf{B}; \ f \mapsto f \circ X$$

is a unital $*$-homomorphism, where $\mathsf{A} = C(E)$ and $\mathsf{B} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.57 A *non-commutative random variable* is a unital $*$-homomorphism j between unital C^* algebras.

A family $(j_t : \mathsf{A} \to \mathsf{B})_{t \in \mathbb{R}_+}$ of non-commutative random variables is a *dilation* of the quantum Feller semigroup T on A if there exists a conditional expectation $\mathbb{E}$ from B onto A such that $T_t = \mathbb{E} \circ j_t$ for all $t \in \mathbb{R}_+$.

The problem of constructing such dilations has attracted the interest of many authors, including Evans and Lewis [13], Accardi et al. [1], Vincent-Smith [31], Kümmerer [19], Sauvageot [27] and Bhat and Parthasarathy [5].

Essentially, one attempts to mimic the functional-analytic proof of Theorem 3.22. Given the appropriate analogue of an initial measure, which is a state μ on the C^* algebra A, the sesquilinear form

$$\mathsf{A}^{\underline{\otimes} n} \times \mathsf{A}^{\underline{\otimes} n} \to \mathbb{C}; \ (a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) \mapsto \mu\big(T_{t_1}(a_1^* \ldots (T_{t_n - t_{n-1}}(a_n^* b_n)) \ldots b_1)\big)$$

must be shown to be positive semidefinite. The key to this is the complete positivity of the semigroup maps. There are many technical issues to be addressed; see [5] for more details.

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