Recent Progress in Optimization of Multiband Electrical Filters

Andrei Bogatyrëv

Abstract The best uniform rational approximation of the *sign* function on two intervals was explicitly found by Russian mathematician E.I. Zolotarëv in 1877. The progress in math eventually led to the progress in technology: half a century later German electrical engineer and physicist W. Cauer on the basis of this solution has invented low- and high-pass electrical filters known today as elliptic or Cauer-Zolotarëv filters and possessing the unbeatable quality. We discuss a recently developed approach for the solution of optimization problem naturally arising in the synthesis of multi-band (analogue, digital or microwave) electrical filters. The approach is based on techniques from algebraic geometry and generalizes the effective representation of Zolotarëv fraction.

1 History and Background

Sometimes the progress in mathematics brings us to the progress in technology. One of such examples is the invention of low- and high-pass electrical filters widely used nowadays is electronic appliances. The story started in year 1877 when E.I. Zolotarëv (1847–1878)—the pupil of P.L. Chebyshëv—has solved a problem of best uniform rational approximation of the function $sign(x)$ on two segments of real axis separated by zero. His solution now called *Zolotarëv fraction* is the analogy of Chebyshëv polynomials in the realm of rational functions and inherits many nice properties of the latter. This work of Zolotarëv who also attended lectures of K. Weierstrass and corresponded to him was highly appreciated by the German scholar. More than 50 years later German electrical engineer, physicist and guru of network synthesis Wilhelm Cauer (1900–1945) has invented electrical filters with the transfer function based on Zolotarëv fraction.

A. Bogatyrëv (✉)
Institute for Numerical Mathematics, Russian Academy of Sciences, Moscow, Russia

   Further development of technologies brings us to more sophisticated optimization problems [13, 14, 20, 21, 23]. In particular, modern gadgets may use several standards of wireless communication like IEEE 806.16, GSM, LTE, GPS and therefore a problem of filtering on several frequency bands arises. Roughly, the problem is this: given the mask of a filter, that is the boundaries of its stop and pass bands, the levels of attenuation at the stopbands and the permissible ripple magnitude at the passbands, to find minimum degree real rational function fitting this mask. The problem reduces to a solution of a series of somewhat more simple minimal deviation problems on several segments similar to the one considered by Zolotarëv. Several equivalent formulations will appear in Section 2.1.

   Those problems turned out to be very difficult from the practical viewpoint because of intrinsic instability of most numerical methods of rational approximation. However, we know how the 'certificate' of the solution (see contribution from Panos Pardalos in this volume) for this particular case looks like: the solution possesses the so-called *equiripple property*, that is behaves like a wave of constant amplitude on each stop or passband. The total number of ripples is bounded from below. In a sense the solution for this problem is rather simple—you just manifest function with a suitable equiripple property. Such behaviour is very unusual for generic rational functions; therefore, functions with equiripple property fill in a variety of relatively small dimension in the set of rational functions of bounded degree. The natural idea is to look for the solution in the 'small' set of the distinguished functions instead of the 'large' set of generic functions. *Ansatz* is an explicit formula with few parameters which allows to parametrize the 'small' set. This Ansatz ideology had been already used to calculate the so-called Chebyshëv polynomials on several segments [4], optimal stability polynomials for explicit multistage *Runge-Kutta* methods [5, 6] and solve some other problems [7]. Recall, e.g., *Bethe Ansatz* for finding exact solutions for Heisenberg antiferromagnetic model. Ansatz for optimal electrical filters is discussed in Section 7.

2   Optimization Problem for Multiband Filter

Suppose we have a finite collection E of disjoint closed segments of real axis $\mathbb{R}$. The set has a meaning of frequency bands and is decomposed into two subsets: $E = E^+ \cup E^-$ which are, respectively, called the passbands E^+ and the stopbands E^-. Both subsets are nonempty. Optimization problem for electrical filter has several equivalent settings [1, 3, 12, 22, 26].

2.1   Four Settings

In each of the listed below cases we minimize certain quantity among real rational functions $R(x)$ of bounded degree $\deg R \leq n$ being the maximum of the degrees of numerator and denominator of the fraction. The goal function may be the following.

2.1.1 Minimal Deviation

$$\frac{\max_{x \in E^+} |R(x)|}{\min_{x \in E^-} |R(x)|} \longrightarrow \min =: \theta^2 \le 1$$

2.1.2 Minimal Modified Deviation

$$\max\{\max_{x \in E^+} |R(x)|, \max_{x \in E^-} 1/|R(x)|\} \longrightarrow \min =: \theta$$

2.1.3 Third Zolotarëv Problem

Minimize θ under the condition that there exist real rational function $R(x)$, $\deg R \le n$, with the restrictions

$$\min_{x \in E_-} |R(x)| \ge \theta^{-1}, \quad \max_{x \in E_+} |R(x)| \le \theta$$

2.1.4 Fourth Zolotarëv Problem

Define the indicator function $S(x) = \pm 1$ when $x \in E^{\pm}$. Find the best uniform rational approximation $R(x)$ of $S(x)$ of the given degree:

$$||R - S||_{C(E)} := \max_{x \in E} |R(x) - S(x)| \to \min =: \mu.$$

It is a good exercise to show that all four settings are equivalent and in particular the value of θ is the same for the first three settings and $1/\mu = (\theta + 1/\theta)/2$ for the fourth one.

2.2 Study of Optimization Problem

Setting 2.1.1 appears in the paper [3] by R.A.-R. Amer, H.R. Schwarz (1964). It was transformed to problem of Section 2.1.2 by V.N. Malozemov [22]. Setting 2.1.3 appears after suitable normalization of the rational function in Section 2.1.1 and essentially coincides with the third Zolotarëv problem [26]. Setting 2.1.4 coincides with the fourth Zolotarëv problem [26] and was studied by N.I. Akhiezer [1]. The latter noticed that already in the classical Zolotarëv case when the set E contains just two components, the minimizing function is not unique. This phenomenon was fully explained in the dissertation of R.-A.R. Amer [3] who decomposed the space of rational functions of bounded deviation (defined in the left-hand side of formula in Section 2.1.1) into classes. Namely, it is possible to fix the sign of the polynomial in the numerator of the fraction on each stopband and fix the sign of denominator polynomial on each passband. Then in the closure of each nonempty

class there is a unique minimum. All mentioned authors established that (local) minimizing functions are characterized by *alternation* (or *equiripple* in terms of electrical engineers) property. For instance, in the fourth Zolotarëv problem the approximation error $\delta(x) := R(x) - S(x)$ of degree n minimizer has $2n + 2$ *alternation* points $a_s \in E$ where $\delta(a_s) = \pm ||\delta||_{C(E)}$ with consecutive change of sign.

3   Zolotarëv Fraction

E.I. Zolotarëv has solved the problem 2.1.4 for the simplest case: $E^{\pm} = \pm[1, 1/k]$, $0 < k < 1$ when $S(x) = sign(x)$. His solution is given parametrically in terms of elliptic functions and its graph (distorted by a pre- composition with a linear fractional map) is shown in Figure 1.

To give an explicit representation for this rational function, we consider a rectangle of size $2 \times |\tau|$:

$$\Pi_{\tau} = \{u \in \mathbb{C}: \quad |Re\, u| \leq 1, 0 < Im\, u < |\tau|\}, \qquad \tau \in i\mathbb{R}^{+}$$

which may be conformally mapped to the upper half plane with the normalization $x_{\tau}(u): \Pi_{\tau}, -1, 0, 1 \longrightarrow \mathbb{H}, -1, 0, 1$. The latter mapping has a closed appearance $x_{\tau}(u) = sn(K(\tau)u|\tau)$ in terms of *elliptic sine* and *complete elliptic integral* $K(\tau) = \frac{\pi}{2}\theta_3^2(\tau)$, both of modulus τ [2]. Zolotarëv fraction has a parametric representation resembling the definition of a classical Chebyshëv polynomial:
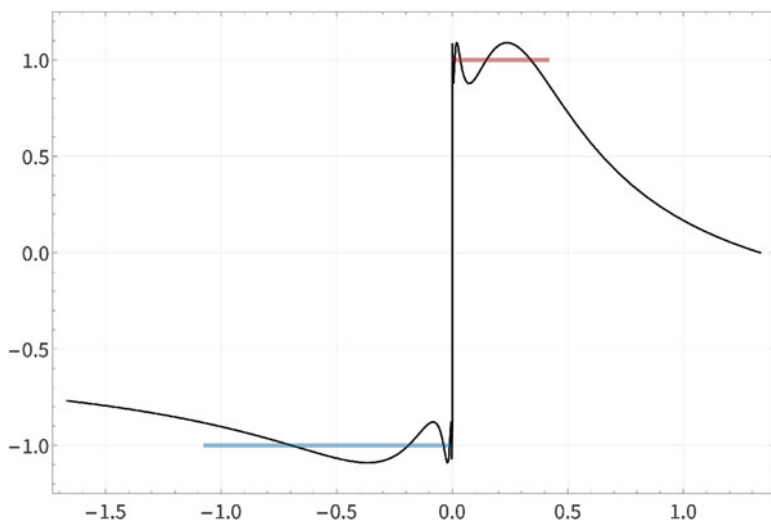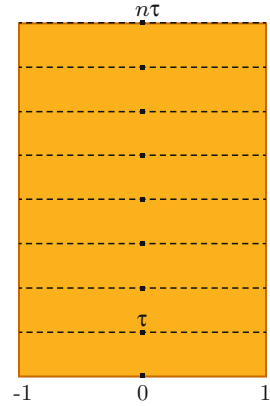

Fig. 1 Graph of Zolotarëv fraction adapted to two segments $E^{\pm}$ of different lengths

Fig. 2 Large rectangle $\Pi_{n\tau}$ composed of n copies of small one Π_τ

$$Z_n(x_{n\tau}(u)) = x_\tau(u).$$

Of course, it takes some effort to prove that Z_n is the rational function of its argument (we face the same difficulty with classical Chebyshev polynomial defined parametrically as $T_n(\cos(u)) := \cos(nu)$). The qualitative graph of Zolotarëv fraction completely follows from Figure 2, for instance its $2n - 2$ critical points correspond to the interior intersection points of the vertical boundaries of the large rectangle $\Pi_{n\tau}$ and horizontal boundaries of smaller rectangles. Alternation points different from critical points of the fraction correspond to four corners of the large rectangle. Zeros/poles of the fraction correspond to $u = l\tau$ with even/odd l.

Remark 1 Zolotarëv fractions share many interesting properties with Chebyshëv polynomials as the latter are the special limit case of the former [9, 10]. For instance, the superposition of suitably chosen Zolotarëv fractions is again a Zolotarëv fraction. They also appear as the solutions to many other extremal problems [15, 16, 18].

4 Projective View

Here we discuss the optimization problem setting which embraces all the formulations we met before in Section 2.1. We do not treat the infinity point both in the domain of definition and the range of rational function as exceptional. Real line extended by a point at infinity becomes a *real projective line* $\mathbb{R}P^1 := \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ which is a topological circle. We consider two collections of disjoint closed segments on the extended real line: E consisting of $m \geq 2$ segments and F of just two segments. The segments of both E and F are of two types: $E := E^+ \cup E^-$; $F := F^+ \cup F^-$.

Definition 1 We introduce the class $\mathcal{R}_n(E, F)$ of real rational functions $R(x)$ of a fixed degree $\deg R = n$ such that $R(E^+) \subset F^+$ and $R(E^-) \subset F^-$.
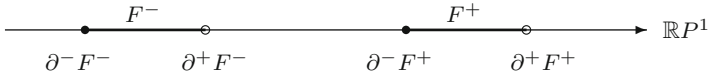
$$\overset{F^-}{\underset{\partial^- F^- \qquad \partial^+ F^-}{\rule{6cm}{0pt}}} \qquad \overset{F^+}{\underset{\partial^- F^+ \qquad \partial^+ F^+}{\rule{6cm}{0pt}}} \longrightarrow \mathbb{R}P^1$$

Fig. 3 The ordering of four endpoints ∂F and their colors

The set of values F modulo *projective (=linear-fractional) transformations* depends on a single value—cross ratio of its endpoints. Suppose the endpoints ∂F are cyclically ordered as follows: $\partial^- F^-$, $\partial^+ F^-$, $\partial^- F^+$, $\partial^+ F^+$—see Figure 3 then the cross ratio of four endpoints we define as follows:

Definition 2

$$\kappa(F) := \frac{\partial^+ F^+ - \partial^+ F^-}{\partial^+ F^+ - \partial^- F^+} : \frac{\partial^- F^- - \partial^+ F^-}{\partial^- F^- - \partial^- F^+} > 1.$$

The classes $\mathcal{R}_n(E, F)$ (possibly empty) and the value of the cross ratio have several easily checked properties:

Lemma 1

1. *Monotonicity.*

$$\mathcal{R}_n(E, F') \subset \mathcal{R}_n(E, F) \qquad once \ F' \subset F.$$

2. *Projective invariance. For any projective transformations* $\alpha, \beta \in PGL_2(\mathbb{R})$,

$$\mathcal{R}_n(\alpha E, \beta F) = \beta \circ \mathcal{R}_n(E, F) \circ \alpha^{-1}.$$

3. *The value* $\kappa(F)$ *is decreasing with the growth of its argument: if* $F' \subset F$ *then* $\kappa(F') > \kappa(F)$.

4.1 Projective Problem Setting

Fix degree $n > 0$ and set $E \subset \mathbb{R}P^1$, find

$$\varkappa(n, E) := \inf\{\kappa(F) : \qquad \mathcal{R}_n(E, F) = \emptyset\}.$$

The idea behind this optimization is the following: we squeeze the set of values F, the functional class $\mathcal{R}_n(E, F)$ diminishes and we have to catch the moment—quantitatively described by the cross ratio $\kappa(F)$—when the class disappears.

Remark 2

1) In problem formulation 2.1.3 the set $F^+ = [-\theta, \theta]$ and the set $F^- = [1/\theta, -1/\theta]$; $\kappa(F) = \left(\frac{1}{2}(\theta + 1/\theta)\right)^2$. In setting 2.1.4 the sets $F^\pm = \pm[1 - \mu, 1 + \mu]$ and $\kappa(F) = \mu^{-2}$.

2) Notice that the cross ratio depends on the order of four participating endpoints and may take six values interchanged by the elements of the so-called unharmonic group.

4.2 Decomposition into Subclasses

Now we decompose each set $\mathcal{R}_n(E, F)$ into subclasses which were first introduced for the problem setting 2.1.1 by R.A.-R. Amer in his PhD thesis [3] in 1964. The construction of these subclasses is purely topological: suppose we identify opposite points of a circle S^1, we get a double cover of a circle identified with real projective line $\mathbb{R}P^1$ by another circle S^1. Now we try to lift the mapping $R(x) : \mathbb{R}P^1 \to \mathbb{R}P^1$ to the double cover of the target space: $\tilde{R}(x) : \mathbb{R}P^1 \to S^1$. There is a topological obstruction to the existence of $\tilde{R}$: the mapping degree or the winding number of $R(x)$ modulo 2. A simple calculation shows that this value is equal to algebraic degree $\deg R \bmod 2$. However this lift exists on any simply connected piece of $\mathbb{R}P^1$. Suppose the segment $E(j)$ is made up of two consecutive segments E_j, E_{j+1} of the set of bands $E = \cup_{j=1}^m E_j$ and the gap between them. The set $F \subset \mathbb{R}P^1$ lifted to the circle S^1 consists of four components cyclically ordered as $F_0^-, F_0^+, F_1^-, F_1^+ \subset S^1$. The mapping $R(x) : E(j) \to \mathbb{R}P^1$ has two lifts to the covering circle S^1 and exactly one of them has values $\tilde{R}(x) \in F_0 := F_0^- \cup F_0^+$ when $x \in E_j$. On the opposite side E_{j+1} of the segment the same function $\tilde{R}(x)$ takes values in the set $F_{\sigma(j)}$ with well-defined $\sigma(j) \in \{0, 1\}$. Totally, the function $R(x)$ defines an element of $\mathbb{Z}_2$ for any two consecutive segments of the set E with the only constraint

$$\sum_{j=1}^m \sigma(j) = \deg R \bmod 2.$$

which defines the element $\Sigma := (\sigma_1, \sigma_2, \ldots, \sigma_{m-1}) \in \mathbb{Z}_2^{m-1}$. All elements $R(x) \in \mathcal{R}_n(E, F)$ with the same value of $\Sigma \in \mathbb{Z}_2^{m-1}$ make up a subclass $\mathcal{R}_n(E, F, \Sigma)$. Again, one readily checks the properties of the new classes:

Lemma 2

1. Monotonicity:

$$\mathcal{R}_n(E, F', \Sigma) \subset \mathcal{R}_n(E, F, \Sigma) \qquad once\ F' \subset F.$$

2. Projective invariance

$$\beta \circ \mathcal{R}_n(E, F, \Sigma) = \mathcal{R}_n(E, \beta F, \beta \Sigma), \qquad \beta \in PGL_2(\mathbb{R}).$$

here projective transformation β acts on Σ component wise: $\sigma(j)$ reverses exactly when β changes the orientation of projective line and the bands E_j, E_{j+1} are of different $\pm$-type. Otherwise—iff $\beta \in PSL_2(\mathbb{R})$ or bands E_j, E_{j+1} are both pass- or stopbands—$\sigma(j)$ is kept intact.

Remark 3 R.-A.R. Amer [3] combines classes $\mathcal{R}_n(E, F, \Sigma)$ and $\mathcal{R}_n(E, F, \beta\Sigma)$ for β reversing the orientation of projective line and conserving the components $F^{\pm}$. This is why he gets twice less number 2^{m-2} of classes.

4.3 Extremal Problem for Classes

Given degree n, set of bands E, and the class Σ—find

$$\varkappa(n, E, \Sigma) := \inf\{\kappa(F): \qquad \mathcal{R}_n(E, F, \Sigma) = \emptyset\}. \tag{1}$$

4.4 Equiripple Property

Definition 3 We say that cyclically ordered (on projective line) points $a_1, a_2, \ldots a_s \subset E$ make up an alternation set for the function $R(x) \in \mathcal{R}_n(E, F)$ iff $R(x)$ maps each of those points to the boundary $\partial F = \partial^+ F \cup \partial^- F$, and any two consecutive points—to different sets $\partial^+ F$, $\partial^- F$ colored black and white in Figure 3.

Theorem 1 *If the value $\varkappa(n, E, \Sigma) > 1$, then the closure of the extremal class $\mathcal{R}_n(E, F, \Sigma)$ contains a unique function $R(x)$ which is characterized by the property of having at least $2n + 2$ alternation points when $R(x)$ is not at the boundary of the class.*

Proof of this theorem and other statements of the current section will be given elsewhere.

5 Problem Genesis: Signal Processing

There are many parallels between analogue and digital electronics, this is why many engineering solutions of the past have moved to the new digital era. In particular, the same optimization problem for rational functions discussed in Section 2.1 arises in the synthesis of both analogue and digital electronic devices.

From the mathematical viewpoint electronic device is merely a linear operator which transforms input signals $x(\cdot)$ to output signals $y(\cdot)$. By signals they mean functions of one continuous or discrete argument: $x(t)$, $t \in \mathbb{R}$ or $x(k)$, $k \in \mathbb{Z}$. For technical simplicity they assume that signals vanish in the 'far past'. Another natural assumption that a device processing a delayed signal gives the same but (equally) delayed output which mathematically means that operator commutes with the time shifts. As a consequence, the operator consists in a (discrete) convolution of the input signal with the certain fixed signal—the response $h(\cdot)$ to (discrete) delta function input:

$$y(t) = \int_{\mathbb{R}} h(t')x(t - t')dt'; \qquad y(k) = \sum_{k' \in \mathbb{Z}} h(k')x(k - k').$$

The causality property means that the output cannot appear before the input and implies that *impulse response* $h(\cdot)$ vanishes for negative arguments. Further restrictions on the *impulse response* follow from the physical construction of the device.

Analogue device is an electric scheme assembled of elements like resistors, capacitors, (mutual) inductances, etc. which is governed by Kirchhoff laws. Digital device is governed by the recurrence relation:

$$y(m) := \sum_{j=0}^{n} p_j x(m - j) + \sum_{j=1}^{n} q_j y(m - j), \qquad m \in \mathbb{Z}. \tag{2}$$

To compute the impulse response, we use the Fourier transform of continuous signals and Z-transform of digital ones (here we do not discuss any convergence):

$$\hat{x}(\omega) := \int_{\mathbb{R}} x(t) \exp(i\omega t)dt; \quad \omega \in \mathbb{H} \qquad \hat{x}(z) := \sum_{k \in \mathbb{Z}} x(k)z^k, \quad z \in \mathbb{C}. \tag{3}$$

Using the explicit relation (2) for digital device and its Kirchhoff counterpart for analogue ones we observe that the images of input signals are merely multiplied by *rational functions* $\hat{h}(\cdot)$ of appropriate argument. Since the impulse response is real valued, its image—also called the *transfer function*—has the symmetry

$$\hat{h}(-\bar{\omega}) = \overline{\hat{h}(\omega)}, \qquad \hat{h}(\bar{z}) = \overline{\hat{h}(z)}, \qquad \omega, z \in \mathbb{C}.$$

In practice we can physically observe the absolute value of the transfer function: if we 'switch on' a harmonic signal of a given frequency as the input one, then after certain transition process the output signal will also become harmonic, however with a different amplitude and phase. The magnification of the amplitude as a function of frequency is called the magnitude response function and it is exactly equal to the absolute value of transfer function of the device.

Multiband filtering consists in constructing a device which almost keeps the magnitude of a harmonic signal with the frequency in the passbands and almost eliminates signals with the frequency in the stopbands. We use the word 'almost' since the square of the magnitude response is a rational function on the real line (for analogue case):

$$|\hat{h}(\omega)|^2 = \hat{h}(\omega)\hat{h}(-\omega) = R(\omega^2), \qquad \omega \in \mathbb{R}. \tag{4}$$

At best we can talk of approximation of an indicator function which is equal 0 at the stopbands and 1 at the passbands. For certain reasons discussed, e.g., by W. Cauer, uniform (or Chebyshev) approximation is preferable for this practical problem. So we immediately arrive at the fourth Zolotarev problem taking the square of frequency as a new variable. For the digital case we get a similar problem set on the segments of the unit circle which can be transformed to the problem on a real line.

Note that the reconstruction of the *transfer function* $\hat{h}(\cdot)$ from the *magnitude response* is not unique: we have to solve Equation (4) given its right-hand side, which has some freedom. This freedom is used to meet another important restriction on the image of impulse response which is prescribed by the causality: $\hat{h}(\cdot)$ can only have poles in the lower half plane $-\mathbb{H}$ of complex variable ω for the analogue case or strictly outside the unit circle of variable z for the digital case.

Minimal deviation problem in any of the given above settings is just an intermediate step to the following problem of great practical importance. *Find minimal degree filter meeting given filter specifications* like the boundaries of the pass- and stopbands attenuation at the stopbands and allowable ripple amplitude at the passbands. The degree of the rational function $\hat{h}$ is directly related to the complexity of device structure, its size, weight, cost of production, energy consumption, cooling, etc.

6 Approaches to Optimization

There are several major approaches for the practical solution of optimization problem of multiband electrical filter. Three of those are discussed below. Along with them we would single out a computationally efficient Caratheodory-Fejer method (known also as AAK-approximation) solving the problem in the class of quasirational functions which may be further truncated to rational at a cost of extra error [19].

6.1 Remez-Type Methods

Direct numerical optimization is usually based on Remez-type methods. This is a group of algorithms specially designed for uniform rational approximation

[17, 21, 24, 25]. They iteratively build the necessary alternation set for the error function of approximation. Unfortunately the intrinsic instability of Remez algorithms does not allow to get high degree solutions and therefore sophisticated filter specifications. For instance, standard double precision accuracy 10^{-15} used, e.g., in MATLAB does not allow to get solutions of degree n greater than 15–20. We know an example when approximation of degree $n \approx 2000$ required mantissa of 150,000 decimal signs for stability of intermediate computations. Writing just one number of this precision requires 75 standard pages—this is the volume of a typical PhD thesis. Another problem of this group of algorithms is the choice of initial approximation. The set of suitable starting points may have infinitesimal volume.

6.2 Composite Filters

Practical approach of engineers is to decompose complicated problem into many simple ones and solve them one by one. In case of filter synthesis they use a battery of single passband (say, Cauer) filters. This approach is very reliable: it always gives working solutions which however are far from being optimal. We get a substantial rise in the order of filter, and therefore complexity of its structure and the downgrading of many consumer properties.

6.3 Ansatz Method

Is based on an explicit analytical formula for the solution generalizing formula for Zolotarëv fractions. However this formula contains unknown parameters, both continuous and discrete which have to be evaluated given the input data of the problem. Of all approaches this one is the least studied from the algorithmic viewpoint and its usage is restrained by involved mathematical apparatus [8]. Nonetheless it copes with very involved filter specifications: narrow transition bands, large number of working bands, their different proportions, high degree of solution.

A detailed comparison of three approaches has been made in [11].

7 Novel Analytical Approach

The idea behind this approach utilizes the following observation: *Almost all—with very few exceptions—critical points of the extremal function have values in some 4-element set* Q. Indeed, the equiripple property claims that a degree n solution has $2n + 2$ alternation points, those in the interior of E inevitably being critical. Their values belong to the set $Q := \pm\theta, \pm 1/\theta$ in the settings (1), (2), (3) or $\pm 1 \pm \mu$ in

setting (4) or ∂F for the projective setting. This number is roughly equal to the total number $2n - 2$ of critical points of a degree n rational function. The number $g - 1$ of exceptional critical points is counted as

$$g - 1 = \sum_{x:\, R(x) \notin \mathbb{Q}} B(R, x) + \sum_{x:\, R(x) \in \mathbb{Q}} \left[\frac{1}{2} B(R, x) \right], \qquad (5)$$

here the summation is taken over points of the Riemann sphere; $[\cdot]$ is the integer part of a number and $B(R, x)$ is the branching number of the holomorphic map R at the point x. The latter value equals zero in all regular points x including simple poles of $R(x)$, or the multiplicity of the critical point of $R(x)$ otherwise.

Mentioned above exceptional property of extremal rational functions may be rewritten in a form of a generalized Pell-Abel functional equation and eventually gives the desired few-parametric representation of the solution [8] for the normalized $\mathbb{Q} = \{\pm 1, \pm 1/k(\tau)\}$

$$R(x) = \mathrm{sn} \left(\int_e^x d\zeta + A(e) \,\Big|\, \tau \right). \qquad (6)$$

Here $\mathrm{sn}(\cdot | \tau)$ is the *elliptic sine* of the *modulus* τ related to the value of the deviation (depending on the setting it is μ, θ or $\kappa(F)$); $d\zeta$ is a *holomorphic differential* on the unknown beforehand *hyperelliptic curve*

$$M = M(\mathbf{e}) = \{(w, x) \in \mathbb{C}^2 : \quad w^2 = \prod_{s=1}^{2g+2} (x - e_s)\}, \qquad \mathbf{e} = \{e_s\}_{s=1}^{2g+2}. \qquad (7)$$

This curve has branching at the points $e \in \mathbf{e}$ where $R(x)$ takes values from the exceptional set $\mathbb{Q}$ *with odd multiplicity*. One can show that the genus g of the curve (7) equals to the above defined number (5) of exceptional critical points plus 1. The arising surface is not arbitrary: it bears a holomorphic differential $d\zeta$ whose periods lie in the rank 2 periods lattice of elliptic sine. The phase shift $A(e)$ is some quarter period of $\mathrm{sn}(\cdot | \cdot)$.

Algebraic curves of this type are not new to mathematicians: they are so-called *Calogero-Moser curves* and describe the dynamics of points on a torus interacting with the Weierstrass potential $\wp(u)$.

8 Examples of Filter Design

We give several examples of optimal magnitude response functions from different classes, all of them are computed by Sergei Lyamaev. Figure 4 shows the solution of fourth Zolotarëv problem with the set E consisting of 30 bands. The solution contains no poles in the transition bands and may be transformed to the transfer
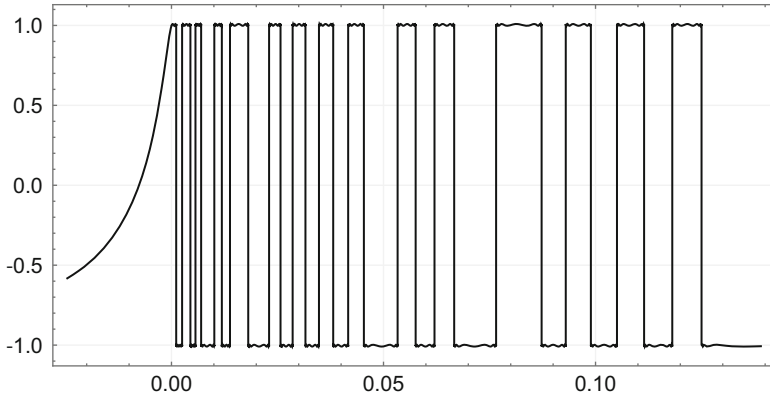
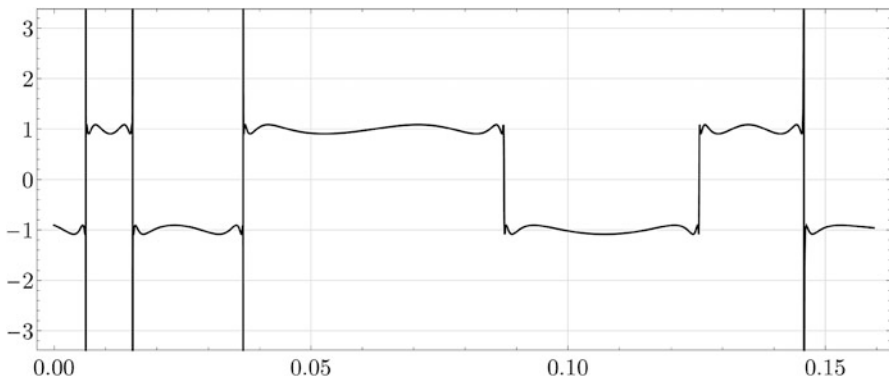**Fig. 4** Optimal magnitude response function with 30 work bands

Fig. 5 The minimizer for the fourth Zolotarëv problem on 7 bands with class Σ admitting poles in some transition bands

function of the multiband filter. Figure 5 shows the solution of the problem with seven working bands. Its class Σ admits poles in the transitions and the function cannot be used for the filter synthesis, which does not exclude other possible applications.

Figures 6 and 7 represents a magnitude response function of the so-called double notch filter which eliminates input signal in the narrow vicinities of two given frequencies. Shown here optimal filter has degree $n = 16$ while same specification composite filter has degree $n = 62$.
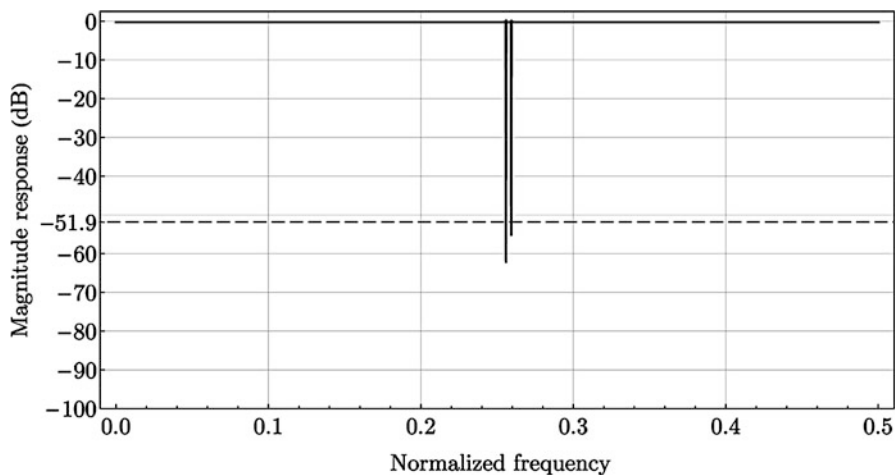
Fig. 6 Optimal *double notch* filter eliminating noise at two given frequencies. Log scale on the vertical axis (amplification)
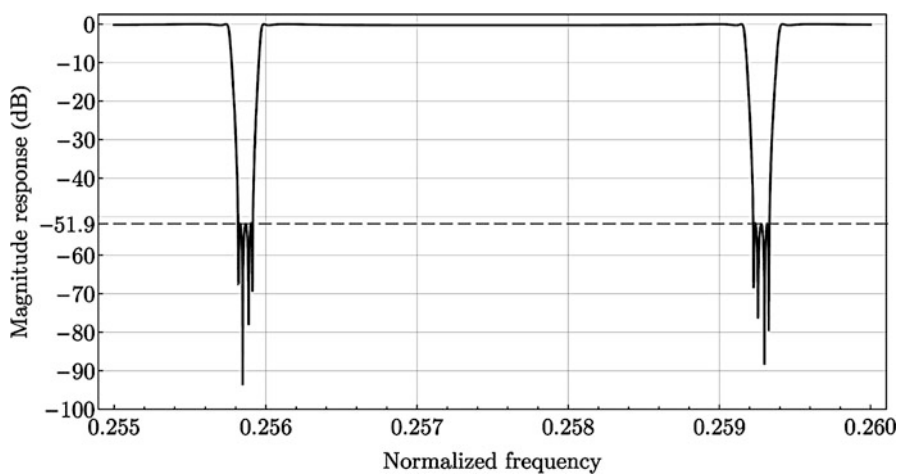

Fig. 7 Magnification of the previous figure at two cutoff frequencies

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