Chapter 6
Appendix: Asymptotic Estimates and Series

Abstract In this appendix, the definitions of symbols O, o, $\sim$ and asymptotic expansions met in the book are shortly given.

6.1 Estimates of Functions

Let functions $f(z)$ and $g(z)$ be defined on a set $\mathbb{D}$ of the complex numbers, $\mathbb{C}$, or the real numbers, $\mathbb{R}$, and let a be a point of accumulation of $\mathbb{D}$.

Notation 1. We write

$$f(z) = O(g(z)) \quad \text{as} \quad z \to a \tag{6.1}$$

if there exists a neighborhood U of the point a and a constant C such that

$$|f(z)| \leq C|g(z)| \quad \text{for any} \quad z \in U \cap \mathbb{D}. \tag{6.2}$$

Notation 2. One writes

$$f(z) = O(g(z)) \tag{6.3}$$

if there exists a constant C such that the inequality

$$|f(z)| \leq C|g(z)| \tag{6.4}$$

holds for all $z \in \mathbb{D}$.

Notation 3. The notation

$$f(z) = o(g(z)) \quad \text{as} \quad z \to a \tag{6.5}$$

means that

$$\lim_{z \to a} \frac{f(z)}{g(z)} = 0. \tag{6.6}$$

G. I. Mikhasev and H. Altenbach, *Thin-walled Laminated Structures*,
Advanced Structured Materials 106, https://doi.org/10.1007/978-3-030-12761-9_6

Notation 4. If $f(z) = O(g(z))$ and $f(z) = o(g(z))$ hold simultaneously as $z \to a$, we write

$$f(z) \sim g(z) \quad \text{as} \quad z \to a. \tag{6.7}$$

The notations $O(g(z))$ and $o(g(z))$ define the class of functions which satisfy estimations (6.3) and (6.5), respectively. We list here some rules for operations with these symbols (classes of functions). As $z \to a$ and $z \in \mathbb{D}$, there are valid the following properties:

$$\begin{gathered} o(g(z)) + o(g(z)) = o(g(z)), \quad o(g(z)) + O(g(z)) = O(g(z)), \\ o(g(z)) \times o(f(z)) = o(g(z) \times f(z)), \quad o(g(z)) \times O(f(z)) = o(g(z) \times f(z)), \\ O(o(g(z))) = o(g(z)), \quad o(O(g(z))) = o(g(z)), \\ o(o(g(z))) = o(g(z)), \quad o(g(z)) = O(g(z)). \end{gathered} \tag{6.8}$$

The prove of some of the above relations as well as a large number of examples may be found in De Bruijn (1970); Nayfeh (1973); Olver (1974); Bauer et al (2015).

6.2 Asymptotic Series

Consider a sequence of functions $u_n(z)$, $n = 0, 1, 2, \ldots$, defined on $\mathbb{D}$ and let a be a point of accumulation of $\mathbb{D}$.

Definition 6.1. The sequence $u_n(z)$ is said to be *asymptotic* as $z \to a$, if for any integer $n \geq 0$,

$$u_{n+1}(z) = o(u_n(z)), \quad \text{as} \quad z \to a. \tag{6.9}$$

For example, the sequence $u_n(z) = F(z)(z-a)^m$ as $z \to a$, where $F(z)$ is an arbitrary function bounded on the set $\mathbb{D}$, is the asymptotic one. Similar sequence appear in Eqs. (3.105). Indeed, the sequence

$$u_n(\varepsilon; \xi, s) = \varepsilon^{n/2} \chi_n(\xi, s) \exp\left\{ \mathrm{i} \left(\varepsilon^{-1/2} p\xi + \frac{1}{2} b\xi^2 \right) \right\} \tag{6.10}$$

is the asymptotic as $\varepsilon \to 0$ for any fixed ξ, s.

Definition 6.2. Let the function $f(z)$ be defined on $\mathbb{D}$ and the sequence $u_n(z)$ is asymptotic as $z \to a$, then the series

$$f(z) \cong \sum_{n=0}^{\infty} a_n u_n(z) \quad \text{as} \quad z \to a \tag{6.11}$$

is called an asymptotic expansion of $f(z)$ in the Poincaré sense by means of the asymptotic sequence $u_n(z)$ if there are constants a_n such that for any integer $N \geq 0$

$$f(z) - \sum_{n=0}^{N} a_n u_n(z) = o(u_N(z)) \tag{6.12}$$

or

$$f(z) - \sum_{n=0}^{N} a_n u_n(z) = O(u_{N+1}(z)) \tag{6.13}$$

as $z \to a$.

If the function $f(z)$ is expanded into asymptotic series (6.11) by means of the asymptotic sequence $u_n(z)$, then the coefficients a_n in (6.11) are determined in a unique way; in other words, expansion (6.11) is unique.

We note that an asymptotic series may diverge. Asymptotic series may be summed, multiplied by functions, differentiated and integrated under special assumptions. Basic properties of asymptotic series and operations on them are given in Jahnke et al (1960); Evgrafov (1961); De Bruijn (1970); Nayfeh (1973); Olver (1974); Erdèlyi (2010); Bauer et al (2015).

References

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