Chapter 4
Free Vibrations of Elastic Laminated Beams, Plates and Cylindrical Shells

Abstract In this chapter, based on the equivalent single layer model for thin laminated members, natural modes and corresponding eigenfrequencies for laminated elastic beams plates and cylindrical shells are studied taking into account shears. At first, elastic vibrations of laminated beams are analyzed in Sect. 4.1, the emphasis being made on non-uniformly stressed beams contacting with an elastic inhomogeneous medium. Then, in Sect. 4.2, the eigenmodes and frequencies of elastic rectangular plates are analyzed for two variants of boundary conditions: if all edges are simply supported and have diaphragms preventing shears, the boundary-value problem is solved in the explicit form; and if one of edges is free of a diaphragm, the solution of a corresponding boundary-value problem is constructed in the form of the superposition of the main stress-strain state and the edge effect integrals accounting for the edge shears. Section 4.3 is devoted to vibrations of a circular cylindrical shell of an arbitrary length with constant geometrical and physical parameters. In Sect. 4.4, the localized natural modes for a medium-length laminated cylinder is investigated. And finally, Sect. 4.5 contains the problem on free localized vibrations of a laminated cylindrical shell under axial forces no-uniformly distributed in the circumferential direction. In the last two sections, natural modes are constructed by using the asymptotic method. In all problems, the effect of shears on the natural frequencies is analyzed. Examples on free vibrations of laminated cylinders and panels assembled from different materials are considered.

4.1 Laminated Beams

In this section, we study free elastic vibrations of laminated beams. Particular attention will be paid to the problem on free vibrations of non-homogeneous beams with low reduced shear modulus. We will call a beam non-homogeneous if it has geometrical and/or physical parameters dependent of an axial coordinate, or if it is non-uniformly pre-stressed by compressive or tensile forces. Geometrically inhomogeneous beams are beams with the cross-sectional sizes (width, high, radius)

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varying along the axis. Physically non-uniform beams are beams in which the material properties (elastic moduli, material density) depend on the axial coordinate. This heterogeneity can be induced by the action of external physical fields (temperature, magnetic field, etc.). Beams with functionally graded materials (FGM) along the beam axis are often considered as well. If the beam is in contact with an inhomogeneous elastic medium, then the dynamic reaction of the beam is also nonuniform along its axis. Within the framework of any deformation model for a non-uniform beam and regardless of the nature of inhomogeneity, the differential equations governing vibrations of similar beams contain variable coefficients, which significantly complicates the problem.

It should be noted that despite the complexity of the problems, vibrations of inhomogeneous beams were studied by many researchers. But for all that, a majority from numerous studies refer to isotropic single layer beams. Cranch and Adler (1956) and Suppiger and Taleb (1956) were probably the first who in 1956 investigated free bending vibrations of isotropic beams with variable section. Assuming the linear (Cranch and Adler, 1956) or exponential (Suppiger and Taleb, 1956) law of variation of the cross-section along the beam axis, they constructed exact solutions for beams with different boundary conditions. Later, applying different approximate analytical or numerical methods, a numerous investigations on free vibrations of isotropic beams with variable section, including tapered ones and beams with steppered sections, were carried out (s., among others, Conway and Dubil, 1965; Carnegie and Thomas, 1967; Sanger, 1968; Goel, 1976; Roy and Ganesan, 1994; Zhou and Cheung, 2000, 2001; Naguleswaran, 2002; Ece et al, 2007; Firouz-Abadi et al, 2007; Jaworski and Dowell, 2008). Free vibration analysis of geometrically no-uniform beams subjected to the axial compression or tension were made by Sato (1980); Naguleswaran (2003); Kukla and Zamojska (2007). The effect of uniform and non-uniform elastic foundations on natural frequencies and modes was examined by Lee and Ke (1990); Wang (1991). Bending vibrations of FGM beams with variation of material properties were studied in (Murin et al, 2010; Huang and Li, 2010; Alshorbagy et al, 2011; Mohanty and Rout, 2012).

As for laminated beams, there are only a few papers considering vibrations taking into account initial axial stresses or response of a surrounding medium or foundation. Li et al (2008, 2016) investigated free vibration and buckling behaviors of axially loaded laminated composite beams having arbitrary lay-up. Using the dynamic stiffness method (Li et al, 2008) and based on a unified higher-order shear deformation beam theory (Li et al, 2016), they analyzed the influences of axial forces, shear deformation and rotary inertia on the natural frequencies, buckling loads and mode shapes. Using a three-node shear flexible beam element, Patel et al (1999) studied nonlinear free flexural vibrations of laminated orthotropic beams resting on a two parameter elastic foundation. Similar problem was considered by Jafari-Talookolaei and Ahmadian (2007). Using FEM on the basis of Timoshenko beam theory, they investigated free vibrations of a cross-ply laminated composite beam on elastic Pasternak foundation. The effect of viscoelastic support on free vibrations of laminated fiberglass beam was examined by Koutsawa and Daya (2007). Large amplitude free vibration analysis of laminated composite thin beams on linear and nonlinear elastic foundations was presented by Malekzadeh and Vosoughi (2009); Baghani et al (2011).

In the aforementioned papers, composite laminated beams were assumed to be shear deformable. However, axial stresses (Li et al, 2008, 2016) and elastic/viscoelastic properties of foundations (Patel et al, 1999; Jafari-Talookolaei and Ahmadian, 2007; Koutsawa and Daya, 2007; Malekzadeh and Vosoughi, 2009; Baghani et al, 2011) were considered to be constant along the beam axis. Apparently, Farghaly and Gadelrab (1995); Dong et al (2005) are among the few available studies in which laminated beams are geometrically heterogeneous in the axial direction. Based on the first order shear deformation theory, they performed vibration analysis of stepped laminated composite Timoshenko beams. We also refer readers to the reviews (Hajianmaleki and Qatu, 2013; Sayyad and Ghugal, 2017), which give some insight of state of the art on dynamics of laminated elastic beams.

4.1.1 Governing Equation

Let us consider a laminated beam consisting of N elastic laminas. It is assumed that the beam is compressed by the axial force F° and/or rest on an elastic foundation with the modulus of substrate reaction c_f. The beam is characterized by the total thickness $h = \sum_{j=1}^{N} h_j$, bending stiffness EI and linear density ρ_l. If the beam cross section has a rectangular form with hight h and width b, then $I = bh^3/12$. In the common case, $F^\circ, \rho_\mathrm{l}, c_\mathrm{f}$ may be functions of the coordinate α_1 $(0 \leq \alpha_1 \leq L)$. We apply here again the ESL theory stated in Chapt. 2. Taking into account the response of elastic foundation and the dependence of the axial force on α_1, Eq. (2.153) governing dynamics of a multi-layered beam is rewritten as

$$EI\eta_3\left(1-\frac{\theta h^2}{\beta}\frac{\partial^2}{\partial\alpha_1^2}\right)\frac{\partial^4\chi}{\partial\alpha_1^4}-\frac{\partial}{\partial\alpha_1}\left[F^\circ\left(1-\frac{h^2}{\beta}\frac{\partial^2}{\partial\alpha_1^2}\right)\right]\frac{\partial\chi}{\partial\alpha_1}$$
$$+c_\mathrm{f}\left(1-\frac{h^2}{\beta}\frac{\partial^2}{\partial\alpha_1^2}\right)\chi+\rho_\mathrm{l}\left(1-\frac{h^2}{\beta}\frac{\partial^2}{\partial\alpha_1^2}\right)\frac{\partial^2\chi}{\partial t^2}=0, \tag{4.1}$$

where the reduced Young's modulus E and shear parameters β, θ are calculated by equations derived in Chapt. 2 with $\nu = \nu_k = 0$.

For the Winkler foundation, the spring constant c_f is only influenced by the elastic properties of the foundation. Assuming the alternative model represented in Chapt. 2, s. Eq. (2.152), then

$$c_\mathrm{f} = \alpha_\mathrm{f} b\pi n/L, \quad \alpha_\mathrm{f} = \frac{2E_\mathrm{f}(1-\nu_\mathrm{f})}{(1+\nu_\mathrm{f})(3-4\nu_\mathrm{f})}, \tag{4.2}$$

where n is the wave number in the function $\chi = \chi_0 \sin \pi n\alpha_1/L$ describing the beam response and E_f, ν_f are the Young's modulus and Poison's ratio of the foundation.

Remark 4.1. Equation (4.1) may be used if E, I, β, θ and η_3 are functions of α_1. The error of the equation depends on the index of variation of these functions by α_1. The higher this index is, the larger the error of Eq. (4.1).

4.1.2 Simply Supported Beam with Constant Parameters

Let the beam edges be simply supported and all parameters, including $F^\circ, \rho_0, c_\mathrm{f}$, be constants. Then the solution of (4.1) satisfying the boundary conditions (3.3) or (3.4) has the simple form

$$\chi = \chi_0 \sin \frac{\pi n \alpha_1}{L}\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{4.3}$$

where L is the beam length, n is the number of waves and ω is the natural frequency.

The substitution of (4.3) into (4.1) results in the natural frequency

$$\omega = \frac{1}{\sqrt{\rho_1}} \sqrt{\frac{EI\eta_3\pi^4 n^4(1+\theta K n^2)}{L^4(1+Kn^2)} + \frac{F^\circ \pi^2 n^2}{L^2} + c_\mathrm{f}}, \tag{4.4}$$

where

$$K = \frac{\pi^2 h^2}{\beta L^2}.$$

If the foundation spring constant is represented by (4.2), then

$$c_\mathrm{f} = \frac{\alpha_\mathrm{f} b \pi n}{L},$$

and for the Winkler foundation c_f is a constant independent of n.

If $F^\circ > 0$, then the beam is stretched, and for $F^\circ < 0$, it is compressed. In the last case, it is assumed that $|F^\circ| < F^*_\mathrm{cr}$, where

$$F^*_\mathrm{cr} = \max_n \left\{ \frac{\pi^2 n^2 EI\eta_3(1+\theta K n^2)}{L^2(1+Kn^2)} + \frac{c_\mathrm{f} L^2}{\pi^2 n^2} \right\} \tag{4.5}$$

is the critical buckling force. For $c_\mathrm{f} = 0$, it coincides with Eq. (3.11) derived in Chapt. 3. The increase of the tensile force F° and/or the spring constant c_f leads to the growth of the natural frequencies for any number n. In contrast, increasing the compressive force F° results in decreasing the eigenfrequencies; herewith, $\omega \to 0$ as $|N^\circ| \to N^*_\mathrm{cr}$.

Other important conclusions are the following:

a) the incorporation of the shear parameter K into the ESL beam model leads to the reduction of the natural frequencies and
b) the effect of K on the natural frequencies is weak for low-frequency vibrations and, in particular, for very long beams, but it becomes noticeable for higher modes (for large n).

Below, it will be shown that the conclusion b) becomes not valid for a medium-length laminated cylindrical shell.

4.1.3 Vibrations of Pre-stressed Beams on Elastic Foundation

Let $F^\circ, c_\mathrm{f}, \rho_\mathrm{l}$ be functions of α_1. The parameter β depends on the correlation between the reduced Young's and shear moduli E, G and may vary in a wide range. We consider here the case when $G \sim h_* E$, then $\beta \sim h_*$, where $h_* = h/L$ is a small parameter (the beam is assumed to be long). The parameter θ is also small. So, for a single layer beam $\theta = 1/85$, and for a multi-layered one it may be much less. Here, it is assumed that $\theta \sim h_*^\varsigma$, $1/2 < \varsigma < 1$. We introduce some assumptions concerning the elastic foundation and axial stress resultant. Let the foundation be *soft* and the axial force be sufficiently weak so that the following relations hold

$$c_\mathrm{f}(\alpha_1) = h_* \frac{Eb\eta_3}{12L}\, k(x), \quad F^\circ = h_*^2 \frac{LEb\eta_3}{12}\, f_1(x), \tag{4.6}$$

where $x = \alpha_1/L$ is a dimensionless coordinate. If $f_1(x) > 0$ for any $x \in [0,1]$, then the force F° is extensional in any point of the beam; when $f_1(x) < 0$ in some points from the segment $[0,1]$, the force F° is compressive in this points, but in this case it is assumed that $\max_x |f_1(x)| < f_\mathrm{cr}$, where f_cr is the critical value resulting in buckling of the beam (s. Chapt. 3).

In the case of free vibrations, the displacement function χ may be found in the form of

$$\chi = LX(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{4.7}$$

where ω is the natural frequency. Let us introduce a dimensionless parameter λ and the characteristic time t_c

$$\lambda = t_\mathrm{c}^2 \omega^2, \quad t_\mathrm{c} = \sqrt{\frac{12\rho_\mathrm{lm} L^2}{h_* Eb\eta_3}}, \tag{4.8}$$

where $\rho_\mathrm{lm} = \max \rho_\mathrm{l}(Rx)$ is a maximum value of the reduced linear density for a nonhomogeneous beam.

Then Eq. (4.1) is rewritten as follows

$$\begin{aligned} -h_*^{3+\varsigma}\tau \frac{\mathrm{d}^6 X}{\mathrm{d}x^6} + h_*^2 \frac{\mathrm{d}^4 X}{\mathrm{d}x^4} - h_* \frac{\mathrm{d}}{\mathrm{d}x}\left[f_1(x)\left(1 - h_*\kappa \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\frac{\mathrm{d}X}{\mathrm{d}x}\right] \\ + k(x)\left(1 - h_*\kappa\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)X - \lambda r(x)\left(1 - h_*\kappa\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)X = 0, \end{aligned} \tag{4.9}$$

where

$$\tau = h_*^{1-\varsigma}\theta\beta^{-1}, \qquad \kappa = h_*\beta^{-1}, \qquad r(x) = \rho_\mathrm{l}(Lx)\rho_\mathrm{lm}^{-1}. \tag{4.10}$$

It is assumed that κ, τ, $f_1(x)$, $k(x)$, $r(x) \sim 1$ as $h_* \to 0$. Equation (4.9) is the singular perturbed differential equation with variable coefficients. In common case, it does not admit a solution in the explicit form. However, from all variety of eigenforms, one can construct an asymptotic solution of a high variability and satisfying the condition $\mathrm{d}X/\mathrm{d}x \sim h_*^{-1/2}$ at $h_* \to 0$.

We apply the Wentzel-Kramers-Brillouin method (WKB-method) and seek a solution in the form of series

$$X = \sum_{j=0}^{\infty} h_*^{j/2} X_j(x) \exp\left\{h_*^{-1/2} \int g(x)(d)x\right\},$$
$$\lambda = \lambda_0 + h_*\lambda_1 + \dots, \tag{4.11}$$

where X_j, $g(x)$ are infinitely differentiable functions of $x \in [0,1]$. It should be noted that a similar asymptotic approach has been applied by Firouz-Abadi et al (2007) to study free vibrations of an isotropic single layer Euler-Bernoulli beam of variable-cross-section with and without axial forces. They gave a compact third-order WKB-approximation for the mode shapes and found the corresponding natural frequencies.

Let us substitute (4.11) into Eq. (4.9) and equate coefficients at the same powers of $h_*^{1/2}$. Then we arrive at the series of equations. In the zeroth-order approximation (at $j = 0$), one has

$$\mathcal{F}(g, x)X_0 = 0. \tag{4.12}$$

where

$$\mathcal{F}(g, x) \equiv g^4 - f_1(x)g^2(1 - \kappa g^2) + k(x)(1 - \kappa g^2) - \lambda_0 r(x)(1 - \kappa g^2). \tag{4.13}$$

We will find the natural frequencies satisfying the inequality

$$\lambda_0 r(x) > k(x) \tag{4.14}$$

for any $x \in [0,1]$. Then, resolving the equation $\mathcal{F}(g, x) = 0$ with respect to g, one obtains

$$g_{1,2} = \pm \mathrm{i}\varphi_1(x), \quad g_{3,4} = \pm\varphi_2(x), \tag{4.15}$$

$$\varphi_1(x) = \sqrt{\frac{\kappa\lambda_0 r - \kappa k - f_1 + \sqrt{(\kappa k - f_1 - \kappa\lambda_0 r)^2 + 4(\lambda_0 r - k)}}{2(1 + \kappa f_1)}},$$
$$\varphi_2(x) = \sqrt{\frac{-(\kappa\lambda_0 r - \kappa k - f_1) + \sqrt{(\kappa k - f_1 - \kappa\lambda_0 r)^2 + 4(\lambda_0 r - k)}}{2(1 + \kappa f_1)}}, \tag{4.16}$$

where $\varphi_1(x), \varphi_2(x) > 0$ for any $x \in [0,1]$.

In the first-order approximation ($j = 1$), we get the following equation

$$\mathcal{F}(g, x)X_1 + \mathcal{G}[g(x), x]X_0' + \left[\frac{1}{2}\mathcal{G}' + \kappa k' g - \kappa\lambda_0 r' g\right] X_0 = 0, \tag{4.17}$$

where the prime means the differentiation by x, and

$$\mathcal{G}[g(x), x] = \frac{\partial \mathcal{F}(g, x)}{\partial g}. \tag{4.18}$$

Owing to (4.15) and (4.16), $\mathcal{F}[g_i(x), x)] \equiv 0$ and Eq. (4.17) results in the differential equation by X_0 which has the following general solution

$$X_0 = \frac{c}{\sqrt{|\mathcal{G}[g(x), (x)]|}} \exp\left[\kappa \int g(\lambda_0 r' - k')\mathrm{d}x\right] \tag{4.19}$$

with an arbitrary constant c.

Considering the higher-order approximations ($j \geq 2$), one can get a sequence of differential equations by X_{j-1} with parameters λ_{j-1}. Let us interrupt this process and consider only the first two approximations. Taking into account (4.15) and (4.16), the general solution of the differential equation (4.9) may be written as follows:

$$\begin{aligned} X_0 = &\frac{c_1}{\sqrt{|\mathcal{G}_1(x)|}} \left\{ \cos\left[h_*^{-1/2} \int_0^x \varphi_1(x)\mathrm{d}x + I_1(x) \right] + O\left(h_*^{1/2}\right) \right\} \\ &+ \frac{c_2}{\sqrt{|\mathcal{G}_1(x)|}} \left\{ \sin\left[h_*^{-1/2} \int_0^x \varphi_1(x)\mathrm{d}x + I_1(x) \right] + O\left(h_*^{1/2}\right) \right\} \\ &+ \frac{c_3}{\sqrt{|\mathcal{G}_2(x)|}} \left\{ \exp\left[-h_*^{-1/2} \int_0^x \varphi_2(x)\mathrm{d}x - I_2(x) \right] + O\left(h_*^{1/2}\right) \right\} \\ &+ \frac{c_4}{\sqrt{|\mathcal{G}_2(x)|}} \left\{ \exp\left[h_*^{-1/2} \int_1^x \varphi_2(x)\mathrm{d}x + I_2(x) \right] + O\left(h_*^{1/2}\right) \right\}, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} I_1(x) &= \kappa \int \varphi_1(x)[\lambda_0 r'(x) - k'(x)]\mathrm{d}x, \\ I_2(x) &= \kappa \int \varphi_2(x)[\lambda_0 r'(x) - k'(x)]\mathrm{d}x, \\ \mathcal{G}_1(x) &= \mathcal{G}[\varphi_1(x), x], \quad \mathcal{G}_2(x) = \mathcal{G}[\varphi_2(x), x], \end{aligned} \tag{4.21}$$

and c_i are constants which are found from the boundary conditions.

We assume here the following restrictions

$$\mathcal{G}_1(x) \neq 0, \quad \mathcal{G}_2(x) \neq 0 \tag{4.22}$$

for any $x \in [0, 1]$. The point $x^* \in [0, 1]$, for which $\mathcal{G}_1(x^*) = 0$ or $\mathcal{G}_1(x^*) = 0$, is generally called the turning point. The general solution (4.20) is the superposition of the integrals describing the basic dynamical stress state of the beam. It is interesting to note that the index of variation (see the definition given by Eq. (2.66)) of these basic integrals is equal to $\iota_1 = 1/2$ which coincide with the index of variation for the simple edge effect introduced above in Subsect. 2.1.13 for a shell. However, the integrals composing (4.20) do not depend on the parameter τ which appears at the highest derivative in Eq. (4.9). In other words, the general solution (4.20) defines the basic dynamic stress state of a high variability and does not take into account the special edge effects with the index of variation $\iota = (1+\varsigma)/2 > \iota_1 = 1/2$, where $1/2 < \varsigma < 1$. The omitted integrals define shears in a vicinity of the edges and may be incorporated in the general solution by considering the special edge effect equation

$$-h_*^{1+\varsigma}\,\tau\frac{\mathrm{d}^6X}{\mathrm{d}x^6} + \frac{\mathrm{d}^4X}{\mathrm{d}x^4} = 0 \tag{4.23}$$

and, afterwards, constructing the higher-order approximation at $j = 2$. The edge effect equation (4.23) gives two additional integrals,

$$X_5 = c_5 \exp\left[-h_*^{-\frac{1+\varsigma}{2}}\frac{x}{\sqrt{\tau}}\right], \quad X_6 = c_6 \exp\left[-h_*^{-\frac{1+\varsigma}{2}}\frac{1-x}{\sqrt{\tau}}\right]. \tag{4.24}$$

As seen from (4.10), the behavior of the shear edge effect integrals depends on the correlation between the shear parameters β, θ and the beam dimensions h, L.

In what follows, we disregard corrections due to the shear edge effect integrals and have to choose the basic boundary conditions corresponding to the basic stress state. As an example, we will consider the boundary conditions of the rigid clamping group (3.28) and (3.29). For this group, the basic boundary conditions are the following:

$$X_0' = 0, \quad X_0 - h_*\kappa X_0'' = 0 \quad \text{at} \quad x = 0,\, 1. \tag{4.25}$$

The substitution of the general solution (4.20) into (4.25) results in the homogeneous system of algebraic equations with respect to constants c_i $(i = 1, ..., 4)$:

$$\mathbf{A}\mathbf{C}^{\mathrm{T}} = 0, \tag{4.26}$$

where $\mathbf{C} = (c_1, c_2, c_3, c_4)$ is the three-dimensional vector, and $\mathbf{A}$ is the 4×4 - matrix with the elements

$$a_{11} = \frac{1+\kappa\varphi_1^2(0)}{\sqrt{|\mathcal{G}_1(0)|}}\cos[I_1(0)], \qquad a_{12} = \frac{1+\kappa\varphi_1^2(0)}{\sqrt{|\mathcal{G}_1(0)|}}\sin[I_1(0)],$$

$$a_{13} = \frac{1-\kappa\varphi_2^2(0)}{\sqrt{|\mathcal{G}_2(0)|}}\exp\left[-I_2(0)\right], \qquad a_{14} = 0,$$

$$a_{21} = -\frac{\varphi_1(0)}{\sqrt{|\mathcal{G}_1(0)|}}\sin[I_1(0)], \qquad a_{22} = \frac{\varphi_1(0)}{\sqrt{|\mathcal{G}_1(0)|}}\cos[I_1(0)],$$

$$a_{23} = -\frac{\varphi_2(0)}{\sqrt{|\mathcal{G}_2(0)|}}\exp\left[-I_2(0)\right], \qquad a_{24} = 0,$$

$$
\begin{aligned}
a_{31} &= \frac{1+\kappa\varphi_1^2(1)}{\sqrt{|\mathcal{G}_1(1)|}}\cos[\Theta_1(1)], & a_{32} &= \frac{1+\kappa\varphi_1^2(0)}{\sqrt{|\mathcal{G}_1(0)|}}\sin[\Theta_1(0)],\\
a_{34} &= 0, & a_{34} &= \frac{1-\kappa\varphi_2^2(1)}{\sqrt{|\mathcal{G}_2(1)|}}\exp\,[\Theta_2(1)],\\
a_{41} &= -\frac{\varphi_1(1)}{\sqrt{|\mathcal{G}_1(1)|}}\sin[\Theta_1(1)], & a_{42} &= \frac{\varphi_1(1)}{\sqrt{|\mathcal{G}_1(1)|}}\cos[\Theta_1(0)],\\
a_{43} &= 0, & a_{44} &= \frac{\varphi_2(1)}{\sqrt{|\mathcal{G}_2(1)|}}\exp\,[I_2(1)],
\end{aligned}
\tag{4.27}
$$

depending on the eigenvalue λ_0. In Eqs. (4.27)

$$
\Theta_1(x) = \frac{1}{h_*^{1/2}}\int_0^x \varphi_1(x)\mathrm{d}x + I_1(x). \tag{4.28}
$$

The transcendental equation $\det \mathbf{A} = 0$ serves for determining the series of unknown eigenvalues $\lambda_0 = \lambda_0^{(n)}$, $n = 1, 2, \ldots$.

Consider the particular case when the beam and foundation are uniform, and the axial stress resultant is a function of α_1. Then $r = 1, k$ are constants, $f_1 = f_1(x)$, and $I_1 = I_2 = 0$ for any $x \in [0, 1]$. For this case the equation $\det \mathbf{A} = 0$ is reduced to the following

$$
\tan\left\{h_*^{-1/2}\int_0^1 \varphi_1(x)\mathrm{d}x\right\} = \frac{\delta_{20}\delta_{11}\varphi_{10}\varphi_{21} + \delta_{10}\delta_{21}\varphi_{20}\varphi_{11}}{\delta_{10}\delta_{11}\varphi_{20}\varphi_{21} - \delta_{20}\delta_{21}\varphi_{10}\varphi_{11}}, \tag{4.29}
$$

where

$$
\begin{aligned}
&\delta_{10} = 1+\kappa\varphi_1^2(0), \quad \delta_{11} = 1+\kappa\varphi_1^2(1),\\
&\delta_{20} = 1-\kappa\varphi_2^2(0), \quad \delta_{21} = 1-\kappa\varphi_2^2(1),\\
&\varphi_{10} = \varphi_1(0), \quad \varphi_{11} = \varphi_1(1), \quad \varphi_{20} = \varphi_2(0), \quad \varphi_{21} = \varphi_2(1),
\end{aligned}
\tag{4.30}
$$

and the functions $\varphi_i(x)$ are specified by (4.16). When deriving Eq. (4.29), we have allowed for the following limiting correlations

$$
\begin{aligned}
\lim_{h_*\to 0} h_*^{-j/2}\exp\left\{-h_*^{-1/2}\int_0^1 \varphi_2(x)\mathrm{d}x\right\} &= 0,\\
\lim_{h_*\to 0} h_*^{-j/2}\exp\left\{h_*^{-1/2}\int_1^0 \varphi_2(x)\mathrm{d}x\right\} &= 0
\end{aligned}
\tag{4.31}
$$

valid for any integer $j = 0, 1, \ldots$.

Constants c_i are defined as follows

$$c_2 = -\frac{\delta_{10}\varphi_{20}}{\delta_{20}\varphi_{10}}\, c_1, \quad c_3 = -\sqrt{\left|\frac{\mathcal{G}_2(0)}{\mathcal{G}_1(0)}\right|}\,\frac{\delta_{10}}{\delta_{20}} c_1,$$

$$c_4 = \sqrt{\left|\frac{\mathcal{G}_2(1)}{\mathcal{G}_1(1)}\right|}\,\frac{\varphi_{11}}{\varphi_{21}} \left\{ \sin\left[\frac{1}{h_*^{1/2}} \int\limits_0^1 \varphi_1(x)\mathrm{d}x\right] + \frac{\delta_{10}\varphi_{20}}{\delta_{20}\varphi_{10}} \cos\left[\frac{1}{h_*^{1/2}} \int\limits_0^1 \varphi_1(x)\mathrm{d}x\right]\right\} c_1. \tag{4.32}$$

To analyse the effect of the shear parameter κ and variable axial force on the natural frequencies we will present an example.

Example 4.1. Let $f_1 = 1 + \epsilon x$ be the linear function of x, where $\epsilon > -1$. It is seen from Eqs. (4.16) that $f_1 < \kappa(\lambda_0 - k)$. We remind that eigenvalues defined by Eq. (4.29) have to satisfy the inequality, s. Eq. (4.14),

$$\lambda_0^{(n)} > k, \quad n = 1, 2, \ldots \tag{4.33}$$

Then the first natural frequency $\omega = \sqrt{\lambda_0^{(1)}}\, t_c^{-1}$ with $\lambda_0^{(1)}$ satisfying (4.33) might be higher than one or several the lowest eigenfrequencies. Table 4.1 displays the first five eigenvalues $\lambda_0^{(n)}$ satisfying (4.33) versus the shear parameter κ for $k = 1$, $\epsilon = 1, h_* = 0.01$. One can see that the influence of the shear parameter κ on the first eigenvalue $\lambda_0^{(1)}$ is weak, but it increases together with the number n. The series of eigenvalues $\lambda_0^{(n)}$ for $n = 1, 2, \ldots, 5$ and different values of a parameter ϵ is shown in Table 4.2. The calculations were performed at $\kappa = 1, k = 1, h_* = 0.01$. As seen that for any fixed number n each eigenfrequency $\lambda_0^{(n)}$ growths together with a parameter ε characterizing the rate of inhomogeneity of the axial force, this frequency increment being greater for a large number n.

It is well known that growing the compressive pre-buckling axial force leads to very quick decreasing the lowest eigenfrequency. Thus, one may conclude that the first

Table 4.1 Eigenvalues $\lambda_0^{(n)}$ vs. shear parameter κ.

κ	0.0	0.5	1.0	2.0
$\lambda_0^{(1)}$	1.225	1.198	1.186	1.173
$\lambda_0^{(2)}$	2.052	1.851	1.771	1.701
$\lambda_0^{(3)}$	3.928	3.085	2.825	2.621
$\lambda_0^{(4)}$	7.552	5.004	4.390	3.944
$\lambda_0^{(5)}$	13.858	7.662	6.477	5.668

Table 4.2 Series of eigenvalues $\lambda_0^{(n)}$ vs. parameter ϵ.

n	1	2	3	4	5
$\epsilon = 1$					
$\lambda_0^{(n)}$	1.186	1.771	2.825	4.390	6.477
$\epsilon = 2$					
$\lambda_0^{(n)}$	1.231	1.945	3.214	5.082	7.560
$\epsilon = 3$					
$\lambda_0^{(n)}$	1.275	2.110	3.582	5.739	8.587

eigenvalue $\lambda_0^{(1)}$ defined by our asymptotic procedure may do not equal the lowest natural frequency for the axially compressed laminated beam.

4.2 Laminated Plates

Consider a laminated rectangular plate with thickness h and sides $0 \leq \alpha_1 \leq L_1$ and $0 \leq \alpha_2 \leq L_2$. The plate is pre-stressed by the shear forces yielding in-plane stresses $T_{11}^\circ, T_{22}^\circ, T_{12}^\circ$. The governing equations for free vibrations of a pre-stressed plate resting on an elastic foundation may be easily obtained from Eqs. (3.23) by introducing additional terms accounting the inertia forces and response of an elastic foundation

$$D\left(1 - \frac{\theta h^2}{\beta}\Delta\right)\Delta^2\chi - \left(\Delta_T - c_\mathrm{f} - \rho_0 h\frac{\partial^2}{\partial t^2}\right)\left(1 - \frac{h^2}{\beta}\Delta\right)\chi = 0, \quad (4.34)$$

were c_f is the spring constant for the elastic foundation and

$$\Delta_T = T_{11}^\circ\frac{\partial^2}{\partial\alpha_1^2} + 2T_{12}^\circ\frac{\partial^2}{\partial\alpha_1\partial\alpha_2} + T_{22}^\circ\frac{\partial^2}{\partial\alpha_2^2}. \quad (4.35)$$

The above equations should be supplemented by the equation

$$\frac{1-\nu}{2}\frac{h^2}{\beta}\Delta\phi = \phi, \quad (4.36)$$

for the shear function ϕ and the boundary conditions as well. We will consider here only the simple support group including the boundary conditions (2.111) or (2.113).

For the first variant of the boundary conditions (when all the edges have a diaphragm inhibiting relative shear)

$$\chi = \Delta\chi = \Delta^2\chi = \frac{\partial\phi}{\partial\alpha_i} = 0 \quad \text{at} \quad \alpha_i = 0, L_i, \quad i = 1, 2, \quad (4.37)$$

one can set $\phi = 0$. For the second variant (diaphragm is absent at least on the one edge $\alpha_1 = 0$)

$$\left(1 - \frac{h^2}{\beta}\Delta\right)\chi = 0, \quad \frac{\partial^2}{\partial\alpha_1^2}\left(1 - \frac{h^2}{\beta}\Delta\right)\chi = 0,$$
$$\left(\frac{\partial^2}{\partial\alpha_1^2} + \nu\frac{\partial^2}{\partial\alpha_2^2}\right)\chi - (1-\nu)\frac{\partial^2\phi}{\partial\alpha_1\alpha_2} = 0,$$
$$2\frac{\partial^2\chi}{\partial\alpha_1\partial\alpha_2} + \frac{\partial^2\phi}{\partial\alpha_1^2} - \frac{\partial^2\phi}{\partial\alpha_2^2} = 0 \quad \text{at} \quad \alpha_1 = 0 \tag{4.38}$$

the function ϕ turns out to be coupled to the displacement function χ and should be taken into account when constructing the edge effects.

4.2.1 Simply Supported Plate with Diaphragm on Edges

At first, we will consider variant (4.37) of the boundary conditions. Let all coefficients in Eq. (4.34) be constants, and the shear stress resultant T_{12}° is equal to zero. Then the solution of the linear boundary-value problem (4.34), (4.37) is easily found as

$$\chi = \chi_0 \mathrm{e}^{\mathrm{i}\omega t} \sin\frac{\pi n\alpha_1}{L_1} \sin\frac{\pi m\alpha_2}{L_2}, \tag{4.39}$$

where n, m are numbers of semi-waves in the α_1- and α_2-directions, respectively, ω is the natural frequency, and χ_0 is a constant. The substitution of (4.39) into Eq. (4.34) leads to the following formula for the frequency

$$\omega^2 = \frac{\pi^4 D}{\rho_0 h L_2^4}\Lambda, \tag{4.40}$$

where

$$\Lambda = \frac{\delta_{nm}^2(1 + \theta K\delta_{nm})}{1 + K\delta_{nm}} + t_1^{\circ}e^2n^2 + t_2^{\circ}m^2 + k_{\mathrm{f}},$$
$$K = \frac{\pi^2h^2}{\beta L_2^2}, \quad \delta_{nm} = e^2n^2 + m^2, \quad e = \frac{L_2}{L_1}, \quad t_i^{\circ} = \frac{L_2^2}{\pi^2 D}T_{ii}^{\circ} \tag{4.41}$$

The equation for k_{f} depends on the accepted model for the elastic foundation. For the Winkler foundation model can be assumed

$$k_{\mathrm{f}} = \frac{L_2^4}{\pi^4 D}c_{\mathrm{f}} \tag{4.42}$$

and for the model represented by Eq. (2.137) one has

$$k_{\mathrm{f}} = \frac{L_2^3 \alpha_{\mathrm{f}}}{\pi^3 D}\, \delta_{nm}^{1/2}, \tag{4.43}$$

where α_{f} is defined by (4.2).

It is obvious that for large numbers n, m, the Winkler model gives understated natural frequencies when comparing to the model represented by Eq. (2.152). It is also seen that the tensile initial stresses ($T_{ii}^{\circ} > 0$) raise eigenfrequencies and the compressive ones ($T_{ii}^{\circ} < 0$) reduce them. In the last case, the magnitudes $|T_{ii}^{\circ}|$ are to be less of the critical buckling values (s. Chapt. 3). Assuming the shear parameter K to be small, formula (4.41) may be rewritten in the following form

$$\Lambda = \delta_{nm}^2 \left[1 + \delta_{nm}^{-2}(t_1^{\circ} e^2 n^2 + t_2^{\circ} m^2 + k_{\mathrm{f}}) - K(1-\theta)\delta_{nm} + O\left(K^2\right)\right]. \tag{4.44}$$

It shows that ignoring shear results in overstating values for the natural frequencies.

4.2.2 Simply Supported Plate Without Diaphragm on Edges

Now, we consider the combination of the simple support conditions (4.37) and (4.38), herewith, the edges $\alpha_1 = 0, L_1$ (without diaphragm) satisfy conditions (4.38), and the edges $\alpha_2 = 0, L_2$ (with the diaphragm) to Eqs. (4.37). In this case, the boundary-value problem (4.34), (4.36)-(4.38) does not admit the explicit form of a solution. It may be found by using some numerical method. For instance, a solution may be represented by an infinite series of beam functions or by the sine- and cosine-series expansions in α_1 and α_2. But we, assuming the shear parameter K as a small one, will apply to the asymptotic approach and construct a solution for low-frequency vibrations in the form of the superposition of the main stress state and the edges effect integrals. This approach will permit us to obtain a simple asymptotic equation for eigenfrequencies and evaluate the effect of shear inside of the plate and in a neighbourhood of the edges as well.

Consider the case when $T_{ij}^{\circ} = c_{\mathrm{f}} = 0$. Let a parameter

$$\mu^2 = \frac{h^2}{\beta R^2}. \tag{4.45}$$

be small, where R is the characteristic size (one of the lengthes L_1, L_2 or $(L_1 L_2)^{1/2}$). The required functions satisfying (4.37) are south in the form:

$$\chi = R\, X(x_1)\, \sin\frac{\pi m x_2}{l_2}\, \mathrm{e}^{\mathrm{i}\omega t}, \qquad \phi = \mu^{v_1} R S(x_1)\, \cos\frac{\pi m x_2}{l_2}\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{4.46}$$

where $x_i = \alpha_i/R$, $l_i = L_i/R$, $v_1 > 0$, $S, X \sim 1$ at $\mu \to 0$, and ω is the natural frequency.

The substitution of (4.46) into (4.34) and (4.36) yields

$$
\begin{gathered}
-\theta\mu^2\left(\frac{\mathrm{d}^6X}{\mathrm{d}x_1^6}-3\delta_m^2\frac{\mathrm{d}^4X}{\mathrm{d}x_1^4}+3\delta_m^4\frac{\mathrm{d}^2X}{\mathrm{d}x_1^2}-\delta_m^6X\right)\\
+\frac{\mathrm{d}^4X}{\mathrm{d}x_1^4}-2\delta_m^2\frac{\mathrm{d}^2X}{\mathrm{d}x_1^2}+\delta_m^4X-\lambda X+\mu^2\lambda\left(\frac{\mathrm{d}^2X}{\mathrm{d}x_1^2}-\delta_m^2X\right)=0
\end{gathered}
\tag{4.47}
$$

and

$$
\frac{\mathrm{d}^2S}{\mathrm{d}x_1^2}=\left(\frac{1}{\mu^2}\frac{2}{1-\nu}+\delta_m^2\right)S. \tag{4.48}
$$

Here

$$
\delta_m=\frac{\pi m}{l_2},\quad \lambda=\frac{\omega^2}{\omega_c^2},\quad \omega_\mathrm{c}^2=\frac{D}{\rho_0 hR^4}, \tag{4.49}
$$

where ω_c is the characteristic frequency. The boundary conditions (4.38) for $X(x_1)$ and $S(x_1)$ on the edges $x_1=0,l_1$ become as follows

$$
X-\mu^2\left(\frac{\mathrm{d}^2X}{\mathrm{d}x_1^2}-\delta_m^2X\right)=0,\quad \frac{\mathrm{d}^2X}{\mathrm{d}x_1^2}-\mu^2\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left(\frac{\mathrm{d}^2X}{\mathrm{d}x_1^2}-\delta_m^2X\right)=0 \tag{4.50}
$$

$$
\frac{\mathrm{d}^2X}{\mathrm{d}x_1^2}-\nu\delta_m^2X+\mu^2(1-\nu)\delta_m\frac{\mathrm{d}S}{\mathrm{d}x_1}=0, \tag{4.51}
$$

$$
2\delta_m\frac{\mathrm{d}X}{\mathrm{d}x_1}+\mu^2\left(\frac{\mathrm{d}^2S}{\mathrm{d}x_1^2}+\delta_m^2S\right)=0. \tag{4.52}
$$

Although a parameter θ is small, we assume here that $\theta\sim 1$. Consider Eq. (4.48). It has the following general solution

$$
S(x_1)=c_1\mathrm{e}^{-\frac{1}{\mu}\gamma x_1}+c_2\mathrm{e}^{-\frac{1}{\mu}\gamma(l_1-x_1)}, \tag{4.53}
$$

where c_1,c_2 are constants, and

$$
\gamma=\sqrt{\frac{2}{1-\nu}+\mu^2\delta_m^2}. \tag{4.54}
$$

Function (4.53) is the superposition of the two integrals which specify the shear edge effects near the ends $x_1=0$ and $x_1=l_1$. But apart from these integrals there are another pair of the edge effect integrals which embrace more narrow regions near the plate edges. These integrals are defined from an additional equation which is easily derived from Eq. (4.47). Let $\mathrm{d}z/\mathrm{d}x_1\sim\mu^{-\iota}$, where $\iota>0$. The asymptotic analysis of all summands in Eq. (4.47) gives $\iota=1$, the basic terms leading to the following additional equation

$$
\theta\mu^2\frac{\mathrm{d}^6X}{\mathrm{d}x_1^6}-\frac{\mathrm{d}^4X}{\mathrm{d}x_1^4}=0. \tag{4.55}
$$

It is obvious that only two integrals of this equation have the properties of the edge effect integrals. Their superposition gives the following general solution

$$X^{(e)} = c_3 \mathrm{e}^{-\frac{1}{\mu\sqrt{\theta}} x_1} + c_4 \mathrm{e}^{-\frac{1}{\mu\sqrt{\theta}} (l_1 - x_1)}, \tag{4.56}$$

where c_3, c_4 are arbitrary constants. It is seen that due to the smallness of θ, function (4.56) decreases faster than integral (4.53).

We seek a solution of the boundary-value problem (4.47), (4.50)-(4.52) in the following form

$$X = X^{(m)}(x_1) + \mu^{\upsilon_2} X^{(e)}(x_1), \quad X^{(m)},\, X^{(e)} \sim 1, \tag{4.57}$$

$$\lambda = \lambda_0 + \mu\lambda_1 + \ldots, \tag{4.58}$$

where $X^{(m)}$ is also expanded into the series

$$X^{(m)} = X_0(x_1) + \mu X_1(x_1) + \ldots \tag{4.59}$$

with functions X_i satisfying the condition $X_i' \sim X_i$. Here and below, the prime $\{'\}$ means the differentiation with respect to x_1.

Let us substitute (4.57) into the boundary conditions (4.51), (4.52) and compare the main terms. Taking into account the estimates $X_i' \sim X_i$, $\left(X^{(e)}\right)' \sim \mu^{-1} X^{(e)}$, $S' \sim \mu^{-1} S$, one gets the indexes of intensity for the functions describing edge effects: $\upsilon_1 = 2$ and $\upsilon_2 = 3$. The substitution of (4.57) - (4.59) into Eq. (4.47) and the boundary conditions (4.50)-(4.52) results in the sequence of the boundary-value problems. Let us consider them step by step.

In the zeroth-order approximation, one has the homogeneous boundary-value problem

$$\mathbf{L}_0 X_0 \equiv \frac{\mathrm{d}^4 X_0}{\mathrm{d}x_1^4} - 2\delta_m^2 \frac{\mathrm{d}^2 X_0}{\mathrm{d}x_1^2} + \delta_m^4 X_0 - \lambda_0 X_0 = 0 \tag{4.60}$$

$$X_0(0) = X_0(l_1) = X_0''(0) = X_0''(l_1) = 0, \tag{4.61}$$

which has the following nontrivial solution

$$X_0 = A \sin \frac{\pi n x_1}{l_1}, \quad \lambda_0 = (\delta_n^2 + \delta_m^2)^2, \quad \delta_n = \frac{\pi n}{l_1}. \tag{4.62}$$

Note that the boundary conditions (4.61) were derived from (4.50).

Keeping in mind the edge integrals (4.53) and solution (4.62), the boundary conditions (4.52) in the zeroth-order approximation results in the following equations

$$2\delta_m X_0' + \frac{2}{1-\nu}\left[c_1 \mathrm{e}^{-\frac{1}{\mu}\sqrt{\frac{2}{1-\nu}} x_1} + c_2 \mathrm{e}^{-\frac{1}{\mu}\sqrt{\frac{2}{1-\nu}} (l_1 - x_1)}\right] = 0 \quad \text{at} \quad x_1 = 0,\, l_1 \tag{4.63}$$

which give the formulae for constants

$$c_1 = -(1-\nu)\delta_n \delta_m A, \quad c_2 = (-1)^{n+1}(1-\nu)\delta_n \delta_m A. \tag{4.64}$$

In the first-order approximation, one gets the nonhomogeneous differential equation

$$\mathbf{L}_0 X_1 = \lambda_1 X_0. \tag{4.65}$$

and the nonhomogeneous boundary conditions at $x_1 = 0,\ l_1$

$$X_1 = 0, \quad X_1'' - \frac{1}{\theta^2}\left[c_3 e^{-\frac{1}{\mu\sqrt{\theta}}x_1} + c_4 e^{-\frac{1}{\mu\sqrt{\theta}}(l_1 - x_1)}\right] = 0,$$

$$X_1'' - \nu\delta_m^2 X_1 + \frac{1}{\theta}\left[c_3 e^{-\frac{1}{\mu\sqrt{\theta}}x_1} + c_4 e^{-\frac{1}{\mu\sqrt{\theta}}(l_1 - x_1)}\right]$$

$$-(1-\nu)\delta_m \frac{2}{1-\nu}\left[c_1 e^{-\frac{1}{\mu}\sqrt{\frac{2}{1-\nu}}x_1} + c_2 e^{-\frac{1}{\mu}\sqrt{\frac{2}{1-\nu}}(l_1 - x_1)}\right] = 0. \tag{4.66}$$

Taking Eqs. (4.64) into account, the last two conditions (4.66) written at $x_1 = 0, l_1$ result in the equations for constants

$$c_3 = -\frac{\sqrt{2(1-\nu)^3}\theta^2\delta_n\delta_m^2 A}{1+\theta}, \quad c_4 = \frac{(-1)^n\sqrt{2(1-\nu)^3}\theta^2\delta_n\delta_m^2 A}{1+\theta}. \tag{4.67}$$

Then the first two equations from (4.66) give the nonhomogeneous boundary conditions for X_1

$$\begin{aligned} &X_1(0) = X_1(l_1) = 0, \\ &X_1''(0) = -\frac{\sqrt{2(1-\nu)^3}\,\delta_n\delta_m^2 A}{1+\theta}, \\ &X_1''(l_1) = \frac{(-1)^n\sqrt{2(1-\nu)^3}\,\delta_n\delta_m^2 A}{1+\theta}. \end{aligned} \tag{4.68}$$

Problem (4.65), (4.68) is the nonhomogeneous boundary-value problem *on spectrum*. The existence condition for a solution of this problem produces the following formula for the correction λ_1

$$\lambda_1 = -\frac{4\sqrt{2(1-\nu)^3}\,\delta_n^2\delta_m^2}{l_1(1+\theta)}. \tag{4.69}$$

Then the solution of the boundary-value problem (4.65), (4.68) will be the following

$$\begin{aligned} X_1(x_1) &= a_1 \sin\delta_n x_1 + a_2\cos\delta_n x_1 + a_3 e^{r_{mn}x_1} + a_4 e^{-r_{mn}x_1} \\ &+ \frac{\lambda_1 A}{4\delta_n(\delta_n^2 + \delta_m^2)}x_1 \cos\delta_1 x_1, \end{aligned} \tag{4.70}$$

where $r_{mn} = \sqrt{2\delta_m^2 + \delta_m^2}$, and constants a_i are determined from the boundary conditions (4.68).

Let the characteristic size R be equal L_2. Then, when breaking the procedure of seeking the functions X_i and parameters λ_i, the approximate equation for natural frequencies may be represented as

$$\omega^2 = \frac{D\pi^4}{\rho_0 h L_2^4}\Lambda, \quad \Lambda = \delta_{nm}^2 \left\{1 - \mu \frac{4\sqrt{2(1-\nu)^3}\delta_n^2\delta_m^2}{e(1+\theta)\pi^4\delta_{nm}^2} + O(\mu^2)\right\}, \tag{4.71}$$

where δ_{nm}, e are determined by Eqs. (4.41). We note that the small parameter is proportional to the shear one (s. Eqs. (4.41) and (4.45)): $\mu^2 = K/\pi^2$. Then the asymptotic formula for the dimensionless frequency parameter Λ may be rewritten as

$$\Lambda = \delta_{nm}^2 \left\{1 - K^{1/2}\frac{4\sqrt{2(1-\nu)^3}\ n^2m^2}{\pi e^3(1+\theta)\delta_{nm}^2} + O(K)\right\} \tag{4.72}$$

One can compare it with the analogous Eq. (4.44). In Eq. (4.72), the shear induced correction generated by the edge effects has the order $K^{1/2}$, whereas the similar correction for simply supported plates with diaphragm, s. Eq. (4.44), is a value of the order K. Thus, when comparing these two cases, one can conclude: if the plate edges are free of diaphragm, then the eigenmodes contain additional components accounting the edge shear and called the edge effect integrals, these integrals may give more lower eigenfrequencies than transverse shear within the plate.

4.3 Simplest Problems on Free Vibrations of Thin Cylindrical Shells

In this section we will consider the class of the simplest boundary-value problems describing free linear vibrations of elastic laminated cylindrical shells. In all problems, the geometrical and physical parameters of layers and a shell in whole are assumed to be constants so that any natural mode defines a system of waves distributed evenly over the shell surface. The objective is to study the influence of different boundary conditions and shear as well on the natural frequencies and corresponding eigenmodes.

Let us consider a thin laminated cylindrical shell composed of N transversally isotropic elastic layers. Studying free vibrations, we assume $q_i = q_\mathrm{n} = 0$ in the governing equations (2.61)-(2.63). For linear vibrations, the required functions may be represented in the form

$$\{\hat{u}_i, \psi_i, w\} = R\{U_i(\alpha_1,\alpha_2), \Psi_i(\alpha_1,\alpha_2), W(\alpha_1,\alpha_2)\}\exp(\mathrm{i}\omega t), \tag{4.73}$$

where $i = 1, 2$, ω is the natural frequency, and R is the characteristic dimension of the shell. We substitute (4.73) into Eqs. (2.61)-(2.63) and omit nonlinear terms. As a result, one obtains the following linear differential equations

$$
\begin{aligned}
&\frac{\partial^2 U_1}{\partial \alpha_1^2} + \frac{1-\nu}{2}\frac{\partial^2 U_1}{\partial \alpha_2^2} + \frac{1+\nu}{2}\frac{\partial^2 U_2}{\partial \alpha_1 \partial \alpha_2} + \nu k_{22}\frac{\partial W}{\partial \alpha_1} + \frac{\rho_0 \omega^2}{\tilde{E}} U_1 = 0,\\
&\frac{1+\nu}{2}\frac{\partial^2 U_1}{\partial \alpha_1 \partial \alpha_2} + \frac{1-\nu}{2}\frac{\partial^2 U_2}{\partial \alpha_1^2} + \frac{\partial^2 U_2}{\partial \alpha_2^2} + \frac{\partial (k_{22} W)}{\partial \alpha_2} + \frac{\rho_0 \omega^2}{\tilde{E}} U_2 = 0,\\
&\eta_2 \frac{\partial (\triangle W)}{\partial \alpha_1} - \eta_1 \left(\frac{\partial^2 \Psi_1}{\partial \alpha_1^2} + \frac{1+\nu}{2}\frac{\partial^2 \Psi_2}{\partial \alpha_1 \partial \alpha_2} + \frac{1-\nu}{2}\frac{\partial^2 \Psi_1}{\partial \alpha_2^2} \right) + \frac{12 q_{44}}{\tilde{E} h^3} \Psi_1 = 0,\\
&\eta_2 \frac{\partial (\triangle W)}{\partial \alpha_2} - \eta_1 \left(\frac{\partial^2 \Psi_2}{\partial \alpha_2^2} + \frac{1+\nu}{2}\frac{\partial^2 \Psi_1}{\partial \alpha_1 \partial \alpha_2} + \frac{1-\nu}{2}\frac{\partial^2 \Psi_2}{\partial \alpha_1^2} \right) + \frac{12 q_{44}}{\tilde{E} h^3} \Psi_2 = 0,\\
&\frac{h^2}{12} \triangle \left[\eta_3 \triangle W - \eta_2 \left(\frac{\partial \Psi_1}{\partial \alpha_1} + \frac{\partial \Psi_2}{\partial \alpha_2} \right) \right]\\
&\quad + k_{22} \left(\nu \frac{\partial U_1}{\partial \alpha_1} + \frac{\partial U_2}{\partial \alpha_2} + k_{22} W \right) - \frac{\rho_0 \omega^2}{\tilde{E}} W = 0
\end{aligned}
\tag{4.74}
$$

with $\tilde{E} = E/(1-\nu^2)$. The system of differential equations (4.74) may be used to study free vibrations of a shell of any length for any number of waves in the axial and circumferential directions. However, they turn out to be too inconvenient and cumbersome in the common case. The selection of governing equations depends on the class of problems under consideration. So, the above equations (4.74) may be used for studying free vibrations of a very long cylindrical shell with formation of long waves. However, to analyze vibrations with a large number of minor waves although in the one direction, it is more convenient to apply to the simplified equations of the technical shell theory (2.77), (2.85), (2.87).

Let us now apply to the variant of the technical shell theory. Assuming

$$
\chi = \tilde{\chi}(\alpha_1, \alpha_2) \mathrm{e}^{\mathrm{i}\omega t}, \quad F = \tilde{F}(\alpha_1, \alpha_2) \mathrm{e}^{\mathrm{i}\omega t}, \quad \phi = \tilde{\phi}(\alpha_1, \alpha_2) \mathrm{e}^{\mathrm{i}\omega t}, \tag{4.75}
$$

Eqs. (2.77), (2.85), (2.87) are reduced to the following ones

$$
\begin{aligned}
&D \left(1 - \frac{\theta h^2}{\beta} \triangle \right) \triangle^2 \tilde{\chi} + k_{22} \frac{\partial^2 \tilde{F}}{\partial \alpha_1^2} - \rho_0 h \omega^2 \left(1 - \frac{h^2}{\beta} \triangle \right) \tilde{\chi} = 0,\\
&\triangle^2 \tilde{F} - E h k_{22} \frac{\partial^2}{\partial \alpha_1^2} \left(1 - \frac{h^2}{\beta} \triangle \right) \tilde{\chi} = 0, \qquad \frac{1-\nu}{2}\frac{h^2}{\beta} \triangle \tilde{\phi} = \tilde{\phi}.
\end{aligned}
\tag{4.76}
$$

The systems of differential equations (4.74) and (4.76) should be supplemented by the boundary conditions (2.93)-(2.108) and (2.110)-(2.118), respectively. The classification of integrals for governing equations analogous to (4.74) as well as their detailed analysis for thin isotropic single-layer shells may be found in Gol'denveizer et al (1979); Mikhasev and Tovstik (2009).

4.3.1 Long Simply Supported Cylinder with Diaphragm on Edges

Let a lengthy cylindrical shell be circular, then $k_{22} = 1/R$ is a constant. From all variants of the boundary conditions, we consider here the simply supported edges with diaphragm. In terms of displacements and stress resultants these conditions are the following (s. Chapt. 2)

$$w = \hat{u}_2 = \psi_2 = \hat{M}_{11} = T_{11} = \hat{L}_{11} = 0 \quad \text{at} \quad \alpha_1 = 0, L. \tag{4.77}$$

Keeping in mind (4.73), we rewrite them in the terms of displacements

$$\begin{aligned} W = U_2 = \Psi_2 = 0,& \\ \eta_3\left(\frac{\partial^2 W}{\partial \alpha_1^2} + \nu\frac{\partial^2 W}{\partial \alpha_2^2}\right) - \eta_2\left(\frac{\partial \Psi_1}{\partial \alpha_1} + \nu\frac{\partial \Psi_2}{\partial \alpha_2}\right) = 0,& \\ \frac{\partial U_1}{\partial \alpha_1} + \nu\frac{\partial U_2}{\partial \alpha_2} + \frac{\nu W}{R} = 0,& \\ \eta_2\left(\frac{\partial^2 W}{\partial \alpha_1^2} + \nu\frac{\partial^2 W}{\partial \alpha_2^2}\right) - \eta_1\left(\frac{\partial \Psi_1}{\partial \alpha_1} + \nu\frac{\partial \Psi_2}{\partial \alpha_2}\right) = 0 \quad \text{at} \quad \alpha_1 = 0, L.& \end{aligned} \tag{4.78}$$

As seen, the above boundary conditions are satisfied by the following functions

$$\begin{aligned} U_1 &= U_1^\circ \cos\frac{\pi n \alpha_1}{L}\cos\frac{m\alpha_2}{R}, \\ U_2 &= U_2^\circ \sin\frac{\pi n \alpha_1}{L}\sin\frac{m\alpha_2}{R}, \\ W &= W^\circ \sin\frac{\pi n \alpha_1}{L}\cos\frac{m\alpha_2}{R}, \\ \Psi_1 &= \Psi_1^\circ \cos\frac{\pi n \alpha_1}{L}\cos\frac{m\alpha_2}{R}, \\ \Psi_2 &= \Psi_2^\circ \sin\frac{\pi n \alpha_1}{L}\sin\frac{m\alpha_2}{R}, \end{aligned} \tag{4.79}$$

where n is a number of semi-waves in the axial direction, m is a number of waves in the circumferential direction, and $U_i^\circ, W^\circ, \Psi_i^\circ$ are constant values.

The substitution of (4.79) into Eqs. (4.74) yields the system of algebraic equations

$$\mathbf{A}\mathbf{X}^{\mathrm{T}} = 0, \tag{4.80}$$

where $\mathbf{X} = (U_1^\circ, U_2^\circ, W^\circ, \Psi_1^\circ, \Psi_2^\circ)$ is the vector, and $\mathbf{A}$ is the 5×5 matrix with the elements a_{ij}

$$a_{11} = -\delta_n^2 - \frac{1-\nu}{2}m^2 + (1-\nu^2)\frac{\omega^2}{\omega_0^2}, \qquad a_{12} = \frac{1+\nu}{2}\delta_n m,$$

$$a_{13} = \nu\delta_n, \quad a_{14} = a_{15} = 0, \quad a_{21} = \frac{1+\nu}{2}\delta_n m,$$

$$a_{22} = -\frac{1-\nu}{2}\delta_n^2 - m^2 + (1-\nu^2)\frac{\omega^2}{\omega_0^2}, \quad a_{23} = -m, \quad a_{24} = a_{25} = 0,$$
$$a_{31} = a_{32} = 0, \quad a_{33} = -\eta_2\delta_n(\delta_n^2 + m^2),$$
$$a_{34} = \eta_1\left(\delta_n^2 + \frac{1-\nu}{2}m^2\right) + \frac{q_{44}R^2\eta_3}{D},$$
$$a_{35} = -\frac{\eta_1(1+\nu)}{2}\delta_n m, \quad a_{41} = a_{42} = 0, \quad a_{43} = -\eta_2 m(\delta_n^2 + m^2),$$

$$a_{44} = -\frac{\eta_1(1+\nu)}{2}\delta_n m, \quad a_{45} = \eta_1\left(m^2 + \frac{1-\nu}{2}\delta_n^2\right) + \frac{q_{44}R^2\eta_3}{D},$$
$$a_{51} = -\frac{\nu}{1-\nu^2}\delta_n, \quad a_{52} = \frac{m}{1-\nu^2},$$
$$a_{53} = \varepsilon^8(\delta_n^2 + m^2)^2 + \frac{1}{1-\nu^2} - \frac{\omega^2}{\omega_0^2},$$
$$a_{54} = -\frac{\varepsilon^8\eta_2\delta_n}{\eta_3}(\delta_n^2 + m^2), \quad a_{55} = \frac{\varepsilon^8\eta_2 m}{\eta_3}(\delta_n^2 + m^2), \tag{4.81}$$

where

$$\delta_n = \frac{\pi n}{l}, \quad l = \frac{L}{R}, \quad \varepsilon^8 = \frac{h^2\eta_3}{12(1-\nu^2)R^2}, \quad \omega_0^2 = \frac{E}{\rho_0 R^2}. \tag{4.82}$$

Here, ε is a small parameter and ω_0 is the characteristic frequency.

The equation

$$\det \mathbf{A} = 0 \tag{4.83}$$

serves as the existence condition of a nontrivial solution of the homogeneous system (4.80). In the general case, it is the cubic equation with respect to the required frequency parameter $\Lambda = (1-\nu^2)\omega^2\omega_0^{-2}$. It will be used below in Chapt. 5 to study free vibrations of viscoelastic laminated shells containing MRE. As a particular case, we consider the axisymmetric vibrations for which $m = U_2^\circ = \Psi_2^\circ = 0$. Then, the cubic equation (4.83) degenerates into the quadratic one:

$$\Lambda^2 - \left(1 + \delta_n^2 + \mu_1\delta_n^4 r_n\right)\Lambda + \delta_n^2\left(1 - \nu^2 + \mu_1\delta_n^4 r_n\right) = 0, \tag{4.84}$$

where

$$\mu_1 = (1-\nu^2)\varepsilon^8, \quad r_n = \frac{\pi^2 + \theta K\delta_n^2}{\pi^2 + K\delta_n^2}, \quad K = \frac{\pi^2 h^2}{\beta R^2}, \quad \theta = 1 - \frac{\eta_2^2}{\eta_1\eta_3}. \tag{4.85}$$

For any fixed number n, there are two the positive roots

$$\Lambda = \Lambda_j = \frac{1}{2}\left\{1 + \delta_n^2 + \mu_1\delta_n^4 r_n - (-1)^j\left[(1 - \delta_n^2 + \mu_1\delta_n^4 r_n)^2 + 4\nu^2\delta_n^2\right]^{1/2}\right\}, \tag{4.86}$$

where $j = 1, 2$. Then the natural frequencies corresponding to the axially symmetric longitudinal and bending vibrations accounting transverse shear are defined as

$$\omega_j = \sqrt{\frac{E\Lambda_j}{\rho_0 R^2(1-\nu^2)}},$$

where ω_1 is the eigenfrequency of predominantly longitudinal vibrations, and ω_2 relates to bending vibrations. It is obviously, for the fixed n, $\omega_1 > \omega_2$.

The amplitudes of axial, normal and shear displacements are coupled by means of equations

$$U_1^\circ = -\frac{\nu\delta_n}{\Lambda - \delta_n^2}W^\circ, \quad \Psi_1^\circ = \frac{\eta_2 K\delta_n^3}{\eta_1(\pi^2 + K\delta_n^2)}W^\circ. \tag{4.87}$$

As seen from Eq. (4.86), $\Lambda_j - \delta_n^2 \neq 0$ for any n. When $K \to 0$, Eq. (4.86) gives the frequency parameter for an isotropic shell without taking into account shears. Because a parameter θ is small, it may be concluded that the incorporation of the shear parameter K into the shell model results in the reduction of the natural frequencies for any δ_n, the influence of the shear parameter K on eigenfrequencies being very weak for modes with small parameter δ_n and becoming essential at large δ_n and, particularly, for modes of bending vibrations with very large number of waves n in the axial direction (and/or for a very short cylindrical shell). This conclusion is confirmed by calculations performed at $m = 0, \nu = 0.4, \varepsilon = 0.2$. Figure 4.1 shows the parameters Λ_1 and Λ_2 corresponding to the axially symmetric longitudinal and bending vibrations, respectively, versus a wave parameter δ_n. Figure 4.2 demonstrates the behavior of the frequency parameter Λ_2 corresponding the bending modes as the function of δ_n for different values of K varying from 0 to 0.6. It is seen, the larger value of δ_n is, the higher effect of the shear parameter on eigenfrequencies of flexural vibrations becomes. Similar computations of the parameter Λ_1 corresponding to the longitudinal modes show that this effect is negligibly small. For instance, curves Λ_1 versus δ_n presented in Fig. 4.1 practically merge in the range of variation of δ_n form 0 to 40.

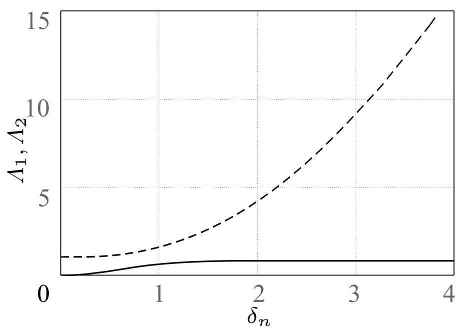
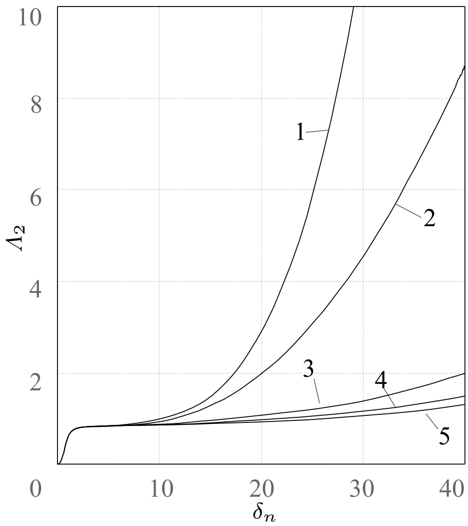


Fig. 4.1 Frequency parameters Λ_1 (dotted line) and Λ_2 (solid line) vs. parameter δ_n at $K = 0$.

Fig. 4.2 Frequency parameter Λ_2 vs. δ_n at different values of K: 1 - $K = 0$, 2 - $K = 0.02$, 3 - $K = 0.2$, 4 - $K = 0.4$, 5 - $K = 0.6$.



4.3.2 Medium-length Cylindrical Shells with Simply Supported Edges

In this subsection, we consider a medium-length cylindrical shell with simply-supported edges with and without diaphragm. The boundary conditions written in terms of the displacement and stress functions are the following:

- for the edges $\alpha_1 = 0, \alpha_1 = L$ with diaphragm (SSD boundary conditions)

$$\tilde{\chi} = \Delta\tilde{\chi} = \Delta^2\tilde{\chi} = \frac{\partial\tilde{\phi}}{\partial\alpha_1} = 0, \quad \frac{\partial^2\tilde{F}}{\partial\alpha_2^2} = 0, \quad \frac{\partial^2\tilde{F}}{\partial\alpha_1^2} = 0, \tag{4.88}$$

- for the edges without diaphragm (SSF boundary conditions)

$$\begin{gathered}\left(1 - \frac{h^2}{\beta}\Delta\right)\tilde{\chi} = 0, \quad \frac{\partial^2}{\partial\alpha_1^2}\left(1 - \frac{h^2}{\beta}\Delta\right)\tilde{\chi} = 0, \\ \left(\frac{\partial^2}{\partial\alpha_1^2} + \nu\frac{\partial^2}{\partial\alpha_2^2}\right)\tilde{\chi} - (1-\nu)\frac{\partial^2\tilde{\phi}}{\partial\alpha_1\alpha_2} = 0, \\ 2\frac{\partial^2\tilde{\chi}}{\partial\alpha_1\partial\alpha_2} + \frac{\partial^2\tilde{\phi}}{\partial\alpha_1^2} - \frac{\partial^2\tilde{\phi}}{\partial\alpha_2^2} = 0,\end{gathered} \tag{4.89}$$

$$\frac{\partial^2\tilde{F}}{\partial\alpha_2^2} = 0, \quad \frac{\partial^2\tilde{F}}{\partial\alpha_1\alpha_2} = 0. \tag{4.90}$$

4.3.2.1 Shell with Diaphragm on Edges: Solution in the Explicit Form

Variant (4.88) of the boundary conditions allows to write down a solution of Eqs. (4.76) in the explicit form

$$\tilde{\chi} = \chi_0 \, sin\frac{\pi n \alpha_1}{L} sin\frac{m\alpha_2}{R}, \quad \tilde{F} = F_0 \, sin\frac{\pi n \alpha_1}{L} sin\frac{m\alpha_2}{R}, \tag{4.91}$$

where n, m are positive integers. Inserting (4.92) into Eqs. (4.76) gives

$$\omega^2 = \frac{\varepsilon^8 \pi^4 E \varDelta_{nm}}{R^2 \rho_0}, \tag{4.92}$$

where

$$\begin{aligned} \varDelta_{nm} &= \left(\frac{1 + \theta K \delta_{nm}}{1 + K \delta_{nm}} \right) \delta_{nm}^2 + \frac{n^4}{l^4 \pi^4 \varepsilon^8 \delta_{nm}^2}, \qquad K = \frac{\pi^2 h^2}{\beta R^2}, \\ \delta_{nm} &= \left(\frac{n^2}{l^2} + \frac{m^2}{\pi^2} \right), \qquad l = \frac{L}{R}. \end{aligned} \tag{4.93}$$

As seen from Eqs. (4.92), (4.93), the effect of the shear parameter K on the natural frequencies remains the same as for the laminated plates (s. Subsect. 4.1.2): the transverse shears leads to some reduction of all natural frequencies when compare them with eigenfrequencies at $K = 0$.

4.3.2.2 Shell without Diaphragm on Edges: Asymptotic Solution

Consider the boundary conditions (4.89), (4.90) corresponding to the case when diaphragm at both edges are absent. The boundary-value problem (4.76), (4.89), (4.90) does not admit the explicit form of a solution, but this problem on low-frequency vibrations is identical to the boundary-value problem on buckling of a medium-length cylindrical shell under external pressure considered in Subsubsect. 3.2.1.3 (s. Chapt. 3) and may be solved by the same asymptotic approach.

As in Subsubsect. 3.2.1.3, we assume that $G \sim h_*^{3/2} E$. Then $K/\pi^2 = \varepsilon^2 \kappa$, where $\kappa \sim 1$. Intending to study low-frequency vibrations, we seek the required functions $\tilde{\chi}, \tilde{F}, \tilde{\phi}$ in the form of

$$\begin{aligned} \tilde{\chi} &= RX(x) \sin\left(\varepsilon^{-1} p \varphi\right), \\ \tilde{F} &= \varepsilon^4 E h R^2 \varPhi(x) \sin\left(\varepsilon^{-1} p \varphi\right), \\ \tilde{\phi} &= RS(x) \cos\left(\varepsilon^{-1} p \varphi\right), \end{aligned} \tag{4.94}$$

where $p \sim 1$, $x = \alpha_1/R, \varphi = \alpha_2/R$. Then the governing equations (4.76) are rewritten as follows

$$\begin{aligned}\varepsilon^4(1-\varepsilon^2\kappa\theta\varDelta_\varepsilon)\varDelta_\varepsilon^2 X + \frac{\mathrm{d}^2\varPhi}{\mathrm{d}x^2} - \varLambda(1-\varepsilon^2\kappa\varDelta_\varepsilon)X &= 0,\\ \varepsilon^4\varDelta_\varepsilon^2\varPhi - \frac{\mathrm{d}^2}{\mathrm{d}x^2}(1-\varepsilon^2\kappa\varDelta_\varepsilon)X &= 0,\end{aligned} \tag{4.95}$$

$$\frac{1-\nu}{2}\kappa_1\varepsilon^2\varDelta_\varepsilon S = S, \tag{4.96}$$

where

$$\varLambda = \frac{\rho_0 R^2\omega^2}{\varepsilon^4 E}, \quad \varDelta_\varepsilon = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \varepsilon^{-2}p^2,$$

and the boundary conditions (4.89), (4.90) at $x = 0, l$ take the form

$$\begin{aligned}(1-\varepsilon^2\kappa_1\varDelta_\varepsilon)X = 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}(1-\varepsilon^2\kappa_1\varDelta_\varepsilon)X &= 0,\\ \left(\varepsilon^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \nu p^2\right)X + \varepsilon(1-\nu)p\frac{\mathrm{d}S}{\mathrm{d}x} &= 0,\\ 2\varepsilon p\frac{\mathrm{d}X}{\mathrm{d}x} + \varepsilon^2\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} + p^2 S &= 0,\\ \varPhi = 0, \qquad \frac{\mathrm{d}\varPhi}{\mathrm{d}x} &= 0.\end{aligned} \tag{4.97}$$

Omitting details for construction of the asymptotic solution of the boundary-value problem (4.95)-(4.97), we outline here only the resultant equations. The shear function S is defined as

$$S = \varepsilon\left\{a_1\exp\left(-\frac{\vartheta_{\mathrm{s}}x}{\varepsilon}\right) + a_2\exp\left[-\frac{\vartheta_{\mathrm{s}}(l-x)}{\varepsilon}\right]\right\}, \tag{4.98}$$

where

$$\vartheta_{\mathrm{s}} = \sqrt{\frac{2}{(1-\nu)\kappa_1} + p^2}, \quad a_1 = -\frac{2\pi n p A}{l\left(p^2+\vartheta_{\mathrm{s}}^2\right)}, \quad a_2 = (-1)^n a_1. \tag{4.99}$$

The displacement and stress functions $X, \varPhi$ and eigenvalue $\varLambda$ as well are evaluated as

$$\begin{aligned}X = X^{(\mathrm{m})} + X^{(\mathrm{e})}, \quad \varPhi = \varPhi^{(\mathrm{m})} + \varPhi^{(\mathrm{e})},\\ X^{(\mathrm{m})} = X_0 + \varepsilon X_1 + O\left(\varepsilon^2\right), \quad \varPhi^{(\mathrm{m})} = \varPhi_0 + \varepsilon\varPhi_1 + O\left(\varepsilon^2\right),\end{aligned} \tag{4.100}$$

$$\varLambda = \varLambda_0 + \varepsilon\varLambda_1 + O\left(\varepsilon^2\right), \tag{4.101}$$

where the superscript (m) denotes functions corresponding to the main stress-strain state with the zeroth index of variation $\iota_1 = 0$ in the axial direction, and functions with the superscript (e) are the integrals of edge effects. All the required functions are determined by the following equations

$$\begin{aligned}
X_0 &= A\sin\frac{\pi n x}{l}, \quad X_1 = -\frac{\Lambda_1 p^6 l^3 A}{4\pi^3 n^3}\, x\,\cos\frac{\pi n x}{l},\\
X^{(\mathrm{e})} &= \varepsilon\left[b_1 \mathrm{e}^{-\frac{r_1}{\varepsilon}\,x} + b_2 \mathrm{e}^{-\frac{r_1}{\varepsilon}\,(l-x)} + O(\varepsilon)\right],\\
\Phi_j &= \frac{1+\kappa p^2}{p^4}\frac{\mathrm{d}^2 X_j}{\mathrm{d}x^2}, \quad \Phi^{(\mathrm{e})} = \frac{\kappa_1}{\varepsilon^2}X^{(\mathrm{e})},\\
b_1 &= -\frac{2\pi n(1-\nu)\vartheta_{\mathrm{s}} p^2 A}{l r_1^2\left[1+(1-\nu)p^2\kappa_1\right]\left[p^2+\vartheta_{\mathrm{s}}^2\right]}, \quad b_2 = (-1)^n b_1,
\end{aligned} \tag{4.102}$$

and the frequency parameters Λ_0, Λ_1 are the following:

$$\begin{aligned}
\Lambda_0(p;n) &= \frac{\pi^4 n^4}{l^4 p^4} + \frac{p^4(1+\theta\kappa p^2)}{1+\kappa p^2},\\
\Lambda_1(p;n) &= \frac{8(-1)^{(n+1)}\pi^4 n^3(1-\nu)\kappa_1\vartheta_{\mathrm{s}}}{l^5 p^2\left[1+(1-\nu)p^2\kappa_1\right]\left(p^2+\vartheta_{\mathrm{s}}^2\right)},
\end{aligned} \tag{4.103}$$

where n is a number of semi-waves in the axial direction of the shell, and $\kappa_1 \equiv \kappa$ is the shear parameter.

Contrary to the problem on buckling of a shell studied in Subsubsect. 3.2.1.3, there here is no need to minimize $\Lambda_0(p;n)$ over a parameter p and a number n. The only requirement for a parameter p is the following: it has to be of the order of the unit ($p \sim 1$) and chosen in such a way that $m = \varepsilon^{-1}p$ is a natural number. When minimizing $\Lambda_0(p;n)$ over p at fixed n, we obtain the eigenvalue

$$\Lambda_0^\circ = \min_p \Lambda_0(p;n) = \Lambda_0(p^\circ;n) \tag{4.104}$$

and its correction $\Lambda_1^\circ = \Lambda_1(p^\circ;n)$ corresponding to eigenfrequencies from the lowest part of spectrum at $n \sim 1$.

Finally, one can write out the asymptotic formula for the natural frequencies

$$\omega^\circ = \varepsilon^2\sqrt{\frac{E\Lambda_0^\circ}{R^2\rho_0}}\left[1+\varepsilon k_s + O\left(\varepsilon^2\right)\right], \quad k_s = \frac{\Lambda_1^\circ}{2\Lambda_0^\circ}. \tag{4.105}$$

It is necessary to distinguish the effect of parameters κ and κ_1 on eigenfrequencies. A parameter κ shows the total influence of the transverse shears on the main stress-state of a shell and the zeroth approximation for natural frequencies as well; as seen from (4.103), it reduces all frequencies when comparing them with ones obtained on the base of the model ignoring shears. And a parameter κ_1 gives the impact of shears generated only by boundary conditions and the edge effect integrals; its influence has a local character and depend on a number of semi-waves in the axial direction. If n is an odd number, then $\varepsilon\Lambda_1$ gives the positive correction for Λ_0, and this correction becomes negative for even n. It should be noted that the natural modes constructed above do not contain the classical (simple) edge effect integrals with the index of variation $\iota_1 = 1/2$, but they comprise the edge effect integrals (see above the functions $S(x)$ and $X^{(e)}(x)$) with the smaller index of variation, $\iota_1 = 1/4$.

It is of interesting to compare formula (4.105) with Eqs. (4.92), (4.93) predicting eigenfrequencies for a medium-length cylinder with the simply supported edges supplied with diaphragms. We assume $n = 1, m = \varepsilon^{-1}p^\circ$, then Eqs. (4.92), (4.93) give the following asymptotic formulas

$$\omega^* = \varepsilon^2\sqrt{\frac{E\Lambda_0^\circ}{R^2\rho_0}}[1 + \varepsilon^2 k_s^* + O(\varepsilon^4)], \quad k_s^* = \frac{\Lambda_2^*}{2\Lambda_0^\circ}, \tag{4.106}$$

where Λ_2^* is calculated by

$$\Lambda_2^* = \frac{2\pi^2 n^2 p^2 + 3\pi^2\theta\kappa n^2 p^4}{l^2(1+\kappa p^2)} - \frac{\pi^2 n^2 p^4}{l^2(1+\kappa p^2)^2} - \frac{2\pi^6 n^6}{l^6 p^6}$$

at $p = p^\circ$. It is seen that (4.105) and (4.106) coincide only in the zeroth approximation, and the next approximations give corrections of different orders. In (4.105), the first correction of an order $O(\varepsilon)$ is generated by the non-classical edge effects, whereas the first correction in (4.106) is more less and not related to any edge effects.

Example 4.2. As an example, we consider the five-layered cylindrical shell of the radius and length $R = L = 0.9$ m assembled from laminas which are made of different materials:

- the first (innermost) layer (thickness $h_1 = 0.5$ mm) is the ABS-plastic SD-0170,
- the fifth (outermost) layer (thickness $h_5 = 0.5$ mm) is made of silicon nitrate (ceramic),
- the second and fourth layers are of the same thicknesses $h_2 = h_4 = 3.0$ mm and made of epoxy,
- the third soft layer of the thickness h_3 is alloy-foam.

All materials are assumed as elastic ones with properties given in Example 3.7 (s. Chapt. 3). Table 4.3 shows the influence of the soft alloy-foam core on the parameters m^*, m°, p° and the lowest frequencies ω^*, ω° for the SSD and SSF boundary conditions. Here, ω^* is calculated by (4.92), (4.93) which may be rewritten as

Table 4.3 Wave numbers m^*, m°, parameters p°, $\Lambda_0^\circ, \Lambda_1^\circ$ and the lowest frequencies ω^*, ω° for the 5-layered cylindrical shell for the two variants of boundary conditions (SSD, SSF) vs. thickness h_3 of the alloy-foam core.

h_3, mm	m^*	ω^*, Hz	p°	m°	Λ_0°	Λ_1°	ω°, Hz
20	6	634	1.84	6	17.82	0.42	628
25	5	614	1.87	6	16.99	0.63	611
30	5	593	1.90	6	16.19	0.80	596
35	5	577	1.94	6	15.46	0.92	582
38	5	569	1.96	6	15.07	0.97	576

$$\omega^* = \frac{\varepsilon^4\pi^2}{K}\sqrt{\frac{E}{\rho_0}\,\Delta^*_{nm}}, \quad \Delta^*_{nm} = \min_{n,m}\Delta_{nm}(n,m) = \Delta_{nm}(1,m^*).$$

The increase of the soft core thickness h_3 (at fixed thicknesses of other layers) results in the decrease of the first natural frequency for both variants of boundary conditions. This effect is explained by some reduction of the reduced Young's modulus with increasing h_3. Also, the correction $\varepsilon\Lambda_1^\circ$ generated by the edge shears turns out to be small, although it increases together with h_3. When comparing results for different boundary conditions, one can conclude: overlapping diaphragm on the edges increases the lowest eigenfrequency.

4.4 Free Low-frequency Localized Vibrations of Medium-length Cylindrical Shells

In this section, we will study free vibrations of elastic, medium-length, non-circular cylindrical shells or panels. It is assumed that the Young's and shear moduli are also functions of the circumferential coordinate. As follows from study (Mikhasev et al, 2014), similar inhomogeneity of physical properties takes place if a laminated shell is assembled from highly polarized MREs and/or placed in magnetic field. It has been also shown (Mikhasev et al, 2014), that the eigenmodes of MRE-based sandwich shells are very affected by applied magnetic field and may be characterized by strong localization in some area on the shell surface. Here, using the asymptotic Tovstik's method (Tovstik, 1983) stated in Subsect. 3.2.2, we will give the formal construction of these modes and find the corresponding natural frequencies. We note that the problem will be considered in the elastic statement, and viscoelastic properties of layers composing the shell will not be taken into account. The effect of viscoelastic properties of MREs on both free and forced vibrations will be studied in detail in the next chapter.

Let us introduce the dimensionless magnitudes by the following equations

$$\begin{aligned} &\alpha_1 = Rs, \quad \alpha_2 = R\varphi, \quad R_2 = \frac{R}{k_2(\varphi)}, \\ &\tilde{\chi} = R\chi_*, \quad \tilde{F} = \varepsilon^4 E^\circ h R^2 \Phi_*, \quad \Lambda = \frac{\rho R^2\omega^2}{\varepsilon^4 E^\circ}, \end{aligned} \tag{4.107}$$

where E° is the characteristic value of the Young's modulus. We make also the following assumptions for the elastic modulus and shear parameter as well

$$E = E^\circ d(\varphi) = E^\circ[1 + \varepsilon d_1(\varphi)], \quad \frac{K}{\pi^2} = \varepsilon^2\kappa_0(\varphi), \tag{4.108}$$

where $d_1, \kappa_0 \sim 1$ as $\varepsilon \to 0$. We note that the last estimate (4.108) for K holds if $G \sim h_*^{3/2}E$. The reduced Poisson's ratio ν and a parameter η_3 are assumed to be weakly dependent on coordinates and considered here as constants and parameter θ is taken as a very small one.

Taking into account (4.107), (4.108) and above assumptions as well, the first two equations from (4.76) are rewritten as

$$\varepsilon^4 d(\varphi)\Delta^2\chi^* + k_2(\varphi)\frac{\partial^2\Phi^*}{\partial s^2} - \Lambda[1-\varepsilon^2\kappa(\varphi)\Delta]\chi^* = 0,$$
$$\varepsilon^4\Delta^2\Phi^* - k_2(\varphi)\frac{\partial^2}{\partial s^2}[1-\varepsilon^2\kappa(\varphi)\Delta]\chi^* = 0, \qquad (4.109)$$

where $d(\varphi), \kappa_0(\varphi)$ are real functions of an angle φ.

Remark 4.2. Equations (4.76) have been derived on the supposition that the Young's and shear moduli as well as Poisson's ratio are constant for all layers. If they are functions of the curvilinear coordinates α_1, α_2, the governing equations like (4.76) and (4.109) will contain additional terms which however do not give the contribution into the asymptotic solution to be constructed below. Also, when deriving Eqs. (4.109) from Eqs. (4.76), we have omitted the operator $\Delta^3\tilde{\chi}$ because of the smallness of the shear parameter $K\theta$.

Consider here the simplest variant of boundary conditions

$$\chi^* = \Delta\chi^* = \Delta^2\chi^* = \Phi^* = \Delta\Phi^* = 0 \quad \text{at} \quad s = 0, l \qquad (4.110)$$

corresponding to the simply supported edges with diaphragm. Let $\varphi = \varphi_0$ be the weakest generatrix in the neighbourhood of which one occurs localization of eigenmodes. The required eigenmodes and eigenvalues are approximated by the following series (Tovstik, 1983; Mikhasev and Tovstik, 2009)

$$\chi^* = \sin\frac{\pi n s}{l}\sum_{j=0}^{\infty}\varepsilon^{j/2}\chi_j(\zeta)\exp\left\{\imath\left(\varepsilon^{-1/2}p\zeta + 1/2b\zeta^2\right)\right\},$$
$$\Phi^* = \sin\frac{\pi n s}{l}\sum_{j=0}^{\infty}\varepsilon^{j/2}\Phi_j(\zeta)\exp\left\{\imath\left(\varepsilon^{-1/2}p\zeta + 1/2b\zeta^2\right)\right\}, \qquad (4.111)$$

$$\Lambda = \Lambda_0 + \varepsilon\Lambda_1 + \dots. \qquad (4.112)$$

where $\zeta = \varepsilon^{-1/2}(\varphi - \varphi_0)$, p is a real wave parameter, b is an imaginary parameter so that $\Im\, b > 0$ and χ_j, Φ_j are polynomials in ζ.

The functions $\kappa_0(\varphi), k_2(\varphi), d_1(\varphi)$ are expanded into series in the neighborhood of the generatrix $\varphi = \varphi_0$. In particular,

$$\kappa_0(\varphi) = \kappa_0(\varphi_0) + \varepsilon^{1/2}\kappa_0'(\varphi_0)\zeta + \frac{1}{2}\varepsilon\kappa_0''(\varphi_0)\zeta^2 + \dots \qquad (4.113)$$

All unknown parameters and functions appeared in (4.111), (4.112) are found in such a way as in Subsect. 3.2.2. We outline here only the principal equations. The substitution of (4.111), (4.112) into Eqs. (4.109) produces the sequence of algebraic equations

$$\sum_{j=0}^{\varsigma} \mathbf{L}_j \boldsymbol{X}^{\mathrm{T}}_{\varsigma-j}, \quad \varsigma = 0, 1, 2, \ldots, \tag{4.114}$$

where $\mathbf{X}_j = (\chi_j, \varPhi_j)$ are two-dimensional vectors, the superscript T denotes transposition and $\mathbf{L}_0$ is the 2×2 matrix with the elements

$$\begin{aligned} l_{11} &= p^4 - \varLambda_0[1 + \kappa_0(\varphi_0)p^2], \qquad & l_{12} &= -k_2(\varphi_0)\pi^2 n^2 l^{-2}, \\ l_{21} &= k_2(\varphi_0)[1 + \kappa_0(\varphi_0)p^2]\pi^2 n^2 l^{-2}, \qquad & l_{22} &= p^4 \end{aligned} \tag{4.115}$$

and the matrix operators $\mathbf{L}_j$ for $j \geq 1$ are expressed in terms of the matrix $\mathbf{L}_0$ by Eqs. (3.111), where $\mathbf{L}_* \equiv 0$ and

$$\mathbf{N} = -\varLambda_1 + d_1(\varphi_0)p^4. \tag{4.116}$$

Considering the homogeneous system of algebraic equations (4.114) at $\varsigma = 0$, one obtains

$$\varPhi_0 = -\frac{g_n^{1/2}(\varphi_0)}{p^4}[1 + p^2\kappa_0(\varphi_0)], \tag{4.117}$$

$$\varLambda_0 = f(p, \varphi_0) = \frac{g_n(\varphi_0)}{p^4} + \frac{p^4}{1 + \kappa_0(\varphi_0)p^2}, \tag{4.118}$$

where

$$g_n(\varphi_0) = \pi^4 n^4 l^{-4} k_2^2(\varphi_0). \tag{4.119}$$

As seen from (4.117), $p \neq 0$. The compatibility condition for system (4.114) at $\varsigma = 1$ implies the equations

$$f_p = 0, \quad f_\varphi = 0, \tag{4.120}$$

which may be rewritten as follows

$$\kappa_0(\varphi_0)p^{10} + 2p^8 - 2g_n(\varphi_0)\kappa_0^2 p^4 - 4g_n(\varphi_0)\kappa_0 p^2 - 2g_n(\varphi_0) = 0, \tag{4.121}$$

$$g_n'(\varphi_0)[1 + \kappa_0(\varphi_0)p^2] - p^{10}\kappa_0'(\varphi_0) = 0, \tag{4.122}$$

where the subscript p, φ denote the partial derivatives of a function with respect to the corresponding variables p, φ_0, and the prime ($'$) means differentiation with respect to φ_0. These equations allow to find the wave number p° and the weakest generatrix $\varphi_0 = \varphi_0^\circ$. Finally, the compatibility condition for system (4.114) at $\varsigma = 2$ yields the following equations

$$f_{pp}b^2 + 2f_{p\varphi}b + f_{\varphi\varphi} = 0, \tag{4.123}$$

$$\lambda_1 = -\mathrm{i}(m + 1/2)(f_{pp}b + f_{p\varphi}) + p^4 d_1(\varphi_0), \tag{4.124}$$

$$\chi_0 = \mathcal{H}_m(z), \quad z = [f_{\varphi\varphi}f_{pp}^{-1} - f_{p\varphi}f_{pp}^{-1}]^{1/4}\zeta, \tag{4.125}$$

where $\mathcal{H}_m(z)$ is the Hermite polynomial of the mth degree. In Eqs. (4.123)-(4.125), the second derivatives of f with respect to p and φ_0 are calculated at $p = p^\circ$, and $\varphi_0 = \varphi_0^\circ$.

Equation (4.123) is used for definition of b. It may be seen that the inequality $\Im b > 0$ holds if the second differential of the function f at point $p = p^\circ$, $\varphi_0 = \varphi_0^\circ$ is a positive definite quadratic form, i.e.

$$\mathrm{d}^2 f = f_{pp}^\circ \mathrm{d}p^2 + 2 f_{p\varphi}^\circ \mathrm{d}p\mathrm{d}\varphi_0 + f_{\varphi\varphi}^\circ \mathrm{d}\varphi_0^2 > 0. \tag{4.126}$$

The superscribe $^\circ$ denotes that the function f and its partial derivatives are calculated at $p = p^\circ, \varphi_0 = \varphi_0^\circ$. The conditions (4.120), (4.126) indicate that only eigenmodes corresponding to the lowest spectrum are considered here. For the inequality (4.126) to be hold, a solution of Eq. (4.120) should be chosen in such a way that $f_{pp}^\circ = f_{pp}(p^\circ, \varphi_0^\circ) > 0$. To determine the parameter Λ_ς and functions $\chi_\varsigma(\zeta)$, $\Phi_\varsigma(\zeta)$ appearing in (4.111), (4.112) for $\varsigma \geq 1$, one must consider responding system of nonhomogeneous equations (4.114) in the $(\varsigma + 2)$nd approximation. However, the formal procedure for constructing these functions is no longer for $\varsigma \geq 4$ because the correction introduced by appropriate approximations into solution (4.111) at the sixth step is of the order ε^2, which is the same as the error of the governing equations (4.76).

Consider two particular cases.

A) Let $k_2 = k_2(\varphi)$ (noncircular shell or panel) and κ_0, $d_1 = 0$ are constants. Here the weakest line $\varphi = \varphi_0^\circ$ is the generatrix with the minimum curvature and found from the conditions

$$k_2'(\varphi_0^\circ) = 0, \quad k_2''(\varphi_0^\circ) > 0, \tag{4.127}$$

and the natural frequency and parameter b are determined by equations

$$\begin{aligned} \omega &= \omega_c \omega^*, \quad \omega^* = (f^\circ)^{1/2}\left[1 + \varepsilon \Xi + O(\varepsilon^2)\right] \\ \Xi &= \frac{(1+2m)\pi^2 n^2 \sqrt{f_{pp}^\circ k_2''(\varphi_0^\circ)}}{4 l^2 f^\circ (p^\circ)^2}, \\ b^\circ &= \frac{\mathrm{i}\,\pi^2 n^2}{l^2 (p^\circ)^2}\sqrt{\frac{k_2''(\varphi_0^\circ)}{f_{pp}^\circ}}, \end{aligned} \tag{4.128}$$

where $\omega_c = \varepsilon^2 R^{-1}(E^\circ/\rho)^{1/2}$ is the characteristic frequency and ω^* is the dimensionless frequency parameter.

B) If k_2 is constant (circular shell or panel), and the shear parameter $\kappa(\varphi)$ is a function, then the weakest line is the one at which the reduced shear parameter K approaches the local maximum:

$$\kappa_0'(\varphi_0^\circ) = 0, \quad \kappa_0''(\varphi_0^\circ) < 0. \tag{4.129}$$

As follows from Eqs. (2.59), (4.93), conditions (4.129) are equivalent to the ones of the local minimum for the reduced shear modulus G. Here, one obtains the following equations for the dimensionless parameters Ξ and b°

$$\Xi = \frac{1}{2f^\circ}\left[\frac{(1+2m)(p^\circ)^3\sqrt{-f^\circ_{pp}\kappa''_0(\varphi^\circ_0)}}{2[1+(p^\circ)^2\kappa_0(\varphi^\circ_0)]} + d_1(\varphi^\circ_0)(p^\circ)^4\right],$$
$$b^\circ = \frac{\mathrm{i}\,(p^\circ)^3}{1+(p^\circ)^2\kappa_0(\varphi^\circ_0)}\sqrt{-\frac{\kappa''_0(\varphi^\circ_0)}{f^\circ_{pp}}} \tag{4.130}$$

If we ignore the shear deformations (assuming $\kappa_0 = 0$), then Eqs. (4.128), (4.130) are reduced to analogues equations obtained before for the Kirchhoff-Love theory-based thin elastic isotropic shell (Mikhasev and Tovstik, 2009).

Equations (4.128) and (4.130) show that increasing the parameter $k''_2(\varphi^\circ_0)$ or $\kappa''(\varphi^\circ_0)$ results in increasing the correction $\omega^* - \omega^*_0$ for the natural frequency, where $\omega^*_0 = (f^\circ)^{1/2}$, and leads to growing the power of localization of eigenmodes.

4.5 Localized Vibrations of a Cylindrical Shell Pre-stressed by Distributed Axial Forces

In this section, we will study free localized vibrations of a thin, axially prestressed, multi-layered circular cylindrical shell consisting of N transversely isotropic layers (Mikhasev and Zgirskaya, 2001; Korchevskaya et al, 2004; Korchevskaya and Mikhasev, 2006; Mikhasev, 2017). It is assumed that simply supported edges are under action of a nonuniform axial forces $T^\circ_{11}(\alpha_2)$ as shown in Fig. 3.11. The governing equations describing free vibrations of the pre-stressed laminated cylindrical shell is readily obtained from Eqs. (2.160) by introducing the inertia term into the first equation

$$\frac{Eh^3\eta_3}{12(1-\nu^2)}\left(1-\frac{\theta h^2}{\beta}\Delta\right)\Delta^2\chi + \frac{1}{R}\frac{\partial^2 F}{\partial\alpha_1^2} + T^\circ_{11}(\alpha_2)\frac{\partial^2}{\partial\alpha_1^2}\left(1-\frac{h^2}{\beta}\Delta\right)\chi$$
$$+\rho h\frac{\partial^2}{\partial t^2}\left(1-\frac{h^2}{\beta}\Delta\right)\chi = 0, \tag{4.131}$$
$$\Delta^2 F - \frac{Eh}{R}\frac{\partial^2}{\partial\alpha_1^2}\left(1-\frac{h^2}{\beta}\Delta\right)\chi = 0, \quad w = \left(1-\frac{h^2}{\beta}\Delta\right)\chi.$$

Here, R is the radius of the reference surface of the laminated shell, and other notations are as above. In terms of the displacement and stress functions, the boundary conditions for simply supported edges are as follows

$$\chi = \triangle\chi = \triangle^2\chi = F = \triangle F = 0. \tag{4.132}$$

Inhomogeneity of the axial force T°_{11} results in the appearance of an area at the shell surface with large compressive axial stresses. If the axial stress resultant turns out to be sufficiently large and reaches the critical buckling value T_{11}^*, then, as shown in Chapt. 3, the shell buckles in the neighbourhood of the weakest generatrix $\alpha_2 = \alpha_2^\circ$, where $\max_{\alpha_2} T_{11}^\circ(\alpha_2) = T_{11}^*$. But if $T_{11}^\circ(\alpha_2) < T_{11}^*$ for any α_2, then the pre-buckling compressive forces distorts the natural modes and may result in strong localization of some ones. To study these modes, we use the same asymptotic approach as in Subsect. 3.3.3.

To take into account the influence of the shear parameter in the zeroth order approximation, we assume the following relations

$$\frac{K}{\pi^2} = \mu^2 \kappa, \quad \frac{K\theta}{\pi^2} = \mu^3 \tau, \quad \kappa, \tau \sim 1 \,\text{as} \quad \mu \to 0, \tag{4.133}$$

which are valid for a sufficiently thin shell with the reduced shear modulus $G \sim h_* E$. Here

$$K = \frac{\pi^2 h^2}{R^2 \beta}, \quad \mu^4 = \frac{h^2 \eta_3}{12 R^2 (1 - \nu^2)} \tag{4.134}$$

The required functions χ and Φ are sought in the form

$$\chi = R\hat{\chi}(s, \varphi) \sin \omega t, \qquad F = \mu^2 E h R \hat{\Phi}(s, \varphi) \sin \omega t. \tag{4.135}$$

Then, Eqs. (4.131) can be rewritten as follows

$$\begin{aligned} \mu^4 (1 - \mu^3 \tau \triangle) \triangle^2 \hat{\chi} + \mu^2 \frac{\partial^2 \hat{\Phi}}{\partial s^2} + \mu^2 t_1(\varphi) \frac{\partial^2}{\partial s^2} (1 - \mu^2 \kappa \triangle) \hat{\chi} \\ - \Lambda (1 - \mu^2 \kappa \triangle) \hat{\chi} = 0, \\ \mu^2 \triangle^2 \hat{\Phi} - \frac{\partial^2}{\partial s^2} (1 - \mu^2 \kappa \triangle) \hat{\chi} = 0, \end{aligned} \tag{4.136}$$

where

$$s = \frac{\alpha_1}{R}, \quad \varphi = \frac{\alpha_2}{R}, \quad l = \frac{L}{R}, \quad t_1(\varphi) = \frac{T_{11}^\circ(R\varphi)}{\mu^2 E h}, \quad \Lambda = \frac{R^2 \rho}{E} \omega^2, \tag{4.137}$$

and the boundary conditions for functions $\hat{\chi}, \hat{\Phi}$ will be

$$\hat{\chi} = \triangle \hat{\chi} = \triangle^2 \hat{\chi} = \hat{\Phi} = \triangle \hat{\Phi} = 0. \tag{4.138}$$

The problem is to find a positive value of Λ for which the system of equations (4.136) has a nontrivial solution satisfying the boundary conditions (4.138).

4.5.1 Asymptotic Solution

A formal asymptotic solution of the boundary-value problem (4.136), (4.138) is constructed in the following form, s. Eqs. (3.164) and (3.165),

$$\hat{\chi} = \sin \frac{r_m s}{\mu} \chi_m(\xi, \mu), \tag{4.139}$$

$$\begin{aligned} \chi_m &= \sum_{j=0}^{\infty} \mu^{j/2} \chi_{mj}(\xi) \exp\left[\mathrm{i}\left(\mu^{-1/2} p\xi + \frac{1}{2} b\xi^2\right)\right], \\ \Lambda &= \Lambda_0 + \mu\Lambda_1 + \mu^2\Lambda_2 + \dots . \end{aligned} \tag{4.140}$$

where ($\hat{\chi} \Rightarrow \hat{\Phi}, \chi_m \Rightarrow \Phi_m, \chi_{mj} \Rightarrow \Phi_{mj}$)

$$\begin{aligned} &\xi = \mu^{-1/2}(\varphi - \varphi_0), \quad \Im b > 0, \\ &|\chi_{mj}|,\ |\Phi_{mj}|,\ \Lambda_j,\ p,\ |b|,\ r_m = \frac{\mu \pi m}{l} \sim 1 \quad \text{as} \quad \mu \to 0, \end{aligned} \tag{4.141}$$

and $\chi_{mj}(\xi), \Phi_{mj}(\xi)$ are polynomials in ξ. Here, $\varphi = \varphi_0$ is a weakest generatrix which is unknown. Functions (4.139), (4.140) approximate the eigenmodes localized in a vicinity of the line $\varphi = \varphi_0$.

The substitution of Eqs. (4.139)-(4.141) into Eqs. (4.136) produces the sequence of algebraic equations

$$\sum_{k=0}^{j} \mathbf{L}_k \mathbf{X}_{j-k} = 0, \quad j = 0, 1, 2, \dots \tag{4.142}$$

where $\mathbf{X}_j = (\xi_{mj}, \Phi_{mj})^{\mathrm{T}}$, and $\mathbf{L}_0$ is the 2×2 matrix with the elements

$$\begin{aligned} &l_{11} = (r_m^2 + p^2)^2 - [1 + \kappa(r_m^2 + p^2)][r_m^2 t_1(\varphi_0) + \Lambda_0], \\ &l_{12} = -r_m^2, \quad l_{21} = r_m^2[1 + \kappa(r_m^2 + p^2)], \quad l_{22} = (r_m^2 + p^2)^2, \end{aligned} \tag{4.143}$$

and the matrix operators $\mathbf{L}_j$ for $j \geq 1$ are expressed by the matrix $\mathbf{L}_0$ in the same way as in Sect. 3.2, s. Eqs. (3.111), but now the operator $\mathbf{N}$ is the 2×2 matrix with the unique nonzero element ($n_{12} = n_{21} = n_{22} = 0$)

$$n_{11} = \tau(r_m^2 + p^2)^3 - \Lambda_1[1 + \kappa(r_m^2 + p^2)]. \tag{4.144}$$

The sequence of Eqs. (4.142) serves to determine all unknown functions and parameters in (4.139) and (4.140). Because the procedure for seeking these magnitudes is the same as in Subsect. 3.3.2, we omit transitional calculations here and give only the principle equations. Considering the homogeneous system of algebraic equations (4.142) for $j = 0$, one obtains the zeroth-order approximation for the frequency parameter

$$\Lambda_0 = f(p, r_m, \varphi_0) = \frac{(r_m^2 + p^2)^2}{[1 + \kappa(r_m^2 + p^2)]} + \frac{r_m^4}{(r_m^2 + p^2)^2} - t_1(\varphi_0) r_m^2. \tag{4.145}$$

Holding a number m (and thus, a parameter r_m) fixed, we minimize the function (4.145) over p and φ. The necessary conditions of this minimum are the following equations

$$\frac{\partial f}{\partial p} = 0, \qquad \frac{\partial f}{\partial \varphi_0} = 0 \tag{4.146}$$

which serve for a determination of p° and φ_0°. When solving Eqs. (4.146), three different cases appear

- $r_m > z_0$ (case A),
- $r_m < z_0$ (case B),
- $r_m \approx z_0$, (case C),

were z_0 is a root of the algebraic equation

$$-2(1 + \kappa r_m z)^2 + z^4(2 + \kappa r_m z) = 0 \tag{4.147}$$

with respect to z. Equation (4.147) contains a parameter κ accounting for shears in the sandwich. If shears are disregarded ($\kappa = 0$), its root is $z_0 = 1$.

At first, we consider the cases A) and B). For $r_m > z_0$ (case A), we derive

$$\Lambda_0^\circ = \min_{p,\varphi_0} f(p, r_m, \varphi_0) = 1 - t_1(\varphi_0^\circ) r_m^2 + \frac{r_m^4}{1 + \kappa r_m^2}, \quad p^\circ = 0, \tag{4.148}$$

and for $r_m < z_0$ (case B), one has

$$\begin{aligned} \Lambda_0^\circ &= \min_{p,\varphi_0} f(p, r_m, \varphi_0) = \frac{z_0^2 r_m^2}{1 + \kappa r_m z_0} + \frac{r_m^2}{z_0^2} - t_1(\varphi_0^\circ) r_m^2, \\ p^\circ &= \sqrt{r_m(z_0 - r_m)}. \end{aligned} \tag{4.149}$$

Note that Eqs . (4.148), (4.149) are identical at $r_m = z_0$. For both cases, the weakest generatrix $\varphi = \varphi_0^\circ$ is determined from the following conditions

$$t_1'(\varphi_0^\circ) = 0, \quad t_1''(\varphi_0^\circ) < 0. \tag{4.150}$$

Now, a solution of the homogeneous system of equations (4.142) at $j = 0$ may be written as

$$\mathbf{X}_0 = P_0(\xi)\mathbf{Y}_0, \tag{4.151}$$

where $P_0(\xi)$ is an unknown polynomial in ξ, and $\mathbf{Y}_0 = (1, -l_{11}/l_{12})$ is the vector.

In the first-order approximation ($j = 1$), one has the non-homogeneous system of equations (4.142). When taking Eqs. (4.146) into account, this system turns into identities. Consider the non-homogeneous system (4.142) in the second order approximation ($j = 2$). The compatibility condition for this system generates the formula

$$b = \mathrm{i}\sqrt{f_{\varphi\varphi}/f_{pp}} \tag{4.152}$$

and the equation for P_0 is

$$\frac{\mathrm{d}^2 P_0}{\mathrm{d}\xi^2} + \mathrm{i}b\left(2\xi\frac{\mathrm{d}P_0}{\mathrm{d}xi}\right) + \frac{2\Lambda_1}{f_{pp}}P_0 + I_{A(B)} = 0, \tag{4.153}$$

where

$$I_A = \frac{2\tau r_m^6}{f_{pp}(1+\kappa r_m^2)}P_0 \quad \text{at} \quad r_m > z_0 \text{ (case A)} \tag{4.154}$$

$$I_B = \frac{2\tau r_m^3 z_0^3}{f_{pp}(1+\kappa r_m z_))}P_0 \quad \text{at} \quad r_m < z_0 \text{ (case B)} \tag{4.155}$$

If $r_m = z_0$, then $I_A = I_B$. For both cases

$$P_0(\xi) = \mathcal{H}_n\left(\sqrt{f_{\varphi\varphi}/f_{pp}}\xi\right). \tag{4.156}$$

Now we can calculate the complex parameter b characterizing the rate of the amplitude decrement far from the generatrix $\varphi = \varphi_0^\circ$. If $r_m > z_0$ (case A), then

$$b = \mathrm{i}\sqrt{\frac{r_m^4(1+\kappa r_m^2)^2[-t_1''(\varphi_0^\circ)]}{2r_m^4(2+\kappa r_m^2) - 4(1+\kappa r_m^2)^2}}, \tag{4.157}$$

and for $r_m > z_0$ (case B), one obtains

$$b = \mathrm{i}\sqrt{\frac{r_m(1+\kappa r_m^2)^3[-t_1''(\varphi_0^\circ)]}{4(z_0 - r_m)[8 + 9\kappa r_m z_0 + 3(\kappa r_m z_0)^2]}}. \tag{4.158}$$

It can be seen that

$$\lim_{r_m \to z_0} |b| = +\infty$$

for both cases (A) and (B). Thus, requirement (4.141) for b does not hold if a root r_m is close to z_0. We will not consider the higher-order approximations because system (4.131) does not contain some terms which affect the third and subsequent approximations.

Now we can write equations for the set of eigenvalues. If $r_m > z_0$, we derive

$$\begin{aligned}\Lambda^{(n,m)} &= 1 - t_1(\varphi_0^\circ)r_m^2 + \frac{r_m^4}{1+\kappa r_m^2} \\ &+ \mu\left\{\frac{(1+2n)\sqrt{-2t''(\varphi_0^\circ)[r_m^4(2+\kappa r_m^2) - 2(1+\kappa r_m^2)^2]}}{2(1+\kappa r_m^2)}\right. \\ &+ \left.\frac{\tau r_m^6}{1+\kappa r_m^2}\right\} + O(\mu^2),\end{aligned}$$

and for $r_m < z_0$ one has

$$\begin{aligned}\Lambda^{(n,m)} &= \frac{z_0^2 r_m^2}{1+\kappa r_m z_0} + \frac{r_m^2}{z_0^2} - t_1(\varphi_0^\circ) r_m^2 \\ &+ \mu \left\{ \frac{(1+2n)\sqrt{-t''(\varphi_0^\circ) r_m^3 (z_0 - r_m)[8 + 9\kappa r_m z_0 + 3(\kappa r_m z_0)^2]}}{(1+\kappa r_m^2)^3} \right. \\ &+ \left. \frac{\tau r_m^3 z_0^3}{1+\kappa r_m^2} \right\} + O(\mu^2).\end{aligned}$$

The corresponding eigenmodes will be the following: if $r_m > z_0$, then

$$\chi^{(n,m)} = \sin\frac{r_m s}{\mu} \exp\left\{\frac{ib(\varphi-\varphi_0^\circ)^2}{2\mu}\right\} \left\{\mathcal{H}_n\left[\sqrt{\frac{ib}{\mu}}(\varphi-\varphi_0^\circ)\right] + O(\mu^{1/2})\right\}, \tag{4.159}$$

and for $r_m < z_0$, one obtains

$$\begin{aligned}\chi^{(n,m)} &= \sin\frac{r_m s}{\mu} \exp\left\{\frac{\mathrm{i}}{\mu}\left[\sqrt{r_m(z_0 - r_m)}(\varphi-\varphi_0^\circ)\right]\right\} \\ &\times \exp\left\{\frac{ib(\varphi-\varphi_0^\circ)^2}{2\mu}\right\}\left\{\mathcal{H}_n\left[\sqrt{\frac{ib}{\mu}}(\varphi-\varphi_0^\circ)\right] + O(\mu^{1/2})\right\}.\end{aligned} \tag{4.160}$$

It may be seen that the eigenmodes (4.159) and (4.160) are different for the cases (A) and (B). If $r_m > z_0$ (case A), the eigenfunctions decay exponentially without oscillations ($p^\circ = 0$), and for $r_m < z_0$ (case B) the localized eigenmodes have a large number (of the order μ^{-1}) of waves. If r_m is close to z_0, then Eqs. (4.159) and (4.160) are not applicable. The case (C), when $r_m \simeq z_0$, deserves the special consideration.

4.5.2 Reconstruction of Asymptotic Solution

Let parameter r_m be close to a root z_0 of Eq. (4.147). In this case, a solution of the boundary-value problem (4.136) and (4.138) is found again in the form of (4.139). The substitution of (4.139) into Eqs. (4.136) results in the following system of ordinary differential equations

$$\begin{aligned}(1-\mu\tau\triangle_m)\triangle_m^2\chi_m - r_m\Phi_m - (t_1 r_m^2 + \Lambda)(1-\kappa\triangle_m)\chi_m - \Lambda = 0, \\ \triangle_m^2\Phi_m + r_m^2(1-\kappa\triangle_m)\chi_m = 0,\end{aligned} \tag{4.161}$$

where

$$\triangle_m = \mu^2\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} - r_m^2 \tag{4.162}$$

is the differential operator.

Consider Eq. (4.147) again. At $r_m = z_0$, it is reduced to the following algebraic equation

$$\kappa r_m^6 + 2r_m^4 - 2(1+\kappa r_m^2)^2 = 0. \tag{4.163}$$

Let $r_m = r_*$ be its root. We introduce the following estimations

$$\begin{aligned} r_m &= r_* + \tilde{\mu} r', \quad \Lambda = \Lambda_* + \tilde{\mu}^2 \Lambda', \quad \varphi - \varphi_0^\circ = \tilde{\mu}\eta, \\ t_1(\varphi) &= t_1(\varphi_0^\circ) + \frac{1}{2}\tilde{\mu}^2 t_1''(\varphi_0^\circ)\eta^2 + \ldots \end{aligned} \tag{4.164}$$

where $r', \Lambda' \sim 1$ as $\tilde{\mu} \to 0$, and

$$\tilde{\mu} = \mu^{2/3} = \left[\frac{h^2 \eta_3}{12R^2(1-\nu^2)}\right]^{1/6} \tag{4.165}$$

is a new small parameter.

We will seek a solution of Eqs. (4.161) in the form of series

$$\chi_m = \sum_{k=0}^{\infty} \tilde{\mu}^k \chi_m^{(k)}(\eta), \quad \Phi_m = \sum_{k=0}^{\infty} \tilde{\mu}^k \Phi_m^{(k)}(\eta), \tag{4.166}$$

where

$$\chi_m^{(k)}, \Phi_m^{(k)} \sim 1, \quad \text{and} \quad \chi_m^{(k)}, \Phi_m^{(k)} \to 0 \quad \text{as} \quad \eta \to \pm\infty. \tag{4.167}$$

In the zeroth- and first-order approximations, Eqs. (4.161) turn into identities if the following condition holds

$$\Lambda_* = 1 - t_1(\varphi_0^\circ) r_*^2 + \frac{r_*^4}{1+\kappa r_*^2}. \tag{4.168}$$

Note that Eq. (4.168) coincides with Eqs. (4.148) and (4.149) at $r_m = r_* = z_0$. Equation (4.168) gives the zeroth-order approximation for the eigenvalue Λ. The eigenfunctions $\chi_m^{(0)}$ and $\Phi_m^{(0)}$ remain undefined at this step.

Let us consider the second-order approximation. When taking Eq. (4.168) into consideration, one gets the following equation with respect to $\chi_m^{(0)}$

$$a_4 \frac{\mathrm{d}^4 \chi_m^{(0)}}{\mathrm{d}\eta^4} + a_2(r') \frac{\mathrm{d}^2 \chi_m^{(0)}}{\mathrm{d}\eta^2} + [a_0(r') - a_\eta \eta^2 - \Lambda' a_\lambda]\chi_m^{(0)} = 0, \tag{4.169}$$

where

$$\begin{aligned} a_4 &= 1 + \frac{\kappa}{r_*^2} + \frac{3}{r_*^4}, \quad a_2(r') = -4r_* r' + 2\kappa r_* r' - \frac{4r'}{r_*}, \\ a_0(r') &= (r')^2 \left[6r_*^2 - 1 - \kappa r_*^2 \left(5 + \frac{r_*^2}{1+\kappa r_*^2}\right)\right], \\ a_\eta &= \frac{1}{2} r_*^2 (1+\kappa r_*^2) t_1''(\varphi_0^\circ), \quad a_\lambda = (1+\kappa r_*^2). \end{aligned}$$

The problem is to find such values of $r', \Lambda'(r')$ which satisfy the following condition

$$\chi_m^{(0)} \to 0 \quad \text{as} \quad \eta \to \pm\infty. \tag{4.170}$$

Applying Fourier transform

$$\chi_m^{(0)}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \chi^{\mathrm{F}}(\tilde{\omega}) \mathrm{e}^{\mathrm{i}\tilde{\omega}\eta} \mathrm{d}\tilde{\omega}, \tag{4.171}$$

we come to the second order equation for function χ^{F}

$$\frac{\mathrm{d}^2\chi^{\mathrm{F}}}{\mathrm{d}x^2} + \left\{ \tilde{\Lambda} - [x^4 + 2\gamma x^2 + \gamma^2 Q(\kappa)] \right\} \chi^{\mathrm{F}} = 0, \tag{4.172}$$

where

$$\begin{aligned}
x &= \frac{\tilde{\omega}}{\alpha(\kappa)}, \quad \gamma = C(\kappa) r', \quad \tilde{\Lambda} = \Lambda' \left\{ \frac{1 + \kappa r_*^2}{(r_*^4 + \kappa r_*^2 + 3)[-t_1''(\varphi_0^\circ)]^{1/2}} \right\}^{1/3}, \\
\alpha(\kappa) &= \left[-\frac{t_1''(\varphi_0^\circ) r_*^6 (1 + \kappa r_*^2)}{2(r_*^4 + \kappa r_*^2 + 3)} \right]^{1/6}, \\
C(\kappa) &= \frac{2 + 2r_*^4 - \kappa r_*^4}{r_* \left[-\frac{1}{2} t_1''(\varphi_0^\circ)(1 + \kappa r_*^2)(r_*^4 + \kappa r_*^2 + 3)^2 \right]^{1/3}}, \\
Q(\kappa) &= 1 + \frac{2A(\kappa)\alpha^2(\kappa)}{C^2(\kappa) t_1''(\varphi_0^\circ) r_*^2 (1 + \kappa r_*^2)}, \\
A(\kappa) &= \frac{1 - (1 - \kappa) r_*^2 (6 + 5\kappa r_*^2)}{1 + \kappa r_*^2} + \frac{(2 + 2r_*^4 - \kappa r_*^4)^2}{r_*^2 (r_*^4 + \kappa r_*^2 + 3)}.
\end{aligned}$$

For each γ, there is a countable set of values $\tilde{\Lambda}_j (j = 0, 1, \ldots)$ of $\tilde{\Lambda}$ for which there exist non-trivial solutions of Eq. (4.172) such that

$$\chi^{\mathrm{F}} \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{4.173}$$

It may be seen from Eq. (4.172) that the eigenvalues $\tilde{\Lambda}_j$ depend on both the fixed value of the shear parameter κ and the axial stress resultant t_1. In Fig. 4.3, the first two eigenvalues $\tilde{\Lambda}_0$ and $\tilde{\Lambda}_1$ versus a parameter γ are presented for $\kappa = 0.5$ and $t_1(\varphi) = 0.5(1 + \cos\varphi)$. As seen from Fig. 4.3, for parameters accepted above, the function $\tilde{\Lambda}$ has the minimum value $\tilde{\Lambda}_0 \approx 0.924$ at $\gamma \approx -0.380$. Here $r_* \approx 1.220$ and $\Lambda_* \approx 0.782$, and applying Eqs. (4.173) one gets $\Lambda'_{\min} \approx 0.553$, and $r' \approx -0.217$. Then, the wave parameter r_m from Eq. (4.164) and the minimum eigenvalue Λ will be as follows

$$r_m \approx 1.22 - 0.217\varepsilon^{2/3}, \quad \Lambda_{\min} \approx 0.782 + 0.553\varepsilon^{4/3}. \tag{4.174}$$

Table 4.4 shows parameters $r_m, z_0, \Lambda_*, \Lambda'$, and Λ_{min} versus κ for the case (C) when $r_m \approx z_0$. It may be seen that increasing the shear parameter κ leads to a decrease of the minimum natural frequency of the laminated cylindrical shell.

Table 4.4 Minimum eigenvalue Λ versus κ at $r_m \approx z_0$ (after Mikhasev, 2017).

κ	r_m	z_0	Λ_*	Λ'	$\Lambda_{\min}$
0.037	0.993	1.014	0.990	0.590	1.005
0.100	1.017	1.039	0.972	0.586	0.986
0.250	1.077	1.102	0.917	0.575	0.931
0.400	1.142	1.171	0.843	0.563	0.857
0.500	1.186	1.220	0.782	0.553	0.796
0.600	1.229	1.271	0.710	0.539	0.723

Example 4.3. We consider a three-layered cylindrical shell with radius $R = 150$ mm and length $L = 450$ mm. The first and third layers have the thickness $h_1 = h_3 = 0.3$ mm and are made of aluminium with the Young's modulus $E_1 = E_3 = 70,3$ GPa, Poisson's ratio $\nu_1 = \nu_3 = 0.345$, and density $\rho_1 = \rho_3 = 2.7 \cdot 10^{-6}$ kg/mm^3, and the second one is an epoxy matrix with $h_2 = 0.8$ mm, $E_2 = 3,45$ GPa, $\nu_2 = 0.3$ and $\rho_2 = 1.2 \cdot 10^{-6}$ kg/mm^3. The dimensionless axial membrane stress resultant is assumed as follows

$$t_1(\varphi) = \frac{1}{2}(1 + \delta \cos\varphi). \tag{4.175}$$

Then the generatrix $\varphi = \varphi_0^\circ = 0$ will be the weakest one.

Figure 4.4 shows the dependence of the zeroth-order approximation of the eigenvalue Λ_0 upon both the shear parameter κ and parameter δ at $m = 20$ ($r_m = 1.3$). In this case $r_m > z_0$ and all calculations were performed by equations corresponding to the variant (A). It may be seen that the eigenvalue Λ_0 is the monotonically decreasing function of both the axial force (in a neighborhood of the weakest generatrix) and the shear parameter κ.

Figure 4.5 demonstrates the nonlinear behavior of the relative correction Λ_1/Λ_0 for the eigenvalue Λ at varying the shear parameter κ for different values of δ. As accepted, the increase in parameter ϱ characterizing inhomogeneity of loading

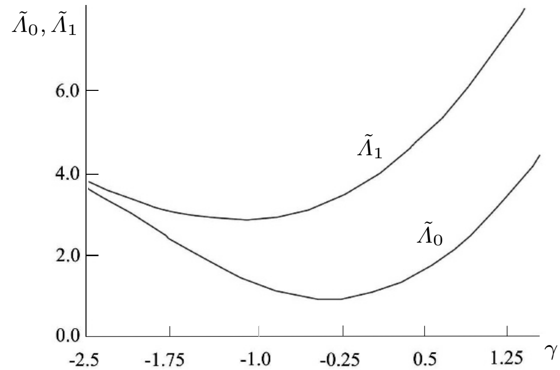


Fig. 4.3 First two eigenvalues $\tilde{\Lambda}_0, \tilde{\Lambda}_1$ vs. parameter γ (after Mikhasev, 2017).

involves the increase in the correction Λ_1/Λ_0 for any fixed κ. But for any fixed δ, there exists the maximum of Λ_1/Λ_0 being the function of κ. Approximately at $\kappa > 0.65$, the influence of inhomogeneity in loading on the natural frequencies becomes negligible.

Example 4.4. Let us consider again the three-layered cylindrical shell with the same geometrical and physical parameters as in the previous Example. In Table 4.5, the dependence of the parameters $\Im b$, Λ_0 (or Λ_* at $r_m \approx z_0$), and Λ_1/Λ_0 (or Λ'/Λ_* for $r_m \approx z_0$) on the wave parameter r_m found by two different asymptotic approaches is presented. The calculations have been performed at $\kappa = 0.5$ for the nonuniform dimensionless stress resultant $t_1(\varphi) = 0.5(1 + \cos\varphi)$. It may be seen that Λ_1/Λ_0 decreases and $\Im b$ increases as $r_m \to z_0 = 1.077$.

All the problems on free vibrations of laminated beams, plates and cylindrical shells considered in this chapter have revealed the general feature for the ESL model taking into account transverse shears: the incorporation of shears into the shell model reduces all natural frequencies, this effect being stronger for eigenmodes with a large number of waves and weaker for modes having a small number of waves. Since the eigenmodes for low-frequency vibrations of thin medium-length cylindrical shells are characterized by a large number of waves in the circumferential direction, than the shear induced lowering of natural frequencies may be too significant for these modes (corresponding to low-frequency vibrations). The outcomes obtained in this chapter, including the derived equations for natural frequencies, will be used below to study free and forced vibrations of laminated thin-walled structures assembled from the viscoelastic smart materials (MREs and ERCs).

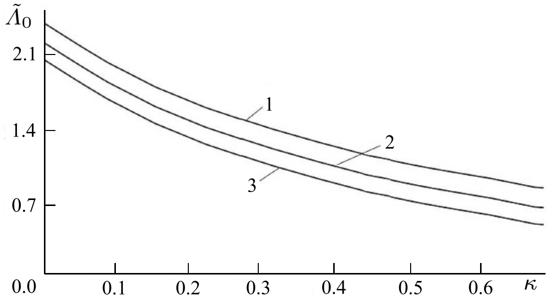


Fig. 4.4 Zero approximation Λ_0 of the eigenvalue Λ vs. the shear parameter κ. $\delta = 0.8$ - curve 1, $\delta = 1$ - curve 2, $\delta = 1.2$ - curve 3 (after Mikhasev, 2017).

Table 4.5 Parameters $\Im b$, Λ_0 (or Λ_*), Λ_1/Λ_0 (or Λ'/Λ_*) vs. r_m (after Mikhasev, 2017).

Cases	r_m	$\Im b$	$\Lambda_0(\Lambda_*)$	$\Lambda_1/\Lambda_0(\Lambda'/\Lambda_*)$
B	0.844	0.285	0.575	1.117
B	0.909	0.347	0.656	0.942
B	0.974	0.448	0.741	0.752
C	1.077	-	0.917	0.627
A	1.360	1.588	1.490	0.552
A	1.490	1.026	1.949	0.564

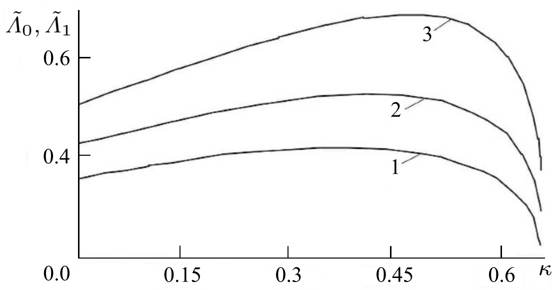


Fig. 4.5 Normalized correction Λ_1/Λ_0 vs. the shear parameter κ.
$\delta = 0.8$ - curve 1,
$\delta = 1$ - curve 2,
$\delta = 1.2$ - curve 3
(after Mikhasev, 2017).

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