Chapter 1
Introduction

Abstract Laminates and sandwiches belong to lightweight structures of rather thin cross sections in comparison with the other structural dimensions. Both have a layered structure. The first one are composed of many layers (in modern structures up to 40 - 60) each of them have as usual the same thickness and properties. The second one are composed of three layers and in classical applications the outer layers are made of uniform (homogeneous) materials, while the inner layer consists either of a soft, relatively continuous material (different foams) or of a structurally complicated, inhomogeneous material (cellular fillers, corrugations). However, in multilayered structures each layer is a composite material itself. A short introduction into the modelling of composite structures is given in Chapt. 1. In Sect. 1.1 some general formulation approaches of plate and shell theories are presented. In Sect. 1.2 an introduction to composite modelling is given. Section 1.3 is devoted to modeling of laminated and sandwich plates and shells.

1.1 Derivation Approaches for Theories of Plates and Shells

Modeling and calculation of three- and multilayered structures is a complicated problem of the mechanics of deformable solid bodies. Since they are as usual thin in on direction (thickness) they belong to the so-called surface structures. In the classical sense two families of structures can be distinguished: plates and shells. Both families are characterized by the assumption that the thickness is smaller in comparison with other spatial dimensions and this allows to approximate the three-dimensional solid mechanics problem by a two-dimensional. Within the geometrical linear theory for isotropic plates the in-plane and the out-of-plane behavior can be decoupled. With respect to the shell curvature such decoupling for shells is not possible without additional assumptions.

Let us present at first the derivation approaches for the governing equations of the theories of plates and shells. In Sects. 1.2 and 1.3 the special cases of laminate and sandwich structures will be discussed starting with the description of composite ma-

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terials and a brief introduction of averaging methods resulting in effective properties of composites.

One of the basic problems in engineering mechanics is the analysis of the strength, the vibration behavior and the stability of structural elements with the help of a structural model (Altenbach and Meenen, 2008). In this context, structural models are approximations of a general continuum theory. The following classification of structural models can be given

- by certain geometrical (spatial) dimensions,
- by certain applied loads and
- by the use of kinematical and/or statical hypotheses approximating their mechanical behavior.

Structural elements and models for their analysis can be categorized into three main classes, depending on the ratio of their characteristic dimensions. The first class is the class of three-dimensional structural elements, which can be defined as follows:

Definition 1.1 (Three-dimensional structural element).
A three-dimensional structural element has three spatial dimensions of the same order, no predominant dimension exists.

Typical examples of geometrically simple, compact structural elements in the theory of elasticity are cube, prism, cylinder, sphere, etc. The second basic class is the class of two-dimensional structural elements which can be defined as follows:

Definition 1.2 (Two-dimensional structural element).
Two-dimensional structural elements are bodies, which have two spatial dimensions of comparable size, and a third spatial dimension, the so-called thickness, which is at least one order of magnitude smaller.

Typical examples of two-dimensional structural elements in civil engineering and structural mechanics are membrane, disc, plate, shell, folded structure, etc. It should be noted that the applied loading results in various sub-classes: for plane structures one should distinguish the in-plane and the out-of-plane loading cases; for curved structures only in some special cases such split makes sense. The last class is related to the one-dimensional structural elements which can be defined as follows:

Definition 1.3 (One-dimensional structural element).
Two spatial dimensions, which can be related to the cross-section, have a comparable size. The third dimension, which is related to the length of the structural element, is at least one order of magnitude larger than the size of the cross-section dimensions.

Typical examples in engineering mechanics are rod, truss, beam, torsion bar, etc. Like in the case of two-dimensional structural elements the applied loading allows to distinguish special cases (tension/compression, bending, torsion).

In general, it is possible to introduce other classes. For example, in shipbuilding, thin-walled structural elements are often used. These are thin-walled light-weight structures with a special profile and they require an extension of the classical one-dimensional structural models:

Definition 1.4 (Quasi-onedimensional structural elements).
If the spatial dimensions are of significantly different order and the thickness of the profile is small in comparison to the other cross-section dimensions, and the cross-section dimensions are much smaller in comparison to the length of the structure one can introduce quasi-onedimensional structural elements.

Suitable theories for the analysis of quasi-onedimensional structural elements are the thin-walled beam theory (Vlasov-Theory) and the semi-membrane theory or generalized beam theory (Altenbach et al, 1994, 2018). Typical thin-walled cross-section profiles are closed cross-section profiles, open cross-section profiles, open-closed cross-section profiles, etc.

Here the focus is on the second and third class of structural elements. Since the characteristic length in thickness direction is much smaller than the characteristic length in the surface direction, for a two-dimensional structures it is tempting to look for procedures that eliminate the thickness dimension (reduction of the coordinates). From the mathematical point of view it is obvious that instead of a three-dimensional coupled partial differential equations, one can analyze a two-dimensional problem, which is described by two spatial coordinates only. These coordinates represent a surface in three-dimensional space, and a procedure has to be developed that maps the real behavior in thickness direction onto the mechanical behavior of the surface. The transition from the three-dimensional to the two-dimensional problem is non-trivial, but once a two-dimensional theory has been obtained, the solution effort decreases significantly and the possibilities to solve problems analytically are increased (Altenbach and Meenen, 2008). One-dimensional theories are here presented as special cases of the two-dimensional one.

During the last 50 years various scientific papers, textbooks, monographs and proceedings on the state of the art and recent developments in the plate and shell theories were published, for example, in Altenbach et al (2016, 2010); Grigolyuk and Seleznev (1973); Libai and Simmonds (1998); Naghdi (1972); Reissner (1985); Rothert (1973). In addition, new developments were discussed on conferences and courses, s. Altenbach and Eremeyev (2011); Altenbach and Mikhasev (2014); Jaiani and Podio-Guidugli (2008); Kienzler et al (2004), among others. From these publications one can conclude that for the formulation of any plate or shell theory there are two starting points:

- the reduction technique, which starts from the equations of three-dimensional (3D) continuum and develops approximate two-dimensional (2D) continuum theories; and
- the direct approach, which starts from a rigorous 2D continuum theory (deformable surface)

If one starts from the 3D continuum theory, the following approaches can be distinguished:

- the use of hypotheses to approximate the three-dimensional equations (e.g. by introducing these hypotheses into the principle of virtual displacements),
- the use of mathematical approaches, such as series expansions, special functions or asymptotic integration, or

- the formulation of consistent theories

All these approaches have their own advantages and disadvantages, and it is difficult to argue what is the best method for deriving a plate or shell theory. Additionally, in many cases different derivation methods result in identical or similar sets of governing equations.

Theories which are based on hypotheses are preferred by engineers because of their simplicity. For example, there is a huge number of theories which are based on displacement assumptions. Note that the three displacements in the classical three-dimensional continuum are split into in-plane displacements and transverse displacement (deflection). Probably the first theory of plates based on displacement assumptions was presented by Kirchhoff (1850). Kirchhoff used similar hypotheses for the kinematics as in the Euler-Bernoulli beam theory. He ignored the in-plane displacements and with the deflection w which was assumed to be independent from the thickness coordinate he got the following kinematical constraints: no transverse shear and no thickness changes. The final version of his theory he presented, for example, in Kirchhoff (1883)

$$D\triangle\triangle w = q,$$

where the bending stiffness[1] which is assumed to be constant

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

is a combination of material parameters (E is the Young's modulus and ν is the Poisson's ratio) and a property of the geometry (h is the plate thickness). $\triangle$ denotes the Laplace operator ($\triangle = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ with $\boldsymbol{\nabla}$ as the Hamilton (nabla) operator) and q the transverse load. It is interesting that Kirchhoff's approach has shown immediately that any approximation results in difficulties. Kirchhoff' final equation was a partial differential equation of fourth order for the deflection. But it was well-known that one has satisfy three boundary conditions in the general case. Kirchhoff solved this problem introducing a combination of the transverse shear force and the torsion moment (Kirchhoff's Ersatzkraft) and special edge forces. Kirchhoff's theory failed if we have thick plates or sandwiches since the constraints of the kinematics are no more valid. The theory was seriously improved about 100 years later (s., for example, Reissner, 1944, 1945; Hencky, 1947; Mindlin, 1951). In the various improved theories, similar to Timoshenko's beam theory additional degrees of freedom (cross-section rotations) were introduced, so that transverse shear was considered in an approximate sense. Such type of theory is named first order shear deformation theory. The introduction of independent rotations is in some cases not enough, since it is assumed that any cross-section will be plane before and after deformation. To solve this problem, Ambartsumyan (1970) introduced an additional distribution function in the thickness direction. A less restrictive approximation was proposed by Levinson (1980) and Reddy (1984b), among others. These refined theories which named third order shear

[1] Note that the term stiffness always means a combination of material parameters and geometrical characteristics.

deformation theories can be understood as theories that introduce additional degrees of freedom, or as some part of a power series expansion. The first suggestion of this type was done by Lo et al (1977). A generalization of the power series approach was given in Meenen and Altenbach (2001).

An alternative approach considering assumptions for the stress state was suggested by Reissner (1944, 1945). It can be shown that Mindlin's and Reissner's plate theories contain partly identical equations, but the coefficients take slightly different values and their physical interpretation is not the same. The similarities are so great that in the Finite Element references as usual the name *Reissner-Mindlin element* is used.

Pure mathematical approaches are mostly based on power series, trigonometric functions, on special functions, asymptotic integration, etc. (s., e.g., Kienzler, 1982; Preußer, 1984; Reissner, 1985; Vekua, 1985; Touratier, 1991). The mathematical approaches are very helpful if one wants to check the accuracy of the given approximation. A nice comparison of the different approximations in the series approach is given in Kienzler (2002) where first time was shown a new approach based on consistent formulations. An actual reference for the consistent approach and the comparison with other approaches is given in Kienzler and Schneider (2016); Schneider et al (2014).

The direct approach is based on the a priori introduction of a two-dimensional deformable surface. This approach was applied by Green et al (1965); Palmow and Altenbach (1982); Rothert (1973); Zhilin (1976), among others. The main advantage of these theories is that their derivation does not rely on assumptions or series expansions and is mathematically and physically as strong and exact as the three-dimensional continuum mechanics. This approach is still under discussion, since the application is not trivial, and a relationship between the constitutive laws of the deformable surface and the corresponding three-dimensional body has to be found.

The development of shell theories was similar. One has to distinguish

- theories based on hypotheses (s., for example, Aron, 1874; Novozhilov, 1970; Donnell, 1976; Love, 1906; Mushtari and Galimov, 1961),
- theories formulated with help of mathematical techniques (Vekua, 1985),
- theories introducing deformable surfaces (Naghdi, 1972, among others)

Details will be not discussed here.

1.2 Modeling of Composites

Development and applications of composite materials and structural elements composed of composite materials have been very rapid in the last decades. The motivation for this development is the significant progress in material science and technology of the composite constituents. In addition, the requirements for high performance materials are not only in aircraft and aerospace structures. The increasing performance of composites is also related to the development of very powerful experimental equipments and numerical methods. With the development of composite materials a new material design is possible that allows an optimal material composition in connection with the structural design. In addition, with the application of electrorheological, magnetorheological, etc. materials as layers in laminates one can suppress vibrations, prevent buckling, among others.

A useful and correct application of composite materials requires a close interaction of different engineering disciplines such as structural design and analysis, material science, mechanics of materials, process engineering, etc. The main topics of composite material research and technology are

- investigation of all characteristics of the constituents and the composite materials,
- material design and optimization for the given working conditions,
- development of analytical modeling and solution methods for determining material and structural behavior,
- development of experimental methods for material characteristics, stress and deformation states, failure,
- modeling and analysis of creep, damage, and life prediction,
- development of new and efficient fabrication and recycling procedures, among others.

1.2.1 Preliminary Remarks and Definitions

In material science the following classification of structural materials is given

- metals,
- ceramics, and
- polymers.

Sometimes there are more classes but we will limited us to these three classes. They are related to different application fields. It is difficult to give an assessment of the advantages and disadvantages of these basic material classes, because each of them covers whole groups of materials within which the range of properties is often as broad as the differences between the material classes. Some obvious characteristic properties can be identified (Altenbach et al, 2018):

- Mostly metals are of medium to high density. They have good thermal stability and can be made corrosion-resistant by alloying. Metals have useful mechanical characteristics and it is moderately easy to shape and join these materials. For this reason metals became the preferred structural engineering material, they posed less problems to the designer than either ceramic or polymer materials.
- Ceramic materials have great thermal stability and are resistant to corrosion, abrasion and other forms of attack. They are very rigid but mostly brittle and can only be shaped with difficulty.
- Polymer materials (plastics) are of low density, have good chemical resistance but lack thermal stability. They have poor mechanical properties, but are easily fabricated and joined. Their resistance to environmental degradation, e.g. the photomechanical effects of sunlight, is moderate.

Let us introduce some basic definitions with respect to the material behavior.

Definition 1.5 (Homogeneous material behavior).
A material is called homogeneous if its properties are the same at every point and therefore independent of the location.

Homogeneity is associated with the scale of modeling or the so-called representative volume and the definition describes the averaged material behavior on a macroscopic (phenomenological) level. On the microscopic level materials can be described as homogeneous, quasi-homogeneous, inhomogeneous or heterogeneous.

Definition 1.6 (Quasi-homogeneous material behavior).
A material is quasi-homogeneous if its effective (averaged) properties are the same at every point.

Definition 1.7 (Inhomogeneous material behavior).
A material is inhomogeneous if its properties depend on the location but there is only one phase.

Definition 1.8 (Heterogeneous material behavior).
A material is heterogeneous if its properties depend on the location but there are two or more phases.

In addition, the material behavior can be dependent on the loading direction.

Definition 1.9 (Isotropic material behavior).
A material is isotropic if its properties are independent of the orientation, they do not vary with direction.

Definition 1.10 (Anisotropic material behavior).
If the properties are changing with the loading direction the material behavior is called anisotropic.

A general anisotropic material has no planes or axes of material symmetry. Special cases of material symmetries are orthotropy (three orthogonal planes of symmetry), transverse isotropy (three orthogonal planes of symmetry and one axis of symmetry in one of the planes of symmetry), among others.

Definition 1.11 (Monolithic material).
If a material contains one constituent or one single phase only, the material is called monolithic.

The above mentioned classes of materials are in many cases on the macroscopic level more or less monolithic, homogeneous and isotropic.

1.2.2 Composite Materials

The group of materials which can be defined as composite materials is extremely large.

Definition 1.12 (Composite material).
A composite material (or shortened to composite) is any material that is a combination of two or more constituent materials and has material properties derived from the individual constituents. The constituents can be from the same material class or different classes.

In dependence of fabrication the properties may have the combined characteristics of the constituents or they are substantially different. Sometimes the material properties of a composite may exceed those of the constituents. The definition of composite materials include:

- reinforced concrete and masonry,
- composite wood such as plywood,
- reinforced plastics, such as fibre-reinforced polymer (long or short fibres) or fiberglass,
- ceramic matrix composites (composite ceramic and metal matrices),
- metal matrix composites and
- other advanced composite materials.

In many cases composites have some excellent properties like low weight in combination with high strength and stiffness which is necessary in modern structural design.

The simplest case of a composite is an assembly of two materials of same or different nature. The special class of reinforced plastics is related to one discontinuous material, called the reinforcement, and another material, mostly less stiff and weaker, continuous and called the matrix. In this case the properties of the composite depend on (Altenbach et al, 2018):

- the properties of the constituents,
- the geometry of the reinforcements, their distribution, orientation and concentration usually measured by the volume fraction or fiber volume ratio and
- the nature and quality of the matrix-reinforcement interface.

The prediction of the interface properties is up to now a problematic task. The properties of the fibres and the matrix can be measured separately, but the interface does not exist separately. As usual the properties of the interface are computed by inververse problems. Models of the interface behavior are presented in Hill (1963, 1964); Gurtin and Murdoch (1975); Murdoch (2005); Hashin (1991) among others. An overview on interface modeling is given, for example, in Nazarenko et al (2018a,b).

Summarizing the aspects defining a composite as a mixture of two or more distinct constituents or phases it must be considered that all constituents have to be present in reasonable proportions that the constituent phases have quite different properties from the properties of the composite material and that man-made composites are produced by combining the constituents by various means (Altenbach et al, 2018). Figure 1.1 shows typical examples of composites with different types of reinforcement. The reinforcement can be more or less regular or chaotic. Composites can be classified by their form and distribution of the constituents (Fig. 1.2). The

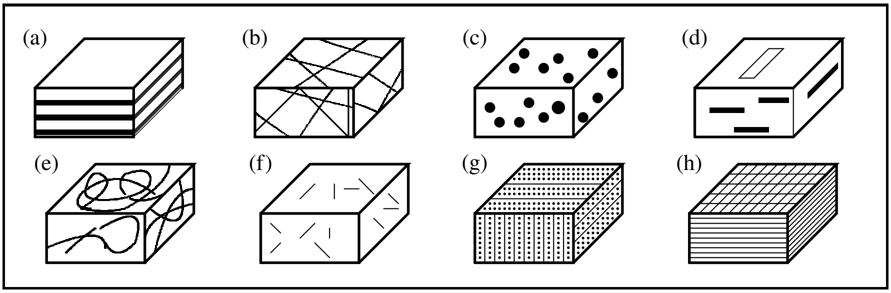

Fig. 1.1 Examples of composite materials with different forms of constituents and distributions of the reinforcements. (a) Laminate with uni- or bidirectional layers, (b) irregular reinforcement with long fibres, (c) reinforcement with particles, (d) reinforcement with plate strapped particles, (e) random arrangement of continuous fibres, (f) irregular reinforcement with short fibres, (g) spatial reinforcement, (h) reinforcement with surface tissues as mats, woven fabrics, etc. (Altenbach et al, 2018, with courtesy of Springer Publisher).

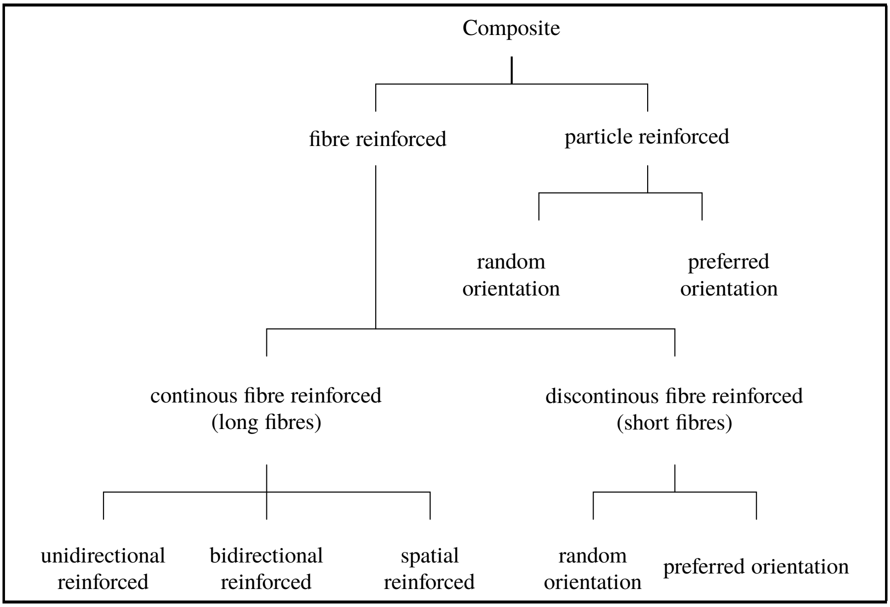

Fig. 1.2 Classification of composites (Altenbach et al, 2018, with courtesy of Springer Publisher).

reinforcement constituent can be described, for example, as fibrous or particulate. The fibres are assumed to be long (size of the structural element) or short (in comparison to the structural element's dimension). Long fibres are mostly arranged in uni- or bidirectional reinforcements, but also irregular reinforcements by long fibres are possible. The arrangement and the orientation of the fibres determine the mechanical properties of composites including the type of anisotropy. Particulate reinforcements can be spherical, platelet or of any regular or irregular geometry. Their arrangement may be random or regular with preferred orientations. In many practical applications particulate reinforced composites are considered to be randomly oriented and the mechanical properties are quasi-homogeneous and isotropic. In the case of mold injection manufacturing the particle orientation over the cross-section is partly in the flow direction, partly orthogonal to the flow direction, and partly chaotic (Gupta and Wang, 1993; Saito et al, 1998, 2000) and it was established that the structural elements can show anisotropic behavior (Altenbach et al, 2003, 2005). The preferred orientation in the case of long fibre composites is unidirectional for each layer or lamina. In this case, we have in each layer an transversely-isotropic material behavior. With the variation of the fibre angle in each layer one gets finally a laminate with anisotropic stiffness properties.

Composite materials can also be classified by the nature of their constituents. According to the nature of the matrix material we have organic, mineral or metallic matrix composites (Altenbach et al, 2018):

- Organic matrix composites are polymer resins with fillers. The fibres can be mineral (glass, etc.), organic (aramid, etc.) or metallic (aluminium, etc.).
- Mineral matrix composites are ceramics with metallic fibres or with metallic or mineral particles.
- Metallic matrix composites are metals with mineral or metallic fibres.

Fibre reinforced polymer resins can be used only in a low temperature range up to 200^0 to 300^0 C. The two basic classes of resins are thermosets and thermoplastics. Typical thermoset matrices include Epoxy, Polyester, Polyamide and Vinyl Ester, among popular thermoplastics are Polyethylene, Polystyrene and Polyether-ether-ketone (PEEK). Ceramic based composites can also be used in a high temperature range up to 1000^0 C and metallic matrix composites in a medium temperature range.

Polymer matrix composites are characterized by relatively low costs, simple manufacturing and high strength. Their main drawbacks are the low working temperature, high coefficients of thermal and moisture expansion and, in certain directions, low elastic properties. Polymer matrix composites are usually reinforced by fibres to improve such mechanical characteristics as stiffness, strength, etc. Fibres can be made of different materials (glass, carbon, aramid, etc.). Glass fibres are widely used because their advantages include high strength, low costs, high chemical resistance, etc., but their elastic modulus is very low and also their fatigue strength. Graphite or carbon fibres have a high modulus and a high strength and are very common in aircraft components. The functional requirements of fibres and matrices in a fibre reinforced polymer matrix composite can be summarized as follows:

- fibres should have a high modulus of elasticity and a high ultimate strength,
- fibres should be stable and retain their strength during handling and fabrication,
- the variation of the mechanical characteristics of the individual fibres should be low, their diameters uniform and their arrangement in the matrix is more or less regular,
- matrices have to bind together the fibres and protect their surfaces from damage,
- matrices have to transfer stress to the fibres by adhesion and/or friction and

- matrices have to be chemically compatible with fibres over the whole working period.

In some new applications more and more elastomers are used as a material of a sandwich or laminate layer. An elastomer is a polymer characterized by viscoelastic properties that means it shows viscose (time-dependent) and elastic (spontaneous) behavior. Sometimes, such behavior is named rubber-like behavior. An elastomer has very weak intermolecular forces, and the Young's modulus is low and the failure strain is high if we compare with other materials. Elastomers are amorphous polymers. At ambient temperatures, such rubbers are relatively soft and the Young's modulus $E \approx 3$ MPa. The deformability is high. In structures discussed later especially elastomeric layers are used as damping and insulating elements. In these cases electro- or magnetorheological elastomers consist of polymeric matrix with embedded micro- or nano-sized polarizable or ferromagnetic particles. In some application instead of elastomers are used electro- or magnetorheological fluids.

1.2.3 Volume Fibre Fraction

The fibre length, their orientation, their shape and their material are main factors which contribute to the mechanical performance of a composite. Their volume fraction usually lies between 0.3 and 0.7. The matrix materials generally have low mechanical properties as compared to fibres, but they influence many characteristics of the composite such as the transverse and shear moduli, the strength, the thermal resistance and expansion, etc.

The most important factor which determines the mechanical behavior of a composite material is the proportion of the matrix and the fibres expressed by their volume or weight fraction. These fractions can be established for a two phase composite in a simple way. The volume V of the composite is made from a matrix volume V_m and a fibre volume V_f

$$V = V_\mathrm{f} + V_\mathrm{m} \tag{1.1}$$

Then the following relations hold

$$v_\mathrm{f} = \frac{V_\mathrm{f}}{V}, \qquad v_\mathrm{m} = \frac{V_\mathrm{m}}{V} \tag{1.2}$$

with

$$v_\mathrm{f} + v_\mathrm{m} = 1, \qquad v_\mathrm{m} = 1 - v_\mathrm{f}$$

as the fibre and the matrix volume fractions, respectively. In a similar way the weight or mass fractions of fibres and matrix can be defined. The mass M of the composite is made from M_f and M_m

$$M = M_\mathrm{f} + M_\mathrm{m}$$

and

$$m_{\mathrm{f}} = \frac{M_{\mathrm{f}}}{M}, \qquad m_{\mathrm{m}} = \frac{M_{\mathrm{m}}}{M} \tag{1.3}$$

with

$$m_{\mathrm{f}} + m_{\mathrm{m}} = 1, \qquad m_{\mathrm{m}} = 1 - m_{\mathrm{f}}$$

m_{f} and m_{m} are the mass fractions of fibres and matrices, respectively. With the relation between volume, mass and density $\rho = M/V$, we can link the mass and the volume fractions

$$\begin{aligned} \rho = \frac{M}{V} = \frac{M_{\mathrm{f}} + M_{\mathrm{m}}}{V} &= \frac{\rho_{\mathrm{f}} V_{\mathrm{f}} + \rho_{\mathrm{m}} V_{\mathrm{m}}}{V} \\ = \rho_{\mathrm{f}} v_{\mathrm{f}} + \rho_{\mathrm{m}} v_{\mathrm{m}} &= \rho_{\mathrm{f}} v_{\mathrm{f}} + \rho_{\mathrm{m}} (1 - v_{\mathrm{f}}) \end{aligned} \tag{1.4}$$

Starting from the total volume of the composite (1.1) we obtain

$$\frac{M}{\rho} = \frac{M_{\mathrm{f}}}{\rho_{\mathrm{f}}} + \frac{M_{\mathrm{m}}}{\rho_{\mathrm{m}}}$$

and

$$\rho = \frac{1}{\dfrac{m_{\mathrm{f}}}{\rho_{\mathrm{f}}} + \dfrac{m_{\mathrm{m}}}{\rho_{\mathrm{m}}}} \tag{1.5}$$

The equations of this subsection can be easily extended to multi-phase composites. Mass fractions are easier to measure in material manufacturing, but volume fractions appear in the theoretical equations for effective moduli. Therefore, it is helpful to have simple expressions for shifting from one fraction to the other. The volume fractions are the base of computing the material parameters of a reinforced composites. The averaged Young's modulus, shear modulus or Poisson's ratio can be expressed using rheological models combining the fibre and matrix properties with the help of parallel connection, connection in series or improved formulaes. The last one are based as usual on fitting experimental data. Some of these expressions are discussed in detail in Altenbach et al (2018).

The quality of a composite material decreases with increase in porosity. The volume of porosity should be less than 5 % for a medium quality and less than 1 % for a high quality composite. If the density is measured experimentally (ρ_{exp}) and calculated with (1.5) (ρ_{theor}), the volume fraction of porosity is given by

$$v_{\mathrm{por}} = \frac{\rho_{\mathrm{theor}} - \rho_{\mathrm{exp}}}{\rho_{\mathrm{theor}}} \tag{1.6}$$

1.2.4 Modeling of Structures Composed of Composites

Composite materials consist of two or more constituents and the modeling, analysis and design of structures composed of composites are different from conventional

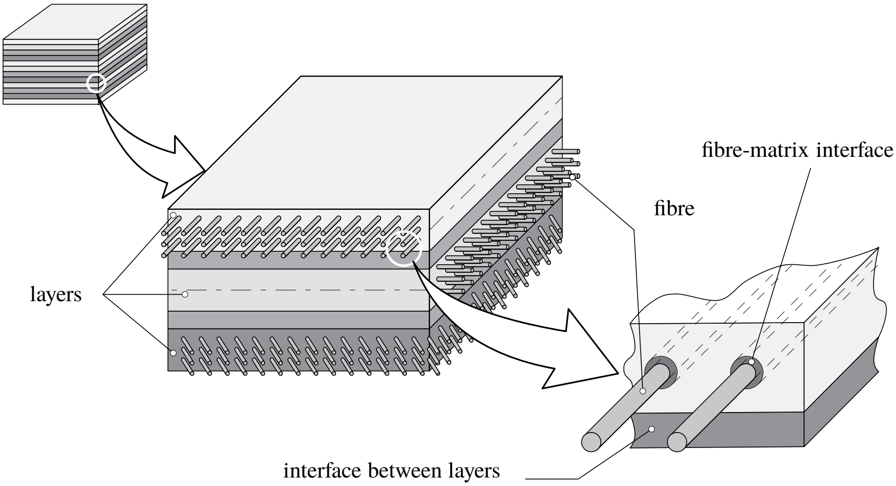

Fig. 1.3 Laminated plates - levels of modeling.

materials such as steel. For example, if we have a laminated structure there are two levels of modeling (Fig. 1.3).

At the micro-mechanical level the average properties of a single reinforced layer (a lamina or a ply) have to be determined from the individual properties of the constituents, the fibres and matrix, and may be the fibre-matrix interface. The average characteristics include the elastic moduli, the thermal and moisture expansion coefficients, etc. The micro-mechanics of a lamina does not consider the internal structure of the constituent elements, but recognizes the heterogeneity of the ply. The micro-mechanics is based on some simplifying approximations. These concern the fibre geometry and packing arrangement, so that the constituent characteristics together with the volume fractions of the constituents yield the average characteristics of the lamina. Note that the averaged properties are derived by considering the lamina to be quasi-homogeneous.

The calculated values of the averaged properties of a lamina provide the basis to predict the macrostructural properties. At the macro-mechanical level, only the averaged properties of a lamina are considered and the microstructure of the lamina is ignored. In some case the interfaces between the layers are taken into account. The properties along and perpendicular to the fibre direction, these are the principal directions of a lamina, are recognized and the so-called on-axis stress-strain relations for a unidirectional lamina can be developed. Loads may be applied not only on-axis but also off-axis and the relationships for stiffness and flexibility, for thermal and moisture expansion coefficients and the strength of an angle ply can be determined. A laminate is a stack of laminae. Each layer of fibre reinforcement can have various orientation and in principle each layer can be made of different materials. Knowing the macro-mechanics of a lamina, one develops the macro-mechanics of the laminate. Averaged stiffness, flexibility, strength, etc. can be determined for the whole laminate. The structure and orientation of the laminae in prescribed sequences to a laminate lead to significant advantages of composite materials when compared to a conventional monolithic material. In general, the mechanical response of laminates is anisotropic.

One very important group of laminated composites are sandwich structures. They as usual consist of two thin faces (the skins or sheets) sandwiching a core (Fig. 1.4). The faces are made of high strength materials having good properties under tension such as metals or fibre reinforced laminates while the core is made of lightweight materials such as foam, resins with special fillers, called syntactic foam, having good properties under compression. Sandwich composites combine lightness and flexural stiffness.

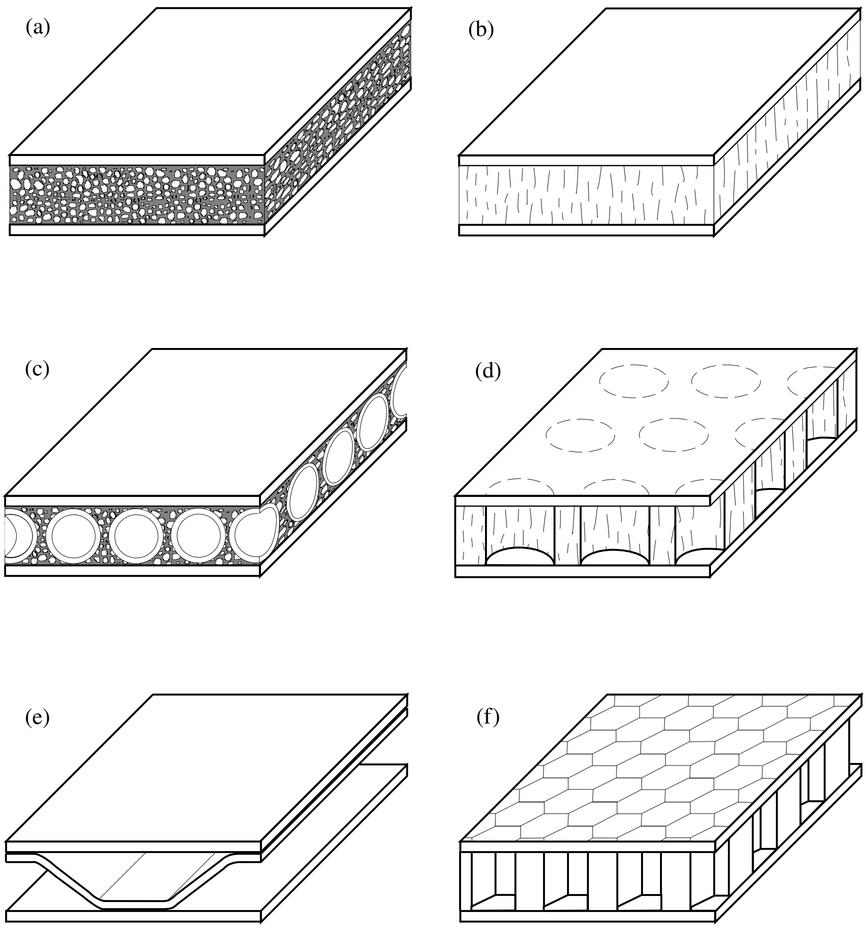

Fig. 1.4 Sandwich materials with solid and hollow cores. (a) foam core, (b) balsa wood core, (c) foam core with fillers, (d) balsa wood core with holes, (e) folded plates core and (f) honeycomb core (Altenbach et al, 2018, with courtesy of Springer Publisher).

In contrast to classical sandwiches in photovoltaic applications we have a opposite situation: thick stiff skin layers and a very thin and very weak core layer (Fig. 1.5). A detailed discussion of the specific properties and the mechanical analysis is given, for example, in Aßmus (2019).

When the micro- and macro-mechanical analysis for laminae and laminates or sandwiches are carried out, the global behavior of laminated composite materials is known. The last step is the modeling on the structure level and to analyze the global behavior of a structure made of composite material. By adapting the classical tools of structural analysis on anisotropic elastic structure elements the analysis of simple structures as beams or plates may be achieved by analytical methods, but for more general boundary conditions and/or loading and for complex structures, numerical methods are used.

Summarizing the different size scales of mechanical modeling structure elements composed of fibre reinforced composites it must be noted that, independent of the different possibilities to formulate beam, plate or shell theories, three modeling levels must be considered (Altenbach et al, 2018):

- *The microscopic level*, where the average mechanical characteristics of a lamina have to be estimated from the known characteristics of the fibres and the matrix material taking into account the fibre volume fracture and the fibre packing arrangement. The micro-mechanical modeling leads to a correlation between constituent properties and average composite properties. In general, simple mixture rules are used in engineering applications. If possible, the average material characteristics of a lamina should be verified experimentally. On the micro-mechanical level a lamina is considered as a quasi-homogeneous orthotropic material.
- The *macroscopic level*, where the effective (average) material characteristics of a laminate have to be estimated from the average characteristics of a set of laminae taking into account their stacking sequence. The macro-mechanical modeling leads to a correlation between the known average laminae properties and effective laminate properties. On the macro-mechanical level a laminate is considered generally as an equivalent single layer element with a quasi-homogeneous, anisotropic material behavior.

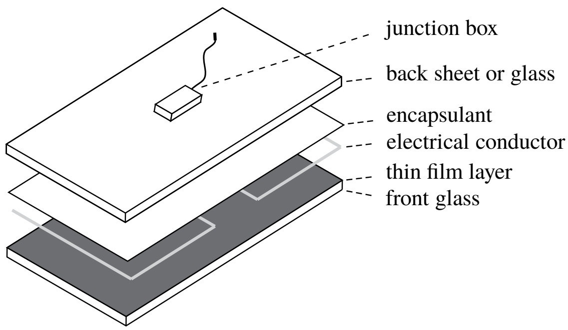

Fig. 1.5 Components of thin film solar module - *anti-sandwich* (Schulze et al, 2012; Weps et al, 2013).

- The *structural level*, where the mechanical response of structural members like beams, plates, shells etc. have to be analyzed taking into account possibilities to formulate structural theories of different order.

1.2.5 Material Characteristics of the Constituents

The optimal design and the analysis of structural elements requires a detailed knowledge of the material properties, which depend on the nature of the constituent materials but also on manufacturing. For structures made of composites as usual we have a more complicated situation. The list of composite materials is numerous but available standards and specifications are rare. The properties of each material used for both reinforcements and matrices of composites are extremely diversified. Structural design based on composite materials requires detailed knowledge about the material properties of the singular constituents of the composite and the fabrication of the composites for optimization of the material in the frame of structural applications and also detailed codes for modeling and analysis are necessary.

Let us focus on fibre reinforced composites with polymer resins. Material tests of the constituents of composites are in many cases a complicated task and so the material data in the literature are limited (Altenbach et al, 2018, and the references therein). In engineering applications the averaged data for a lamina are often tested to avoid this problem and in order to use correct material characteristics in structural analysis. The main properties for the estimation of the material behavior are

- density ρ,
- Young's modulus E, Poisson's ratio ν, shear modulus G,
- ultimate strength σ_{u} and
- thermal expansion coefficient α.

The material can be made in bulk form or in the form of fibres. To estimate properties of a material in the form of fibres, the fibre diameter d can be important.

The estimate of electro- and magneto-rheological properties is more complicated and will be not discussed here.

1.3 Modeling of Laminated Structures: Different Approaches

Many theories have been developed to model the mechanical behavior of laminated thin-walled structures. The most accurate models are based on the three-dimensional elasticity theory. However, this approach leads to complex problems of analysis the stress-strain state and rigid-body motions (s., among many others Shakeri et al, 2006; Saviz et al, 2007; Malekzadeh et al, 2009; Kulikov and Plotnikova, 2013) and since the computational effort is great this approach has found limited applications in the engineering practice. Taking into account thinness of the beams, plates and shell (in comparison to its other dimensions) researchers make simplifying assumptions, called hypotheses, which result in two- or one-dimensional representation of a shell with some predictable and reasonable accuracy.

Let us focus our attention at first to plates and shells. Theories for beams can be established in a similar manner. The 2D theories for thin laminated plates and shells may be divided into two basic models: the equivalent single layer (ESL) model and the layer-wise (LW) model. A short review of these models will be presented below. Later in this monograph the ESL model will be preferred. The ESL theories may be classified, for example, as in Qu et al (2013): classical shell theory (CST), the first-order shear deformation theory (FSDT), and the higher-order shear deformation theory (HSDT).

The CST is based on the Kirchhoff-Love hypotheses (Kirchhoff, 1850; Love, 1888). In the original paper of Kirchhoff the following two hypotheses are mentioned as the base of his theory

- straight line normal to the undeformed middle surface remains straight and normal to the deformed middle surface,
- the elements of the midplane during the deformation have no dilatation.

The first hypothesis results in neglecting the transverse shear strains. Considering the second hypothesis, in classical Kirchhoff theory one gets only an equation for the deflection. Love introduced also the in-plane displacements for the midplane of the shell.

Depending on different assumptions related to the strain-displacement, constitutive and equilibrium equations the CSTs may be conventionally subdivided into theories named by Ambartsumyan (1970), Donnell (1976), Flügge (1973), Mushtari and Galimov (1961), Love (1906), Mindlin (1951), Novozhilov (1970), Reissner (1944), Sanders (1959), Vlasov (1944), etc. All these approaches lead to three differential equations w.r.t. three unknowns. Surveys on the classical theories, initially derived for isotropic plates and shells, may be found in the monographs of Leissa (1973); Reddy (2004), and in a early work of Naghdi (1956). Obviously, the first studies on mechanical behavior of laminated plates and shells were performed in the framework of the CSTs (Reissner and Stavsky, 1961; Stavsky, 1961; Dong et al, 1962; Yang et al, 1966; Whitney and Leissa, 1969; Ambartsumyan, 1970; Bert, 1976, 1980). These approaches neglecting shear deformations have been shown to be adequate for the static analysis of thin laminates (Pagano, 1969, 1970). Considering dynamic problems for layered composite shells, such theories may be exploited in the low-frequency range (Qu et al, 2013), but they result in errors up to 30% in the prediction of large natural frequencies (Reddy, 2004). For thicker plates and shells pliable in shear as well as for thin layered structures consisting of a *soft* layer(s), the classical theories become inadequate even for predicting static deflections and stresses.

The first attempts to overcome the shortcomings of the CSTs were made by Reissner (1945, 1952); Lurie (1947); Hildebrand et al (1949) and Mindlin (1951). They proposed the so-called FSDTs in accordance to which the deflection w is independent of the normal coordinate z and the in-plane displacements u_1, u_2 of the middle surface are linear functions of z. These theories take into account the transverse shear strain components which are constant along the thickness. A detailed description of these theories may be found in Reissner (1975); Reissner and Wan (1982). In fact, these approaches may be considered as the development of the Timoshenko's beam theory (Timoshenko, 1921). Later, the extension of the FSDTs to laminated plates has been performed by Yang et al (1966); Whitney and Pagano (1970); Sun (1971), and a number of similar theories for thin and moderately thick laminated shells have been developed by Dong et al (1962); Dong and Tso (1972); Hsu and Wang (1970); Zukas and Vinson (1971); Reddy (1984a); Qatu (1999); Toorani and Lakis (2000); Auricchio and Sacco (2003). Aforementioned theories result, as a rule, in five coupled equations w.r.t. five unknowns. Recently Wang et al (2018) proposed a simple first-order shear deformation shell theory which contains only four unknowns and can be regarded as an enhanced CST with the consideration of the effects of transverse shear deformation and rotary inertia terms. The main defect of the FSDTs (as well as of classical shell theories) is that the traction conditions at the shell surfaces are violated and so, it requires shear correction factors (Reissner, 1944; Mindlin, 1951). The problem is that the shear correction factors are difficult to determine for arbitrary laminated plates and shells because they depend on the geometrical and lamination parameters, loading and boundary conditions as well (among many others, s. Srinivas et al, 1970; Chow, 1971; Whitney, 1973; Bert, 1973; Wittrick, 1987; Vlachoutsis, 1992).

In spite of the above mentioned drawbacks of the FSDTs, several improvements are suggested to study numerous applied problems on mechanical behavior of laminated shells. Qatu (1999) studied free vibrations of laminated simply supported cylindrical shells. Taking into account transverse shear deformation and rotary inertia effects as well, Toorani and Lakis (2001) considered the coupled problem on free vibrations of anisotropic laminated cylindrical shell partially or completely filled with liquids. Wang et al (2002) investigated the propagation of waves in orthotropic laminated spherical shells. Free vibrations of thick laminated anisotropic non-circular cylindrical shells were analyzed by Ganapathi and Haboussi (2003). The effect of transverse shear and rotary inertia on waves in laminated piezoelectric cylindrical shells in thermal environment was examined by Dong and Wang (2007). Ribeiro (2009) studied the effect of membrane inertia and shear deformation on geometrically nonlinear vibrations of open cylindrical laminated shells. Using a unified variational formulation based on the FSDT, Qu et al (2013) considered free, steady-state and transient vibrations of composite laminated shells of revolution subjected to various combinations of boundary conditions.

Further improvements of the shear deformable theories were based on quadratic, cubic and higher expansions at least of the in-plane displacements u_1, u_2 in terms of the transverse coordinate z. These theories are named higher-order shear deformation theories (HSDTs). First of all, we refer to Whitney and Sun (1973, 1974). They proposed a second-order theory in which the transverse displacement w is assumed as a linear function of the thickness coordinate z and the in-plane displacements u_1, u_2 of the reference surface are expanded as quadratic functions of z. This approach results in eight coupled equations w.r.t. eight unknowns. Due to the large number of dependent unknown magnitudes, this theory is results in more computational effort then the FSDTs. Furthermore, this second-order theory requires a correction to the transverse shear stiffness. In contrast to this theory, Reddy (1984a); Reddy and Liu (1985) developed a third-order but more simple shear deformation theory of laminated plates and shells. Although, this theory is based on a displacement field in which the in-plane displacements u_1, u_2 are expanded as cubic functions of z and the normal deflection w is constant through the thickness, it contains only five unknowns as in FSDTs but requires no shear correction factors. To date, there is a wide variety of higher-order theories (s. among many others Librescu et al, 1987; Librescu and Khdeir, 1988; Grigolyuk and Kulikov, 1988b; Mallikarjuna and Kant, 1993, 2002; Batra and Vidoli, 2002b; Ganapathi et al, 2002; Khare et al, 2003; Swaminathan and Ragounadin, 2004; Khare and Rode, 2005; Balah and Al-Ghemady, 2005; Tovstik and Tovstik, 2007; Amabili, 2015; Tovstik and Tovstik, 2017; Shi et al, 2018). In addition, we refer to the so-called *New HSDTs* proposed by Karama et al (2009); Aydogdu (2009); Mantari et al (2011a,b). In these theories, the transverse displacement is assumed to be independent of the thickness coordinate z, and the in-plane displacements of the reference surface are expanded as a combination of exponential and polynomial functions of z. Most of the well known shear deformable theories available in literature, including the aforementioned ones, were developed as particular cases. A common property of these theories is that they lead to a system of differential equations for five unknowns and comply with the traction-free boundary conditions on the top and bottom surfaces of the laminated plate/shell. Recently, Viola et al (2013) proposed a general variant of HSDTs which contains nine independent displacement parameters and unifies most of the known higher-order theories due to the incorporation of general shear functions.

In high accurate layer-wise theories (LWTs), accounting the zig-zag effects, each layer is considered as a shell with interface boundary conditions guaranteing the continuity of the displacement or/and stress fields. The early investigations in this direction, which were performed by Hsu and Wang (1970); Cheung and Wu (1972); Srinivas (1973); Sun and Whitney (1973); Bolotin and Novichkov (1980); Murakami (1986); Barbero et al (1990); Cho et al (1991); Gaudenzi et al (1995); Carrera (1998a,b), have shown the superiority of layer-wise models over ESL ones. Indeed, the LWTs provide more realistic kinematics of multi-layered plates and shells and turn out to be more accurate and effective to predict local effects (Reddy and Robbins, 1994; Carrera, 2001; Batra and Vidoli, 2002a; Khare et al, 2003; Demasi, 2009), high frequency response (Braga and Rivas, 2005; Oh, 2007) and formulate shell finite elements (Moreira et al, 2006; Yasin and Kapuria, 2013; Wu et al, 2018; Kordkheili and Soltani, 2018). To date, there are a lot of papers proposing improved and refined variants of layer-wise models (among many others, s. Sahoo and Singh, 2014; Naumenko and Eremeyev, 2014; Iurlaro et al, 2015; Naumenko and Eremeyev, 2017; Carrera et al, 2015; Akoussan et al, 2017; Shi et al, 2018; Flores et al, 2018) and studying topical problems on mechanical behavior of laminated plates and shells based on these models. Thus, Starovoitov and Leonenko (2010) analyzed free and resonant vibrations of circular sandwich plate using the zig-zag theory, Cetkovic (2015) studied thermo-mechanical bending of laminated composite and sandwich plates subjected to mechanical load and non-uniform temperature field, and in more recent papers (Nikbakht et al, 2017) employed the full layer-wise method to analyze the elastic bending of functionally graded plates up to yielding, Treviso et al (2017) used the refined zig-zag theory (RZT) in the framework of shell elements for vibration analysis of laminated and sandwich shells and shown that the RZT element can be effectively used to reduce the computational costs of dynamic simulations of laminated structures, and Moita et al (2018) developed a simple and efficient finite element model to examine damped vibrations of multilayered sandwich plates and shells with a viscoelastic core sandwiched between functionally graded material layers, and including piezoelectric layers.

Despite the variety of layer-wise shell theories, they have not gained wide popularity in modeling practical shell vibration problems because of extreme complexity of theoretical formulations and high computational costs. There are a few examples when only layer-wise theories yields in correct results as shown in Schulze et al (2012); Weps et al (2013); Aßmus (2019). It should be noted that each model of laminated plates and shells has its advantages as well as disadvantages (Reddy, 1993), and the correct choice of the theory depends on many factors, such as the shell geometry, the number of layers, the material of which each layer is made, as well as the loads. Another point affecting the choice of the shell model is the expected variability of the displacements, strains and stresses. So, if vibrations or buckling are accompanied by formation of a large number of short waves/dents, then the full system of governing equations is, as a rule, simplified and reduced to the shallow shell equations with a less number of unknowns (Grigolyuk and Kulikov, 1988b).

The above literature review does not pretend to be complete. We refer readers to the survey articles by Grigolyuk and Kulikov (1988a); Kapania (1989); Kapania and Raciti (1989a,b); Soldatos and Timarci (1993); Altenbach (1998); Toorani and Lakis (2000); Reddy (1993); Reddy and Wang (2000); Carrera (2002, 2003a); Qatu (2002); Qatu et al (2010); Atteshamuddin et al (2015); Caliri et al (2016), and books by Qatu (2004); Reddy (2004); Gorshkov et al (2005). One should also mention Carrera (2003b); Demasi et al (2017) where a unified formulation is proposed and models, types and classes of theories for laminated plates and shells are described.

Completing the short overview of existing theories for laminated shells, we draw attention to the approach developed by Grigolyuk and Kulikov (1988b) which will be used below. This ESL theory is based on the generalized kinematical hypotheses of Timoshenko for the in-plane displacements u_1, u_2 of the reference surface and the parabolic distribution of transverse shear stresses through both each layer thickness and the entire shell thickness. It complies with the traction-free boundary conditions on the top and bottom surfaces of a laminated shell and guaranties the continuity of transverse shear stresses in the direction of the thickness coordinate z. Recently, this model was adapted by Mikhasev et al (2011) to laminated cylindrical shells composed of smart materials (magnetorheological elastomers and electrorheological composites). The choice of this theory can be explained by the following items:

- if the stress and strain state of a shell has a large variability, even if by one coordinate, then the full system of differential equations w.r.t. five unknowns is readily simplified and reduced to three equations written in terms of the displacement, stress and shear functions χ, Φ, ϕ;

- the governing equations written in terms of χ, Φ, ϕ completely coincide with similar equations derived by Tovstik and Tovstik (2007, 2017) from the 3D theory of elasticity, the assumed model unifying the simple equations of the Mushtary-Donnell-Vlasov technical theory and the HSDT equations;
- this theory is simple enough for prediction of the mechanical behavior of multi-layered shells, including smart structures;
- the accuracy of the governing equations has been verified by finite element simulations (s. Mikhasev et al, 2001; Korchevskaya et al, 2004; Mikhasaev et al, 2004).

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