SPRINGER BRIEFS IN ECONOMICS

Dan S. Felsenthal
Hannu Nurmi

Voting Procedures Under a Restricted Domain

An Examination of the (In)Vulnerability of 20 Voting Procedures to Five Main Paradoxes

Springer

SpringerBriefs in Economics

More information about this series at http://www.springer.com/series/8876

Dan S. Felsenthal · Hannu Nurmi

Voting Procedures Under a Restricted Domain

An Examination of the (In)Vulnerability of 20 Voting Procedures to Five Main Paradoxes

Springer

Dan S. Felsenthal

Hannu Nurmi
Department of Philosophy, Contemporary
History and Political Science
University of Turku
Turku, Finland

ISSN 2191-5504 ISSN 2191-5512 (electronic)
SpringerBriefs in Economics
ISBN 978-3-030-12626-1 ISBN 978-3-030-12627-8 (eBook)
https://doi.org/10.1007/978-3-030-12627-8

Library of Congress Control Number: 2019933905

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

Voting paradoxes are a fascinating subject. Like all paradoxes, they are likely to invoke suspicion and disbelief; things that are supposed to work in an orderly, predictable and 'nice' fashion yield surprising, counterintuitive, and 'nasty' outcomes. Voting paradoxes have the distinction among the other paradoxes that they are related to man-made institutions. Hence, one would expect that they can be avoided by suitable re-drafting of institutions. It turns out, unfortunately, that basically all voting institutions are plagued with some paradoxes. Hence, the choice between voting procedures is a kind of balancing act where an effort to avoid certain types of paradoxes leads to the choice of procedures that are associated with other kinds of paradoxes.

In this booklet, we study a limited number of paradoxes and a pretty large, but still limited, number of procedures. This work complements two preceding booklets by the same authors, viz. *Monotonicity Failures Afflicting Procedures for Electing a Single Candidate* (2017) and *Voting Procedures for Electing a Single Candidate. Proving Their (In)Vulnerability to Various Voting Paradoxes* (2018). In the former text, we analyzed a class of paradoxes where, *ceteris paribus*, voters increase, or decrease, the support of some candidates thereby causing these candidates to be worse off, or better off, respectively. In the latter text, our focus was on a larger set of voting paradoxes. The setting in the two booklets was general: the choices ensuing from various procedures were considered in all possible preference environments. In other words, no restrictions whatsoever were imposed regarding how different are the voters' opinions about the candidates or policy alternatives under examination. A procedure was classified as vulnerable to a voting paradox if even one distribution of opinions—however unlikely from a practical point of view—is found so that the paradox occurs. The present booklet takes a more nuanced look at the procedures and the paradoxes associated with them. More specifically, we focus on opinion distributions where the procedures under investigation initially lead to seemingly stable (in a specific sense) outcomes. We then study how this starting point is reflected in the vulnerability of the procedures to various voting paradoxes.

The present booklet contains eight chapters. Seven of these are joint works by the two authors, while Chap. 4 is authored by Dan S. Felsenthal alone. In the course of working on the chapters, we have incurred intellectual debts to many scholars of whom we would here like to mention Felix Brandt and Stefan Napel. This debt is gratefully acknowledged. We are most grateful to Dr. Martina Bihn—the Editorial Director of Business/Economics and Statistics at Springer—for her efficient handling of the production process and, above all, for her unfailing support and encouragement not only in the production of this booklet but of several earlier works of the authors. Thanks are also due to Springer staff member Judith Kripp for smooth cooperation over the months preceding the publication of this booklet, as well as to project manager Kokila Durairaj of Scientific Publishing Services, Chennai, India, and her team for making the publication of this booklet in its present form possible.

Jerusalem, Israel Dan S. Felsenthal
Turku, Finland Hannu Nurmi

The news of the passing away of Dan S. Felsenthal came just a few days after this booklet was sent to production. I lost a brilliant co-author and a good friend.

Hannu Nurmi

Contents

Chapter 1
Introduction

Abstract Voting paradoxes occur in particular profile domains. For the avoidance of the paradoxes it is therefore important to know if the profiles typically encountered in practice are of such nature that the paradoxes are very unlikely or downright impossible. Ever since the publication of Arrow's theorem, the role of domain restrictions has been appreciated. However, the earlier studies have mainly focused on conditions for rational collective choices through pairwise majority comparisons. In those studies the single-peaked preferences have been found to be an important type of preference similarity that guarantees complete and transitive collective outcomes. This booklet introduces and analyzes a similar preference restriction, viz., the existence in the profile of a Condorcet winner that is elected by the procedure under study. We examine the possibilities of various voting procedures to end up with voting paradoxes under these restricted domains. Thereby we refine the results that establish the vulnerability of some procedures to various kinds of voting paradoxes.

Keywords Voting paradoxes · Single-peaked preferences · Restricted domains · Arrow's theorem

The study of voting procedures has a long albeit discontinuous history. Ideas have been invented, developed, discussed, applied, discarded to be then left into oblivion—sometimes to be re-discovered or re-invented by subsequent generations of scholars and practitioners (see Black, 1958; Colomer, 2013; Hägele & Pukelsheim, 2008; McLean & Urken, 1995; Riker, 1961; Szpiro, 2010; Tangian, 2014). As a continuous research tradition the study of voting procedures has been subsumed under the more general area of Collective Choice (or Social Choice) with origins in the late 1940s. Undoubtedly the best-known result in this field is Arrow's theorem which has been called the Arrow Impossibility Theorem or sometimes Arrow's General Possibility Theorem. Its first version appears in Arrow's doctoral thesis in 1951, but its final polished form appeared in 1963 (Arrow, 1963). It states the incompatibility of a few social choice desiderata. This was to become the standard line of argumentation in the abstract social choice theory.

Let us assume that there are n individuals (voters) in the decision making body and that each one of them is endowed with complete (or connected) and transitive binary preference relation over a set of k candidates (or other decision alternatives,

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_1

such as policies). In its most commonly repeated version Arrow's theorem says that the following four desiderata cannot be satisfied simultaneously by any social choice rule:

1. *Unrestricted domain*: The procedure results in a collective decision under any *n*-tuple of individual preferences.
2. *Independence of irrelevant alternatives*: For any pair of alternatives *x* and y, the collective ranking of *x vis-à-vis y* depends only on the relative ranking of these two alternatives in the individual rankings. So if the two alternatives are ranked in the same way in two profiles by the same voters, their collective ranking must be the same in those rankings.
3. *Pareto property*: If for some *x* and *y*, the former is preferred to the latter by *all* voters, then *y* is not ranked first in the collective ranking.
4. *Non-dictatorship*: There is no voter whose ranking would in all profiles coincide with the collective ranking regardless of the rankings of the other voters.

Arrow's seminal theorem states that these four conditions are not compatible in any social welfare function, i.e., a rule that assigns for any *n*-tuple of individual rankings a complete and transitive collective preference relation. The theorem soon gave rise to a voluminous literature assessing the relevance of the above desiderata 1–4 (see e.g., Kelly, 1978).

The significance of unrestricted domain was in fact dealt with already before the publication of Arrow's theorem by Black in the context of delineating conditions under which the pairwise majority comparison method would end up with an outcome that would be stable in the sense of not being defeatable by the other alternatives in pairwise majority comparison (Black, 1948). Black's discovery is that if the individual preferences can be represented as single-peaked utility curves over a single dimension along which the decision alternatives are located, the median location defeats all the others by a majority.

So, if one can be sure that the collective decisions to be taken in a voting body can always be represented as an aggregation of points situated in a continuum over which the voters have single-peaked preferences, then we can rest assured that the exhaustive pairwise majority comparison yields a complete and transitive ranking over the candidates. Pretty soon it turned out (cf., Sen & Pattanaik, 1969) that single-peakedness of preferences was just one of many 'similarity conditions' that are sufficient to guarantee a majority rule equilibrium. So many in fact that in 1973 Kramer concluded that "thus the search for additional 'similarity' conditions on individual preferences is, in a sense, over" (Kramer, 1973, p. 286). Instead, Kramer raised the issue of whether single-dimensional decisions are typical, or even common, in decision making. His finding is that when dealing with multi-dimensional policy spaces even a very small degree of heterogeneity in tastes or opinions is incompatible with the notion that the pairwise majority comparisons would inevitably lead to an equilibrium outcome.

What these early results suggest is that the types of profiles one is faced with in collective decision making are a highly significant determinant of the success of one's decision making apparatus when the success means the satisfaction of various criteria of goodness. This booklet introduces a heretofore unexplored restriction on preference profiles suggested by one of us (DSF), viz., one where the starting point in assessing a voting procedure is a profile where a Condorcet winner exists and is elected by the procedure under examination.[1] So, our profile restriction says that we focus on profiles where a Condorcet winner not only exists, but also coincides with the choice ensuing from the procedure when applied to the profile. The set of profiles is thus a subset of the Condorcet domains which are characterized by the existence of a Condorcet winner.

Our focus is on some paradoxical results that can occur when applying voting procedures to specific profiles. Analyzing the restricted domain just defined enables us to resort to a type of *a fortiori* reasoning. To wit, suppose that a procedure is known to be vulnerable to a specific voting paradox, called the No-Show paradox.[2] Having the additional information that the paradox may be encountered only when the preference profile does not contain an intuitively stable outcome (Condorcet winner) may be useful in deciding the value of the procedure in various settings. In similar vein, knowing that the vulnerability to this paradox may extend to situations where a Condorcet winner exists may be equally important in showing that even the stable outcomes do not provide a rescue from the possibility of a paradoxical outcome.

Our booklet does not provide information about the likelihood of paradoxical outcomes ensuing from different voting procedures other than in a roundabout way (for a more direct approach to the likelihood estimation, see Gehrlein & Lepelley, 2017). We are dealing with possibilities rather than with frequencies. If our results show that a specific procedure is vulnerable to a given paradox only when the preference profile includes no stable outcome to start with, and if we know that our decisions will be made predominantly in circumstances where the profiles contain a stable outcome, then we need not worry too much about that particular paradox occurring in practice. In contrast, if we show that the vulnerability extends to both stable and unstable profiles, then the strive to sail clear of the paradox may encourage us to look for other procedures to avoid the paradox.

This booklet complements our previous studies on voting paradoxes launched by our first booklet on monotonicity failures (cf., Felsenthal & Nurmi, 2017) and continued in a slightly more comprehensive booklet on voting paradoxes in unrestricted domains (cf., Felsenthal & Nurmi, 2018). Although we have endeavoured to keep the present booklet as self-contained as possible, its reader would undoubtedly benefit by glancing over these two previous works.

---

[1]The Condorcet winner is a candidate that would defeat all the others if pairwise majority comparisons were conducted and the voters voted according to their preferences.

[2]The No-Show paradox occurs when a group of voters with identical preferences would be better off abstaining than voting according to its preferences, *ceteris paribus*.

References

Arrow, K. J. (1963). *Social choice and individual values* (2nd ed.). New York: Wiley.
Black, D. (1948). On the rationale of group decision making. *Journal of Political Economy, 56,* 23–34.
Black, D. (1958). *The theory of committees and elections*. Cambridge: Cambridge University Press.
Colomer, J. (2013). Ramon Llull: From 'ars electionis' to social choice theory. *Social Choice and Welfare, 40,* 317–328.
Felsenthal, D. S., & Nurmi, H. (2017). *Monotonicity failures afflicting procedures for electing a single candidate*. Cham, Switzerland: Springer.
Felsenthal, D. S., & Nurmi, H. (2018). *Voting procedures for electing a single candidate: Proving their (in)vulnerability to various voting paradoxes*. Cham, Switzerland: Springer.
Gehrlein, W. V., & Lepelley, D. (2017). *Elections, voting rules and paradoxical outcomes*. Cham, Switzerland: Springer.
Hägele, G., & Pukelsheim, F. (2008). The electoral systems of Nicholas of Cusa in the *Catholic Concordance* and beyond. In G. Christianson, T. M. Izbicki, & C. M. Bellitto (Eds.), *The church, the councils & reform—The legacy of the fifteenth century* (pp. 229–249). Washington, DC: Catholic University of America Press.
Kelly, J. (1978). *Arrow impossibility theorems*. New York: Academic Press.
Kramer, G. H. (1973). On a class of equilibrium conditions for majority rule. *Econometrica, 41,* 285–297.
McLean, I., & Urken, A. B. (Eds.). (1995). *Classics in social choice*. Ann Arbor, MI: The University of Michigan Press.
Riker, W. H. (1961). Voting and the summation of preferences: An interpretive bibliographical review of selected developments in the last decade. *American Political Science Review, 55,* 900–911.
Sen, A. K., & Pattanaik, P. (1969). Necessary and sufficient conditions for rational choice under majority decision. *Journal of Economic Theory, 1,* 178–202.
Szpiro, G. (2010). *Numbers rule. The vexing mathematics of democracy from Plato to the present*. Princeton: Princeton University Press.
Tangian, A. (2014). *Mathematical theory of democracy*. Heidelberg: Springer.

Chapter 2
20 Voting Procedures Designed to Elect a Single Candidate

Abstract 20 voting procedures for electing a single candidate are introduced and briefly commented upon. The procedures fall into three classes in terms of the type of voter input and Condorcet consistency: non-ranked procedures, ranked procedures that are not Condorcet-consistent and ranked ones that are Condorcet-consistent. The first class consists of four procedures, the second consists of seven procedures and the third class consists of nine procedures.

Keywords Non-ranked voting procedures · Ranked procedures · Condorcet-consistent procedures

2.1 Non-ranked Voting Procedures

There are four main voting procedures for electing a single candidate where voters do not have to rank-order the candidates.

2.1.1 Plurality Voting (aka First Past the Post) Procedure

This is the most common procedure for electing a single candidate, and is used, *inter alia*, for electing the members of the House of Commons in the UK and the members of the House of Representatives in the US. Under this procedure every voter casts one vote for a single candidate and the candidate obtaining the largest number of votes is elected.

This chapter is largely based on Felsenthal (2012, pp. 24–31) and on Felsenthal and Nurmi (2018, Chap. 3).

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_2

2.1.2 Plurality with Runoff Voting Procedure

Under the usual version of this procedure up to two voting rounds are conducted. In the first round each voter casts one vote for a single candidate. In order to win in the first round a candidate must obtain either a special plurality (usually at least 40% of the votes) or an absolute majority of the votes. If no candidate is declared the winner in the first round then a second round is conducted. In this round only the two candidates who obtained the highest number of votes in the first round participate, and the one who obtains the majority of votes wins. This too is a very common procedure for electing a single candidate and it is currently used for electing the President in 40 countries, *inter alia*, in Argentina, Austria, Brazil, Finland, France, India, Portugal, Romania, Russia, Turkey and Ukraine. In France it is also used to elect the members of the legislature, and in Israel it is used to elect mayors and was used to elect the Prime Minister in the 1996, 1999, and 2001 elections.

This procedure can also be viewed as a procedure where voters rank-order all the competing alternatives and visit the ballot box only once—but there are up to two counting rounds. If in the first counting round there exists an alternative which is ranked first by an absolute majority of the voters then this alternative is declared the winner. But if no alternative is ranked first by an absolute majority of the voters then: (1) one selects the two alternatives which received more votes in the first count than each of the other alternatives; suppose these are alternatives x and y. (2) One then inspects all ballots where neither x nor y were listed first to determine in how many of these ballots x is preferred to y and in how many y is preferred to x. These numbers are then added to the number of ballots in which x and y were listed first to determine the ultimate winner.

2.1.3 Approval Voting (Brams & Fishburn, 1978, 1983)

Under this procedure every voter has a number of votes which is equal to the number of competing candidates, and every voter can cast one vote or no vote for every candidate. The candidate obtaining the largest number of votes is elected. So far this procedure has not been used in any public elections but is already used by several professional associations and universities in electing their officers.

2.1.4 Successive Elimination (Farquharson, 1969)

This procedure is common in parliaments when voting on alternative versions of bills. According to this procedure voting is conducted in a series of rounds. In each round two alternatives compete; the one obtaining fewer votes is eliminated and the other competes in the next round against one of the alternatives which has not yet been eliminated. If there are n candidates, $n-1$ pairwise votes are taken. The alternative winning in the last round is the ultimate winner.

2.2 Ranked Voting Procedures That Are Not Condorcet-Consistent

Seven ranked procedures under which every voter must rank-order all competing candidates—but which do not ensure the election of a Condorcet Winner when one exists—have been proposed, as far as we know, during the last 250 years. These procedures are described below. Only one of these procedures (Alternative Vote) is used currently in public elections.

2.2.1 Borda's Count (Black, 1958; Borda, 1784)

This voting procedure was proposed by Jean Charles de Borda in a paper he delivered in 1770 before the French Royal Academy of Sciences entitled 'Memorandum on election by ballot' ('Mémoire sur les élections au scrutin'). According to Borda's procedure each candidate, x, is given a score equal to the sum of voters who prefer x to each of the other alternatives, and the candidate with the largest score is elected. Thus the Borda winner can be viewed also as the candidate who occupies the highest position, *on average*, in the rankings of the voters. Equivalently, under Borda's procedure each candidate x gets no points for each voter who ranks x last in his/her preference ordering, 1 point for each voter who ranks x second-to-last in his/her preference ordering, and so on, and $n-1$ points for each voter who ranks x first in his/her preference ordering (where n is the number of candidates). The candidate with the largest number of points is elected. Thus if all v voters have linear preference orderings among the n candidates then the total number of points obtained by all candidates is equal to the number of voters multiplied by the number of paired comparisons among the candidates, i.e., to $vn(n-1)/2$.

2.2.2 Alternative Vote (aka Instant Runoff)

This is the version of the *Single Transferable Vote* (STV) procedure (independently proposed by Carl George Andrae in Denmark in 1855 and by Thomas Hare in England in 1857) for electing a single candidate. It works as follows. In the first step one verifies whether there exists a candidate who is ranked first by an absolute majority of the voters. If such a candidate exists s/he is declared the winner. If no such candidate exists then, in the second step, the candidate who is ranked first by the smallest number of voters is deleted from all ballots and thereafter one again verifies whether there is now a candidate who is ranked first by an absolute majority of the voters. The elimination process continues in this way until a candidate who is ranked first by an absolute majority of the voters is found. The Alternative Vote procedure is used in electing the President of the Republic of Ireland, the Australian House of Representatives, as well as the mayors in some municipal elections in the US.

2.2.3 Coombs's Method (Coombs, 1964, pp. 397–399; Coombs, Cohen, & Chamberlin, 1984; Straffin, 1980)

This procedure was proposed by the psychologist Clyde H. Coombs in 1964. It is similar to Alternative Vote except that the elimination in a given round under Coombs's method involves the candidate who is ranked last by the largest number of voters (instead of the candidate who is ranked first by the smallest number of voters as under Alternative Vote).

2.2.4 Negative Plurality Elimination Rule (NPER) (Lepelley, Moyouwou, & Smaoui, 2018)

This procedure is similar to Coombs's procedure with one difference: whereas under Coombs's procedure a candidate who is ranked first by an absolute majority of the voters is elected even if this candidate is also ranked last by a plurality of the voters, under NPER such a candidate is deleted from all ballots. In other words, under NPER one deletes in each round the candidate who is ranked last by the largest number of voters until only one candidate remains—who is declared the winner.

2.2.5 Bucklin's Method (Hoag & Hallett, 1926, pp. 485–491; Tideman, 2006, p. 203)

This voting system can be used for single-member and multi-member district elections. It is named after James W. Bucklin of Grand Junction, Colorado, who first promoted it in 1909. In 1913 the US Congress prescribed (in the Federal Reserve Act of 1913, section 4) that this method be used for electing district directors of each Federal Reserve Bank.

Under Bucklin's method voters rank-order the competing candidates. The vote count starts like in the Alternative Vote method. If there exists a candidate who is ranked first by an absolute majority of the voters s/he is elected. Otherwise the number of voters who ranked a given candidate in second place are added to the number of voters who ranked him/her first, and if now there exists a candidate supported by a majority of voters s/he is elected. If not, the counting process continues in this way by adding for each candidate his/her third, fourth, …, and so forth rankings, until a candidate is found who is supported by an absolute majority of the voters. If two or more candidates are found to be supported by a majority of voters in the same counting round then the one supported by the largest majority is elected.[1]

2.2.6 Range Voting (Smith, 2000)

According to this procedure the suitability (or level of performance) of every candidate is assessed by every voter and is assigned a (cardinal) grade (chosen from a pre-specified range) reflecting the candidate's suitability or level of performance in the eyes of the voter. The candidate with the highest average grade is the winner. This procedure is currently championed by Warren D. Smith (see http://rangevoting.org) and used to elect the winner in various sport competitions.

2.2.7 Majority Judgment (Balinski & Laraki, 2007a, 2007b, 2011)

According to this proposed procedure, the suitability (or level of performance) of every candidate is assessed by every voter and is assigned an ordinal grade (chosen from a pre-specified range) reflecting the candidate's suitability or level of performance in the eyes of the voter. The candidate with the highest median grade is the winner.

---

[1]However, it is unclear how a tie between two candidates, say *a* and *b*, ought to be broken under Bucklin's procedure when both *a* and *b* are supported in the same counting round by the same number of voters and this number constitutes a majority of the voters. If one tries to break the tie between *a* and *b* in such an eventuality by performing the next counting round in which all other candidates are also allowed to participate, then it is possible that the number of (cumulated) votes of another candidate, *c*, will exceed that of *a* and *b*.

To see this, consider the following simple example. Suppose there are 18 voters who must elect one candidate under Bucklin's procedure and whose preference orderings among four candidates, *a*, *b*, *c*, *d* are as follows: seven voters with preference ordering $a > b > c > d$, eight voters with preference ordering $b > a > c > d$, one voter with preference ordering $d > c > a > b$, and two voters with preference ordering $d > c > b > a$. None of the candidates constitutes the top preference of a majority of the voters. However, both *a* and *b* constitute the top + second preference by a majority of voters (15). If one tries to break the tie between *a* and *b* by performing the next (third) counting round in which *c* and *d* are also allowed to participate, then *c* will be elected (with 18 votes), but if only *a* and *b* are allowed to participate in this counting round then *b* will be elected (with 17 votes).

So which candidate ought to be elected in this example under Bucklin's procedure? As far as we know, Bucklin did not supply an answer to this question.

2.3 Ranked Voting Procedures That Are Condorcet-Consistent[2]

All the nine voting procedures described in this section require that voters rank-order all competing candidates. Under all these procedures a Condorcet Winner, if one exists, is elected. The procedures differ from one another regarding which candidate gets elected when a Condorcet Winner does not exist.

2.3.1 The Minimax Procedure

Condorcet specified that the Condorcet Winner (whom he called 'the majority candidate') ought to be elected if one exists. However, according to Black (1958, pp. 174–175, 187) Condorcet did not specify clearly which candidate ought to be elected when the *social preference ordering*[3] contains a top cycle. Black (1958, p. 175) suggests that "It would be most in accordance with the spirit of Condorcet's … analysis … to discard all candidates except those with the minimum number of majorities against them and then to deem the largest size of minority to be a majority, and so on, until one candidate had only actual or deemed majorities against each of the others." In other words, the procedure attributed by Black to Condorcet when cycles exist in the social preference ordering is a *Minimax procedure*[4] since it chooses that candidate whose worst loss in the paired comparisons is the least bad. This procedure is also known in the literature as the *Simpson–Kramer rule* (see Kramer, 1977; Simpson, 1969).

2.3.2 Dodgson's Procedure (Black, 1958, pp. 222–234; McLean & Urken, 1995, pp. 288–297)

This procedure is named after the Rev. Charles Lutwidge Dodgson, aka Lewis Carroll, who proposed it in 1876. It elects the Condorcet Winner when one exists. If no Condorcet Winner exists it elects that candidate who requires the fewest number

---

[2]We list here only deterministic procedures. For a Condorcet-consistent probabilistic procedure see, for example, Felsenthal and Machover (1992). We also do not list here two Condorcet-consistent deterministic procedures proposed by Tideman (1987) and by Schultze (2003) because we do not consider satisfying (or violating) the independence-of-clones property, which is the main reason why these two procedures were proposed, to be associated with any voting paradox. (A phenomenon where candidate x is more likely to be elected when two clone candidates, y and y', exist, and where x is less likely to be elected when, *ceteris paribus*, one of the clone candidates withdraws, does not seem to us surprising or counter-intuitive). Except for Black's (1958) hybrid procedure, which is well-known, we do not analyze any other hybrid procedure.

[3]This is the preference ordering of the majority of the voters with respect to any pair of alternatives.

[4]Young (1977, p. 349) prefers to call this procedure 'Minimax function'.

of switches (i.e., inversions of two adjacent candidates) in the voters' preference orderings in order to make him or her the Condorcet Winner.

2.3.3 Nanson's Method (McLean & Urken, 1995, Chap. 14; Nanson, 1883)

Nanson's method is a recursive elimination based on Borda's method. In the first step one calculates for each candidate his/her Borda score. In the second step the candidates whose Borda score does not exceed the average Borda score of the candidates in the first step are eliminated from all ballots and revised Borda scores are computed for the remaining candidates. The elimination process is continued in this way until one candidate is left. If a (strong) Condorcet Winner exists then Nanson's method elects him/her.[5]

2.3.4 Borda's Elimination Rule (BER) (Baldwin, 1926)

This procedure is similar to Nanson's procedure in the sense that in each voting round one computes the Borda score of the participating candidates in that round. However, in contrast to Nanson's procedure one deletes in each round only the candidate(s) with the lowest Borda score. The elimination process continues in this way until only two candidates remain and the one with the higher Borda score is the ultimate winner. If a Condorcet winner exists in the initial profile s/he will be elected under BER.

2.3.5 Copeland's Method (Copeland, 1951)

Every candidate x gets one point for every paired comparison with another candidate y in which an absolute majority of the voters prefer x to y, and half a point for every

[5]Although Nanson's procedure satisfies the strong Condorcet condition, i.e., it always elects a candidate who beats every other candidate in paired comparisons, this procedure may not satisfy the weak Condorcet condition which requires that if there exist(s) candidate(s) who is (are) unbeaten by any other candidate then this (these) candidate(s)—and only this (these) candidate(s)—ought to be elected. For an example of violation of the weak Condorcet condition by Nanson's procedure see Niou (1987). Niou shows that when the set of Nanson winners consists of two candidates, one of them may not satisfy the weak Condorcet condition, while the other Nanson winner does. The following profile (where the symbol > means "is preferred to") shows that the Nanson winner may be distinct from those candidates that satisfy the weak Condorcet condition (Nurmi, 1989, p. 202): one voter: $a > b > c > d > e$, one voter: $a > d > b > c > e$, one voter: $a > d > e > b > c$, one voter: $b > c > e > d > a$, two voters: $c > e > d > b > a$. Here the Nanson winner is c, but the only candidate satisfying the weak Condorcet condition is a.

paired comparison in which the number of voters preferring x to y is equal to the number of voters preferring y to x. The candidate obtaining the largest sum of points is the winner.

2.3.6 Black's Method (Black, 1958, p. 66)

According to this method one first performs all paired comparisons to verify whether a Condorcet Winner exists. If such a winner exists then s/he is elected. Otherwise the winner according to the Borda count (see above) is elected.

2.3.7 Kemeny's Method (Kemeny, 1959; Kemeny & Snell, 1960; Young, 1988, 1995; Young & Levenglick, 1978)

Kemeny's method (aka *Kemeny–Young rule* or *Kemeny's median*) specifies that up to $n!$ possible social preference orderings should be examined (where n is the number of candidates) in order to determine which of these is the "most likely" true social preference ordering.[6] The selected "most likely" social preference ordering according to this method is the one where the number of pairs (V, y), where V is a voter and y is a candidate such that V prefers x to y, and y is ranked below x in the social preference ordering is maximized. Given the voters' various preference orderings, Kemeny's procedure can also be viewed as finding the most likely (or the best predictor, or the best compromise) true social preference ordering, called the *median preference ordering*, i.e., that social preference ordering S that minimizes the sum, over all voters i, of the number of pairs of candidates that are ordered oppositely by S and by the ith voter.[7]

---

[6]Tideman (2006, pp. 187–189) proposes two heuristic procedures that simplify the need to examine all $n!$ preference orderings.

[7]According to Kemeny (1959) the distance between two preference orderings, R and R', is the number of pairs of candidates (alternatives) on which they differ. For example, if $R = a > b > c > d$ and $R' = d > a > b > c$, then the distance between R and R' is 3, because they agree on three pairs [$(a > b)$, $(a > c)$, $(b > c)$] but differ on the remaining three pairs, i.e., on the preference ordering between a and d, b and d, and between c and d. Similarly, if R'' is $c > d > a > b$ then the distance between R and R'' is 4 and the distance between R' and R'' is 3. According to Kemeny's procedure the most likely social preference ordering is that R such that the sum of distances of the voters' preference orderings from R is minimized. Because this R has the properties of the median central measure in statistics it is called the *median preference ordering*. The median preference ordering (but not the *mean preference ordering* which is that R which minimizes the sum of the squared differences between R and the voters' preference orderings) will be identical to the possible social preference ordering W which maximizes the sum of voters that agree with all paired comparisons implied by W.

2.3.8 Schwartz's Method (Schwartz, 1972, 1986)

Thomas Schwartz's method is based on the notion that a candidate x deserves to be listed ahead of another candidate y in the social preference ordering if and only if x beats or ties with some candidate that beats y, and x beats or ties with all candidates that y beats or ties with. The Schwartz set (from which the winner should be chosen) is the smallest set of candidates who are unbeatable by candidates outside the set. The Schwartz set is also called *GOCHA* (*Generalized Optimal Choice Axiom*).

2.3.9 Young's Method (Young, 1977)

According to Fishburn's (1977, p. 473) informal description of Young's procedure "[it] is like Dodgson's in the sense that it is based on altered profiles that have candidates who lose to no other candidate under simple majority. But unlike Dodgson, Young deletes voters rather than inverting preferences to obtain the altered profiles. His procedure suggests that we remain most faithful to Condorcet's Principle if the choice set consists of alternatives that can become simple majority non-losers with removal of the fewest number of voters."

Exercises for Chapter 2

Problem 2.1 Consider the following profile:

No. of voters	Preference ordering
5	$a > c > b > d$
4	$b > d > a > c$
2	$c > d > a > b$

Determine the winners according to Bucklin's, Minimax, Plurality with Runoff, and Copeland's procedures.

Problem 2.2 Consider the following profile:

No. of voters	Preference orderings
10	$d > a > b > c$
7	$b > c > a > d$
7	$c > a > b > d$
4	$d > c > a > b$
1	$b > a > c > d$

Determine the winners according to Dodgson's, Minimax, Nanson's, and Kemeny's procedures.

Problem 2.3
Construct a profile where the Plurality, Plurality with Runoff and Copeland winners differ from each other.

Problem 2.4
Show by way of an example that the Borda winner may differ from the winner ensuing from Copeland's procedure.

Problem 2.5
An Absolute Winner is a candidate who is ranked first by more than 50% of the voters. Determine whether each of the following voting procedures always elects the Absolute Winner when one exists: Successive Elimination, Coombs's procedure, Borda count, Plurality Voting, Alternative Vote, Bucklin's procedure.

Answers to Exercises of Chapter 2

Problem 2.1
The Bucklin Winner is c (on the second round of computing with 7 voters placing it either first or second);
The Minimax Winner is a (with a maximum of 6 votes against it);
The Plurality with Runoff Winner is a (in the runoff with b);
The Copeland Winners are a and c (each defeats two other candidates).

Problem 2.2
The winner according to Dodgson's procedure is d (needing just 3 preference inversions to become a Condorcet Winner);
The Minimax winner is also d whose maximal loss (15) is smallest;
The winner according to Nanson's procedure is c (which beats a 18:11 in the second count);
The most likely (transitive) social preference orderings according to Kemeny are $a > b > c > d$ and $c > a > b > d$ (each supported by the largest number (95) of pairs fitting these social preference orderings), so here according to Kemeny there is a tie between a and c.

Problem 2.3
4 voters: $x > z > y$
3 voters: $y > z > x$
2 voters: $z > y > x$
Plurality Voting elects x, the Plurality with Runoff elects y (after the runoff between x and y) and Copeland's procedure elects z since it defeats x 5:4 and y 6:3, while y defeats only x and x defeats no candidate.

Problem 2.4

Consider, e.g., the following profile:

5 voters: $x > y > z$

4 voters: $y > z > x$

Here the Borda scores of x, y and z are 10, 13 and 4, respectively. Copeland's procedure results in x since it defeats both y and z, while y defeats z and z defeats no other candidate. Thus, y wins under the Borda count, while x wins when Copeland's procedure is in use.

Problem 2.5

Of the procedures listed only one, viz., the Borda count, may not end up with an Absolute Winner when one exists. An example of this is shown in the answer to Problem 2.4.

References

Baldwin, J. M. (1926). The technique of the Nanson preferential majority system. *Proceedings of the Royal Society of Victoria, 39,* 42–52.

Balinski, M., & Laraki, R. (2007a). A theory of measuring, electing and ranking. *Proceedings of the National Academy of Sciences of the United States of America (PNAS), 104*, 8720–8725.

Balinski, M., & Laraki, R. (2007b). *Election by majority judgement: Experimental evidence, (mimeograph)*. Paris: Ecole Polytechnique, Centre National De La Recherche Scientifique, Laboratoire D'Econometrie, Cahier No. 2007–28. Downloadable from http://tinyurl.com/65q6cg.

Balinski, M., & Laraki, R. (2011). *Majority judgment: Measuring, ranking, and electing*. Cambridge, MA: MIT Press.

Black, D. (1958). *The theory of committees and elections*. Cambridge: Cambridge University Press.

Brams, S. J., & Fishburn, P. C. (1978). Approval voting. *American Political Science Review, 72,* 831–847.

Brams, S. J., & Fishburn, P. C. (1983). *Approval voting*. Boston: Birkhäuser.

Coombs, C. H. (1964). *A theory of data*. New York: Wiley.

Coombs, C. H., Cohen, J. L., & Chamberlin, J. R. (1984). An empirical study of some election systems. *American Psychologist, 39,* 140–157.

Copeland, A. H. (1951). *A 'reasonable' social welfare function, mimeographed*. University of Michigan, Department of Mathematics, Seminar on Applications of Mathematics to the Social Sciences.

de Borda, J.-C. (1784) [1995]. Mémoire sur les élections au scrutin. *Histoire de l'Academie Royale des Sciences année* 1781 (pp. 651–665). Paris. Reprinted and translated in I. McLean & A. B. Urken (1995), *Classics of social choice* (pp. 83–89). Ann Arbor, MI: University of Michigan Press.

Farquharson, R. (1969). *Theory of voting*. New Haven, CT: Yale University Press.

Felsenthal, D. S. (2012). Review of paradoxes afflicting procedures for electing a single candidate. In D. S. Felsenthal & M. Machover (Eds.), *Electoral systems: Paradoxes, assumptions, and procedures* (Ch. 3, pp. 19–92). Berlin Heidelberg: Springer.

Felsenthal, D. S., & Machover, M. (1992). After two centuries should Condorcet's voting procedure be implemented? *Behavioral Science, 37,* 250–273.

Felsenthal, D. S., & Nurmi, H. (2018). *Voting procedures for electing a single candidate—Proving their (in)vulnerability to various voting procedures*. Cham, Switzerland: Springer.

Fishburn, P. C. (1977). Condorcet social choice functions. *SIAM Journal on Applied Mathematics, 33,* 469–489.

Hoag, C. G., & Hallett, G. H. (1926). *Proportional representation*. New York: The Macmillan Co.
Kemeny, J. G. (1959). Mathematics without numbers. *Daedalus, 88,* 577–591.
Kemeny, J. G., & Snell, J. L. (1960). *Mathematical models in the social sciences*. Boston: Ginn.
Kramer, G. H. (1977). A dynamical model of political equilibrium. *Journal of Economic Theory, 16,* 310–333.
Lepelley, D., Moyouwou, I., & Smaoui, H. (2018). Monotonicity paradoxes in three-candidate elections using scoring elimination rules. *Social Choice and Welfare, 50,* 1–33.
McLean, I., & Urken, A. B. (1995). *Classics of social choice*. Ann Arbor: University of Michigan Press.
Nanson, E. J. (1883). Methods of elections. *Transactions and Proceedings of the Royal Society of Victoria, 19*, 197–240. Also in I. McLean & A. B. Urken (Eds.), *Classics of social choice* (Ch. 14, pp. 321–359, 1995). University of Michigan Press.
Niou, E. M. S. (1987). A note on Nanson's rule. *Public Choice, 54,* 191–193.
Nurmi, H. (1989). On Nanson's method. In O. Borg, O. Apunen, H. Hakovirta, & J. Paastela (Eds.), *Democracy in the modern world. Essays for Tatu Vanhanen*, series A (Vol. 260, pp. 199–210). Tampere: Acta Universitatis Tamperensis.
Schultze, M. (2003). A new monotonic and clone-independent single-winner election method. *Voting Counts, 17*, 9–19. Downloadable from: http://www.macdougall.org.uk/VM/ISSUE17/I17P3.PDF.
Schwartz, T. (1972). Rationality and the myth of the maximum. *Noûs, 6,* 97–117.
Schwartz, T. (1986). *The logic of collective choice*. New York: Columbia University Press.
Simpson, P. B. (1969). On defining areas of voter choice: Professor Tullock on stable voting. *Quarterly Journal of Economics, 83,* 478–490.
Smith, W. D. (2000). *Range voting*. Downloadable from: http://www.math.temple.edu/~wds/homepage/rangevote.pdf.
Straffin, P. D. (1980). *Topics in the theory of voting*. Boston: Birkhäuser.
Tideman, T. N. (1987). Independence of clones as a criterion for voting rules. *Social Choice and Welfare, 4,* 185–206.
Tideman, N. (2006). *Collective decisions and voting: The potential for public choice*. Aldershot, Hampshire, England: Ashgate Publishing Ltd.
Young, H. P. (1977). Extending Condorcet's rule. *Journal of Economic Theory, 16,* 335–353.
Young, H. P. (1988). Condorcet's theory of voting. *American Political Science Review, 82,* 1231–1244.
Young, H. P. (1995). Optimal voting rules. *Journal of Economic Perspectives, 9,* 51–63.
Young, H. P., & Levenglick, A. (1978). A consistent extension of Condorcet's election principle. *SIAM Journal of Applied Mathematics, 35,* 283–300.

Chapter 3
The (In)Vulnerability of 20 Voting Procedures to Lack of Monotonicity in a Restricted Domain

Abstract This chapter focuses on the possibility that some well-known voting procedures lead to specific types of monotonicity paradoxes in preference profiles that are characterized by the presence and election of a Condorcet winner. Moulin's (Journal of Economic Theory 45:53–64, 1988) theorem establishes the incompatibility of Condorcet-consistency and invulnerability to the No-Show paradox in voting procedures when there are more than three alternatives to be chosen from. We ask whether this conclusion would also hold in the proper subset of profiles distinguished by the property that a Condorcet winner exists and is elected in the initial profile. Our focus is on 20 voting procedures designed to elect a single candidate. These procedures include both Condorcet-consistent and non-consistent rules. The former are, however, only briefly touched upon because their invulnerability to most types of monotonicity violations in the restricted domain is obvious.

Keywords Elections · Non-monotonicity · No-show paradox · Condorcet-consistency · Fixed electorates · Variable electorates

3.1 Introduction

A fundamental rationale of holding elections is that the voters should find it in their interest to reveal their preferences on the issues to be decided upon or on the candidates running for various kinds of offices. This is what 'going to the people' is commonly thought to be. The underlying idea is that the more voters support an alternative, the better chances it has to succeed. Indeed, the very notion of success in elections is based on this rationale. We know that under any voting procedure there are settings where some group of voters might ensure an outcome that it prefers to the one ensuing from its voting according to its true preferences (i.e., sincerely) (cf., Gibbard, 1973; Satterthwaite, 1975), but the requirements for succeeding by voting as if one's preferences were different from what they actually are, make strategic (i.e., insincere) voting risky: it might backfire by producing an outcome worse than that resulting from sincere voting, *ceteris paribus* (cf., Slinko & White, 2014), it might be computationally demanding (cf., Bartholdi, Tovey, & Trick, 1989) and/or

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_3

costly in terms of coordination expenses. Moreover, for many voters the act of voting is the primary (or even the only) way of expressing political opinions. Hence, the possibility of expressing one's views is likely to overcome strategic considerations even in those situations where the chances of getting a better outcome by strategizing are relatively straightforward (e.g., Ross Perot's supporters in the 1992 and Ralph Nader's supporters in the 1996 and 2000 US presidential elections). Be the proportions of sincere (expressive) and strategic voters in general what they may, we focus on the way various voting procedures handle the submitted ballots in determining the election winners and losers. In particular, we study the procedures from the viewpoint of the above-mentioned rationale: the larger the support, the larger the chance of winning.

This rationale has been discussed extensively in the social choice theory, especially after Fishburn's (1982) seminal article on monotonicity paradoxes, but it was touched upon and elaborated much earlier by election practitioners. In his *Proportional Representation in Ireland*, Meredith (1913) discusses the Single Transferable Vote (STV) system. This system was dealt with later on in a simpler version which is devised for electing a single winner. This version is known as Alternative Vote (aka Hare's system). Meredith (1913, pp. 92–93) provides the following illustrative example of the peculiarities of STV:

> The order in which candidates are eliminated may also make a serious difference in the result [of STV elections]. Suppose that D (Nationalists), M (Ind. Unionists), and Z (Unionists) are three continuing candidates, and that one seat remains to be filled. The quota is, say, 800, and D has 410 votes, M 400, and Z 500. Then M is eliminated, and his votes may be supposed to be transferred to Z, who is declared elected. But if D were eliminated before M, we may easily suppose that his votes would go to M, who would be elected. The injustice of the result appears even more striking when we reflect that, if D had 11 votes less, his supporters would have succeeded in returning M instead of Z, as they desired to do.

The last sentence discusses what would today be called a No-Show paradox. That these issues were more widely discussed is apparent from Meredith's (1913, p. 93) footnote which states:

> The Report of the Royal Commission, par. 76, states that a case was put to the Commission 'to show that with a certain disposition on the part of the electors, the representation of a party might be so much at the mercy of the order of elimination, that while it would only obtain one seat with 19,000 votes of its own, it would obtain two with 18,000 because in the latter case the order of elimination would be reversed'.

Some 70 years later, the same issues were taken up by Fishburn and Brams (1983) in a delightful and instructive article on paradoxes of preferential voting. The term *No-Show paradox* was introduced in this article. Perhaps the most important theoretical result in this field was achieved by Moulin (1988) who established the theoretical incompatibility between two commonly held social choice desiderata: *Condorcet-consistency*[1] and *invulnerability to the No-Show paradox*.[2] The result says that in

---

[1]The (In)Vulnerability of the various voting procedures to the Inconsistency (aka Reinforcement) paradox will be analyzed in Chap. 4.

[2]The (In)Vulnerability of the various voting procedures to the No-Show paradox will be analyzed in Chap. 5.

settings involving more than three candidates any procedure that always elects the Condorcet winner when one exists can lead to an outcome such that some voters would be better off (i.e., would get a result they prefer to the present one) by abstaining than by voting according to their true preferences.

The No-Show paradox is one in the class of choice set variability paradoxes which are associated with unexpected or counterintuitive changes in choice resulting from a given voting procedure as a result of some changes in the electorate. These changes may involve adding or removing unanimous voter groups from the initial electorate. These paradoxes are called *variable electorate paradoxes*. Another class of similar paradoxes involve preference changes within a given initial electorate. These are related to the changes in outcomes that result when the position of a given alternative is either improved or deteriorated. These paradoxes are called *fixed electorate paradoxes*. If the initial winner's position is improved, with no other changes made in the voters' preferences, i.e., *ceteris paribus*, and yet the outcome is the victory of another alternative, then we have an instance of *upward monotonicity failure* or non-monotonicity. If, on the other hand, an alternative which is not the winning one in the initial preference profile becomes one when its position is deteriorated, then we have an instance of what is called *downward monotonicity failure* (Miller, 2017). In similar vein, Felsenthal and Tideman (2013, 2014) distinguish two special classes of variable electorate paradoxes: P-TOP and P-BOT. The former occurs in a given profile when in the initial profile using a given procedure alternative x wins, but becomes a non-winner by the same procedure if the initial electorate is augmented with a group of voters that rank x first in their preferences. The P-BOT paradox, in turn, occurs when in the initial profile using a given procedure x wins, but another alternative, y, wins by the same procedure if the electorate is augmented by adding a group of unanimous voters all of whom rank y at the bottom of their preference ordering. (For a more extensive discussion of the paradoxes mentioned above, see Felsenthal (2012), Felsenthal and Nurmi (2016, 2017, 2018) and Nurmi (2004)).

Moulin's (1988) result states that all Condorcet-consistent procedures may—under suitable profiles—lead to the No-Show paradox, i.e., monotonicity violation in variable electorates. But supposing one knows something about the profiles to be encountered by using a given Condorcet-consistent procedure, can this knowledge help in assessing the practical importance of the paradox? Can one think of some restrictions regarding the allowable preference profiles that would eliminate the possibility of encountering the No-Show paradox? This chapter deals with one such restriction and pursues its implications to the vulnerability of various procedures to monotonicity paradoxes in fixed and variable electorates. The restriction we focus upon is the *existence and election of a Condorcet winner in the initial profile*. The question we ask is: are the procedures known to be vulnerable to the monotonicity paradoxes in fixed or variable electorates also vulnerable to the same paradoxes when the initial profiles contain a Condorcet winner which is at the same time the winner that results from the procedure we are examining? Phrased in another way, supposing that we know that the initial preference profiles we are going to encounter in practice always (or nearly always) include and elect a Condorcet winner, do we have to worry about the possibility of a monotonicity paradox occurring at all?

This is the question we now set out to investigate in connection with 14 voting rules which are known to be vulnerable to at least one type of monotonicity failure in unrestricted domains. Of course, the five voting rules that are invulnerable to any kind of monotonicity paradox (i.e., Plurality, Approval Voting, Borda count, Range Voting, and Minimax) need not be discussed in the present context as they obviously are invulnerable under the specified profile restriction as well.

The rest of the chapter is organized as follows. We start by discussing the analyzed voting procedures in the light of eventual upwards monotonicity failures, first in fixed and then in variable electorates. Thereafter we investigate the susceptibility of these procedures to downward monotonicity failures, again first in fixed and then in variable electorates. The final section sums up the findings and discusses their implications for institution design.

3.2 The (In)Vulnerability of the Investigated Procedures to Upward Monotonicity Failure

3.2.1 Fixed Electorates

3.2.1.1 Condorcet-Consistent Procedures

In all the 10 Condorcet-consistent procedures described in Chap. 2 (Successive Elimination, Minimax, Dodgson's, Black's, Copeland's, Kemeny's, Nanson's, BER, Schwartz's, Young's) the Condorcet winner is elected whenever it exists in a profile. Hence moving a Condorcet winner, say x, up, *ceteris paribus*, in some preferences of the initial profile P to form P′, implies that also P′ has a Condorcet winner, viz., x. Therefore all Condorcet-consistent procedures elect x in P′ as well. Therefore no upward monotonicity failure can occur in Condorcet-consistent procedures in fixed electorates.

3.2.1.2 Totally Invulnerable Non-Condorcet-Consistent Procedures

The six voting procedures Plurality Voting, Approval Voting, Borda, Bucklin's, Range Voting and Majority Judgment described in Chap. 2 are invulnerable to upward monotonicity failure in general under fixed electorates and hence they are also invulnerable to upward monotonicity failure in fixed electorates under a restricted domain where a Condorcet winner exists and is elected in the initial profile. For explanation why these six voting procedures are generally invulnerable to upward monotonicity failure under fixed electorates see Felsenthal and Nurmi (2018).

3.2.1.3 Plurality with Runoff

This procedure is invulnerable to upward monotonicity failure in a fixed electorate when the restricted domain assumption is applied for the following reasons.

Suppose that x is elected in profile P and that x is the Condorcet winner in P as well. Let now profile P′ be formed so that x is ranked higher by some voter group G in P, *ceteris paribus*. If x won in the first round in P, it will obviously do so in P′ as well since no other alternative has the number of its first ranks increased. If, on the other hand, x won in the second round in P, its runoff contestant in P′ may be different than in P. Nonetheless x will win the runoff contest in P′ as well since after its position is improved from what it is in P, *ceteris paribus*, it will remain the Condorcet winner. Hence it will, by definition, beat any alternative it is confronted with in P′. Thus, the Plurality with Runoff procedure is upward monotonic in profiles where the plurality runoff winner coincides with the Condorcet winner in the initial profile.

3.2.1.4 Alternative Vote

The Alternative Vote (AV) procedure is invulnerable to the upward monotonicity failure in fixed electorate when the domain is restricted, i.e., when there is a Condorcet winner who is elected in the initial profile. This is so for the following reasons. Suppose that x is the Condorcet and AV winner in the initial profile P. Let now x's position be improved in some voters' preference ordering, *ceteris paribus*, and call the ensuing profile P′. This means that x remains the Condorcet winner in P′ as well since it defeats all those alternatives it beats in P by a majority of votes. Its first-rank count (i.e., the number of voters ranking it first) is at least the same in P′ as in P. All other alternatives have at most the same first-rank count in P′ as in P. Hence, if x wins in P on the first counting round, it does so in P′ as well. If it wins in P in some subsequent count, it may do so also in P′ since being the Condorcet winner both in P and in P′ and the AV winner in P guarantees that the number of voters ranking x first in P′ will never be the smallest, so in the final count x must beat by majority of votes whichever alternative(s) confront(s) it in P′. In other words, although it is of course possible that as a result of some voters improving in P′ the position of x in their preference ordering that the order in which candidates are eliminated in P′ will be different than in P, the combination that x is the Condorcet winner in both P and P′ as well as the AV winner in P guarantees that the number of voters ranking x first (either immediately or eventually) must constitute an absolute majority—and hence AV is invulnerable to upward monotonicity violation in fixed electorates in the restricted domain.

3.2.1.5 Coombs's Procedure

This procedure is vulnerable to the upward monotonicity failure under fixed electorate when a Condorcet winner exists and is elected in the initial profile. To see this, consider the following example.

Suppose a group of 42 voters whose preference orderings among four candidates, *a*–*d*, are as follows:[3]

10 voters: $a > d > c > b$
6 voters: $a > d > b > c$
4 voters: $b > a > c > d$
7 voters: $b > c > a > d$
8 voters: $b > c > d > a$
3 voters: $c > a > d > b$
4 voters: $d > a > b > c$

Here *a* is the Condorcet winner and is elected under Coombs's procedure. (As no candidate is ranked first by an absolute majority of the voters, candidate *b*, who is ranked last by the plurality of the voters, is deleted from all ballots. As still no candidate is ranked first by an absolute majority of the voters, candidate *c*, who is now ranked last by a plurality of the voters, is deleted from all ballots. Thereafter candidate *a* is ranked first by an absolute majority of the voters [30] and is therefore elected under Coombs's procedure).

Now suppose that, *ceteris paribus*, the eight $b > c > d > a$ voters change their preference ordering to $b > c > a > d$ thereby *increasing a*'s position in their preference ordering. However, as a result candidate *b* will be elected according Coombs's procedure thereby demonstrating its vulnerability to the upward monotonicity failure when a Condorcet winner exists and is elected in the initial profile. (As no candidate is ranked first by an absolute majority of the voters, candidate *d*, who is ranked last by the plurality of the voters, is deleted from all ballots. As still no candidate is ranked first by an absolute majority of the voters, candidate *a*, who is now ranked last by a plurality of the voters, is deleted from all ballots. Thereafter candidate *b* is ranked first by an absolute majority of the voters [29] and is therefore elected under Coombs's procedure).

3.2.1.6 The Negative Plurality Elimination Rule (NPER) Procedure

The same example used to demonstrate the vulnerability of Coombs's procedure to upward monotonicity failure in fixed electorate when a Condorcet winner exists and is elected in the initial profile can be used also to demonstrate the same type of vulnerability of the NPER procedure.

[3]When the notation $a > b$ is used with respect to a single voter or with respect to a group of voters it means that the voter(s) rank(s) candidate *a* ahead of candidate *b*. When it is used with respect to the entire electorate it means that a majority of the voters rank *a* ahead of *b*.

3.2.1.7 Bucklin's Procedure

This procedure cannot display the upward monotonicity failure in fixed electorate when a Condorcet winner exists in the initial profile and is elected according to Bucklin's procedure. This is so because if the Condorcet winner was elected initially then the Condorcet winner will be elected, *a fortiori*, (i.e., either with a larger majority and/or at an earlier stage) when some voters elevate the Condorcet winner in their preference ordering and all other voters' preference orderings remain the same.

3.2.1.8 Majority Judgment

Under the Majority Judgment (MJ) procedure the improvement of the Condorcet winner x's position (grade), *ceteris paribus*, changes no grades of other alternatives and, consequently, their median grades remain the same as in the initial profile, while x's median grade is at least as high as before the improvement. Hence, if x was the MJ winner in the initial profile, it is the MJ winner in the profile that is formed by improving its grade, *ceteris paribus*. Therefore MJ is invulnerable to upward monotonicity failure in fixed electorates in the restricted domain.

3.2.2 Variable Electorates

3.2.2.1 Condorcet-Consistent Procedures

Let x and P be like in 3.2.1.1 above and let P′ be formed by introducing an additional group G of g voters each ranking x first in their preference ordering. Since x is the Condorcet winner in P, x defeats all alternatives it defeats in P also in P′. It is thus the Condorcet winner in P′ as well. Thus all 10 Condorcet-consistent procedures described in Chap. 2 and listed in 3.2.1.1 elect x in P and in P′. We therefore conclude that all these Condorcet-consistent procedures are also invulnerable to the upward monotonicity failure in variable electorates when there is a Condorcet winner in the initial profile.

3.2.2.2 Alternative Vote (AV) and Plurality with Runoff Procedures

Suppose that x is the AV and Condorcet winner in P and that P′ is formed by adding a group G of g identically-minded voters all ranking x at the top of their preference rankings. This means:

(1) that x's count of top preferences increases by g from what it is in P and,
(2) that x is the Condorcet winner in P′.

All other alternatives have the same number of top preferences in P and in P′. If x wins on the first count in P, it does so in P′ as well. If it wins in P in some subsequent count, it does so in P′ as well because the order in which alternatives are being eliminated in P′ is the same as in P. This follows from the fact that—since x is not eliminated in P at any stage—those voters with x at the top of their rankings in P and in P′ have no influence over the sequence of eliminations. Since x in P′ has a larger number of first rank positions, and since it was not eliminated in P, it is not eliminated in P′ either. Hence AV is also invulnerable to the upward monotonicity failure in variable electorates when the initial profile contains a Condorcet winner which is simultaneously the AV winner.

The argument is very similar, but simpler, with respect to the Plurality with Runoff procedure. If the Condorcet winner, x, is the Plurality Runoff winner in P, it means that it has the first or second largest count of first ranks in P. Adding g voters with x ranked first in their preference orderings increases this count by g, while no other candidate has an increased count of top ranks in moving from P to P′. Hence x wins in P′ either on the first round or in the runoff contest.

3.2.2.3 Coombs's Procedure

This procedure is vulnerable to upward monotonicity failure in variable electorate when the initial profile contains a Condorcet winner which is simultaneously the Coombs winner. To see this consider the following example (taken from Felsenthal & Nurmi, 2017, Sect. 4.8.1, pp. 53–54).

Suppose there are 42 voters who must elect one of four candidates, a, b, c, or d under Coombs's method, and that their ranking of the candidates are as follows:

7 voters: $a > c > d > b$
6 voters: $a > d > b > c$
3 voters: $b > a > c > d$
7 voters: $b > c > a > d$
9 voters: $b > c > d > a$
4 voters: $c > a > d > b$
6 voters: $d > a > b > c$

Here a is the Condorcet winner and is elected under Coombs's procedure. (Since none of the candidates is ranked first by a majority of the voters, candidate c is eliminated in the first round under Coombs's method, candidate b is eliminated in the second round, and thereafter candidate a is elected).

Now suppose that, *ceteris paribus*, three additional voters join the electorate whose ranking is $a > c > b > d$. Although the number of voters who rank a first has now increased, still none of the candidates is ranked first by an absolute majority of the voters. So according to Coombs's method candidate d is eliminated in the first counting round, candidate a is eliminated in the second counting round, whereupon candidate b is elected thus demonstrating the susceptibility of Coombs's method to upward monotonicity failure also in variable electorate when a Condorcet winner exists and is elected in the initial profile.

3.2.2.4 The Negative Plurality Elimination Rule (NPER) Procedure

The same example used to demonstrate the vulnerability of Coombs's procedure to upward monotonicity failure in variable electorate when a Condorcet winner exists and is elected in the initial profile can be used also to demonstrate the same type of vulnerability of the NPER procedure.

3.2.2.5 Bucklin's Procedure

This procedure is vulnerable to upward monotonicity failure in variable electorate when the initial profile contains a Condorcet winner which is elected by Bucklin's procedure. To see this consider the following example (taken from Felsenthal & Nurmi, 2017, Sect. 5.3.1, pp. 66–67).

Suppose there are five voters who must elect one out of six candidates a, b, c, d, x, y under Bucklin's method, and that their preference orderings are as follows:

1 voter: $a > c > x > b > d > y$
1 voter: $a > d > x > b > c > y$
1 voter $b > d > x > a > c > y$
1 voter: $b > y > x > a > c > d$
1 voter: $c > y > x > a > b > d$

Here x is the Condorcet winner and is elected under Bucklin's procedure. (As no candidate constitutes the top preference of an absolute majority of the voters, nor of the top and second preferences of the majority of the voters, x constitutes the third preference of all voters and hence is elected according to Bucklin's procedure).

Now suppose that, *ceteris paribus*, two additional voters whose preference orderings are $x > y > a > b > c > d$ join the electorate, thereby, presumably, strengthening x's position. However, in fact, candidate y will now be elected according to Bucklin's procedure—thereby demonstrating the vulnerability of this procedure to upward monotonicity failure in variable electorates—because this candidate constitutes the second preference of a majority (4) of voters, whereas for x to continue being the winner one must take into consideration not only the voters' top two preferences but also their third preference.

3.2.2.6 Majority Judgment (MJ) Procedure

This procedure is also vulnerable to the upward monotonicity failure in variable electorate when the initial profile contains a Condorcet winner that is elected by MJ.

To see this consider the following example (taken from Felsenthal & Nurmi, 2017, Sect. 5.4.1, p. 69).

Suppose that three voters, V_1, V_2 and V_3, grade two candidates, *x* and *y*, on an ordinal scale ranging between A (lowest) and D (highest), as follows:

Candidate/voter	V_1	V_2	V_3	Median grade
x	B	D	D	D
y	C	C	C	C

Since the median grade (D) of candidate *x* exceeds that of candidate *y*, candidate *x* (who is also the Condorcet winner) is elected according to the MJ procedure.

Now suppose that, *ceteris paribus*, two additional voters, V_4 and V_5, join the electorate assigning to candidates *x* and *y* the ranks of B and A, respectively. As a result we get:

Candidate/voter	V_1	V_2	V_3	V_4	V_5	Median grade
x	B	D	D	B	B	B
y	C	C	C	A	A	C

Here *y* is the winner according to the MJ procedure because *y*'s median grade is higher than that of *x*; thus the two additional voters caused their more favorite candidate, *x*, not to be elected even though they awarded *x* a higher grade than they awarded *y*—which demonstrates the susceptibility of the MJ procedure to the upward monotonicity failure in variable electorates when a Condorcet winner exists and is elected in the initial profile.

However, it should be noted that a necessary condition for the MJ procedure to display upward monotonicity failure when there are only two candidates is that the grade awarded by the additional voters to the initial (Condorcet) winner must be lower than his/her median grade in the initial profile. The same is true of its vulnerability to downward monotonicity failure in variable electorates, as will be demonstrated in 3.3.2.14 below.

3.3 The (In)Vulnerability of the Investigated Procedures to Downward Monotonicity Failure

3.3.1 Fixed Electorates

3.3.1.1 Condorcet-Consistent Procedures

Let P be a profile where candidate *x* is the Condorcet winner. Let now profile P′ be formed so that the position of another candidate, *y*, is lowered *vis-à-vis* some other alternatives, *ceteris paribus*, and call the resulting profile P′. Can *y* be the winner in P′? No, since the lowering of *y* does not change *x*'s status and it remains the Condorcet winner in P′. Hence all 10 Condorcet-consistent methods are invulnerable to downward monotonicity failure in fixed electorates under the constraint we have been making (i.e., that the initial profile contains a Condorcet winner which is elected).

3.3.1.2 Totally Invulnerable Non-Condorcet-Consistent Procedures

The six voting procedures Plurality, Approval Voting, Borda, Bucklin's, Range Voting and Majority Judgment described in Chap. 2 are invulnerable to downward monotonicity failure in general under fixed electorates and hence they are also invulnerable to downward monotonicity failure in fixed electorates under a restricted domain where a Condorcet winner exists and is elected in the initial profile. For explanation why these six voting procedures are generally invulnerable to downward monotonicity failure under fixed electorates see Felsenthal and Nurmi (2018).

3.3.1.3 Plurality with Runoff Procedure

Following is an example under the Plurality with Runoff procedure showing it is vulnerable to downward monotonicity failure in fixed electorate even when the domain is restricted, viz., a Condorcet winner exists and is elected in the initial profile.

Suppose that the initial profile consists of 25 voters with the following preference ordering among four candidates, *a*–*d*:

6 voters: $a > b > d > c$
5 voters: $b > c > a > d$
4 voters: $c > a > d > b$
3 voters: $d > c > a > b$
7 voters: $d > a > c > b$

Here *a* is the Condorcet winner and is elected under the Plurality with Runoff procedure (*b* and *c* are eliminated in the first counting round and thereafter *a* beats *d* 15:10).

As *d* has not been elected in the initial profile, suppose now that, *ceteris paribus*, the three $d > c > a > b$ voters change their preference ordering to $c > d > a > b$, thereby decreasing further *d*'s support. As a result *a* and *b* will be eliminated in the first counting round and thereafter *d* beats *c* (13:12) and becomes the winner (which is a more preferable outcome for the three voters who changed their preference ordering)—thereby demonstrating the vulnerability of the Plurality with Runoff procedure to the downward monotonicity failure in fixed electorate even in a restricted domain.

3.3.1.4 Alternative Vote Procedure

The same example used in Sect. 3.3.1.3 can also be used to show the vulnerability of the Alternative Vote procedure to downward monotonicity failure in fixed electorate even when the domain is restricted, viz., a Condorcet winner exists and is elected in the initial profile.

Let us repeat below the original preference distribution of the 25 voters in Sect. 3.3.1.3 among candidates, *a–d*:

6 voters: $a > b > d > c$
5 voters: $b > c > a > d$
4 voters: $c > a > d > b$
3 voters: $d > c > a > b$
7 voters: $d > a > c > b$

Here *a* is the Condorcet winner and is elected under the Alternative Vote procedure (*c* and *b* are eliminated in the first and second counting rounds, respectively, and thereafter *a* beats *d* 15:10).

As *d* has not been elected in the initial profile, suppose now that, *ceteris paribus*, the three $d > c > a > b$ voters change their preference ordering to $c > d > a > b$, thereby decreasing further *d*'s support. As a result *b* and *a* will be eliminated in the first and second counting rounds, respectively, and thereafter *d* beats *c* (13:12) and becomes the winner (which is a more preferable outcome for the three voters who changed their preference ordering)—thereby demonstrating the vulnerability of the Alternative Vote procedure to downward monotonicity failure in fixed electorate even in a restricted domain.

3.3.1.5 Coombs's Procedure

Following is an example under Coombs's procedure showing it is vulnerable to downward monotonicity failure in fixed electorate even when the domain is restricted, viz., a Condorcet winner exists and is elected in the initial profile. (This example is due, in part, to Nicolaus Tideman; see Sect. 5.3.5 and Example 5.3.1.1 in Felsenthal & Nurmi, 2018, pp. 58–59).

Suppose that the initial profile consists of 45 voters with the following preference ordering among three candidates, *a–c*:

1 voter: $a > b > c$
10 voters: $a > c > b$
22 voters: $b > c > a$
10 voters: $c > a > b$
2 voters: $c > b > a$

Here *b* is the Condorcet winner and is elected under Coombs's procedure. (As no candidate is ranked first by a majority of voters, candidate *a*, who is ranked last by a plurality of the voters, is deleted from all ballots. Thereafter b is ranked first by a majority of the voters (23) and is elected according to Coombs's procedure).

However, as c has not been elected, suppose now that, *ceteris paribus*, 11 of the 22 $b > c > a$ voters change their preference ordering to $b > a > c$ thus downgrading further c's position. As a result b, who is ranked last by a plurality of the voters, will be eliminated in the first counting round according to Coombs's procedure, and thereafter c will beat a (23:22) and become the Coombs winner—thereby demonstrating Coombs's procedure vulnerability to downward monotonicity failure in fixed electorate when a Condorcet winner exists and is elected in the initial profile.

3.3.1.6 The Negative Plurality Elimination Rule (NPER) Procedure

The same example used to demonstrate the vulnerability of Coombs's procedure to downward monotonicity failure in fixed electorate when a Condorcet winner exists and is elected in the initial profile can be used also to demonstrate the same type of vulnerability of the NPER procedure.

3.3.2 Variable Electorates

3.3.2.1 Totally Invulnerable Condorcet-Consistent and Non-Consistent Procedures

Five of the investigated procedures (Minimax, Plurality Voting, Approval Voting, Borda and Range Voting) whose description appears in Chap. 2 are invulnerable to downward monotonicity failure in general and hence also to this type of failure when a Condorcet winner exists and is elected in the initial profile. For an explanation see Felsenthal and Nurmi (2018).

3.3.2.2 Black's Procedure

This procedure is vulnerable to the downward monotonicity failure in variable electorate under the restricted domain assumption, i.e., when a Condorcet winner exists (and therefore elected) in the initial profile. To see this consider the following example (due to Felsenthal & Nurmi, 2017, Sect. 5.6.3, p. 75).

Suppose there are nine voters who must elect one of four candidates a, b, c, d under Black's method, and that their preference orderings are as follows: five voters $b > c > d > a$; four voters $c > d > a > b$.

As b is ranked first by an absolute majority of the voters and, hence, is the Condorcet winner, b will be elected under Black's procedure.

Now assume that three additional voters whose preference ordering is $a > d > b > c$ join the electorate. As a result, the social preference ordering becomes cyclical

($a > b > c > d > a$); so, according to Black's procedure one uses Borda's method to determine the winner. According to Borda's method the number of points awarded to candidates a, b, c, d is 13, 18, 22, and 19, respectively. Despite the fact that c was not elected in the original electorate and was ranked last by the additional voters who joined the electorate, c nevertheless is elected in the expanded electorate according to Black's procedure, thus demonstrating the vulnerability of this procedure to the downward monotonicity failure in variable electorate.

3.3.2.3 Kemeny's Procedure

This procedure is vulnerable to the downward monotonicity failure in variable electorate under the restricted domain assumption. To see this consider the following example (due to Felsenthal & Nurmi, 2017, Sect. 5.7.3, pp. 77–78).

Assume there are 11 voters who must select one out of four candidates, a, b, c, d, and that their preference orderings among these candidates are as follows:

5 voters: $d > b > c > a$
3 voters: $a > d > c > b$
3 voters: $a > d > b > c$

Here, a (who is ranked first by an absolute majority of the voters) is the Condorcet winner, so a is elected according to Kemeny's procedure.

Now suppose that, *ceteris paribus*, four additional voters whose preference ordering is $b > c > a > d$ join the electorate. As a result we obtain that the social preference ordering becomes cyclical ($a > d > b > c > a$) and, hence, according to Kemeny's procedure, the most likely (transitive) social preference ordering is $d > b > c > a$ because the sum (57) associated with the pairwise comparisons of this social preference ordering is highest. Thus, according to Kemeny's method d will be elected—thereby demonstrating the vulnerability of that method to the downward monotonicity failure in variable electorate.

3.3.2.4 Nanson's Procedure

This procedure is vulnerable to the downward monotonicity failure in variable electorates under the restricted domain assumption. To see this consider the following example.[4]

Suppose there are 24 voters whose preference ordering among four candidates, a, b, c, d are as follows:

7 voters: $a > d > c > b$
2 voters: $b > a > d > c$

[4]This example refutes the statement made by Felsenthal and Tideman (2013, p. 71, fn 10) according to which Nanson's and Dodgson's methods are invulnerable to downward monotonicity failure in variable electorates.

4 voters: $b > c > a > d$
4 voters: $d > a > c > b$
2 voters: $d > c > a > b$
5 voters: $d > c > b > a$

Here *a* is the Condorcet winner and as Nanson's procedure is Condorcet-consistent *a* is elected.

Now suppose that, *ceteris paribus*, eight additional voters whose preference ordering is $b > c > a > d$ join the electorate. As a result the social preference ordering contains a top cycle ($a > d > c > b > a$) and according to Nanson's method one eliminates candidates *a* and *b* (whose Borda score does not exceed the average Borda score of all four candidates) and thereafter candidate *d* beats candidate *c* and is elected. However, note that here *d* has not been elected initially and was ranked last by the additional voters and despite this has been elected in the enlarged electorate—thereby demonstrating the vulnerability of Nanson's method to downward monotonicity failure in variable electorates when a Condorcet winner exists (and is elected) initially.

3.3.2.5 Borda Elimination Rule (BER)

The same example used to demonstrate the vulnerability of Nanson's procedure to downward monotonicity failure in variable electorate under our restricted domain assumption is applicable under BER as well.

3.3.2.6 Successive Elimination Procedure

This procedure is vulnerable to the downward monotonicity failure in variable electorates under the restricted domain assumption. To see this consider the following example (due to Felsenthal & Nurmi, 2017, Sect. 5.2.3, p. 65).

Assume there are nine voters whose preference ordering in the initial profile among four candidates, *a*–*d*, is the same as in Sect. 3.3.2.2. Assume further that:

(1) The first voting round is between *a* and *b*, the second voting round is between the winner of the first round and *c*, and the third round is between the winner of the second round and *d*.
(2) All voters vote sincerely in all voting rounds.

Accordingly, as *b* is ranked first by an absolute majority of the voters (and hence is also the Condorcet winner), this candidate will be the ultimate winner regardless of the order in which the various candidates are voted upon.

Now assume that, *ceteris paribus*, three additional voters whose preference ordering is $a > d > b > c$ join the electorate. As a result, *a* will beat *b* (7:5) in the first round, *c* will beat *a* (9:3) in the second round, and *c* will beat *d* (9:3) in the third round—and thus *c* becomes the ultimate winner, thereby demonstrating the susceptibility of the Successive Elimination procedure to downward monotonicity failure in variable electorates when a Condorcet winner exists and is elected initially.

3.3.2.7 Young's Procedure

This procedure is totally invulnerable to downward monotonicity failure. This is explained as follows by Pérez (2001, p. 614); see also Felsenthal and Nurmi (2017, Sect. 5.9.3, p. 82).

Suppose we start with a profile P of voters' preferences and find the Young winner in it by counting the number of voter removals needed to make each alternative the Condorcet winner. Suppose further that we find that candidate x is associated with a minimum number, say k, of voter removals and is thus the Young winner. Now suppose that we remove a voter who has placed x at the bottom of his/her ranking. Let us denote the resulting profile by P′. If for any such P′ profile Young's method results in a winner other than x, then we must conclude that Young's method is vulnerable to the P-BOT paradox. So let us suppose that y is elected by Young's method in P′, i.e., that the P-BOT paradox occurs. This means that y needed the minimal number of voter removals, say s, to become the Condorcet winner. This number, s, must be strictly less than $k-1$ since in P′ alternative x would need this number $(k-1)$ of removals to become the Condorcet winner. The fact that $s < k-1$ implies that in P alternative y would have needed $s + 1$ removals to become the Condorcet winner. This is strictly less than k which was the number x needed in P to become the Condorcet winner. Hence x could not be the Young winner in P which contradicts the assumption that s/he was. This contradiction leads to the conclusion that examples demonstrating the vulnerability of Young's procedure to downward monotonicity failure cannot be constructed.

3.3.2.8 Copeland's Procedure

This procedure is vulnerable to the downward monotonicity failure in variable electorate under the restricted domain assumption. To see this consider the following example (due to Felsenthal & Nurmi, 2017, Sect. 5.5.3, p. 73).

Assume there are nine voters whose preference ordering in the initial profile among four candidates, *a*–*d*, is the same as in Sect. 3.3.2.2. As b is ranked first by an absolute majority of the voters (and is therefore also the Condorcet winner), b will be elected under Copeland's procedure.

Now assume that, as in the second part of Sect. 3.3.2.2, three additional voters whose preference ordering is $a > d > b > c$ join the electorate. As a result, the social preference ordering becomes cyclical ($a > b > c > d > a$); thus, according to Copeland's procedure only c and d (each with two points) should belong to the choice set. Hence c—who was not elected in the original electorate and who is ranked last by the additional voters—now belongs to the choice set according to Copeland's procedure, thereby demonstrating the susceptibility of Copeland's procedure to the downward monotonicity failure when a Condorcet winner exists initially.

3.3.2.9 Schwartz's Procedure

This procedure is vulnerable to downward monotonicity failure in variable electorates under the restricted domain assumption. To see this consider the following example (due to Felsenthal & Nurmi, 2017, Sect. 5.8.3, p. 80) where we use a somewhat revised definition of the downward monotonicity failure. According to this, a downward monotonicity failure occurs whenever candidate x is the *unique* winner in the original electorate and, after an additional group of voters ranking another candidate y at the *bottom* of their preference orderings joins the electorate, the outcome is a tie that includes candidate y. (This modified definition was used in 3.3.2.8 as well).

Suppose there are nine voters who must elect one of four candidates a, b, c, d under Schwartz's method, and that their preference orderings are as in the first part of Sect. 3.3.2.2. As b is ranked first by an absolute majority of the voters, b will be elected under Schwartz's procedure.

Now assume that, as in the second part of Sect. 3.3.2.2, three additional voters whose preference ordering is $a > d > b > c$ join the electorate. As a result the social preference ordering becomes cyclical ($a > b > c > d > a$); according to Schwartz's procedure, all four candidates should belong to the choice set—including c, who was not elected in the original electorate and was ranked last by the additional voters who joined the electorate, thereby demonstrating the vulnerability of Schwartz's procedure to downward monotonicity failure when a Condorcet winner exists (and is elected) initially.

3.3.2.10 Dodgson's Procedure

This procedure is vulnerable to downward monotonicity failure in variable electorates under the restricted domain assumption. To see this consider the following example (due to Felsenthal & Nurmi, 2017, Sect. 4.9.3, p. 57).

Suppose there are nine voters who must elect one of four candidates a, b, c, d under Dodgson's method, and that their preference orderings are as in the first part of Sect. 3.3.2.2. As b is ranked first by an absolute majority of the voters, b will be elected under Dodgson's procedure.

Now suppose that, as in the second part of Sect. 3.3.2.2, three additional voters whose preference ordering is $a > d > b > c$ join the electorate. As a result the social preference ordering contains a top cycle and according to Dodgson's method the candidate who can become a Condorcet winner with fewest preference inversions should be the winner. In the augmented electorate this candidate is c because this candidate needs only that three voters invert their preference ordering from $b > c$ to $c > b$ whereas each of the remaining candidates needs more preference inversions in order to become a Condorcet winner. However, note that here c has not been elected initially and was ranked last by the additional voters and despite this has been elected in the enlarged electorate—thereby demonstrating the vulnerability of Dodgson's method to downward monotonicity failure in variable electorates when a Condorcet winner exists initially.

3.3.2.11 Plurality with Runoff and Alternative Vote Procedures

These procedures are vulnerable to downward monotonicity failure in variable electorates under the restricted domain assumption. To see this consider the following example (due to Felsenthal & Maoz, 1992, Example 5, p. 119; see also Felsenthal & Nurmi, 2017, Sect. 4.7.2, pp. 52–53).

Suppose there are 19 voters whose rankings of three candidates, *a*, *b*, and *c*, are as follows:

4 voters: $a > b > c$
1 voter: $a > c > b$
2 voters: $b > a > c$
4 voters: $b > c > a$
3 voters: $c > a > b$
5 voters: $c > b > a$

The social preference ordering is $b > c > a$, i.e., *b* is the Condorcet winner. Under the Plurality with Runoff and Alternative Vote methods, candidate *a* is eliminated after the first round and *b* is elected in the second round.

As candidate *c* has not been elected, suppose now that, *ceteris paribus*, two additional voters whose ranking is $a > b > c$ join the electorate (thereby further downgrading *c*). As a result *b* is eliminated in the first round, and *c* is elected in the second round—thus demonstrating the susceptibility of the Plurality with Runoff and the Alternative Vote methods to downward monotonicity failure when a Condorcet winner exists and is elected initially.

3.3.2.12 Coombs's and Negative Plurality Elimination Rule (NPER) Procedures

These procedures are totally invulnerable to downward monotonicity failure in variable electorate. This is so, as explained by Felsenthal and Nurmi (2017, Sect. 4.8.3, p. 55), because under these methods one eliminates sequentially the candidates who are ranked last by the largest number of voters. So if candidate *z* was not elected originally, then *z* can certainly not be elected under the Coombs's and NPER methods if additional voters who rank *z* last join the electorate.

3.3.2.13 Bucklin's Procedure

This procedure is vulnerable to downward monotonicity failure in variable electorate under the restricted domain assumption. To see this consider the following example (due to Felsenthal & Nurmi, 2017, Sect. 5.3.3, p. 68).

Suppose there are nine voters who must elect one of four candidates *a*, *b*, *c*, or *d* under Bucklin's method, and that their preference orderings are as in the first part of Sect. 3.3.2.2. As *b* is ranked first by an absolute majority of the voters, *b* (who is also the Condorcet winner) will be elected under Bucklin's procedure.

Now suppose that, as in the second part of Sect. 3.2.2.2, three additional voters whose preference ordering is $a > d > b > c$ join the electorate. As a result, *c*—who was not elected in the original electorate and who is ranked last by the additional voters—is now elected under Bucklin's method because the number of voters (9) who rank *c* as their top and second preference exceeds the number of voters who rank any of the other candidates in their top and second preference, thereby demonstrating the susceptibility of Bucklin's procedure to downward monotonicity failure in variable electorate when a Condorcet winner is elected in the initial profile.

3.3.2.14 Majority Judgment (MJ) Procedure

This procedure is vulnerable to downward monotonicity failure in variable electorate under the restricted domain assumption. To see this consider the following example (adapted from Felsenthal & Machover, 2008, p. 329; see also Felsenthal & Nurmi, 2017, Sect. 5.4.3, pp. 70–71).

Suppose that five voters, V_1–V_5, grade two candidates, *x* and *y*, on an ordinal scale ranging between A (lowest) and F (highest), as follows:

Candidate/voter	V_1	V_2	V_3	V_4	V_5	Median grade
x	A	D	E	E	F	E
y	B	C	F	F	F	F

Since the median grade (F) of candidate *y* is higher than that of candidate *x*, candidate *y* (who is also the Condorcet winner) is elected according to the MJ procedure.

Given that candidate *x* has not been elected, suppose now that, *ceteris paribus*, two additional voters, V_6 and V_7, join the electorate, assigning to candidates *x* and *y* the same (or similar) grades as those assigned by voter V_1 (i.e., the lowest grade to *x* and a higher grade to *y* not exceeding C). As a result, we get:

Candidate/voter	V_1	V_2	V_3	V_4	V_5	V_6	V_7	Median grade
x	A	D	E	E	F	A	A	D
y	B	C	F	F	F	B	B	C

Here *x* is the winner according to the MJ procedure because *x*'s median grade (D) is higher than that of *y*; the two additional voters thus caused their less favorite candidate, *x*, to be elected even though they assigned to *x* the lowest grade possible—which demonstrates the susceptibility of the MJ procedure to downward monotonicity failure in variable electorates when a Condorcet winner exists in the initial profile.

3.4 Conclusions

The results of this paper are summarized in Table 3.1, where "+" and "−" indicate, respectively, that the procedure represented by the row is or is not vulnerable to the type of monotonicity violation represented by the column in the restricted domain that has been our focus. To reiterate, we have focused on initial profiles where the winners produced by various procedures coincide with the Condorcet winner. We feel that these domains are of some interest in institution design and analysis because it is often argued that profiles where the pairwise majorities cycle are less common in practice than profiles where a Condorcet winner exists. Without taking firm stand on this argument it seems to us that it is important to know which kinds of environments are likely to be associated with various kinds of voting paradoxes. Of course, this is not to play down the importance of the existence results pertaining to the paradoxes. If it turns out that certain types of cultures are associated with significantly greater likelihood of encountering paradoxes, one could seek procedures that are less likely to produce those paradoxes. The settings where a Condorcet winner exists and coincides with the choice ensuing from a given procedure give the impression of stability; after all there seems to be two ways of coming up with the same outcome. If it then turns out that the basic rationale of voting as expressed in various types of monotonicity is thereby undermined, one should of course be more worried than if the opposite holds, i.e., if the coincidence of outcomes rules out voting paradoxes.

Some highlights of Table 3.1 are worth pointing out:

- All except two of the Condorcet-consistent procedures (Minimax and Young's procedure) are vulnerable only to downward monotonicity failures in variable electorates.
- The Minimax, Plurality, Approval Voting, Borda and Range Voting procedures are the only systems invulnerable to all four types of monotonicity violations in general and hence also under the domain constraint we have imposed.[5]
- Most monotonicity violations in Table 3.1 are associated with variable electorates and downward monotonicity. In fact only eight of the systems examined (Minimax, NPER, Coombs's, Young's, Plurality, Approval Voting, Borda and Range Voting) seem immune to these kinds of violations.

[5]Although Young's procedure too is shown in Table 3.1 to be invulnerable to all four types of monotonicity failure under the domain constraint imposed, it is vulnerable to upward monotonicity failure in variable electorate when the initial social preference ordering contains a majority cycle. For an example see Felsenthal and Nurmi (2017, Sect. 5.9.2, pp. 81–82).

Table 3.1 (In)Vulnerability of 20 voting procedures to monotonicity failures in a restricted domain

	Upward monotonicity failures		Downward monotonicity failures	
Procedure\electorate	Fixed electorates	Variable electorates	Fixed electorates	Variable electorates
Minimax	–	–	–	–
Black	–	–	–	+
Kemeny	–	–	–	+
Nanson	–	–	–	+
Successive Elimination	–	–	–	+
Young	–	–	–	–
Copeland	–	–	–	+
Schwartz	–	–	–	+
Dodgson	–	–	–	+
Borda Elimination	–	–	–	+
Plurality with Runoff	–	–	+	+
Alternative Vote	–	–	+	+
Coombs	+	+	+	–
Negative Plurality Rule	+	+	+	–
Bucklin	–	+	–	+
Majority Judgment	–	+	–	+
Plurality	–	–	–	–
Approval Voting	–	–	–	–
Borda	–	–	–	–
Range Voting	–	–	–	–

Notes A + sign indicates that a procedure is vulnerable to the specified paradox; A – sign indicates that a procedure is invulnerable to the specified paradox

Exercises for Chapter 3

Problem 3.1

Construct a preference profile with no Condorcet winner showing that the Alternative Vote procedure is non-monotonic in a fixed electorate.

Problem 3.2

Show that Nanson's procedure and Borda Elimination (BER aka Baldwin's) procedure may lead to different choices when a Condorcet winner does not exist in the initial profile.

Problem 3.3
Nanson proved that his procedure necessarily elects the Condorcet winner if one exists. Does the same apply to BER? If it does, explain the reason; if not, provide an example where BER does not end up with the Condorcet winner when one exists.

Problem 3.4
Show that Coombs's procedure is vulnerable to the strong No-Show paradox when the social preference ordering in the initial profile contains a majority cycle.

Problem 3.5
A procedure is called*Maskin monotonic*if whenever a candidate, say x, wins in an initial profile, it also wins when its support is increased or kept the same with respect to every other candidate. (N.B. there is no *ceteris paribus* clause in this definition). Show that the Plurality Voting procedure is not Maskin monotonic (Maskin, 1999).

Problem 3.6
Show that the Borda count is not Maskin monotonic.

Answers to Exercises of Chapter 3

Problem 3.1
Consider the following profile:

9 voters: $a > b > c$
8 voters: $b > c > a$
7 voters: $c > a > b$

Here there is no Condorcet winner. Once c has been eliminated, a wins. Suppose now that two of the voters whose preference ordering is *bca* lift a to the top of their ranking, *ceteris paribus*. The resulting profile is

11 voters: $a > b > c$
6 voters: $b > c > a$
7 voters: $c > a > b$

Now b is eliminated, whereupon c wins.

Problem 3.2
Consider the following profile:

8 voters: $a > b > c$
5 voters: $b > c > a$
7 voters: $c > a > b$

Here a wins by Nanson's method since both b and c have a lower than average Borda score. On the other hand, BER eliminates first b, whereupon c defeats a and becomes the winner.

Problem 3.3
BER eliminates at each counting round the candidate with the lowest Borda score. This cannot be the Condorcet winner since the Condorcet winner has always a strictly larger than average Borda score. Thus, the Condorcet winner survives all counting rounds and wins according to the BER procedure.

Problem 3.4
Consider the following profile:

5 voters: $a > b > c$
5 voters: $b > c > a$
6 voters: $c > a > b$
3 voters: $c > b > a$

Here the social preference ordering contains a majority cycle: a defeats b, b defeats c and c defeats a. Candidate b wins once a has been eliminated. If the three last mentioned voters abstain, b is eliminated first and then c wins. This is obviously preferred to b by the abstainers as c is their top-ranked candidate.

Problem 3.5
Consider the following profile:

2 voters: $a > b > c > d$
1 voter: $b > c > a > d$
1 voter: $c > b > a > d$
1 voter: $d > c > b > a$

Here a wins. Now lifting b ahead of c and d in the last two voters' preferences does not change a's position *vis-à-vis* the others. Yet, after the change b becomes the Plurality winner.

Problem 3.6
Consider the following profile:

5 voters: $a > b > c$
4 voters: $b > c > a$
4 voters: $c > a > b$

Here a wins. Improve now a's position in one voter's ranking and move c ahead of b in the first and second groups' preferences. This does not deteriorate a's position. Yet, after the change c wins because these changes result in the following profile (where c wins according to Borda's count procedure).

5 voters: $a > c > b$
4 voters: $c > b > a$
3 voters: $c > a > b$
1 voter: $a > c > b$.

References

Bartholdi, J. J., Tovey, C. A., & Trick, M. A. (1989). The computational difficulty of manipulating an election. *Social Choice and Welfare, 6,* 227–241.

Felsenthal, D. S. (2012). Review of paradoxes afflicting procedures for electing a single candidate. In D. S. Felsenthal & M. Machover (Eds.), *Electoral systems: Paradoxes, assumptions, and procedures* (pp. 19–91). Heidelberg: Springer.

Felsenthal, D. S., & Machover, M. (2008). The majority judgment voting procedure: A critical evaluation. *Homo Oeconomicus, 25,* 319–333.

Felsenthal, D. S., & Maoz, Z. (1992). Normative properties of four single-stage multi-winner electoral procedures. *Behavioral Science, 37,* 109–127.

Felsenthal, D. S., & Nurmi, H. (2016). Two types of participation failure under nine voting methods in variable electorates. *Public Choice, 168,* 115–135.

Felsenthal, D. S., & Nurmi, H. (2017). *Monotonicity failures afflicting procedures for electing a single candidate*. Cham, Switzerland: Springer.

Felsenthal, D. S., & Nurmi, H. (2018). *Voting procedures for electing a single candidate: Proving their (in)vulnerability to various voting paradoxes*. Cham, Switzerland: Springer.

Felsenthal, D. S., & Tideman, N. (2013). Varieties of failure of monotonicity and participation under five voting methods. *Theory and Decision, 75,* 59–77.

Felsenthal, D. S., & Tideman, N. (2014). Interacting double monotonicity failure with direction of impact under five voting methods. Mathematical Social Sciences, *67*, 57–66.

Fishburn, P. C. (1982). Monotonicity paradoxes in the theory of elections. *Discrete Applied Mathematics, 4,* 119–134.

Fishburn, P. C., & Brams, S. J. (1983). Paradoxes of preferential voting. *Mathematics Magazine, 56,* 207–214.

Gibbard, A. (1973). Manipulation of voting systems: A general result. *Econometrica, 41,* 587–601.

Maskin, E. (1999). Nash equilibrium and welfare optimality. *Review of Economic Studies, 66,* 23–38.

Meredith, J. C. (1913). *Proportional representation in Ireland*. Dublin: Edward Ponsonby, Ltd. (Reprint from the collection of the University of California Libraries).

Miller, N. R. (2017). Closeness matters: Monotonicity failure in IRV elections with three candidates. *Public Choice, 173,* 91–108.

Moulin, H. (1988). Condorcet's principle implies the no-show paradox. *Journal of Economic Theory, 45,* 53–64.

Nurmi, H. (2004). Monotonicity and its cognates in the theory of choice. *Public Choice, 121,* 25–49.

Pérez, J. (2001). The strong no show paradoxes are a common flaw in Condorcet voting correspondences. *Social Choice and Welfare, 18,* 601–616.

Satterthwaite, M. A. (1975). Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social choice functions. *Journal of Economic Theory, 10,* 187–217.

Slinko, A., & White, S. (2014). Is it ever safe to vote strategically? *Social Choice and Welfare, 43,* 403–427.

Chapter 4
The (In)Vulnerability of 20 Voting Procedures to the Inconsistency Paradox (aka Reinforcement Paradox) in a Restricted Domain

Abstract This chapter focuses on the possibility that some well-known voting procedures are vulnerable to the Inconsistency paradox even in preference profiles that are characterized by a restricted domain where a Condorcet winner exists and is elected in each disjoint subset of voters but not in their union. Our focus is on 15 voting procedures known to be vulnerable to the Inconsistency paradox in unrestricted domains. These procedures include 10 Condorcet-consistent and 5 Condorcet-non-consistent rules. The former are, however, only briefly touched upon because their invulnerability to the Inconsistency paradox in the restricted domain is obvious.

Keywords Voting paradoxes · Inconsistency voting paradox · Restricted domains · Condorcet-consistent procedures · Condorcet non-consistent procedures · Variable electorates

4.1 Introduction

Young (1974, p. 44) has defined a voting procedure, *f*, to be *consistent* "if two disjoint subsets of voters V and V' would choose the same alternative using *f*, then their union should also choose this alternative using *f*." Thus the *Inconsistency paradox* (aka Reinforcement or Multiple Districts paradox) is a situation where the same candidate, *x*, is elected under a given voting procedure in each separate district but some other candidate, *y*, is elected if, *ceteris paribus*, the various districts are amalgamated into a single district.

In most examples appearing in the literature demonstrating the vulnerability of 14 well-known voting procedures to the Inconsistency paradox, the social preference ordering of the voters[1] among the competing candidates in at least one of the districts contains a top cycle (see, for example, Felsenthal & Nurmi, 2018). We were therefore interested to find out which of these 14 procedures would still be vulnerable to the

This chapter was composed by the first-named author of this booklet.

[1]This is the preference ordering of the majority of the voters. The voters constituting the majority may not be the same for all pairs of candidates.

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_4

Inconsistency paradox when the profile of the voters' preference ordering among the competing candidates in each separate district is restricted so that a Condorcet winner exists and is elected in every district.

The rest of the chapter is organized as follows. In the next section we first explain briefly why all the Condorcet-consistent as well as some non-Condorcet consistent procedures are invulnerable to the Inconsistency paradox under our restricted domain, and thereafter we demonstrate the vulnerability of the remaining non-Condorcet-consistent procedures to the Inconsistency paradox even under our restricted domain.[2] The final section sums up the findings and discusses their implications for institution design.

4.2 The (In)Vulnerability of the Various Procedures to the Inconsistency Paradox Under the Restricted Domain Assumption

4.2.1 The Condorcet-Consistent Procedures

If x is a Condorcet winner existing under a given voting procedure in every disjoint subset of voters, then x must be the Condorcet winner also in the union of these subsets. Since the 10 procedures described in Sects. 2.1.4, 2.3.1–2.3.9 in Chap. 2 as well as Borda's Elimination Rule (BER aka Baldwin's procedure) are all Condorcet-consistent, it follows that under these procedures not only a Condorcet winner is elected in every disjoint subset of voters in which it exists, but also that it exists—and therefore is elected—also in the union of these subsets. Hence all the 11 Condorcet-consistent procedures listed above are invulnerable to the Inconsistency paradox under our restricted domain assumption.

4.2.2 Totally Invulnerable Non-Condorcet-Consistent Procedures

The four non-Condorcet-consistent procedures Plurality Voting, Approval Voting, Borda's procedure and the Range Voting procedures described in Chap. 2 are generally invulnerable to the Inconsistency paradox and hence also to this paradox under our restricted domain assumption.[3]

[2]The description of all the 20 voting procedures analyzed in this chapter appears in Chap. 2.

[3]For an explanation why these four procedures are generally invulnerable to the Inconsistency paradox see Felsenthal and Nurmi (2018, Chaps. 4–5).

4.2.3 Plurality with Runoff and the Alternative Vote (AV) Procedures

These two non-Condorcet-consistent procedures are vulnerable to the Inconsistency paradox even under our restricted domain assumption. To see this, consider the following example. Suppose there are two districts, I and II. In district I there are 16 voters whose preference orderings among three candidates, *a*, *b*, *c*, are as follows:

5 voters: $a > b > c$
4 voters: $b > a > c$
4 voters: $c > a > b$
3 voters: $c > b > a$

Here *a* is the Condorcet winner and is elected under both the Plurality with Runoff and AV procedures. (As no candidate is ranked first by an absolute majority of the voters, candidate *b* is deleted from all ballots in the first count and thereafter *a* is ranked first by an absolute majority of the voters [9] and is elected).

In district II there are 13 voters whose preference orderings among the same three candidates are as follows:

4 voters: $a > c > b$
3 voters: $b > a > c$
3 voters: $b > c > a$
3 voters: $c > a > b$

Here *a* is also the Condorcet winner and is elected under both the Plurality with Runoff and the AV procedures (*c* is eliminated in the first counting round and thereafter *a* is ranked first by an absolute majority of the voters [7] and is elected).

However, when the two districts are amalgamated into a single district we obtain the following distribution of the 29 voters among the three candidates:

5 voters: $a > b > c$
4 voters: $a > c > b$
7 voters: $b > a > c$
3 voters: $b > c > a$
7 voters: $c > a > b$
3 voters: $c > b > a$

As no candidate is ranked first by an absolute majority of the voters, candidate *a*, the Condorcet winner, (who is ranked first by the fewest voters) is deleted from all ballots in the first count under both the Plurality with Runoff and AV procedures and thereafter candidate *b* is ranked first by an absolute majority of the voters (15) and is elected—thus demonstrating the vulnerability of both the Plurality with Runoff and the AV procedures to the Inconsistency paradox even under our restricted domain.

4.2.4 Coombs's Procedure

This non-Condorcet-consistent procedure too is vulnerable to the Inconsistency paradox even under our restricted domain assumption. To see this, consider the following example. Suppose there are two districts, I and II. In district I there are 42 voters whose preference ordering among four candidates, *a*, *b*, *c* and *d* are as follows:

10 voters: $a > d > c > b$
6 voters: $a > d > b > c$
4 voters: $b > a > c > d$
7 voters: $b > c > a > d$
9 voters: $b > c > d > a$
2 voters: $c > a > d > b$
4 voters: $d > a > b > c$

Here candidate *a* is the Condorcet winner and is elected according to Coombs's procedure, as follows:

As no candidate is ranked first by an absolute majority of the voters, the candidate who is ranked last by the plurality of the voters (*b*) is deleted from all ballots. As none of the remaining candidates is as yet ranked first by a majority of voters, the candidate who is now ranked last by the plurality of the voters (*c*) is deleted from all ballots and thereafter candidate *a*—who is the Condorcet winner—is ranked first by an absolute majority of the voters (29) and is therefore elected according to Coombs's procedure.

In district II there are 41 voters whose preference ordering among the same four candidates are as follows:

21 voters: $a > b > c > d$
20 voters: $d > c > b > a$

Here too *a* is the Condorcet winner and is elected immediately according to Coombs's procedure as this candidate is ranked first by an absolute majority of the voters.

However, when the two districts are amalgamated into a single district we obtain the following distribution of the 83 voters' preference orderings among the four candidates:

21 voters: $a > b > c > d$
6 voters: $a > d > b > c$
10 voters: $a > d > c > b$
4 voters: $b > a > c > d$
7 voters: $b > c > a > d$
9 voters: $b > c > d > a$
2 voters: $c > a > d > b$
4 voters: $d > a > b > c$
20 voters: $d > c > b > a$

Here no candidate is ranked first by an absolute majority of the voters so the candidate who is ranked last by the plurality of the voters (*d*) is deleted from all ballots. Since none of the remaining candidates is as yet ranked first by an absolute majority of the voters, the candidate who is now ranked last by the plurality of the voters (*a*) is deleted from all ballots and thereafter *b* is ranked first by an absolute majority of the voters (51) and is therefore elected according to Coombs's procedure—thus demonstrating the vulnerability of this procedure to the Inconsistency paradox even under our restricted domain.

4.2.5 The Negative Plurality Elimination Rule (NPER)

This non-Condorcet-consistent procedure too is vulnerable to the Inconsistency paradox even under our restricted domain assumption. To see this, consider the following example. Suppose there are two districts, I and II. In district I there are 42 voters whose preference ordering among four candidates, *a*, *b*, *c* and *d* are as in district I in Sect. 4.2.4, and in district II there are 41 voters whose preference ordering among the same four candidates are as in district II in Sect. 4.2.4.

In district I candidate *a* is the Condorcet winner and is elected according to the NPER procedure, as follows: The candidate who is ranked last by the plurality of the voters (*b*) is deleted from all ballots. Thereafter the candidate who is now ranked last by the plurality of the voters (*c*) is deleted from all ballots and thereafter candidate *d* is deleted from all ballots thus candidate *a*—who is the Condorcet winner—is the only survivor and is therefore elected according to NPER procedure.

In district II candidate *a* is also the Condorcet winner and is elected according to the NPER procedure, as follows:

The candidate who is ranked last by the plurality of the voters (*d*) is deleted from all ballots. Thereafter the candidate who is now ranked last by the plurality of the voters (*c*) is deleted from all ballots and thereafter candidate *b* is deleted from all ballots thus candidate *a*—who is the Condorcet winner—is the only survivor and is therefore elected according to NPER procedure.

However, when the two districts are amalgamated into a single district we obtain the same distribution of the 83 voters' preference orderings among the four candidates as in the second part of Sect. 4.2.4. As we saw, candidate *d* is ranked last by the plurality of voters. Once it is eliminated from all ballots, candidate *a* is eliminated. Of the remaining two candidates *c* is eliminated next, thus leaving *b* to be the winner.

4.2.6 Bucklin's Procedure

This non-Condorcet-consistent procedure too is vulnerable to the Inconsistency paradox even under our restricted domain assumption. To see this, consider the following example. Suppose there are two districts, I and II. In district I there are three voters whose preference ordering among three candidates, *a*, *b*, and *c* are as follows:

1 voter: $a > c > b$
1 voter: $b > a > c$
1 voter: $c > a > b$

In district II there are seven voters whose preference ordering among the same three candidates are as follows:

4 voters: $a > b > c$
2 voters: $b > c > a$
1 voter: $c > b > a$

A Condorcet winner (*a*) exists and is elected in both districts according to Bucklin's procedure. However, when the two districts are amalgamated into a single district one obtains the following distribution of 10 voters' preferences:

4 voters: $a > b > c$
1 voter: $a > c > b$
1 voter: $b > a > c$
2 voters: $b > c > a$
1 voter: $c > a > b$
1 voter: $c > b > a$

None of the voters is ranked first by an absolute majority of the voters. However, *b* is ranked first or second by more voters (8) than each of the other two candidates and is therefore elected according Bucklin's procedure—thus demonstrating the vulnerability of this procedure to the Inconsistency paradox even in our restricted domain.

4.2.7 *The Majority Judgment (MJ) Procedure*

This non-Condorcet-consistent procedure too is vulnerable to the Inconsistency paradox even under our restricted domain assumption. To see this, consider the following example. Suppose there are two districts, I and II. In district I there are three voters, V_1, V_2, and V_3, who rank three candidates, *x*, *y*, and *z*, on an ordinal scale from A (lowest) to E (highest) as follows:

Candidate\voter	V_1	V_2	V_3	Median
x	E	C	B	C
y	A	D	A	A
z	B	A	D	B

Here candidate *x* is the Condorcet winner (as it is preferred by the majority of voters over each of the other two candidates) and is elected according to the MJ procedure because its median rank (C) is highest.

In district II there are seven voters, V_4–V_{10}, who rank the same three candidates on the same ordinal scale as follows:

Candidate\voter	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	Median
x	E	E	E	E	B	B	B	E
y	A	A	A	A	C	C	A	A
z	D	D	D	D	A	A	D	D

Here too candidate *x* is the Condorcet winner and is elected according to the MJ procedure.

However, when the two districts are amalgamated into a single district we get the following rankings of the three candidates by the 10 voters:

Candidate\voter	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	Median
x	E	C	B	E	E	E	E	B	B	B	C
y	A	D	A	A	A	A	A	C	C	A	A
z	B	A	D	D	D	D	D	A	A	D	D

Here *x* is still the Condorcet winner. However, as now candidate *z* has the highest median grade (D), this candidate, rather than the Condorcet winner *x*, is elected, thus demonstrating the vulnerability of the MJ procedure to the Inconsistency paradox even under our restricted domain assumption.

4.3 Conclusion

We have focused on well-known voting procedures which are, in general, vulnerable to the Inconsistency paradox, because we wanted to investigate whether these procedures would also be vulnerable to this paradox under a restricted domain assumption where a Condorcet winner exists and is elected in every disjoint subset of voters. We explained why all Condorcet-consistent procedures are invulnerable to the Inconsistency paradox under our restricted domain assumption, and demonstrated that six well-known non-Condorcet-consistent procedures are vulnerable to this paradox even under our restricted domain assumption. We feel that our restricted domain assumption is of some interest in institution design and analysis because it is often argued that profiles where the pairwise majorities cycle are less common in practice than profiles where a Condorcet winner exists. Without taking a firm stand on this argument, it seems to us that it is important to know which kinds of environments are likely to be associated with various kinds of voting paradoxes. Of course, this is not to play down the importance of the existence results pertaining to the paradoxes. If it turns out that certain types of cultures are associated with significantly greater likelihood of encountering paradoxes, one could seek voting procedures that are less likely to produce those paradoxes. This is certainly the case with respect to the Inconsistency paradox which cannot occur under any Condorcet-consistent procedure where our restricted domain assumption is satisfied.

Exercises for Chapter 4

Problem 4.1
Compose an example profile where the Plurality with Runoff procedure ends up with consistent choices in the restricted domain.

Problem 4.2
Consider Bucklin's procedure and a situation involving two districts. Assume that in district I all voters are unanimous that candidate x is to be ranked first. How large must the electorate of district I be in proportion to the whole electorate to guarantee the victory of x in the electorate at-large when Bucklin's procedure is used?

Problem 4.3
Describe the similarities and differences between the Inconsistency and the No-Show paradox.

Answers to Exercises of Chapter 4

Problem 4.1
The simplest example is undoubtedly one where both districts have the same Absolute Winner (i.e., a candidate who is ranked first by an absolute majority of the voters) which is thereby elected. In the union of the two districts, the elected candidate is also the Absolute Winner and is therefore elected.

Problem 4.2
If district I constitutes more than 50% of the entire electorate, the victory of x is thereby guaranteed. Obviously, this is just a sufficient condition for x's election.

Problem 4.3
Each instance of the P-TOP paradox can be seen as an instance of Inconsistency since the P-TOP paradox occurs whenever a group of voters electing x is joined by another group of unanimous voters all ranking x first and as a result some candidate y is elected. Hence the added group is better off not joining the electorate. This situation is obviously identical with a situation where the initial profile represents district I and the added unanimous group stands for district II. So, all instances of the P-TOP paradox represent Inconsistency. However, the converse is not true: there

are instances of Inconsistency where the groups that join are not unanimous. These are not instances of the P-TOP paradox.

References

Felsenthal, D. S., & Nurmi, H. (2018). *Voting procedures for electing a single candidate: Proving their (in)vulnerability to various voting paradoxes*. Cham, Switzerland: Springer.
Young, H. P. (1974). An axiomatization of Borda's rule. *Journal of Economic Theory, 9,* 43–52.

Chapter 5
The (In)Vulnerability of 20 Voting Procedures to the No-Show Paradox in a Restricted Domain

Abstract The No-Show paradox occurs whenever a group of identically-minded voters is better off abstaining than by voting according to its preferences. Moulin's (Journal of Economic Theory 45:53–64, 1988) result states that if one wants to exclude the possibility of the No-Show paradox, one has to resort to procedures that do not necessarily elect the Condorcet winner when one exists. This paper examines 10 Condorcet-consistent and 10 Condorcet-non-consistent procedures in a restricted domain, viz., one where there exists a Condorcet winner who is elected in the original profile and the profile is subsequently modified by removing a group of voters with identical preferences. The question asked is whether the No-Show paradox can occur in these settings. It is found that only 2 of the 10 Condorcet-consistent procedures investigated (Minimax and Schwartz's procedure) are invulnerable to the No-Show paradox, whereas only 3 of the 10 non-Condorcet-consistent ranked procedures investigated (Coombs's, the Negative Plurality Elimination Rule, and the Majority Judgment procedures) are vulnerable to this paradox in the restricted domain. In other words, for a No-Show paradox to occur when using Condorcet-consistent procedures it is not, in general, necessary that a top Condorcet cycle exists in the original profile, while for this paradox to occur when using (ranked) non-Condorcet-consistent procedures it is, almost always, necessary that the original profile has a top cycle.

Keywords Condorcet-consistency · Domain restrictions · No-Show paradox · Voting paradoxes · Voting procedures

5.1 Introduction

The theory of voting is known for its many apparently negative results that amount to demonstrating the impossibility of satisfying several social choice desiderata. Arrow's (1963) impossibility theorem is the best-known result of this kind, but it is by no means the only one. In fact, the incompatibility of two or more desirable

This chapter is based mainly on Felsenthal and Nurmi (2018).

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_5

properties is one standard method of expressing findings that are related to the study of voting rules; if one wishes that one's rules always behave in a plausible way in all circumstances, then one has to be prepared for the possibility that the behavior is not plausible in another sense.

One important result on voting rules was achieved by Moulin (1988). It states that if the number of alternatives under consideration is at least 4 and the number of voters is at least 25, then no Condorcet-consistent rule is compatible with the property known as Participation.[1] A rule satisfies Participation if any group of voters with the same preference ranking over the alternatives is under no circumstances better off abstaining than voting according to its preferences. Following Fishburn and Brams (1983), situations where Participation is not satisfied are called *No-Show paradoxes*. Condorcet-consistent rules share the defining property that in all circumstances they always result in the Condorcet winner being the sole alternative chosen, whenever there is a Condorcet winner in the profile under investigation. Condorcet winner, in turn, is an alternative that—according to the voters' preferences—would be preferred to any other alternative by a majority of voters.

Moulin's result has subsequently been refined and augmented (see e.g., Brandt, Geist, & Peters, 2017; Brandt, Hofbauer, & Strobel, 2018; Felsenthal & Nurmi, 2017; Pérez, 1995, 2001), but the basic incompatibility between the two social choice desiderata, Condorcet-consistency and Participation, remains intact. Our main interest here is to find out what this incompatibility would mean in terms of the design of voting institutions. In other words, under what kinds of circumstances can we expect the incompatibility to materialize? More specifically, if one adopts a Condorcet-consistent voting system, how likely is it that an instance of the No-Show paradox will be encountered?

A straight-forward way to address these questions is to construct a probability model of the process generating the preference profiles and to determine those giving rise to the No-Show paradox. The relevance of the models depends on the degree in which they mimic the process that underlies the emergence of the profiles of the decision making body under examination.[2] Our goal is more modest. We aim at determining the effect of one important profile characteristic, viz., the presence or absence of a Condorcet winner, on the possibility of the No-Show paradox. Our main problem is to determine whether various Condorcet-consistent, as well as various ranked Condorcet non-consistent procedures, are vulnerable to the No-Show paradox in the restricted domain characterized by the presence—and the election—of the Condorcet winner in the initial profile. Probability and simulation results suggest that the probability of a Condorcet winner existing in randomly generated preference profiles is in general higher than the probability of majority cycles (see, e.g., Gehrlein, 1983;

---

[1] Brandt et al. (2017) corrected Moulin's result by showing that there exists no Condorcet-consistent rule which satisfies Participation when there are at least 12 (rather than 25) voters and 4 alternatives.

[2] Brandt et al. (2018) examined the incidence of the No-Show paradox displayed by three Condorcet-consistent procedures (Black's, Minimax and Tideman's rule) using Ehrhart theory and extensive computer simulations. They found that for a small number of alternatives (4) the probability that these procedures display the No-Show paradox is negligible and as the number of alternatives increases (up to 30) the No-Show paradox becomes much more likely.

Gehrlein & Lepelley, 2011, 2017). So, should the absence of a Condorcet winner be a necessary prerequisite of the No-Show paradox, this would significantly diminish the practical importance of Moulin's theorem. In what follows it will, however, be seen that the No-Show paradoxes are possible in the presence of a Condorcet winner for nearly all those 10 Condorcet-consistent systems that we will focus upon, whereas the No-Show paradox cannot occur under 4 of the 6 ranked non-Condorcet-consistent procedures investigated when a Condorcet winner is present and elected in the original profile.

5.2 Assumptions and Definitions

We shall focus on rules that aggregate individual opinions into collective ones in the following sense. Each individual is assumed to be endowed with a complete (or connected) and transitive preference relation (ranking) over the decision alternatives (candidates, policies, etc.). We denote the set of individuals (voters) by N and assume that it consists of n voters. The rules that specify the aggregation process are set-valued social choice correspondences so that for each n-tuple of individual preference rankings (called the *preference profile*), the rule indicates the set of chosen alternatives, the winners. The set of alternatives is denoted by A and it contains k elements. We assume that the rules are anonymous so that the number, not the identity, of the voters having each preference ranking determines the decision outcome when the rule is applied.

Our basic tool in the analysis of the preference profiles is the *pairwise comparison matrix* which contains k rows and k columns so that the element in cell (i, j) indicates the number of individuals strictly preferring alternative i to alternative j. The k entries along the main diagonal are left blank. By completeness of the individual preference relations we can assume that each non-diagonal entry is non-empty. In all our examples the individual preferences are not only complete and transitive, but also strict, meaning that if x is preferred to y by an individual, this implies that y is not preferred to x by that same individual. From this it follows that the pairwise comparison matrix is reciprocal, i.e., the sum of entries (i, j) and (j, i) is always n.

5.3 Examples Demonstrating the Possibility of No-Show Paradox Under Eight Condorcet-Consistent and Three Non-Condorcet-Consistent Procedures When a Condorcet Winner Exists in the Initial Profile

In the following examples we use notation such as '3 voters $a > b > c$' to denote three voters having (transitive) preference ordering among alternatives a, b, c such that they prefer alternative a to b, b to c, and hence also a to c. The descriptions of all the voting procedures listed in this as well as in subsequent sections appear in Chap. 2.

5.3.1 Black's Procedure

Suppose an initial profile with 23 voters whose preference orderings are as follows:

3 voters: $a > c > b > d$
3 voters: $a > c > d > b$
6 voters: $b > d > a > c$
6 voters: $c > a > d > b$
5 voters: $c > d > b > a$

In this initial profile a is the Condorcet winner. Now, if *ceteris paribus*, the six $c > a > d > b$ voters abstain then the social preference ordering becomes cyclical ($a > c > b > d > a$)[3] in which case the winner is determined according to Borda's procedure. This winner is c (with a Borda score of 27) which is a preferable result for the six abstainers than the election of a. Therefore Black's procedure is vulnerable to the No-Show paradox even when the initial profile contains a Condorcet winner.

5.3.2 Kemeny's Procedure

The example used in Sect. 5.3.1 to demonstrate the vulnerability of Black's procedure to the No-Show paradox under our restricted domain assumption applies to Kemeny's procedure too. In the initial profile a is the Condorcet winner and is elected under Kemeny's procedure. If the six $c > a > d > b$ voters abstain the social preference ordering becomes cyclical in which case the transitive social preference ordering according to Kemeny's procedure is $c > b > d > a$ (with the highest sum of 58 pairwise voter agreements with this possible social preference ordering), i.e., c is elected according to Kemeny's procedure which is preferable for the six abstainers to the election of a. It is therefore concluded that Kemeny's procedure too is vulnerable to the No-Show paradox even if a Condorcet winner exists (and is elected) in the initial profile.

5.3.3 Nanson's and the BER (Baldwin's) Procedures

Let the initial profile be the same 23 voters and 4 candidates as under Black's procedure. In this initial profile a is the Condorcet winner and is therefore elected under

[3]Read: the majority of voters prefer a to c, c to b, b to d, and d to a.

both Nanson's and Baldwin's procedures. Now, if *ceteris paribus*, the six $c > a > d > b$ voters abstain then the Borda scores of a, b, c, d in the first count are 24, 26, 27, and 25, respectively, with an average score of 25.5—so according to Nanson's procedure both a and d are deleted and thereafter c beats b (11:6) and becomes the Nanson winner—which is preferable for the six abstainers. Similarly under the BER procedure: after the first counting round a (whose Borda score is 24) is deleted, in the second counting round d (whose Borda score is 14) is deleted, and in the third counting round c beats b (11:6). Thus both Nanson's and the BER procedures are vulnerable to the No-Show paradox even under our restricted domain assumption.

5.3.4 Successive Elimination Procedure

Suppose the initial profile has seven voters with the following preference orderings (cf., Felsenthal & Nurmi, 2017, pp. 64–65). Suppose further that under this procedure the elimination is conducted as follows: first candidate a competes against b and the loser is eliminated; thereafter the winner competes against c.

2 voters: $a > b > c$
2 voters: $b > c > a$
1 voter: $c > a > b$
2 voters: $c > b > a$

Accordingly, in this initial profile b is the Condorcet winner. Now suppose that, *ceteris paribus*, the two $c > b > a$ voters abstain. In this case a would be elected in the first stage, and thereafter c will beat a and will become the final winner—which is preferable for the abstainers to the election of b. This demonstrates the vulnerability of the Successive Elimination procedure to the No-Show paradox even in (initial) profiles containing a Condorcet winner.

5.3.5 Young's Procedure

Let the initial profile be one with 49 voters whose preference orderings are as follows (Felsenthal, 2012, pp. 87–88; Nurmi, 2012, pp. 266–267):

11 voters: $b > a > d > e > c$
10 voters: $e > c > b > d > a$
10 voters: $e > d > a > b > c$
10 voters: $a > c > d > b > e$
2 voters: $e > c > d > b > a$
2 voters: $e > d > c > b > a$
2 voters: $c > b > a > d > e$
1 voter: $d > c > b > a > e$

1 voter: $a > b > d > e > c$

Here d is the Condorcet winner. Now suppose that, *ceteris paribus*, the 10 $e > d > a > b > c$ voters abstain. As a result in the diminished profile candidate e needs the removal of only 12 voters in order to become the Condorcet winner (the 10 $a > c > b > d > e$ voters and the 2 voters whose top preference is c), whereas each of the other candidates needs more removals in order to become a Condorcet winner. So according to Young's procedure e becomes the winner in the reduced profile—which is preferable for the abstainers to the election of d. Thus Young's procedure can display the No-Show paradox even under the restricted domain where a Condorcet winner exists in the initial profile.

5.3.6 Copeland's Procedure

Consider the initial profile with 21 voters whose preference orderings are as follows:

3 voters: $a > c > b > d$
3 voters: $a > c > d > b$
5 voters: $b > d > a > c$
5 voters: $c > a > d > b$
5 voters: $c > d > b > a$

Here a is the Condorcet winner. Now suppose that the five $c > a > d > b$ voters decide to abstain. As a result c becomes the Copeland winner which is preferable for the five abstainers to the election of a. Thus, the No-Show paradox can occur when using Copeland's procedure even in profiles containing a Condorcet winner.

5.3.7 Dodgson's Procedure

We start with the initial profile where there are 19 voters whose preference orderings are as follows:

5 voters: $d > b > c > a$
4 voters: $d > a > b > c$
4 voters: $b > c > a > d$
3 voters: $a > d > c > b$
3 voters: $a > d > b > c$

Here a is the Condorcet winner. Now suppose that, *ceteris paribus*, the four $d > a > b > c$ voters decide to abstain. As a result, in the reduced profile it would take for d only three preference switches to become a Condorcet winner (if three $b > c > a > d$ voters change their preference ordering to $b > c > d > a$) whereas each of the other candidates needs more than three preference switches in order to become a Condorcet winner. So according to Dodgson's procedure *d* becomes the winner in the reduced profile which is a preferable outcome for the four abstainers.

5.3.8 Coombs's and the Negative Plurality Elimination Rule (NPER) Procedures

Consider the initial profile with 19 voters whose preference orderings are as follows (Felsenthal, 2012, p. 78; Nurmi, 2012, pp. 266–267):

5 voters: $d > b > c > a$
4 voters: $d > a > b > c$
4 voters: $b > c > a > d$
3 voters: $a > d > c > b$
3 voters: $a > d > b > c$

In the initial profile *a* is the Condorcet winner and although Coombs's and the NPER procedures do not necessarily elect a Condorcet winner when one exists, *a* is nevertheless elected under Coombs's and the NPER procedures in this example. (As in the initial profile there is no candidate who is ranked first by the majority of voters, candidate *c*—who is ranked last by the plurality of voters—is eliminated from all ballots according to Coombs's and the NPER procedures, thereafter candidate *b* is eliminated from all ballots, and thus finally candidate *a*, the Condorcet winner, becomes the candidate listed first in the ballots of the majority of voters and is therefore declared the winner according to Coombs's and the NPER procedures).

Now suppose that, *ceteris paribus*, the four $d > a > b > c$ voters abstain. As in the reduced 15-voter profile there is no candidate who is ranked first by the majority of voters, candidate *a* (who is ranked last by the plurality of voters) is eliminated from all ballots according to Coombs's and the NPER procedures. Thereafter candidate *d* is ranked first by a majority of voters and hence is elected according to Coombs's procedure—which is a preferable outcome for the four $d > a > b > c$ abstainers to the election of *a* (the Condorcet winner in the original profile). After the elimination of *a* in the first counting round, candidates *c* and *b* are eliminated in the second and third counting rounds, respectively, according to the NPER procedure, so *d* becomes the winner according to this procedure too.

5.3.9 The Majority Judgment Procedure

The MJ procedure is vulnerable to No-Show paradox also under the restricted domain assumption. To see this consider the following example.[4]

[4]This example refutes the statement made by Felsenthal and Nurmi (2018) that the MJ procedure is invulnerable to the No-Show paradox under the restricted domain assumption.

Suppose there are seven voters, V_1–V_7 who rank originally two candidates, *x* and *y*, on a scale from A (lowest) to F (highest), as follows:

Alternative/voter	V_1	V_2	V_3	V_4	V_5	V_6	V_7	Median
x	A	A	C	F	B	F	C	C
y	B	B	B	E	D	D	A	B

Here *x* is the Condorcet winner and is elected according to the MJ procedure because its median rank is higher than *y*'s.

Now suppose that, *ceteris paribus*, voters V_1 and V_2, who prefer the election of *y*, decide to abstain. As a result one obtains the following distribution of grades among the two candidates:

Alternative/voter	V_3	V_4	V_5	V_6	V_7	Median
x	C	F	B	F	C	C
y	B	E	D	D	A	D

As a result of this abstention *y* wins—which would be a preferable outcome for the two abstainers than the original election of *x*.

5.4 Proofs Regarding the Impossibility of the No-Show Paradox Under Two Condorcet-Consistent and Three Non-Condorcet-Consistent Procedures When a Condorcet Winner Exists and Is Elected in the Initial Profile

5.4.1 Minimax Procedure

The Minimax procedure is one of two Condorcet-consistent procedures investigated which is invulnerable to the No-Show paradox when a Condorcet winner exists in the initial profile (the second procedure is Schwartz's as explained in 5.4.2 below). This can be seen from the following argument. Denote by $n(x, y)$ the number of voters strictly preferring *x* to *y* in profile P and denote by $n'(x, y)$ the number of voters strictly preferring *x* to *y* in profile P′. These profiles are now defined. Let *c* be the Condorcet winner in the original profile P, and let a group G consisting of *g* voters with identical preferences and strictly preferring another alternative *x* to *c* leave P, so that the remaining electorate constitutes profile P′. With *g* identically minded voters now abstaining, *x*'s support in all pairwise comparisons involving alternatives that G ranks lower than *x* (including, *inter alia*, *c*) diminishes by *g* votes, while all other *x*'s pairwise comparisons remain the same. Since G ranks x higher than c, for all those alternatives z that differ from c and x, if $n(c, z) - n'(c, z) = g$ then also $n(x, z) - n'(x, z) = g$, but not conversely, i.e., there is at least one alternative, w, such that $n(x, w) - n'(x, w) = g$, but $n(c, w) - n'(c, w) = 0$. It then follows that if min $n(c, z) >$ min $n(x, z)$, so must be min $n'(c, z) >$ min $n'(x, z)$. Thus the Minimax procedure cannot lead to a No-Show paradox when the initial profile contains a Condorcet winner.

5.4.2 Schwartz's Procedure

Let c be the Condorcet winner in the original profile P. By abstaining, *ceteris paribus*, and thereby creating profile P′, a group G of g like-minded voters can, at most, bring about a change in the choice outcome either (i) by replacing c with another Condorcet winner, say x, or (ii) by creating a multi-member choice set. In case (i) the outcome cannot be better for G since in P it prefers c to x. In case (ii) the Schwartz set consists of at least one candidate, say d, that is regarded worse than c by G, for otherwise d that is not in the Schwartz set in P would not be in the Schwartz set in P′. Hence, abstaining cannot bring about a better outcome for the abstainers in the restricted domain.

5.4.3 The Plurality with Runoff Procedure

Suppose x is the Condorcet and Plurality with Runoff winner in profile P and a group G consisting of g voters with the same preference ordering decides to abstain. Obviously, if x is their first-ranked alternative, they cannot benefit from abstaining. So assume that x is their second- or lower-ranked alternative. Now, to make a difference, G's abstaining has to change one of the runoff contestants, while the first-round support of the others remains as it was in the original profile. Suppose the runoff contestants in the original profile were x and w, while in the reduced profile they are x and z (x will have to be one of the runoff contestants in the reduced profile, since G did not rank it first. Hence x's plurality count remains the same as in the original profile). Can z now be preferred to x by G and at the same time defeat x in the second round of the reduced profile election? Now, if z was preferred to x by G in P, then it must be those voters not in G (i.e., voters in N-G) that turned the pairwise victory of z over x into a victory of x over z since x was the elected Condorcet winner in P. These voters are not abstaining in P′. Hence they still guarantee the victory of x over z in P′. So z cannot be the runoff winner in the reduced profile. Thus, the only way G can make a difference by abstaining is to bring about an outcome that is worse than x for G. Therefore, the Plurality with Runoff procedure is invulnerable to the No-Show paradox if the initial profile contains a Condorcet winner which at the same time is also the Plurality Runoff winner. (The possibility that the winner in P′ is found already on the first round cannot be a result of a successful abstaining of G since by abstaining G only affects the absolute and relative plurality count of alternatives that are lower than top in G's ranking. Suppose that G prefers v to the original Condorcet and Plurality with Runoff winner, x. Then by abstaining it can at most make x the first-round winner in P′ because either x's plurality count now exceeds 50% or make no difference at all).

5.4.4 The Alternative Vote Procedure

The Alternative Vote procedure is also invulnerable to the No-Show paradox when the initial profile contains a Condorcet winner which is the Alternative Vote winner as well. This is so for the following reasons.

Let x be the Alternative Vote and Condorcet winner in the initial profile P. Suppose a group G of g identically-minded voters who prefer some other candidate to x leaves P, *ceteris paribus*, and denote the remaining reduced profile by P′. Can the ensuing Alternative Vote winner in P′ be preferred to x by G? No, for the following reason. If x is elected only on the basis of the top-ranked alternatives in P (that would make x the Absolute Winner in P and in P′), then a removal of G maintains x's winning position *a fortiori* since x is not ranked first by G and the threshold of the required majority is smaller in P′ than in P. If, on the other hand, x is elected under the Alternative Vote procedure in P after removing (sequentially) some other alternative(s) which is (are) ranked first by a smaller number of voters than x, then the removal of G may decrease (due to the decreased required majority threshold) the number of candidates that must be removed in P′ before some candidate is ranked first by a majority of the voters. Can this candidate be different than x, say z, who is preferred by G over x? No. The fact that x, the Condorcet winner, is elected in P implies that a majority of voters in P preferred x to z. These voters do not belong to G and none of them is inclined to abstain in P′. All those individuals ranking x higher than z in P have the same preference in P′, while strictly fewer (namely g fewer) individuals rank z higher than x in P′ than in P. At the same time, the relative positions of all alternatives (including x and z) remain precisely the same in P and P′ among those voters who are not members of G. Since the voters preferring x to z continue to constitute the majority in P′ too, it is not possible that z be elected in P′.

5.4.5 Bucklin's Procedure

This voting procedure, too, is invulnerable to the No-Show paradox when the initial profile contains a Condorcet winner which is also the Bucklin winner. This is so for the following reasons.

Let x be the Bucklin and Condorcet winner in the initial profile P. Suppose a group G of g identically-minded voters who prefer some other candidate to x leaves P, *ceteris paribus*, and denote the remaining reduced profile by P′. Can the ensuing

Bucklin winner in P′ be preferred to *x* by G? No, for the following reason. If *x* is elected in P because it is ranked first by an absolute majority of the voters then this would make *x* also the Absolute Winner in P′. So a removal of G maintains *x*'s winning position *a fortiori* since *x* is not first ranked by G and the threshold of required majority is smaller in P′ than in P. If, on the other hand, *x* is elected under Bucklin in P after inclusion of lower than first ranked alternatives, then the removal of G may change the number of ranks that have to be taken into account in determining the Bucklin winner in P′. To wit, the majority threshold may be reached at an earlier stage by, say, *z*. Can *z* be preferable to *x* by G? No, it cannot since all alternatives ranked higher than *x* by G (including *z*) have equal or smaller first, second, etc., rank counts in P′ than in P. In other words, whatever advantage *z* gets in terms of shifting the number of ranks considered in order to find the winner in P′ is offset by the advantage accruing to *x* since the removal of G improves *x*'s relative standing *vis-à-vis z*.

5.5 Proofs Regarding the General Impossibility of the No-Show Paradox Under Four Non-Condorcet-Consistent Procedures

5.5.1 Plurality Voting

The Plurality Voting procedure is generally invulnerable to the No-Show paradox since the selected alternative, say *x*, which by definition is ranked first by the plurality of voters, can be changed to another winner, say *y*, only if some voters originally ranking *x* first, abstain. This is because the abstaining of any other voters only increases *x*'s plurality margin with respect to those candidates ranked first by the abstaining voters. Also those originally ranking *x* first cannot benefit from abstaining since thereby they decrease *x*'s plurality count, possibly even rendering *x* a non-winner. Thus, no voters can benefit from abstaining under the Plurality Voting procedure.

5.5.2 Approval Voting Procedure

The Approval Voting procedure is generally invulnerable to the No-Show paradox for the same reasons that the Plurality voting procedure is generally invulnerable to this paradox assuming that the improvement of a candidate's position does not change its approvability status, i.e., the candidates approved initially will remain approved after improvement and the same holds for disapproved candidates.

5.5.3 Borda's Procedure

The Borda procedure is not susceptible to the No-Show paradox because the winning alternative under Borda's procedure is the alternative whose sum in the pairwise comparison matrix is largest. Therefore any single voter who abstains decreases by 1 the entries in the pairwise comparison matrix that fit his/her preference ordering. Thus, for example, if there are four alternatives, *a*, *b*, *c*, *d*, and, *ceteris paribus*, a voter whose preference ordering is $a > b > c > d$ abstains, then the entries in the cells *(a, b), (a, c), (a, d), (b, c), (b, d)* and *(c,d)* in the pairwise comparison matrix decrease by 1 each—consequently the sum in row *a* (the most preferred alternative) is decreased by 3, the sum in row *b* (the second most preferred alternative) is decreased by 2, the sum in row *c* (the third most preferred alternative) is decreased by 1, and the sum of row *d* (the least preferred alternative) is not changed. So a voter whose preference ordering is $a > b > c > d$ is not only unable to benefit by abstaining, but may even obtain a worse outcome by doing so as there is an increasing probability that the less preferable an alternative is, the more likely it may end up as the selected alternative because the decrease in the sum of its row becomes increasingly smaller.

5.5.4 Range Voting Procedure

If *x* is the Range Voting winner in a profile, no voter ranking *x* first can improve the outcome by abstaining since by so doing s/he decreases the score of *x* thereby possibly making it a non-winner. The same applies to his/her second ranked candidate: by giving this candidate the second-largest number of points s/he might turn a non-winning candidate into a winner, and so on. So, whatever the distribution of points sums over candidates, the voter cannot benefit from abstaining when compared with voting according to his/her true preferences. Furthermore, if *x* is the Range Voting winner, no candidate *y* can become the winner in circumstances where a group of voters ranking *y* last joins the electorate. This is because *y* receives less value from the new entrants than any other alternative including *x*. Therefore, *y* cannot become the winner in the new profile. (This does not say that *x* remains the winner in the new profile, only that *y* isn't).

5.6 Concluding Remarks

Condorcet winners are usually considered to be relatively stable outcomes and hence the profile changes required to upset those outcomes are of considerable interest. Given Moulin's (1988) seminal result on the incompatibility, in general, of Condorcet-consistency and invulnerability to the No-Show paradox, we considered it worthwhile to examine whether this incompatibility is associated with only those profiles where a majority cycle—and hence a relatively unstable original setting—prevails. On the other hand, we were also interested in examining whether several ranked non-Condorcet-consistent procedures which are vulnerable, in general, to the No-Show paradox, would also exhibit this paradox when a Condorcet winner is present and elected in the initial profile. Our results reported above may seem a bit surprising: all Condorcet-consistent procedures examined except two (Minimax and Schwartz's procedure) are vulnerable to the No-Show paradox when a Condorcet winner exists in the initial profile, while all the non-Condorcet-consistent procedures examined except three (Coombs's, the NPER and the MJ procedures) are not vulnerable to this paradox in a restricted domain where a Condorcet winner exists and is elected in the initial profile. So it seems that under one type of voting procedures the existence (and election) of a Condorcet winner in the initial profile almost always guarantees a stable outcome, while most (seemingly more desirable) election procedures—which guarantee the election of a Condorcet winner if one exists in the initial profile—do not necessarily guarantee a stable outcome.

Exercises for Chapter 5

Problem 5.1
The strong No-Show paradox occurs in a profile where a group of identically-minded voters is not only better off abstaining, *ceteris paribus*, but gets its most preferred candidate elected, whereas by voting according to its preferences some less preferred candidate wins. Are there any instances of the strong No-Show paradox among the preceding examples?

Problem 5.2
A candidate, x, is said to be Pareto-dominated in a given a preference profile if *all* voters prefer some other candidate y to x. Show by way of an example that the Successive Elimination procedure may nevertheless lead to a Pareto dominated candidate being elected.

Problem 5.3
Given the example just constructed, would it be correct to state that (a) the elected candidate is necessarily a Condorcet loser and/or that (b) the elected candidate is never the first-ranked by any voter, and/or (c) every voter would have been better off had the voting agenda been different?

Problem 5.4
Show by way of an example that the Plurality with Runoff procedure is vulnerable to the No-Show paradox when there is no Condorcet winner in the original profile.

Problem 5.5
Show by way of an example that the Plurality with Runoff procedure is vulnerable to the No-Show paradox in the Condorcet domain when the Condorcet winner exists but is NOT elected in the initial profile.

Problem 5.6
How does the Alternative Vote procedure perform in the above two settings?

Answers to Exercises of Chapter 5

Problem 5.1
All profiles discussed in Sects. 5.3.1–5.3.8 exhibit the vulnerability of the respective procedures to the strong No-Show paradox.

Problem 5.2
Consider the following profile

1 voter: $a > b > d > c$
1 voter: $b > d > c > a$
1 voter: $d > c > a > b$

and the agenda of pairwise majority votes: (i) b versus d, (ii) the winner versus a, (iii) the winner versus c. Here c wins, but is Pareto-dominated by d.

Problem 5.3
(a) No, it is not; (b) yes, it is correct; (c) yes, it is correct.

Problem 5.4
Consider the following profile

6 voters: $a > b > c$
5 voters: $b > c > a$
4 voters: $c > a > b$

Here a wins. Now remove two $b > c > a$ voters, then c wins.

Problem 5.5
Consider the following profile

8 voters: $a > c > b$
5 voters: $b > c > a$
4 voters: $c > b > a$

Here c is the Condorcet winner, but b wins the runoff against a. Now, remove five $a > c > b$ voters and c becomes the winner.

Problem 5.6
In the same way as the Plurality with Runoff procedure, as is always the case in three-candidate contests.

References

Arrow, K. J. (1963). *Social choice and individual values* (2nd ed.). New York: Wiley.

Brandt, F., Geist, C., & Peters, D. (2017). Optimal bounds for the no show paradox via SAT solving. *Mathematical Social Sciences, 90,* 18–27.

Brandt, F., Hofbauer, J., & Strobel, M. (2018). Exploring the no-show paradox for Condorcet extensions using Ehrhart theory and computer simulations. In *Proceedings of the 7th International Workshop on Computational Social Choice (COMSOC)*, Troy, NY, USA, June 25–27, 2018.

Felsenthal, D. S. (2012). Review of paradoxes afflicting procedures for electing a single candidate. In D. S. Felsenthal & M. Machover (Eds.), *Electoral systems: Paradoxes, assumptions, and procedures* (pp. 19–91). Heidelberg: Springer.

Felsenthal, D. S., & Nurmi, H. (2017). *Monotonicity failures afflicting procedures for electing a single candidate*. Cham, Switzerland: Springer.

Felsenthal, D. S., & Nurmi, H. (2018). The no-show paradox under a restricted domain. *Homo Oeconomicus*. https://doi.org/10.1007/s41412-018-00079-w.

Fishburn, P. C., & Brams, S. J. (1983). Paradoxes of preferential voting. *Mathematics Magazine, 56,* 207–214.

Gehrlein, W. V. (1983). Condorcet's paradox. *Theory and Decision, 15,* 161–197.

Gehrlein, W. V., & Lepelley, D. (2011). *Voting paradoxes and group coherence: The Condorcet efficiency of voting rules*. Berlin: Springer.

Gehrlein, W. V., & Lepelley, D. (2017). *Elections, voting rules and paradoxical outcomes*. Berlin: Springer.

Moulin, H. (1988). Condorcet's principle implies the no show paradox. *Journal of Economic Theory, 45,* 53–64.

Nurmi, H. (2012). On the relevance of theoretical results to voting system choice. In D. S. Felsenthal & M. Machover (Eds.), *Electoral systems: Paradoxes, assumptions, and procedures* (pp. 255–274). Heidelberg: Springer.

Pérez, J. (1995). Incidence of no show paradoxes in Condorcet choice functions. *Investigaciones Economicas, XIX,* 139–154.

Pérez, J. (2001). The strong no show paradoxes are a common flaw in Condorcet voting correspondences. *Social Choice and Welfare, 18,* 601–616.

Chapter 6
Which of 20 Voting Procedures Satisfy or Violate the Subset Choice Condition (SCC) in a Restricted Domain?

Abstract The relative desirability of a voting procedure is assessed, *inter alia*, by verifying which axioms, or postulates, it satisfies or violates. One of these axioms is the *subset choice condition* (SCC). This axiom requires that if a candidate, x, is elected under a given voting procedure, f, in a profile consisting of n voters and k competing candidates (n, $k > 1$), then x ought to be elected by f also in such profiles over any proper subset of candidates that contain x and that preserve the pairwise preference relations of the original profile. Most known voting procedures violate, in general, this axiom. However, we were interested to find out which voting procedures satisfy or violate this axiom under a *restricted domain assumption* where a Condorcet winner exists and is elected in the initial profile by the investigated voting procedure. It turns out that, obviously, all conceivable Condorcet-consistent voting procedures satisfy SCC under this restricted domain assumption. However, most known non-Condorcet-consistent procedures continue to violate SCC even under the restricted domain assumption.

Keywords Elections · Subset choice condition · Condorcet winner · Condorcet-consistency · Domain restriction · Preference profile

6.1 Introduction

A common way to analyze voting procedures is to study the properties of the social choice correspondences that underlie those procedures (for surveys, see e.g., Felsenthal & Nurmi, 2018; Nurmi, 1987; Richelson, 1979; Riker, 1982; Straffin, 1980). The typical starting point of the analysis is the preference profile, i.e., a set consisting of complete and transitive preference relations of individual voters over a set of candidates (or other decision alternatives, as the case may be). Given a preference profile, each voting procedure singles out a set of candidates as the winners. Similarly, the social choice desiderata single out, for each profile, a set of subsets of candidates that are in accordance with each desideratum. For example, for any given profile, the well-known Condorcet-consistency desideratum singles out the set consisting of the Condorcet winner as the only allowable outcome. If the set of winners and the set of

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_6

allowable voting outcomes always coincide,—i.e., in any conceivable profile—the voting rule satisfies the desideratum. Otherwise it does not. In the former (latter, respectively) case, it is customary to say that the rule is compatible (incompatible) with the criterion. So, to prove that a voting rule does not satisfy a performance criterion (desideratum) it is sufficient to show that a preference profile can be constructed where the winners produced by the rule do not coincide with the set of candidates allowed by the criterion. To prove the contrary, viz., that the rule satisfies a criterion, one must provide a sustainable argument that in any conceivable profile the outcomes allowed by the criterion are always the same as those resulting from the application of the voting rule to the profile under examination.

The comparisons based on these types of criteria result in dichotomies: either a rule satisfies a criterion or it does not. Anyone who has worked on demonstrating the incompatibilities of criteria and voting rules knows that for some rule-criterion pairs the demonstration is quite easy, while for others it can be very difficult. This suggests that one should look at the circumstances which make incompatibilities—when they are theoretically possible—more likely. It is, after all, well-known that all preference profiles do not emerge in real world voting bodies. Hence, the unrestricted domain assumption seems occasionally irrelevant for comparing voting rules. Without taking a stand on the general relevance or irrelevance of the unrestricted domain assumption, we shall here focus on the significance of a specific, but in our view plausible, domain restriction, viz., that the profiles be such that *a Condorcet winning candidate exists and, moreover, is elected in the initial profile by the procedure under examination*. Profiles satisfying this restriction have thus resulted in an intuitively stable—and for many authors, plausible—outcome. We ask whether this outcome affects the possibility of avoiding an instability that results from reducing the candidate set, e.g., by short-listing of candidates or other devices. Hence in this chapter our focus is on *the subset choice condition* (SCC), which, in general, is violated by most voting procedures.

6.2 The Subset Choice Condition

The Subset Choice Condition (SCC) belongs to choice set variance criteria that are related to changes in social choice resulting from changes in preference profiles. Some voting outcomes are seemingly plausible or reasonable while others are more or less bizarre. Best-known of the choice set variance criteria are Monotonicity and its cognates. SCC relates to choices in a candidate set and its various subsets. Given a profile P of n voters over the set A of k candidates and a voting rule f, let x be the winner specified by f. Take now any subset A' of A that includes x. Then SCC amounts to the requirement that x be the winner in A' as well. In other words, SCC requires that whenever someone wins in a large set, s/he also wins in every such subset of the large set that includes him/her. SCC is known under other names as well. To wit, Chernoff (1954, p. 429) introduced it as *individual choice postulate* 4, Aizerman and Malishevski (1981, p. 1033) call it *heritage*, Aleskerov (1999, p. 27) the *heredity condition*, and Sen (1970, p. 17) calls it *property alpha*; (see also Plott, 1976, p. 550).

It turns out that a vast majority of voting procedures, especially those based on mere preference relations, fail on SCC. In his study Nurmi (1987, pp. 104–106) concluded that only the Approval Voting procedure satisfies SCC, provided that not only the preference relations but also the sets of approved alternatives remain the same in all sets of candidates considered. The set of procedures considered by Nurmi was, however, smaller than the one we focus on here (13 versus 20).

Our focus is, therefore, not on whether voting procedures are compatible with SCC in general, but whether they satisfy the SCC in the restricted domains characterized by (i) the existence of a Condorcet winner in the initial profile, and (ii) the coincidence of the choice set determined by the voting rule and the profile under study with the Condorcet winner. Of course those procedures that are compatible with SCC in general, are also compatible with SCC in the restricted domain.

Our results will be reported in the following sections starting with the Condorcet-consistent voting procedures, whereupon we focus on other ranked voting procedures and finally discuss some systems that require a richer voter input than just preference relations.

6.3 The Condorcet-Consistent Voting Procedures

No Condorcet-consistent procedure is vulnerable to the SCC paradox under the restricted domain where there is a Condorcet winner and it is elected in the initial profile. To wit, let P be the initial profile, A the set of candidates and x the Condorcet winner in it. Then all Condorcet-consistent procedures, by definition, elect x in P. Let now A' be a proper subset of A and P' the profile obtained from P by restricting it to A'. In other words, in P' the preference relations over all pairs of candidates that belong to A' are the same as in P. Therefore, since x defeats all the other candidates according to P in A, it beats all its contestants in A' as well (with the same margins). Hence, x remains the Condorcet winner in A'. As we are here dealing with just Condorcet-consistent procedures, all these must elect x in A' as well. Hence, no matter how A' is formed as a reduction of A, the Condorcet winner in A will be elected in A' as well, assuming that the initial Condorcet winner x is included in A'.

This means that the 10 well-known Condorcet-consistent procedures described in Chap. 2, i.e., Successive Elimination, Minimax, Copeland's, Dodgson's, Schwartz's, Nanson's, BER (Baldwin's), Black's, Young's and Kemeny's procedures are compatible with SCC under our restricted domain assumption.

6.4 Seven Non-Condorcet-Consistent Procedures Violating SCC Generally and Under the Restricted Domain Assumption[1]

6.4.1 Plurality Voting

The (simple) Plurality Voting procedure is incompatible with SCC not only in general but in the restricted domain as well. This is demonstrated by the following 13 voters whose preference orderings among 4 candidates, *a*–*d*, are as follows:

5 voters: $a > b > c > d$
3 voters: $b > a > c > d$
2 voters: $c > a > d > b$
2 voters: $d > b > a > c$
1 voter: $b > d > a > c$

Here *a* is the Plurality and Condorcet winner. Yet, in the candidate subset $\{a, b, c\}$, *b* is the Plurality winner.

6.4.2 Plurality with Runoff

The Plurality with Runoff procedure fails on the SCC criterion as well as shown by the following 31-voter, 4-candidate profile.

7 voters: $d > a > c > b$
6 voters: $c > a > d > b$
5 voters: $b > d > a > c$
3 voters: $a > b > c > d$
3 voters: $a > c > b > d$
2 voters: $a > d > b > c$
2 voters: $b > c > a > d$
2 voters: $d > c > a > b$
1 voter: $c > b > a > d$

In this profile *a* is the Condorcet winner and, after the runoff contest with *d*, also the Plurality with Runoff winner. With *b* absent from all ballots, however, the runoff contestants are *c* and *d*, whereupon the latter becomes the winner.

[1]The description of the seven voting procedures in this section appears in Chap. 2.

6.4.3 Alternative Vote

The following example shows that the Alternative Vote (AV) procedure is incompatible with SCC. To wit, suppose there are 31 voters whose preference ordering among 4 candidates, *a*–*d*, are as follows:

3 voters: $a > b > c > d$
3 voters: $a > c > b > d$
2 voters: $a > d > b > c$
3 voters: $b > c > a > d$
5 voters: $b > d > a > c$
6 voters: $c > a > d > b$
1 voter: $c > b > a > d$
2 voters: $d > c > a > b$
6 voters: $d > a > c > b$

Here *a* is the Condorcet and AV winner (*c* and *d* are eliminated in the first and second counting rounds, respectively, whereupon *a* defeats *b* with 22 votes to 9). With *b* absent from all ballots, however, *a* is eliminated in the first counting round, whereupon *c* defeats *d* with 16 votes to 15.

6.4.4 Coombs's Procedure

Coombs's procedure is similarly incompatible with SCC in the restricted domain. The following 42-voter, 4-candidate example shows this:

7 voters: $a > c > d > b$
6 voters: $a > d > b > c$
3 voters: $b > a > c > d$
7 voters: $b > c > a > d$
9 voters: $b > c > d > a$
4 voters: $c > a > d > b$
6 voters: $d > a > b > c$

Here *a* is both the Condorcet and Coombs winner. However, with *d* absent from all ballots, the Coombs winner is *b*, in violation of SCC.

6.4.5 The Borda Count and the Negative Plurality Elimination Rule (NPER)

The previous example can also be used to show incompatibility of the Borda Count and the NPER procedures with SCC since the Borda, NPER and Coombs's winners coincide both in the initial 4-candidate and in the reduced 3-candidate (with *d* removed from all ballots) profile.

6.4.6 Bucklin's Procedure

The last item in our set of seven procedures that are non-Condorcet-consistent is Bucklin's procedure. It is also incompatible with SCC as shown in the following example involving seven voters and four candidates (*a*–*d*).

2 voters: $a > d > b > c$
1 voter: $b > c > a > d$
2 voters: $c > a > d > b$
1 voter: $c > d > a > b$
1 voter: $d > c > a > b$

With five first and second ranks *c* is the Bucklin winner. It is also the Condorcet winner. However, with *a* out of the race, the Bucklin winner is *d*—in violation of SCC.

Thus all the above-mentioned seven procedures that are non-Condorcet-consistent are not only incompatible with SCC in general but also in the restricted domains where a Condorcet winner exists in the initial profile and is elected by the procedure under examination.

6.5 Three Non-Condorcet-Consistent Procedures Which Satisfy SCC

Some voting procedures require more information from the voters than their preference relations in order to determine winning candidates. Perhaps the best known of such procedures are Approval Voting, Range Voting (RV) and Majority Judgment (MJ).[2] Since the winners in all these procedures are determined by scores or grades that are not directly determined by the rankings with respect to other alternatives, it can be concluded that they are in general compatible with SCC. In the case of Approval Voting, the MJ and RV procedures, each candidate's number of approval votes, median and mean grades, respectively, remain the same in the subsets as in the initial set of candidates. Hence in these three procedures the winners remain the same in all subsets to which they belong. This obviously also holds for those restricted domains we are focusing upon.

[2]For description of these voting procedures see Chap. 2.

6.6 Concluding Remarks

We have seen that all Condorcet-consistent procedures are compatible with SCC in the restricted domain we have focused upon. In this sense, then, the domain restriction saves them from the incompatibility that holds in an unrestricted domain. For seven non-Condorcet-consistent procedures studied here, no similar rescue is in sight: all of them are incompatible with SCC not only in unrestricted, but also in our restricted domain. However, the three non-Condorcet-consistent procedures Approval Voting, RV and MJ, are always compatible with SCC.

Exercises for Chapter 6

Problem 6.1
Construct a profile showing that the Plurality Voting procedure does not satisfy SCC when there is no Condorcet winner.

Problem 6.2
Construct a profile with the following properties:

(i) there is a Condorcet winner, say z;
(ii) z is not elected under the Plurality with Runoff procedure;
(iii) z is elected by the Plurality with Runoff procedure in all proper subsets of candidates containing z.

Problem 6.3
Construct a similar example for the Alternative Vote procedure.

Problem 6.4
Consider the following procedure called Borda with (single) Runoff: given any profile over a set of candidates, the Borda score of each candidate is computed, whereupon those two candidates with the largest score are selected for runoff. In this binary comparison, the candidate ranked higher than the other by more voters is elected. Is this procedure Condorcet-consistent?

Problem 6.5
Does the preceding Borda with (single) Runoff procedure satisfy SCC?

Answers to Exercises for Chapter 6

Problem 6.1

3 voters: $x > y > z$
2 voters: $y > z > x$
2 voters: $z > x > y$

There is no Condorcet winner in this profile, x is the Plurality Voting winner, but z defeats x in the subset $\{x, z\}$.

Problem 6.2

4 voters: $x > z > y$
3 voters: $y > z > x$
2 voters: $z > y > x$

Here z is the Condorcet winner, but is not elected in the Plurality Runoff contest. Yet, z is elected in both proper subsets it belongs to.

Problem 6.3
Since the Alternative Vote and Plurality with Runoff procedures are equivalent in three-candidate contests, the preceding example applies here too.

Problem 6.4
No, it is not. See the following profile:

3 voters: $x > y > z > w$
2 voters: $x > z > y > w$
2 voters: $y > z > w > x$
2 voters: $z > y > w > x$

Here x is the (Absolute) Condorcet winner, but y and z are the runoff contestants (in which y wins). Thus x is not elected by the Borda with (single) Runoff procedure.

Problem 6.5
No, it does not. The preceding example demonstrates this. There y wins, but would not win in the $\{x, y\}$ subset.

References

Aizerman, M. A., & Malishevski, V. (1981). General theory of best variants choice: Some aspects. *IEEE Transactions on Automatic Control, AC26,* 1030–1040.
Aleskerov, F. T. (1999). *Arrovian aggregation models*. Dordrecht: Kluwer.
Chernoff, H. (1954). Rational selection of decision functions. *Econometrica, 22,* 422–443.
Felsenthal, D. S., & Nurmi, H. (2018). *Voting procedures for electing a single candidate: Proving their (in)vulnerability to various voting paradoxes*. Cham, Switzerland: Springer.
Nurmi, H. (1987). *Comparing voting systems*. Dordrecht: D. Reidel.
Plott, C. R. (1976). Axiomatic social choice theory: An overview and interpretation. *American Journal of Political Science, 20,* 511–596.
Richelson, J. T. (1979). Comparative analysis of social choice functions: I, II, III: A summary. *Behavioral Science, 24,* 355.
Riker, W. H. (1982). *Liberalism against populism*. San Francisco: W. H. Freeman.
Sen, A. K. (1970). *Collective choice and social welfare*. San Francisco: Holden-Day.
Straffin, P. D. (1980). *Topics in the theory of voting*. Boston: Birkhäuser.

Chapter 7
The (In)Vulnerability of 20 Voting Procedures to the Preference Inversion Paradox in a Restricted Domain

Abstract Responsiveness to electoral opinions is one of the hallmarks of democratic governance. We focus on a particularly strong type of unresponsiveness, viz., one where the complete inversion of all preferences in the electorate is accompanied with no change in the electoral outcome. It is known that the possibility of this extreme type of unresponsiveness, known as the Preference Inversion paradox or Reversal Bias, is associated with many voting rules. We set out to find out whether the paradox can be encountered when using various procedures under a restricted Condorcet domain, viz., one where a Condorcet winner exists and is elected by the procedure under study.

Keywords Responsiveness of voting rules · Preference inversion paradox · Restricted Condorcet domain · Reversal bias

7.1 Introduction

Government by the people presupposes a reasonable resemblance between the voter opinions and decisions made on behalf of the people and in its name. This resemblance takes on degrees and varies from issue to issue, but its existence and desirability, while sometimes doubted, is seldom downright denied. Yet, it is difficult to say how much discrepancy there exists in general between the popular opinions and the government's decisions. To the extent it exists, it can easily be attributed to the fact that we are dealing with two different actors: the electorate and the government. In this chapter we shall approach the discrepancy from a rather specific and technical perspective, viz., we look at the difference to be found between the popular opinions and the outcomes reached through the aggregation of those opinions. In other words, the question we are asking is: how responsive are various voting rules to changes in the opinions of the electorate?

Since the changes in popular opinion may take on many forms, we shall here focus on some special patterns that we deem of major importance. First, the change may be related to increasing the support of the winning candidate or decision alternative, whereupon we could ask how does this real or hypothetical increase in support

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_7

translate into the collective decisions. Is the increase in the support of the winner, *ceteris paribus*, always accompanied with outcomes that maintain the status of the winner? Or is it possible that some increase in support turns existing winners into non-winners? These questions have been dealt with in other chapters of this book, especially in Chaps. 3 and 5. Second, the changes in opinion may take the form of augmenting the electorate with new voters. We may then ask whether the addition of voters ranking the existing winner first, *ceteris paribus*, inevitably leads to outcomes that retain the winner or whether such an addition could undo the winner. Alternatively, we can ask whether addition of voters all ranking the same candidate, say x, last, *ceteris paribus*, could make x the winner in the augmented electorate. Both of these scenarios would exhibit a type of unresponsiveness called the *No-Show Paradox*. These were discussed in Chap. 5. Here we deal with a specific and *prima facie* dramatic form of unresponsiveness known as the *Preference Inversion Paradox*, also known as the *Reversal Bias* (Saari & Barney, 2003). The paradox occurs whenever there is profile, say R, over a fixed set of candidates and a voting procedure so that the procedure results in the same outcome when applied to R and its reversal *Rev(R)*. By reversal we mean that the ranking of every voter is inverted, e.g., a ranking where a is preferred to b and b is preferred to c is turned into a ranking where c is preferred to b and the latter to a.

The pioneering work in this field has been done by Saari (1999, 2001, 2018). Saari and Barney's (2003) article also proves several results on preference inversion. Notably, in three-candidate elections, the only positional procedure that is invulnerable to the Inversion paradox is the Borda count. The following profile demonstrating this by listing the preference orderings of 14 voters was devised by Saari and Barney (2003, p. 18).[1]

4 voters: $y > z > x$
3 voters: $z > y > x$
4 voters: $x > z > y$
3 voters: $x > y > z$

Let the weight given to the first, second and third ranked candidates be w_1, w_2 and w_3, where $w_1 = 1$, $w_3 = 0$ and w_2 takes on some value between these two (the end points included). Assuming that the collective ranking is determined by the sum of weights assigned by voters to the three candidates, any positional method is then uniquely characterized by the value of w_2. For example, $w_2 = 0$ characterizes the Plurality Voting procedure, $w_2 = ½$ the Borda Count and $w_2 = 1$ the Antiplurality procedure. In the above profile the sums of scores are then $7w_1$, $4w_1 + 6w_2$ and $3w_1 + 8w_2$, respectively for x, y and z.

The reversal profile is as follows.

4 voters: $x > z > y$
3 voters: $x > y > z$

[1]In all the following examples a notation such as "3 voters: $a > b > c$" means a group of 3 voters who prefer alternative a to b, b to c, and hence also a to c.

4 voters: $y > z > x$
3 voters: $z > y > x$

The sum scores are precisely the same as in the original profile for x, y and z. It is easy to see that the Plurality Voting procedure exhibits the Inversion paradox since the initial profile gives x the largest sum and so does the reversed profile. Similarly, the Antiplurality procedure assigns z the largest sum score both in the initial and in the reversed profile. The only way to avoid the preference Inversion paradox is to have the three sum scores equal to each other. In other words, $7w_1 = 4w_1 + 6w_2 = 3w_1 + 8w_2$ which amounts to $w_1 = 2w_2$. The latter weight assignment characterizes uniquely the Borda Count.

Saari and Barney (2003) provide an illuminating theoretical analysis of the vulnerability of voting procedures to various forms of reversal bias. Some forms focus on profiles where the procedure amounts to the identical rankings of the first r candidates in the initial and in the reversed profiles. For voting procedures electing a single winner the most interesting form is one where the winner under a procedure is the same in the profile and its reversal. Our focus is solely on this form. Furthermore, instead of discussing the general vulnerability of the procedures to the Preference Inversion paradox, we shall focus on a specific domain, viz., one where (i) there is a Condorcet winner and (ii) it is elected by the procedure under study in the initial profile. Obviously, for procedures that are Condorcet-consistent, i.e., where a Condorcet winner is elected when it exists in a profile, (i) and (ii) are reduced to (i).

Before analyzing the vulnerability of 20 voting procedures to the Preference Inversion paradox (or to the *top-winner reversal bias* in Saari and Barney's terminology), a couple of general remarks are in order. First, if a procedure is proven to be invulnerable to the Preference Inversion paradox in general, it is, *ipso facto*, also invulnerable to it in the restricted Condorcet domain that we are focusing upon. Second, if a voting procedure is found vulnerable to the Preference Inversion paradox in a specific setting involving a fixed number of voters and candidates, it is thereby vulnerable to the paradox. However, it is possible that a procedure is vulnerable to the Preference Inversion paradox in the sense just mentioned, but at the same time it may not be vulnerable to it in the restricted Condorcet domain. This special domain is the primary focus of this chapter.

In the remainder of this chapter the procedures analyzed are not described. For their description the reader is referred to Chap. 2. We shall first analyze the Condorcet-consistent procedures, i.e., those procedures that always end up with the Condorcet winner—and only it—being elected when a Condorcet winner exists in the profile under study. Since we, by assumption, study a subset of those domains where a Condorcet winner exists, our restriction stating that the Condorcet winner be elected is trivially satisfied. We then study procedures that do not necessarily end up with the Condorcet winner when one exists in the profile under investigation. However, our assumption is that the Condorcet winner is elected in the profile by the procedure under study. Lastly, some concluding remarks are presented.

7.2 Condorcet-Consistent Procedures

7.2.1 Minimax

Suppose there are nine voters whose preference ordering among four candidates, w, x, y, z, are as follows:

3 voters: $z > y > x > w$
1 voter: $x > z > y > w$
2 voters: $w > x > z > y$
3 voters: $w > y > x > z$

Here w is the (strong) Condorcet winner. Upon reversal of preferences w becomes the Condorcet loser but is nevertheless elected under the Minimax procedure. Thus the Minimax procedure is vulnerable to the Preference Inversion paradox.

7.2.2 Young's Procedure

Young's procedure is vulnerable to the Preference Inversion paradox. This can be shown with the same example used to show the vulnerability of the Minimax procedure to this paradox in Sect. 7.2.1.

7.2.3 Dodgson's Procedure

Dodgson's procedure is also vulnerable to the Preference Inversion paradox. This can be shown with the following example.

10 voters: $z > y > x > w$
8 voters: $w > x > z > y$
7 voters: $w > y > x > z$
4 voters: $y > x > z > w$

Since w is the (strong) Condorcet winner in the initial profile it will be elected in Dodgson's procedure. Upon reversal of the profile w is the only candidate who needs only that two voters (any two of the 15 voters ranking w last in the reversed profile) to move w from their bottom to their top preference (i.e., a total of six binary preference switches) in order for w to become the Condorcet winner. Other candidates need more binary preference switches in order to become the Condorcet winner. Thus, the result obtained by Saari and Barney (2003, p. 28) to the effect that Dodgson's method is vulnerable to the Preference Inversion paradox holds also in the restricted Condorcet domain.

7.2.4 Successive Elimination

Since in the restricted domain there is a Condorcet winner, say x, it will inevitably be elected in the Successive Elimination procedure. Upon inversion of the profile, however, x becomes the Condorcet loser. Hence x will be defeated in any pairwise comparison where it appears and, thus, cannot be elected. Therefore, we conclude that the Successive Elimination procedure is invulnerable to the inversion of preferences paradox under the restricted domain assumption. This refines Saari and Barney's (2003, p. 18) finding according to which agenda procedures are vulnerable to the Preference Inversion paradox: for the paradox to occur there cannot be a Condorcet winner in the initial profile.

7.2.5 Black's, Copeland's, Kemeny's, Nanson's, Baldwin's and Schwartz's Procedures

These six Condorcet-consistent procedures are invulnerable to the Preference Inversion paradox under the restricted domain assumption because they are invulnerable in general to this paradox (cf., Felsenthal & Nurmi, 2018, Ch. 6).

7.3 Ranked Non-Condorcet-Consistent Procedures

7.3.1 Plurality Voting

The Plurality Voting procedure is vulnerable to the Preference Inversion paradox. To see this consider the following example.

Suppose there are seven voters whose preference orderings among three candidates, x, y, z, are as follows:

2 voters: $x > z > y$
2 voters: $x > y > z$
3 voters: $z > y > x$

Here x is the (strong) Condorcet winner. It is obviously also the Plurality winner. Upon reversal of preferences x is again the Plurality winner. Thus, Plurality Voting is vulnerable to the Preference Inversion paradox, also in the restricted Condorcet domain. Thus, Saari and Barney's (2003, p. 18) result applies to the restricted domain as well.

7.3.2 Approval Voting

The Approval Voting procedure is vulnerable to the Preference Inversion paradox under the restricted domain assumption. To see this, consider the following example.

Suppose there are seven voters whose preference orderings among five candidates, v, w, x, y, z are as follows:

2 voters: $x > v > z > w > y$
2 voters: $x > w > y > v > z$
3 voters: $z > w > v > y > x$

Suppose further that the first four voters approve of their top two preferences and that the last three voters approve only of their top preference. Then x is both the (strong) Condorcet winner as well as the Approval Voting winner with four votes (candidates v, w, y, and z get two, two, zero and three votes, respectively).

Now, if *ceteris paribus*, the seven voters invert their preferences we obtain the following distribution of preferences:

2 voters: $y > w > z > v > x$
2 voters: $z > v > y > w > x$
3 voters: $x > y > v > w > z$

As the first four voters continue to approve their first two preferences while the last three voters continue to approve only their top preference one obtains that x remains the Approval Voting winner (with three votes), while v, w, y and z obtain only two approval votes each. We conclude, then, that the Approval Voting is vulnerable to the Preference Inversion paradox not only in unrestricted domains, as shown by Saari and Barney (2003, p. 19), but also in the restricted Condorcet domain.

7.3.3 Plurality with Runoff

This procedure is invulnerable to the Preference Inversion paradox under the restricted domain assumption. To wit, if x is the Condorcet and the Plurality with Runoff winner, it by definition defeats any candidate by a majority. Now, if the rankings are reversed, x becomes the Condorcet loser. Hence it cannot have the majority of first ranks in the reversed profile. If it makes it to the runoff anyway, it will be defeated by whichever candidate it is confronted with in the second round by virtue of being the Condorcet loser. Hence, it cannot win in the reversal profile. This finding can be compared with the result of Saari and Barney (2003, pp. 27–28) in unrestricted domains. They show that basically all positional runoff procedures are vulnerable to the Preference Inversion paradox. We have just seen that while this is in general the case, this does not hold for Plurality with Runoff in restricted Condorcet domains where the Condorcet winner is elected in the initial profile.

7.3.4 Alternative Vote

This procedure too is invulnerable to the Preference Inversion paradox under the restricted domain assumption for the same reason that the Plurality with Runoff procedure is invulnerable to this paradox. Stated in another way, the fact that a Condorcet winner translates into a Condorcet loser in the reversal profile excludes its election under Alternative Vote since this voting procedure never elects a Condorcet loser.

7.3.5 Coombs's and the Negative Plurality Elimination Rule (NPER) Procedures

Coombs's and the NPER procedures are invulnerable to the Preference Inversion paradox under the restricted Condorcet domain assumption. To wit, if candidate x wins under Coombs's or under the NPER procedures and is the Condorcet winner, then it becomes the Condorcet loser in the reversed profile. However, as Coombs's and the NPER procedures never elect a Condorcet loser they cannot elect x in the reversed profile, hence these procedures are invulnerable to the Preference Inversion paradox under the restricted Condorcet domain assumption.

7.3.6 The Borda Count

This procedure is invulnerable to the Preference Inversion paradox under the restricted domain assumption as it is always invulnerable to this paradox except for the case where all candidates' Borda scores are equal (Saari & Barney, 2003).

7.3.7 Range Voting

The Range Voting procedure is vulnerable to the Preference Inversion paradox also under the restricted Condorcet domain assumption. Here is an example demonstrating this.

Suppose there are nine voters, V_1–V_9, who assign the following cardinal grades on a scale from 0–13 to three candidates, x, y, z, as follows:

Candidate\voter	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	Sum
x	7	7	7	4	4	0	0	0	0	29
y	1	1	1	1	1	7	7	1	1	21
z	0	0	0	0	0	1	1	13	13	28

Here candidate *x* is the Condorcet winner and is elected under the Range Voting procedure. Now suppose that all voters invert their preference ordering, i.e., each voter now assigns the (previous) highest grade to the candidate to whom s/he previously assigned the lowest grade and vice versa. As a result we obtain the following distribution of grades:

Candidate\voter	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	Sum
x	0	0	0	0	0	7	7	13	13	40
y	1	1	1	1	1	0	0	1	1	7
z	7	7	7	4	4	1	1	0	0	31

Although now *z* is the Condorcet winner, *x* is nevertheless elected again because it has the highest sum score—thus demonstrating the vulnerability of the Range Voting procedure to the Preference Inversion paradox also under the restricted Condorcet domain assumption.

7.3.8 *Majority Judgment*

The Majority Judgment procedure is vulnerable to the Preference Inversion paradox also under the restricted Condorcet domain assumption. Here is an example demonstrating this.

Suppose there are 3 voters, V_1–V_3, who assign the following ordinal grades on a scale from A (lowest) to G (highest) to three candidates, *x*, *y*, and *z*:

Candidate\voter	V_1	V_2	V_3	Median grade
x	A	G	B	B
y	E	E	C	E
z	F	C	A	C

Here *y* is both the Condorcet winner (it is preferred by V_1 and V_3 over *x*, and by V_2 and V_3 over *z*) and the Majority Judgement winner.

Now suppose that, *ceteris paribus*, the three voters invert the ordinal grades they awarded to the three candidates. As a result we obtain the following distribution of grades:

Candidate\voter	V_1	V_2	V_3	Median grade
x	F	C	B	C
y	E	E	A	E
z	A	G	C	C

So we see that despite the inversion of grades y remains the Majority Judgment winner thus showing that the procedure is vulnerable to the Preference Inversion paradox even under the restricted Condorcet domain assumption.

7.3.9 Bucklin's Procedure

The Bucklin procedure is invulnerable to the Preference Inversion paradox under the restricted domain assumption if there is a unique Bucklin winner, i.e., no tie exists among any of the candidates. This is so for the following reasons.

Let the initial profile be R and its inversion be R'. Denote by *Nx(i)* the number of voters placing x in the i'th position in their rankings and let n be the number of voters and k the number of candidates. Let now I be the smallest value of i such that $\sum_{i=1}^{I} z(i)$ is larger than $n/2$ for some z. Candidate z is then the Bucklin winner if and only if $\sum_{i=1}^{I} Nz(i) > n/2$ and $\sum_{i=1}^{I} Nz(i) > \sum_{i=1}^{I} Ny(i)$ for all other candidates y. In words, the number of voters placing z in or above (to the left of) the I'th rank is larger than the majority and at the same time larger than the number of voters placing any other candidate in or above the rank I where I is the highest (the left-most) position where this holds for any candidate. In the restricted Condorcet domain we assume that there is a Condorcet winner in the profile and that it coincides with the *unique* Bucklin winner.

Take now any candidate, say w, who is not a Bucklin winner. The fact that z is the Bucklin winner in profile R means that strictly fewer voters place w than z in the rank I or higher in R. Consequently, strictly more voters assign w a higher rank than z in the reversal profile R'. This means that for positions $i = I + 1, \ldots, k$ (i.e. below I in R), $\sum_{I+1}^{k} Nz(i) < n/2$, for the Bucklin winner z and $\sum_{I+1}^{k} Nz(i) < \sum_{I+1}^{k} Ny(i)$ for all other candidates y.

Now, since we assume a restricted domain where z is not only the Bucklin but also the Condorcet winner in R, it implies that z is the Condorcet loser in R'. This means that a majority of voters in R' rank all other candidates except the Bucklin winner in R higher (to the left) in their ordering in R'. Hence, the Bucklin winner z in R cannot reach the majority threshold in R' at a higher level (more left) than the candidates that are not Bucklin winners in R. Therefore, Bucklin's procedure is invulnerable to the Preference Inversion paradox in cases where there is a unique Bucklin winner and this candidate is also the Condorcet winner.

The above conclusion has to be qualified in cases where there are several tied Bucklin winners z with identical $\sum_{i=1}^{I} Nz(i)$ values. Under a particular tie-breaking procedure, it may happen that the Preference Inversion paradox can occur under Bucklin's procedure also under the restricted Condorcet domain assumption. (For possible ways of resolving ties under Bucklin's procedure see Felsenthal and Nurmi (2018, fn 1, p. 19)). To see this, consider the following example:

Suppose there are nine voters whose preference ordering among nine candidates $a, b, c, d, e, f, g, h, x$ are as follows:

1 voter: $a > b > c > d > x > e > f > g > h$
1 voter: $h > a > b > c > x > d > e > f > g$
1 voter: $g > h > a > b > x > c > d > e > f$
1 voter: $f > g > h > a > x > b > c > d > e$
1 voter: $e > f > g > h > x > a > b > c > d$
1 voter: $d > e > f > g > x > h > a > b > c$
1 voter: $c > d > e > f > x > g > h > a > b$
1 voter: $b > c > d > e > x > f > g > h > a$
1 voter: $x > a > b > c > d > e > f > g > h$

Here x is the Condorcet winner (x beats each of the other candidates 5:4). Now, each of candidates a, b, c accumulates five votes—an absolute majority—in the first four counting rounds. So, if one attempts to break this tie by continuing to the 5th counting round—without deleting any other candidate—then x wins in the 5th counting round with an accumulated nine votes, more than any other candidate. x will also win under this tie-breaking assumption if all voters invert their preferences. In this eventuality each of candidates h, g, f accumulates five votes—an absolute majority—in the first four counting rounds. But if this tie is to be broken by continuing to a 5th counting round without deleting any other candidate, then x wins in the 5th counting round with 8 votes—more votes than any other candidate accumulates up to (and including) the 5th counting round.

7.4 Concluding Remarks

The Preference Inversion paradox differs from several other paradoxes in that it is difficult to see how a voter could benefit from its occurrence. In other words, it does not seem to have any implications regarding strategic behavior. Still, it is genuinely counter-intuitive: it is surprising that there are voting procedures that result in the same outcome in a given preference profile and in its reversal. One would intuitively expect that when every voter changes his/her mind as completely as possible, the same would also happen in the election results, i.e., they would also change as completely as possible or at the very least some significant change in the outcomes would be observed. The fact that the paradox can occur in some systems but not in others would seem to speak in favor of the latter. This holds, by the same token, when the starting profile is one where an intuitively stable winner, the Condorcet winner, exists and is elected by the procedure under consideration. In the preceding we have shown that some procedures that are vulnerable to the paradox in unrestricted domains are not vulnerable in the restricted Condorcet domains. Yet, there are several procedures that remain vulnerable also in the restricted Condorcet domain.

Exercises for Chapter 7

Problem 7.1
Assume that a *strong Condorcet winner* exists in the initial profile, i.e., an alternative which is ranked first by an absolute majority of the voters. Is the Plurality Voting procedure vulnerable to the Preference Inversion Paradox in such a domain? What about the Plurality with Runoff procedure?

Problem 7.2
Consider the following profile:

7 voters: $x > y > z$
4 voters: $y > x > z$
5 voters: $z > x > y$

Suppose that this electorate is augmented with a set of voters whose preferences "cancel out" in the sense that half of the augmented set has some given ranking over the three candidates and the other half has the reversal of this ranking. Can z be rendered the Plurality winner by adding such a set of voters? If it can, describe the profile. If it cannot, explain why this is impossible.

Problem 7.3
Consider the following profile:

5 voters: $x > y > z$
4 voters: $y > x > z$
3 voters: $z > x > y$

Add now the following group of four voters:

2 voters: $z > x > y$
2 voters: $y > x > z$

What happens to the Condorcet, Plurality Voting and Borda Count winners when the original profile is augmented with these four voters whose preferences cancel out?

Problem 7.4
Consider the profile devised by Saari and Barney presented in Sect. 7.1. Compute the positional sum scores and the ensuing collective ranking of x, y and z when $w_2 = 2/3$ both in the original and inverted profiles.

Problem 7.5
Considering again the same profile, compute the positional sum scores and the corresponding collective ranking for x, y and z when $w_2 = 1/10$ both in the original and the inverted profiles.

Answers to Exercises of Chapter 7

Problem 7.1
When there exists a strong Condorcet winner in the initial profile, it is not possible that this candidate would be positioned last in more than half of the electorate. Hence it cannot be the strong Condorcet winner in the reversal profile, but it can be the last ranked by more voters than any other candidate in the initial profile. So, the Plurality Voting procedure is vulnerable to the Preference Inversion paradox even in the domain considered here. In fact, an example of this is presented in 7.3.1.

In the Plurality with Runoff procedure the strong Condorcet winner is obviously elected in the initial profile, but it will be defeated by any other candidate in the runoff in the inverted profile by virtue of being the Condorcet loser there. Hence the Plurality with Runoff procedure is not vulnerable to the Preference Inversion paradox.

Problem 7.2
Yes, it can. Add the following profile:

3 voters: $z > x > y$
3 voters: $y > x > z$

By adding this subset of voters z becomes the Plurality winner with eight votes.

Problem 7.3
In the original profile x is the Plurality Voting, Borda Count and Condorcet winner. Adding the 4-voter profile results in y becoming the Plurality Voting winner, while x remains the Borda Count and Condorcet winner.

Problem 7.4
By construction, the scores—and therefore the collective ranking—is the same in the original and in the inverted profile. The collective ranking is $z > y > x$ with scores 7, 8 and 25/3, respectively for x, y and z.

Problem 7.5
Again, by construction, the scores—and therefore the collective ranking—is the same in both the original and in the inverted profile. The collective ranking is $x > y > z$ with scores 7, 23/5 and 19/5, respectively for x, y and z.

References

Felsenthal, D. S., & Nurmi, H. (2018). *Voting procedures for electing a single candidate. Proving their (in)vulnerability to various voting paradoxes*. Cham: Springer.

Saari, D. G. (1999). Explaining all three-alternative voting outcomes. *Journal of Economic Theory, 87,* 313–355.

Saari, D. G. (2001). *Chaotic elections! A mathematician looks at voting*. Providence, RI: American Mathematical Society.

Saari, D. G. (2018). *Mathematics motivated by the social and behavioral sciences*. Philadelphia: Society for Industrial and Applied Mathematics.

Saari, D. G., & Barney, S. (2003). Consequences of reversing preferences. *The Mathematical Intelligencer, 25*(4), 17–31.

Chapter 8
Summary

Abstract This chapter concludes the analysis of the 20 voting procedures in terms of 5 voting paradoxes in restricted domains characterized by the existence of a Condorcet winner which at the same time is elected by the procedure under investigation. The restricted domain provides a perspective to how much difference various profile types make in terms of the possibility of encountering a voting paradox. In this analysis we contrast the general (unrestricted) domain with one where the initial outcome is stable. We illustrate the problems involved in the choice of an appropriate procedure by discussing the recent proposal for electoral reform suggested by Maskin and Sen.

Keywords Restricted domain · Condorcet winner · Maskin-Sen proposal · Social preference ordering

In this booklet we surveyed the (in)vulnerability to 5 well-known voting paradoxes[1] of 20 voting procedures designed to elect a single candidate in a *restricted domain, i.e., given that a Condorcet winner exists and is elected by the analyzed procedure*. This analysis should be distinguished from the analysis we conducted in our 2018 booklet (see Felsenthal & Nurmi, 2018) where we surveyed the (in)vulnerability in an *unrestricted domain* of 18 voting procedures to the same 5 voting paradoxes analyzed in this booklet plus some additional ones (which we ignored in the present booklet because they are not applicable in our restricted domain setting).

So what is the practical, as well as the theoretical difference, between our analysis in Felsenthal and Nurmi (2018) and the analysis in the present booklet? The analysis conducted in Felsenthal and Nurmi (2018) is useful for any person who wishes to evaluate the susceptibility to many voting paradoxes of various voting procedures in an *unrestricted domain*, i.e., without knowing in advance whether the *social preference ordering* would contain or would not contain cycles. In contrast, the analysis conducted in this booklet assumes that the social preference ordering is *unlikely* to contain cycles under any of the analyzed procedures (which, it is argued,

[1]These 5 paradoxes are: various types of Monotonicity failure, the Inconsistency (or Reinforcement) paradox, the No-Show paradox, the violation of the Subset Choice Condition (SCC) and the Preference Inversion paradox—analyzed in this booklet in Chaps. 3, 4, 5, 6, and 7, respectively.

D. S. Felsenthal and H. Nurmi, *Voting Procedures Under a Restricted Domain*, SpringerBriefs in Economics, https://doi.org/10.1007/978-3-030-12627-8_8

is indeed a common phenomenon)[2] and, moreover, that a Condorcet winner exists and is elected by the analyzed procedure. This theoretical (and somewhat hypothetical) type of analysis is useful for any person who wishes to know whether, *ceteris paribus*, there are any measures that can be taken in order to prevent the occurrence of some of the five analyzed paradoxes.

The summary of results of our analysis in this booklet is presented in Table 8.1.

In order to be able to compare the results of Table 8.1 with some of the results in Felsenthal and Nurmi (2018), we wish to refer the reader, as a clarifying example, to an OP-ED article authored by Eric Maskin and Amartya Sen (two Nobel Laureates in Economic Sciences) published in the New York Times on 28 April 2016 and entitled “How Majority Rule Might Have Stopped Donald Trump”.[3] The authors of this article proposed, *inter alia*, that the Plurality Voting procedure, used in the US in various types of elections designed to elect a single candidate, should be replaced by a Condorcet-consistent and ranked electoral procedure so that: (i) if a Condorcet winner would be found to exist in the social preference ordering s/he would be elected, or else (ii) if a Condorcet winner does not exist then the winner would be determined (using the rankings of the voters) according to the Plurality with Runoff procedure.

Now, Table 8.1 shows to which of the 5 voting paradoxes analyzed in this booklet the 10 Condorcet-consistent procedures, as well as the Plurality Voting and the Plurality with Runoff procedures, are (in)vulnerable *given that a Condorcet winner exists and is therefore elected*. However, according to the Maskin-Sen proposal if a Condorcet winner does not exist the winner must be determined according to the Plurality with Runoff procedure. In order to see the (in)vulnerability of this procedure to our five analyzed paradoxes when a Condorcet winner does not exist, i.e., in an *unrestricted domain*, it is necessary to look up the results reported in Felsenthal and Nurmi (2018, p. 45, Table 4.1) where it transpires that this procedure is vulnerable to all five voting paradoxes analyzed in this booklet.[4]

It should also be noted from Table 8.1 that the Plurality Voting procedure (which, as stated by Maskin and Sen, is currently the common procedure used in the US for electing a single candidate and which they advocate should be replaced), performs quite well: it is vulnerable to only two of the five voting paradoxes (violation of SCC and the Preference Inversion paradox) when a Condorcet winner exists and is elected. Moreover, it does better (or no worse) than 9 of the 10 Condorcet-consistent

[2] There are various estimates as to the relative frequency of finding cyclical majorities in the social preference ordering as a function of the number of voters and competing alternatives. These estimates are based on various theoretical assumptions and computer simulations, as well as on some laboratory experiments and limited actual election results conducted under some voting procedures. These estimates, in general, seem to be quite low.

[3] This article can be viewed in the following website: https://www.nytimes.com/2016/05/01/opinion/sunday/how-majority-rule-might-have-stopped-donald-trump.html.

[4] In contrast, note that the Plurality with Runoff procedure is invulnerable to two of the five paradoxes analyzed in this booklet, i.e., to the No-Show and to the Preference Inversion paradoxes.

Table 8.1 Summary—(in)vulnerability of 20 voting procedures to five paradoxes in a restricted domain

Monotonicity Failure Paradoxes					Other Paradoxes				Total +
Procedures	Upward monotonicity fixed electorate	Upward monotonicity variable electorate	Downward monotonicity fixed electorate	Downward monotonicity variable electorate	Consistency	No-Show	SCC	Preference Inversion	
Plurality voting	–	–	–	–	–	–	+	+	2
Plurality with runoff	–	–	+	+	+	–	+	–	4
Approval voting	–	–	–	–	–	–	–	+	1
Successive elimination	–	–	–	+	–	+	–	–	2
Borda	–	–	–	–	–	–	+	–	1
Alternative vote	–	–	+	+	+	–	+	–	4
Coombs	+	+	+	–	+	+	+	–	6
NPER	+	+	+	–	+	+	+	–	6
Bucklin	–	+	–	+	+	–	+	–*	4
Range voting	–	–	–	–	–	–	–	+	1
Majority judgment	–	+	–	+	+	+	–	+	5
Minimax	–	–	–	–	–	–	–	+	1
Dodgson	–	–	–	+	–	+	–	+	3
Nanson	–	–	–	+	–	+	–	–	2
BER (Baldwin)	–	–	–	+	–	+	–	–	2
Copeland	–	–	–	+	–	+	–	–	2
Black	–	–	–	+	–	+	–	–	2
Kemeny	–	–	–	+	–	+	–	–	2
Schwartz	–	–	–	+	–	–	–	–	1
Young	–	–	–	–	–	+	–	+	2
Total +	2	4	4	12	6	11	7	7	53

Notes Columns 2–5 pertain to all possibilities of the monotonicity paradox while columns 6–9 pertain to the remaining four paradoxes

*The Bucklin procedure may be vulnerable to the Preference Inversion paradox if there is a particular type of tie. See Sect. 7.3.9

A + sign indicates that a procedure is vulnerable to the paradox

A – sign indicates that a procedure is invulnerable to the paradox procedures (most of which are vulnerable to one type of monotonicity failure).[5] In contrast, as can be seen from the results reported by Felsenthal and Nurmi (2018, p. 45, Table 1), when a Condorcet winner does not exist, or when it exists but is not elected,—i.e., in an unrestricted domain—the Plurality Voting procedure is vulnerable to the same two paradoxes indicated in Table 8.1 plus one additional and very serious paradox, viz., the possibility that a Condorcet loser[6] is elected. However, the type and number of (in)vulnerabilities to paradoxes in an unrestricted domain associated with the Plurality Voting and Condorcet-consistent procedures (cf., Felsenthal & Nurmi, 2018, Tables 4.1 and 6.1) reveals some of the typical tradeoffs one must ponder upon when replacing one voting procedure with another: the suggested procedure may rectify a major flaw existing in another procedure but is associated with other failures that do not afflict the existing procedure. Whether in such a situation the suggested procedure can be considered an improvement over the existing one depends ultimately on the weight one assigns to various criteria including vulnerabilities to voting paradoxes.

Reference

Felsenthal, D. S., & Nurmi, H. (2018). *Voting procedures for electing a single candidate: Proving their (in)vulnerability to various voting paradoxes*. Cham, Switzerland: Springer.

[5]The only Condorcet-consistent procedure in Table 8.1 which seems to have an advantage over the Plurality Voting procedure is Minimax, which is vulnerable to only the Preference Inversion paradox.

[6]The Condorcet loser is a candidate that would be defeated by all the others if pairwise majority comparisons were conducted and the voters voted according to their preferences.