Any F-Transform Is Defined by a Powerset Theory

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Abstract. Relationships between powerset theories and F-transforms are investigated. Both these methods represent strong tools in fuzzy sets theory and applications. Although both methods deal with similar objects, both these methods use different tools and, so far, the relationship between the two methods has not been investigated. The aim of this paper is to show that there is a strong relationship between the two methods. Namely, arbitrary lower or upper F-transform of lattice-valued fuzzy sets can be derived from a special powerset theory and, conversely, there exists a special class of powerset theories, such that maps defined by these powerset theories are lower or upper F-transforms. These results allow, among other things, to extend the range of methods and tools that are used in both theories.

1 Introduction

In fuzzy set theory there are two important methods which are frequently used both in theoretical research and applications. These methods are the *powerset theory and the F-transform*. Both these methods were, in full details and theoretical backgrounds, introduced relatively recently and, in the present, both methods represent very strong tools in the theory and applications.

The powerset structures are widely used in algebra, logic, topology and also in computer science. The standard example of a powerset structure $P(X) = \{A : A \subseteq X\}$ and the corresponding extension of a mapping $f : X \to Y$ to the map $f_P^{\to} : P(X) \to P(Y)$ is widely used in almost all branches of mathematics and their applications, including computer science. For illustrative examples of possible applications see, e.g., the introductory part of the paper of [24]. Because the classical set theory can be considered to be a special part of the fuzzy set theory, introduced by [26], it is natural that powerset objects associated with fuzzy sets were soon investigated as generalizations of classical powerset objects. The first approach was done again by Zadeh [26], who defined $[0,1]^X$ to be a new powerset object $Z(X)$ instead of $P(X)$ and introduced the new powerset operator $f_Z^{\to} : Z(X) \to Z(Y)$, such that for $s \in Z(X), y \in Y$,

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$$f_Z^{\rightarrow}(s)(y) = \bigvee_{x, f(x)=y} s(x).$$

A lot of papers were published about Zadeh's extension and its generalizations, see, e.g., [5,10,11,14,21–24]. Zadeh's extension was for the first time intensively studied by Rodabaugh in [21] for lattice-valued fuzzy sets. This paper was, in fact, the first real attempt to uniquely derive the powerset operator $f_Z^{\rightarrow}$ from $f_P^{\rightarrow}$ and not only explicitly stipulate them. The works of Rodabaugh gave very serious basis for further research of powerset objects and operators. That new approach to the powerset theory was based on application of the *theory of monads in clone form*, introduced by Manes [9]. A special example of monads in clone form was introduced by Rodabaugh [23] as a special structure describing powerset objects. In the papers [10] and [11] we presented some examples of powerset theories based on fuzzy sets which are generated by monads in clone form.

Another important method which was recently introduced in the fuzzy set theory is the F-transform. This theory was in lattice-valued form introduced by Perfilieva [19] and elaborated in many other papers (see, e.g., [16–18,20]). Analogically as the powerset operator $f_P^{\rightarrow} : P(X) \rightarrow P(Y)$, F-transform is a special transformation map $F : \mathcal{L}^X \rightarrow \mathcal{L}^Y$, that transforms $\mathcal{L}$-valued fuzzy sets defined in the set X to $\mathcal{L}$-valued fuzzy sets defined in another set Y.

Fuzzy transforms represent new methods that have been successfully used in signal and image processing [1,2,5], signal compressions [16], numerical solutions of ordinary and partial differential equations [7,25], data analysis [3,4,18] and many other applications.

Although both methods deal also with the same object, i.e. $\mathcal{L}$-valued fuzzy sets, in general, both these methods use different tools and, so far, the relationship between the two methods has not been investigated. The aim of this paper is to show that, in fact, there is a very strong relationship between the two methods. We show, that arbitrary F-transform of $\mathcal{L}$-valued fuzzy sets defined by a fuzzy partition can be derived from a special powerset theory and, conversely, there exists a special class of powerset theories, such that maps defined by these powerset theories are F-transforms. This result allows, among other things, to extend the range of methods and tools that are used in both theories.

2 Preliminaries

A principal structure used in the paper is a *complete residuated lattice* (see e.g. [9,15]), i.e. a structure $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0_L, 1_L)$ such that $(L, \wedge, \vee)$ is a complete lattice, $(L, \otimes, 1_L)$ is a commutative monoid with operation $\otimes$ isotone in both arguments and $\rightarrow$ is a binary operation which is residuated with respect to $\otimes$, i.e.

$$\alpha \otimes \beta \leq \gamma \quad \text{iff} \quad \alpha \leq \beta \rightarrow \gamma.$$

Recall that a negation of an element a in $\mathcal{L}$ is defined by $\neg a = a \rightarrow 0_L$.

A special example of a residuated lattice $\mathcal{L}$ is a MV-algebra, i.e., a structure $\mathcal{L} = (L, \oplus, \otimes, \neg, 0_L, 1_L)$ satisfying the following axioms:

(i) $(L, \otimes, 1_L)$ is a commutative monoid,
(ii) $(L, \oplus, 0_L)$ is a commutative monoid,
(iii) $\neg\neg x = x$, $\neg 0_L = 1_L$,
(iv) $x \oplus 1_L = 1_L$, $x \oplus 0_L = x$, $x \otimes 0_L = 0_L$,
(v) $x \oplus \neg x = 1_L, x \otimes \neg x = 0_L$,
(vi) $\neg(x \oplus y) = \neg x \otimes \neg y$, $\neg(x \otimes y) = \neg x \oplus \neg y$,
(vii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

for all $x, y \in X$.

If we put

$$x \vee y = (x \oplus \neg y) \otimes y, \quad x \wedge y = (x \otimes \neg y) \oplus y, \quad x \to y = \neg x \oplus y,$$

then, $(L, \wedge, \vee, 0_L, 1_L)$ is a distributive lattice and $(L, \wedge, \vee, \otimes, \to, 0_L, 1_L)$ is a residuated lattice. MV-algebra is called a complete algebra, if that lattice is a complete lattice.

MV-algebras have their origin in algebraic analysis of Łukasiewicz logic by Chang in [6] and represent a generalization of Boolean algebras. A standard example of a MV-algebra is the *Lukasiewicz algebra* $\mathcal{L}_{Ł} = ([0,1], \oplus, \otimes, \neg, 0, 1)$, where

$$x \otimes y = 0 \vee (x + y - 1), \quad \neg x = 1 - x, \quad x \oplus y = 1 \wedge (x + y).$$

If $\mathcal{L}$ is a complete residuated lattice, a $\mathcal{L}$-fuzzy set in a crisp set X is a map $f : X \to L$. f is a non-trivial $\mathcal{L}$-fuzzy set, if f is not identical to the zero function. The core of a $\mathcal{L}$-fuzzy set f in a set X is defined by $core(f) = \{x \in X : f(x) = 1_L\}$.

We recall some basic facts about F-transforms. An F-transform in a form introduced by Perfilieva [20] is based on the so called fuzzy partitions on the crisp set. Unless otherwise stated, by $\mathcal{L}$ we denote the complete residuated lattice $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0_L, 1_L)$.

Definition 1. *Let X be a set. A system $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ of normal $\mathcal{L}$-fuzzy sets in X is a fuzzy partition of X, if $\{core(A_\lambda) : \lambda \in \Lambda\}$ is a partition of X. A pair $(X, \mathcal{A})$ is called a space with a fuzzy partition. The index set of $\mathcal{A}$ will be denoted by $|\mathcal{A}|$.*

In [12,13] we introduced the category **SpaceFP** of spaces with fuzzy partitions. In the paper we consider the modified version of this category.

Definition 2. *The category* **SpaceFP** *is defined by*

1. *Fuzzy partitions $(X, \mathcal{A})$, as objects,*
2. *Morphisms $(g, \sigma) : (X, \{A_\lambda : \lambda \in \Lambda\}) \to (Y, \{B_\omega : \omega \in \Omega\})$, such that*
   (a) $g : X \twoheadrightarrow Y$ and is $\sigma : \Lambda \twoheadrightarrow \Omega$ are surjective mappings,
   (b) $\forall \lambda \in \Lambda$, $A_\lambda(x) = B_{\sigma(\lambda)}(g(x))$, for each $x \in X$.
3. *The composition of morphisms in* **SpaceFP** *is defined by $(h, \tau) \circ (g, \sigma) = (h \circ g, \tau \circ \sigma)$.*

Objects of the category **SpaceFP** represent ground structures for a fuzzy transform, firstly proposed by Perfilieva [19] and, in the case where it is applied to $\mathcal{L}$-fuzzy sets with $\mathcal{L}$-valued partitions, in [20].

Definition 3. *Let $(X, \mathcal{A})$ be a space with a fuzzy partition $\mathcal{A} = \{A_\lambda : \lambda \in |\mathcal{A}|\}$.*

1. *An upper F-transform with respect to the space $(X, \mathcal{A})$ is a function $F^{\uparrow}_{X,\mathcal{A}} : \mathcal{L}^X \to \mathcal{L}^{|\mathcal{A}|}$, defined by*

$$f \in \mathcal{L}^X, \lambda \in |\mathcal{A}|, \quad F^{\uparrow}_{X,\mathcal{A}}(f)(\lambda) = \bigvee_{x \in X} (f(x) \otimes A_\lambda(x)).$$

2. *A lower F-transform with respect to the space $(X, \mathcal{A})$ is a function $F^{\downarrow}_{X,\mathcal{A}} : \mathcal{L}^X \to \mathcal{L}^{|\mathcal{A}|}$, defined by*

$$f \in \mathcal{L}^X, \lambda \in |\mathcal{A}|, \quad F^{\downarrow}_{X,\mathcal{A}}(f)(\lambda) = \bigwedge_{x \in X} (A_\lambda(x) \to f(x)).$$

3 Powerset Theories in the Category SpaceFP

In what follows, by $CSLAT(\vee)$ or $CSLAT(\wedge)$ we denote the category of complete $\vee$- or $\wedge$-semilattices as objects, respectively, with $\vee$- or $\wedge$-preserving maps as morphisms. If there is no need to distinguish between $\vee$ and $\wedge$, we will only write $CSLAT$. The standard definition of powerset theories was presented by Rodabaugh [23].

Definition 4. *Let* **K** *be a ground category. Then* $\mathbf{T} = (T, \to, V, \eta)$ *is called* $CSLAT$**-powerset theory in K**, *if*

1. *$T : \mathbf{K} \to CSLAT$ is an object-mapping,*
2. *for each morphism $f : A \to B$ in* **K**, *there exists $f^{\to}_T : T(A) \to T(B)$ in $CSLAT$,*
3. *There exists a concrete functor $V : \mathbf{K} \to Set$, such that η determines in Set for each $A \in \mathbf{K}$ a mapping $\eta_A : V(A) \to T(A)$,*
4. *For each $f : A \to B$ in* **K**, *$f^{\to}_T \circ \eta_A = \eta_B \circ V(f)$.*

In the paper we deal with powerset theories in the category **SpaceFP** which satisfy additional properties, typical for fuzzy sets structures. Two types of these powerset theories are introduced in the following definitions.

Definition 5. *A structure $\mathbf{T} = (T, \to, V, \eta)$ is called a $\mathcal{L}^{\vee}$-powerset theory in the category* **SpaceFP**, *if*

1. **T** *is a $CSLAT(\vee)$-powerset theory in the category* **SpaceFP**,
2. *For each object $(X, \mathcal{A}) \in$* **SpaceFP**,
   (a) *there exists a $\bigvee$-preserving embedding $i_{(X,\mathcal{A})} : T(X, \mathcal{A}) \hookrightarrow \mathcal{L}^{|\mathcal{A}|}$,*
   (b) *for each $x \in V(X, \mathcal{A})$ there exists $\alpha \in |\mathcal{A}|$, such that $core(i_{(X,\mathcal{A})}(\eta_{(X,\mathcal{A})}(x))) = \{\alpha\}$,*

(c) there exists an external operation $\star : \mathcal{L} \times T(X, \mathcal{A}) \to T(X, \mathcal{A})$*, such that* $i_{(X,\mathcal{A})}(\alpha \star f) = \alpha \otimes i_{(X,\mathcal{A})}(f)$*, for each* $f \in T(X, \mathcal{A}), \alpha \in \mathcal{L}$*.*

If $\mathcal{L} = (L, \oplus, \otimes, \neg, 0_L, 1_L)$ is a complete MV-algebra, we can also define the $\mathcal{L}^{\wedge}$*-powerset theory in the category* **SpaceFP**.

Definition 6. *A structure* $\mathbf{S} = (S, \to, W, \mu)$ *is called a* $\mathcal{L}^{\wedge}$*-powerset theory in the category* **SpaceFP***, if*

1. $\mathbf{S}$ *is a* $CSLAT(\wedge)$*-powerset theory in the category* **SpaceFP***,*
2. *For each object* $(X, \mathcal{A}) \in$ **SpaceFP***,*
   (a) there exists a $\bigwedge$*-preserving embedding* $j_{(X,\mathcal{A})} : S(X, \mathcal{A}) \hookrightarrow \mathcal{L}^{|\mathcal{A}|}$*,*
   (b) for each $x \in W(X, \mathcal{A})$ *there exists* $\alpha \in |\mathcal{A}|$*, such that* $core(j_{(X,\mathcal{A})}(\mu_{(X,\mathcal{A})}(x))) = \{\alpha\}$*,*
   (c) there exists an external operation $+ : \mathcal{L} \times S(X, \mathcal{A}) \to S(X, \mathcal{A})$*, such that* $j_{(X,\mathcal{A})}(\alpha + f) = \alpha \oplus j_{(X,\mathcal{A})}(f)$*, for each* $f \in S(X, \mathcal{A}), \alpha \in \mathcal{L}$*.*

Let us consider the following examples of the $\mathcal{L}^{\vee}$-and $\mathcal{L}^{\wedge}$-powerset theory.

Example 1. Let $\mathcal{U} = \{\tau_{(X,\mathcal{A})} : (X, \mathcal{A}) \in \mathbf{SpaceFP}\}$ be a system of $\mathcal{L}$-valued similarity relations defined on sets $|\mathcal{A}|$, such that for arbitrary morphism $(f, \sigma) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ in the category **SpaceFP**, $\tau_{(X,\mathcal{A})}(\alpha, \beta) = \tau_{(Y,\mathcal{B})}(\sigma(\alpha), \sigma(\beta))$ holds for arbitrary $\alpha, \beta \in |\mathcal{A}|$. Moreover, let the following condition holds for arbitrary $(X, \mathcal{A})$:

$$\alpha, \beta \in \mathcal{L}, \quad \tau_{(X,\mathcal{A})}(\alpha, \beta) = 1_L \Leftrightarrow \alpha = \beta.$$

For arbitrary morphism $(f, \sigma) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ in **SpaceFP**, we set

$$V(X, \mathcal{A}) = |\mathcal{A}|, \quad V(f, \sigma) = \sigma,$$

$$T(X, \mathcal{A}) = \{g \in \mathcal{L}^{|\mathcal{A}|} : g \text{ is extensional with respect to } \tau_{(X,\mathcal{A})}\} \hookrightarrow \mathcal{L}^{|\mathcal{A}|},$$

$$(f, \sigma)^{\to}_T : T(X, \mathcal{A}) \to T(Y, \mathcal{B}), \quad (f, \sigma)^{\to}_T(g)(\beta) = \bigvee_{\alpha \in |\mathcal{A}|} g(\alpha) \otimes \tau_{(Y,\mathcal{B})}(\beta, \sigma(\alpha)),$$

$$\eta_{(X,\mathcal{A})} : V(X, \mathcal{A}) = |\mathcal{A}| \to T(X, \mathcal{A}), \quad \eta_{(X,\mathcal{A})}(\alpha)(\beta) = \tau_{(X,\mathcal{A})}(\alpha, \beta).$$

It is clear that $T(X, \mathcal{A})$ is a complete $\bigvee$-semilattice and $(f, \sigma)^{\to}_T(g)$ is also extensional with respect to $\tau_{(Y,\mathcal{B})}$. Then, $\mathbf{T} = (T, \to, V, \eta)$ is the $\mathcal{L}^{\vee}$-powerset theory called *powerset theory defined by* $\mathcal{U}$. In fact, we set

$$\alpha \in \mathcal{L}, g \in T(X, \mathcal{A}), \quad \alpha \star g = \alpha \otimes g.$$

It can be proven simply that $\alpha \otimes g$ are elements of $T(X, \mathcal{A})$ and the following diagram commutes,

$$\begin{array}{ccc} |\mathcal{A}| & \xrightarrow{\ \sigma\ } & |\mathcal{B}| \\ \downarrow{\scriptstyle \eta_{(X,\mathcal{A})}} & & \downarrow{\scriptstyle \eta_{(Y,\mathcal{B})}} \\ T(X, \mathcal{A}) & \xrightarrow{(f,\sigma)^{\to}_T} & T(Y, \mathcal{B}) \end{array}$$

Hence, $(T, \to, V, \eta)$ is a $\mathcal{L}^{\vee}$-powerset theory. □

Example 2. Let $\mathcal{L}$ be a complete MV-algebra. Let $\mathcal{U} = \{\tau_{(X,\mathcal{A})} : (X,\mathcal{A}) \in$ **SpaceFP**$\}$ be the same sets of similarity relation as in the Example 1.

For arbitrary morphism $(f,\sigma) : (X,\mathcal{A}) \to (Y,\mathcal{B})$ we set

$$S(X,\mathcal{A}) = T(X,\mathcal{A}),$$
$$(f,\sigma)_S^{\rightarrow} : S(X,\mathcal{A}) \to S(Y,\mathcal{B}), \quad (f,\sigma)_S^{\rightarrow}(g)(\beta) = \bigwedge_{\alpha\in|\mathcal{A}|} \neg\tau_{(Y,\mathcal{B})}(\sigma(\alpha),\beta) \oplus g(\alpha),$$
$$\alpha \in \mathcal{L}, g \in S(X,\mathcal{L}), \quad \alpha + g := \alpha \oplus g.$$

It can be proven that $(f,\sigma)_S^{\rightarrow}$ is defined correctly, i.e., $(f,\sigma)_S^{\rightarrow}(g) \in S(Y,\mathcal{B})$, for arbitrary $g \in S(X,\mathcal{A})$. In fact, for $\beta,\omega \in |\mathcal{B}|$, we have

$$\tau_{(Y,\mathcal{B})}(\sigma(\alpha),\beta) \geq \tau_{(Y,\mathcal{B})}(\beta,\omega) \otimes \tau_{(Y,\mathcal{B})}(\sigma(\alpha),\omega) \quad \Rightarrow$$
$$\tau_{(Y,\mathcal{B})}(\sigma(\alpha),\beta) \to g(\alpha) \leq \tau_{(Y,\mathcal{B})}(\beta,\omega) \otimes \tau_{(Y,\mathcal{B})}(\sigma(\alpha),\omega) \to g(\alpha) =$$
$$\tau_{(Y,\mathcal{B})}(\beta,\omega) \to (\tau_{(Y,\mathcal{B})}(\sigma(\alpha),\omega) \to g(\alpha)) \quad \Rightarrow$$
$$(\tau_{(Y,\mathcal{B})}(\sigma(\alpha),\beta) \to g(\alpha)) \otimes \tau_{(Y,\mathcal{B})}(\beta,\omega) \leq \tau_{(Y,\mathcal{B})}(\sigma(\alpha),\omega) \to g(\alpha) \quad \Rightarrow$$
$$(f,\sigma)_S^{\rightarrow}(g)(\beta) \otimes \tau_{(Y,\mathcal{B})}(\beta,\omega) \leq (f,\sigma)_S^{\rightarrow}(g)(\omega),$$

and $(f,\sigma)_S^{\rightarrow}(g)$ is extensional with respect to $\tau_{(Y,\mathcal{B})}$ and $\alpha \oplus g \in S(X,\mathcal{A})$. In fact, for arbitrary $\beta,\omega \in \mathcal{L}$, we have

$$(\neg\alpha \to g(\beta)) \otimes \neg\alpha \otimes \tau_{(X,\mathcal{A})}(\beta,\omega) \leq g(\beta) \otimes \tau_{(X,\mathcal{A})}(\beta,\omega) \leq g(\omega),$$

and it follows that

$$(\alpha \oplus g(\beta)) \otimes \tau_{(X,\mathcal{A})}(\beta,\omega) = (\neg\alpha \to g(\beta)) \otimes \tau_{(X,\mathcal{A})}(\beta,\omega) \leq$$
$$\neg\alpha \to (\omega) = \alpha \oplus g(\omega).$$

Therefore, $\mathbf{S} = (S, \to, V, \eta)$ is the $\mathcal{L}^{\wedge}$-powerset theory, where η is the same as in the previous Example. □

As we mentioned in the Introduction, our goal is to show that the classical F-transform $F_{X,\mathcal{A}} : \mathcal{L}^X \to \mathcal{L}^{|\mathcal{A}|}$ defined by the space with a fuzzy partition $(X,\mathcal{A})$ can be derived from a powerset theory and, vice versa, that each suitable powerset theory $\mathbf{T}$ in the category **SpaceFP**, defines for arbitrary $(X,\mathcal{A}) \in$ **SpaceFP** the map $T_{[X,\mathcal{A}]} : \mathcal{L}^{V(X,\mathcal{A})} \to T(X,\mathcal{A})$, which can be represented by the F-transform $F_{V(X,\mathcal{A}),\mathcal{B}}$ defined by (possible different) space with a fuzzy partition $(V(X,\mathcal{A}),\mathcal{B})$.

Let us introduce the definition of the map defined by a $\mathcal{L}^{\vee}$- or $\mathcal{L}^{\wedge}$-powerset theories.

Definition 7. *1. Let* $\mathbf{T} = (T, \to, V, \eta)$ *be a* $\mathcal{L}^{\vee}$*-powerset theory in the category* **SpaceFP**. *For* $(X,\mathcal{A}) \in$ **SpaceFP**, *the map defined by* $\mathbf{T}$ *is*

$$T^{[X,\mathcal{A}]} : \mathcal{L}^{V(X,\mathcal{A})} \to T(X,\mathcal{A}),$$
$$f \in \mathcal{L}^{V(X,\mathcal{A})}, \quad T^{[X,\mathcal{A}]}(f) := \bigvee_{x\in V(X,\mathcal{A})} \eta_{(X,\mathcal{A})}(x) \star f(x) \in T(X,\mathcal{A}).$$

2. Let $\mathcal{L}$ be a complete MV-algebra and let $\mathbf{S} = (S, \rightarrow, W, \mu)$ be a $\mathcal{L}^{\wedge}$-powerset theory in the category **SpaceFP**. *For $(X, \mathcal{A}) \in$* **SpaceFP**, *the map defined by* $\mathbf{S}$ *is*

$$S_{[X,\mathcal{A}]} : \mathcal{L}^{W(X,\mathcal{A})} \rightarrow S(X, \mathcal{A}),$$
$$f \in \mathcal{L}^{W(X,\mathcal{A})}, \quad S_{[X,\mathcal{A}]}(f) := \bigwedge_{x \in W(X,\mathcal{A})} \neg\eta_{(X,\mathcal{A})}(x) + f(x) \in T(X, \mathcal{A}).$$

In the next theorem we prove that lower and upper F-transforms are derived from powerset theories. We show that for arbitrary space with a fuzzy partition $(X, \mathcal{A})$, the upper F-transform $F^{\uparrow}_{X,\mathcal{A}} : \mathcal{L}^X \rightarrow \mathcal{L}^{|\mathcal{A}|}$ is identical to the map $T^{[X,\mathcal{A}]}$ defined by a $\mathcal{L}^{\vee}$-powerset theory $\mathbf{T}$. An analogical result we can obtain for lower F-transform $F^{\downarrow}_{X,\mathcal{A}}$, which is identical to the map $T_{[X,\mathcal{A}]}$.

Theorem 1. *There exists the powerset theory $\mathbf{T} = (T, \rightarrow, V, \eta)$ of the category* **SpaceFP**, *such that*

1. $\mathbf{T}$ *is $\mathcal{L}^{\vee}$-powerset theory.*
2. *If $\mathcal{L}$ is a complete MV-algebra, then $\mathbf{T}$ is also $\mathcal{L}^{\wedge}$-powerset theory,*
3. *For each $(X, \mathcal{A}) \in$* **SpaceFP**,

$$T^{[X,\mathcal{A}]} = F^{\uparrow}_{X,\mathcal{A}}, \quad T_{[X,\mathcal{A}]} = F^{\downarrow}_{X,\mathcal{A}}.$$

Proof. Let $(f, \sigma) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ be a morphism in the category **SpaceFP**.

(1) We define

$$T : \mathbf{SpaceFP} \rightarrow CSLAT(\vee), \quad V : \mathbf{SpaceFP} \rightarrow Set,$$
$$T(X, \mathcal{A}) = \mathcal{L}^{|\mathcal{A}|}, \quad V(X, \mathcal{A}) = X,$$
$$(f, \sigma)^{\rightarrow}_T = T(f, \sigma) : T(X, \mathcal{A}) \rightarrow T(Y, \mathcal{B}), \quad V(f, \sigma) = f,$$
$$g \in T(X, \mathcal{A}), \quad (f, \sigma)^{\rightarrow}_T(g) = \sigma^{\rightarrow}_Z(g) \in T(Y, \mathcal{B}),$$

where $\sigma^{\rightarrow}_Z$ is the Zadeh's extension of the map $\sigma : |\mathcal{A}| \rightarrow |\mathcal{B}|$ to the map $\mathcal{L}^{|\mathcal{A}|} \rightarrow \mathcal{L}^{|\mathcal{B}|}$. The ordering on the set $T(X, \mathcal{A})$ is point-wise and it is clear that $T(X, \mathcal{A})$ is a complete $\bigvee$-semilattice and $\sigma^{\rightarrow}$ is $\bigvee$-preserving map.

We define the map $\eta_{(X,\mathcal{A})} : X \rightarrow T(X, \mathcal{A})$ by

$$x \in X, \alpha \in |\mathcal{A}| \quad \eta_{(X,\mathcal{A})}(x)(\alpha) = A_{\alpha}(x),$$

where $\mathcal{A} = \{A_{\alpha} : \alpha \in |\mathcal{A}|\}$. We show that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \eta_{(X,\mathcal{A})}}\downarrow & & \downarrow{\scriptstyle \eta_{(Y,\mathcal{B})}} \\ T(X, \mathcal{A}) & \xrightarrow{(f,\sigma)^{\rightarrow}_T} & T(Y, \mathcal{B}). \end{array}$$

In fact, for $x \in X, \beta = \sigma(\alpha) \in |\mathcal{B}|$, we have

$$(f,\sigma)^{\rightarrow}_{T}(\eta_{(X,\mathcal{A})}(x))(\beta) = \sigma^{\rightarrow}_{Z}(\eta_{(X,\mathcal{A})}(x))(\beta) = \bigvee_{\alpha,\sigma(\alpha)=\beta} \eta_{(X,\mathcal{A})}(x)(\alpha) =$$

$$\bigvee_{\alpha,\sigma(\alpha)=\beta} A_\alpha(x) = \bigvee_{\alpha,\sigma(\alpha)=\beta} B_{\sigma(\alpha)}(f(x)) = B_\beta(f(x)) = \eta_{(Y,\mathcal{B})}(f(x))(\beta).$$

Hence, $\mathbf{T} = (T, \rightarrow, V, \eta)$ is the $CSLAT(\vee)$-powerset theory. To prove that $\mathbf{T}$ is the $\mathcal{L}^{\vee}$-powerset theory, we define the external operation $\star$ by

$$\alpha \in \mathcal{L}, \omega \in |\mathcal{A}|, g \in \mathcal{L}^{|\mathcal{A}|}, \quad (\alpha \star g)(\omega) := \alpha \otimes g(\omega).$$

Moreover, we have

$$core(\eta_{(X,\mathcal{A})}(x)) = \{\alpha \in |\mathcal{A}| : A_\alpha(x) = 1_L\} = \{u_{\mathcal{A}}(x)\},$$

where $u_{\mathcal{A}} : X \rightarrow |\mathcal{A}|$ is the map defined by $u_{\mathcal{A}}(x) = \alpha \Leftrightarrow x \in core(A_\alpha)$. Hence, the condition (b) is also satisfied. Finally, for the map $T^{[X,\mathcal{A}]}$ defined by $\mathbf{T}$, for arbitrary $h \in \mathcal{L}^X, \alpha \in |\mathcal{A}|$ we have

$$T^{[X,\mathcal{A}]}(h)(\alpha) = (\bigvee_{x\in X} \eta_{(X,\mathcal{A})}(x) \star h(x))(\alpha) = \bigvee_{x \in X} \eta_{(X,\mathcal{A})}(x)(\alpha) \otimes h(x) =$$

$$\bigvee_{x\in X} A_\alpha(x) \otimes h(x) = F^{\uparrow}_{X,\mathcal{A}}(h)(\alpha).$$

Hence, $T^{[X,\mathcal{A}]} = F^{\uparrow}_{X,\mathcal{A}}$.

(2) Let $\mathcal{L}$ be the complete MV-algebra. For arbitrary morphism $(f,\sigma) : (X,\mathcal{A}) \rightarrow (Y,\mathcal{B})$, the set $T(X,\mathcal{A}) = \mathcal{L}^{|\mathcal{A}|}$ is also complete $\bigwedge$-semilattice. Since any complete MV-algebra is completely distributive ([8]), the map $\sigma^{\rightarrow}_{Z}$ is $\bigwedge$-preserving map, as follows from

$$\sigma^{\rightarrow}_{Z}(\bigwedge_{j\in J} h_j)(\beta) = \bigvee_{\alpha,\sigma(\alpha)=\beta} (\bigwedge_{j\in J} h_j(\alpha)) = \bigwedge_{j\in J} (\bigvee_{\alpha_j,\sigma(\alpha_j)=\beta} h_j(\alpha_j)) = \bigwedge_{j\in J} \sigma^{\rightarrow}_{Z}(h_j)(\beta).$$

Hence, the object function T from the previous case is also the object function $T : \mathbf{SpaceFP} \rightarrow CSLAT(\wedge)$ and $\mathbf{T} = (T, \rightarrow, V, \eta)$ can be consider to be also the $CSLAT(\wedge)$-powerset theory in the category **SpaceFP**. To prove that $\mathbf{T}$ is also $\mathcal{L}^{\wedge}$-powerset theory, we need to change only the definition of the external operation $+$ as follows:

$$g \in T(X,\mathcal{A}), \alpha \in \mathcal{L}, \omega \in |\mathcal{A}|, \quad (\alpha + g)(\omega) := \alpha \oplus g(\omega).$$

Then, for the map $T_{[X,\mathcal{A}]}$ defined by $\mathbf{T}$, for arbitrary $h \in \mathcal{L}^X, \alpha \in |\mathcal{A}|$ we have

$$T_{[X,\mathcal{A}]}(h)(\alpha) = (\bigwedge_{x\in X} \neg\eta_{(X,\mathcal{A})}(x) + h(x))(\alpha) = \bigwedge_{x\in X} \neg\eta_{(X,\mathcal{A})}(x)(\alpha) \oplus h(x) =$$

$$\bigwedge_{x\in X} \neg A_\alpha(x) \oplus h(x) = \bigwedge_{x\in X} A_\alpha(x) \rightarrow h(x) = F^{\downarrow}_{X,\mathcal{A}}(h)(\alpha).$$

Hence, $T_{[X,\mathcal{A}]} = F^{\downarrow}_{X,\mathcal{A}}$. □

In the next theorem we deal with the converse problem: Is it true that for arbitrary $\mathcal{L}^{\vee}$-powerset theory of the category **SpaceFP**, the map $T^{[X,\mathcal{A}]}$ defines an F-transform? The answer is "yes" and it allows to derive new types of F-transform maps $F : \mathcal{L}^X \to \mathcal{L}^{|\mathcal{A}|}$, where the transformed map $F(g)$ could have some additional properties.

Theorem 2. *Let* $\mathbf{T} = (T, \to, V, \eta)$ *be an arbitrary* $\mathcal{L}^{\vee}$*-powerset theory in the category* **SpaceFP**. *Then for arbitrary space with a fuzzy partition* $(X, \mathcal{A}) \in$ **SpaceFP** *there exists another space with a fuzzy partition* $(V(X,\mathcal{A}), \mathcal{B}) \in$ **SpaceFP**, *such that the following diagram commutes*

$$\begin{array}{ccc} & \mathcal{L}^{V(X,\mathcal{A})} & \\ {\scriptstyle T^{[X,\mathcal{A}]}}\swarrow & & \searrow{\scriptstyle F^{\uparrow}_{V(X,\mathcal{A}),\mathcal{B}}} \\ T(X,\mathcal{A}) & \xhookrightarrow{\ i_{(X,\mathcal{A})}\ } & \mathcal{L}^{|\mathcal{A}|} \end{array}$$

Proof. For arbitrary $\alpha \in |\mathcal{A}|, x \in V(X,\mathcal{A})$, we set $B_\alpha(x) = i_{(X,\mathcal{A})}(\eta_{(X,\mathcal{A})}(x))(\alpha)$. Then $(V(X,\mathcal{A}), \mathcal{B})$ is a space with a fuzzy partition, where $\mathcal{B} = \{B_\alpha : \alpha \in |\mathcal{A}|\}$ is a fuzzy partition, as simply follows from the properties of η. Then, for $\alpha \in |\mathcal{A}|, g \in \mathcal{L}^{V(X,\mathcal{A})}$, we have

$$i_{(X,\mathcal{A})}.T^{[X,\mathcal{A}]}(g)(\alpha) = i_{(X,\mathcal{A})}\Big(\bigvee_{x \in V(X,\mathcal{A})} \eta_{(X,\mathcal{A})}(x) \star g(x)\Big)(\alpha) =$$

$$\bigvee_{x \in V(X,\mathcal{A})} i_{(X,\mathcal{A})}\eta_{(X,\mathcal{A})}(x)(\alpha) \otimes g(x) = \bigvee_{x \in V(X,\mathcal{A})} B_\alpha(x) \otimes g(x) = F^{\uparrow}_{V(X,\mathcal{A}),\mathcal{B}}(g)(\alpha).$$

□

An analogical result we obtain for lower F-transform. The proof is similar and will be omitted.

Theorem 3. *Let* $\mathcal{L}$ *be a complete MV-algebra and let* $\mathbf{S} = (S, \to, W, \mu)$ *be an arbitrary* $\mathcal{L}^{\wedge}$*-powerset theory in the category* **SpaceFP**. *Then for arbitrary space with a fuzzy partition* $(X, \mathcal{A}) \in$ **SpaceFP** *there exists another space with a fuzzy partition* $(W(X,\mathcal{A}), \mathcal{B}) \in$ **SpaceFP**, *such that the following diagram commutes*

$$\begin{array}{ccc} & \mathcal{L}^{W(X,\mathcal{A})} & \\ {\scriptstyle S_{[X,\mathcal{A}]}}\swarrow & & \searrow{\scriptstyle F^{\downarrow}_{W(X,\mathcal{A}),\mathcal{B}}} \\ S(X,\mathcal{A}) & \xhookrightarrow{\ i_{(X,\mathcal{A})}\ } & \mathcal{L}^{|\mathcal{A}|} \end{array}$$

To illustrate the meaning of the preceding theorems, we show upper and lower F-transforms generated by the Theorems 2 and 3 from the $\mathcal{L}^{\vee}$- and $\mathcal{L}^{\wedge}$-powerset theories from the Examples 1 and 2, respectively.

Recall that for an arbitrary set X and an $\mathcal{L}$-valued similarity relation δ on the set X, a function $g \in \mathcal{L}^X$ is called the *extensional core of a function* $f \in \mathcal{L}^X$ with respect to δ, if

1. $\forall x \in X, \quad g(x) \le f(x)$,
2. g is extensional with respect to δ,
3. if $h \in \mathcal{L}^X$ is extensional with respect to δ, $h \le f$, then $g \ge h$.

Example 3. Let $\mathbf{T} = (T, \to, V, \eta)$ be the $\mathcal{L}^{\vee}$-powerset theory in the category **SpaceFP** from the Example 1. Then, according to the proof of the Theorem 2, for arbitrary $(X, \mathcal{A}) \in$ **SpaceFP**, the set $\mathcal{B} = \{\tau_{(X,\mathcal{A}}(\alpha, -) : \alpha \in |\mathcal{A}|\}$ is a fuzzy partition on $|\mathcal{A}|$, such that

$$T^{[X,\mathcal{A}]} = F^{\uparrow}_{|\mathcal{A}|,\mathcal{B}} : \mathcal{L}^{|\mathcal{A}|} \to \mathcal{L}^{|\mathcal{A}|},$$
$$g \in \mathcal{L}^{|\mathcal{A}|}, \quad F^{\uparrow}_{|\mathcal{A}|,\mathcal{B}}(g)(\omega) = \bigvee_{\alpha \in |\mathcal{A}|} g(\alpha) \otimes \tau_{(X,\mathcal{A})}(\alpha, \omega) = \widehat{g}(\omega).$$

It is clear that $\widehat{g}$ is the extensional hull of g with respect to $\tau_{(X,\mathcal{A})}$. Therefore, in that case, the upper F-transform $F^{\uparrow}_{|\mathcal{A}|,\mathcal{B}}$ represents the extensional hull transformation. □

Example 4. Let $\mathcal{L}$ be a complete MV-algebra and let $\mathbf{S} = (S, \to, V, \eta)$ be the $\mathcal{L}^{\wedge}$-powerset theory in the category **SpaceFP** from the Example 2. Then, according to the proof of the Theorem 3, for arbitrary $(X, \mathcal{A}) \in$ **SpaceFP**, the set $\mathcal{B} = \{\tau_{(X,\mathcal{A}}(\alpha, -) : \alpha \in |\mathcal{A}|\}$ is a fuzzy partition on the set $|\mathcal{A}|$, such that

$$T_{[X,\mathcal{A}]} = F^{\downarrow}_{|\mathcal{A}|,\mathcal{B}} : \mathcal{L}^{|\mathcal{A}|} \to \mathcal{L}^{|\mathcal{A}|},$$
$$g \in \mathcal{L}^{|\mathcal{A}|}, \quad F^{\downarrow}_{|\mathcal{A}|,\mathcal{B}}(g)(\beta) = \bigwedge_{\alpha \in |\mathcal{A}|} \tau_{(X,\mathcal{A})}(\alpha, \beta) \to g(\alpha) = \underline{g}(\beta).$$

It can be proven that $\underline{g}$ is the extensional core of g with respect to $\tau_{(X,\mathcal{A})}$. In fact, analogously as in the Example 2, we can prove that $\underline{g}$ is extensional with respect to $\tau_{(X,\mathcal{A})}$, $\underline{g} \le g$ and $\underline{g}$ is the largest extensional map with these properties. Therefore, in that case, the lower F-transform $F^{\downarrow}_{|\mathcal{A}|,\mathcal{B}}$ represents the extensional core transformation. □

4 Conclusions

F-transforms of lattice-valued fuzzy sets and powerset theories in fuzzy structures are frequently used tools in the fuzzy set theory and applications. Although these theories seem to be independent from the point of view of methods used, there exist deep relationships between these theories. We proved that arbitrary F-transform of $\mathcal{L}$-valued fuzzy sets defined by a fuzzy partition can be derived from a special powerset theory defined on the set of all $\mathcal{L}$-valued fuzzy sets and, conversely, there exists a special class of powerset theories, such that maps defined by these powerset theories are F-transforms. Using these relations, we can define new types of F-transforms and we can use, for example, new methods in the F-transform theory, including the theory of monads in special categories, which are typical tools in the powerset theories.

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