Fractal Dimension Analysis of Urban Morphology Based on Spatial Correlation Functions

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Abstract A number of mathematical models in urban studies are associated with spatial correlation functions. However, the theory and method of spatial correlation modeling have not been developed for urban morphology. Based on power-law urban density models, a density–density correlation function can be constructed for urban modeling. Using scaling analysis and spectral analysis, we can derive a set of fractal parameter equations, which can be employed to explore urban form and growth. The main results and findings are as follows: first, if urban density follows power law, the spatial correlation function and its energy spectral density will follow scaling law; second, the reasonable numerical ranges of fractal parameters can be derived by the ideas from multifractals; third, the spatial correlation modeling can be generalized to spatial autocorrelation and gravity models. As an example, the analytical process is applied to the city of Beijing in China to show how to use this method. A conclusion can be drawn that the scaling analysis, spectral analysis, and spatial correlation analysis can be integrated into a new framework to form the 3S analysis for urban morphology.

Keywords Fractals · Multifractals · Urban form · Scaling analysis · Spectral analysis · Spatial correlation analysis

1 Introduction

If a system has typical scales which can be represented by some characteristic length, it can be described with conventional mathematical methods based on calculus, linear algebra, or probability theory and statistics. However, urban growth and form represent a type of scaling phenomena, which bears no characteristic scale and cannot be effectively described by the traditional measures and mathematical theories. Fractal

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geometry provides a powerful tool for scale-free analysis, resulting in a number of interesting and revealing studies on urban systems (e.g., Ariza-Villaverde et al. 2013; Arlinghaus 1985; Arlinghaus and Arlinghaus 1989; Batty 1995; Batty and Longley 1994; Benguigui and Daoud 1991; De Keersmaecker et al. 2003; Frankhauser 1998, 2014; Thomas et al. 2010; Murcio et al. 2015). The fractal concept is on the basis of scaling symmetry, and symmetry suggests some invariance in transformation process. This suggests that self-similarity is equivalent to invariance under contraction or dilation (Mandelbrot 1989). In geometry, fractal property can be abstracted as power-law relations between linear sizes and the corresponding measurements. A city can be empirically treated as a random fractal system that possesses self-similar or self-affine structure (e.g., Batty 2005, 2008; Benguigui et al. 2000; Feng and Chen 2010; Frankhauser 1994; Thomas et al. 2007; Thomas and Frankhauser et al. 2008; White and Engelen 1993, 1994). In urban morphology, the spatial relation between the radius (a scale) from a city center and the corresponding urban density (a measurement) may follow the inverse power law (Batty and Longley 1994; Batty and Xie 1999; Frankhauser 1994; Makse et al. 1995). In many cases, a power law implies fractal or scaling. The typical inverse power function is Smeed's model on traffic network (Smeed 1963), which can be employed to describe urban density distribution and estimate the fractal dimension of urban form (Batty and Longley 1994).

However, the key process of fractal modeling in urban studies is not yet clear and remains to be further explored. Due to scale-free property, fractal phenomena cannot be described with traditional measures such as length, area, and density. It should be characterized by fractal dimension and the related scaling exponents. Fractal dimension can be defined from two angles of view. One is the logarithmic relation between the linear sizes of fractal copies and the corresponding entropy functions, and the other is the power-law relation between the linear sizes and the corresponding correlation functions. The inverse power law of urban density proved to be a special spatial correlation function (Takayasu 1990). Urban morphology and systems of cities have been modeled by using the ideas from spatial correlation (Chen and Jiang 2010; Makse et al. 1995, 1998). Spatial correlation is one of the most important ways of modeling both urban form and urban systems, and the correlation equations can be solved by scaling transform. In particular, a general spatial correlation function of cities can be constructed on the basis of urban density function (Chen 2011; Chen and Jiang 2010). If the correlation function is converted into energy spectrum by means of Fourier transform, the urban density will become spectral density (Chen 2008, 2013). Thus, spatial correlation analysis can be turned into spectral analysis and vice versa, and the two analytical processes are associated with scaling analysis (Chen 2009). A problem is how to integrate the scaling analysis, spectral analysis, and spatial correlation analysis into a logic framework to make a new process of spatial analysis.

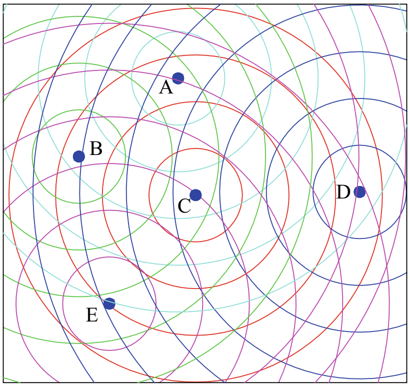
This work is devoted to developing new methodological framework for urban analysis and revealing the theoretical relations between different fractal parameters of urban form. In previous studies, several problems have been solved (Chen 2008, 2010, 2013; Chen and Jiang 2010). First, a new analytical process is preliminarily developed for fractal cities. Second, a number of fractal parameter relations are theoretically derived. Third, the proper ranges of fractal parameters of urban morphology are partially revealed. This is an integrated study. The main new points are as below: First, the previous framework of models is improved according to new thinking. Second, the spatial correlation modeling is generalized to spatial interaction and spatial autocorrelation analysis. Third, a new concise 3S-based case study on urban form is presented to illustrate how to use the 3S analytical process. Fourth, based on the empirical analysis, the scaling break phenomenon of wave spectrums is revealed and associated with self-affine urban growth. The rest parts are organized as follows. In Sect. 2, scaling analysis, spectral analysis, and spatial correlation analysis will be combined into a new approach, which is based on the inverse power law on urban density. In Sect. 3, the national capital of China, Beijing, is taken as example to make an empirical analysis. In Sect. 4, the related questions are discussed, and the theoretical models are further developed. Finally, the discussion is concluded by summarizing the main points of this studies.

2 Theoretical Models

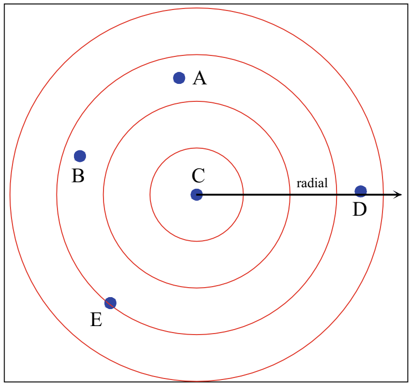
2.1 Basic Postulates

Urban morphology should be researched through proper concept of geographical space. If we examine a city by means of fractal methods based on a digital map or a remotely sensed image, the dimension of the embedding space of the city fractal is regarded as $d = 2$. In fact, a fractal city is always defined in a 2-dimension geographical space (Batty and Longley 1994; Frankhauser 1994). There are three common approaches to estimating the fractal dimension values of urban patterns, that is, box-counting methods (Benguigui et al. 2000; Chen 2012; Feng and Chen 2010; Shen 2002), area–radius scaling (Batty and Longley 1994; Chen 2010; Frankhauser 1998; White and Engelen 1993), and area–perimeter scaling (Batty and Longley 1994; Wang et al. 2005). Each method has its strong and weak points. If we want to research the spatial patterns of urban land use, the box-counting method is the best way of evaluating fractal dimension. However, if we want to explore the dynamic process of urban growth, the area–radius scaling is the best approach to estimating fractal dimension because this procedure is more consistent with the relation between urban core and periphery than other methods. In theory, the area–radius scaling is equivalent to and can be replaced by the density–radius scaling. The density–radius modeling is a good approach to research spatial correlation processes and patterns of city development. If we model 2-dimension spatial correlation through 1-dimension space, the density–radius relation is effective for scaling analysis (Fig. 1).

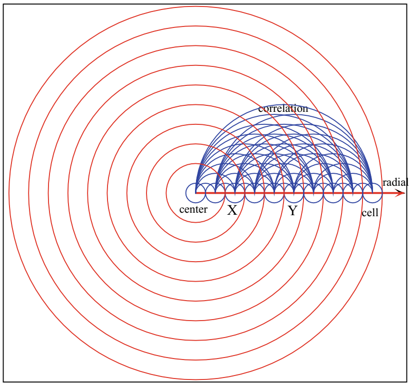
To explore the density–radius scaling relation of urban growth, the concepts of urban form and urban density should be clarified before the theoretical models are presented. Based on a 2-dimensional space, *urban form* can be defined as the spatial pattern of elements, which compose the city in terms of its networks, buildings, and



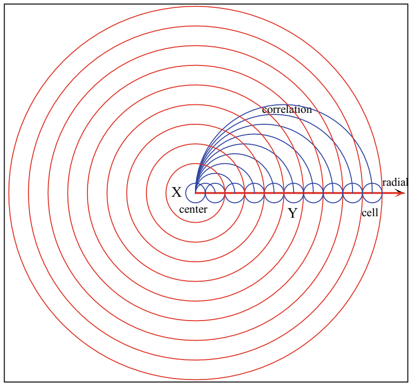
a. 2-dimension correlation



b. 1-dimension correlation



c. Point-point correlation



d. One-point correlation

Fig. 1 The sketch maps of three types of spatial correlation processes (by Chen 2013). *Note* As schematic diagrams, only five urban elements are taken into account. The concentric circles are employed to compute average density of urban land use or traffic networks. The space between two immediate large circles forms a ring, the small circles falling between two circles represent cells along a radial line, and the arcs indicate spatial correlation between two cells. Subgraphs: **a** For the 2-dimension spatial correlation, we need five sets of concentric circles to determine spatial scaling; **b** For the 1-dimension spatial correlation, we need one set of concentric circles to compute average density; **c** The point–point correlation suggests density–density correlation; **d** The one-point correlation suggests central correlation spaces (Batty and Longley 1994). Thus, *urban density* refers to the number of inhabitants, buildings, roads, streets, and so on, in given urbanized area. The urban density has different connotations and denotations for different spatial measurements. If we examine urban population distribution, the urban density implies urban population density; if we investigate the patterns of urban land uses, the urban density implies urban land use density; if we study the spatial structure of transport network, the urban density suggests the urban road density. Suppose that the fractal dimension of a city's spatial form will be evaluated by the remote sensing data for the purpose of exploring urban growth. The number of cells, $N(r)$, within the radius, r, from the city center can be used to measure the urbanized area. A cell is defined as the smallest image-forming unit or a pixel in a digital map. If the relation between measurement $N(r)$ and linear size r follow a power law such as $N(r) = N_0 r^D$, where D and N_0 are two parameters, then the urban morphology can be treated as a random fractal pattern, and the scaling exponent D is just the fractal dimension, which is termed *radial dimension* (Frankhauser 1998; Frankhauser and Sadler 1991). As an alternative way, the density–radius scaling can be employed to evaluate fractal dimension. In other words, we can examine the scaling relation between the radius r from a city center and the corresponding urban density $\rho(r)$. If the relation follow an inverse power law, and if the value of the scaling exponent falls between 0 and 1, then the urban growth can be regarded as a self-organized process. Self-organization of urban evolution results in a self-similar pattern with a fractional dimension value ranging from 0 to 2 (Chen 2010). The fractal morphology of cities can be described with a set of spatial correlation functions, which are equivalent to the corresponding entropy functions.

2.2 Spatial Correlation Functions

Scientific theoretical construction is often based on simple and clear prototype. First of all, for simplicity and without loss of generality, let us consider a monocentric city. For a fractal city with a growth core as the circle center, the area of concentric circles of radius r is $A(r) = \pi r^2$, where π denotes the circular constant. The marginal urban density $\rho(r)$ at distance r from the urban center can be expressed as (Batty and Longley 1994; Makse et al. 1995)

$$\rho(r) = \frac{\mathrm{d}N(r)}{\mathrm{d}A(r)} = \rho_1 r^{D_f - d} = \rho_1 r^{-a}, \tag{1}$$

where $\rho_1 = DN_0/(2\pi)$ refers to a proportionality coefficient, $a = d - D_f$ denotes the scaling exponent of density distribution ($a > 0$), d is the Euclidean dimension of the embedding space indicative of geographical space, and D_f is the *radial dimension* of urban morphology ($D_f < d$) (Frankhauser 1998). Equation (1) is identical in form to Smeed's model on traffic network density (Batty and Longley 1994; Smeed 1963). As there is no mathematical definition in Eq. (1) for the city center, we can specially define a central density $\rho(0) = \rho_0$ for the location $r = 0$. The ρ_0 value can be obtained by observation in empirical analyses. The urban density function is actually defined in a 1-dimension space based on the idea of statistical average, but it reflects the geographical information of a 2-dimension space (Chen 2010). In theory, the fractal dimension values of urban form (D_f) falls between 0 and 2, and the scaling exponent of urban density (a) varies from 0 to 1. In positive studies, sometimes we have D_f > 2, and thus the scaling exponent $a > 1$.

In terms of scaling notion, a fractal model is in fact based on a correlation function. A monofractal model is based on simple central correlation function, and a multifractal model is based on complex density–density correlation function. The former can be termed one-point correlation function, and the latter can be treated as point–point correlation function (Chen 2013). So, fractal analysis is usually related with correlation analysis. The global fractal dimension in multifractal theory is what is called generalized correlation dimension (Chen 2013; Chen and Jiang 2010; Grassberger and Procaccia 1983). Suppose that there exist two points on a radial from the city center, X and Y, on a digital map (Fig. 1c). Based on Eq. (1), the density–density correlation function can be expressed as

$$C(r) = \int_{-\infty}^{\infty} \rho(x)\rho(x+r)\mathrm{d}x = 2\rho_1^2 \int_0^{\infty} x^{D_f-d}(x+r)^{D_f-d}\mathrm{d}x, \tag{2}$$

where x denotes the distance of the first point (X) from the city center, and r represents the distance of the second point (Y) from the first point (X). Note that the central location ($x = 0$) is a discontinuity point in the density function. It is easy to demonstrate that the spatial correlation function follows the scaling law. The fractal essence is the scaling symmetry, which indicates the invariance under contraction or dilation transform (Mandelbrot 1989). For a function $f(x)$, if we contract or dilate the argument x by a constant scale factor λ, the function will not change in structure, but vary in size, and thus the result is $f(\lambda x) = \lambda^{\alpha} f(x)$, where α is a scaling exponent. In this case, the function conforms to the scaling law. Let $x = \xi y$, where ξ is a scale factor ($x \geq 0$, $y \geq 0$). A parameter relation can be derived by a scaling analysis as follows:

$$C(\xi r) = 2\rho_1^2 \int_0^{\infty} x^{D_f-d}(x+\xi r)^{D_f-d}\mathrm{d}x = \xi^{2H} C(r), \tag{3}$$

where $H = D_f - d + 1/2$, and H is the generalized Hurst exponent. Equation (3) is in fact a functional equation. In the theory of R/S analysis, the Hurst exponent is a scaling exponent associated with the autocorrelation coefficient of the time series (Hurst et al. 1965). The R/S analysis is normally termed "rescaled range analysis" (Feder 1988). This is a statistical method developed by Harold Edwin Hurst and his coworkers to analyze long-term continuous or regular records of natural phenomena (Hurst et al. 1965). It can also be employed to analyze the long orderly spatial series with proportional spacing. The rescaled range is a statistical measure of the variability of a time/space series based on two main measures/variables: one is the standard deviation (S), and the other, the range (R) of the data set, i.e., the difference between the highest and lowest values. From the slope of the logarithmic linear relation between the ratio of $R(\tau)$ to $S(\tau)$ and the time lag or spatial displacement τ, we can obtain a useful parameters, the Hurst exponent (H). Concretely speaking, for the increment series $\Delta x(i)$ of a space/time series $x(i)$, H is the scaling exponent of the ratio $R(\tau)/S(\tau)$ versus time lag or spatial displacement (τ) ($i = 1, 2, 3, \ldots; \tau = 1, 2, \ldots, i$). In other words, H is defined by the power function $R(\tau)/S(\tau) = (\tau/2)^H$ (Chen 2010; Feder 1988; Hurst et al. 1965). The exponent value falls in between 0 and 1, i.e., $0 \leq H \leq 1$. The value of $H = 1/2$ indicates a Brownian motion, while the values of $H \neq 1/2$ suggests the fractional Brownian motion (fBm) (Feder 1988; Mandelbrot 1983). Obviously, the solution to the above functional equation is

$$C(r) = C_1 r^{2(D_f - d)+1} = C_1 r^{2H} = C_1 r^b, \tag{4}$$

where C_1 refers to a proportionality coefficient related with the parameter ρ_1 in Eq. (1), and $b = 2H$ is a scaling exponent indicating fractal dimension.

A correlation dimension model can be derived from the spatial correlation function. Equation (4) represents a correlation function defined in a 1-dimension space, differing from the spatial correlation function defined in a 2-dimension space (Chen and Jiang 2010). For the latter, the scaling exponent is just the correlation dimension, but for the former, the scaling exponent is less than the correlation dimension. By dimensional analysis, we have

$$N(r) \propto rC(r) = r^{2-2(d-D_f)} = r^{2(D_f-1)} = r^{D_c}, \tag{5}$$

where $N(r)$ denotes the number of cells (pixels) within the radius of r from the center, and D_c is the density–density correlation dimension, or the *point-point correlation dimension* of urban density. Equation (5) gives a fractal dimension relation as below:

$$D_c = 2(D_f - 1) = b + 1, \tag{6}$$

which is important for our understanding fractal dimension. For the monocentric cities, the radial dimension can be directly calculated by the area–radius scaling (Batty and Longley 1994; Frankhauser 1998). However, for the polycentric cities, the area–radius scaling relations often break down and the radial dimension cannot be effectively estimated in a simple way. In this case, the spectral analysis is helpful due to the filter function of Fourier transform (Chen 2010).

A special case of the urban density–density correlation is the central correlation, which can be treated as "one-point correlation". Based on a 1-dimension space, the one-point correlation indicates the spatial correlation between a given point, e.g., a city's central point, and other points around the point, while the point–point correlation implies the spatial correlation between one point on a circle and another point on another circle around the center of a city (Chen 2013). Suppose that there is a radial line from the city center to the periphery. The density–density correlation implies the spatial relation between any two points on the radial line. If we fix one point to the center of city, we will have $x = 0$ (Fig. 1d). Thus, the integral in the point–point correlation function, Eq. (2), will vanish, and it is reduced to a special one-point correlation function as follows:

$$C_0(r) = \rho_0 \rho_1 r^{D_f - d} = \rho_0 \rho(r), \quad (7)$$

in which $C_0(r)$ denotes the one-point correlation measurement. This indicates that, if we fix one point in the central point of a city, the one-point correlation function is just proportional to the urban density function, and thus the correlation integral is proportional to the cell number within the radius of r from the center, $N(r)$. The radial dimension of cities is actually the one-point correlation dimension, that is, $D_0 = D_f$, where D_0 refers to the *one-point correlation dimension*. Therefore, $a = d - D_f = d - D_0$ indicates the scaling exponent of the one-point correlation. In this contexture, the one-point correlation represents central correlation, and the one-point correlation dimension suggests the spatial relation and interaction between a city's core and its periphery.

If a city takes on a polycentric pattern, it cannot be described by a simple monofractal model. In this case, the density–radius scaling should be substituted with wave-spectrum scaling of cities (Chen 2010, 2013). By means of Fourier transform, a correlation function can be converted into energy spectrum function and vice versa. The Fourier transform of the density–density correlation function also follows the scaling law. Based on Eq. (4), the scaling property of Fourier transform can be demonstrated as below:

$$S(\xi k) = \int_{-\infty}^{\infty} C(r) e^{-i2\pi\xi k r} \mathrm{d}r = \xi^{-2(D_f - 1)} S(k), \quad (8)$$

where k denotes the wave number, which bears an analogy with the frequency in time series analysis. Thus, we have

$$S(k) \propto k^{-2(D_f - 1)} = k^{-\beta}, \quad (9)$$

in which $\beta = 2(D_f - 1) = 2H + 1$ refers to the spectral exponent. Note that $2H = 2(D_f - d) + 1$ and $D_c = 2(D_f - 1) = b + 1$. According to Eq. (6), a parameter relation can be gotten as below:

$$\beta = D_c = 2(D_f - 1), \quad (10)$$

which suggests that the spectral exponent equals the density–density correlation dimension. Equation (10) has been empirically verified by Chen (2010). Comparing Eq. (5) with Eq. (9) yields the relation between the spectral density and the correlation function such as

$$N(r) \propto rC(r) \propto \frac{1}{S(k)}. \tag{11}$$

This implies that the pixel number of urban land use varies inversely as the spectral density.

For the one-point correlation, the cosine transformation relation between the correlation function and energy spectrum is not effective. The Fourier transform of the one-point correlation function can be expressed as

$$F(k) = \int_{-\infty}^{\infty} C_0(r)e^{-i2\pi kr}\mathrm{d}r = \rho_0\rho_1 \int_{-\infty}^{\infty} r^{D_0-d}e^{-i2\pi kr}\mathrm{d}r. \tag{12}$$

The equation proved to follow the scaling law under contraction or dilation, that is,

$$F(\xi k) = \rho_0\rho_1 \int_{-\infty}^{\infty} r^{D_0-d}e^{-i2\pi\xi kr}\mathrm{d}r = \xi^{-(D_0-d+1)}F(k). \tag{13}$$

This indicates the scaling relation of the energy spectrum as below

$$S_0(\xi k) = |F(\xi k)|^2 = \xi^{-2(D_0-d+1)}|F(k)|^2 = \xi^{-2(D_0-1)}S_0(k). \tag{14}$$

The solution to functional Eq. (14) is

$$S_0(k) \propto k^{-2(D_0-1)} = k^{-\beta_0}. \tag{15}$$

Apparently, the spectral exponent of the one-point correlation equals that of the point–point correlation, i.e., $\beta_0 = \beta$. To sum up, we have the following useful parameter relation:

$$\beta = 2(D_f - 1) = D_c = 2(D_0 - 1) = \beta_0, \tag{16}$$

which is equal to $b + 1$. Based on the parameter relationships shown in Eq. (16), more fractal dimension equations can be revealed for spatial analysis of urban morphology.

2.3 Fractal Parameter Equations

The fractal parameters based on wave-spectrum scaling are defined at the macro-level of urban morphology. macro-mathematical regularity is associated with micro-dynamic mechanism. It is necessary to investigate the spatial autocorrelation of urban growth at the micro-level. The macro-level is based on the urban density function, while the micro-level is based on the urban density increment function. An integral of Eq. (1) in the 2-dimension space yields the area–radius scaling relation (Batty and Longley 1994)

$$N(r) = N_1 r^{D_f}, \tag{17}$$

in which N_1 is a proportionality coefficient. From Eq. (17), an average density formula can be derived as follows (Batty and Longley 1994; Chen 2013; Longley et al. 1991):

$$\rho^*(r) = \frac{N(r)}{A(r)} = \frac{N_1}{\pi} r^{D_f - 2} \propto \frac{\mathrm{d}N(r)}{\mathrm{d}A(r)} = \rho(r), \tag{18}$$

where $\rho^*(r)$ refers to the average density within a radius of r from a city center, and $A(r) = \pi r^2$ represents the area within the circle of radius r. This indicates that the average density $\rho^*(r)$ is in proportion to the marginal density $\rho(r)$. Further, let us consider the variance of the density increment. Due to the symmetry of urban density function (from $-\infty$ to 0 then to ∞), the mean value of the density increment can be regarded as zero. So, the variance can be defined as

$$V(r) = \int_{-\infty}^{\infty} [\Delta\rho(x) - 0]^2 \mathrm{d}x = \int_{-\infty}^{\infty} [\rho(x + r) - \rho(x)]^2 \mathrm{d}x. \tag{19}$$

A scaling analysis of the above variance function yields

$$V(\xi r) = \rho_1^2 \int_0^{\infty} [(x + \xi r)^{D_f - 2} - x^{D_f - 2}]^2 \mathrm{d}x = \xi^{2H} V(r), \tag{20}$$

where $2H = 2(D_f - d) + 1$ has been given above. This suggests a scaling ratio such as $C(r)/V(r) = C(\xi r)/V(\xi r)$. The solution to functional Eq. (20) is

$$V(r) \propto r^{2H}. \tag{21}$$

If the spatial series of urban density changes is a white noise, the density increment $\Delta\rho(x) = \rho(x + r) - \rho(x)$ can be regarded as a Brownian motion, namely, random walk. Thus, we have $H = 1/2$, and this indicates $D_f = 2$. However, an urban density increment series is not a white noise. Because D_f value ranges from 0 to 2, the H value should vary from $-3/2$ to $1/2$. If so, the spatial process is always treated as an fBm process (Feder 1988; Peitgen et al. 2004). Of course, the H value comes between 0 and 1, and this will be clarified next.

Then, the self-affine fractal dimension can be derived by means of dimensional analysis. The method of dimensional analysis is useful in the theoretical studies of human geography (Haynes 1975; Haggett et al. 1977). Comparing Eq. (21) with Eq. (4) displays that the variance of density increment is in the proportion to the spatial correlation function, i.e., $V(r) \propto C(r)$. The square root of the variance is just the standard deviation

$$s(r) \propto r^H. \tag{22}$$

If the radial dimension D_f falls between 1.5 and 2, and the embedding space dimension is $d = 2$, the scaling exponent $a = d - D_f$ will come between 0 and 0.5, and thus the Hurst exponent $H = 1/2 - a$ will also fall between 0 and 0.5. According to the principle of dimensional consistency, a proportional relation appears as below:

$$\rho(r) \propto \frac{1}{s(r)} \propto r^{-H}. \tag{23}$$

Substituting Eq. (23) into Eq. (18) yields the following relation:

$$N(r) \propto \rho(r)A(r) \propto \frac{\pi r^2}{s(r)} \propto r^{2-H} = r^{D_s}. \tag{24}$$

Thus, another parameter equation can be obtained as follows:

$$D_s = 2 - H, \tag{25}$$

which denotes a special fractal dimension, termed *profile dimension* of urban morphology (Chen 2008). In literature, the parameter D_s is treated as the self-affine record dimension of the random walk (Feder 1988).

The above mathematical derivation and theoretical analysis result in a series of fractal parameter relations. These equations form a useful framework for the fractal study of urban systems. In terms of Eq. (3) or Eq. (20), if $d = 2$ as given, then we have

$$D_f = H + \frac{3}{2}, \tag{26}$$

which can also be derived from the dimensional analysis based on the point–point correlation function. The density function is in a proportion to r^{-a}, and squaring the function gives r^{-2a}. The integral of the squared density function is proportional to r^{1-2a}, and the second root of r^{1-2a} produces $r^{1/2-a}$. This indicates that $H = 1/2 - (d - D_f) = D_f - 3/2$ for $d = 2$. The result is the same as those proceeding from the scaling analysis of correlation function as well as variance function. Apparently, the scaling analysis and dimensional analysis reach the same goal by different routes.

A pair of useful fractal parameter relations can be further derived. First, substituting Eq. (25) into Eq. (26) yields an equation as follows:

$$D_f + D_s = \frac{7}{2}, \tag{27}$$

which gives the relation between the self-similar fractal dimension and self-affine fractal dimension of cities (Chen 2010). Second, integrating Eq. (16) into Eq. (26) or Eq. (27) yields another equation such as

$$\beta = 5 - 2D_s = 2H + 1, \tag{28}$$

which gives the relations between spectral exponent, Hurst exponent, and self-affine fractal dimension (Chen 2013; Feder 1988; Takayasu 1990). So far, we have had a set of fractal parameter relations, which comprises Eqs. (6), (16), (25), (26), (27), and (28).

The parameters and parameter equations can be integrated into a logic framework of fractal dimensions and relations. This framework represents a system of fractal parameters and scaling relations (Fig. 2). If the radial dimension D_f is calculated, then the one-point correlation dimension D_0, the point–point correlation dimension D_c, the wave spectral exponent β, the profile dimension D_s, the Hurst exponent H, and so on, can be estimated by the parameter equations (Table 1). Using this set of fractal parameters, we can make a systematic analysis of urban growth and form. To evaluate the radial dimension, we should select an urban center of concentric circles on a digital map. Given a center of circles, a radial dimension value is determined, and other fractal parameters will be in scale with the radial dimension in value. The valid ranges of different fractal parameters are as below: $0 \le D_f = D_0 \le 2$, $0 \le D_c = \beta \le 2$, $0 \le D_s \le 2$, $0 \le \beta \le 3$, $0 \le H \le 1$. In light of multifractal geometry, the one-point correlation dimension value must be greater than the point–point correlation dimension value, that is, $\beta = D_c \le D_f = D_0$. All these parameters are defined at two different levels: the parameters D_0, D_f, D_c, and β are at the macro-level, and D_s and H are at the micro-level. If the relation between macro and micro-levels of a city is overlooked, the acceptable domain of radial dimension will fall between 1 and 2. If $D_f > 2$ as given, then $\beta = D_c > D_f$; while if $D_f < 1$, then we will have $\beta = D_c < 0$. The two cases are illogic in multifractal theory. If we take the relation between macro and micro-levels into consideration, the feasible domain of the radial dimension will range from 1.5 to 2. When $D_f < 1.5$, we will have $D_s > 2$ and $H < 0$, and these values

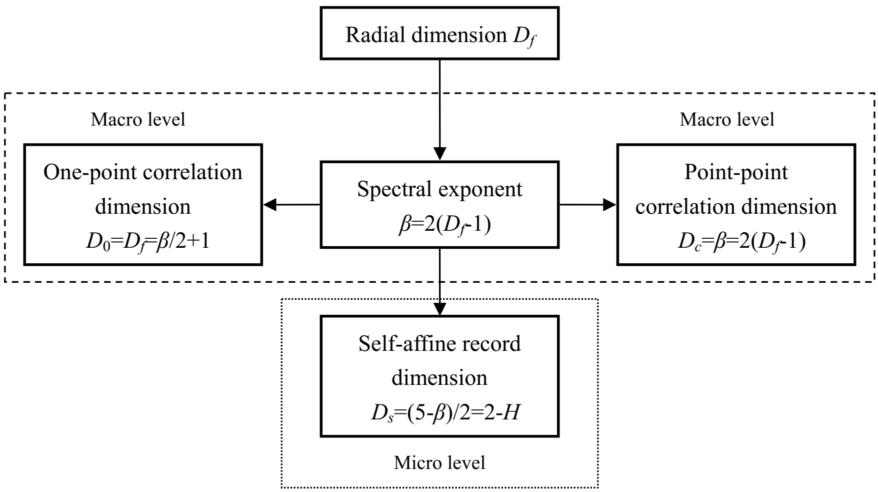


Fig. 2 A schematic diagram of the relationships between five fractal parameters for urban spatial analyses (by Chen 2013)

Table 1 The numerical relationships between radial dimension, correlation dimension, profile dimension, spectral exponent, and Hurst exponent

Radial dimension, one-point correlation dimension		Point–point correlation dimension, spectral exponent		Self-affine record dimension	Hurst exponent
D_f	D_0	D_c	β	D_s	H
0.5	**0.5**	−1	−1	3	−1
0.75	**0.75**	−0.5	−0.5	2.75	−0.75
1	**1**	**0**	**0**	2.5	−0.5
1.1	**1.1**	**0.2**	**0.2**	2.4	−0.4
1.2	**1.2**	**0.4**	**0.4**	2.3	−0.3
1.3	**1.3**	**0.6**	**0.6**	2.2	−0.2
1.4	**1.4**	**0.8**	**0.8**	2.1	−0.1
1.5	**1.5**	**1**	**1**	**2**	**0**
1.6	**1.6**	**1.2**	**1.2**	**1.9**	**0.1**
1.7	**1.7**	**1.4**	**1.4**	**1.8**	**0.2**
1.8	**1.8**	**1.6**	**1.6**	**1.7**	**0.3**
1.9	**1.9**	**1.8**	**1.8**	**1.6**	**0.4**
2	**2**	**2**	**2**	**1.5**	**0.5**
2.25	2.25	2.5	2.5	1.25	**0.75**
2.5	2.5	3	**3**	**1**	**1**

Note The proper numerical ranges of the fractal parameters are as follows: $0 < D_f$, D_0, $D_c < 2$; $0<\beta<3$; $0<H<1$; $-1<C_\Delta<1$. If a value exceeds the limiting sphere, the result will be meaningless. See Chen (2010, 2013)

are absurd. All in all, the reasonable range of both the radial dimension and profile dimension is from 1.5 and 2, that is, $1.5 \leq D_f, D_s \leq 2$.

2.4 New Analytical Framework for Urban Morphology

An analytical process of spatial correlation analysis based on scaling analysis and spectral analysis can be developed for urban studies. At macro-level, spatial correlation analysis of urban morphology can be made through Eqs. (1) and (4). Smeed's model, Eq. (1), is a one-point correlation function. However, its integral form, Eq. (4), is a point–point correlation function. What is more, the spatial autocorrelation coefficient defined in the 1-dimension space is associated with Hurst's exponent (Feder 1988), and can be expressed as

$$C_\Delta = 2^{2H-1} - 1 = 2^{2(d-D_s)-1} - 1, \tag{29}$$

Table 2 The autocorrelation coefficients at the micro-level and autocorrelation functions at the macro-level of cities

Radial dimension (D_f)	micro-level	macro-level	
	Autocorrelation coefficient (C_Δ)	One point spatial autocorrelation function [$C_0(r)$]	Point–point spatial autocorrelation function [$C(r)$]
0.5	−0.875	$r^{-1.5}$	$r^{-2.0}$
0.75	−0.823	$r^{-1.25}$	$r^{-1.5}$
1.0	−0.75	$r^{-1.0}$	$r^{-1.0}$
1.1	−0.713	$r^{-0.9}$	$r^{-0.8}$
1.2	−0.670	$r^{-0.8}$	$r^{-0.6}$
1.3	−0.621	$r^{-0.7}$	$r^{-0.4}$
1.4	−0.565	$r^{-0.6}$	$r^{-0.3}$
1.5	**−0.5**	$\boldsymbol{r^{-0.5}}$	$\boldsymbol{r^{0}}$ **= Constant**
1.6	**−0.426**	$\boldsymbol{r^{-0.4}}$	$\boldsymbol{r^{0.2}}$
1.7	**−0.340**	$\boldsymbol{r^{-0.3}}$	$\boldsymbol{r^{0.4}}$
1.8	**−0.242**	$\boldsymbol{r^{-0.2}}$	$\boldsymbol{r^{0.6}}$
1.9	**−0.129**	$\boldsymbol{r^{-0.1}}$	$\boldsymbol{r^{0.8}}$
2.0	**0**	$\boldsymbol{r^{0} = 1}$	$\boldsymbol{r^{1.0}}$
(2.25)	0.414	($r^{0.25}$)	($r^{1.5}$)
(2.5)	1	($r^{0.50}$)	($r^{2.0}$)

Note The bold numerals represent the proper scale of the fractal parameters of urban form. See Chen (2010, 2013)

in which C_Δ denotes a one-order autocorrelation coefficient, and $d = 2$ is the Euclidean dimension of the embedding space in which urban morphology is investigated. For a certain radial dimension value, we will get a corresponding autocorrelation coefficient or correlation function by means of fractal dimension equations (Table 2).

Using the concepts from multifractal, we can derive a reasonable interval of fractal dimension values of urban morphology. Theoretical correlation analysis shows that there are two special radial dimension values: $D_f = 1.5$ and $D_f = 2$. For $D_f = 1.5$, the density–density spatial correlation function will become a constant, that is, $C(0) = const$. Thus, the correlation function will be independent of the distance r. For $D_f < 1.5$, the density–density spatial correlation function will be inversely proportional to the distance r. This indicates that the spatial centripetal force for concentration growth surpasses the centrifugal force for urban deconcentration growth. In this instance, the major way of city development is inward space filling. For $D_f > 1.5$, the density–density spatial correlation function will be directly proportional to the distance r. This indicates the urban spatial centrifugal force overrides urban centripetal force. In this case, the main way of city development is outward spatial expansion. According to the ideas from multifractal theory, the density–density correlation dimension D_c must be less than the central correlation dimension D_f, that is $D_c \leq D_f = D_0$ (Chen 2011, 2013). However, if $D_f > 2$, we will have $D_c > D_f$. This relation violates the multifractal dimension principle. An inference is that, if $D_f < 1.5$, we will have a thin city growth (undergrowth); and if $D_f \geq 2$, we will have a fat city growth (overgrowth); if and only if the radial dimension falls between 1.5 and 2, urban growth will be consistent with a theoretical fractal growth at various scales.

As indicated above, the spectral exponent proved to be a density correlation dimension. Based on this result, the theoretical inferences from the spatial correlation analysis can be summarized as follows. If $\beta = D_c < 1$, we will have $D_f < 1.5$, thus the spatial correlation intensity is directly proportional to the distance r between two places. This suggests that city development is prone to filling in vacant space (e.g., spare land, open space) inwards. In this case, the focus of urban planning is the internal structure. If $\beta = D_c > 1$, we will have $D_f > 1.5$, and the spatial correlation intensity is inversely proportional to the distance r between two locations. This means that city development is inclined to growing outwards, and suburbs or even exurbs are gradually occupied by various buildings. In this instance, the focus of urban planning is external space. If $\beta = D_c = 1$, we will have $D_f = 1.5$, and the density–density correlation intensity has nothing to do with distance. Under these circumstances, urban evolution will fall into a self-organized critical state, in which the power-law distributions will emerge (Bak 1996). The self-organized criticality (SOC) is indeed a revealing concept in the theoretical research of cities (Batty and Xie 1999; Chen and Zhou 2006, 2008; Portugali 2000). The radial dimension $D_f = 2$ represents another special value of fractal dimension for urban morphology. If $D_f = 2$, we will have $C_0(0) = 1$, and this implies that the central correlation is independent of distance r. The interaction between city center and any urban place is the same as one another. This seems to be unaccountable in theory. The value $D_f = 2$ suggests that the density at one place equals that at another place. On the other hand, if $D_f = 2$, the autocorrelation coefficient $C_\Delta = 0$, and the fBm process will change to a random walk. The process of space filling within the urbanized area will stop due to the absence of vacant land.

In short, different fractal parameters are in fact related with one another. Each parameter value has its reasonable range (Table 1). The intersection of the proper ranges of the parameters D_0, D_c, D_s, H, and β suggests the feasible domain of the radial dimension D_f. Corresponding to the domain, all these fractal parameters make sense and accord with each other. If the fractal dimension values exceed the bounds ($D_f < 1.5$ or $D_f > 2$), the logic relations between different fractal parameters will be broken. In spite of the scaling nature of fractals, a fractal dimension is a measurement with characteristic length. If a city's fractal dimension value is too high (e.g., $D_f > 2$) or too low (e.g., $D_f < 1$), the possible problems of urban growth must be examined. Too low fractal dimension value indicates that the geographical space is not well developed for a city, while too high fractal dimension value implies that the geographical space may be overly filled. Both empirical and theoretical studies show that the advisable fractal dimension value of urban form is around $D_f = 1.7$ (Batty and Longley 1994; Chen 2010).

3 Case Study

3.1 Methodology

Based on the above-shown fractal parameter equations, the scaling analysis, spectral analysis, and spatial correlation analysis can be integrated into a logical framework to form a new analytical process for urban morphology. The new framework can be termed "3S analyses" method of urban geography (Fig. 3). The variables in the theoretical spatial correlation function are continuous ones. However, in practice, the spatial sampling is a discrete rather than a continuous process. So, the correlation functions must be discretized for empirical analyses. Consequently, spatial distance and displacement in the practical correlation functions are discrete variable and parameter. In fact, all quantitative analysis relates to three worlds: the *real world* (e.g., a city), the *mathematical world* (e.g., logical deduction), and *computational world* (e.g., algorithms and measurements) (Casti 1996). The theoretical derivations based on continuous variables are fulfilled in the mathematical world in the above section. The object of study, a Chinese city, is examined in the real world. The empirical analysis based on discrete variables will be made in the computational world in the following part.

A density–density correlation function can be defined for urban analysis either in the 1-dimension Euclidean space or in the 2-dimension Euclidean space. In other words, two approaches can be adopted to construct the point–point correlation functions: one is to construct the correlation in the 1-dimension space, and the other, to construct the correlation in the 2-dimension space. A practicable approach is to implement a 2-dimension correlation analysis through a 1-dimension space. It is indeed simpler to make a correlation analysis of cities in the 1-dimension geographical space than in the 2-dimension geographical space. The construction method and the related analytical process of spatial correlation function based on the 2-dimension space have been illustrated for systems of cities (Chen and Jiang 2010). This paper focuses on the density–density correlation function defined in a 1-dimension space. Now, let us consider a city as a system (a city's system rather than a system of cities), which comprises N geographical elements. The urban elements can be abstracted as "cells" or "pixels" on a digital map. If the density–density correlation function is defined in a 1-dimension space, we need a set of concentric circles for spatial measurements. In contrast, if the correlation function is constructed in a 2-dimension space, then we will need N sets of concentric circles for spatial data processing (Fig. 1).

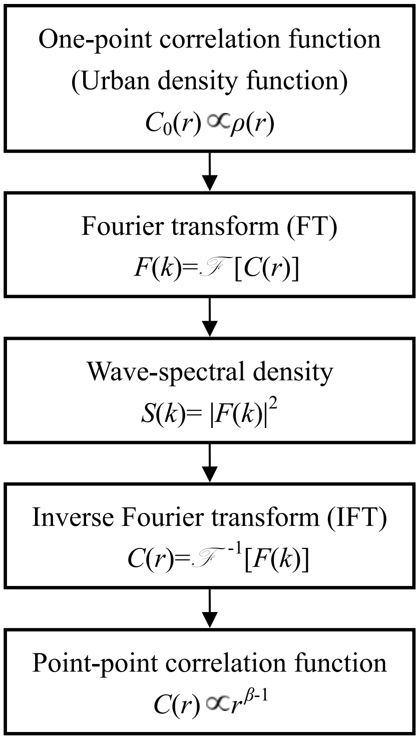


Fig. 3 A flow diagram of spatial correlation analysis based on scaling and spectral analyses (by Chen 2013)

The analytical process of the 1-dimension spatial correlation for 2-dimension geographical space can be divided into six steps. **Step 1**: Identify the center of mass (centroid) of an urban agglomeration. The center of a Western city is always inside the central business district (CBD). For Chinese cities, however, things are complicated. **Step 2**: Draw a set of concentric circles. The center of the circles should be the centroid of the urban cluster. The interval between the concentric circles is equal. That is to say, if we draw a radial line from the common center of these concentric circles to the periphery, the intersections of the circles and the radial are uniformly distributed over the straight line. The space between two circles can be treated as a "ring", and the "width" of the rings are the intervals, which may be very small. **Step 3**: Calculate the average density for each ring. This process is not only a spatial measurement but also a spatial mapping. As soon as the average density of urban element distribution between two circles is computed, the geographical information in the irregular 2-dimension space is mapped into the regular 1-dimension space. Then, the set of concentric circles can be converted into a continuous series of cells (Fig. 1). If the interval between two concentric circles is narrow enough, the size of these cells will be small enough. **Step 4**: Estimate the radial dimension by the radius–density scaling. As indicated above, we can evaluate the central correlation dimension by Eq. (1) in theory, but in empirical studies, it is Eq. (17) instead of

Eq. (1) that is suitable for determining the radial dimension. A problem is that the density series is too sensitive to stochastic perturbation in spatial measurements to give reliable results. **Step 5**: Compute the spectral exponent using the fast Fourier transform (FFT) and wave-spectrum scaling. In theory, the spectral analysis is based on Eq. (9), but in practice, the spectral analysis can be made by means of Eq. (15). As demonstrated above, the scaling exponent of wave-spectrum relation is actually the density–density correlation dimension. **Step 6**: Carry out spatial analysis for the urban morphology. Using the fractal parameter relations (Fig. 2), we can indirectly work out varied parameter values besides the spectral exponent and radial dimension. These parameters include the Hurst exponent and the profile dimension. With the help of the technology roadmap (Fig. 3), we can implement deeper spatial correlation analysis for urban form and growth.

3.2 Study Area, Datasets, and Results

The method of the 3S analysis and the related theory proposed above can be applied to the city of Beijing, the national capital of China. Beijing is a well-known megacity in the world, with an urban population of more than 16 million in 2010 (by the sixth census). The datasets were extracted from the remotely sensed images (Fig. 4). We have datasets of urban land use density for 11 years from 1984 to 2009. It has been shown that there are two approaches to estimating the radial dimension of urban morphology: one is the area–radius scaling based on Eq. (17), and the other is the energy spectrum scaling based on Eq. (9). The former is a direct approach, while the latter is an indirect approach. If urban growth is isotropic, the radius-area scaling relation will be well fitted to urban density data. Unfortunately, Beijing is of anisotropic growth, and the scaling relation cannot be directly fitted to the observational data. In this case, the energy spectrum scaling relation can be used as alternative approach.

The idea of spatial energy spectrum is defined in the mathematical world, based on continuous variable such as infinite spatial distance. However, in an empirical analysis for the real world, urbanized area is limited and variables are in discrete format. In this instance, the energy spectral density, $S(k)$, is substituted by the wave spectral density, $W(k)$, which is actually defined in the computational world. The wave spectrum is on the base of energy spectrum, and spectral density is as below:

$$W(k) = \frac{1}{N} S(k), \tag{30}$$

in which N refers to the number of data point of urban density indicative of the length of the sample path. For simplicity, the number is taken as $N = 64$ in this example. Thus, the energy spectrum relation is substituted by the wave-spectrum relation as follows:

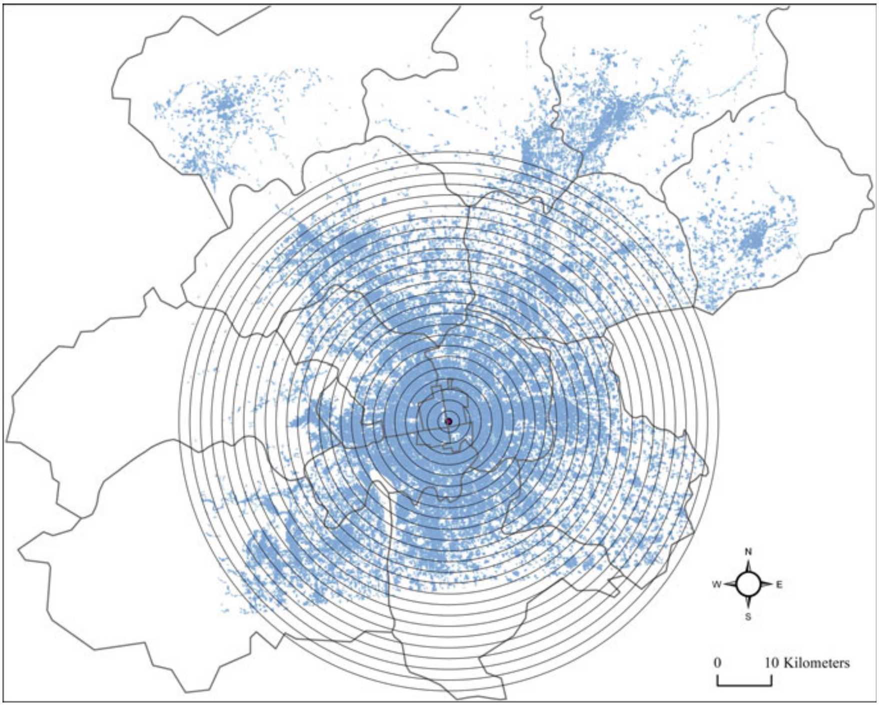


Fig. 4 A sketch map of wave spectral analysis for urban morphology of Beijing, the national capital of China (2009)

$$W(k) \propto k^{-\beta^*}, \tag{31}$$

where β^* denotes the predicted value of the wave spectral exponent. In light of Eq. (10), a fractal parameter can be derived from the wave-spectrum scaling relation such as

$$D_f^* = 1 + \frac{\beta^*}{2}, \tag{32}$$

which is termed "image dimension" of urban morphology (Chen 2013). A large number of mathematical experiments suggest an empirical relation between the radial dimension and its image dimension of a city as below

$$D_f = 1 + \frac{2}{5} D_f^*, \tag{33}$$

which is in fact an approximate relation (Chen 2010). The radial dimension reflects the space filling extent of the whole urban field, while the image dimension reflects the space filling degree of the central part of a city. Both the two parameters mirror the core-periphery relation of urban growth and form.

The procedure of data processing and spatial analysis of Beijing's morphology is as follows. **First, calculate the average urban density of Beijing in each year**. Based on remote sensing images, the center of Beijing city can be identified, and the urban land use density can be estimated by means of concentric circles. According to the requirement of the FFT algorithm, the number of circles is set as $N = 64$. Through ArcGIS technique, we can complete this computation. **Second, work out the wave spectral density**. FFT can be employed to turn Beijing's urban density into the spectral density. Today, it is easy to conduct this numerical transformation. Through Matlab or even MS Excel, we can fulfill this task of data conversion. **Third, fit the wave-spectrum scaling relation to the spectral density datasets**. The ordinary least squares (OLS) method can be utilized to make wave-spectrum analyses. For example, for Beijing in 1998, the wave-spectrum relation is $W(k) = 0.0001199k^{-1.7381}$. The goodness of fit is about $R^2 = 0.9770$, and the spectral exponent is estimated as $\beta^* = 1.7381$(Fig. 5). **Fourth, evaluate the radial dimension**. For the abovementioned example, according to Eq. (32), the image dimension is $D_f^* \approx 1.7381/2 + 1 \approx 1.8690$. Further, in light of Eq. (33), the radial dimension is $D_f \approx 1 + 2 * 1.8690/5 \approx 1.7476$. The rest may be done in the similar way (Table 3). **Fifth, estimate the other fractal parameters by the related formulae**. Using the fractal parameter equations presented in above section, we can figure out the spatial correlation dimension, D_c, the profile dimension, D_s, the Hurst exponent, H, and the autocorrelation coefficient, C_Δ. For the example abovementioned, we have $D_c = 1.7381$, $D_s = 1.6310$, $H = 0.3690$, and $C_\Delta = -0.1660$. The others may be treated by analogy (Table 4).

A discovery is that the wave-spectrum relations of Beijing's urban morphology do not follow scaling law globally. If we fit the power law relation between wave number and spectral density to the observational data, the data points cannot match the trend lines very well (Fig. 5). This suggests that the estimated values of spectral exponents are biased to some extent. In fact, the scattered points form two scaling ranges on double logarithmic wave-spectrum plots. The first scaling range falls between the wave numbers 1 and 9, and the second scaling range falls between the wave numbers 10 and 32 (Fig. 6). Thus, we have two spectral exponents for each year, and this suggests bi-fractal structure. Bi-fractals were found in the studies on both urban form and traffic networks (Benguigui and Daoud 1991; White and Engelen 1993, 1994). The essence of bi-fractals rests with self-affine growth. If urban growth is isotropic, no bi-fractal structure appears. However, if urban growth is anisotropic, bi-fractal structure will emerge. In short, anisotropic growth may lead to self-affine process, which in turn lead to bi-fractal pattern. The wave-spectrum relations imply that Beijing's urban growth bears self-affinity, which proceeds from anisotropic development in the last 30 years. In this case, all the fractal parameters can be estimated through bi-scaling ranges (Table 3).

Now, the spatiotemporal information of Beijing's city development can be revealed, and the main points are as below. **First**, the global radial dimension (D_f) falls between 1.3 and 2.2, while the local radial dimensions may be greater than 2

Table 3 The global and local fractal parameters of Beijing's urban form from 1984 to 2009

Year	Global parameters				Local parameters for core				Local parameters for periphery			
	β^*	R^2	D_s	D_f^*	β^*	R^2	D_s	D_f^*	β^*	R^2	D_s	D_f^*
1984	1.8472	0.9335	1.5764	1.9236	2.7643	0.9964	1.1178	2.3822	1.0628	0.9701	1.9686	1.5314
1988	1.6669	0.9059	1.6665	1.8335	2.6839	0.9968	1.1581	2.3419	0.8240	0.9837	2.0880	1.4120
1989	1.6160	0.9131	1.6920	1.8080	2.5673	0.9955	1.2164	2.2836	0.8664	0.9824	2.0668	1.4332
1991	2.1487	0.9029	1.4257	2.0743	3.1759	0.9861	0.9121	2.5879	0.6385	0.8468	2.1808	1.3192
1992	1.6437	0.9453	1.6782	1.8218	2.3580	0.9871	1.3210	2.1790	1.0337	0.9904	1.9832	1.5168
1994	1.5885	0.9334	1.7057	1.7943	2.3473	0.9849	1.3263	2.1737	0.9173	0.9849	2.0413	1.4587
1998	1.7381	0.9770	1.6310	1.8690	2.1872	0.9858	1.4064	2.0936	1.3463	0.9910	1.8269	1.6731
1999	1.4523	0.8353	1.7738	1.7262	2.6673	0.9869	1.1663	2.3337	0.4892	0.8914	2.2554	1.2446
2001	1.5111	0.8945	1.7444	1.7556	2.4900	0.9878	1.2550	2.2450	0.7538	0.9747	2.1231	1.3769
2006	1.5143	0.9075	1.7428	1.7572	2.4289	0.9930	1.2855	2.2145	0.7841	0.9807	2.1079	1.3921
2009	1.3926	0.8338	1.8037	1.6963	2.5882	0.9942	1.2059	2.2941	0.4199	0.8256	2.2900	1.2100

Table 4 The global fractal parameters and the derived scaling exponents of Beijing's urban form from 1984 to 2009

Year	β^*	D_s	D_f^*	D_f	D_c	H	C_Δ
1984	1.8472	1.5764	1.9236	1.7694	1.8472	0.4236	−0.1005
1988	1.6669	1.6665	1.8335	1.7334	1.6669	0.3335	−0.2062
1989	1.6160	1.6920	1.8080	1.7232	1.6160	0.3080	−0.2337
1991	2.1487	1.4257	2.0743	1.8297	2.1487	0.5743	0.1085
1992	1.6437	1.6782	1.8218	1.7287	1.6437	0.3218	−0.2188
1994	1.5885	1.7057	1.7943	1.7177	1.5885	0.2943	−0.2481
1998	1.7381	1.6310	1.8690	1.7476	1.7381	0.3690	−0.1660
1999	1.4523	1.7738	1.7262	1.6905	1.4523	0.2262	−0.3159
2001	1.5111	1.7444	1.7556	1.7022	1.5111	0.2556	−0.2874
2006	1.5143	1.7428	1.7572	1.7029	1.5143	0.2572	−0.2858
2009	1.3926	1.8037	1.6963	1.6785	1.3926	0.1963	−0.3436

and less than 1. This suggests that the peak area of urban density distribution is not near the city center. What is more, the average values of the local radial dimensions is close to the corresponding global radial dimension. This indicates that we can make wave spectral analysis using the approximate global spectral exponents. **Second**, the self-affine fractal dimension (D_s) went up and up, while the self-similar fractal dimension (D_f) went down and down over years. Generally speaking, the self-similar fractal dimension value of urban form goes up gradually due to urban space filling. The fractal dimension curve of urban growth is a logistic curve or quadratic logistic curve. However, for Beijing, the radial dimension change cannot be described with logistic function. In contrast, the self-affine profile dimension can be described with a quadratic logistic function. This indicates that it is the self-affine dimension rather than the self-similar dimension that can effectively reflect the space filling of Beijing because of anisotropic growth. **Third**, the Hurst exponent is less than 0.5, i.e., $H < 1/2$. This suggests that the spatial autocorrelation of Beijing urban growth is negative, and urban pattern is of heterogeneity. **Fourth**, the case in 1991 is exceptional. In this year, the radial dimension value is less than the correlation dimension values, that is, $D_f > D_c$, and the Hurst exponent is greater than 0.5. Of course, this may come from the quality of remote sensing image.

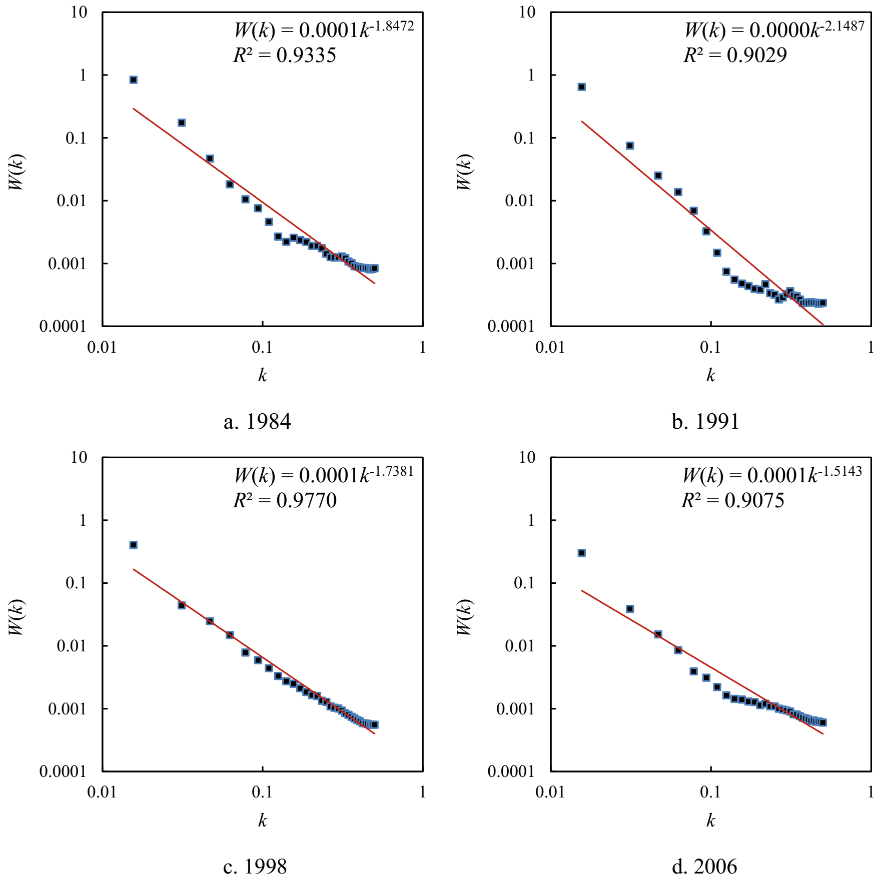


Fig. 5 The global wave-spectrum scaling relations of Beijing's urban morphology in four years (examples). *Note* The global scaling includes all the data points. For Beijing, the global wave-spectrum relation cannot be well fitted to the observational data. In this instance, the estimated spectral exponent values are biased

4 Questions and Discussion

4.1 Methodological Outline

A methodology of 3S analysis for cities has been presented, and the analytical process comprises scaling analysis, spectral analysis, and spatial correlation analysis. The key is the spatial correlation function. The spatial correlation is an important process in urban evolution. The scaling analysis and spectral analysis can be used to find the parameter solutions for a nonlinear correlation equation. The functions of the 3S method are as below: **First, the 3S analysis can be utilized to examine the spatial**

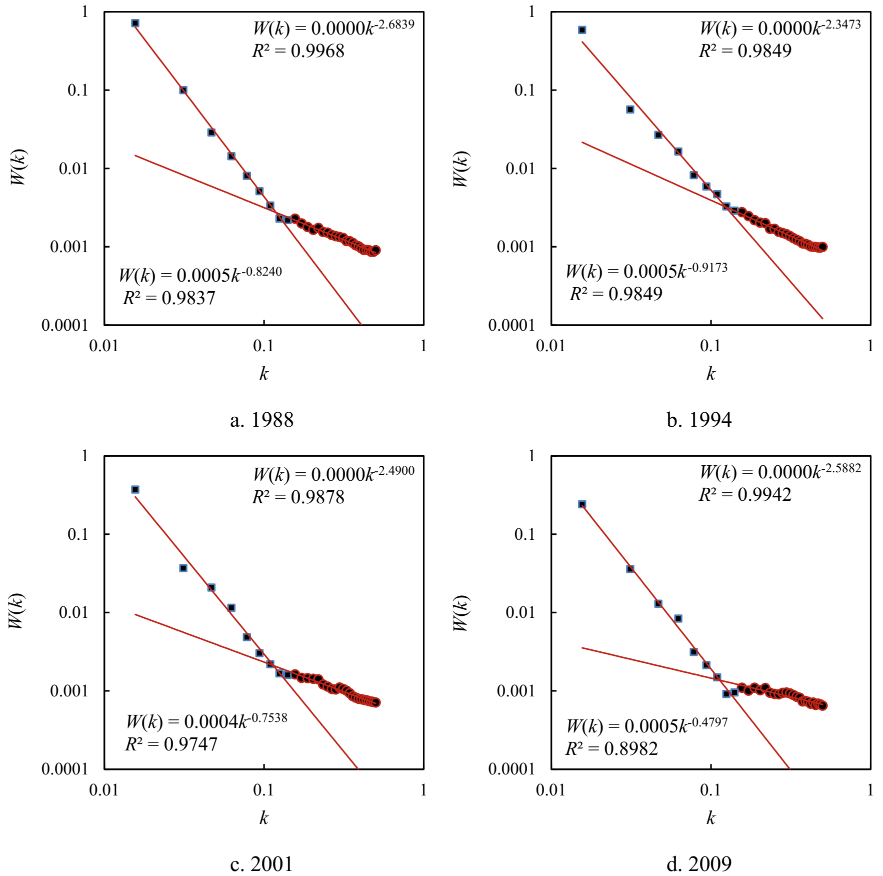


Fig. 6 The local wave-spectrum scaling relations of Beijing's urban morphology in four years (examples). *Note* The local scaling includes partial contiguous data points. On a log–log plot, the data points form two straight line segments, representing two scaling ranges. This suggests bi-fractals, which originated from self-affine process

interaction between urban components. The spatial correlation functions can be used to describe the patterns of local interactions of urban parts at the micro-level. Based on spatial correlation, spectral analysis can be used to explore the energy distribution of spatial interactions between different urban units. **Second, the 3S analysis can be used to investigate the relationships between different levels of a city's system**. The scaling analysis is important for geographers to understand the links between global and local levels. A scaling process involves various scales, ranging from the global level to the local level of urban structure. If a city evolves from a simple state into a complex critical state by self-organization, the power-law patterns will emerge, and the scaling analysis can be used to reveal the parameter relations in fractals, rank-size rule, and the law of allometric growth. The spectral analysis can be employed to examine the association of the spatial correlation (global level) with the autocorrelation (local level) of a city. **Third, the 3S analysis can be adopted to research the connection between structure of urban morphology and dynamics of urban growth**. Due to the absence of continuous sampling records within certain period, it is hard to model the spatial dynamics of urban evolution. Fortunately, the spatial structure of a city always reflects the dynamic information of city development. Combining correlation analysis and spectral analysis, we can bring to light the geographical information of urban spatial dynamics.

Spatial dimension is one of conundrums for mathematical modeling in scientific research. Reducing the spatial dimension makes it easy to model the correlation of urban elements. In this work, the correlation function is defined in the 1-dimension space, reflecting the geographical information in the 2-dimension space because the fractal dimension values fall between 1 and 2. All the fractal parameters based on the radial dimension display the 2-dimension spatial information of urban form and growth. If a correlation function is defined in the 2-dimension Euclidean space, it will cause three main problems as below: **First**, **mathematical modeling**. The spatial correlation cannot be directly constructed by the urban density function, Eq. (1), which is simple and fundamental in urban studies. **Second**, **spectral analysis**. It is difficult to make wave spectral analysis based on Fourier transform, which is helpful for us to reveal geographical spatial order and rules. **Third**, **computational effort**. The task quantity of numerical computation will be large if the pixels are too many. The common principle of choosing a model or method is the maximum ratio of "output" (explanation and prediction effects) to "input" (variable number, parameter number, computational effort). In many cases, the output–input ratio of fractal models do not decrease if we use the correlation function defined in the 1-dimension space to replace that defined in the 2-dimension space (Fig. 1).

The advantages and disadvantages of a method are relative, and sometimes an advantage becomes disadvantage under different conditions. One of the deficiencies of the spatial correlation analysis mainly rests with the model's mathematical structure. As indicated above, the urban density function based on monocentric cities is essentially defined in the 1-dimension Euclidean space. Although the fractal parameters based on the 1-dimension spatial correlation function reflect the 2-dimension spatial information, they cannot bring to light ALL the geographical processes and patterns of urban morphology. The shortcomings of the radial dimension and the related parameters are as below: First, these parameters cannot reflect the spatial correlation in all directions. The correlational direction is limited to the 1-dimension linear space of a city. Second, the parameters cannot effectively mirror the spatial morphology of heterogeneous cities. If a city has a number of growth centers, the power-law relation of urban density distribution usually breaks. Third, the parameters cannot reveal all the components of a city's system. Even in a monocentric city, not all of the elements' distribution density obeys the power law. Smeed's model can be used to describe the spatial density of the traffic network within a metropolitan area (Batty and Longley 1994; Smeed 1963). In empirical studies, the inverse power law cannot be well fitted to urban population density and urban land use density data (Chen 2010).

Multifractal theory provides a powerful tool for spatial correlation of urban morphology. If the effect of the 1-dimension correlation function is not good because of polycentric growth of cities, or if we want to explore the spatial correlation based on the 2-dimension space, two approaches can be considered. One is to employ the 2-dimension-based correlation function (Chen and Jiang 2010), and the other is to utilize multifractal geometry (Grassberger 1985; Hentschel and Procaccia 1983; Mandelbrot 1999; Stanley and Meakin 1988). The second approach is well known for fractal scientists, but it is difficult to grasp the ideas from multifractals. In fact, the generalized correlation dimension of multifractals is based on correlation function. Based on the functional box-counting method, the generalized dimension is formulated as

$$D_q = \frac{1}{q-1} \lim_{\varepsilon \to 0} \frac{\ln \sum_{i=1}^{n} \sum_{j=1}^{n} P_i(\varepsilon)^q}{\ln \varepsilon}, \tag{34}$$

in which D_q denotes the qth order correlation dimension, $P(\varepsilon)$ is the growth probability of fractal elements in the ith box with linear size ε at the mth level ($m = 1, 2, 3, \ldots$), generally, $\varepsilon = 1, 1/2, 1/4, \ldots, 1/2^{m-1}$, thus $n = 1/\varepsilon$, and $N(\varepsilon) = n * n$ is the number of the boxes at the given level. The parameter q is termed "moment order" in statistics ($-\infty < q < +\infty$). If $q = 0$, $D_q = D_0$ denotes the *capacity dimension*; if $q = 1$, $D_q = D_1$ indicates the *information dimension*; if $q = 2$, $D_q = D_2$ represents the *correlation dimension*. Capacity dimension tells us whether or not there exists a fractal element in a box, information dimension tells how many fractal elements can be found in a nonempty box, and correlation dimension tells us how many fractal elements can be found around a given fractal element (within certain distance). In theory, $q(-\infty, +\infty)$ represents a continuous variable. But in empirical analysis, q is often a discrete sequence, and we can set an ordered set of numbers such as $q = \ldots, -100, -99, \ldots, -2, -1, 0, 1, 2, \ldots, 99, 100, \ldots$. Thus, we will have a pair of multifractal spectrums for urban analysis, including generalized dimension spectrum (D_q vs. q) and local dimension spectrum ($f(\alpha(q))$ vs. $\alpha(q)$) (Chen and Wang 2013). In practice, the conventional box-counting method can be substituted with the grid methods of fractal dimension estimation (Frankhauser 1998). One of the special grid methods is just the functional box-counting method, which is useful in urban studies on fractals (Chen and Wang 2013; Feng and Chen 2010).

4.2 Model Generalization

Spatial correlation modeling is related to spatial autocorrelation and spatial interaction models in human geography. There are two important classical theories of spatial analysis for geography: one is spatial autocorrelation analysis (Cliff and Ord 2009; Haggett et al. 1977), and the other, spatial interaction modeling (Haggett et al. 1977;

Wilson 2010). Spatial autocorrelation coefficient can be defined by correlation function, and spatial interaction models are associated with gravity models. The spatial correlation analysis can be generalized to the gravity models and spatial autocorrelation functions. As indicated above, the spatial correlation function can be linked to the gravity models defined at the micro-level. Based on the power-law density function of cities, Eq. (1), a micro-gravity model can be constructed as follows:

$$I(x, r) = K\frac{\rho(x)\rho(x+r)}{r^b} = \frac{\rho_1^2}{r^b}[x(x-r)]^{D_f-d}, \tag{35}$$

where K denotes the gravity coefficient. According to the above definition, if $r = 0$ as given, then $\rho(0) = \rho_0$, which can be observed but cannot be predicted by Eq. (1). For the interaction between the city center and the periphery area, Eq. (35) will change to

$$I_0(r) = K\frac{\rho(0)\rho(r)}{r^b} = K\frac{\rho_0\rho_1}{r^b}r^{D_f-d} = kr^{D_f-d-b}, \tag{36}$$

in which the proportionality coefficient is $k = K\rho_0\rho_1$. In fact, Eq. (36) is a distance-decay function based on the power-law gravity model. Obviously, the distance-decay function is not in the linear proportionality to the urban density function. This suggests that, where power law distribution is concerned, urban density does not merely result from spatial interaction, and the gravity decay is much faster than urban density decay. Integrating Eq. (35) yields a spatial correlation function based on the gravity model as below:

$$F(r) = K\int_{-\infty}^{\infty}\frac{\rho(x)\rho(x+r)}{r^b}\mathrm{d}x = 2K\rho_1^2\int_0^{\infty}\frac{x^{D_f-d}(x+r)^{D_f-d}}{r^b}\mathrm{d}x, \tag{37}$$

which follows the scaling law. A scaling analysis of Eq. (37) can be made as follows:

$$\begin{aligned} F(\xi r) &= 2K\rho_1^2\int_0^{\infty}\frac{x^{D_f-d}(x+\xi r)^{D_f-d}}{(\zeta r)^b}\mathrm{d}x \\ &= 2K\rho_1^2\int_0^{\infty}\frac{(\xi y)^{D_f-d}(\xi y+\xi r)^{D_f-d}}{(\xi r)^b}\mathrm{d}(\xi y) \\ &= \xi^{2(D_f-d)+1-b}2K\rho_1^2\int_0^{\infty}\frac{y^{D_f-d}(y+r)^{D_f-d}}{r^b}\mathrm{d}y \\ &= \xi^{2(D_f-d)+1-b}F(r). \end{aligned} \tag{38}$$

A solution to this functional equation is

$$F(r) = F_1 r^{2(D_f-d)+1-b}, \tag{39}$$

where F_1 refers to the proportionality constant associated with the parameters K and ρ_1. This suggests that the spatial correlation function based on the power law gravity model still follows the power law. However, its scaling exponent is significantly less than the power exponent of the density–density correlation function. The Fourier transform of the gravity-based correlation function, Eq. (39), also follows the scaling law, that is,

$$\begin{aligned} S(\xi k) &= \int_{-\infty}^{\infty} F(r)e^{-i2\pi\xi kr}\mathrm{d}r \\ &= C_1\int_{-\infty}^{\infty} r^{1-2(d-D_f)-b}e^{-i2\pi\xi kr}\mathrm{d}r \\ &= \xi^{-2+2(d-D_f)+b}C_1\int_{-\infty}^{\infty}(\xi r)^{1-2(d-D_f)-b}e^{-i2\pi k(\xi r)}\mathrm{d}(\xi r) \\ &= \xi^{-2(D_f-1)+b}S(k), \end{aligned} \tag{40}$$

which suggests a spectral exponent such as $\beta = 2(D_f - 1) - b$. Due to $\beta > 0$, we have $b < 2$.

In urban geography, there are two types of density distributions. One is the spatial distribution with characteristic scales, which can be described by Clark's model and its variants (Batty and Longley 1994; Clark 1951); the other is the distribution without a characteristic scale, which can be described with Smeed's model and the similar models (Batty and Longley 1994; Smeed 1963). The former can be termed scale distribution, and the latter, termed scale-free distribution or scaling distribution. One of the typical scale distributions is exponential distribution, and the mathematical model for this distribution is the well-known Clark's model (Clark 1951). The model can be expressed as

$$\rho(r) = \rho_0 e^{-r/r_0}, \tag{41}$$

where ρ_0 refers to the proportionality constant indicative of the urban density of central location, and r_0 to the characteristic radius of urban density distribution. Based on Eq. (41), the density–density correlation function can be constructed as

$$\begin{aligned} C(r) &= \int_{-\infty}^{\infty}\rho(x)\rho(x+r)\mathrm{d}x \\ &= 2\rho_0^2\int_0^{\infty} e^{-x/r_0}e^{-(x+r)/r_0}\mathrm{d}x \\ &= -r_0\rho_0^2 e^{-r/r_0}\int_0^{\infty} e^{-2x/r_0}\mathrm{d}(-2x/r_0) \\ &= r_0\rho_0^2 e^{-r/r_0}, \end{aligned} \tag{42}$$

which implies that $C(r) = r_0\rho_0\rho(r)$. This suggests that, for the exponential distribution, the spatial correlation function is linearly proportional to the density function.

In other words, the exponential density function is in essence a spatial correlation function. This type of correlation function does not follow the scaling law. The micro-gravity model based on the exponential density distribution is as follows:

$$I(x, r) = K\frac{\rho(x)\rho(x+r)}{r^b} = K\rho_0^2\frac{e^{-(2x+r)/r_0}}{r^b}. \tag{43}$$

If x is fixed to the city center, Eq. (43) will be reduced to a gamma function

$$I_0(r) = K\frac{\rho(0)\rho(r)}{r^b} = K\rho_0^2 r^{-b}e^{-r/r_0}, \tag{44}$$

which is another distance-decay function. This suggests the gravity decay is significantly faster than urban density decay.

Spatial autocorrelation can be theoretically associated with spatial correlation analysis. Based on centralized variable, the spatial autocorrelation function of urban density can be transformed into a spatial correlation function. Suppose that urban density is modeled by Eq. (1) or Eq. (41). The formula of spatial autocorrelation function is as below:

$$C(x, r) = \frac{\sum_{x-r}(\rho(x)-\bar{\rho})(\rho(x+r)-\bar{\rho})}{\sum_{x}(\rho(x)-\bar{\rho})}. \tag{45}$$

where ρ bar denotes arithmetic mean of urban density. If the variable is centralized, the average value equals 0; if the variable is further standardized, the variance equals 1. Based on standardized density variable, Eq. (45) can be reduced to

$$C(x, r) = \sum_{x-r}(\rho(x)\rho(x+r)). \tag{46}$$

For the continuous variables x and r, Eq. (45) can be expressed as integral form, and we have a generalized spatial correlation function as follows:

$$C(x, r) = \lim_{R\to\infty}\frac{1}{R-r}\int_0^{R-r}\rho(x)\rho(x+r)\mathrm{d}x, \tag{47}$$

where R represents the length of sample path measured by urban maximum radius (Liu and Chen 2007). This implies that the spatial autocorrelation function is actually a spatial correlation function.

5 Conclusions

Fractals suggest the optimum structure of physical and human systems, and a fractal object can occupy its space in the most efficient way. Using the ideas from fractals to plan or design cities will possibly help human beings make the best of geographical space. The preconditions of applying fractal geometry to city design and planning are as below: the theoretical framework of fractal cities must be constructed, the mechanism of emergence of urban evolution must be revealed, and the self-organizing city planning methods must be developed. Many important mathematical models of cities can be associated with some type of spatial correlation functions. Spatial correlation is a ubiquitous process in urban evolution. This paper is a theoretical and methodological research of fractal cities by using ideas from scaling, symmetry, correlation, and fractal dimension. The scaling analysis, spectral analysis, and spatial correlation analysis are integrated into a 3S framework for urban studies. The theoretical standpoint of this work is monocentric cities, but possibly the conclusions can be generalized to polycentric cities and even systems of cities. The main points of this paper can be summarized as follows.

First, a new analytical process termed 3S analysis of urban morphology can be developed by means of fractal parameters. Geographical analysis is mainly spatial analysis. The traditional concept of geographical space is based on distance, and the spatial measurement and mathematical modeling are chiefly based on characteristic lengths. Now, new space concept of geographical systems based on fractal dimension has emerged, and the corresponding description and modeling methods are principally based on scaling notion. A series of fractal parameters of cities can be combined with each other by the radial dimension and the spectral exponent. The fractal parameters comprise the one-point correlation dimension (self-similar dimension), the point–point correlation dimension, the Hurst exponent, the self-affine record dimension, and the spatial autocorrelation coefficient. Thus, based on the inverse power law of urban density, scaling analysis, spectral analysis, and spatial correlation analysis can be integrated into a new analytical framework of urban morphology.

Second, the kernel method of the 3S analysis of urban morphology is the scaling analysis, despite the fact that the model foundation is correlation function. The 3S analytical framework comprises scaling analysis, spectral analysis, and spatial correlation analysis. The spatial correlation analysis forms the basis of this analytical process because many important models of cities are or associated with correlation function. The scaling analysis is the critical technology. A great number of nonlinear equations cannot be solved by the conventional approaches. However, if a nonlinear equation is based on scale-free relations, a parameter solution such as fractal dimension relation can be found for it through scaling analysis. The parameter solution can displace the common variable solution. A correlation function is always connected with Fourier transform, and thus the spectral analysis can be employed to open new ways of finding solutions to urban problems and simplify the analytical procedure of urban studies.

Third, the proper ranges of fractal parameter values can be found by using the 3S analysis. The radial dimension of urban morphology is the central correlation dimension (one-point spatial correlation dimension), while the scaling exponent based on the wave spectral density is the density–density correlation dimension (point–point correlation dimension). Using the 3S analysis, we can derive a set of equations of fractal parameters. Using these formulae, we can convert a fractal parameter such as spectral exponent and central correlation dimension into another parameter such as the radial dimension and point–point correlation dimension. The conversion relations reveal the reasonable numerical ranges of the fractal parameters. In particular, the reasonable scale of the radial dimension value ranges from 1.5 to 2. If the value of the radial dimension is greater than 2, the relation between the central correlation and the density–density correlation will fall into confusion. On the other hand, if the radial dimension value is less than 1, the density–density correlation dimension will become invalid. If the value of the radial dimension is less than 1.5, the relation between the macro-level and the micro-level of urban structure will be inharmonious. If and only if the radial dimension comes between 1.5 and 2, various fractal parameters of cities will be logical and thus valid meantime, and this suggests that varied relations of urban morphology become consistent with one another.

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