The Lebesgue Constants of Fourier Partial Sums

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Abstract We give a brief overview of the results on the behavior of the Lebesgue constants for various partial sums of multiple Fourier series. In addition, we establish a new property of the Lebesgue constants concerning its partly increasing behavior.

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1 Introduction

Estimates for partial sums of Fourier series play an important role in many areas of analysis. The norms of the corresponding operators are called the Lebesgue constants. They have numerous applications: in the study of convergence and summability of Fourier series, in approximation and interpolation theories, and even in the study of the stability of homogeneous polynomials; for some of these applications, see, e.g., [1, 3, 13, 15, 41]. When dealing with uniform convergence, the operators are considered in L^1 or, equivalently, in the space of continuous functions C. In dimension one, the situation is clear: for the N-th partial sum, the norm differs from $(4/\pi^2)\ln N$ by a bounded value ([18]; see, e.g., [46, Sect. 2.12]). This topic becomes much more complicated in the multivariate case. The main reason is obvious: contrary to the univariate case, in several variables

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there are numerous ways of ordering the partial sums. This leads to various types of convergence (or divergence), and the difference between some of them is drastic. Given a set $B \subset \mathbb{R}^n$, the Fourier partial sum generated by $B \cap \mathbb{Z}^n$ is defined as

$$S_B(x; f) := \sum_{k \in B \cap \mathbb{Z}^n} \widehat{f}(k) e^{i\langle k,x\rangle},$$

where $f \in L^1(\mathbb{T}^n)$, with $\mathbb{T}^n = [-\pi, \pi)^n$, $\langle k, x\rangle = k_1x_1 + \ldots + k_nx_n$, and

$$\widehat{f}(k) := (2\pi)^{-n} \int_{\mathbb{T}^n} f(x) e^{-i\langle k,x\rangle} dx$$

is the k-th Fourier coefficient of f. The norm of the Fourier operator S_B is called the B-th Lebesgue constant and in the most of the regular situations it can be expressed as

$$\begin{aligned}
\Lambda_B &:= \sup_{\|f\|_{C(\mathbb{T}^n)} \le 1} \|S_B(\cdot; f)\|_{C(\mathbb{T}^n)} \\
&= \sup_{\|f\|_{L^1(\mathbb{T}^n)} \le 1} \|S_B(\cdot; f)\|_{L^1(\mathbb{T}^n)} \\
&= (2\pi)^{-n} \int_{\mathbb{T}^n} \left| \sum_{k \in B \cap \mathbb{Z}^n} e^{i\langle k,x\rangle} \right| dx.
\end{aligned}$$

The core of the theory is specifying the set B and obtaining estimates for the corresponding Lebesgue constants, as sharp as possible. In the study of the B-th Lebesgue constants the compact or non-compact set $B \subset \mathbb{R}^n$ usually depends on a scalar or vectorial parameter N. More precisely, if D is a fixed set, then we study various ND as B. We use notation L_N (instead of Λ_{ND}) for the corresponding Lebesgue constants, while the underlying D is assumed to be known. Last but not least, the sum under the absolute value is called the Dirichlet kernel generated by B. The main feature of the univariate case is that the Dirichlet kernel can be expressed as a relatively simple function, while in the multivariate case this is possible only in certain trivial situations.

A very detailed survey on the Lebesgue constants is given in [29] but it deals mostly with linear methods of summation of the Fourier series than with partial sums. There are other surveys (see, e.g., [16]) but some of them, including [16], have the Lebesgue constants as a section only and in addition, they are outdated now. This makes somewhat difficult to see the state of affairs just with the partial sums. We do this in Sects. 2 and 3 of this paper, without unnecessary details. Note that some new important results have been recently published and we discuss them here.

However, we do not discuss the related problems in more general settings, say, for spherical harmonics or even for compact Lie groups. We only note that the interesting though very specific problem of the behavior of Lebesgue constants for

Lie groups has been studied in the 1970s by Dreseler and later by G. Travaglini and his colleagues (see, e.g., [19]). Furthermore, it turned out that the main open problem for the trigonometric case (see Sect. 2.2) was affirmatively solved in [28] for the case of spherical harmonic expansions. We also do not discuss the results concerning the L^p norms, with $p > 1$ (see, e.g., [31]).

In Sect. 4 we establish a new result: the multidimensional Lebesgue constant is a partly increasing function of N.

2 Lebesgue Constants Generated by the Homothety of a Fixed Set

As mentioned, the sets B are mostly constructed by N-dilations of a fixed set D. Different geometrical properties of D imply very important differences in the behavior of the corresponding partial sums and, as a consequence, very different convergence and approximation properties of the corresponding Fourier series.

2.1 Cubic Partial Sums

If D is a cube with faces parallel to the coordinate hyperplanes, its N-homotheties give for L_N the n-th power of the univariate asymptotic, which results in the growth $\ln^n N$, with the remainder terms dominated by $\ln^{n-1} N$. The situation is very similar if D is a parallelepiped with faces parallel to the coordinate hyperplanes. It is natural that it is N_j-dilated in each direction, $j = 1, 2, \dots, n$. Then L_N is asymptotically equal to $(\frac{4}{\pi^2})^n \ln N_1 \dots \ln N_n$. This asymptotic looks natural, since what is anticipated here is nothing more than the product of the univariate estimates. But even in this case there exists Fefferman's remarkable result [17], which gives an example of a continuous function with everywhere rectangularly divergent partial sums. Considering more general objects within the scope of "polyhedral" case, one can see many non-trivial problems. We will overview them below in Sect. 3.

2.2 Spherical Partial Sums

The case of spherical partial sums, where D is a ball centered at the origin, is completely different. If the polyhedral case is one of the poles of possible estimates of the Lebesgue constants, the lowest one of logarithmic nature, the spherical one is the other pole, with the largest, in a sense, possible bound $CN^{\frac{n-1}{2}}$. What we have there is bilateral power estimate: there are positive constants C_1 and C_2 such that

$$C_1 N^{\frac{n-1}{2}} \le L_N \le C_2 N^{\frac{n-1}{2}}. \tag{1}$$

The lower estimate is known from [21], the upper one was obtained in [22] and simultaneously and independently in Babenko's preprint [3]. The publication of [22] was apparently the reason why Babenko never converted his preprint into a regular paper though a different method involving the Riemann zeta-function and number theory arguments was used by Babenko.

There are many ways to prove (1); the reader can find them and other details in [29] and in the recent books [41] and [23]. The main question posed in [3] (see also [38]) was about the existence of the limit $\lim_{N\to\infty} L_N N^{-\frac{n-1}{2}}$. This question is still open. See other interesting results by Babenko in the same direction in [4].

Attempts to find which sets D are similar to the ball as far as the Lebesgue constants are concerned had been undertaken long ago. First of all, let us mention Yudin's general upper estimate $L_N \le CN^{\frac{n-1}{2}}$ [42] for starlike sets D having finite upper Minkowski measure:

$$\limsup_{\varepsilon\to 0} \frac{\operatorname{mes}\{x : \rho(x, \partial D) < \varepsilon\}}{\varepsilon} < \infty,$$

where $\rho(x, \partial D) := \inf_{y\in\partial D} \rho(x, y)$ and $\rho(x, y)$ is the distance between two points $x, y \in \mathbb{R}^n$. Somewhat less general results but given in more transparent geometric terms can be found in [33] and [44].

In the general lower estimate in the following Theorem 1 (obtained in[27] and as a particular case in [30]), conditions are less restrictive than in the earlier paper [12] and in the later paper [11]. Moreover, the conditions in [27] are local.

Theorem 1 *Let the boundary of a domain D contain a simple (non-intersecting) piece of a surface of smoothness $n/2+1$ in which there is at least one point with non-vanishing principal curvatures. Then there exists a positive constant C depending only on D such that $L_N \ge CN^{\frac{n-1}{2}}$ for large N.*

This result shows that the presence of one boundary "curved" point is sufficient for the Lebesgue constants L_N to be of the growth $N^{\frac{n-1}{2}}$. A similar two-dimensional result is obtained in [20] without smoothness assumptions but for a convex set D. A very general lower estimate in [43] is also of interest.

2.3 Hyperbolic Partial Sums

Since the publication of Babenko's paper [2], linear means with harmonics in "hyperbolic crosses"

$$\begin{aligned} B := ND = \Gamma(N, \gamma) \\ = \{k \in \mathbb{Z}^n : h(N, k, \gamma) = \prod_{j=1}^{n} \left(\frac{|k_j|}{N}\right)^{\gamma_j} \le 1, \quad \gamma_j \ge 1, j = 1, \ldots, n\} \end{aligned}$$

have attracted much attention of approximation and Fourier analysts. The exact degree of growth for the Lebesgue constants of hyperbolic crosses is the same as that for the spherical case, that is $L_N \asymp N^{\frac{n-1}{2}}$. This fact was established in the two-dimensional case independently by Belinsky [6] and by Yudin and Yudin [44], and afterwards it was generalized to the case of arbitrary dimension in [26]. It should be mentioned that these results were proved by step-by-step transition from sums to the corresponding integrals. However, it is by no means surprising if we recall that such a cross does not contain points with non-vanishing curvature (cf. Theorem 1) and that the points of the coordinate hyperplanes do not contribute much to the estimate though their cardinality is infinite. The situation may change if the cross is rotated and then dilated. There are two principal cases there: the "rational turn" and "irrational turn." More precisely, the following results were obtained in [10] (the two-dimensional results were obtained earlier by a different method in [7, 8]).

Let

$$L_j(x) = l_{j1}x_1 + \ldots + l_{jn}x_n, \qquad j = 1, 2, \ldots, n,$$

be linear forms with nonsingular coefficient matrix

$$\Lambda = \{l_{jk}\}, \quad 1 \le j, k \le n, \quad \det \Lambda \neq 0,$$

and

$$B = \{x \in \mathbb{R}^n : \prod_{j=1}^{n} |L_j(x)| \le N^n\}.$$

We call the matrix Λ *rational* if each row of this matrix consists of integers, possibly up to a common factor. In the contrary case, the matrix is said to be *irrational.*

Theorem 2 *The following two statements hold.*

1) *If the matrix Λ is rational, then $L_N \asymp N^{\frac{n-1}{2}}$.*
2) *If Λ is irrational, then there exists an integer N_0 such that the operator L_N is unbounded for all $N > N_0$.*

One can see that this theorem does not deal with all the hyperbolic crosses. Indeed, the proof is based on certain results in geometric number theory and such results are not valid for all crosses. Let us present the most important ingredient for such type results (see, e.g., [37, Th. 3.1.3]), a theorem on bounded linear projections in L^1. In our setting it reads as follows:

If the operator of taking partial sums with respect to some dilation of a given set is bounded, then this set may be represented as a finite union of co-sets of discrete subgroups of the lattice $\mathbb{Z}^n$.

3 Polyhedral Partial Sums

Theorem 1 shows that even one boundary point with non-vanishing curvature affects the rate of growth of the Lebesgue constants. Therefore, the polyhedral case, where D is everywhere flat, illustrates, in a sense, the case of flatness versus curvature.

Of course, many cases in this section can be related to the previous one, where the N-dilations of a polyhedron D are considered. However, the polyhedral case delivers interesting situations when B is constructed not by means of dilations. Therefore, all polyhedral cases are given in one section.

3.1 General Estimates

In this case, there exist two positive constants C_1 and C_2, $C_1 < C_2$, such that for each polyhedron D we have $C_1 \ln^n N \le L_N \le C_2 \ln^n N$. Actually this was proved by Belinsky [6]; nothing new was added in later publications [5, 34]. Thus, we see an essential difference between this case and the spherical one. In the latter case, the Lebesgue constants are of power growth, the worst possible, in a sense, while the former is the best possible estimate one can achieve for partial sums generated by a non-trivial set. We are going to concentrate on two important problems which are essentially of "polyhedral" nature.

Note first that for strips, the results similar to those in Theorem 2 were obtained in [9]. Let us also note that in the bilateral logarithmic estimates for the Lebesgue constants in [45], the constant C_1 in the lower estimate is absolute (absolute constants may depend only on the dimension), while the upper estimate is given as an absolute constant times the number of sides of the polygon D. Recently, this result has been essentially refined in [24] as follows.

Theorem 3 *If B is a convex polyhedron such that* $[0, M_1] \times \ldots \times [0, M_n] \subset B \subset [0, N_1] \times \ldots \times [0, N_n]$, *then*

$$C_1 \prod_{j=1}^{n} \ln(M_j + 1) \le L_B \le C_2\, s \prod_{j=1}^{n} \ln(N_j + 1),$$

where s is the size of the triangulation of B.

As in some other results, a number theory technique was the key tool in the proof of Theorem 3.

3.2 Intermediate Growth

The following question is quite natural.

Can partial sums be defined by sets for which the norms of the corresponding operators have an intermediate rate of growth between the classical power ($N^{\frac{n-1}{2}}$) and logarithmic ($\ln^n N$) rates of growth with respect to the N-dilations of these sets?

Some trivial solutions were suggested in [43], where an intermediate growth is achieved by Cartesian product of the two mentioned cases. Of course, this is possible only for dimension three and higher. Thus, the interesting cases to consider are in dimension two. These have been done by Podkorytov in [35] (similar but weaker results were given in [45]). It is clear that the boundary cannot have points of non-vanishing curvature—otherwise the maximal order of growth $N^{(n-1)/2}$ is immediately achieved. On the other hand, the Lebesgue constants for any polyhedron enjoy the logarithmic estimates. Thus, the only chance for an intermediate growth might be achieved by a "polyhedron" with an infinite number of specially organized sides.

Let C_1 and C_2 denote, as above, positive constants such that $C_1 < C_2$. Then the following result [35] is valid.

Theorem 4 *The following two statements hold.*

1) *For any $p > 2$ there exists a compact convex set D for which $C_1 \ln^p N \le L_N \le C_2 \ln^p N$, $N \ge 2$.*
2) *For any $p \in (0, 1/2)$ and $\alpha > 1$ there exists a compact, convex set D for which $C_1 N^p \ln^{-\alpha p} N \le L_N \le C_2 N^p \ln^{2-2p} N$, $N \ge 2$.*

3.3 Asymptotics

The next question also seems to be very natural.

Is it possible to write a certain asymptotic relation instead of the bilateral logarithmic estimate?

Some partial cases were investigated by Daugavet [14], Kuznetsova [25], Skopina [39]. For example, Kuznetsova generalized Daugavet's result as follows.

Theorem 5 *Let*

$$B := B_{N_1,N_2} = \{(k_1, k_2) : |k_1|/N_1 + |k_2|/N_2 \le 1\}.$$

The asymptotic equality

$$L_N = 32\pi^{-4} \ln N_1 \ln N_2 - 16\pi^{-4} \ln^2 N_1 + O(\ln N_2)$$

holds uniformly with respect to all natural N_1, N_2, and $l = \frac{N_2}{N_1}$.

The case $l = 1$ is the mentioned result of Daugavet. What differentiates both these results from many others is that dilations of a fixed domain are not taken. This is a source of additional difficulties, and nothing is known for noninteger l as well as for the case of higher dimensions.

As for the case where dilations of a fixed domain are considered, an unexpected result was obtained again by Podkorytov [36]. He has shown that there are two main cases. The first one, the aforementioned asymptotic results of Theorem 5 may be referred to, deals with polygons (we are discussing two-dimensional results) with integral or rational slopes of sides. In this case one can show that the estimates change insignificantly if one considers the corresponding integrals instead of the sums, that is, the Fourier transform $\widehat{\chi}_{ND}$ of the indicator function of the N-dilation of the corresponding set D. In other words, the Dirichlet kernel is well approximated by $\widehat{\chi}_{ND}$. This circumstance allows one to obtain the logarithmic asymptotics, namely L_N is equivalent to both $\ln^2 N$ and $\int_{\mathbb{T}^2} |\widehat{\chi}_{ND}(x)|\,dx$ (see [39]).

In the second case, that is, when at least one slope is irrational, the situation changes qualitatively: the upper limit and the lower limit of the ratio of L_N and $\ln^2 N$, as $N \to \infty$, may be different. In other words, in this case the behavior of the Fourier transform of the indicator function of ND is not representative of the behavior of the corresponding partial sums. The quantitative estimate of this phenomenon was given in [36]. The main shortcoming of that work is that this result is true only for a very scarce number of cases. This uncertainty was partially removed by Nazarov and Podkorytov [32].

4 Partial Increasing of Lebesgue Constants

In this section we assume again that $B = ND$, where $N \in [0, \infty)$ is a continuous parameter and $D \subset \mathbb{R}^n$ is the closure of an open bounded star domain with respect to the origin.

In case of $n = 1$ and $D = [-1, 1]$, Szegö [40] (Fejér in [18] for large N; see also [46, Ch. 2, Ex. 24]) proved that L_N is increasing in N. However, an extension of this result to the multivariate Lebesgue constants is unknown. Here, we prove the following weakened version of Szegö's theorem.

Theorem 6 *For any $d \in \mathbb{N}$ and any $N \in [0, \infty)$, the following inequalities hold:*

$$L_{N/d} \le L_N \le L_{dN}. \tag{2}$$

Proof For any continuous and 2π-periodic in each variable function f on $\mathbb{R}^n$, we consider the Fourier operator

$$S_{ND}(x; f) = S_{ND}(x_1, \dots, x_n; f) = (2\pi)^{-n} \int_{\mathbb{T}^n} f(t) \sum_{k \in ND \cap \mathbb{Z}^n} e^{i\langle k, x-t\rangle} dt.$$

Next, for a fixed $d \in \mathbb{N},\ d > 1$, and a continuous and 2π-periodic in each variable function f on $\mathbb{R}^m$, we define an averaging linear operator by the formula

$$Q_{ND,d}(x; f) := d^{-n} \sum_{s_1=1}^{d-1} \dots \sum_{s_n=1}^{d-1} S_{ND}(x_1+2\pi s_1/d, \dots, x_n+2\pi s_n/d; f). \quad (3)$$

Then the integral representation for $Q_{ND}(\cdot, f)$ is given by the formula

$$Q_{ND,d}(x; f) = (2\pi)^{-n} \int_{\mathbb{T}^n} f(t) \sum_{l \in (N/d)D \cap \mathbb{Z}^n} e^{i\, d\langle l, x-t\rangle} dt. \quad (4)$$

Indeed, for any $s = (s_1, \dots, s_n)$ and $k \in ND \cap \mathbb{Z}^n$, we obtain by simple calculation

$$d^{-n} \sum_{s_1=1}^{d-1} \dots \sum_{s_n=1}^{d-1} e^{(2\pi/d)i\langle k,s\rangle} = \begin{cases} 1, \ (1/d)k \in \mathbb{Z}^n, \\ 0, \ (1/d)k \notin \mathbb{Z}^n. \end{cases} \quad (5)$$

Since D is a star domain with respect to the origin, the vector $l = (1/d)k$ belongs to $(N/d)D$. Therefore, (4) follows from (5).

Furthermore, we see from (4) that for a continuous function $f(t) = \varphi(dt)$ on $\mathbb{R}^n$, where φ is 2π-periodic in each variable, the following representation holds:

$$Q_{ND,d}(x; \varphi(d\cdot)) = (2\pi)^{-n} \int_{\mathbb{T}^n} \varphi(t) \sum_{l \in (N/d)D \cap \mathbb{Z}^n} e^{i(d\langle l,x\rangle - \langle l,t\rangle)} dt. \quad (6)$$

In addition, the following relations follow from (3):

$$\|Q_{ND}\| := \sup_{\|f\|_{C(\mathbb{T}^n)} \le 1} \|Q_{ND}(\cdot; f)\|_{C(\mathbb{T}^n)} \le \sup_{\|f\|_{C(\mathbb{T}^n)} \le 1} \|S_{ND}(\cdot; f)\|_{C(\mathbb{T}^n)} = L_N. \quad (7)$$

Then we obtain from (6) and (7)

$$\begin{aligned} L_N \ge \|Q_{ND}\| &\ge \sup_{\|\varphi\|_{C(\mathbb{T}^n)} \le 1} \|Q_{ND}(\cdot; \varphi(d\cdot))\|_{C(\mathbb{T}^n)} \\ &= \sup_{\|\varphi\|_{C(\mathbb{T}^n)} \le 1} \|S_{(N/d)D}(\cdot; \varphi)\|_{C(\mathbb{T}^n)} = L_{N/d}. \end{aligned}$$

Hence (2) is established. □

Remark 7 We say that a function $h : [0, \infty) \to \mathbb{R}^1$ is *partly increasing* if for any $d \in \mathbb{N}$ and $N \in [0, \infty)$, the following inequality holds:

$$h(N/d) \le h(N). \quad (8)$$

Theorem 6 is equivalent to the statement that L_N is partly increasing in N. It is obvious that an increasing function is partly increasing. The following counterexample shows that the converse of this statement is not valid. The question as to whether $L_N := L_{ND}$ is an increasing function of N for certain domains D in $\mathbb{R}^n$ remains open.

Example 8 For fixed numbers $\alpha > 0$ and $N_0 > 0$ we define a function

$$h_0(N) := \begin{cases} N^\alpha + (1-2^{-\alpha})(2m-1)^\alpha N_0^\alpha, & N \in [(2m-2)N_0, (2m-1)N_0],\ m \in \mathbb{N}, \\ N^\alpha, & N \in [(2m-1)N_0, 2mN_0],\ m \in \mathbb{N}. \end{cases}$$

The function h_0 is not increasing in some neighborhoods of the points $(2m-1)N_0$, $m \in \mathbb{N}$. Next, it is easy to verify that (8) holds for $N \in [0, 2N_0)$. Then (8) can be proved for all $N \in [0, \infty))$ by induction.

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References

1. L. Aizenberg, N. Tarkhanov, Stable expansions in homogeneous polynomials, Complex analysis and dynamical systems III. Contemp. Math. **455**, 1–22 (2008). Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI
2. K.I. Babenko, Approximation by trigonometric polynomials in a certain class of periodic functions of several variables. Dokl. Akad. Nauk SSSR **132**, 982–985 (1960) (Russian). English transl. in Soviet Math. Dokl. **1**, 672–675 (1960)
3. K.I. Babenko, On the mean convergence of multiple Fourier series and the asymptotics of the Dirichlet kernel of spherical means. Inst. Prikl. Mat. Akad. Nauk SSSR, Moscow, Preprint No. 52 (1971) (Russian)
4. K.I. Babenko, On the asymptotics of the Dirichlet kernel of spherical means of multiple Fourier series. Dokl. Akad. Nauk SSSR **243**, 1097–1100 (1978) (Russian). English transl. in Soviet Math. Dokl. **19**, 1457–1461 (1978)
5. S.P. Baiborodov, Lebesgue constants of polyhedra. Mat. Zametki **32**, 817–822 (1982) (Russian). English transl. in Math. Notes **32**, 895–898 (1982)
6. E. Belinsky, Behavior of the Lebesgue constants of certain methods of summation of multiple Fourier series, in *Metric Questions of the Theory of Functions and Mappings* (Naukova Dumka, Kiev, 1977), pp. 19–39 (Russian)
7. E. Belinsky, Some properties of hyperbolic partial sums of Fourier series and Fourier integrals. Dokl. Acad. Sci. Ukraine Ser. A **10**, 869–870 (1978) (Russian)
8. E. Belinsky, Some properties of hyperbolic partial sums, in *Theory of Functions and Mappings* (Naukova Dumka, Kiev, 1979), pp. 28–36 (Russian)
9. E. Belinsky, On the growth of Lebesgue constants of partial sums generated by certain unbounded sets, in *Theory of Mappings and Approximation of Functions* (Naukova Dumka, Kiev, 1983), pp. 18–20 (Russian)

10. E. Belinsky, E. Liflyand, Behavior of the Lebesgue constants of hyperbolic partial sums. Mat. Zametki **43**, 192–196 (1988) (Russian). English transl. in Math. Notes **43** (1988), 107–109
11. L. Brandolini, Fourier transform of characteristic functions and Lebesgue constants for multiple Fourier series. Colloquium Math. **LXV**, 51–59 (1993)
12. M. Carenini, P.M. Soardi, Sharp estimates for Lebesgue constants. Proc. Am. Math. Soc. **89**, 449–452 (1983)
13. L. Colzani, P.M. Soardi, L^p norms of certain kernels on the N-dimensional torus. Trans. Am. Math. Soc. **266**, 617–627 (1981)
14. I.K. Daugavet, On the Lebesgue constants for double Fourier series. Meth. Comput. Leningrad Univ. **6**, 8–13 (1970) (Russian)
15. P. Dencker, W. Erb, Y. Kolomoitsev, T. Lomako, Lebesgue constants for polyhedral sets and polynomial interpolation on Lissajous-Chebyshev nodes. J. Complexity **43**, 1–27 (2017)
16. M. Dyachenko, Some problems in the theory of multiple trigonometric series. Uspekhi Mat. Nauk **47**(5), 97–162 (1992) (Russian). English transl. in Russian Math. Surveys **47**(5), 103–171 (1992)
17. C. Fefferman, On the divergence of multiple Fourier series. Bull. Am. Math. Soc. **77**, 191–195 (1971)
18. L. Fejér, Lebesguessche Konstanten und divergente Fourierreihen. J. Reine Angew. Math. **138**, 22–53 (1910)
19. S. Giulini, G. Travaglini, Sharp estimates for Lebesgue constants on compact lie groups. J. Funct. Anal. **68**, 106–116 (1986)
20. E. Gutierrez Gonzales, A lower bound for two-dimensional Lebesgue constants. Vestnik S.-Peterburg. Univ. Mat. **26**, 119–121 (1993) (Russian). English transl. in Vestnik St. Peterburg Univ. Math. **26**, 69–71 (1993)
21. V.A. Ilyin, Problems of localization and convergence for Fourier series in fundamental systems of the Laplace operator. Uspekhi Mat. Nauk **23**, 61–120 (1968) (Russian). English transl. in Russian Math. Surveys **23** (1968), 59–116
22. V.A. Ilyin, S.A. Alimov, Conditions for the convergence of expansions corresponding to self-adjoint extensions of elliptic operators. I. Diff. Urav. **7**, 670–710 (1971) (Russian). English transl. in Differential Equations **7**, 516–543 (1971)
23. A. Iosevich, E. Liflyand, *Decay of the Fourier Transform: Analytic and Geometric Aspects* (Birkhauser, Basel, 2014)
24. Y. Kolomoitsev, T. Lomako, On the growth of Lebesgue constants for convex polyhedra. Trans. Am. Math. Soc. (to appear). https://doi.org/10.1090/tran/7225; see also https://arxiv.org/abs/1801.00608
25. O. Kuznetsova, The asymptotic behavior of the Lebesgue constants for a sequence of triangular partial sums of double Fourier series. Sib. Mat. Zh. **XVIII**, 629–636 (1977) (Russian). English transl. in Siberian Math. J. **18**, 449–454 (1977)
26. E. Liflyand, Exact order of the Lebesgue constants of hyperbolic partial sums of multiple Fourier series. Mat. Zametki **39**, 674–683 (1986) (Russian). English transl. in Math. Notes **39**, 369–374 (1986)
27. E. Liflyand, Sharp estimates of the Lebesgue constants of partial sums of multiple Fourier series. Trudy Mat. Inst. Im. V. A. Steklova **180**, 151–152 (1987) (Russian). English transl. in Proc. Steklov Math. Inst. **180**, 176–177 (1989)
28. E. Liflyand, Lebesgue constants of Cesàro means for spherical harmonic expansions. Acta Sci. Math. (Szeged) **64**, 215–222 (1998)
29. E. Liflyand, Lebesgue constants of multiple Fourier series. Online J. Anal. Combin. **1**, 112 (2006)
30. E.R. Liflyand, A.G. Ramm, A.I. Zaslavsky, Estimates from below for Lebesgue constants. J. Fourier Anal. Appl. **2**, 287–301 (1996)
31. J. Marshall Ash, L. De Carli, Growth of L^p Lebesgue constants for convex polyhedra and other regions. Trans. Am. Math. Soc. **361**, 4215–4232 (2009)

32. F. Nazarov, A. Podkorytov, On the behavior of the Lebesgue constants for two-dimensional Fourier sums over polygons. Algebra i Analiz **7**, 214–238 (1995) (Russian). English transl. in St.-Petersburg Math. J. **7**, 663–680 (1995)
33. A.N. Podkorytov, On the Lebesgue constants for double Fourier series. Vestnik Leningrad. Univ. Matem. **7**, 79–84 (1977) (Russian). English transl. in Vestnik Leningrad Univ. Math. **10**, 187–192 (1982)
34. A.N. Podkorytov, The order of growth of the Lebesgue constants of Fourier sums over polyhedra. Vestn. Leningr. Univ. Matem. **7**, 110–111 (1982) (Russian)
35. A.N. Podkorytov, Intermediate rates of growth of Lebesgue constants in the two-dimensional case, Numerical methods and questions on the organization of calculations, part 7 notes of scientific seminars of the V.A. Steklov Math. Inst. Leningrad. Branch, Acad. Sci. USSR, Nauka, Leningrad **139**, 148–155 (1984) (Russian). English transl. in J. Soviet Math. **32**, 276–282 (1987)
36. A.N. Podkorytov, Asymptotic behavior of the Dirichlet kernel of Fourier sums with respect to a polygon. Zap. Nauch. Sem. LOMI **149**, 142–149 (1986) (Russian). English transl. in J. Soviet Math. **42**, 1640–1646 (1988)
37. W. Rudin, Fourier analysis on groups, in *Interscience Tracts in Pure and Applied Mathematics*, No. 12 (Interscience Publishers (a division of John Wiley and Sons), New York, 1962)
38. H.S. Shapiro, Lebesgue constants for spherical partial sums. J. Appr. Theory **13**, 40–44 (1975)
39. M.A. Skopina, Lebesgue constants of multiple de la Vallée Poussin sums over polyhedra. Zap. Nauch. Sem. LOMI **125**, 154–165 (1983) (Russian). English transl. in J. Soviet Math. **26**, 2404–2413 (1984)
40. G. Szegö, Über die Lebesgueschen Konstanten bei den Fourierschen Reihen. Math. Zeitschrift **9**, 163–166 (1921)
41. R.M. Trigub, E.S. Belinsky, *Fourier Analysis and Approximation of Functions* (Springer, Netherlands, 2004)
42. V.A. Yudin, Behavior of Lebesgue constants. Mat. Zametki **17**, 401–405 (1975) (Russian). English transl. in Math. Notes **17**, 369–374 (1975)
43. V.A. Yudin, A lower bound for Lebesgue constants. Mat. Zametki **25**, 119–122 (1979) (Russian). English transl. in Math. Notes **25**, 63–65 (1979)
44. A.A. Yudin, V.A. Yudin, Discrete imbedding theorems and Lebesgue constants. Mat. Zametki **22**, 381–394 (1977) (Russian). English transl. in Math. Notes **22**, 702–711 (1977)
45. A.A. Yudin, V.A. Yudin, Polygonal Dirichlet Kernels and growth of Lebesgue constants. Mat. Zametki **37**, 220–236 (1985) (Russian). English transl. in Math. Notes **37**, 124–135 (1985)
46. A. Zygmund, *Trigonometric Series: Vols. I, II*, Second edition, reprinted with corrections and some additions (Cambridge University Press, London, 1968)