

Chapter 2

Introduction to Multi-objective Optimization and Decision-Making Analysis



2.1 Introduction

In today's world, overcoming complicated real-world engineering challenges with big data has become an indispensable need for specialists and researchers in order to make better use of their time and resources. These engineering challenges are mostly addressed as a large-scale, non-convex optimization problem having a nonlinear, mixed-integer nature. As a result, the application of the optimization is significantly increased for solving real-world engineering challenges and achieving an optimal solution. Initially, optimization problems were organized as simple single-objective mathematical models in which one given objective function needed to be minimized or maximized. More precisely, a single-objective optimization problem consists of an individual objective function subject to some specified constraints in such a way that solving this particular optimization problem leads to finding an individual optimal solution. That is to say that the main goal of minimization or maximization of a single-objective optimization problem is to obtain the minimum or maximum value of the corresponding objective function, provided that this value does not violate any specified constraints—an optimal solution.

Conversely, in multi-objective optimization problems (MOOPs), a set of objective functions that are often conflicting should be simultaneously minimized or maximized. In this circumstance, solving a MOOP results in finding a set of different compromise solutions called a Pareto-optimal solution set or non-dominated optimal solution set. With that in mind, only one solution should be chosen from the Pareto-optimal solution set. Unlike a single-objective optimization problem, solving a MOOP is, therefore, composed of three important and completely different steps: (1) formation of a mathematical model; (2) optimization; and (3) decision-making [1]. In the optimization step, the Pareto-optimal solution set is determined; however, in the decision-making step, the most satisfactory solution is chosen from the Pareto-optimal solution set based on the preferences of the decision maker.

To illustrate, consider a two-objective optimization problem for operation of a typical engineering system. Simultaneous minimization of operational costs and maximization of system reliability are two conflicting objectives considered in this optimization problem. This means that minimization of operational costs brings about a reduction in system reliability, and maximization of system reliability gives rise to an increase in operational costs. As a consequence, there is no individual solution that can simultaneously optimize these two conflicting objectives. In this condition, a Pareto-optimal solution set is achieved by solving this two-objective optimization problem. As an instance, take into account two solutions in the Pareto-optimal solution set. If the first solution, in terms of the operational costs, can overcome the second solution—given that the first solution has a lower cost than the second one—this solution with respect to system reliability cannot overcome the second solution—given that the first solution definitely has less reliability than the second one. Put another way, compared to the second solution, the first solution is a non-dominated response/output in terms of operational costs—the lowest operational costs—when it is a dominated response/output with respect to system reliability—the most unfavorable performance. After determination of the Pareto-optimal solution set, or non-dominated solution set, the decision maker, by considering its preferences, should choose the final solution so that a trade-off is made up between operational costs and system performance. Given the preferences of the decision maker, it is also possible that system performance would have a higher degree of significance compared to the operational costs and vice versa. Hence, the preferences of the decision maker dramatically affect the choice of the final solution from the Pareto-optimal solution set.

In the MOOPs, the complexities and difficulties of the solution process are dramatically increased in view of introducing and integrating new concepts compared to single-objective optimization problems. In addition, multi-objective optimization algorithms (MOOAs) are needed to solve the MOOPs. In related literature, many MOOAs have been developed to deal with a wide range of multi-objective optimization problems. However, each MOOA is appropriate for solving only a specific range of MOOPs. The choice of a well-suited MOOA depends first on a full understanding of the MOOP and its characteristics and second on having full knowledge of the architecture and features of the different MOOAs. Due to the different concepts of the optimization in the MOOPs and the diversity and variety of MOOAs, it is thoroughly indispensable to clarify the fundamental concepts of multi-objective optimization and provide a suitable classification for the MOOAs. In this chapter, then, the authors will concentrate on the following targets.

- Target 1: Providing a brief introduction associated with fundamental concepts of optimization in the MOOPs.
- Target 2: Presenting a brief overview pertaining to the classification of the MOOAs.

In this chapter, the authors do not present all of the details related to optimization concepts in the MOOPs, as it is assumed that the reader is already familiar with the elementary concepts of optimization. The main focus, then, will be on the fundamental concepts of optimization in the MOOPs, particularly the fundamental

concepts discussed in the MOOPs that will be widely employed later in this book. Where appropriate, though, the reader will be referred to related studies that cover more details of concepts of optimization in the MOOPs.

The remainder of this chapter is arranged as follows: First, the necessity of using the multi-objective optimization process is reviewed in Sect. 2.2. Then, the fundamental concepts of optimization in the MOOPs are described in Sect. 2.3. In Sect. 2.4, the classification of the MOOAs is addressed from different points of view. A fuzzy satisfying method is also expounded upon in Sect. 2.5. Finally, the chapter ends with a brief summary and some concluding remarks in Sect. 2.6.

2.2 Necessity of Using Multi-objective Optimization

In a very general sense, many objective functions can be employed in real-world engineering problems. These objective functions usually have a conflicting, noncommensurable, and correlated nature with each other. In this way, the integration of objective functions of a MOOP, with the aim of forming a single-objective optimization problem and then employing the developed single-objective solvers, is a common misconception.

The conversion of a MOOP into a single-objective optimization problem causes the decision-making step to be transferred before the optimization step. It is, therefore, very difficult to specify the preferences of the decision maker before the optimization, and it may not match the obtained solution of the single-objective optimization problem with a determined solution of the MOOP that is selected from the Pareto-optimal solution set by the decision maker. Nevertheless, the implementation of the optimization process in a MOOP, without turning it into a single-objective problem, can force the decision-making step to be placed after the optimization step or these two steps to be transformed into a hybrid process. This structure helps the decision maker to better understand the MOOP and be able to make a more knowledgeable choice through the Pareto-optimal solution set with regard to its preferences. Achieving the Pareto-optimal solution set also enables the decision maker to perform a thorough analysis regarding the interdependencies among decision-making variables, objective functions, and constraints. Acquiring knowledge about these interactions can be employed in order to reconsider the mathematical model of the optimization problem with the aim of increasing the chances of determining a solution that not only aligns better with reality but also better matches the preferences of the decision maker. As a result, if an optimization problem consists of multiple conflicting, noncommensurable, and correlated objective functions, the most reasonable strategy is to take advantage of the multi-objective optimization process in order to solve the problem.

2.3 Fundamental Concepts of Optimization in the MOOPs

The concepts of optimization in the MOOPs are different from those in single-objective problems. In this section, the fundamental concepts of optimization in the MOOPs are briefly addressed. As previously mentioned, the optimization process in the MOOPs includes three general steps: (1) formation of a mathematical model; (2) optimization; and, (3) decision-making [1]. The mathematical description of an optimization problem—the formulation of an optimization problem by defining its decision-making variables, objective functions, and constraints—is considered as the first step in the optimization process. The next two steps in the optimization process depend on the structure and characteristics of the problem. Many studies carried out in the context of the optimization process implicitly suppose that the MOOP has been correctly determined. In practice, however, this assumption is not necessarily valid on all occasions. As a consequence, providing a rigorous mathematical model by considering the structure and characteristics of a MOOP can be practically helpful in the optimization process.

2.3.1 Mathematical Description of a MOOP

Technically speaking, a MOOP consists of multiple objective functions in such a way that these functions ordinarily have a conflicting, noncommensurable, and correlated nature with each other. The mathematical description of a MOOP can generally be expressed according to Eqs. (2.1) and (2.2) [1]:

$$\begin{aligned}
 & \underset{x \in X}{\text{Minimize}} \quad F(x) = [f_1(x), \dots, f_a(x), \dots, f_A(x)]; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A\} \\
 & \text{subject to :} \\
 & \quad G(x) = [g_1(x), \dots, g_b(x), \dots, g_B(x)] = 0; \quad \forall \{B \geq 0\}, \forall \{b \in \Psi^B\} \\
 & \quad H(x) = [h_1(x), \dots, h_e(x), \dots, h_E(x)] \leq 0; \quad \forall \{E \geq 0\}, \forall \{e \in \Psi^E\}
 \end{aligned} \tag{2.1}$$

$$\begin{aligned}
 x = & [x_1, \dots, x_v, \dots, x_{NDV}]; \quad \forall \{v \in \Psi^{NDV}, \Psi^{NDV} = \Psi^{NCDV} + NDDV, x \in X\}, \\
 & \forall \{x_v^{\min} \leq x_v \leq x_v^{\max} | v \in \Psi^{NCDV}\}, \\
 & \forall \{x_v \in \{x_v(1), \dots, x_v(w), \dots, x_v(W_v)\} | v \in \Psi^{NDDV}\}
 \end{aligned} \tag{2.2}$$

The explanations associated with the parameters and variables from Eqs. (2.1) and (2.2) were previously defined in Sect. 1.2.1 of Chap. 1. The vector of objective functions addresses the illustration of the vector of decision-making variables and contains the values of the objective functions, as given by Eq. (2.3):

$$z = F(x) = [f_1(x), \dots, f_a(x), \dots, f_A(x)]; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A\} \quad (2.3)$$

The explanations related to the parameters and variables from Eq. (2.3) were formerly described in Sect. 1.2.1 of Chap. 1.

2.3.2 Concepts Associated with Efficiency, Efficient frontier, and Dominance

In this section, efficiency, efficient frontier, and dominance, as main concepts of the MOOP, are thoroughly demonstrated.

Efficiency definition: A vector of decision-making variables $x^{**} \in X$ is efficient in the MOOP, given in Eqs. (2.1) and (2.2), if there is no another vector of decision-making variables like $x^* \in X$ so that $F(x^*) \leq F(x^{**})$ with at least one $f_a(x^*) < f_a(x^{**})$. Otherwise, the vector of decision-making variables $x^{**} \in X$ is inefficient [2].

Efficient frontier definition: The complete set of efficient vectors of decision-making variables is known as the efficient frontier [2].

Dominance definition: A vector of objective functions $F(x^{**}) \in Z$ is non-dominated in the MOOP, given in Eqs. (2.1) and (2.2), if there is no another vector of objective functions like $F(x^*) \in Z$ in such a way that $F(x^*) \leq F(x^{**})$ with at least one $f_a(x^*) < f_a(x^{**})$. Otherwise, the vector of objective functions $F(x^{**}) \in Z$ is dominated or has failed [2]. Put another way, the vector of objective functions $F(x^*) \in Z$ overcomes the vector of objective functions $F(x^{**}) \in Z$ in the MOOP, if the following two conditions are simultaneously met:

Condition 1: All components or elements of the vector of objective functions $F(x^*) \in Z$ are not worse than the corresponding components or elements of the vector of objective functions $F(x^{**}) \in Z$, as given by Eq. (2.4):

$$F(x^*) \leq F(x^{**}) \vee f_a(x^*) \leq f_a(x^{**}); \text{for all objective functions, } \forall \{a \in \Psi^A\} \quad (2.4)$$

Note that the symbol “ \vee ” in Eq. (2.4) represents the operator “or.”

Condition 2: At least one of the components or elements of the vector of objective functions $F(x^*) \in Z$ is better than components or elements of the vector of objective functions $F(x^{**}) \in Z$, as presented by Eq. (2.5):

$$f_k(x^*) < f_k(x^{**}); \text{for at least one objective function, } \forall \{k \in \Psi^A\} \quad (2.5)$$

Given what has been described, it can be found that the definitions of efficiency and dominance are analogous for all practical aims. However, it must be noted that the concepts associated with efficiency and dominance usually refer to the vector of decision-making variables in a feasible decision-making space and vector of objective functions in a feasible objective space, respectively.

2.3.3 Concepts Pertaining to Pareto Optimality

Unlike a single-objective optimization problem, there is no a single solution that can simultaneously optimize all objective functions in a MOOP. The objective functions of a MOOP are often in conflict with each other, and parameters related to some objective functions may not lead to optimality for other objective functions—even though these parameters can sometimes give rise to worse amounts for other objective functions. As a consequence, in contrast to solving a single-objective optimization problem that yields a single solution, solving a MOOP results in finding a set of solutions that represents a trade-off among the different objective functions. These solutions are also known collectively as a Pareto-optimal solution set or a non-dominated optimal solution set. With that in mind, Pareto optimally, weakly, and appropriately Pareto optimal are other fundamental concepts of the MOOPs which are briefly reviewed in this section.

Definition of a Pareto-optimal solution: A vector of decision-making variables $x^{**} \in X$ is a Pareto-optimal solution in the MOOP, given in Eqs. (2.1) and (2.2), if there is no another vector of decision-making variables like $x^* \in X$ so that $F(x^*) \leq F(x^{**})$ and $f_a(x^*) < f_a(x^{**})$ for at least one objective function. Otherwise, the vector of decision-making variables $x^{**} \in X$ is not a Pareto-optimal solution [1–3]. In other words, the vector of decision-making variables $x^{**} \in X$ is a Pareto-optimal solution if there is no another vector of decision-making variables like $x^* \in X$ that can simultaneously satisfy the conditions presented in Eqs. (2.4) and (2.5). That is to say that the vector of decision-making variables $x^{**} \in X$ is a Pareto-optimal solution if there does not exist another vector of decision-making variables like $x^* \in X$ that can improve at least one of the objective functions of the MOOP without worsening other objective functions. The set of Pareto-optimal vectors of decision-making variables is taken into account as $P(X)$. Mutually, a vector of objective functions is a Pareto-optimal solution if the corresponding vector of decision-making variables is a Pareto-optimal solution. In this way, the set of Pareto-optimal vectors of objective functions is considered as $P(Z)$. Many algorithms used for solving multi-objective optimization problems provide solutions that are not Pareto-optimal. These solutions can, however, meet other criteria. One of the most important of these criteria that can be very useful and effective in real-world MOOPs and provide useful information for the decision maker is the weak Pareto-optimal solution, which can be explained as follows:

Definition of a weak Pareto-optimal solution: A vector of decision-making variables $x^{**} \in X$ is a weak Pareto-optimal solution in the MOOP, given in Eqs. (2.1) and (2.2), if there is no another vector of decision-making variables like $x^* \in X$ in such a way that $F(x^*) < F(x^{**})$. Otherwise, the vector of decision-making variables $x^{**} \in X$ is not a weak Pareto-optimal solution [1–3]. In simple terms, the vector of decision-making variables $x^{**} \in X$ is a weak Pareto-optimal solution if there does not exist another vector of decision-making variables like $x^* \in X$ so that the response/output obtained by this vector of decision-making variables in all objective functions of the MOOP is better than the response/output calculated by the vector of decision-making variables $x^{**} \in X$ in the corresponding objective functions. Or, the

vector of decision-making variables $x^{**} \in X$ is a weak Pareto-optimal solution if there is no another vector of decision-making variables like $x^* \in X$ that can simultaneously improve all objective functions of the MOOP. The set of weak Pareto-optimal vectors of decision-making variables is considered as $WP(X)$. Correspondingly, a vector of objective functions represents a weak Pareto-optimal solution if the corresponding vector of decision variables is a weak Pareto-optimal solution. In this case, the set of weak Pareto-optimal vectors of objective functions is taken into account as $WP(Z)$. As a result, the set of Pareto-optimal vectors of decision-making variables belongs to a larger set called the set of weak Pareto-optimal vectors of decision-making variables. Accordingly, if the vector of decision-making variables $x^{**} \in X$ is the Pareto-optimal solution, then this vector is the weak Pareto-optimal solution. However, if the vector of decision-making variables $x^{**} \in X$ is the weak Pareto-optimal solution, then this vector is not necessarily the Pareto-optimal solution. Each of the available responses or outputs in the Pareto-optimal solution set can be classified as either an appropriate or an inappropriate Pareto-optimal response or output. In related literature, there are different definitions for the appropriate Pareto-optimal concept, which are not equivalent. Here, the definition employed for an appropriate or inappropriate Pareto-optimal concept is derived according to Geoffrion [4].

Definition of an appropriate Pareto-optimal solution: A vector of decision-making variables $x^{**} \in X$ is an appropriate Pareto-optimal solution in the MOOP, given in Eqs. (2.1) and (2.2), if this vector is the Pareto-optimal solution and if there is some real number $J > 0$ not only for each objective function a and for each of the other vectors of decision-making variables like $x^* \in X$ satisfying $f_a(x^*) < f_a(x^{**})$, but also that there is at least one objective function k in the MOOP such that $f_k(x^{**}) < f_k(x^*)$ and $\{f_a(x^{**}) - f_a(x^*)\} / \{f_k(x^*) - f_k(x^{**})\} \leq J$. The quotient of the fraction $\{f_a(x^{**}) - f_a(x^*)\} / \{f_k(x^*) - f_k(x^{**})\}$ refers to a compromise in the MOOP; that is, it indicates an increase in the objective function k originating from a decrease in the objective function a . Put simply, a Pareto-optimal solution is an appropriate Pareto-optimal solution if there exists at least one pair of objective functions such that a confined decrease in one objective function is possible only with an increase in the other objective function. The set of appropriate Pareto-optimal vectors of decision-making variables is $AP(X)$. Mutually, a vector of objective functions is an appropriate Pareto-optimal solution if the corresponding vector of decision-making variables is an appropriate Pareto-optimal solution. In this manner, the set of appropriate Pareto-optimal vectors of objective functions is considered as $AP(Z)$.

As a result, the set of appropriate Pareto-optimal vectors of decision-making variables belongs to a larger set called the set of Pareto-optimal vectors of decision-making variables. Therefore, if the vector of decision-making variables $x^{**} \in X$ is the appropriate Pareto-optimal solution, then this vector is the Pareto-optimal solution. However, if the vector of decision-making variables $x^{**} \in X$ is the Pareto-optimal solution, then this vector is not necessarily the appropriate Pareto-optimal solution.

Concepts pertaining to Pareto optimality and relationships among these concepts are demonstrated in Fig. 2.1. In Fig. 2.1, the set of appropriate Pareto-optimal

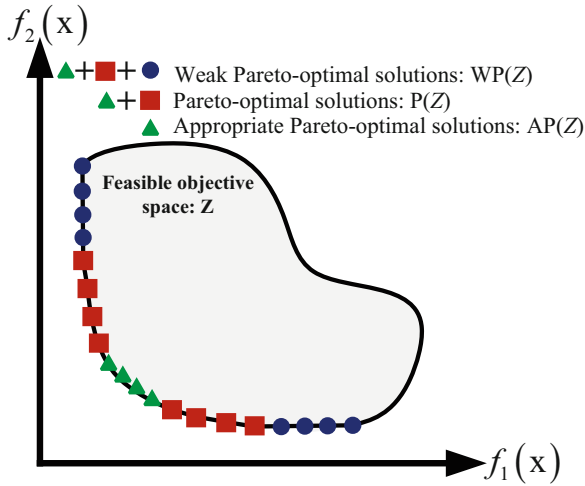


Fig. 2.1 Concepts pertaining to Pareto optimality

responses and outputs is shown with solid green triangles. The set of Pareto-optimal responses and outputs is illustrated as a sum of the solid green triangles and solid red squares. The set of weak Pareto-optimal responses and outputs is depicted as a sum of solid green triangles, solid red squares, and solid blue circles. Also from Fig. 2.1, it can be seen that the set of appropriate Pareto-optimal responses and outputs belongs to a larger set called the set of Pareto-optimal responses and outputs, as previously mentioned. Moreover, it can be seen that the set of Pareto-optimal responses/outputs belongs to a larger set called the set of weak Pareto-optimal responses/outputs, as stated earlier.

2.3.4 Concepts Related to the Vector of Ideal Objective Functions and the Vector of Nadir Objective Functions

Suppose that in the MOOP, given in Eq. (2.1) and (2.2), the objective functions are bounded on the feasible objectives space. In this circumstance, the upper and lower bounds associated with the set of Pareto-optimal responses and outputs in the feasible objectives space can provide very useful information about the MOOP. For this MOOP, the lower bounds related to the set of Pareto-optimal responses and outputs are available in the vector of ideal objective functions— $z^{\text{ideal}} \in \mathfrak{R}^A$ [1–3]. The vector of ideal objective functions is defined using Eq. (2.6):

$$z^{\text{ideal}} = [z_1^{\text{ideal}}, \dots, z_a^{\text{ideal}}, \dots, z_A^{\text{ideal}}]; \quad \forall \{a \in \Psi^A\} \tag{2.6}$$

The component or element a relevant to the vector of ideal objective functions z_a^{ideal} can be obtained by minimizing the objective function a of the MOOP as a single-objective optimization problem bounded by equality and inequality constraints, as given in Eqs. (2.7) and (2.8):

$$\begin{aligned} \text{Minimize}_{x \in X} \quad & F(x) = [f_a(x)]; \quad \forall \{a \in \Psi^A\} \\ \text{subject to:} \quad & \\ G(x) = [g_1(x), \dots, g_b(x), \dots, g_B(x)] = 0; \quad & \forall \{B \geq 0\}, \forall \{b \in \Psi^B\} \\ H(x) = [h_1(x), \dots, h_e(x), \dots, h_E(x)] \leq 0; \quad & \forall \{E \geq 0\}, \forall \{e \in \Psi^E\} \end{aligned} \quad (2.7)$$

$$\begin{aligned} x = [x_1, \dots, x_v, \dots, x_{\text{NDV}}]; \quad & \forall \{v \in \Psi^{\text{NDV}}, \Psi^{\text{NDV}} = \Psi^{\text{NCDV}} + \text{NDDV}, x \in X\}, \\ \forall \{x_v^{\text{min}} \leq x_v \leq x_v^{\text{max}} \mid v \in \Psi^{\text{NCDV}}\}, \quad & \\ \forall \{x_v \in \{x_v(1), \dots, x_v(w), \dots, x_v(W_v)\} \mid v \in \Psi^{\text{NDDV}}\} \end{aligned} \quad (2.8)$$

A vector of objective functions strictly dominated by the vector of ideal objective functions is known as the vector of utopian objective functions— z^{utopian} [1–3]. The vector of the utopian objective functions is defined by Eq. (2.9):

$$z^{\text{utopian}} = [z_1^{\text{utopian}}, \dots, z_a^{\text{utopian}}, \dots, z_A^{\text{utopian}}]; \quad \forall \{a \in \Psi^A\} \quad (2.9)$$

The relationship between the component or element a related to the vector of ideal objective functions and the component or element a relevant to the vector of utopian objective functions is defined using Eq. (2.10):

$$z_a^{\text{utopian}} = z_a^{\text{ideal}} - \varepsilon; \quad \forall \{a \in \Psi^A\} \quad (2.10)$$

In Eq. (2.10), ε is a positive scalar number. For this same MOOP, the upper bounds associated with the set of Pareto-optimal responses and outputs are available in the vector of nadir objective functions— z^{nadir} [1–3]. The vector of nadir objective functions is described using Eq. (2.11):

$$z^{\text{nadir}} = [z_1^{\text{nadir}}, \dots, z_a^{\text{nadir}}, \dots, z_A^{\text{nadir}}]; \quad \forall \{a \in \Psi^A\} \quad (2.11)$$

In MOOPs having a nonlinear nature, there is usually no useful well-recognized process to accurately calculate the vector of nadir objective functions. It is, therefore, generally difficult to precisely capture the components or elements relevant to the vector of nadir objective functions. These components or elements can be approximately estimated by using some decision-making analysis tools, such as the payoff table; however, the estimate resulting from these approaches may not be trustworthy [1].

2.3.5 Concepts Relevant to the Investigation of Pareto Optimality

In related literature, there are several methods generally used to investigate Pareto optimality of the vector of decision-making variables. One of the most well-known methods is to examine the Pareto optimality of the vector of decision-making variables, with the idea of forming an optimization problem [5]. In this regard, Pareto optimality of the vector of decision-making variables $x^{**} \in X$ can be investigated by solving the optimization problem given in Eq. (2.12):

$$\begin{aligned}
 & \text{Maximize}_{x \in X, \gamma} \quad \sum_{a \in \Psi^A} \gamma_a; \quad \forall \{a \in \Psi^A\} \\
 & \text{subject to :} \\
 & f_a(x) + \gamma_a = f_a(x^{**}); \quad \forall \{a \in \Psi^A\} \\
 & \gamma_a \geq 0; \quad \forall \{a \in \Psi^A\} \\
 & G(x) = [g_1(x), \dots, g_b(x), \dots, g_B(x)] = 0; \quad \forall \{B \geq 0\}, \forall \{b \in \Psi^B\} \\
 & H(x) = [h_1(x), \dots, h_e(x), \dots, h_E(x)] \leq 0; \quad \forall \{E \geq 0\}, \forall \{e \in \Psi^E\}
 \end{aligned} \tag{2.12}$$

In Eq. (2.12), both $x \in \mathfrak{R}^{\text{NDV}}$ and $\gamma \in \mathfrak{R}_+^A$ are variables. The coefficients vector γ can be also indicated by using Eq. (2.13):

$$\gamma = [\gamma_1, \dots, \gamma_a, \dots, \gamma_A]; \quad \forall \{a \in \Psi^A\} \tag{2.13}$$

If the value of the objective function of the optimization problem given in Eq. (2.12) is equal to zero, the vector of decision-making variables $x^{**} \in X$ is then Pareto optimal. In other words, the vector of decision-making variables $x^{**} \in X$ is Pareto optimal, provided that all of the components or elements of the coefficients vector γ given in Eq. (2.13) are equal to zero. This strategy can also be employed to generate initial Pareto-optimal solutions for interactive MOOAs. Readers interested in a comprehensive discussion on this strategy are referred to the work by Benson [5].

2.4 Multi-objective Optimization Algorithms

Basically, the process of solving a MOOP in order to find the Pareto-optimal solution set, and then select a final optimal solution from this set, requires information related to the preferences of the decision maker. More precisely, the process of solving a MOOP should be established with regard to the preferences of the decision maker. In this way, the solution process can give rise to finding solutions that have more compatible with the preferences of the decision maker. The decision maker generally has sufficient insight into the MOOP. In addition, the decision maker can provide

information pertaining to the preferences of different objective functions or different solutions in various structures.

In related literature, many MOOAs have been developed for finding the Pareto-optimal solution set and selecting the final optimal solution. Because the classification of these optimization algorithms can be carried out with a view to different criteria, it is a challenging task to provide a well-organized classification for these optimization algorithms. In related literature, different classifications have been reported on the basis of various criteria. MOOAs can be broken down into two types of approaches, according to the role of the decision maker in the solution process [1]: noninteractive and interactive.

2.4.1 Noninteractive Approaches

In a general classification, noninteractive approaches (NIAs) can be divided into four classes: (1) basic; (2) no preference; (3) a priori; and, (4) a posteriori.

2.4.1.1 Basic Approaches

Basic approaches are one of the most well-known and most used approaches for solving MOOPs. In order to employ solutions developed for single-objective optimization, these approaches transform a MOOP into a single-objective problem. Therefore, these approaches cannot actually be taken into account as a MOOA. The weighting coefficient approach and the ε -constraint approach are the most common basic approaches. Because of the widespread use and applicability of these approaches in solving MOOPs, an overview of these approaches is provided next.

Weighting coefficient approach: In the literature, the weighting coefficient approach is one of the simplest and most popular basic approaches for solving a MOOP. In this approach, the objective functions of the MOOP are transformed into a scalar objective function by using weighting coefficients [3, 6]. More precisely, in this approach, the MOOP, given in Eqs. (2.1) and (2.2), is turned into a single-objective optimization problem in accordance with Eqs. (2.14) and (2.15) through the weighting coefficients:

$$\begin{aligned}
 \text{Minimize}_{x \in X} \quad & F(x) = \left\{ \sum_{a \in \Psi^A} \omega_a f_a(x) \right\}; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A, \omega_a \in [0, 1]\} \\
 \text{subject to:} \quad & \\
 G(x) = [g_1(x), \dots, g_b(x), \dots, g_B(x)] = & 0; \quad \forall \{B \geq 0\}, \forall \{b \in \Psi^B\} \\
 H(x) = [h_1(x), \dots, h_e(x), \dots, h_E(x)] \leq & 0; \quad \forall \{E \geq 0\}, \forall \{e \in \Psi^E\}
 \end{aligned} \tag{2.14}$$

$$\begin{aligned}
\mathbf{x} = & [x_1, \dots, x_v, \dots, x_{\text{NDV}}]; \quad \forall \{v \in \Psi^{\text{NDV}}, \Psi^{\text{NDV}} = \Psi^{\text{NCDV} + \text{NDDV}}, \mathbf{x} \in \mathbf{X}\}, \\
& \forall \{x_v^{\min} \leq x_v \leq x_v^{\max} \mid v \in \Psi^{\text{NCDV}}\}, \\
& \forall \{x_v \in \{x_v(1), \dots, x_v(w), \dots, x_v(W_v)\} \mid v \in \Psi^{\text{NDDV}}\}
\end{aligned} \tag{2.15}$$

In these equations, ω_a describes the weighting coefficient corresponding to objective function a of the MOOP, which is usually followed by Eqs. (2.16) and (2.17):

$$0 \leq \omega_a \leq 1; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A\} \tag{2.16}$$

$$\sum_{a \in \Psi^A} \omega_a = 1; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A\} \tag{2.17}$$

In the weighting coefficient approach, the decision maker, by systematically changing the weighting coefficients, solves the single-objective optimization problem organized in Eqs. (2.14) and (2.15). Solving the single-objective optimization problem formed in Eqs. (2.14) and (2.15) for different weighting coefficients results in the estimation of the Pareto-optimal solutions. The solution specified by solving the optimization problem given in Eqs. (2.14) and (2.15) is a weak Pareto-optimal solution if the condition provided in Eq. (2.18) is satisfied:

$$\omega_a > 0; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A\} \tag{2.18}$$

This solution is also the Pareto-optimal solution if it is unique [3]. The weighting coefficient approach is appropriate for a MOOP in which all objective functions are of the same type and have a common scale (e.g., all objective functions are of a cost type with a dollar scale). If objective functions of optimization problem are not of same type and not have a common scale, the use of the weighting coefficient approach is not efficient. In this situation, the recommended strategy for employing the weighting coefficient approach is to normalize the objective functions. Objective function a of this same MOOP is normalized through Eq. (2.19):

$$\tilde{f}_a(\mathbf{x}) = \frac{f_a^{\max}(\mathbf{x}) - f_a(\mathbf{x})}{f_a^{\max}(\mathbf{x}) - f_a^{\min}(\mathbf{x})}; \quad \forall \{a \in \Psi^A\} \tag{2.19}$$

This equation refers to a situation in which the minimization of the objective function a of the MOOP is taken into account. Similarly, if maximization of the objective function a of the MOOP is regarded, Eq. (2.19) should be rewritten according to Eq. (2.20):

$$\tilde{f}_a(\mathbf{x}) = 1 - \frac{f_a^{\max}(\mathbf{x}) - f_a(\mathbf{x})}{f_a^{\max}(\mathbf{x}) - f_a^{\min}(\mathbf{x})}; \quad \forall \{a \in \Psi^A\} \tag{2.20}$$

In Eqs. (2.19) and (2.20), the lower and upper bounds of objective function a of the MOOP are calculated by using a single-objective optimization. More precisely, the upper bound of objective function a , $f_a^{\max}(x)$, is achieved by single-objective maximization of the corresponding objective function. In the same way, the lower bound of objective function a , $f_a^{\min}(x)$, is determined by single-objective minimization of the corresponding objective function. After normalization of the objective functions of the MOOP given in Eqs. (2.1) and (2.2), this optimization problem can be rewritten based on Eqs. (2.21) and (2.22):

$$\begin{aligned} \text{Minimize}_{x \in X} \quad & F(x) = \left\{ \sum_{a \in \Psi^A} \omega_a \cdot \tilde{f}_a(x) \right\}; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A, \omega_a \in [0, 1]\} \\ \text{subject to :} \quad & \\ G(x) = [g_1(x), \dots, g_b(x), \dots, g_B(x)] = 0; \quad & \forall \{B \geq 0\}, \forall \{b \in \Psi^B\} \\ H(x) = [h_1(x), \dots, h_e(x), \dots, h_E(x)] \leq 0; \quad & \forall \{E \geq 0\}, \forall \{e \in \Psi^E\} \end{aligned} \quad (2.21)$$

$$\begin{aligned} x = [x_1, \dots, x_v, \dots, x_{\text{NDV}}]; \quad & \forall \{v \in \Psi^{\text{NDV}}\}, \forall \{\Psi^{\text{NDV}} = \Psi^{\text{NCDV}} + \Psi^{\text{NDDV}}\}, \forall \{x \in X\}, \\ \forall \{x_v^{\min} \leq x_v \leq x_v^{\max} \mid v \in \Psi^{\text{NCDV}}\}, \quad & \\ \forall \{x_v \in \{x_v(1), \dots, x_v(w), \dots, x_v(W_v)\} \mid v \in \Psi^{\text{NDDV}}\} \end{aligned} \quad (2.22)$$

One of the most important strengths of the weighting coefficients approach, making it widely utilized for solving a wide range of MOOPs, is the simplicity of its use. In this approach, one solution can be found through the Pareto-optimal solution set by changing the weighting coefficients. It has been proven, however, that this characteristic is reliable only in convex optimization problems. That is, in non-convex optimization problems, regardless of how the weighing coefficients are chosen, some Pareto-optimal solutions cannot be found. Furthermore, if some objective functions correlate with each other in the MOOPs, changing the weighting coefficients may not lead to finding Pareto-optimal solutions. As a result, the weighting coefficient approach does not have an appropriate performance for these MOOPs. It should be pointed out that the decision maker can employ the weighting coefficient approach either as an a priori approach or as an a posteriori approach.

ϵ -Constraint approach: In related literature, the ϵ -constraint approach is one of the most applicable basic approaches for solving MOOPs [7, 8]. At each step in this approach, one of the objective functions of the MOOP is chosen for optimization, while the remaining objective functions are considered as constraints. This process is repeated for all objective functions of the MOOP. By using the ϵ -constraint approach, the MOOP, again from Eqs. (2.1) and (2.2), is transformed into a single-objective optimization problem in accordance with Eqs. (2.23) and (2.24):

$$\begin{aligned}
& \underset{x \in X}{\text{Minimize}} && F(x) = \{f_a(x)\}; \quad \forall \{a \in \Psi^A\} \\
& && \text{subject to :} \\
& && G(x) = [g_1(x), \dots, g_b(x), \dots, g_B(x)] = 0; \quad \forall \{B \geq 0\}, \forall \{b \in \Psi^B\} \\
& && H(x) = [h_1(x), \dots, h_e(x), \dots, h_E(x)] \leq 0; \quad \forall \{E \geq 0\}, \forall \{e \in \Psi^E\} \\
& && f_k(x) \leq \varepsilon_k^{\max}; \quad \forall \{k \in \Psi^A\}, \forall \{k \neq a\}
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
x = & [x_1, \dots, x_v, \dots, x_{\text{NDV}}]; \quad \forall \{v \in \Psi^{\text{NDV}}, \Psi^{\text{NDV}} = \Psi^{\text{NCDV}} + \text{NDDV}, x \in X\}, \\
& \forall \{x_v^{\min} \leq x_v \leq x_v^{\max} \mid v \in \Psi^{\text{NCDV}}\}, \\
& \forall \{x_v \in \{x_v(1), \dots, x_v(w), \dots, x_v(W_v)\} \mid v \in \Psi^{\text{NDDV}}\}
\end{aligned} \tag{2.24}$$

In Eq. (2.23), ε_k^{\max} represents the upper bound for objective function k of the MOOP. The vector of decision-making variables $x^{**} \in X$ is Pareto optimal if and only if this vector solves the optimization problem organized in Eqs. (2.23) and (2.24) for each objective function of the MOOP, $f_a(x^{**}); \forall \{a \in \Psi^A\}$, while satisfying $\varepsilon_k^{\max} = f_k(x^{**}); \forall \{k \in \Psi^A\}, \forall \{k \neq a\}$ [7, 8]. More precisely, to ensure that Pareto optimality corresponds to the vector of decision-making variables $x^{**} \in X$ —finding one solution from the Pareto-optimal solution set—either the single-objective optimization problem formed in Eqs. (2.23) and (2.24) must be solved by the number of objective functions of the MOOP or one unique solution of the single-objective optimization problem formed in Eqs. (2.23) and (2.24) must be achieved. Nevertheless, in a MOOP, if the weak Pareto-optimal solution is satisfactory, from the perspective of the decision maker, solving the single-objective optimization problem organized in Eqs. (2.23) and (2.24) is sufficient for an objective function to find one solution from the weak Pareto-optimal solution set.

In contrast to the weighting coefficient approach, finding the Pareto-optimal solution set by using the ε -constraint approach does not depend on the convexity or non-convexity of the optimization problem. In other words, the ε -constraint approach has a desirable performance in dealing with convex or non-convex optimization problems.

In practice, the selection of the upper bounds associated with different objective functions of the MOOP has many complexities. These complexities are dramatically expanded by increasing the number of objective functions of the MOOP. The selection of the upper bounds must, therefore, be made meticulously. In this manner, the upper bounds selected for different objective functions of the MOOP must be within the feasible space; otherwise, the single-objective optimization problem formed in Eqs. (2.23) and (2.24) will not have a solution. If maximization of this MOOP is taken into account, then the MOOP is turned into a single-objective optimization problem based on Eqs. (2.25) and (2.26) by using the ε -constraint approach:

$$\begin{aligned}
& \underset{x \in X}{\text{Maximize}} && F(x) = \{f_a(x)\}; \quad \forall \{a \in \Psi^A\} \\
& \text{subject to :} && \\
& G(x) = [g_1(x), \dots, g_b(x), \dots, g_B(x)] = 0; && \forall \{B \geq 0\}, \forall \{b \in \Psi^B\} \\
& H(x) = [h_1(x), \dots, h_e(x), \dots, h_E(x)] \leq 0; && \forall \{E \geq 0\}, \forall \{e \in \Psi^E\} \\
& f_k(x) \geq \varepsilon_k^{\min}; && \forall \{k \in \Psi^A\}, \forall \{k \neq a\}
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
x = & [x_1, \dots, x_v, \dots, x_{\text{NDV}}]; \quad \forall \{v \in \Psi^{\text{NDV}}, \Psi^{\text{NDV}} = \Psi^{\text{NCDV}} + \Psi^{\text{NDDV}}, x \in X\}, \\
& \forall \{x_v^{\min} \leq x_v \leq x_v^{\max} \mid v \in \Psi^{\text{NCDV}}\}, \\
& \forall \{x_v \in \{x_v(1), \dots, x_v(w), \dots, x_v(W_v)\} \mid v \in \Psi^{\text{NDDV}}\}
\end{aligned} \tag{2.26}$$

In Eq. (2.25), ε_k^{\min} describes the lower bound for objective function k of the MOOP. Similar to the weighting coefficient approach, the ε -constraint approach can be utilized by the decision maker either as an a priori approach or as an a posteriori approach.

2.4.1.2 No-Preference Approaches

In no-preference approaches, known as neutral-preference approaches, the preferences of the decision maker are not considered in the process of solving the MOOP. In these approaches, the MOOP is solved by using some relatively simple approaches, at which point the solution is taken at the disposal of the decision maker. The decision maker is also able to accept or reject the specified solution. Non-preference approaches are suitable for situations in which information related to the preferences of the decision maker is not available, or the decision maker does not consider particular preferences. The global criterion approach and neutral-compromise solution approach are the best-known no-preference approaches. Readers interested in a thorough discussion on these approaches are directed to the work by Yu [9] and Wierzbicki [10], respectively.

2.4.1.3 A Priori Approaches

In a priori approaches, the decision maker first determines the information related to his/her preferences, and then solves the MOOP by trying to find a Pareto-optimal solution that can, as much as possible, satisfy his/her preferences. Simply put, in a priori approaches, information related to the preferences of the decision maker is determined before the process of solving the MOOP begins.

A major disadvantage in a priori approaches is that the decision maker is not necessarily aware of the possibilities and restrictions of the MOOP in advance. As a result, it is possible that information about the preferences of the decision maker is overly optimistic or pessimistic. That is to say that the decision maker does not

necessarily know in advance what the response/output is likely to be from the MOOP or how realistic his/her preferences are. In this situation, it is possible that the solution cannot satisfy the decision maker and encourage the decision maker to rectify his/her preferences. The most well-known a priori approaches can be referred to as value function approaches, lexicographic ordering approaches, and goal programming approaches. Readers interested in a comprehensive discussion on these approaches are referred to the work by Keeney and Raiffa [11], Fishburn [12], and Charnes and Cooper [13], respectively.

2.4.1.4 A Posteriori Approaches

The main idea of a posteriori approaches is established on the basis of finding the Pareto-optimal solution set and presenting it to the decision maker with the aim of choosing the final solution through the aforementioned set. More precisely, in a posteriori approaches, the process of solving the MOOP first tries to find the Pareto-optimal solution set. After determination of the Pareto-optimal solution set, this set is taken at the disposal of the decision maker. Finally, the decision maker chooses the most satisfactory solution from the set as the final optimal solution.

One of the strengths of a posteriori approaches, compared to a priori approaches, is that in a posteriori approaches, the Pareto-optimal solution set is completed before being presented to the decision maker. In this way, the decision maker has a complete overview of all solutions, making it easier and more realistic to choose the most satisfactory solution. Nonetheless, the major weakness of a posteriori approaches is their high computational burden. Additionally, the decision maker encounters a very large amount of information in optimization problems with more than two objective functions, which makes analysis of the information a difficult task.

The best-known a posteriori approaches can be referred to as weighted metrics approaches, achievement scalarizing function approaches, approximation approaches, and meta-heuristic MOOAs. Readers interested in a thorough discussion on these approaches are referred to the work by Miettinen [3], Wierzbicki [14], and Ruzika and Wiecek [15], respectively. It is important to be noted that detailed descriptions of some developed meta-heuristic MOOAs by the authors are provided in Chap. 4.

2.4.2 Interactive Approaches

Interactive approaches (IAs) are established on the basis of creating an iterative solution procedure or pattern that consists of different steps. In this approach to finding the most satisfactory solution, different steps of this iterative procedure are repeated and the decision maker progressively determines preference information during the solution process. In other words, after completion of each step of the iterative procedure, the information is taken at the disposal of the decision maker, at

which point the decision maker assesses the information and then may specify additional details. This interactive process is repeated until the stopping criterion is satisfied—and as long as the most satisfactory solution has been specified by the decision maker. In this structure, the decision maker can modify and update his/her preference information. As a consequence, the decision maker straightforwardly directs the process in the IAs.

The main steps of an iterative procedure can be briefly expressed as follows:

- Step one: Initialization (i.e., determine the ideal vector of objective functions and nadir vector of objective vector and present these values to the decision maker).
- Step two: Produce a Pareto-optimal starting point (i.e., some neutral-compromise solution or solution specified by the decision maker that can be taken into account as the starting point).
- Step three: Specify the preference information by the decision maker (i.e., the number of new solutions to be produced).
- Step four: Produce one or more Pareto-optimal solutions by taking into account the preferences specified by the decision maker in the previous step and then showing this Pareto-optimal solution or solutions along with information associated with the MOOP to the decision maker.
- Step five: Select the most satisfactory solution by the decision maker through Pareto-optimal solutions achieved thus far, if multiple Pareto-optimal solutions have been produced in the fourth step. If a Pareto-optimal solution has been produced in the fourth step, this solution is considered as the most satisfactory solution by the decision maker in this step.
- Step six: Stop, if the consent of the decision maker is satisfied by the solution chosen in the fifth step; otherwise, go to the third step.

One of the strengths of the IAs is that the decision maker is able to update his/her preference information in each iteration of the process. Accordingly, by informing the decision maker about interdependencies between the iterative solution procedure and its preferences, the probability of achieving a satisfactory solution that meets the preferences of the decision maker is increased. In other words, because of the establishment of the IAs, based on an iterative procedure that allows the decision maker to specify or update preference information during the process, Pareto-optimal solutions are produced that can satisfy the decision maker. As a result, the structure of the IAs can give rise to a significant reduction in computational burden.

In recent years, a wide range of IAs have been developed for solving MOOPs. Basically, there is no unique IA that has a more preferred performance for solving the MOOPs with different features and structures as well as multiple decision makers compared to other approaches. This means that each approach is generally developed for a specific range of MOOPs and decision makers. In a wide classification, IAs can be broken down into three general classes: (1) compromise-driven or trade-off-based approaches; (2) reference point approaches; and, (3) classification-based approaches. Readers interested in a thorough discussion of these approaches are referred to the work by Branke et al. [1].

2.5 Selection of the Final Solution by Using a Fuzzy Satisfying Method

After solving our MOOP and determining the Pareto-optimal solution set, the next step is to select a flexible and realistic solution from the entire set of candidate solutions that represent a compromise among different objective functions of the MOOP. In related literature, there are several multi-objective decision-making tools for selecting the most satisfactory solution from the Pareto-optimal solution set; keep in mind, though, that a fuzzy satisfying method (FSM) is highly regarded in this situation, owing to its simplicity and similarity to human reasoning [16–19]. In this method, then, a fuzzy membership function, $\Phi_{f_a^n(x)}(\bar{x}_n)$, is defined for each given objective function, $f_a^n(x)$, in any available solution in the Pareto-optimal solution set, \bar{x}_n . The value of this membership function can vary from 0 to 1. The fuzzy membership function demonstrates a numerical description for the satisfaction level of the decision maker regarding the value of objective function a in the available solution n in the Pareto-optimal solution set. The fuzzy membership function with a value of 0, $\Phi_{f_a^n(x)}(\bar{x}_n) = 0$, represents a complete dissatisfaction of the decision maker. On the other hand, the fuzzy membership function with a value of 1, $\Phi_{f_a^n(x)}(\bar{x}_n) = 1$, represents full satisfaction of the decision maker. As a result, higher values of this membership function refer to higher levels of satisfaction of the decision maker regarding the value of objective function a in the available solution n in the Pareto-optimal solution set. Different types of fuzzy membership functions can generally be used by the decision maker, such as linear, convex exponential, concave exponential, piecewise linear, and hyperbolic types. Considering different types of fuzzy membership functions for different objective functions of the MOOP can affect the choice of the final solution through the Pareto-optimal solution set. As an example, suppose that the fuzzy membership function considered by the decision maker for objective function a of our MOOP is convex exponential and the fuzzy membership function regarded by the decision maker for other objective functions is linear. These conditions provide a priority for minimization of objective function a relative to other objective functions. This is due to the fact that a smaller fuzzy membership function in the neighborhood of the upper bound of the objective function a , $f_a^{\max}(x)$, has been assigned by the convex exponential membership function compared with the linear membership function.

Here, the fuzzy membership function considered for all of the existing objective functions in our MOOP is assumed to be a linear membership function. To clarify, the linear membership function corresponds to objective function a of the MOOP that is depicted in Fig. 2.2.

If the minimization of the objective functions of the MOOP is considered, the linear membership function related to objective function a is represented as a descending uniform function (see Fig. 2.2). The mathematical description of the linear membership function shown in Fig. 2.2 can also be set out using Eq. (2.27):

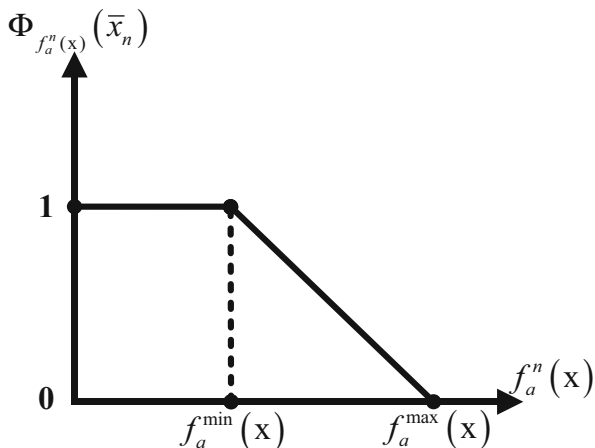


Fig. 2.2 The linear membership function corresponds to the minimization of objective function a of the MOOP

$$\Phi_{f_a^n(x)}(\bar{x}_n) = \begin{cases} 0 & ;\forall\{f_a^n(x) > f_a^{\max}(x)\} \\ \frac{f_a^{\max}(x) - f_a^n(x)}{f_a^{\max}(x) - f_a^{\min}(x)} & ;\forall\{f_a^{\min}(x) \leq f_a^n(x) \leq f_a^{\max}(x)\} \\ 1 & ;\forall\{f_a^n(x) < f_a^{\min}(x)\} \end{cases} \quad (2.27)$$

$;\forall\{A \geq 2\}, \forall\{a \in \Psi^A, n \in \Psi^N\}$

As shown in Fig. 2.2 and formulated in Eq. (2.27), this membership function has both a lower bound, $f_a^{\min}(x)$, and an upper bound, $f_a^{\max}(x)$. These bounds are achieved by using a single-objective optimization. That is to say that the lower and upper bounds of objective function a of the MOOP are calculated by minimizing and maximizing only the corresponding objective function as a single-objective optimization problem, respectively. Similarly, if the maximization of the objective functions of the same MOOP is taken into account, the linear membership function relevant to objective function a is addressed as an ascending uniform function (see Fig. 2.3). The mathematical description of the linear membership function depicted in Fig. 2.3 can also be set out using Eq. (2.28):

$$\Phi_{f_a^n(x)}(\bar{x}_n) = \begin{cases} 1 & ;\forall\{f_a^n(x) > f_a^{\max}(x)\} \\ 1 - \frac{f_a^{\max}(x) - f_a^n(x)}{f_a^{\max}(x) - f_a^{\min}(x)} & ;\forall\{f_a^{\min}(x) \leq f_a^n(x) \leq f_a^{\max}(x)\} \\ 0 & ;\forall\{f_a^n(x) < f_a^{\min}(x)\} \end{cases}$$

$;\forall\{A \geq 2\}, \forall\{a \in \Psi^A, n \in \Psi^N\}$

(2.28)

After describing the membership functions for all objective functions for all available solutions in the Pareto-optimal solutions set, the decision maker must specify the level of desirability of achieving each objective function of the MOOP,

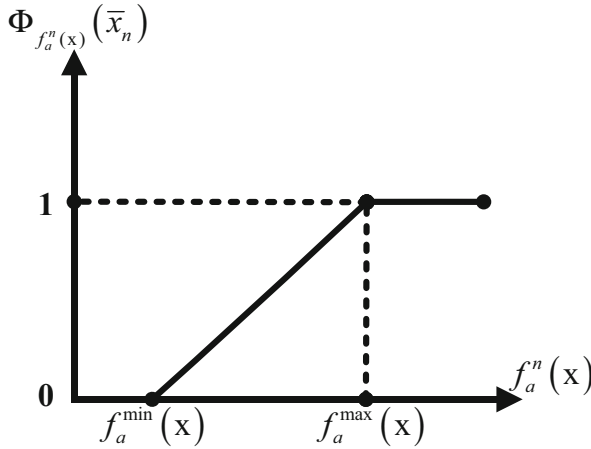


Fig. 2.3 The linear membership function relevant to the maximization of objective function a of the MOOP

$\tilde{\Phi}_{f_a(x)}$. The level of desirability of achieving objective function a of the MOOP, $f_a(x)$, is known as the reference level of achieving the corresponding objective function. After determination of the level of desirability of achieving each objective function of the MOOP, the decision maker should employ a well-suited decision-making analysis tool in order to choose the final optimal compromise solution from the Pareto-optimal solution set. To do this, there are generally many decision-making analysis tools developed using a variety of philosophies and from myriad perspectives. Here, the conservative and distance metric methodologies, as two applicable and well-known decision-making analysis tools, are reviewed and discussed.

2.5.1 Conservative Methodology

In the conservative methodology (CM)—the min-max formulation—conservative decision-making can be achieved by trying to find a solution whose minimum meets the maximum objective function. This means that the decision maker is willing to specify a solution that simultaneously achieves the highest level of satisfaction for all of the objective functions of the MOOP. In this methodology, the final optimal compromise solution is determined from all available solutions in the Pareto-optimal set by solving the optimization problem given in Eq. (2.29):

$$\min_{n \in \Psi^N} \left\{ \max_{a \in \Psi^A} \{ |\tilde{\Phi}_{f_a(x)} - \Phi_{f_a^n(x)}(\bar{x}_n)| \} \right\}; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A, n \in \Psi^N\} \quad (2.29)$$

If the decision maker is willing to achieve the highest level of satisfaction for all of the objective functions of the MOOP, $\tilde{\Phi}_{f_a(x)} = 1; \forall \{a \in \Psi^A\}$, Eq. (2.29) must be rewritten as Eq. (2.30):

$$\min_{n \in \Psi^N} \left\{ \max_{a \in \Psi^A} \{ |1 - \Phi_{f_a^n}(\bar{x}_n)| \} \right\}; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A, n \in \Psi^N\} \quad (2.30)$$

In other words, Eq. (2.30) can also be expressed as Eq. (2.31):

$$\max_{n \in \Psi^N} \left\{ \min_{a \in \Psi^A} \{ |\Phi_{f_a^n}(\bar{x}_n)| \} \right\}; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A, n \in \Psi^N\} \quad (2.31)$$

The CM ensures for the decision maker that all of the objective functions of the MOOP are well optimized. Interested readers are directed to the work by Sakawa and Yano [20] for a comprehensive discussion of the CM.

2.5.2 Distance Metric Methodology

In the distance metric methodology (DM), the final optimal compromise solution is obtained from all available solutions in the Pareto-optimal set by solving the optimization problem given in Eq. (2.32):

$$\min_{n \in \Psi^N} \left\{ \sum_{a \in \Psi^A} \{ |\tilde{\Phi}_{f_a(x)} - \Phi_{f_a^n}(\bar{x}_n)|^u \} \right\}; \quad \forall \{A \geq 2\}, \forall \{a \in \Psi^A, n \in \Psi^N, u \in [1, \infty)\} \quad (2.32)$$

It can be seen that Eq. (2.32) attempts to minimize the u -norm deviations from the values of the reference membership. The u quantity has a value between one and infinity, a value that has already been specified by the decision maker. Because the absolute difference of the level of desirability of achieving objective function a and its fuzzy membership function in available solution n in the Pareto-optimal set $|\tilde{\Phi}_{f_a(x)} - \Phi_{f_a^n}(\bar{x}_n)|$ always has a value between zero and one, a larger value of u represents less sensitivity to reference levels and vice versa. It should be pointed out that if the decision maker is not satisfied by the solution, he/she is able to improve the corresponding solution by updating the levels of desirability of achieving different objective functions of the MOOP, $\tilde{\Phi}_{f_a(x)}$. Interested readers are directed to the work by Chen [21] for a comprehensive discussion of DM.

2.5.3 Step-by-Step Process for Implementing the FSM

As a general result, after solving the MOOP, as given in Eqs. (2.1) and (2.2), and specifying the Pareto-optimal solution set, the implementation of the FSM by the

decision maker to select the final optimal compromise solution through the Pareto-optimal solution set can be expressed using the following step-by-step process:

- Step one: Set the number of objective functions of the MOOP equal to A .
- Step two: Set the counter of the objective function to $a = 1$.
- Step three: Determine the lower and the upper bounds of objective function a of the MOOP by minimizing and maximizing only the corresponding objective function as a single-objective optimization, respectively.
- Step four: Set the number of available solutions in the Pareto-optimal solution set specified by solving the MOOP equal to N .
- Step five: Set the counter of available solutions in the Pareto-optimal solution set to $n = 1$.
- Step six: Calculate the value of the linear membership function associated with objective function a in available solution n in the Pareto-optimal solution set by using Eq. (2.27).
- Step seven: If $n < N$, set $n = n + 1$ and go to step six; otherwise, go to the next step.
- Step eight: If $a < A$, set $a = a + 1$ and go to step three; otherwise, go to the next step.
- Step nine: Specify the level of desirability of achieving each objective function of the MOOP.
- Step ten: Determine the final optimal compromise solution from the Pareto-optimal solution set either by using the CM—min-max formulation given in Eq. (2.29)—or by using the DM—formulation given in Eq. (2.32).

2.6 Conclusions

In this chapter, the authors presented a brief introduction to the multi-objective optimization process. First, the necessity of employing the multi-objective optimization process instead of the single-objective optimization process was justified. Then, the fundamental concepts of optimization in the MOOPs were exhaustively addressed in the five sections: (1) mathematical description of a MOOP; (2) concepts associated with efficiency, efficient frontier, and dominance; (3) concepts pertaining to Pareto optimality; (4) concepts related to the vector of ideal objective functions and the vector of nadir objective functions; and, (5) concepts relevant to Pareto optimality investigation. In addition, a thorough classification was provided for the MOOAs with a focus on the role of the decision maker in the process of solving the MOOP. This classification was broken down into two approaches: NIAs and IAs. The NIAs were also classified into four different classes including basic, no preference, a priori, and a posteriori approaches. Finally, the FSM, as the most preferred multi-objective decision-making tool, was thoroughly described in order to select the final optimal compromise solution from the Pareto-optimal solution set.

Appendix 1: List of Abbreviations and Acronyms

CM	Conservative methodology
DM	Distance metric methodology
FSM	Fuzzy satisfying method
IAs	Interactive approaches
MOOAs	Multi-objective optimization algorithms
MOOPs	Multi-objective optimization problems
NIAs	Noninteractive approaches

Appendix 2: List of Mathematical Symbols

<i>Index:</i>	
a	Index for objective functions running from 1 to A
b	Index for equality constraints running from 1 to B
e	Index for inequality constraints running from 1 to E
k	Index for objective functions running from 1 to A
n	Index for available solutions in the Pareto-optimal solution set running from 1 to N
v	Index for decision-making variables, including the continuous and discrete decision-making variables, running from 1 to NDV, and an index for continuous decision-making variables running from 1 to NCDV and also an index for discrete decision-making variables running from 1 to NDDV
<i>Set:</i>	
Ψ^A	Set of indices of objective functions
Ψ^B	Set of indices of equality constraints
Ψ^E	Set of indices of inequality constraints
Ψ^{NCDV}	Set of indices of continuous decision-making variables
Ψ^{NDDV}	Set of indices of discrete decision-making variables
Ψ^{NDV}	Set of indices of decision-making variables, including the continuous and discrete decision-making variables
Ψ^N	Set of indices of available solutions in the Pareto-optimal solution set
AP(X)	Set of appropriate Pareto-optimal vectors of decision-making variables
AP(Z)	Set of appropriate Pareto-optimal vectors of objective functions
P(X)	Set of Pareto-optimal vectors of decision-making variables
P(Z)	Set of Pareto-optimal vectors of objective functions
\mathfrak{R}^A	A-dimensional set of real numbers
\mathfrak{R}_+^A	A-dimensional set of positive real numbers
\mathfrak{R}^E	E-dimensional set of real numbers
W_v	Set of indices of candidate permissible values of discrete decision-making variable v
WP(X)	Set of weak Pareto-optimal vectors of decision-making variables
WP(Z)	Set of weak Pareto-optimal vectors of objective functions

(continued)

<i>Parameters:</i>	
$f_a^{\max}(x)$	Upper bound of objective function a
$f_a^{\min}(x)$	Lower bound of objective function a
J	Some real number
u	Integer with a value between one and infinity
x_v^{\max}	Upper bound on continuous decision-making variable v
x_v^{\min}	Lower bound on continuous decision-making variable v
X	Nonempty feasible decision-making space
Z	Feasible objective space
ε	A positive scalar number
ε_k^{\max}	Upper bound of objective function k
ε_k^{\min}	Lower bound of objective function k
$\Phi_{f_a(x)}$	Level of desirability of achieving objective function a
<i>Variables:</i>	
$f_a(x)$	Objective function a of the optimization problem or component a of the vector of objective functions
$f_k(x)$	Objective function k of the optimization problem or component k of the vector of objective functions
$\tilde{f}_a(x)$	Normalized value of objective function a of the optimization problem or normalized value of component a of the vector of objective functions
$f_a^n(x)$	Given objective function a in available solution n in the Pareto-optimal solution set
$F(x)$	Vector of objective functions of the optimization problem
$g_b(x)$	Equality constraint b of the optimization problem or component b of the vector of equality constraints
$G(x)$	Vector of equality constraints of the optimization problem
$h_e(x)$	Inequality constraint e of the optimization problem or component e of the vector of inequality constraints
$H(x)$	Vector of inequality constraints of the optimization problem
\bar{x}_n	Solution n in the Pareto-optimal solution set
x_v	Continuous or discrete decision-making variable v
$x_v(w_v)$	Candidate permissible value w of discrete decision-making variable v
x, x^*, x^{**}	Vector of decision-making variables
z	Vector of objective functions
z^{ideal}	Vector of ideal objective functions
z_a^{ideal}	Component or element a relevant to the vector of ideal objective functions
z^{utopian}	Vector of utopian objective functions
z_a^{utopian}	Component or element a relevant to the vector of utopian objective functions
z^{nadir}	Vector of the nadir objective functions
z_a^{nadir}	Component or element a relevant to the vector of nadir objective functions
γ	Coefficient vector
γ_a	Component or element a relevant to the coefficient vector
ω_a	Weighting coefficient relevant to objective function a
$\Phi_{f_a^n(x)}(\bar{x}_n)$	Fuzzy membership function of objective function a in available solution n in the Pareto-optimal solution set

References

1. J. Branke, K. Deb, K. Miettinen, *Multiobjective Optimization: Interactive and Evolutionary Approaches* (Springer, New York, 2008)
2. R.T. Marler, J.S. Arora, Survey of multi-objective optimization methods for engineering. *Struct. Multidiscip. Optim.* **26**(6), 369–395 (2004)
3. K. Miettinen, *Nonlinear Multiobjective Optimization* (Kluwer Academic Publishers, Boston, MA, 1999)
4. A.M. Geoffrion, Proper efficiency and the theory of vector maximization. *J. Math. Anal. Appl.* **22**(3), 618–630 (1968)
5. H.P. Benson, Existence of efficient solutions for vector maximization problems. *J. Optim. Theory Appl.* **26**(4), 569–580 (1978)
6. L. Zadeh, Optimality and non-scalar-valued performance criteria. *IEEE Trans. Autom. Control* **8**(1), 59–60 (1963)
7. Y.Y. Haimes, L.S. Lasdon, D.A. Wismer, On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE Trans. Syst. Man Cybern.* **1**(3), 296–297 (1971)
8. V. Chankong, Y.Y. Haimes, *Multiobjective Decision Making: Theory and Methodology* (Elsevier Science Publishing, New York, 1983)
9. P.L. Yu, A class of solutions for group decision problems. *Manag. Sci.* **19**(8), 936–946 (1973)
10. A.P. Wierzbicki, Reference point approaches, in *Multicriteria Decision Making, Chapter 9*, vol. 21, (Springer, New York, 1999), pp. 237–275
11. R.L. Keeney, H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs* (Wiley, Chichester, 1976)
12. P.C. Fishburn, Lexicographic orders, utilities and decision rules: a survey. *Manag. Sci.* **20**(11), 1442–1471 (1974)
13. A. Charnes, W.W. Cooper, Goal programming and multiple objective optimization: part 1. *Eur. J. Oper. Res.* **1**(1), 39–54 (1977)
14. A.P. Wierzbicki, A mathematical basis for satisficing decision making. *Mathematical Modelling* **3**(5), 391–405 (1982)
15. S. Ruzika, M.M. Wiecek, Approximation methods in multiobjective programming. *J. Optim. Theory Appl.* **126**(3), 473–501 (2005)
16. M. Sakawa, H. Yano, An interactive fuzzy satisfying method using augmented minimax problems and its application to environmental systems. *IEEE Trans. Syst. Man Cybern.* **17**(6), 720–729 (1985)
17. M. Sakawa, K. Yauchi, An interactive fuzzy satisfying method for multiobjective nonconvex programming problems through floating point genetic algorithms. *Eur. J. Oper. Res.* **117**(1), 113–124 (1999)
18. C. Kahraman, *Fuzzy Multi-Criteria Decision Making: Theory and Applications with Recent Developments* (Springer, New York, 2008)
19. M. Kiani-Moghaddam, M. Shivaie, A. Salemnia, M.T. Ameli, Probabilistic multi-objective framework for multiple active power filters planning. *Electr. Power Compon. Syst.* **45**(18), 2062–2077 (2017)
20. M. Sakawa, H. Yano, An interactive fuzzy satisfying method for multi-objective nonlinear programming problems with fuzzy parameters. *Fuzzy Sets Syst.* **30**(3), 221–238 (1989)
21. Y.L. Chen, An interactive fuzzy-norm satisfying method for multi-objective reactive power sources planning. *IEEE Trans. Power Syst.* **15**(3), 1154–1160 (2000)