Upwind-Based Numerical Approximation of a Space-Time Fractional Advection-Dispersion Equation for Groundwater Transport Within Fractured Systems

A. Allwright and A. Atangana

Abstract Modelling groundwater transport in fractured aquifer systems is complex due to the uncertainty associated with delineating the specific fractures along which water and potential contaminants could be transported. The resulting uncertainty in modelled contaminant movement has implications for the protection of the environment, where inadequate mitigation or remediation measures could be employed. To improve the governing equation for groundwater transport modelling, the Atangana–Baleanu in Caputo sense (ABC) fractional derivative is applied to the advection-dispersion equation with a focus on the advection term to account for anomalous advection. Boundedness, existence and uniqueness for the developed advection-focused transport equation is presented. In addition, a semi-discretisation analysis is performed to demonstrate the equation stability in time. Augmented upwind schemes are investigated as they have been found to address stability problems when solute transport is advection-dominated. The upwind-based schemes are developed, and stability analysis conducted, to facilitate the solution of the complex equation. The numerical stability analysis found the upwind Crank–Nicolson to be the most stable, and is thus recommended for use with the ABC fractional advection-dispersion equation.

Keywords Fractional calculus · Atangana–Baleanu fractional derivative · Fractional advection-dispersion equation

A. Allwright · A. Atangana (✉)
Faculty of Agricultural and Natural Sciences, Institute for Groundwater Studies, University of the Free State, Bloemfontein 9301, Free State, South Africa
e-mail: AtanganaA@ufs.ac.za

J. F. Gómez et al. (eds.), *Fractional Derivatives with Mittag-Leffler Kernel*, Studies in Systems, Decision and Control 194,
https://doi.org/10.1007/978-3-030-11662-0_18

1 Introduction

Real-world systems are complex. By definition, a complexity by which any one method is not able to capture all the nuances of that system. It is this that compels science to improve and continuously strive for new methods and approaches, ever endeavouring to reconcile the difference between modelled and observed.

Simulating the transport of particles using the advection-dispersion equation is a real-world problem, where the general discrepancy between modelled and observed is particularly large. This discrepancy lead to the development of the term anomalous diffusion (non-Fickian diffusion), especially when using linear or traditional methods. For this reason, numerous nonlocal approaches have been applied to the advection-dispersion equation to reduce this divergence, ranging from multiple-rate mass transfer method and rate-limited mass transfer, stochastic averaging, continuous-time random walk, to fractal and fractional differential equations [1–9].

Complexity from the perspective of fractional calculus is explored in [10], where fractional differential equations are one method to improve the simulation of real world problems. Fractional calculus is not a new topic, having its original inception in the late 1600s, but the application of fractional derivations to practical problems has steadily increased since the 1970s. With the endeavour to continually improve simulation methods, a progression of fractional derivative definitions have been developed over the years, with definitions including Riemann-Liouville, Caputo, Caputo-Fabrizio, and the latest Atangana–Baleanu [11–21].

The newest fractional derivative definition Atangana–Baleanu, is used to develop an advection-focused fractional transport equation. The suitability of this particular formulation and fractional derivative definition is investigated for the specific real-world system of groundwater transport within fractured aquifers. Modelling groundwater transport in fractured aquifer systems is complex due to the uncertainty associated with demarcating the specific fractures along which water and potential contaminants could be transported along. A result in this uncertainty is the misrepresentation of the expected movement of a potential contaminant in the groundwater system. This potentially increases the impact on the environment because the misrepresented transport could result in inadequate mitigation or remediation measures [22–28].

A faster than expected transport along unknown fractures is referred to as super-advection associated with anomalous advection by [29], and a fractional derivative was applied to the advection term of the advection-dispersion equation to better simulate this phenomena. A fractional derivative was also applied to the time component of the advection-dispersion equation to activate the waiting-time distribution properties as discussed by [30, 31]. A similar approach is followed in developing the Atangana–Baleanu in Caputo sense (ABC) fractional advection-dispersion equation.

2 Advection-Focused Space-Time Fractional Transport Equation with Atangana–Baleanu in Caputo Sense (ABC) Derivative

The one-dimensional, non-reactive fractional advection-dispersion equation with the ABC fractional derivative definition is given by

$$ {}_{0}^{ABC}D_t^{\alpha}\left(c\left(x,t\right)\right) = -v_0{}^{ABC}D_x^{\alpha}\left(c\left(x,t\right)\right) + D_L\frac{\partial^2}{\partial x^2}\left(c\left(x,t\right)\right). \tag{1} $$

The boundedness, existence and uniqueness of the ABC fractional advection-dispersion equation is first determined, using the Picard–Lindelöf theorem, before the numerical approximation in the following sections.

2.1 *Picard–Lindelöf Theorem for Existence and Uniqueness*

Applying the AB integral to both sides of the ABC fractional advection-dispersion equation, we get

$$ \begin{aligned} c\left(x,t\right) - c\left(x,0\right) &= {}_{0}^{AB}I_x^{\alpha}\left(-v_0{}^{ABC}D_x^{\alpha}\left(c\left(x,\tau\right)\right) + D_L\frac{\partial^2}{\partial x^2}\left(c\left(x,\tau\right)\right)\right)d\tau \\ &= \frac{1-\alpha}{AB\left(\alpha\right)}\left(-v_0{}^{ABC}D_x^{\alpha}\left(c\left(x,\tau\right)\right) + D_L\frac{\partial^2}{\partial x^2}\left(c\left(x,\tau\right)\right)\right) \\ &+ \frac{\alpha}{AB\left(\alpha\right)\Gamma\left(\alpha\right)}\int_0^t\left(-v_0{}^{ABC}D_x^{\alpha}\left(c\left(x,\tau\right)\right) + D_L\frac{\partial^2}{\partial x^2}\left(c\left(x,\tau\right)\right)\right)\left(t-\tau\right)^{\alpha-1}d\tau. \end{aligned} \tag{2} $$

Now, we consider a new function $F\left(x,t,c\right)$ to simplify:

$$ F\left(x,t,c\right) = -v_0{}^{ABC}D_x^{\alpha}\left(c\left(x,t\right)\right) + D_L\frac{\partial^2}{\partial x^2}\left(c\left(x,t\right)\right). \tag{3} $$

Thus

$$ c\left(x,t\right) - c\left(x,0\right) = \frac{1-\alpha}{AB\left(\alpha\right)}F\left(x,t,c\right) + \frac{\alpha}{AB\left(\alpha\right)\Gamma\left(\alpha\right)}\int_0^t F\left(x,t,c\right)\left(t-\tau\right)^{\alpha-1}d\tau. \tag{4} $$

Let

$$ C_{\lambda,\beta} = I_{\lambda}\left(t_0\right) \times B_{\beta}\left(x_0\right), $$

where

$$I_\lambda (t_0) = [t_0 - \lambda, t_0 + \lambda],$$

$$B_\beta (x_0) = [x_0 - \beta, x_0 + \beta].$$

The Banach fixed-point theorem is applied by introducing the norm of the supremum (statistical limit of a set) for I_λ,

$$M = \|\varphi\|_\infty = \begin{matrix} sup|\varphi (t)| \\ t \in I_\lambda (t_0). \end{matrix} \tag{5}$$

Considering the practical meaning of $c (x, t)$, it can be assumed that the initial concentration (c_0) will always be greater than subsequent concentrations (c_n) due to advection, dispersion and diffusion processes which reduce the concentration over time and space,

$$\|c\|_\infty < c_0. \tag{6}$$

Considering the max norm for the function $F (x, t, c)$,

$$\|F\|_\infty = \left\| -v \frac{AB (\alpha)}{(1 - \alpha)} \int_0^x \frac{d}{d\tau} c (\tau, x) E_\alpha \left[-\frac{\alpha}{1 - \alpha} (x - \tau)^\alpha \right] d\tau + D_L \frac{\partial^2}{\partial x^2} (c (x, t)) \right\|_\infty. \tag{7}$$

Thus

$$\|F\|_\infty \leq v \frac{AB (\alpha)}{(1 - \alpha)} \left\| \int_0^x \frac{d}{d\tau} c (\tau, x) E_\alpha \left[-\frac{\alpha}{1 - \alpha} (x - \tau)^\alpha \right] d\tau \right\|_\infty + D_L \left\| \frac{\partial^2}{\partial x^2} (c (x, t)) \right\|_\infty. \tag{8}$$

Applying the proven theorem for partial differential Lipschitz condition in [9], the second order derivative is bounded (M_1), thus

$$\|F\|_\infty \leq v \frac{AB (\alpha)}{(1 - \alpha)} \int_0^x \left\| \frac{d}{d\tau} C (\tau, x) \right\|_\infty \left\| E_\alpha \left[-\frac{\alpha}{1 - \alpha} (x - \tau)^\alpha \right] \right\|_\infty d\tau + D_L M_1. \tag{9}$$

The Mittag-Leffler function is bounded because $1 > \alpha > 0$. The first-order derivative is bounded due to its physical meaning being related to the spread of a particle defined by its concentration (M_2). Thus, the derivative is considered at the maximum physical time that is applicable to the existence of the concentration (T_{max}),

$$\|F\|_\infty \leq v \frac{AB (\alpha)}{(1 - \alpha)} M_2 T_{max} + D_L M_1 < \infty. \tag{10}$$

Therefore, the solution is bounded because we obtain a positive constant, such that

$$A = \|F\|_{\infty} = \begin{matrix} sup|F(x, t, c)| \\ t \in C_{\lambda,\beta}. \end{matrix} \tag{11}$$

Let $C_{\lambda,\beta}$ be a set where, $F : C_{\lambda,\beta} \rightarrow C_{\lambda,\beta}$, such that

$$\Gamma\phi(x, t) = c(x, 0) + \frac{1-\alpha}{AB(\alpha)} F(x, t, \phi) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t F(x, \tau, \phi)(t-\tau)^{\alpha-1} d\tau. \tag{12}$$

Thus

$$\begin{aligned} \|\Gamma\phi(x, t) - c(x, 0)\|_{\infty} = &\left\| \frac{1-\alpha}{AB(\alpha)} F(x, t, \phi) \right. \\ &\left. + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t F(x, \tau, \phi)(t-\tau)^{\alpha-1} d\tau \right\|_{\infty} \|\Gamma\phi(x, t) - c(x, 0)\|_{\infty} \\ \le &\frac{1-\alpha}{AB(\alpha)} \|F(x, t, \phi)\|_{\infty} + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|F(x, \tau, \phi)\|_{\infty} d\tau. \end{aligned} \tag{13}$$

The function $F(x, t, c)$ has been shown to be bounded (Eqs. (7)–(11))

$$\left[\|\Gamma\phi(x, t) - c(x, 0)\|_{\infty} \le \frac{1-\alpha}{AB(\alpha)} A + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} A \int_0^t (t-\tau)^{\alpha-1} d\tau \right].$$

The integral is considered at the maximum physical time that is applicable to the existence of the concentration (T_{max})

$$\begin{aligned} \left[\|\Gamma\phi(x, t) - c(x, 0)\|_{\infty} \le \frac{1-\alpha}{AB(\alpha)} A + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} A \frac{T_{max}^{\alpha}}{\alpha} \right] \\ \le \frac{1-\alpha}{AB(\alpha)} A + \frac{AT_{max}^{\alpha}}{AB(\alpha)\Gamma(\alpha)} < \infty. \end{aligned}$$

Therefore Γ is well-posed because we obtain a positive constant.

We want to prove that Γ is Lipschitz

$$\begin{aligned} \|\Gamma\phi_1 - \Gamma\phi_2\|_{\infty} = &\left\| \frac{1-\alpha}{AB(\alpha)} (F(x, t, \phi_1) - F(x, t, \phi_2)) \right. \\ &\left. + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (F(x, \tau, \phi_1) - F(x, \tau, \phi_2))(t-\tau)^{\alpha-1} d\tau \right\|_{\infty} \|\Gamma\phi_1 - \Gamma\phi_2\|_{\infty} \\ &\le \frac{1-\alpha}{AB(\alpha)} \|(F(x, t, \phi_1) - F(x, t, \phi_2))\|_{\infty} \\ &+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|F(x, \tau, \phi_1) - F(x, \tau, \phi_2)\|_{\infty} d\tau. \end{aligned} \tag{14}$$

To achieve this, we first evaluate

$$\begin{aligned}\|F(x,t,\phi_1)-F(x,t,\phi_2)\|_\infty\\=\left\|-v_0^{ABC}D_x^\alpha(\phi_1-\phi_2)+D_L\frac{\partial^2}{\partial x^2}(\phi_1-\phi_2)\right\|_\infty \|F(x,t,\phi_1)-F(x,t,\phi_2)\|_\infty\\ \le v\|_0^{ABC}D_x^\alpha(\phi_1-\phi_2)\|_\infty+D_L\left\|\frac{\partial^2}{\partial x^2}(\phi_1-\phi_2)\right\|_\infty .\end{aligned} \quad (15)$$

Applying the ABC fractional derivative, and the proven theorem for partial differential Lipschitz condition in [9], the second order derivative is bounded (ρ_2^2)

$$\begin{aligned}&\|F(x,t,\phi_1)-F(x,t,\phi_2)\|_\infty\\ \le& v\frac{AB(\alpha)}{(1-\alpha)}\int_0^x\left\|\frac{d}{d\tau}(\phi_1-\phi_2)\right\|_\infty\left\|E_\alpha\left[-\frac{\alpha}{1-\alpha}(x-\tau)^\alpha\right]\right\|_\infty d\tau+D_L\rho^2\|(\phi_1-\phi_2)\|_\infty .\end{aligned} \quad (16)$$

Similarly as in Eq. (9), the Mittag-Leffler function is bounded due to the constrain $1>\alpha>0$, and the first order derivative is bounded as explained (ρ_1). Thus, the derivative is considered at the maximum physical space that is applicable to the concentration (X_{max})

$$\|F(x,t,\phi_1)-F(x,t,\phi_2)\|_\infty\le v\frac{AB(\alpha)}{(1-\alpha)}X_{max}\rho_1\|(\phi_1-\phi_2)\|_\infty+D_L\rho_2^2\|(\phi_1-\phi_2)\|_\infty . \quad (17)$$

Simplifying

$$\begin{aligned}\|F(x,t,\phi_1)-F(x,t,\phi_2)\|_\infty&\le\left(v\frac{AB(\alpha)}{(1-\alpha)}X_{max}\rho_1+D_L\rho_2^2\right)\|(\phi_1-\phi_2)\|_\infty\\&<K_\alpha\|(\phi_1-\phi_2)\|_\infty .\end{aligned} \quad (18)$$

Applying to Eq. (14), we get

$$\begin{aligned}\|\Gamma\phi_1-\Gamma\phi_2\|_\infty\le&\frac{1-\alpha}{AB(\alpha)}K_\alpha\|(\phi_1-\phi_2)\|_\infty\\&+\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}K_\alpha\|(\phi_1-\phi_2)\|_\infty\int_0^t(t-\tau)^{\alpha-1}d\tau .\end{aligned} \quad (19)$$

Applying a similar process as previously, we obtain

$$\begin{aligned}\|\Gamma\phi_1 - \Gamma\phi_2\|_\infty &\le \frac{1-\alpha}{AB(\alpha)} K_\alpha \| (\phi_1 - \phi_2) \|_\infty + \frac{\alpha}{AB(\alpha)\,\Gamma(\alpha)} K_\alpha \| (\phi_1 - \phi_2) \|_\infty \frac{T_{max}^\alpha}{\alpha} \\ &\le \left(\frac{1-\alpha}{AB(\alpha)} K_\alpha + \frac{T_{max}^\alpha}{AB(\alpha)\,\Gamma(\alpha)} K_\alpha \right) \| (\phi_1 - \phi_2) \|_\infty \le V \| (\phi_1 - \phi_2) \|_\infty . \end{aligned} \tag{20}$$

Therefore, Γ is a contraction when $V < 1$, which translates to a condition

$$K_\alpha < \frac{1}{\frac{1-\alpha}{AB(\alpha)} + \frac{T_{max}^\alpha}{AB(\alpha)\Gamma(\alpha)}}. \tag{21}$$

Then $F(x, t, c)$ has a fixed point using the Banach fixed-point theorem and the ABC fractional advection-dispersion equation is bounded and has a unique solution under this condition.

2.2 Semi-discretisation Stability

The stability of the defined fractional advection-dispersion equation is evaluated in time, and thus discretised in time while the concentration in space is considered constant. The forward finite difference approximation in time is applied to the ABC fractional derivative, considered for a specific time (t_n), and the numerical integration of the Mittag-Leffler function is performed in [32]

$${}_0^{ABC}D_t^\alpha \left(c\left(x, t_n\right)\right) = \frac{AB(\alpha)}{(1-\alpha)} \sum_{k=0}^{n-1} \left(c_x^{k+1} - c_x^k\right) \delta_{n,k}^\alpha ,$$

where,

$$\delta_{n,k}^\alpha = (n-k)\, E_{\alpha,2}\left[-\frac{\alpha \Delta t}{1-\alpha} (n-k) \right] - (n-k-1)\, E_{\alpha,2}\left[-\frac{\alpha \Delta t}{1-\alpha} (n-k-1) \right]. \tag{22}$$

Now, substituting back into fractional advection-dispersion equation with the ABC derivative, and applying the assumption of discretisation in time only

$$\frac{AB(\alpha)}{(1-\alpha)} \sum_{k=0}^{n-1} \left(c_x^{k+1} - c_x^k\right) \delta_{n,k}^\alpha = -v_0^{ABC} D_x^\alpha \left(c_x^n\right) + D_L \frac{\partial^2}{\partial x^2} \left(c_x^{n+1}\right). \tag{23}$$

A function $A^{\alpha}_{n,k}$ is applied to simplify

$$\sum_{k=0}^{n-1}\left(c_x^{k+1}-c_x^k\right)A^{\alpha}_{n,k}=-v_0^{ABC}D_x^{\alpha}\left(c_x^n\right)+D_L\frac{\partial^2}{\partial x^2}\left(c_x^{n+1}\right). \tag{24}$$

Reformulating to obtain,

$$\left(c_x^{n+1}-c_x^n\right)A^{\alpha}_{n}+\sum_{k=0}^{n-1}\left(c_x^{k+1}-c_x^k\right)A^{\alpha}_{n,k}=-v_0^{ABC}D_x^{\alpha}\left(c_x^n\right)+D_L\frac{\partial^2}{\partial x^2}\left(c_x^{n+1}\right). \tag{25}$$

Rearranging

$$c_x^{n+1}=c_x^n-\frac{v}{A^{\alpha}_n}{}_0^{ABC}D_x^{\alpha}\left(c_x^n\right)+\frac{D_L}{A^{\alpha}_n}\frac{\partial^2}{\partial x^2}\left(c_x^{n+1}\right)-\frac{1}{A^{\alpha}_n}\sum_{k=0}^{n-1}\left(c_x^{k+1}-c_x^k\right)A^{\alpha}_{n,k}. \tag{26}$$

Equation (26) is the numerical approximation of the ABC fractional advection-dispersion equation with respect to time. Now, the semi-stability can be evaluated defining the following norms

$$(f,g)=\int_{\Omega}(f\cdot g)(x)\,dx,$$

where,

$$\|g\|_0=\sqrt{(g\cdot g)},$$

$$\|g\|_1=\sqrt{\|g\|_0+\varepsilon\|\frac{d^2}{dx^2}g\|_0}.$$

When $n=0$, Eq. (26) becomes

$$c_x^1=c_x^0-\frac{v}{A^{\alpha}_n}{}_0^{ABC}D_x^{\alpha}\left(c_x^0\right)+\frac{D_L}{A^{\alpha}_n}\frac{\partial^2}{\partial x^2}\left(c_x^1\right). \tag{27}$$

Simplifying using functions λ_1 and λ_2,

$$c_x^1=c_x^0-\lambda_{10}^{ABC}D_x^{\alpha}\left(c_x^0\right)+\lambda_2\frac{\partial^2}{\partial x^2}\left(c_x^1\right). \tag{28}$$

Applying the norm with respect to g,

$$\left(c_x^1,g\right)=\left(c_x^0,g\right)-\lambda_1\left({}_0^{ABC}D_x^{\alpha}c_x^0,{}_0^{ABC}D_x^{\alpha}g\right)+\lambda_2\left(\frac{\partial^2}{\partial x^2}c_x^1,\frac{\partial^2}{\partial x^2}g\right). \tag{29}$$

Let $\forall g \in H^1(\Omega),\ g = c_x^1$

$$\left(c_x^1, c_x^1\right) = \left(c_x^0, c_x^1\right) - \lambda_1 \left({}_0^{ABC}D_x^\alpha c_x^0, {}_0^{ABC} D_x^\alpha c_x^1\right) + \lambda_2 \left(\frac{\partial^2}{\partial x^2} c_x^1, \frac{\partial^2}{\partial x^2} c_x^1\right). \tag{30}$$

From the defined norms, the following statement is to be proven,

$$\|c_x^1\|_1 \leq \|c_x^0\|_0.$$

Reformulating in terms of the defined norms

$$\begin{aligned}\left(c_x^1, c_x^1\right) - \lambda_2 \left(\frac{\partial^2 c_x^1}{\partial x^2}, \frac{\partial^2 c_x^1}{\partial x^2}\right) &= \left(c_x^0, c_x^1\right) - \lambda_1 \left({}_0^{ABC}D_x^\alpha c_x^0, {}_0^{ABC} D_x^\alpha c_x^1\right) \|c_x^1\|_1^2 \\ &= \|c_x^0\|_0 \|c_x^1\|_0 - \lambda_1 \|{}_0^{ABC}D_x^\alpha c_x^0\|_0 \|{}_0^{ABC}D_x^\alpha c_x^1\|_0,\end{aligned} \tag{31}$$

where,

$$\left[\left\|{}_0^{ABC}D_x^\alpha c_x^0\right\|_0 = \left\|\frac{AB(\alpha)}{(1-\alpha)} \int_0^x \frac{dc_\tau^0}{d\tau} E_\alpha\left[-\frac{\alpha}{1-\alpha}(x-\tau)^\alpha\right] d\tau\right\|_0\right].$$

Thus,

$$\left\|{}_0^{ABC}D_x^\alpha c_x^0\right\|_0 \leq \frac{AB(\alpha)}{(1-\alpha)} \int_0^x \left\|\frac{dc_\tau^0}{d\tau}\right\|_0 \left\|E_\alpha\left[-\frac{\alpha}{1-\alpha}(x-\tau)^\alpha\right]\right\|_0 d\tau. \tag{32}$$

As before, the Mittag-Leffler function is bounded due to the limited range of α, where $1 > \alpha > 0$. The first-order derivative represents the spread of a particle, defined by its concentration, and is thus bound due to its inherent physical meaning. The derivative is considered at the maximum physical space that is applicable to the concentration (X_{max})

$$\|{}_0^{ABC}D_x^\alpha c_x^0\|_0 \leq \frac{AB(\alpha)}{(1-\alpha)} \int_0^x \left\|\frac{dc_\tau^0}{d\tau}\right\|_0 d\tau \leq \frac{AB(\alpha)}{(1-\alpha)} \theta \|c_x^0\|_0 \int_0^{x_{max}} d\tau \leq \frac{AB(\alpha)}{(1-\alpha)} \theta \|c_x^0\|_0 X_{max}. \tag{33}$$

Substituting back into Eq. (31)

$$\|c_x^1\|_1^2 < \|c_x^0\|_0 \|c_x^1\|_0 - \lambda_1 \left(\frac{AB(\alpha)}{(1-\alpha)} \theta \|c_x^0\|_0 X_{max}\right) \left(\frac{AB(\alpha)}{(1-\alpha)} \theta \|c_x^1\|_0 X_{max}\right). \tag{34}$$

Rearranging

$$\|c_x^1\|_1^2 < \|c_x^0\|_0\|c_x^1\|_0 - \lambda_1\left(\frac{AB\left(\alpha\right)\theta X_{max}}{(1-\alpha)}\right)^2\|c_x^0\|_0\|c_x^1\|_0$$
$$< \left(1-\lambda_1\left(\frac{AB\left(\alpha\right)\theta X_{max}}{(1-\alpha)}\right)^2\right)\|c_x^0\|_0\|c_x^1\|_0. \quad (35)$$

Applying the assumption that

$$\|c_x^1\|_0 \leq \|c_x^1\|_1.$$

Simplifying, the stability condition becomes

$$\|c_x^1\|_1^2 < \left(1-\lambda_1\left(\frac{AB\left(\alpha\right)\theta X_{max}}{(1-\alpha)}\right)^2\right)\|c_x^0\|_0\|c_x^1\|_1\|c_x^1\|_1$$
$$< \left(1-\lambda_1\left(\frac{AB\left(\alpha\right)\theta X_{max}}{(1-\alpha)}\right)^2\right)\|c_x^0\|_0\frac{\|c_x^1\|_1}{\|c_x^0\|_0} < 1-\lambda_1\left(\frac{AB\left(\alpha\right)\theta X_{max}}{(1-\alpha)}\right)^2, \quad (36)$$

where,

$$1-\lambda_1\left(\frac{AB\left(\alpha\right)\theta X_{max}}{(1-\alpha)}\right)^2 < 1\lambda_1\left(\frac{AB\left(\alpha\right)\theta X_{max}}{(1-\alpha)}\right)^2 > 0.$$

The first condition is thus upheld and unconditionally stable.
Secondly, Let $\forall g \in H^1\left(\Omega\right),\ g = c_x^{n+1}$

$$\left(c_x^{n+1}, c_x^{n+1}\right) = \left(c_x^n, c_x^{n+1}\right) - \lambda_1\left({}_0^{ABC}D_x^{\alpha}c_x^n, {}_0^{ABC}D_x^{\alpha}c_x^{n+1}\right) + \lambda_2\left(\frac{\partial^2}{\partial x^2}c_x^{n+1}, \frac{\partial^2}{\partial x^2}c_x^{n+1}\right)$$
$$-\lambda_3\sum_{k=0}^{n-1}\left(\left(c_x^{k+1}, c_x^{n+1}\right) - \left(c_x^k, c_x^{n+1}\right)\right), \quad (37)$$

where,

$$\lambda_3 = \frac{1}{A_n^{\alpha}}A_{n,k}^{\alpha}.$$

From the defined norms, the following statement is to be proven

$$\|c_x^{n+1}\|_1 \leq \|c_x^0\|_0.$$

Reformulating Eq. (37) in terms of the defined norms, we have

$$\left(c_x^{n+1}, c_x^{n+1}\right) - \lambda_2 \left(\frac{\partial^2 c_x^{n+1}}{\partial x^2}, \frac{\partial^2 c_x^{n+1}}{\partial x^2}\right) = \left(c_x^n, c_x^{n+1}\right) - \lambda_1 \left({}_0^{ABC}D_x^{\alpha} c_x^n, {}_0^{ABC}D_x^{\alpha} c_x^{n+1}\right)$$
$$- \lambda_3 \sum_{k=0}^{n-1} \left(\left(c_x^{k+1}, c_x^{n+1}\right) - \left(c_x^k, c_x^{n+1}\right)\right)$$
$$\|c_x^{n+1}\|_1^2 = \|c_x^n\|_0 \|c_x^{n+1}\|_0 - \lambda_1 \|{}_0^{ABC}D_x^{\alpha} c_x^n\|_0 \|{}_0^{ABC}D_x^{\alpha} c_x^{n+1}\|_0$$
$$- \lambda_3 \sum_{k=0}^{n-1} \left(\left(\|c_x^{k+1}\|_0 \|c_x^{n+1}\|_0\right) - \left(\|c_x^k\|_0 \|c_x^{n+1}\|_0\right)\right). \quad (38)$$

Applying Eq. (33), we get

$$\|c_x^{n+1}\|_1^2 \le \|c_x^n\|_0 \|c_x^{n+1}\|_0 - \lambda_1 A \|c_x^n\|_0 \|c_x^{n+1}\|_0 - \lambda_3 \sum_{k=0}^{n-1} \left(\left(\|c_x^{k+1}\|_0 \|c_x^{n+1}\|_0\right) - \left(\|c_x^k\|_0 \|c_x^{n+1}\|_0\right)\right), \quad (39)$$

where,

$$A = \left(\frac{AB(\alpha)\,\theta X_{max}}{(1-\alpha)}\right)^2.$$

Using the inductive method for

$$\|c_x^n\|_0 \le \|c_x^0\|_0,$$

Equation (39) becomes

$$\|c_x^{n+1}\|_1^2 \le \|c_x^0\|_0 \|c_x^{n+1}\|_0 - \lambda_1 A \|c_x^0\|_0 \|c_x^{n+1}\|_0 - \lambda_3 \sum_{k=0}^{n-1} \left(\left(\|c_x^0\|_0 \|c_x^{n+1}\|_0\right) - \left(\|c_x^0\|_0 \|c_x^{n+1}\|_0\right)\right). \quad (40)$$

Reformulating in terms of the defined norms, we have

$$\|c_x^{n+1}\|_1^2 \le \|c_x^0\|_0 \|c_x^{n+1}\|_1 - \lambda_1 A \|c_x^0\|_0 \|c_x^{n+1}\|_1. \quad (41)$$

Rearranging and simplifying

$$\|c_x^{n+1}\|_1^2 \le (1-\lambda_1 A)\, \|c_x^0\|_0 \|c_x^{n+1}\|_1 \|c_x^{n+1}\|_1 \le (1-\lambda_1 A)\, \|c_x^0\|_0, \quad (42)$$

where,

$$[1 - \lambda_1 A < 1],$$

$$\lambda_1 A > 0.$$

The second condition is thus unconditionally stable. This concludes the semi-discretisation analysis for an evolution equation, where the proposed ABC fractional advection-dispersion equation has been found to be stable in time.

3 Upwind Numerical Approximation Schemes

Boundedness, existence and uniqueness has been established for the ABC fractional advection-dispersion equation. Furthermore, the stability in time has been demonstrated for the equation. Numerical schemes for this equation are now explored to facilitate the solution of the complex equation. Upwind-based finite difference schemes are investigated as motivated in [33], where upwind schemes aim to improve the stability of advection-dominated transport [34].

3.1 First-Order Upwind Explicit

The numerical approximation of the ABC fractional derivative with respect to time is considered, with the resulting scheme as [32]:

$$ {}_{0}^{ABC}D_{t}^{\alpha}\left(c\left(x_{m},t_{k}\right)\right)=\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{k=0}^{n-1}\left(c_{m}^{k+1}-c_{m}^{k}\right)\begin{pmatrix}\left(n-k\right)E_{\alpha,2}\left[-\frac{\alpha\Delta t}{1-\alpha}\left(n-k\right)\right]\\-\left(n-k-1\right)E_{\alpha,2}\left[-\frac{\alpha\Delta t}{1-\alpha}\left(n-k-1\right)\right]\end{pmatrix}, \tag{43} $$

where, a function $\delta_{n,k}^{\alpha}$ is applied to simplify,

$$ \delta_{n,k}^{\alpha}=\left(n-k\right)E_{\alpha,2}\left[-\frac{\alpha\Delta t}{1-\alpha}\left(n-k\right)\right]-\left(n-k-1\right)E_{\alpha,2}\left[-\frac{\alpha\Delta t}{1-\alpha}\left(n-k-1\right)\right]. $$

Thus,

$$ {}_{0}^{ABC}D_{t}^{\alpha}\left(c\left(x_{m},t_{k}\right)\right)=\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{k=0}^{n-1}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha}. \tag{44} $$

The upwind finite difference scheme uses a one-sided finite difference in the upstream direction to approximate the advection term of the advection-dispersion equation (assuming $v>0$). Applying the upwind scheme and the ABC fractional derivative with respect to space (explicit) becomes [32]

$$ {}_{0}^{ABC}D_{x}^{\alpha}\left(c\left(x_{m},t_{k}\right)\right)=\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)\begin{pmatrix}\left(m-i\right)E_{\alpha,2}\left[-\frac{\alpha\Delta x}{1-\alpha}\left(m-i\right)\right]\\-\left(m-i-1\right)E_{\alpha,2}\left[-\frac{\alpha\Delta x}{1-\alpha}\left(m-i-1\right)\right]\end{pmatrix}, \tag{45} $$

where, a function $\delta_{m,i}^{\alpha}$ is applied to simplify

$$ {}_{0}^{ABC}D_{x}^{\alpha}\left(c\left(x_{m},t_{k}\right)\right)=\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)\delta_{m,i}^{\alpha}. \tag{46} $$

Substituting into the ABC fractional advection-dispersion equation and using the traditional finite difference approach for the local second-order derivative, we have

$$ \frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{k=0}^{n-1}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha}+v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)\delta_{m,i}^{\alpha} - D_{L}\left(\frac{c_{m+1}^{n-1}-2c_{m}^{n-1}+c_{m-1}^{n-1}}{\left(\Delta x\right)^{2}}\right)=0. \tag{47} $$

Reformulating the following can be obtained

$$ \frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\left(c_{m}^{n}-c_{m}^{n-1}\right)\delta_{n,n-1}^{\alpha}+\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{k=0}^{n-2}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha} +v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\left(c_{m}^{n-1}-c_{m-1}^{n-1}\right)\delta_{m,i}^{\alpha}+v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)\delta_{m,i}^{\alpha} - D_{L}\left(\frac{c_{m+1}^{n-1}-2c_{m}^{n-1}+c_{m-1}^{n-1}}{\left(\Delta x\right)^{2}}\right)=0. \tag{48} $$

Rearranging

$$ c_{m}^{n}\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{n,n-1}^{\alpha}=c_{m}^{n-1}\left(\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{n,n-1}^{\alpha}-v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}-\frac{2D_{L}}{\left(\Delta x\right)^{2}}\right) +c_{m-1}^{n-1}\left(v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}+\frac{D_{L}}{\left(\Delta x\right)^{2}}\right)+c_{m+1}^{n-1}\left(\frac{D_{L}}{\left(\Delta x\right)^{2}}\right)-\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{k=0}^{n-2}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha} - v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)\delta_{m,i}^{\alpha}. \tag{49} $$

The numerical scheme can be simplified using place-keeper functions as follows

$$ ac_{m}^{n}=bc_{m}^{n-1}+dc_{m-1}^{n-1}+fc_{m+1}^{n-1}-g\sum_{k=0}^{n-2}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha}-vg\sum_{i=0}^{m}\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)\delta_{m,i}^{\alpha}, \tag{50} $$

where,

$$a = \frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1};\, b = \frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} - v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} - \frac{2D_L}{(\Delta x)^2};\, d = v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{D_L}{(\Delta x)^2},$$

$$f = \frac{D_L}{(\Delta x)^2};\, g = \frac{AB(\alpha)}{(1-\alpha)}.$$

3.2 First-Order Upwind Implicit

Following the same approach as the explicit upwind numerical approximation, the following is obtained for the implicit upwind scheme

$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{n-1}\left(c_m^{k+1} - c_m^k\right)\delta^{\alpha}_{n,k} + v\frac{AB(\alpha)}{(1-\alpha)}\sum_{i=0}^{m}\left(c_i^n - c_{i-1}^n\right)\delta^{\alpha}_{m,i} - D_L\left(\frac{c_{m+1}^n - 2c_m^n + c_{m-1}^n}{(\Delta x)^2}\right) = 0. \quad (51)$$

Reformulating and rearranging the following can be obtained

$$c_m^n\left(\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} + v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2}\right) = c_{m-1}^n\left(v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} - \frac{D_L}{(\Delta x)^2}\right)$$
$$+c_{m+1}^n\frac{D_L}{(\Delta x)^2} - \frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{n-2}\left(c_m^{k+1} - c_m^k\right)\delta^{\alpha}_{n,k} - v\frac{AB(\alpha)}{(1-\alpha)}\sum_{i=0}^{m}\left(c_i^n - c_{i-1}^n\right)\delta^{\alpha}_{m,i}$$
$$+ c_m^{n-1}\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}. \quad (52)$$

The numerical scheme is simplified by substituting functions as followings

$$hc_m^n = jc_{m-1}^n + fc_{m+1}^n - g\sum_{k=0}^{n-2}\left(c_m^{k+1} - c_m^k\right)\delta^{\alpha}_{n,k} - vg\sum_{i=0}^{m}\left(c_i^n - c_{i-1}^n\right)\delta^{\alpha}_{m,i} + ac_m^{n-1}, \quad (53)$$

where,

$$h = \frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} + v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2};\, j = v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} - \frac{D_L}{(\Delta x)^2}.$$

3.3 First-Order Upwind Crank Nicolson Scheme

The upwind Crank Nicolson finite difference scheme for ABC fractional advection-dispersion equation is now considered [33]. The time component remains the same as with the implicit/explicit upwind schemes, but the space components change to

$$ {}_{0}^{ABC}D_{x}^{\alpha}\left(c\left(x_{m},t_{k}\right)\right)=\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left[0.5\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)+0.5\left(c_{i}^{n}-c_{i-1}^{n}\right)\right)\delta_{m,i}^{\alpha}. \quad (54) $$

Substituting

$$ \frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{k=0}^{n-1}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha}+v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left[0.5\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)+0.5\left(c_{i}^{n}-c_{i-1}^{n}\right)\right]\delta_{m,i}^{\alpha} - D_{L}\left(\frac{c_{m+1}^{n}-2c_{m}^{n}+c_{m-1}^{n}}{\left(\Delta x\right)^{2}}\right)=0. \quad (55) $$

Reformulating and rearranging, the following can be obtained

$$ \begin{aligned} c_{m}^{n}\left(\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{n,n-1}^{\alpha}+0.5v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}+\frac{2D_{L}}{\left(\Delta x\right)^{2}}\right)&=c_{m}^{n-1}\\ \times\left(\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{n,n-1}^{\alpha}+0.5v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}\right)&\\ +c_{m-1}^{n}\left(0.5v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}-\frac{D_{L}}{\left(\Delta x\right)^{2}}\right)+c_{m+1}^{n}\frac{D_{L}}{\left(\Delta x\right)^{2}}+c_{m-1}^{n-1}0.5v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}&\\ -\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{k=0}^{n-2}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha}-v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\sum_{i=0}^{m}\left[0.5\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)+0.5\left(c_{i}^{n}-c_{i-1}^{n}\right)\right]\delta_{m,i}^{\alpha}.& \end{aligned} \quad (56) $$

Simplifying by substituting place-keeper functions

$$ lc_{m}^{n}=mc_{m}^{n-1}+oc_{m-1}^{n}+fc_{m+1}^{n}+pc_{m-1}^{n-1}-g\sum_{k=0}^{n-2}\left(c_{m}^{k+1}-c_{m}^{k}\right)\delta_{n,k}^{\alpha} - vg\sum_{i=0}^{m}\left[0.5\left(c_{i}^{n-1}-c_{i-1}^{n-1}\right)+0.5\left(c_{i}^{n}-c_{i-1}^{n}\right)\right]\delta_{m,i}^{\alpha}, \quad (57) $$

where,

$$l = \frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} + 0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2}; m = \frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} + 0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i},$$

$$o = 0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} - \frac{D_L}{(\Delta x)^2}; p = 0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i}.$$

3.4 First-Order Upwind-Downwind Weighted Scheme (Explicit)

For the upwind-downwind weighted scheme, the upwind and downwind direction for the advection term are both integrated using a weighting factor. The weighting factor of upwind to downwind is defined as θ, where $0 \le \theta \le 1$ [33]. Thus, the advection component is approximated as

$$ {}^{ABC}_{0}D^{\alpha}_{x}\left(c\left(x_m, t_k\right)\right) = \frac{AB(\alpha)}{(1-\alpha)}\sum_{i=0}^{m}\left[\theta\left(c_i^{n-1} - c_{i-1}^{n-1}\right) + (1-\theta)\left(c_{i+1}^{n-1} - c_i^{n-1}\right)\right]\delta^{\alpha}_{m,i}. \tag{58}$$

Substituting this back into the advection-dispersion equation

$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{n-1}\left(c_m^{k+1} - c_m^{k}\right)\delta^{\alpha}_{n,k} + v\frac{AB(\alpha)}{(1-\alpha)}\sum_{i=0}^{m}\left[\theta\left(c_i^{n-1} - c_{i-1}^{n-1}\right) + (1-\theta)\left(c_{i+1}^{n-1} - c_i^{n-1}\right)\right]\delta^{\alpha}_{m,i} - D_L\left(\frac{c_{m+1}^{n-1} - 2c_m^{n-1} + c_{m-1}^{n-1}}{(\Delta x)^2}\right) = 0. \tag{59}$$

Reformulating and rearranging, the following can be obtained

$$\begin{aligned} c_m^n\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} &= c_m^{n-1}\left(\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}\right. \\ &\quad \left. - v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + v(1-\theta)\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} - \frac{2D_L}{(\Delta x)^2}\right) \\ &+ c_{m-1}^{n-1}\left(v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{D_L}{(\Delta x)^2}\right) - c_{m+1}^{n-1}\left(v(1-\theta)\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{D_L}{(\Delta x)^2}\right) \\ &- \frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{n-2}\left(c_m^{k+1} - c_m^{k}\right)\delta^{\alpha}_{n,k} - v\frac{AB(\alpha)}{(1-\alpha)}\sum_{i=0}^{m}\left[\theta\left(c_i^{n-1} - c_{i-1}^{n-1}\right) + (1-\theta)\left(c_{i+1}^{n-1} - c_i^{n-1}\right)\right]\delta^{\alpha}_{m,i}. \end{aligned} \tag{60}$$

Place-keeper functions are used to simplify the explicit upwind-downwind weighted scheme as followings

$$ac_m^n = qc_m^{n-1} + rc_{m-1}^{n-1} - sc_{m+1}^{n-1} - g\sum_{k=0}^{n-2}\left(c_m^{k+1} - c_m^k\right)\delta_{n,k}^{\alpha}$$
$$-vg\sum_{i=0}^{m}\left[\theta\left(c_i^{n-1} - c_{i-1}^{n-1}\right) + (1-\theta)\left(c_{i+1}^{n-1} - c_i^{n-1}\right)\right]\delta_{m,i}^{\alpha}, \quad (61)$$

where,

$$q = \frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha} - v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} + v(1-\theta)\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} - \frac{2D_L}{(\Delta x)^2};$$
$$r = v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} + \frac{D_L}{(\Delta x)^2},$$
$$s = v(1-\theta)\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} + \frac{D_L}{(\Delta x)^2}.$$

3.5 First-Order Upwind-Downwind Weighted Scheme (Implicit)

Correspondingly, both the upwind and downwind directions are considered for the advection term in the implicit upwind-downwind weighted scheme and the space advection component becomes

$${}_{0}^{ABC}D_x^{\alpha}\left(c\left(x_m, t_k\right)\right) = \frac{AB(\alpha)}{(1-\alpha)}\sum_{i=0}^{m}\left[\theta\left(c_i^n - c_{i-1}^n\right) + (1-\theta)\left(c_{i+1}^n - c_i^n\right)\right]\delta_{m,i}^{\alpha}. \quad (62)$$

Substituting

$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{n-1}\left(c_m^{k+1} - c_m^k\right)\delta_{n,k}^{\alpha} + v\frac{AB(\alpha)}{(1-\alpha)}\sum_{i=0}^{m}\left[\theta\left(c_i^n - c_{i-1}^n\right) + (1-\theta)\left(c_{i+1}^n - c_i^n\right)\right]\delta_{m,i}^{\alpha}$$
$$- D_L\left(\frac{c_{m+1}^n - 2c_m^n + c_{m-1}^n}{(\Delta x)^2}\right) = 0. \quad (63)$$

Rearranging

$$c_m^n\left(\frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha} + v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} - v(1-\theta)\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} + \frac{2D_L}{(\Delta x)^2}\right)$$
$$= c_{m+1}^n\left(\frac{D_L}{(\Delta x)^2} - v(1-\theta)\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}\right) + c_{m-1}^n\left(\frac{D_L}{(\Delta x)^2} + v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}\right)$$

$$+ c_m^{n-1} \frac{AB(\alpha)}{(1-\alpha)} \delta_{n,n-1}^{\alpha} - \frac{AB(\alpha)}{(1-\alpha)} \sum_{k=0}^{n-2} \left(c_m^{k+1} - c_m^k\right) \delta_{n,k}^{\alpha}$$

$$- v \frac{AB(\alpha)}{(1-\alpha)} \sum_{i=0}^{m} \left[\theta \left(c_i^n - c_{i-1}^n\right) + (1-\theta) \left(c_{i+1}^n - c_i^n\right)\right] \delta_{m,i}^{\alpha}. \quad (64)$$

Simplifying

$$uc_m^n = vc_{m+1}^n + rc_{m-1}^n + ac_m^{n-1} - g \sum_{k=0}^{n-2} \left(c_m^{k+1} - c_m^k\right) \delta_{n,k}^{\alpha}$$

$$- vg \sum_{i=0}^{m} \left[\theta \left(c_i^n - c_{i-1}^n\right) + (1-\theta) \left(c_{i+1}^n - c_i^n\right)\right] \delta_{m,i}^{\alpha}, \quad (65)$$

where,

$$u = \frac{AB(\alpha)}{(1-\alpha)} \delta_{n,n-1}^{\alpha} + v\theta \frac{AB(\alpha)}{(1-\alpha)} \delta_{m,i}^{\alpha} - v(1-\theta) \frac{AB(\alpha)}{(1-\alpha)} \delta_{m,i}^{\alpha} + \frac{2D_L}{(\Delta x)^2},$$

$$v = \frac{D_L}{(\Delta x)^2} - v(1-\theta) \frac{AB(\alpha)}{(1-\alpha)} \delta_{m,i}^{\alpha}.$$

This concludes the formulation of the numerical approximations schemes to be investigated for the ABC fractional advection-dispersion equation. In the following section, the numerical stability of each scheme will be assessed.

4 Numerical Stability Analysis

The numerical stability analysis is performed using the recursive numerical stability method [33, 35, 36]. The numerical stability for the upwind schemes are evaluated to validate their use in solving the ABC fractional advection-dispersion equation for fracture flow in groundwater systems.

4.1 First-Order Upwind Implicit

Substituting the induction method terms for the developed finite difference first-order upwind (implicit) numerical scheme discussed in Sect. 3.2

$$hc_ne^{jk_im} = jc_ne^{jk_i(m-\Delta m)} + fc_ne^{jk_ix(m+\Delta m)} - g\sum_{k=0}^{n-2}\left(c_{k+1}e^{jk_im} - c_ke^{jk_im}\right)\delta_{n,k}^{\alpha}$$
$$-vg\sum_{i=0}^{m}\left(c_ne^{jk_im} - c_ne^{jk_i(m-\Delta m)}\right)\delta_{m,i}^{\alpha} + ac_{n-1}e^{jk_im}. \quad (66)$$

Expand and simplify

$$hc_n = jc_ne^{-jk_i\Delta m} + fc_ne^{jk_i\Delta m} - g\sum_{k=0}^{n-2}(c_{k+1} - c_k)\,\delta_{n,k}^{\alpha}$$
$$-vg\sum_{i=0}^{m}\left(c_n - c_ne^{-jk_i\Delta m}\right)\delta_{m,i}^{\alpha} + ac_{n-1}. \quad (67)$$

The first procedure for the induction numerical stability analysis requires proving for a set $\forall\, n > 1$, that

$$|c_n| < |c_o|. \quad (68)$$

If $n = 1$, then

$$hc_1 = jc_1e^{-jk_i\Delta m} + fc_1e^{jk_i\Delta m} - vg\sum_{i=0}^{m}\left(c_1 - c_1e^{-jk_i\Delta m}\right)\delta_{m,i}^{\alpha} + ac_0. \quad (69)$$

A subset for m is now considered, where $m = 0$

$$hc_1 = jc_1e^{-jk_i\Delta m} + fc_1e^{jk_i\Delta m} + ac_0. \quad (70)$$

Simplifying and rearranging

$$\frac{c_1}{c_0} = \frac{a}{h - je^{-jk_i\Delta m} - fe^{jk_i\Delta m}}. \quad (71)$$

Taking the norm, the condition for the first induction requirement becomes

$$\frac{|a|}{|h| + |j| + |f|} < 1. \quad (72)$$

The term is expanded using the simplification terms associated with Eq. (53)

$$\frac{\left|\frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha}\right|}{\left|\frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha} + v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} + \frac{2D_L}{(\Delta x)^2}\right| + \left|v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} - \frac{D_L}{(\Delta x)^2}\right| + \left|\frac{D_L}{(\Delta x)^2}\right|} < 1. \quad (73)$$

The following assumption is made

$$\left[v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha} > \frac{D_L}{\left(\Delta x\right)^2}\right].$$

Then, the condition is

$$\frac{\frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha}}{\frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha}+v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}+\frac{2D_L}{(\Delta x)^2}+v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}-\frac{D_L}{(\Delta x)^2}+\frac{D_L}{(\Delta x)^2}} < 1. \tag{74}$$

Simplifying

$$2v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}+\frac{2D_L}{\left(\Delta x\right)^2} > 0. \tag{75}$$

Therefore, under this assumption, the first inductive stability condition for this subset is upheld and unconditionally stable.

The complementary assumption is made, and the condition becomes

$$\frac{\frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha}}{\frac{AB(\alpha)}{(1-\alpha)}\delta_{n,n-1}^{\alpha}+v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}+\frac{2D_L}{(\Delta x)^2}-v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}+\frac{D_L}{(\Delta x)^2}+\frac{D_L}{(\Delta x)^2}} < 1. \tag{76}$$

Simplifying

$$\frac{4D_L}{\left(\Delta x\right)^2} > 0. \tag{77}$$

Thus, the first inductive stability condition for this subset is supported and unconditionally stable under this assumption as well.

A subset for (m) is now considered for all $(m \geq 1)$

$$hc_1 = jc_1e^{-jk_i\Delta m} + fc_1e^{jk_i\Delta m} - vg\sum_{i=0}^{m}\left(c_1 - c_1e^{-jk_i\Delta m}\right)\delta_{m,i}^{\alpha} + ac_0 \tag{78}$$

Simplifying

$$\left(h - je^{-jk_i\Delta m} - fe^{jk_i\Delta m} + vg\left(1 - e^{-jk_i\Delta m}\right)\sum_{i=0}^{m}\delta_{m,i}^{\alpha}\right)c_1 = ac_0. \tag{79}$$

Expanding the summation, and simplifying

$$\sum_{i=0}^{m}\delta_{m,i}^{\alpha} = \left(\left(m\right)E_{\alpha,2}\left[-\frac{\alpha\Delta x}{1-\alpha}\left(m\right)\right]\right) + \left(E_{\alpha,2}\left[-\frac{\alpha\Delta x}{1-\alpha}\right] - \left(-1\right)E_{\alpha,2}\left[-\frac{\alpha\Delta x}{1-\alpha}\left(-1\right)\right]\right). \tag{80}$$

Substituting back into Eq. (77), we get

$$\begin{pmatrix} h - je^{-jk_i \Delta m} - fe^{jk_i \Delta m} + vg\left(1 - e^{-jk_i \Delta m}\right) \cdot \\ \left((m)\, E_{\alpha,2}\left[-\frac{\alpha \Delta x}{1-\alpha}(m)\right] + E_{\alpha,2}\left[-\frac{\alpha \Delta x}{1-\alpha}\right] - (-1)\, E_{\alpha,2}\left[-\frac{\alpha \Delta x}{1-\alpha}(-1)\right]\right) \end{pmatrix} c_1 = ac_0, \tag{81}$$

where the function $\beta_{m,E_{\alpha,2}}$ is used to simplify as follows

$$\left(h - je^{-jk_i \Delta m} - fe^{jk_i \Delta m} + vg\left(1 - e^{-jk_i \Delta m}\right)\beta_{m,E_{\alpha,2}}\right) c_1 = ac_0. \tag{82}$$

Let a function simplify to

$$[\phi = k_i \Delta x],$$

where,

$$[e^{-j\phi} = e^{-jk_i \Delta x}].$$

Remembering Euler's formula for complex numbers, and substituting back into Eq. (80)

$$\left(h - je^{-jk_i \Delta m} - fe^{jk_i \Delta m} + vg\,(1 - cos\phi + isin\phi)\,\beta_{m,E_{\alpha,2}}\right) c_1 = ac_0. \tag{83}$$

Applying a norm and simplifying

$$\left(|h| + |j| + |f| + v|g|\,(2 - 2cos\phi)\,|\beta_{m,E_{\alpha,2}}|\right)|c_1| = |a||c_0|. \tag{84}$$

Rearranging

$$\frac{|c_1|}{|c_0|} = \frac{|a|}{\left(|h| + |j| + |f| + v|g|\,(2 - 2cos\phi)\,|\beta_{m,E_{\alpha,2}}|\right)}. \tag{85}$$

Thus, the condition becomes

$$\frac{|a|}{\left(|h| + |j| + |f| + v|g|\,(2 - 2cos\phi)\,|\beta_{m,E_{\alpha,2}}|\right)} < 1. \tag{86}$$

The term is expanded using the simplification terms associated with Eq. (53)

$$\frac{\left|\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}\right|}{\begin{pmatrix} \left|\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} + v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2}\right| + \left|v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} - \frac{D_L}{(\Delta x)^2}\right| + \left|\frac{D_L}{(\Delta x)^2}\right| \\ +v\left|\frac{AB(\alpha)}{(1-\alpha)}\right|(2 - 2cos\phi)\,|\beta_{m,E_{\alpha,2}}| \end{pmatrix}} < 1. \tag{87}$$

The assumption is made where

$$\left[v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} > \frac{D_L}{(\Delta x)^2}\right].$$

Then, the conditions is

$$2v\frac{AB(\alpha)}{(1-\alpha)}\left(\delta_{m,i}^{\alpha}+(1-cos\phi)\,\beta_{m,E_{\alpha,2}}\right)+\frac{2D_L}{(\Delta x)^2}>0. \tag{88}$$

Under this assumption, the first inductive stability condition for the second subset of m is sustained and unconditionally stable.

The opposite assumption is made, and the condition becomes

$$\frac{4D_L}{(\Delta x)^2}+v\frac{AB(\alpha)}{(1-\alpha)}(2-2cos\phi)\,\beta_{m,E_{\alpha,2}}>0. \tag{89}$$

The first inductive stability condition for the second subset of m is upheld and unconditionally stable under this assumption as well.

The second procedure for the induction numerical stability analysis requires proving for a set $\forall\, n\geq 1$

$$|c_n|<|c_o|. \tag{90}$$

Rearranging Eq. (66) for c_n

$$\left(h+e^{-jk_i\Delta m}-fe^{jk_i\Delta m}+vg\left(1-e^{-jk_i\Delta m}\right)\sum_{i=0}^{m}\delta_{m,i}^{\alpha}\right)c_n=ac_{n-1}-g\sum_{k=0}^{n-2}(c_{k+1}-c_k)\,\delta_{n,k}^{\alpha}. \tag{91}$$

Following a similar simplification process as previously performed

$$\left(h+je^{-jk_i\Delta m}-fe^{jk_i\Delta m}+vg\,(1-cos\phi+isin\phi)\,\beta_{m,E_{\alpha,2}}\right)c_n$$
$$=ac_{n-1}-g\sum_{k=0}^{n-2}(c_{k+1}-c_k)\,\delta_{n,k}^{\alpha}. \tag{92}$$

Applying a norm

$$\left|\left(h+je^{-jk_i\Delta m}-fe^{jk_i\Delta m}+vg\,(1-cos\phi+isin\phi)\,\beta_{m,E_{\alpha,2}}\right)c_n\right|$$
$$=\left|ac_{n-1}-g\sum_{k=0}^{n-2}(c_{k+1}-c_k)\,\delta_{n,k}^{\alpha}\right|. \tag{93}$$

Therefore,

$$|h+je^{-jk_i\Delta m}-fe^{jk_i\Delta m}+vg\,(1-cos\phi+isin\phi)\,\beta_{m,E_{\alpha,2}}||c_n|$$
$$<|a||c_{n-1}|+|g|\left|\sum_{k=0}^{n-2}(c_{k+1}-c_k)\,\delta_{n,k}^{\alpha}\right|. \tag{94}$$

Remembering that it has been proved that for a set $\forall\, n > 1$

$$[|c_{n-1}| < |c_o|].$$

Thus,

$$|h + je^{-jk_i \Delta m} - fe^{jk_i \Delta m} + vg\,(1 - cos\phi + isin\phi)\,\beta_{m,E_{\alpha,2}}||c_n| < |a||c_{n-1}| + |g|\left|\sum_{k=0}^{n-2}(c_{k+1} - c_k)\,\delta_{n,k}^{\alpha}\right| < |a||c_o| + |g|\left|\sum_{k=0}^{n-2}(c_{k+1} - c_k)\,\delta_{n,k}^{\alpha}\right|. \tag{95}$$

Therefore, it can be inferred that

$$|h + je^{-jk_i \Delta m} - fe^{jk_i \Delta m} + vg\,(1 - cos\phi + isin\phi)\,\beta_{m,E_{\alpha,2}}||c_n| < |a||c_o| + |g|\left|\sum_{k=0}^{n-2}(c_{k+1} - c_k)\,\delta_{n,k}^{\alpha}\right|. \tag{96}$$

The remaining summation is considered at the upper limit

$$\left|\sum_{k=0}^{n-2}(c_{k+1} - c_k)\,\delta_{n,k}^{\alpha}\right| < \sum_{k=0}^{n-2}|c_{k+1}|\left(\left|1 - \frac{c_k}{c_{k+1}}\right|\right)\delta_{n,k}^{\alpha}. \tag{97}$$

Subset (k) will follow the same assumption made for a set ($\forall\, n \geq 1$), where

$$\sum_{k=0}^{n-2} c_{k+1}\left(1 - \frac{c_k}{c_{k+1}}\right)\delta_{n,k}^{\alpha} < |c_0|\sum_{k=0}^{n-2}\delta_{n,k}^{\alpha}.$$

Substituting back into Eq. (97)

$$|h + je^{-jk_i \Delta m} - fe^{jk_i \Delta m} + vg\,(1 - cos\phi + isin\phi)\,\beta_{m,E_{\alpha,2}}||c_n| < |a||c_o| + |g||c_0|\sum_{k=0}^{n-2}\delta_{n,k}^{\alpha}. \tag{98}$$

Expanding the summation

$$\sum_{k=0}^{n-2}\delta_{n,k}^{\alpha} = \left((n)\,E_{\alpha,2}\left[-\frac{\alpha\Delta t}{1-\alpha}\,(n)\right]\right) + \left(2E_{\alpha,2}\left[-2\frac{\alpha\Delta t}{1-\alpha}\right] - E_{\alpha,2}\left[-\frac{\alpha\Delta t}{1-\alpha}\right]\right).$$

Substituting back into Eq. (98) and simplifying

$$|h + je^{-jk_i \Delta m} - fe^{jk_i \Delta m} + vg\,(1 - cos\phi + isin\phi)\,\beta_{m,E_{\alpha,2}}||c_n| < |a||c_o| + |g||c_0|\beta_{n,E_{\alpha,2}}, \quad (99)$$

where,

$$\beta_{n,E_{\alpha,2}} = nE_{\alpha,2}\left[-\frac{\alpha \Delta tn}{1-\alpha}\right] + 2E_{\alpha,2}\left[-\frac{2\alpha \Delta t}{1-\alpha}\right] - E_{\alpha,2}\left[-\frac{\alpha \Delta t}{1-\alpha}\right].$$

Simplifying and rearranging

$$\frac{|c_n|}{|c_o|} < \frac{|a| + |g|\beta_{n,E_{\alpha,2}}}{|h| + |j| + |f| + v|g|\,(|1 - cos\phi| + i|sin\phi|)\,|\beta_{m,E_{\alpha,2}}|}. \quad (100)$$

Thus, the condition becomes

$$\frac{|a| + |g|\beta_{n,E_{\alpha,2}}}{|h| + |j| + |f| + v|g|\,(|1 - cos\phi| + i|sin\phi|)\,|\beta_{m,E_{\alpha,2}}|} < 1. \quad (101)$$

The condition is expanded using the simplification terms associated with Eq. (53)

$$\frac{|\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}| + |\frac{AB(\alpha)}{(1-\alpha)}|\beta_{n,E_{\alpha,2}}}{|\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} + v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2}| + |v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} - \frac{D_L}{(\Delta x)^2}| + |\frac{D_L}{(\Delta x)^2}| + v|\frac{AB(\alpha)}{(1-\alpha)}|\,(2 - 2cos\phi)\,|\beta_{m,E_{\alpha,2}}|} < 1. \quad (102)$$

The following assumption is made

$$\left[v\frac{AB\,(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} > \frac{D_L}{(\Delta x)^2}\right].$$

Simplifying

$$2v\left(\delta^{\alpha}_{m,i} + (1 - cos\phi)\,\beta_{m,E_{\alpha,2}}\right) + \frac{2D_L}{(\Delta x)^2} > \beta_{n,E_{\alpha,2}}. \quad (103)$$

Under this assumption, the second inductive stability requirement is found to be true. And, the numerical scheme is conditionally stable, under this condition.

The opposite assumption is made, and the condition becomes

$$2v\,(1 - cos\phi)\,\beta_{m,E_{\alpha,2}} + \frac{4D_L}{(\Delta x)^2} > \beta_{n,E_{\alpha,2}}. \quad (104)$$

Thus, under the opposite assumption, the second inductive stability condition is also found to be appropriate. Moreover, under this condition the numerical scheme is conditionally stable.

This concludes the stability analysis for the implicit upwind scheme for the ABC advection-dispersion equation, where the scheme is stable under the condition stated in Eqs. (103) and (104). These conditions can be simplified to an overall condition

$$min\left(\gamma,\eta\right) > \beta_{n,E_{\alpha,2}},$$

where,

$$\gamma = 2v\left(\delta^{\alpha}_{m,i} + \left(1 - cos\phi\right)\beta_{m,E_{\alpha,2}}\right) + \frac{2D_L}{\left(\Delta x\right)^2},$$

$$\eta = 2v\left(1 - cos\phi\right)\beta_{m,E_{\alpha,2}} + \frac{4D_L}{\left(\Delta x\right)^2}.$$

Under these conditions, the error of the approximation is not propagated throughout the solution, but rather decreases with each time step, as according to the induction method, where for all values of n, $|c_{n+1}| < |c_o|$.

4.2 First-Order Upwind Explicit

The induction method terms for the developed explicit upwind numerical scheme (Sect. 3.1) are substituted as follows

$$ac_n e^{jk_i m} = bc_{n-1}e^{jk_i m} + dc_{n-1}e^{jk_i x(m-\Delta m)} + fc_{n-1}e^{jk_i(m+\Delta m)} - g\sum_{k=0}^{n-2}\left(c_{k+1}e^{jk_i m} - c_k e^{jk_i m}\right)\delta^{\alpha}_{n,k}$$
$$- vg\sum_{i=0}^{m}\left(c_{n-1}e^{jk_i m} - c_{n-1}e^{jk_i(m-\Delta m)}\right)\delta^{\alpha}_{m,i}. \tag{105}$$

The same procedure applied in Sect. 4.1 is followed for the explicit upwind numerical scheme. When $n = 1$, and a subset for m is considered where $m = 0$, the explicit upwind numerical scheme for the ABC fractional advection-dispersion equation is conditionally stable, under the assumption

$$\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta^{\alpha}_{n,n-1} < v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta^{\alpha}_{m,i} + \frac{2D_L}{\left(\Delta x\right)^2}. \tag{106}$$

Following the condition

$$2v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta^{\alpha}_{m,i} + \frac{4D_L}{\left(\Delta x\right)^2} < \frac{2AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta^{\alpha}_{n,n-1}. \tag{107}$$

When a subset for m is considered where $m \geq 1$, the explicit upwind numerical scheme is also conditionally stable under the same assumption, with the condition

$$2v\frac{AB(\alpha)}{(1-\alpha)}\left(\delta^{\alpha}_{m,i}+(1-cos\phi)\,\beta_{m,E_{\alpha,2}}\right)+\frac{4D_L}{(\Delta x)^2}<\frac{2AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}. \quad (108)$$

Therefore, the assumption has been established where $|c_{n-1}| < |c_o|$. The next step for the recursive stability analysis is to use this assumption to demonstrate that for a set $\forall\, n \geq 1$, that

$$|c_n| < |c_o|.$$

Following the equivalent technique as shown in Sect. 4.1, and making the same assumption, the scheme is conditionally stable, with the following condition

$$\frac{AB(\alpha)}{(1-\alpha)}\left(2v\delta^{\alpha}_{m,i}-v\,(2-2cos\phi)\,\beta_{m,E_{\alpha,2}}+\beta_{n,E_{\alpha,2}}\right)+\frac{4D_L}{(\Delta x)^2}<\frac{2AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}. \quad (109)$$

This concludes the stability analysis for the explicit upwind scheme for the ABC advection-dispersion equation, where the scheme is found to be conditionally stable under the assumption made Eq. (106), with conditions Eqs. (107)–(109). The conditions can be simplified into

$$max\,(\lambda,\mu,\rho)<\frac{2AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1},$$

where,

$$\lambda=2v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i}+\frac{4D_L}{(\Delta x)^2};\ \mu=2v\frac{AB(\alpha)}{(1-\alpha)}\left(\delta^{\alpha}_{m,i}+(1-cos\phi)\,\beta_{m,E_{\alpha,2}}\right)+\frac{4D_L}{(\Delta x)^2},$$

$$\rho=\frac{AB(\alpha)}{(1-\alpha)}\left(2v\delta^{\alpha}_{m,i}-v\,(2-2cos\phi)\,\beta_{m,E_{\alpha,2}}+\beta_{n,E_{\alpha,2}}\right)+\frac{4D_L}{(\Delta x)^2}.$$

4.3 First-Order Upwind Crank–Nicolson Scheme

The recursive induction method terms are substituted into the upwind Crank–Nicolson numerical scheme presented in Sect. 3.3

$$\begin{aligned}lc_ne^{jk_im}&=mc_{n-1}e^{jk_im}+oc_ne^{jk_i(m-\Delta m)}+fc_ne^{jk_ix(m+\Delta m)}+pc_{n-1}e^{jk_i(m-\Delta m)}\\&-g\sum_{k=0}^{n-2}\left(c_{k+1}e^{jk_im}-c_ke^{jk_im}\right)\delta^{\alpha}_{n,k}-vg\sum_{i=0}^{m}\begin{bmatrix}0.5\left(c_{n-1}e^{jk_im}-c_{n-1}e^{jk_i(m-\Delta m)}\right)\\+0.5\left(c_ne^{jk_im}-c_ne^{jk_i(m-\Delta m)}\right)\end{bmatrix}\delta^{\alpha}_{m,i}.\end{aligned} \quad (110)$$

In the same way, the method outlined in Sect. 4.1 is employed to determine the numerical stability of the upwind Crank–Nicolson numerical scheme. If $n = 1$, and a subset for m is considered where $m = 0$, then the scheme is unconditionally stable, under the following assumption

$$0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} > \frac{D_L}{(\Delta x)^2}. \tag{111}$$

With the following condition resulting in the unconditional stability

$$\frac{2D_L}{(\Delta x)^2} > 0. \tag{112}$$

If the complementary assumption is made

$$0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} < \frac{D_L}{(\Delta x)^2}. \tag{113}$$

The scheme is conditionally stable, with the following condition

$$v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} < \frac{4D_L}{(\Delta x)^2}. \tag{114}$$

When a subset for m is considered for all $m \geq 1$, under the assumption made in Eq. (111), the scheme is unconditionally stable. Similarly, when the complementary assumption is made, the scheme is conditionally stable.

The next step for the recursive stability analysis was performed as described in Sects. 4.1 and 4.2. The same assumption is made as in Eq. (111), and once more the same unconditionally stable condition was found. Likewise, the opposite condition resulting in the same condition as in Eq. (114).

This completes the stability analysis for the upwind Crank–Nicolson scheme for the ABC advection-dispersion equation, where the scheme is stable under the following condition stated in Eq. (114). Under this single condition, the error of the approximation is not propagated throughout the solution, but rather decreases with each time step, as according to the induction method, where for all values of n, $|c_{n+1}| < |c_o|$.

4.4 Upwind-Downwind Weighted Scheme (Implicit)

Substituting the induction method terms for the developed finite difference implicit upwind-downwind weighted numerical scheme discussed in Sect. 3.5

$$uc_n e^{jk_i m} = vc_n e^{jk_i x(m+\Delta m)} + rc_n e^{jk_i(m-\Delta m)} + ac_{n-1} e^{jk_i m} - g\sum_{k=0}^{n-2}\left(c_{k+1}e^{jk_i m} - c_k e^{jk_i m}\right)\delta_{n,k}^{\alpha}$$

$$- vg\sum_{i=0}^{m}\left[\theta\left(c_n e^{jk_i m} - c_n e^{jk_i(m-\Delta m)}\right) + (1-\theta)\left(c_n e^{jk_i x(m+\Delta m)} - c_n e^{jk_i m}\right)\right]\delta_{m,i}^{\alpha}. \quad (115)$$

A parallel procedure as applied in Sect. 4.1 is followed for the implicit upwind-downwind weighted numerical scheme. When $n = 1$, and a subset for m is considered where $m = 0$, the scheme for the ABC fractional advection-dispersion equation is unconditionally stable when $0.5 \le \theta \le 1$, but conditionally stable when $0 \le \theta < 0.5$, under the assumption

$$\frac{D_L}{(\Delta x)^2} + v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} > v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}. \quad (116)$$

With the condition being

$$(2\theta - 1)\, v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} + \frac{2D_L}{(\Delta x)^2} > 0. \quad (117)$$

If the opposed assumption is made

$$\frac{D_L}{(\Delta x)^2} + v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} < v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha}. \quad (118)$$

The implicit upwind-downwind weighted numerical scheme is conditionally stable, with the following condition

$$v\frac{AB(\alpha)}{(1-\alpha)}\delta_{m,i}^{\alpha} > \left(\delta_{n,n-1}^{\alpha} + v\theta\delta_{m,i}^{\alpha}\right)\frac{AB(\alpha)}{(1-\alpha)} + \frac{D_L}{(\Delta x)^2}. \quad (119)$$

A subset for m is then considered for all $m \ge 1$. The assumption in Eq. (112) is made again, under which the scheme is unconditionally stable when $0.5 \le \theta \le 1$; and conditionally stable when $0 \le \theta < 0.5$, with the condition

$$(2\theta - 1)\frac{AB(\alpha)\, v}{(1-\alpha)}\delta_{m,i}^{\alpha} + (1 - cos\phi)\frac{AB(\alpha)\, v}{(1-\alpha)}\beta_{m,E_{\alpha,2}} + \frac{2D_L}{(\Delta x)^2} > 0. \quad (120)$$

If complementary assumption is made, and the scheme is conditionally stable under the following condition

$$v\frac{AB(\alpha)}{(1-\alpha)}\left(\beta_{m,E_{\alpha,2}} + \delta_{m,i}^{\alpha}\right) > \frac{AB(\alpha)}{(1-\alpha)}\left(\delta_{n,n-1}^{\alpha} + v\theta\delta_{m,i}^{\alpha} + vcos\phi\beta_{m,E_{\alpha,2}}\right) + \frac{D_L}{(\Delta x)^2}. \quad (121)$$

The next step for the recursive stability analysis was performed as described in Sects. 4.1 and 4.2, and first the same assumption is made as in Eq. (116), and once more the scheme is unconditionally stable when $0.5 \leq \theta \leq 1$; and conditionally stable when $0 \leq \theta < 0.5$, with the condition now being

$$2v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\left(\delta_{m,i}^{\alpha}\left(2\theta-1\right)+\left(1-cos\phi\right)\beta_{m,E_{\alpha,2}}\right)+\frac{4D_{L}}{\left(\Delta x\right)^{2}}>\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\beta_{n,E_{\alpha,2}}. \quad (122)$$

In the same way, when the complementary assumption is made, the scheme is conditionally stable, with the condition now being

$$2v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\left(\left(1-\theta\right)\delta_{m,i}^{\alpha}+\left(1-cos\phi\right)\beta_{m,E_{\alpha,2}}\right)-\frac{2D_{L}}{\left(\Delta x\right)^{2}}>\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\left(2\delta_{n,n-1}^{\alpha}+\beta_{n,E_{\alpha,2}}\right). \quad (123)$$

The stability analysis has established that the implicit upwind-downwind weighted scheme is stable under the conditions in Eqs. (119)–(122). Additional conditions are activated when the weighting factor is $0 \leq \theta < 0.5$, as stated in Eqs. (117) and (120). Only under these conditions, the error of the approximation made by the implicit upwind-downwind weighted numerical scheme is not propagated throughout the solution.

4.5 Upwind-Downwind Weighted Scheme (Explicit)

For the stability analysis of the explicit upwind-downwind weighted numerical scheme (Sect. 3.4), the recursive induction terms are substituted as follows

$$\begin{aligned}ac_{n}e^{jk_{i}m} &= qc_{n-1}e^{jk_{i}m}+rc_{n-1}e^{jk_{i}(m-\Delta m)}-sc_{n-1}e^{jk_{i}(m+\Delta m)}-g\sum_{k=0}^{n-2}\left(c_{k+1}e^{jk_{i}m}-c_{k}e^{jk_{i}m}\right)\delta_{n,k}^{\alpha}\\ &\quad -vg\sum_{i=0}^{m}\left[\begin{array}{l}\theta\left(c_{n-1}e^{jk_{i}m}-c_{n-1}e^{jk_{i}(m-\Delta m)}\right)\\ +\left(1-\theta\right)\left(c_{n-1}e^{jk_{i}x(m+\Delta m)}-c_{n-1}e^{jk_{i}m}\right)\end{array}\right]\delta_{m,i}^{\alpha}.\end{aligned} \quad (124)$$

Again, the process outlined in Sect. 4.1 is used to define the numerical stability of the explicit upwind-downwind weighted numerical scheme. When $n = 1$, and a subset for m is considered where $m = 0$, then the scheme is conditionally stable, under the following assumption

$$\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{n,n-1}^{\alpha}+v\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}<2v\theta\frac{AB\left(\alpha\right)}{\left(1-\alpha\right)}\delta_{m,i}^{\alpha}+\frac{2D_{L}}{\left(\Delta x\right)^{2}}, \quad (125)$$

with the condition

$$2v(\theta - 1)\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2} > \frac{2AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}. \tag{126}$$

A subset for m is then considered for all $m \geq 1$, and under the same assumption as in Eq. (121), the scheme is conditionally stable, under the following condition

$$\frac{2AB(\alpha)}{(1-\alpha)}\left(\delta^{\alpha}_{n,n-1} + v\delta^{\alpha}_{m,i} + vcos\phi\beta_{m,E_{\alpha,2}}\right) > \frac{2AB(\alpha)}{(1-\alpha)}\left(2v\theta\delta^{\alpha}_{m,i} + v\beta_{m,E_{\alpha,2}}\right) + \frac{4D_L}{(\Delta x)^2}. \tag{127}$$

The subsequent stage of the recursive stability analysis was performed as described in Sects. 4.1 and 4.2. The same assumption is made as in Eq. (125), and again the scheme is conditionally stable, with the following stability condition

$$\frac{2AB(\alpha)}{(1-\alpha)}\left(\delta^{\alpha}_{n,n-1} + v\delta^{\alpha}_{m,i} + vcos\phi\beta_{m,E_{\alpha,2}}\right) > \frac{AB(\alpha)}{(1-\alpha)}\left(2v\beta_{m,E_{\alpha,2}} + 4v\theta\delta^{\alpha}_{m,i} + \beta_{n,E_{\alpha,2}}\right) + \frac{4D_L}{(\Delta x)^2}. \tag{128}$$

The stability analysis for the explicit upwind-downwind weighted scheme is concluded, where the scheme is conditionally stable under the assumption stated in Eq. (125), with the conditions stated in Eqs. (126)–(128). Only under these conditions, the error of the approximation made by the explicit upwind-downwind weighted scheme is not proliferated throughout the solution.

5 Comparison of Numerical Stability

The stability conditions for the traditional upwind (implicit/explicit), and the new upwind Crank-Nicholson and weighted upwind-downwind (implicit/explicit) numerical schemes are tabulated in Appendix A. The traditional implicit upwind scheme applied to the ABC fractional advection-dispersion equation is conditionally stable under both assumptions made with a single condition for each assumption. There is only one practically applicable assumption for the customary explicit upwind scheme, which has three sub-conditions. The upwind Crank–Nicolson numerical scheme applied to the ABC fractional advection-dispersion equation is unconditionally stable under the first assumption, and has a single condition under the second assumption made. The implicit upwind-downwind weighted numerical scheme is conditionally stable under both assumptions made, but unconditionally stable for the first assumption when the weighting factor is $0.5 \leq \theta \leq 1$. Similar to the explicit upwind scheme, the explicit upwind-downwind weighted numerical scheme has one practically applicable assumption, which has three conditions for stability. Table 1 show the summary of numerical schemes.

Table 1 Summary of numerical schemes

Summary		
Scheme	Assumption	Stability conditions
Implicit upwind	$v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} > \frac{D_L}{(\Delta x)^2}$	$2v\left(\delta^{\alpha}_{m,i} + (1-cos\phi)\,\beta_{m,E_{\alpha,2}}\right) + \frac{2D_L}{(\Delta x)^2} > \beta_{n,E_{\alpha,2}}$
	$v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} < \frac{D_L}{(\Delta x)^2}$	$2v\,(1-cos\phi)\,\beta_{m,E_{\alpha,2}} + \frac{4D_L}{(\Delta x)^2} > \beta_{n,E_{\alpha,2}}$
Explicit upwind	Assumption not made $\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} <$ $v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2}$	$max\,(\lambda, \mu, \rho) < \frac{2AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}$
Upwind Crank-Nicholson	$0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} > \frac{D_L}{(\Delta x)^2}$ $0.5v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} < \frac{D_L}{(\Delta x)^2}$	Unconditionally stable $v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} < \frac{4D_L}{(\Delta x)^2}$
Upwind-downwind weighted scheme (implicit)	$\frac{D_L}{(\Delta x)^2} + v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} >$ $v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i}$	$(2\theta-1)\,v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2} > 0$
		$(2\theta-1)\,\frac{AB(\alpha)v}{(1-\alpha)}\delta^{\alpha}_{m,i}+$ $(1-cos\phi)\,\frac{AB(\alpha)v}{(1-\alpha)}\beta_{m,E_{\alpha,2}} + \frac{2D_L}{(\Delta x)^2} > 0$
		$2v\frac{AB(\alpha)}{(1-\alpha)}\left(\delta^{\alpha}_{m,i}\,(2\theta-1) + (1-cos\phi)\,\beta_{m,E_{\alpha,2}}\right)+$ $\frac{4D_L}{(\Delta x)^2} > \frac{AB(\alpha)}{(1-\alpha)}\beta_{n,E_{\alpha,2}}$
		Unconditionally stable when $0.5 \le \theta \le 1$
		Conditionally stable when $0 \le \theta < 0.5$
	$\frac{D_L}{(\Delta x)^2} + v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} <$ $v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i}$	$v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} > \left(\delta^{\alpha}_{n,n-1} + v\theta\delta^{\alpha}_{m,i}\right)\frac{AB(\alpha)}{(1-\alpha)} + \frac{D_L}{(\Delta x)^2}$
		$v\left(\beta_{m,E_{\alpha,2}} + \delta^{\alpha}_{m,i}\right)\frac{AB(\alpha)}{(1-\alpha)} >$ $2v\left((1-\theta)\,\delta^{\alpha}_{m,i} + (1-cos\phi)\,\beta_{m,E_{\alpha,2}}\right)\frac{AB(\alpha)}{(1-\alpha)}$
		$-\frac{2D_L}{(\Delta x)^2} > \left(2\delta^{\alpha}_{n,n-1} + \beta_{n,E_{\alpha,2}}\right)\frac{AB(\alpha)}{(1-\alpha)}$
	Assumption not made	
Upwind-downwind weighted scheme (explicit)	$\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1} +$ $v\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} <$ $2v\theta\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2}$	$2v\,(\theta-1)\,\frac{AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{m,i} + \frac{2D_L}{(\Delta x)^2} > \frac{2AB(\alpha)}{(1-\alpha)}\delta^{\alpha}_{n,n-1}$
		$\left(\delta^{\alpha}_{n,n-1} + v\delta^{\alpha}_{m,i} + vcos\phi\beta_{m,E_{\alpha,2}}\right)\frac{2AB(\alpha)}{(1-\alpha)} >$ $\left(2v\theta\delta^{\alpha}_{m,i} + v\beta_{m,E_{\alpha,2}}\right)\frac{2AB(\alpha)}{(1-\alpha)} + \frac{4D_L}{(\Delta x)^2}$
		$\left(\delta^{\alpha}_{n,n-1} + v\delta^{\alpha}_{m,i} + vcos\phi\beta_{m,E_{\alpha,2}}\right)\frac{2AB(\alpha)}{(1-\alpha)} >$ $\left(2v\beta_{m,E_{\alpha,2}} + 4v\theta\delta^{\alpha}_{m,i} + \beta_{n,E_{\alpha,2}}\right)\frac{AB(\alpha)}{(1-\alpha)} + \frac{4D_L}{(\Delta x)^2}$

6 Conclusions

An advection-focused fractional transport equation is developed using the ABC fractional derivative. The boundedness, existence and uniqueness is determined using the Picard–Lindelöf theorem. The semi-discretisation stability is evaluated in time, and demonstrated that the developed equation is stable in time. Upwind-based finite difference approximations were developed, and the stability of each was determined. The implicit upwind formulations are found to be more stable than their comparable explicit formulations. The proposed implicit weighted upwind-downwind scheme is more stable than the traditional upwind scheme when the weighting factor is $0.5 \leq \theta \leq 1$, which denotes at least half upwind-weighted or more, and the downwind influence less than half. Of the numerical schemes analysed, the upwind Crank–Nicolson is the most stable numerical scheme, and would be suggested for use with the ABC fractional advection-dispersion equation.

References

1. Koch, D.L., Brady, J.F.: Anomalous diffusion in heterogeneous porous media. Phys. Fluids **31**(5), 965–973 (1988)
2. Schumer, R., Benson, D.A., Meerschaert, M.M., Baeumer, B.: Multiscaling fractional advection-dispersion equations and their solutions. Water Resour. Res. **39**(1), 1–11 (2003)
3. Berkowitz, B., Cortis, A., Dentz, M., Scher, H.: Modeling non-Fickian transport in geological formations as a continuous time random walk. Rev. Geophys. **44**(2), 1–49 (2006)
4. Singha, K., Day-Lewis, F.D., Lane, J.W.: Geoelectrical evidence of bicontinuum transport in groundwater. Geophys. Res. Lett. **34**(12), 1–14 (2007)
5. Zhang, Y., Papelis, C., Young, M.H., Berli, M.: Challenges in the application of fractional derivative models in capturing solute transport in porous media: Darcy-scale fractional dispersion and the influence of medium properties. Math. Probl. Eng. **1**, 1–21 (2013)
6. Neuman, S.P., Tartakovsky, D.M.: Perspective on theories of non-Fickian transport in heterogeneous media. Adv. Water Resour. **32**(5), 670–680 (2009)
7. Zhang, Y., Benson, D.A., Reeves, D.M.: Time and space nonlocalities underlying fractional-derivative models: distinction and literature review of field applications. Adv. Water Resour. **32**(4), 561–581 (2009)
8. Sun, H., Zhang, Y., Chen, W., Reeves, D.M.: Use of a variable-index fractional-derivative model to capture transient dispersion in heterogeneous media. J. Contam. Hydrol. **157**, 47–58 (2014)
9. Allwright, A., Atangana, A.: Fractal advection-dispersion equation for groundwater transport in fractured aquifers with self-similarities. Eur. Phys. J. Plus **133**(2), 1–14 (2018)
10. West, B.J.: Fractional Calculus View of Complexity: Tomorrow's Science. CRC Press, Florida (2016)
11. Oldham, K.B., Spanier, J.: The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic, New York (1974)
12. Herrmann, R.: Fractional Calculus: An Introduction for Physicists. World Scientific Publishing, Singapore (2011)
13. Li, C., Qian, D., Chen, Y.: On riemann-liouville and caputo derivatives. Discret. Dyn. Nat. Soc. **1**, 1–15 (2011)
14. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. **1**(2), 73–85 (2015)
15. Caputo, M., Fabrizio, M.: Applications of new time and spatial fractional derivatives with exponential kernels. Prog. Fract. Differ. Appl. **2**(1), 1–11 (2016)
16. Yépez-Martínez, H., Gómez-Aguilar, J.F.: A new modified definition of Caputo-Fabrizio fractional-order derivative and their applications to the Multi Step Homotopy Analysis Method (MHAM). J. Comput. Appl. Math. **346**, 247–260 (2019)
17. Atangana, A., Baleanu, D.: New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm. Sci. **20**(2), 763–769 (2016)
18. Atangana, A., Gómez-Aguilar, J.F.: A new derivative with normal distribution kernel: theory, methods and applications. Phys. A Stat. Mech. Appl. **476**, 1–14 (2017)

19. Morales-Delgado, V.F., Gómez-Aguilar, J.F., Escobar-Jiménez, R.F., Taneco-Hernández, M.A.: Fractional conformable derivatives of Liouville-Caputo type with low-fractionality. Phys. A Stat. Mech. Appl. **503**, 424–438 (2018)
20. Atangana, A., Gómez-Aguilar, J.F.: Hyperchaotic behaviour obtained via a nonlocal operator with exponential decay and Mittag-Leffler laws. Chaos Solitons Fractals **102**, 285–294 (2017)
21. Sun, H., Hao, X., Zhang, Y., Baleanu, D.: Relaxation and diffusion models with non-singular kernels. Phys. A **468**, 590–596 (2017)
22. Schmelling, S.G., Ross, RR.: Contaminant transport in fractured media: models for decision makers. (EPA Superfund) Issue Paper, Groundwater **28**(2), 272–279 (1989)
23. Zimmerman, D.A., De Marsily, G., Gotway, C.A., Marietta, M.G., Axness, C.L., Beauheim, R.L., Gallegos, D.P.: A comparison of seven geostatistically based inverse approaches to estimate transmissivities for modeling advective transport by groundwater flow. Water Resour. Res. **34**(6), 1373–1413 (1998)
24. Fomin, S., Chugunov, V., Hashida, T.: The effect of non-Fickian diffusion into surrounding rocks on contaminant transport in a fractured porous aquifer. Proc. R. Soc. Lond. A Math. Phys. Eng. Sci. **461**(2061), 2923–2939 (2005)
25. Goode, D.J., Tiedeman, C.R., Lacombe, P.J., Imbrigiotta, T.E., Shapiro, A.M., Chapelle, F.H.: Contamination in fractured-rock aquifers: research at the former naval air warfare center, West Trenton, New Jersey, p. 3074 (2007)
26. Cello, P.A., Walker, D.D., Valocchi, A.J., Loftis, B.: Flow dimension and anomalous diffusion of aquifer tests in fracture networks. Vadose Zone J. **8**(1), 258–268 (2009)
27. Shapiro, A.M.: The challenge of interpreting environmental tracer concentrations in fractured rock and carbonate aquifers. Hydrogeol. J. **19**(1), 9–12 (2011)
28. Masciopinto, C., Palmiotta, D.: Flow and transport in fractured aquifers: new conceptual models based on field measurements. Transp. Porous Media **96**(1), 117–133 (2013)
29. Allwright, A., Atangana, A.: Augmented upwind numerical schemes for a fractional advection-dispersion equation in fractured groundwater systems. Discret. Contin. Dyn. Syst.-Ser. S **1**, 1–14 (2018)
30. Tateishi, A.A., Ribeiro, H.V., Lenzi, E.K.: The role of fractional time-derivative operators on anomalous diffusion. Front. Phys **5**(52), 1–17 (2017)
31. Atangana, A., Gómez-Aguilar, J.F.: Decolonisation of fractional calculus rules: breaking commutativity and associativity to capture more natural phenomena. Eur. Phys. J. Plus **133**, 1–22 (2018)
32. Alkahtani, B.S.T., Koca, I., Atangana, A.: New numerical analysis of Riemann-Liouville time-fractional Schrödinger with power, exponential decay, and Mittag-Leffler laws. J. Nonlinear Sci. Appl. **10**(8), 4231–4243 (2017)
33. Allwright, A., Atangana, A.: Augmented upwind numerical schemes for the groundwater transport advection-dispersion equation with local operators. Int. J. Numer. Methods Fluids **87**, 437–462 (2018)
34. Ewing, R.E., Wang, H.: A summary of numerical methods for time-dependent advection-dominated partial differential equations. J. Comput. Appl. Math. **128**(1), 423–445 (2001)
35. Atangana, A.: On the stability and convergence of the time-fractional variable order telegraph equation. J. Comput. Phys. **293**, 104–114 (2015)
36. Gnitchogna, R., Atangana, A.: New two-step Laplace Adam-Bashforth method for integer a noninteger order partial differential equations. Numer. Methods Partial. Differ. Equ. **1**, 1–20 (2017)