Delay Margin for Robust Stabilization of LTI Delay Systems

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1 Introduction

With a steadily growing interest, over the last two decades or so there have been significant advances in the study of time-delay systems, thanks to the development of analysis methods drawing upon robust control theory, and the development of computational methods in solving *linear matrix inequality (LMI)* problems. In particular, an extraordinary volume of the literature is in existence on stability problems, and various time- and frequency-domain stability analysis approaches have been developed (see, e.g., [8, 16, 17, 21, 25], and the references therein).

Despite the considerable advances on stability studies, stabilization of time-delay systems poses a more difficult problem. The existing work has been largely focused on synthesis problems for systems with a *fixed* delay. Feedback design for such systems can be conducted based on LQR and $\mathcal{H}_\infty$ techniques (see, e.g., [20, 30] and the references therein), via predictor feedback [14, 31], or using LMI-based solutions [6, 17]. On the other hand, fundamental robustness of stabilization in the presence of *uncertain, variable* delays has been seldom investigated. Nor is it clear how the above methods may be extended to address the robust stabilization problem.

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In this vein, particularly noteworthy is the problem of *delay margin* [18], which by nature addresses a system's robust stabilization against uncertain delays and seeks to answer the question: *What is the largest range of delay such that there exists a single feedback controller that can stabilize all the plants subject to delays within the range?* An age-old problem by itself [3, 5], this problem bears a close similarity to the gain margin and phase margin problems, which are two classical stability margin optimization problems solvable analytically by solving a finite-dimensional $\mathcal{H}_\infty$ optimal control problem [4]. Unlike the gain and phase margin however, the delay margin problem proves fundamentally more challenging, due to obstacles in solving infinite-dimensional optimization problems. Indeed, the problem has been open except in isolated cases. In [17, pp. 154], the delay margin was determined for first-order systems achievable by static feedback, while in [27], the delay margin was found for first-order systems when PID controllers are used instead. Other related results concerning stabilizability via delayed feedback can be found in, e.g., [13, 22].

In [10, 18], upper bounds on the delay margin were obtained for general SISO systems subject to an uncertain constant delay. These bounds serve to provide a limit beyond which no single LTI output feedback controller may exist to robustly stabilize the delay plant family within the margin. The results show that this fundamental limit is determined by the unstable poles and nonminimum phase zeros in the plant. In its essence, however, the work of [18] is by and large limited to systems with no more than one unstable pole and nonminimum phase zero, for which the bounds were found to be exact; otherwise, under more general circumstances, the bounds may be crude and pessimistic. Moreover, the analysis in [18] was carried out largely case by case, and for this reason, its technique does not appear readily generalizable. The same can be said of the improvement in [10].

This chapter aims at developing lower bounds on the delay margin. Unlike in [18], which addresses the question when a delay system is *not* stabilizable, we ask when it *is* stabilizable. Thus, the results provide a *guaranteed* range of delay ensuring robust stabilization. Built on small-gain stability conditions, our approach employs rational approximation of delay elements, which enables us to cast the problem as one of finite-dimensional, parameter-dependent $\mathcal{H}_\infty$ optimization; the latter may then be tackled and solved using such analytic interpolation techniques as Nevanlinna-Pick interpolation [1]. This operator-theoretic approach ensures not only that the bounds can be efficiently computed, but also that it can be cohesively extended, and indeed, in a unified manner, to more general classes of systems with more general classes of delays, e.g., systems with time-varying delays. Furthermore, since the approach amounts to solving a standard $\mathcal{H}_\infty$ control synthesis problem, it in fact yields a robustly stabilizing controller that achieves the bounds and guarantees the stabilization for all possible delay values within the bounds.

We consider LTI output feedback controllers. Our contribution is twofold. First, for a SISO system with an arbitrary number of plant unstable poles and nonminimum phase zeros, we provide an explicit bound on the delay margin, which requires computing only the largest real eigenvalue of a constant matrix. Second, we extend our analysis to systems subject to time-varying delays, which yield similar bounds. In both cases, which are unified in our interpolation approach, the results not only are computationally attractive, but shed useful conceptual insights; when specialized to more specific cases, e.g., to plants with one unstable pole and one nonminimum phase zero, they furnish analytical expressions exhibiting explicit dependence of the bounds on the pole and zero, showing how fundamentally unstable poles and nonminimum phase zeros may limit the range of delays over which a plant may be robustly stabilized by a LTI controller. It should be emphasized nonetheless that the results and conclusions presented herein address only the limitation of LTI controllers in stabilizing time-delay systems. More general controllers with varying degrees of implementation complexity, such as linear periodic controllers [19], nonlinear periodic controllers [7], and nonlinear adaptive controllers [15, 23] can be constructed to lend an infinite delay margin, allowing a LTI delay plant to be stabilized for arbitrarily long uncertain delays.

The notation used throughout this chapter is fairly standard. Let $\mathbb{R}$ be the space of real numbers, $\mathbb{R}^n$ the space of n-dimensional real vectors, and $\mathbb{R}^n_+$ the n-dimensional space of positive real numbers. For any complex number z, we denote its conjugate by $\bar{z}$. For any complex vector x, we denote its transpose by x^T and its conjugate transpose by x^H. Similarly, for any complex matrix A, A^H denotes its conjugate transpose. The largest real eigenvalue of a matrix A will be written as $\sigma_{\max}(A)$, and if A is a Hermitian matrix, its largest eigenvalue will be written as $\bar{\lambda}(A)$. We write $A \geq 0$ if A is nonnegative definite, and $A > 0$ if it is positive definite. The symbol $\otimes$ denotes the Kronecker product. Let $\mathbb{C}_- := \{s : \ \mathrm{Re}(s) < 0\}$, $\mathbb{C}_+ := \{s : \ \mathrm{Re} > 0\}$, and $\bar{\mathbb{C}}_+ := \{s : \ \mathrm{Re} \geq 0\}$ be the open left and the open right-half of the complex plane, and the closed right-half of the complex plane, respectively. For any stable transfer function matrix $G(s)$, define its $\mathcal{H}_\infty$ norm by $\|G(s)\|_\infty = \sup_\omega \bar{\sigma}\left(G(j\omega)\right)$, where $\bar{\sigma}(\cdot)$ stands for the largest singular value. For any unitary vectors $u, v \in \mathbb{C}^n$, we denote the principal angle between the directions spanned by u and v as $\cos\angle(u, v) = \left|u^H v\right|$.

We note that subsequent to this chapter, an extended version reporting the results herein has appeared in [24], which develops in full the approach with extensions to multi-input multi-output delay systems. We refer to [24] as well for all the proofs of the results in this chapter.

2 Bounds on Delay Margin of SISO Systems

In this section, we present the results for SISO delay systems, which consist of a general lower bound on the delay margin that amounts to computing an eigenvalue problem. We also present explicit bounds for more specialized cases, which exhibit the dependence of the delay margin on the plant's unstable poles and nonminimum phase zeros.

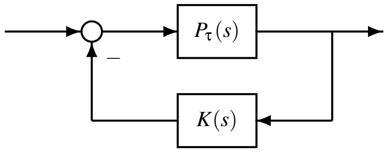

Fig. 1 Standard feedback control structure

The Delay Margin Problem

We consider the feedback control system depicted in Fig. 1, in which $P_\tau(s)$ represents a family of plants subject to an unknown delay τ, with $P_0(s)$ being the delay-free plant:

$$P_\tau(s) = e^{-\tau s} P_0(s), \quad \tau \geq 0. \tag{1}$$

Suppose that $P_0(s)$ is stabilized by a certain finite-dimensional LTI controller $K(s)$. By continuity, $K(s)$ can stabilize $P_\tau(s)$ for sufficiently small $\tau > 0$. But how large may τ be, before the system loses closed-loop stability?

The *delay margin problem* seeks to answer the above question, which amounts to computing

$$\tau^* = \sup \{\nu : \ K(s) \text{ stabilizes } P_\tau(s), \ \forall \tau \in [0, \ \nu)\}.$$

In other words, we want to determine the largest delay range within which $P_\tau(s)$ can be stabilized by a finite-dimensional LTI controller $K(s)$. Note that for $K(s)$ to stabilize $P_\tau(s)$, it is both necessary and sufficient that

$$1 + P_\tau(s)K(s) \neq 0, \quad \forall s \in \bar{\mathbb{C}}_+.$$

Under the condition that $P_0(s)$ is stabilized by $K(s)$, this condition is equivalent to

$$1 + T_0(s)\left(e^{-\tau s} - 1\right) \neq 0, \quad \forall s \in \bar{\mathbb{C}}_+, \tag{2}$$

where $T_0(s) = P_0(s)K(s)\,(1 + P_0(s)K(s))^{-1}$ is the system's complementary sensitivity function. It is clear that there exists some stabilizing $K(s)$ for all $\tau \in [0, \ \bar{\tau}]$ if

$$\sup_{\tau \in [0, \ \bar{\tau}]} \inf_{K(s)} \left\| T_0(s)(e^{-\tau s} - 1) \right\|_\infty < 1. \tag{3}$$

Define

$$\phi_{\bar{\tau}}(\omega) = \sup_{\tau \in [0, \ \bar{\tau}]} |e^{-j\omega\tau} - 1| = \begin{cases} 2\sin(\omega\bar{\tau}/2), & |\omega\bar{\tau}| \leq \pi, \\ 2, & |\omega\bar{\tau}| > \pi. \end{cases} \tag{4}$$

Evidently, the condition (3) holds whenever

$$\inf_{K(s)} |T_0(j\omega)\phi_{\bar{\tau}}(\omega)| < 1, \quad \forall \omega \in \mathbb{R}. \tag{5}$$

Unfortunately, the problems in (3), (5) and the delay margin problem itself all pose a formidable challenge, for they all require solving infinite-dimensional optimization problems due to the presence of the weighting function $\left(e^{-\tau s}-1\right)$.

One instrumental step in our approach is to construct a parameter-dependent rational approximation

$$w_\tau(s) = \frac{b_\tau(s)}{a_\tau(s)} = \frac{b_q(\tau s)^q + \cdots + b_1(\tau s) + b_0}{a_q(\tau s)^q + \cdots + a_1(\tau s) + a_0}, \tag{6}$$

such that

$$\phi_\tau(\omega) \le |w_\tau(j\omega)|, \quad \forall \omega \in \mathbb{R}. \tag{7}$$

We require that $w_\tau(s)$ be stable and have no nonminimum phase zero, excluding the origin where $w_\tau(s)$ might have a zero, that is $w_\tau(0) = 0$. This latter condition may be imposed to ensure a close-fit of $|w_\tau(j\omega)|$ to $\phi_\tau(\omega)$ at low frequencies. Note that under this requirement, with no loss of generality, it is necessary that $a_i > 0$ for $i = 0, 1, \ldots, q$, and $b_i > 0$ for $i = 1, \ldots, q$. Some of specific, low-order approximants in this spirit can be found in, e.g., [9, 24, 28]:

$$w_{1\tau}(s) = \tau s, \tag{8}$$

$$w_{2\tau}(s) = \frac{\tau s}{1 + \tau s/3.465}, \tag{9}$$

$$w_{3\tau}(s) = \frac{1.216\tau s}{1 + \tau s/2}, \tag{10}$$

$$w_{4\tau}(s) = \frac{\tau s(2 \times 0.2152^2 \tau s + 1)}{(0.2152\tau s + 1)^2}, \tag{11}$$

$$w_{5\tau}(s) = \frac{\tau s}{1 + \tau s/2}\frac{0.1791(\tau s)^2 + 0.7093\tau s + 1}{0.1791(\tau s)^2 + 0.5798\tau s + 1}, \tag{12}$$

$$w_{6\tau}(s) = \frac{\tau s}{1 + \tau s/2}\frac{0.02952(\tau s)^4 + 0.210172(\tau s)^3 + 0.70763(\tau s)^2 + 1.3188\tau s + 1}{0.02952(\tau s)^4 + 0.191784(\tau s)^3 + 0.64174(\tau s)^2 + 1.195282\tau s + 1}. \tag{13}$$

Note that $w_{6\tau}(s)$, with a highest order, betters all other $w_{i\tau}(s)$, for $i = 1, \ldots, 5$. Figure 2 shows the magnitude responses of these rational functions.

With the rational approximant alluded to above, we may then attempt to compute

$$\underline{\tau} = \sup\left\{\tau \ge 0 : \inf_{K(s)} \|T_0(s) w_\tau(s)\|_\infty < 1\right\}, \tag{14}$$

which, unlike in (5), amounts to solving a finite-dimensional $\mathcal{H}_\infty$ optimal control problem, nonetheless parameterized by a nonnegative parameter $\tau \ge 0$; for a different

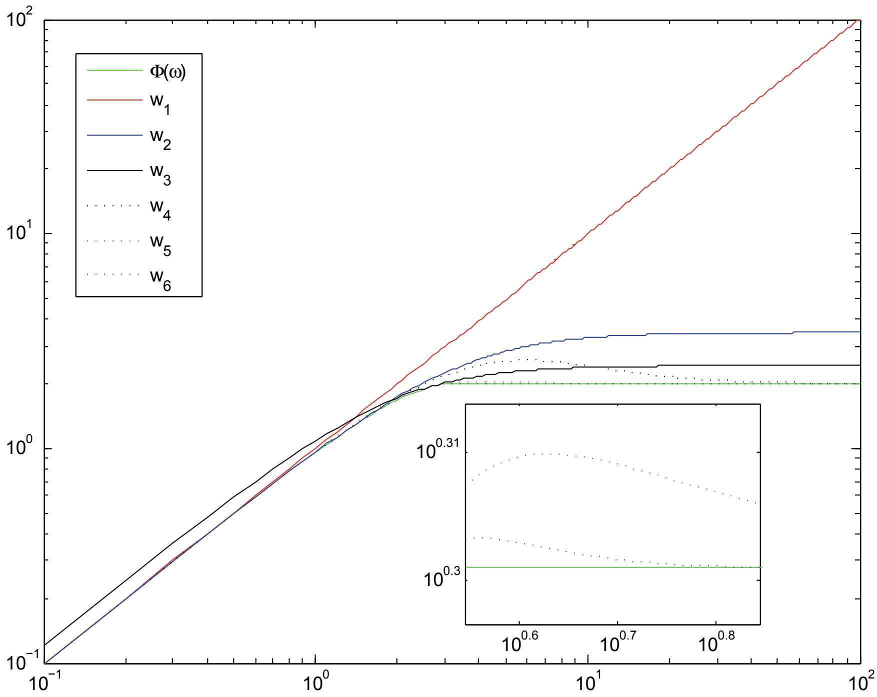

Fig. 2 Rational approximation for $\phi(\omega)$

$w_{i\tau}(s)$, a corresponding $\mathcal{H}_\infty$ problem is solved, resulting in a different $\underline{\tau}_i$. Clearly, the condition (5) holds for $\phi_{\underline{\tau}}(\omega)$ whenever

$$\inf_{K(s)} \left\| T_0(s) w_{\underline{\tau}}(s) \right\|_\infty < 1. \tag{15}$$

Note that $\phi_\tau(\omega)$ is monotonically increasing with $\tau \geq 0$ within the range of $0 \leq \omega\tau \leq \pi$. As such, $\underline{\tau}$ serves as a lower bound on the delay margin τ^*, and in turn provides a range guaranteeing the stabilizability of $P_\tau(s)$: there exists a controller $K(s)$ that can stabilize $P_\tau(s)$ for all $\tau \in [0,\ \underline{\tau})$.

A Computational Formula

We compute the lower bound $\underline{\tau}$ with a general rational approximant given in (6), by casting the problem (14) into one of the Nevanlinna-Pick interpolation [1]. The following result illustrates this point.

Theorem 1 *Let $p_i \in \mathbb{C}_+$, $i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then for any $w_\tau(s)$ in (6),*

$$\underline{\tau} = \sigma_{\max}^{-1}\left(\begin{bmatrix} -\Phi_0^{-1}\Phi_1 & \cdots & -\Phi_0^{-1}\Phi_{2q-1} & -\Phi_0^{-1}\Phi_{2q} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}\right),$$

where $\Phi_0 = \begin{bmatrix} Q_p & b_0 \\ b_0 & a_0^2 Z^{-1} \end{bmatrix}$,

$$\Phi_k = \begin{cases} \begin{bmatrix} 0 & b_k (D_p^H)^k \\ b_k D_p^k & \sum_{l=0}^{k} a_l a_{k-l} D_p^l Z^{-1} (D_p^H)^{k-l} \end{bmatrix}, & k = 1, \ \ldots, \ q, \\ \operatorname{diag}\left(0, \ \sum_{l=k-q}^{q} a_l a_{k-l} D_p^l Z^{-1} (D_p^H)^{k-l}\right), & k = q+1, \ \ldots, \ 2q, \end{cases}$$

$$Z = Q_p + Q_{zp}^H Q_z^{-1} Q_{zp}, \quad D_p = diag\,(p_1, \ \ldots, \ p_n)\,,$$

$$Q_z = \left[\frac{1}{z_i + \bar{z}_j}\right], \quad Q_p = \left[\frac{1}{\bar{p}_i + p_j}\right], \quad Q_{zp} = \left[\frac{1}{z_i - p_j}\right].$$

We note that Theorem 1 can be extended to accommodate multiple poles and zeros in $P_0(s)$, using a more sophisticated result on the mixed Nevanlinna-Pick and Carathéodory-Fejér interpolation problem [26, 29]. Imaginary poles and zeros can also be incorporated in the analysis as boundary interpolation constraints [1, 2]. For technical simplicity, however, we choose not to address such poles and zeros herein.

In view of Theorem 1, a lower bound $\underline{\tau}$ on the delay margin can be found by solving rather efficiently an eigenvalue problem, which guarantees that $P_\tau(s)$ can be stabilized by a certain LTI controller $K(s)$ for all $\tau \in [0, \ \underline{\tau})$. Since $\underline{\tau}$ corresponds to an optimal $\mathcal{H}_\infty$ optimization problem, a robustly stabilizing controller can be synthesized accordingly. Indeed, to synthesize this robustly stabilizing controller $K(s)$, it suffices to solve the standard $\mathcal{H}_\infty$ control problem in (15), once $\underline{\tau}$ is computed according to Theorem 1. This gives rise to an optimal controller $K(s)$ depending on $\underline{\tau}$. In this vein, it is worth pointing out that a lower order $w_\tau(s)$, such as those given in (8)–(13), can be particularly desirable, since they potentially result in low-order controllers.

Special Cases

A number of special cases are further examined in this section. The first result concerns the circumstance where $P_0(s)$ has only a single unstable pole. In this case, an explicit lower bound is obtained which exhibits how the plant unstable pole may confine the delay margin.

Corollary 1 *Suppose that $P_0(s)$ has only one unstable pole $p \in \mathbb{C}_+$, and no non-minimum phase zero. Then for any $w_\tau(s)$ in (6) with $b_0 < a_0$,*

$$\underline{\tau} = \frac{\lambda_{\min}}{p}, \tag{16}$$

where

$$\lambda_{\min} = \min\left\{\lambda > 0 : \sum_{k=0}^{q}(b_k - a_k)\lambda^k = 0\right\}. \tag{17}$$

In particular, if $w_\tau(s) = w_{i\tau}(s)$ for $w_{i\tau}(s)$, $i = 1, \ldots, 6$ given in (8)–(13), then we have $\underline{\tau} = \underline{\tau}_i$, with

(1) $\underline{\tau}_1 = 1/p$; (2) $\underline{\tau}_2 \approx 1.406/p$; (3) $\underline{\tau}_3 \approx 1.397/p$; (4) $\underline{\tau}_4 \approx 1.5582/p$; (5) $\underline{\tau}_5 \approx 1.7008/p$; (6) $\underline{\tau}_6 \approx 1.722/p$.

In other words, for plants with a sole unstable pole, it suffices to solve the smallest positive real root of a polynomial.

While Corollary 1 shows a varying degree of conservatism in the various lower bounds resulted from their respective approximants $w_{i\tau}(s)$, it is interesting to observe that $w_{5\tau}(s)$ and $w_{6\tau}(s)$, despite being only a third-order and a fifth-order approximant respectively, provide rather accurate estimates of the true delay margin; in these cases, $\underline{\tau}_5 = 1.7008/p$, and $\underline{\tau}_6 = 1.722/p$, respectively, as opposed to the exact delay margin $\tau^* = 2/p$, obtained in [18]. Note however that the exact delay margin $\tau^* = 2/p$ may not be attainable in a realistic sense, for the robustly stabilizing controller corresponding to τ^* will result in an arbitrarily small loop bandwidth [18] and thus will be hardly of use. In practice, one must then accept to find a robustly stabilizing controller for a smaller range of delay, which will be even closer to the lower bounds obtained herein.

More generally, Corollary 1 can be extended to systems containing nonminimum phase zeros as well, as demonstrated by the following result.

Corollary 2 *Suppose that $P_0(s)$ has only one unstable pole $p \in \mathbb{C}_+$ and distinct nonminimum phase zeros $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$. Let*

$$M = \prod_{i=1}^{m}\left|\frac{z_i - p}{\bar{z}_i + p}\right|.$$

Then for any $w_\tau(s)$ in (6) with $b_0 < Ma_0$,

$$\underline{\tau} = \frac{\lambda_{\min}}{p}, \tag{18}$$

where

$$\lambda_{\min} = \min\left\{\lambda > 0 : \sum_{k=0}^{q}(b_k - Ma_k)\lambda^k = 0\right\}. \tag{19}$$

Furthermore, for $w_\tau(s) = w_{i\tau}(s)$, $i = 1,\ 2,\ 3$ *given in (8)–(10), we have* $\underline{\tau} = \underline{\tau}_i$*, with*

$$(1)\ \ \underline{\tau}_1 = \frac{M}{p};\quad (2)\ \ \underline{\tau}_2 = \frac{M}{(1-0.289M)p};\quad (3)\ \ \underline{\tau}_3 = \frac{M}{(1.216-0.5M)p}.$$

Evidently, Corollary 2 shows that in the presence of nonminimum phase zeros, the range of delay with guaranteed stabilizability will be further shrunk. This is consistent with the finding of [18], which shows that it is less likely to stabilize a delay plant containing nonminimum phase zeros. The explicit relations given in (1)–(3) of Corollary 2 show that $\underline{\tau}$ is a monotonically increasing function of M. In the limit when $M \to 0$, stabilization is rendered impossible. This scenario occurs when the plant has a pair of closely located unstable pole and nonminimum phase zero. Note also that for the fourth, fifth, and sixth order approximants $w_{4\tau}(s)$, $w_{5\tau}(s)$, and $w_{6\tau}(s)$, similar yet more complex expressions of $\underline{\tau}$ can be found explicitly in terms of M, by more tedious calculations.

3 Systems with Time-Varying Delays

With an added advantage, the interpolation approach can be expanded to analyze linear systems with time-varying delays. Consider the system

$$\begin{cases} \dot{x} = Ax + Bu(t-\tau(t)), \\ y = Cx. \end{cases} \tag{20}$$

It is customary to confine the time-varying delay $\tau(t)$ to a given range $[0,\ \bar{\tau})$, i.e.,

$$0 \le \tau(t) \le \bar{\tau}, \tag{21}$$

and bound the variation rate $\dot{\tau}(t)$ as,

$$|\dot{\tau}(t)| \le \delta < 1. \tag{22}$$

Let $P_0(s) = C(sI-A)^{-1}B$ be the transfer function of the delay-free system. We want to find a LTI controller $K(s)$ so as to stabilize the delay system (20) by way of the output feedback $u(s) = K(s)y(s)$ within a region defined by $(\bar{\tau},\ \delta)$.

Rate-Independent Bound

It is readily recognized that the closed loop system can be represented by Fig. 3, in which Δ is a linear time-varying operator such that

$$\Delta u(t) = u(t-\tau(t)).$$

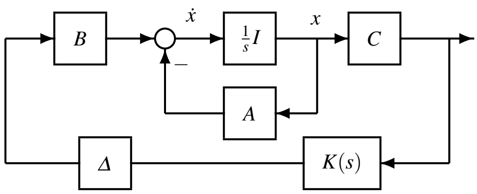

Fig. 3 Feedback system with time-varying input delay

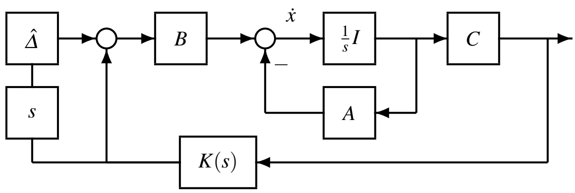

Fig. 4 Small-gain setup of systems with time-varying delay

By employing the *model transformation* [8],

$$u(t - \tau(t)) = u(t) - \int_{t-\tau(t)}^{t} \dot{u}(\sigma)\mathrm{d}\sigma,$$

the system can be transformed into the one depicted in Fig. 4, where

$$\hat{\Delta}x = -\int_{t-\tau(t)}^{t} x(\sigma)\mathrm{d}\sigma. \tag{23}$$

It is well-known [8] that the system in Fig. 3, i.e., the original system (20) with the controller $K(s)$, is stable whenever the system in Fig. 4 is stable. Thus, by applying the small-gain condition developed in [12, 32], we conclude that $K(s)$ stabilizes the system (20) if it stabilizes $P_0(s)$ and the small-gain condition

$$\|\bar{\tau}sT_0(s)\|_\infty < 1 \tag{24}$$

holds. As a result, in much the same manner, a lower bound on $\bar{\tau}$ can be found by solving the $\mathcal{H}_\infty$ optimization problem in (24), which will guarantee the existence of a controller $K(s)$ that can stabilize the system (20) for all $\tau(t) \in [0,\ \bar{\tau})$ regardless of δ. Evidently, this problem coincides with that in (14), with $w_\tau(s) = \bar{\tau}s$. The following result is thus clear.

Theorem 2 *Let $p_i \in \mathbb{C}_+, i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+, i = 1, \ldots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then the system (20) can be stabilized by some $K(s)$ for all $\tau(t) \in [0,\ \bar{\tau})$ with*

$$\bar{\tau} = \bar{\lambda}^{-\frac{1}{2}}\left(Q_{p1}^{-\frac{1}{2}}(Q_{p2} + Q_{zp}^H Q_z^{-1} Q_{zp})Q_{p1}^{-\frac{1}{2}}\right),$$

where

$$Q_z = \left[\frac{1}{z_i + \bar{z}_j}\right], \quad Q_{p1} = \left[\frac{1}{\bar{p}_i + p_j}\right], \quad Q_{p2} = \left[\frac{\bar{p}_i p_j}{\bar{p}_i + p_j}\right], \quad Q_{zp} = \left[\frac{p_j}{z_i - p_j}\right].$$

Rate-Dependent Bound

More generally, it is possible to employ more elaborate approximations of the time-varying operator Δ. It may also be useful to incorporate the delay variation rate in the approximation. One such approximation scheme is suggested in [11], which stipulates that for any $\bar{\tau} \geq 0$ and $0 \leq \delta < 1$, $K(s)$ can stabilize the system (20) whenever

$$|T_0(j\omega)\psi_\epsilon(j\omega)| < 1, \qquad \forall \omega \in \mathbb{R}, \tag{25}$$

where $\psi_\epsilon(s)$ is a stable rational function meeting the condition

$$|\psi_\epsilon(j\omega)| \geq \sqrt{\frac{2}{2-\delta}}\, \phi_\tau(\omega) + \epsilon$$

and hence can be constructed so that

$$|\psi_\epsilon(j\omega)| = \sqrt{\frac{2}{2-\delta}}\, |w_\tau(j\omega)| + \epsilon,$$

for any $\epsilon > 0$ and any rational function $w_\tau(s)$ given in (6) and satisfying (7). Since $\epsilon > 0$ can be made arbitrarily small, the condition (25) is met whenever

$$\inf_{K(s)} \|T_0(s) w_\tau(s)\|_\infty < \sqrt{\frac{2-\delta}{2}}\,. \tag{26}$$

As a consequence, the stabilizability of the system (20) can also be ascertained using the same interpolation approach. The following result extends Theorem 1 to systems described by (20), with time-varying delays.

Theorem 3 *Let $p_i \in \mathbb{C}_+$, $i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then the system (20) can be stabilized by some $K(s)$ for all $\tau(t) \in [0, \bar{\tau})$, $|\dot{\tau}(t)| \leq \delta$ if*

$$\bar{\tau} = \sigma_{\max}^{-1}\left(\begin{bmatrix} -\Phi_0^{-1}\Phi_1 & \cdots & -\Phi_0^{-1}\Phi_{2q-1} & -\Phi_0^{-1}\Phi_{2q} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}\right),$$

where for any $w_\tau(s)$ *in (6),* Φ_k, $k = 1, \ldots, 2q$ *are defined as in Theorem 1, and* Φ_0 *is given by*

$$\Phi_0 = \begin{bmatrix} \frac{2-\delta}{2} Q_p & b_0 \\ b_0 & a_0^2 Z^{-1} \end{bmatrix},$$

with Q_p *and* Z *defined in Theorem 1 as well.*

Analogously, explicit bounds can be obtained for more special cases. The following corollary summarizes the time-varying counterparts to Corollaries 1 and 2.

Corollary 3 *Suppose that* $P_0(s)$ *is minimum phase and has only one unstable pole* $p \in \mathbb{C}_+$*. Define*

$$N = \sqrt{\frac{2-\delta}{2}}\,.$$

Then the system (20) can be stabilized by some $K(s)$ *for all* $\tau(t) \in [0, \bar{\tau})$ *with* $\bar{\tau} = \bar{\tau}_i, \ i = 1, \ldots, 4$*, where*

$$(1)\ \bar{\tau}_1 = 1/p; \quad (2)\ \bar{\tau}_2 = \frac{N}{(1-0.289N)p}; \quad (3)\ \bar{\tau}_3 = \frac{N}{(1.216-0.5N)p};$$
$$(4)\ \bar{\tau}_4 = \frac{10.81 - 4.654N - \sqrt{116.9 - 57.32N}}{(N-2)\,p}.$$

Additionally, suppose also that $P_0(s)$ *has distinct nonminimum phase zeros* $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$*. Let* M *be defined in Corollary 2. Then the system (20) can be stabilized by some* $K(s)$ *for all* $\tau(t) \in [0, \bar{\tau})$ *with* $\bar{\tau} = \bar{\tau}_i, \ i = 1, \ldots, 4$*, where*

$$(1)\ \bar{\tau}_1 = M/p; \quad (2)\ \bar{\tau}_2 = \frac{N}{(1-0.289N)p}M; \quad (3)\ \bar{\tau}_3 = \frac{N}{(1.216-0.5N)p}M;$$
$$(4)\ \bar{\tau}_4 = \frac{10.81 - 4.654N - \sqrt{116.9 - 57.32N}}{(N-2)\,p}M.$$

Theorems 2 and 3 differ from each other due to the incorporation of the variation rate δ, which results from the difference between (24) and (26). Similarly, in Corollary 3, the bound $\bar{\tau}_1$ is derived using the condition (24), while $\bar{\tau}_2$, $\bar{\tau}_3$ and $\bar{\tau}_4$ are obtained using (26), together with $w_{2\tau}(s)$, $w_{3\tau}(s)$ and $w_{4\tau}(s)$, respectively. Among the rate-dependent bounds, $\bar{\tau}_i, i = 2, 3, 4$ become progressively less conservative. Compared to the rate-independent $\bar{\tau}_1$, they may or may not be advantageous depending on the value of δ. It is also worth noting that for $\delta = 0$, the condition (26) reduces to (15), and hence the results in this section all recover the bounds for LTI systems presented in Sect. 3.

4 Examples

We now consider a number of illustrating examples. Example 1 presents a system with a constant delay, while Example 2 addresses systems with a time-varying delay. In both examples, we assume that the plant is excited by a unit step input.

Example 1 Consider the plant

$$P_0(s) = \frac{0.1(s-10)(s-0.1659)}{(s-0.1081)(s^2+0.2981s+0.06281)}. \tag{27}$$

This system has an unstable pole $p = 0.1081$ and two nonminimum phase zeros $z_1 = 10,\ z_2 = 0.1659$. Using the approximant $w_{6\tau}(s)$, we find $\underline{\tau}_6 = 2.0741$, achievable by the optimal controller $K(s)$ solving (15):

$$\begin{aligned} K(s) = &\frac{10643(s+0.9643)(s^2+0.2981s+0.06282)}{(s+1044)(s+30.13)(s+1.134)(s+0.7959)} \\ &\times \frac{(s^2+1.965s+1.109)(s^2+1.167s+1.651)}{(s+0.617)(s^2+1.27s+1.647)}. \end{aligned} \tag{28}$$

The closed-loop output response is plotted in Fig. 5 for $\tau = 0.6,\ 1,\ 1.5,\ 2$, respectively. For $\tau = 2$, Fig. 6 shows the state responses of the system (27). Clearly, the system is internally stable. Moreover, Fig. 7 shows the magnification of the output responses for $t \in [0,\ 25]$, exhibiting the typical undershoot behavior of nonminimum phase systems.

Example 2 The following system, given in state-space form, contains a time-varying delay $\tau(t)$:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -599 & 600 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-\tau(t)), \\ y(t) &= \begin{bmatrix} 1 & -3 \end{bmatrix} x(t). \end{aligned} \tag{29}$$

The delay-free part of the system has an unstable pole $p = 1$ and a nonminimum phase zeros $z = 3$. The time-varying delay under consideration is described by

$$\tau(t) = \alpha(1-\sin(\beta t))$$

for some $\alpha > 0,\ \beta > 0$. It is evident that $0 \le \tau(t) \le 2\alpha$, and $\delta = \alpha\beta$. From Theorem 2, we assert that the system (29) is robustly stabilizable regardless of β whenever $\alpha < 0.25$. For all $\alpha \in [0,\ 0.25)$, the system (29) can be stabilized by the feedback controller $K(s)$ solving the $\mathcal{H}_\infty$ optimal control problem in (24):

$$K(s) = -1.0273 \times 10^8 \frac{(s+600)(s+100)}{(s+3.2\times 10^6)(s+1640)(s+5.853)}. \tag{30}$$

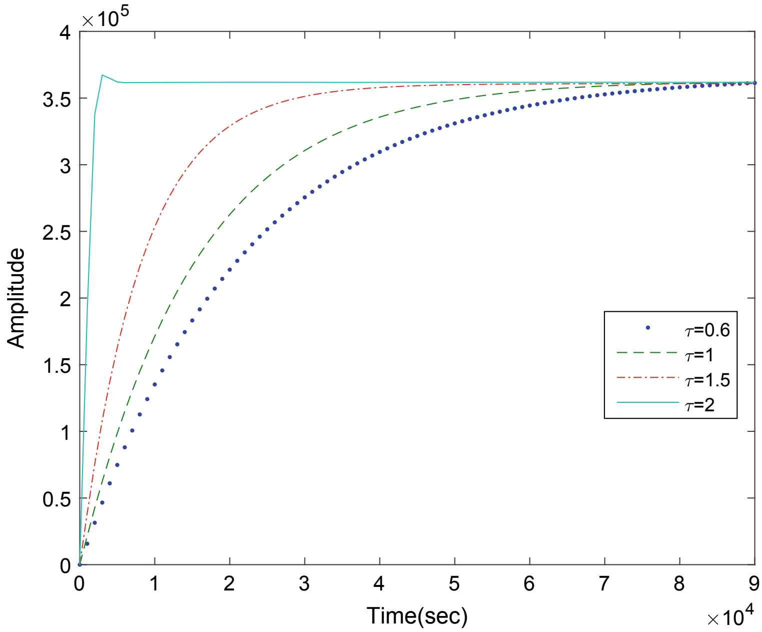

Fig. 5 Step response of the system (27) with controller (28)

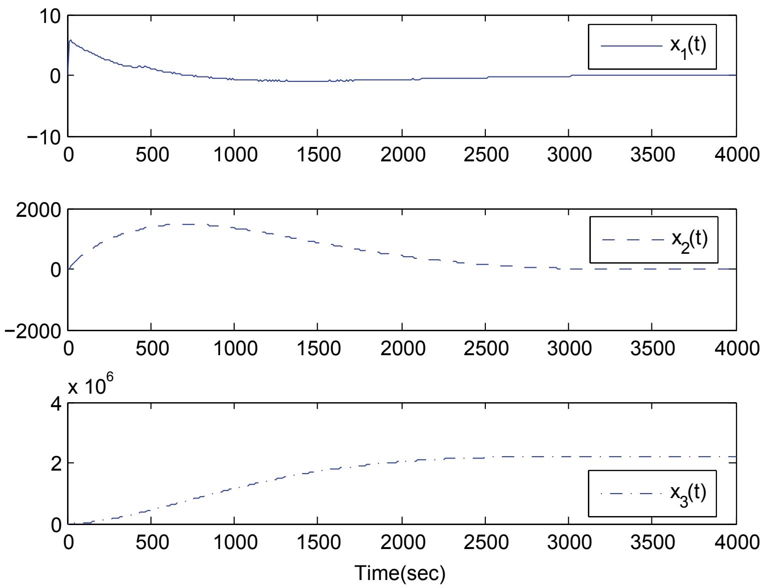

Fig. 6 State response of the system (27) with controller (28) for $\tau = 2$

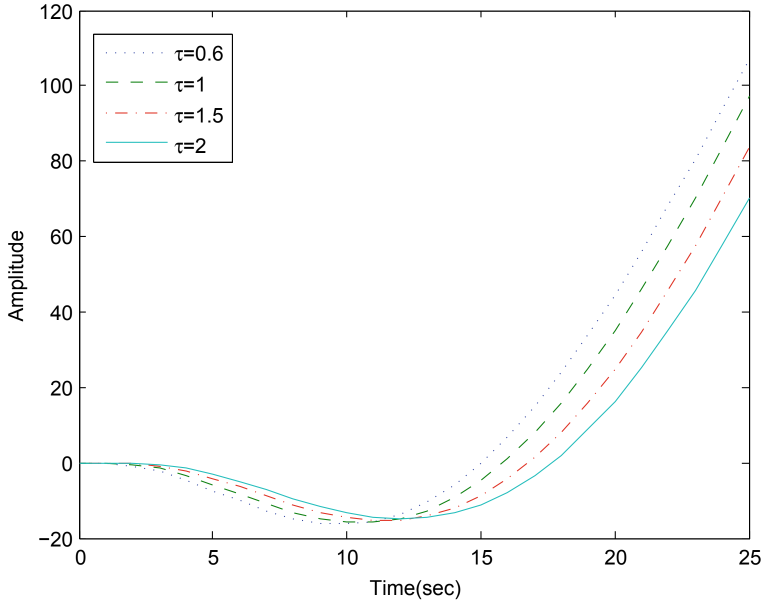

Fig. 7 Step response of the system (27) with controller (28) $t \in [0,\ 25]$

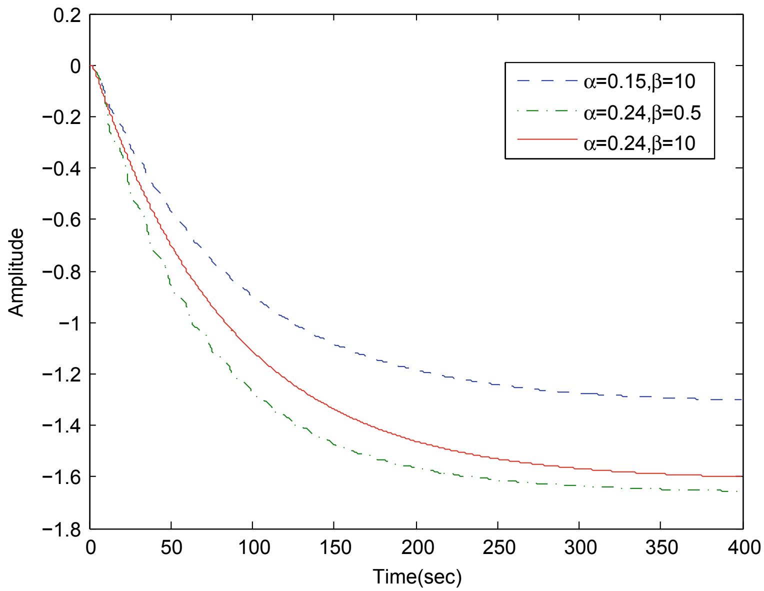

Fig. 8 Output response of the system (29) with controller (30)

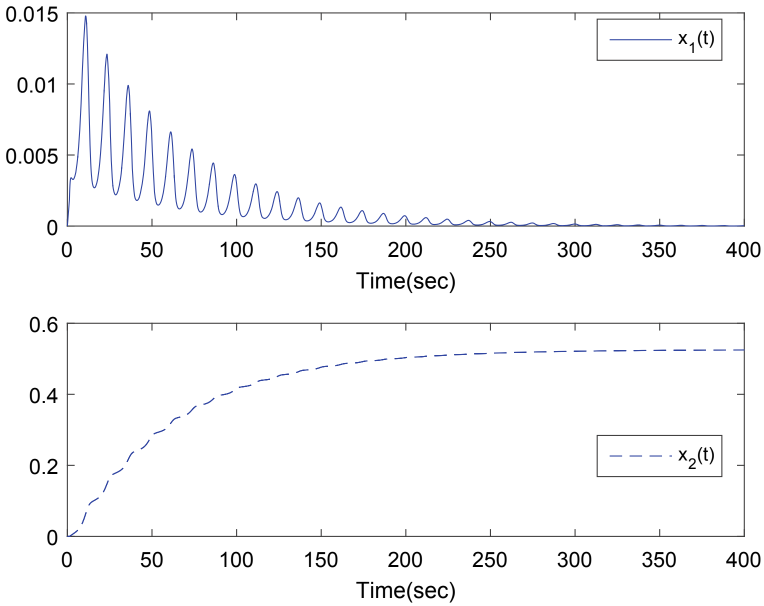

Fig. 9 State response of the system (29) with controller (30) for $\alpha = 0.24$, $\beta = 0.5$

Figure 8 shows the stable output responses of the closed-loop system for various combinations of α within the interval $[0,\ 0.25)$ and arbitrarily selected β. For $\alpha = 0.24$, $\beta = 0.5$, Fig. 9 shows the state responses.

5 Conclusion

In this chapter we have studied the delay margin and delay robust stabilization problems for linear delay systems. Our solutions seek to ascertain the existence of a finite-dimensional LTI output feedback controller that can robustly stabilize an entire family of plants subject to uncertain, possibly time-varying delays within a given range. Built on small-gain stability conditions, we employed analytic interpolation and rational approximation techniques to develop bounds on the delay margin. The development has led to a unified interpolation-based approach, applicable to SISO systems with constant and time-varying delays. The results consist of readily computable bounds on the delay margin of SISO systems, within which a delay plant is guaranteed to be stabilizable. The bounds can in general be computed by solving an eigenvalue problem. For more special plants admitting, e.g., only one unstable pole, explicit results are found which show how unstable poles and nonminimum phase zeros may fundamentally confine the range of delay allowed for robust stabilization.

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