Proportional-Retarded (PR) Protocol for a Large Scale Multi-agent Network with Noisy Measurements; Stability and Performance

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1 Introduction

Study of multi-agent systems has attracted tremendous attention especially in the past decade, with applications involving robotic networks [1], traffic flow dynamics [2], human-machine interactions and collaborative human-robot systems [3]. While such systems can enjoy rich information flow amongst the agents with the network interconnectivity, distributed nature of the agents and the need to utilize advanced technologies to tailor these agents inevitably bring about a number of unique challenges to the design and control of multi-agent systems. One key challenge is the presence of time delays in the network dynamics [4], which may arise due to various reasons including agents' actuation times, the need to use a communication medium to enable the agents to exchange information, and necessary computation times to process and interpret large stream of data. The presence of time delay in a dynamical system often imports undesirable characteristics, including poor performance, oscillatory response, and instability [5]. Nevertheless, if carefully engineered, time delay can also be used as a vehicle to craft the dynamic response, including fast stabilization [6–11].

In the context of multi-agent networks however, use of delays as a design parameter to achieve fast stabilization is under-explored although this is of high interest [12, 13]. One opportunity in this endeavor is to utilize reliable computational tools to approximate the rightmost eigenvalues of the dynamics [14], or to use those tools to

This work was supported in part by the US National Science Foundation Award 1536397.

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G. Valmorbida et al. (eds.), *Delays and Interconnections: Methodology, Algorithms and Applications*, Advances in Delays and Dynamics 10,
https://doi.org/10.1007/978-3-030-11554-8_16

tune the controller gains via optimization schemes [15]. Other ideas to achieve fast stabilization include strategically removing certain links between some of the agents to expedite consensus reaching [16], or re-designing their coupling strengths [17].

Recent results on linear time-invariant (LTI) single-input single-output (SISO) systems [9, 10] indicate that analytical tuning rules can be developed with certain classes of controllers to optimize the spectrum of the closed-loop system. Specifically, in [18, 19], authors analytically designed Proportional-Retarded (PR) protocols that can assign a closed-loop system's spectral abscissa to a user-defined location on the complex plane. These results point out opportunities also for large scale LTI network control problems see, e.g., [20–27] for studies utilizing PR controllers in network settings.

In this chapter we seek to develop distributed PR-based protocols for a benchmark large-scale LTI consensus system. The main objective is to utilize Lambert W functions to analyze the stability of the system in terms of PR protocol parameters. For this, we first take advantage of standard decomposition tools to break down the corresponding characteristic equation into subsystems and treat each subsystem stability one by one. This result provides a transparent understanding in terms of which specific eigenvalue of the graph Laplacian underlying the network governs directly the stability of the entire consensus system. Furthermore, it connects with our recent study in [28] where, with each subsystem being in a particular form, we utilized some inherent features of Lambert W functions to tune the PR protocol without any approximation while shifting the spectrum of the subsystems all at once, thereby yielding fast stabilization. With the novelty of the results pertaining to this tuning approach left to [28], here we summarize some key findings from the cited work for the completeness of the presentation. Overall, in an undirected network, the proposed approach as we demonstrate is scalable and easy to implement, even in the presence of signals with high-frequency noise components.

The rest of the chapter is organized as follows. Section 2 describes the consensus dynamics under analysis and states the problem formulation in light of the Lambert W function. Section 3 starts with a useful factorization of the system that enables a comprehensive study of the stability of the complete network using dimensional analysis. Section 4 summarizes some results without proofs from [28] regarding the design of the spectral abscissa of the system ensuring fast consensus. Section 5 verifies the findings via the analysis of a challenging numerical example. Finally some concluding remarks and further directions on research are given in Sect. 6.

2 Preliminaries and Problem Formulation

In the following we consider a system with n identical agents whose dynamics is captured by the integrator plant

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where $x_i(t)$ is the state of the ith agent and $u_i(t)$ is the control input by which agent i communicates with the rest of the agents. The communication topology of the network is described by an undirected weighted graph $\mathcal{G} = (N, E, A)$ where $N = \{1, 2, \ldots, n\}$ is the set of nodes, $E \subset N \times N$ is the set of edges (communication channels), and $A = [a_{ij}]$ is the weighted adjacency matrix. We assume that each edge has an associated weight $a_{ij} = a_{ji}$, also known as the coupling strength, where the indices $(i, j) \in E$ indicate that agent $i \in N$ receives information either instantaneously or with delay from agent $j \in N$ whenever $a_{ij} > 0$.

Let $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ be the Laplacian matrix with

$$l_{ij} = \begin{cases} \sum_{m=1,m\neq i}^{n} a_{im} & i = j, \\ -a_{ij} & i \neq j, \end{cases} \tag{2}$$

then L is symmetric $l_{ij} = l_{ji}$ and accepts the diffusive property $\sum_{j=1}^{n} l_{ij} = 0$. Hence, from the spectral theorem for Hermitian matrices [29], its eigenvalues are real.

The control objective, as proposed in [28], is to achieve agreement of the states amongst all the agents of the network. To this end, we consider that the agents are coupled via the Proportional-Retarded (PR) protocol originally developed for SISO systems [11],

$$u_i(t) = k_p \sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)] - k_r \sum_{j=1}^{n} a_{ij}[x_j(t-h) - x_i(t-h)]. \tag{3}$$

Here, k_p and k_r determine respectively the strength of the proportional and retarded actions, and $h > 0$ is an intentional delay induced as part of the input with the aim of obtaining a delayed term by which high-frequency measurement noise is attenuated. We say that protocol (3) solves the consensus problem if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, for all $i, j \in N$.

Note that the introduction of the retarded part in the PR protocol mimics a pure derivative action, thus improving transient response but being insensitive to measurement noise. To see this, observe that (3) can be written in terms of the entries of L as: $u_i(t) = -\sum_{j=1}^{n} \ell_{ij}[k_p x_j(t) - k_r x_j(t-h)]$. Introducing the null term $\pm k_r x_j(t)$ into the above and defining $\tilde{k}_p \equiv k_p - k_r$ and $\tilde{k}_r \equiv h k_r$ we obtain the following alternative representation

$$u_i(t) = -\sum_{j=1}^{n} \ell_{ij}\Big[\tilde{k}_p x_j(t) + \tilde{k}_r \frac{1}{h} \int_{t-h}^{t} \dot{x}_j(\tau) d\tau\Big]. \tag{4}$$

Hence, the proposed protocol performs an averaged derivative action [11] distributed throughout the network by which high-frequency noise components are attenuated without relying on measurements or approximations of $\dot{x}_j(t)$.

Let $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^\top$ be the stack vector of the states at all nodes and $\{A_0, A_1\} = \{-k_p, k_r\} \cdot L$, then system (1) with (3) can be conveniently expressed in matrix form as

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-h), \tag{5}$$

whose stability properties are defined by the location of the *characteristic roots* of the function

$$f(s, k_p, k_r) = \det(sI - A_0 - A_1 e^{-sh}) = 0, \tag{6}$$

which is also known as the *characteristic equation* of system (5). Let Γ be the collection of all characteristic roots satisfying (6) and define the spectral abscissa

$$\gamma^* = \max\{\mathbb{R}(s) \mid s \in \Gamma\}. \tag{7}$$

Then, $\gamma^* < 0$ implies that the spectrum of the system, Γ, lies in the open left-half of the complex plane, thus leading to the following definition [10, 30].

Definition 1 The system (5) is exponentially stable if and only if the spectral abscissa is strictly negative.

Remark on the solution of DDEs via the Lambert *W* function: A Lambert W function is any function $W : \mathbb{C} \to \mathbb{C}$ satisfying

$$W(z)e^{W(z)} = z, \tag{8}$$

for all $z \in \mathbb{C}$. Due to the fact that W is multi-valued, it possesses infinitely-many branches [31]. For a Delay-Differential Equation (DDE), such as the one in (1) with (3), each of these branches can be associated to an element of its spectrum; i.e., to an eigenvalue. In particular, the Lambert W function is useful for the stability analysis and control of LTI-TDS represented by DDEs [32]. For example, the principal branch W_0 can be employed to find the system's dominant root s_0. Then, if $\mathbb{R}(s_0)$ is negative, we can conclude that the system is stable.[1] Computation of W_0 follows from the Lagrange inversion theorem [31] as the series expansion

$$W_0(z) = \sum_{r=1}^{\infty} \frac{(-r)^{r-1}}{r!} z^r. \tag{9}$$

Moreover, W_0 may also be defined as the only branch of W that is analytic at 0.[2] In addition, the Lambert W function can also be extended to design feedback controllers placing s_0 at a desired position by using numerical procedures [33].

[1] It is worthy of mention that the radius of convergence of the series is e^{-1}. For practical computation, the reader is referred to [31] where additional asymptotic formulae can be found considering all the branches of the Lambert W function.

[2] The scalar Lambert W function is available as embedded function in MATLAB, see the function *lambertw*.

3 Stability of the Network

Next, we present a decomposition of system (5) by which stability conditions can be derived using the Lambert W function [33] and the D-subdivision method [34].

Factorization of the Consensus Dynamics

Let us begin with a modal transformation that rotates the vector fields of system (5) aiming at obtaining a diagonal representation of it. Similar transformations have been widely utilized in the context of time delay systems, see for example [35–38]. Here, we will say that both systems are equivalent if they share the same stability properties in terms of their spectrum.

Proposition 1 *Let $\{\lambda_1, \ldots, \lambda_n\}$ be the set of eigenvalues of L ordered increasingly. Then, system (5) is equivalent to the diagonal system*

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{\Lambda}_0 \boldsymbol{\xi}(t) + \boldsymbol{\Lambda}_1 \boldsymbol{\xi}(t-h), \tag{10}$$

where $\{\boldsymbol{\Lambda}_0, \boldsymbol{\Lambda}_1\} = \{-k_p, k_r\} \cdot \boldsymbol{\Lambda}$, *and* $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$.

Proof Since the graph is undirected the Laplacian matrix is symmetric, hence the Schur's theorem [29] guarantees the existence of a nonsingular orthogonal matrix $U \in \mathbb{R}^{n\times n}$, such that the following representation holds: $L = U\boldsymbol{\Lambda}U^{-1}$. Introducing the change of variable $\mathbf{x}(t) = U\boldsymbol{\xi}(t)$, system (5) is reduced to the diagonal form (10), which can be treated as a set of n decoupled subsystems with dynamics

$$\dot{\xi}_m(t) = -\lambda_m k_p \xi_m(t) + \lambda_m k_r \xi_m(t-h), \quad m = 1, \ldots, n. \tag{11}$$

The characteristic equation of any subsystem of the form (11) is

$$f_m(s, k_p, k_r) = s + \lambda_m k_p - \lambda_m k_r e^{-sh} = 0. \tag{12}$$

The fact that the matrix coefficients of systems (5) and (10) share the same set of eigenvalues implies that each $f_m(s, k_p, k_r)$ in (12) is a factor of $f(s, k_p, k_r)$ in (6); i.e.,

$$f(s, k_p, k_r) = \prod_{m=1}^{n} \left[s + \lambda_m k_p - \lambda_m k_r e^{-sh}\right] = 0. \tag{13}$$

Then, the spectra and thus the stability properties of (10) and (5) are equivalent.
□ ■

Proposition 2 *The system (5) is exponentially stable if and only if*

$$\gamma^* = \max\left\{\gamma_m^*\right\}_{m=1}^{n} < 0, \tag{14}$$

where

$$\gamma_m^* = \mathbb{R}\left(h^{-1} W_0\left(\lambda_m k_r h e^{\lambda_m k_p h}\right) - \lambda_m k_p\right), \tag{15}$$

and W_0 is the principal branch of the Lambert W function.

Proof Consider the mth factor $f_m(s, k_p, k_r)$ in (12). Multiplying both sides of this equation by $he^{\lambda_m k_p h}$ yields

$$h\big(s + \lambda_m k_p\big)e^{h\left(s+\lambda_m k_p\right)} = \lambda_m k_r h e^{\lambda_m k_p h}. \tag{16}$$

Comparing (8) and (16), we can see that $h\big(s + \lambda_m k_p\big) = W(\lambda_m k_r h e^{\lambda_m k_p h})$. Solving the above equation for s leads to the solution

$$s = h^{-1} W\big(\lambda_m k_r h e^{\lambda_m k_p h}\big) - \lambda_m k_p. \tag{17}$$

Then, Eq. (15) follows from the real part of (17) using the principal branch W_0. As per Definition 1, the exponential stability of the system is equivalent to (14). □ ■

Assuming that agents are connected, matrix L has a zero eigenvalue $\lambda_1 = 0$ corresponding to the consensus state, and with $\ell_{ij} > 0$, its remaining eigenvalues $\lambda_2, \ldots, \lambda_n$ are positive [39]. Therefore, while ignoring the case of $m = 1$ since this corresponds to the consensus state $s = 0$, we have the following corollary.

Corollary 1 *Let γ_m^* be the spectral abscissa corresponding to the* mth *subsystem (11). Then, if $\gamma_m^* < 0$ for all $m = 2, \ldots, n$, system (5) is exponentially stable around the consensus state of the network.*

Proof Note that $\gamma_m^* < 0$ for all $m = 2, \ldots, n$ guarantees $\gamma^* < 0$. □ ■

Corollary 1 states that separately analyzing the stability of the individual subsystems is equivalent to analyzing the stability of the complete system.

Decomposition of the Space of Parameters

The above discussion indicates that the stability analysis of system (5) can be performed by studying a finite set of subsystems with reduced complexity. With this in mind, using the D-subdivision method, we next decompose the space of controller parameters to study the stability switches of each subsystem (11) with the aim of obtaining a complete stability picture of the system. First, we transform the characteristic Eq. (12) into a dimensionless form. To this end, let the quasipolynomial $f_m(s, k_p, k_r)$ be scaled by h and introduce the time-scaled Laplace operator $\tilde{s} = hs$, this then transforms (12) into

$$hf_m(\tilde{s}/h, k_p, k_r) = \tilde{s} + \lambda_m k_p h - \lambda_m k_r h e^{-\tilde{s}} = 0. \tag{18}$$

Note that the new quasipolynomial retains the stability properties of the original one but with a time delay transformed to unity and where h is now acting as a gain in the system. Defining the lumped gains $\rho_p = \lambda_m k_p h$ and $\rho_r = \lambda_m k_r h$ and the scaled function $\tilde{f}(\tilde{s}, \rho_p, \rho_r) = hf_m(\tilde{s}/h, k_p, k_r)$, we can recast (18) as

$$\tilde{f}(\tilde{s}, \rho_p, \rho_r) = \tilde{s} + \rho_p - \rho_r e^{-\tilde{s}} = 0. \tag{19}$$

Remark 1 Observe that under this transformation, all factors $f_m(s, k_p, k_r)$ in (12) share the uniform structure (19). Hence, for the sake of generality, we temporarily drop the index m associated with the mth eigenvalue.

Following the same logic as in Proposition 2, we multiply (19) by a factor $e^{(\tilde{s}+\rho_p)}$ and obtain $(\tilde{s} + \rho_p)e^{(\tilde{s}+\rho_p)} = \rho_r e^{\rho_p}$ with which, using the real part of the principal branch W_0 of the Lambert W function, we find the spectral abscissa

$$\tilde{\gamma}^* = \mathbb{R}(W_0\,(\rho_r e^{\rho_p}) - \rho_p). \tag{20}$$

As per Corollary 1, stability of any subsystem of the form (11) follows from (20) if and only if $\tilde{\gamma}^* < 0$. Moreover, a stability switch can only occur if some characteristic roots cross the imaginary axis. Therefore, we next search for the lumped crossing points $(\rho_p^\sharp, \rho_r^\sharp)$ and the corresponding scaled crossing frequencies $\tilde{\omega} = \omega h$ such that

$$\tilde{f}(j\tilde{\omega}, \rho_p^\sharp, \rho_r^\sharp) = 0. \tag{21}$$

Due to symmetry of the characteristic roots with respect to the real axis, we can consider only nonnegative frequencies.

Proposition 3 *For a given $\tilde{\omega} \neq k\pi$, $k \in \mathbb{N}$ the corresponding lumped crossing point $(\rho_p^\sharp, \rho_r^\sharp)$ is given by*

$$(\rho_p^\sharp, \rho_r^\sharp) = \left(-\tilde{\omega}\cos(\tilde{\omega})\big/\sin(\tilde{\omega}), -\tilde{\omega}\big/\sin(\tilde{\omega})\right). \tag{22}$$

Moreover, any point on the line

$$\rho_p^\sharp - \rho_r^\sharp = 0, \tag{23}$$

is also a lumped crossing point.

Proof Collecting real and imaginary parts of (21) and some algebraic manipulations generate (22), which are well defined for $\tilde{\omega} \neq k\pi$, $k \in \mathbb{N}$. Corresponding to a root on the origin of the complex plane, (23) satisfies (21) as $\tilde{\omega} \to 0$. □ ■

Let $(\rho_p^\sharp, \rho_r^\sharp)$ be a lumped crossing point and define

$$\tilde{C} = \left\{(\rho_p^\sharp, \rho_r^\sharp) \mid \tilde{\omega} \in [0, \infty), \tilde{\omega} \neq k\pi, k \in \mathbb{N}\right\}. \tag{24}$$

The collection of points in (24) generates almost-everywhere smooth curves [40] known as stability crossing boundaries. Then, $\tilde{C}$ decomposes the lumped space of parameters $\tilde{D} = \{(\rho_p, \rho_r) \in \mathbb{R} \times \mathbb{R}\}$ into a finite number of regions. Since the spectrum of system (5), Γ, behaves continuously with respect to small variations of the lumped parameters, each of these regions is characterized by the same number of unstable roots ν. Let us denote each region by $\tilde{D}(\nu)$, thus,

$$\tilde{D} = \bigcup_{\nu=0}^{\infty} \tilde{D}(\nu), \tag{25}$$

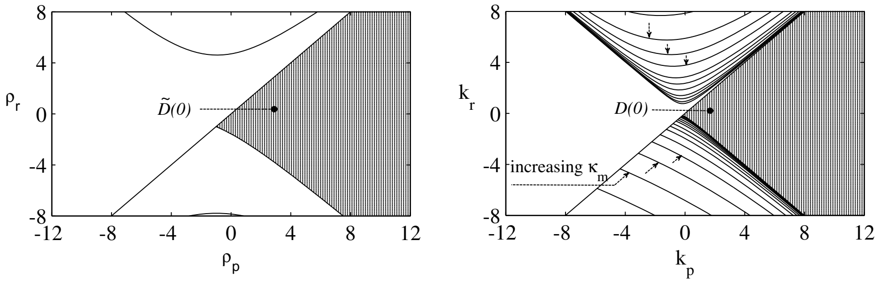

Fig. 1 (Left panel) Stability map obtained with the quasipolynomial (19), the stability crossing boundaries $\widetilde{C}$ are shown in solid lines and the (ρ_p, ρ_r) pairs satisfying $\tilde{\gamma}^* < 0$ are depicted with isolated points. (Right panel) The stability crossing boundaries C_m obtained with Proposition 4 are shown in solid lines and the (k_p, k_r) pairs satisfying $\check{\gamma}^* < 0$ are depicted with isolated points

forms a partition of the lumped space of parameters. Here, $\tilde{D}(0)$ is referred to as the lumped stability domain.

From Proposition 3, we compute the stability crossing boundaries depicted in solid line in Fig. 1 (Left panel). The stability condition $\tilde{\gamma}^* < 0$ is next tested, with $\tilde{\gamma}^*$ in (20), using the embedded function *lambertw* in MATLAB and sweeping both ρ_p and ρ_r. The isolated points in Fig. 1 (Left panel) correspond to (ρ_p, ρ_r) pairs where the stability condition holds. Here, the stability domain is given by

$$\tilde{D}(0) = \{(\rho_p, \rho_r) \mid \tilde{\gamma}^* < 0\}, \tag{26}$$

whose outlook, shaped by $\tilde{C}$, remains invariant with respect to both the eigenvalues of the Laplacian and the amount of induced delay.

To determine the impact of λ_m and h in the stability properties of the system in the original coordinates (k_p, k_r), we now consider a network with an infinite number of agents $(n \to \infty)$ represented by an undirected graph. According to the spectral theorem for Hermitian matrices [29], all the eigenvalues of L are real. Without loss of generality, let the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}^\infty$ be ordered increasingly and define the merged parameter $\kappa_m = \lambda_m h$. Once again, we ignore $\lambda_1 = 0$ as explained above, and therefore $\kappa_m > 0$. Then, the following proposition is well defined.

Proposition 4 *For a given $\kappa_m > 0$, $m = 2, 3, \ldots$, and $\tilde{\omega} \neq k\pi$, $k \in \mathbb{N}$ the corresponding crossing point $(k_p^\sharp, k_r^\sharp)$ is given by*

$$(k_p^\sharp, k_r^\sharp) = \left(-\kappa_m^{-1}\tilde{\omega}\cos(\tilde{\omega})\big/\sin(\tilde{\omega}), -\kappa_m^{-1}\tilde{\omega}\big/\sin(\tilde{\omega})\right). \tag{27}$$

Moreover, any point on the line

$$k_p^\sharp - k_r^\sharp = 0, \tag{28}$$

is also a crossing point.

Proof The result follows directly from Proposition 3. □ ■

Since $n \to \infty$ and $\ell_{ij} = \ell_{ji}$ is a free parameter, the eigenvalues of L are allowed to take any real value, then $\kappa_m \in (0, \infty)$ and $\kappa_{m+1} > \kappa_m$ with $m = 2, 3, \ldots$. Considering a fixed value of κ_m, the crossing points $(k_p^\sharp, k_r^\sharp)$ define the stability crossing boundary

$$C_m = \left\{(k_p^\sharp, k_r^\sharp) \mid \tilde{\omega} \in [0, \infty), \tilde{\omega} \neq k\pi, k \in \mathbb{N}\right\}. \tag{29}$$

Associated with κ_m, let us define the stability domain

$$D_m(0) = \left\{(k_p, k_r) \mid \check{\gamma}_m^* < 0\right\}, \tag{30}$$

where $\check{\gamma}_m^* = h\gamma_m^*$ follows from (15) and is given by

$$\check{\gamma}_m^* = \mathbb{R}\left(W_0\left(\kappa_m k_r e^{\kappa_m k_p}\right) - \kappa_m k_p\right). \tag{31}$$

Corollary 1 states that the stability of all subsystems in (12) implies the stability of the complete network. Moreover, as per Proposition 2, the stability of the complete network implies

$$\check{\gamma}^* = \max\left\{\check{\gamma}_m^*\right\}_{m=2}^n < 0. \tag{32}$$

Conversely, condition (32) implies the stability of the complete network. From Proposition 4, we compute the stability crossing boundaries depicted in solid line in Fig. 1 (Right panel) with several values of $\kappa_m \to \infty$. Condition $\check{\gamma}^* < 0$ in (32) is next tested, using the embedded function *lambertw* in MATLAB and sweeping both k_p and k_r. The isolated points in Fig. 1 (Right panel) corresponds to (k_p, k_r) pairs where the stability condition of (32) holds. Here, the stability domain is given by

$$D(0) = \bigcap_{m=2}^{n} D_m(0), \tag{33}$$

where $D(0)$ is the stability domain of the overall system (5). Note that $D_2(0) \supset D_3(0) \supset \cdots \supset D_n(0)$, therefore (33) reduces to

$$D(0) = D_n(0). \tag{34}$$

Since $D_n(0)$ is related to κ_n, and κ_n is related to λ_n, we can conclude that ensuring the stability of the subsystem that corresponds to the largest eigenvalue of the Laplacian will ensure the stability of the complete system. That is,

$$D(0) = \left\{(k_p, k_r) \mid \check{\gamma}_n^* < 0\right\}. \tag{35}$$

The above result is formalized in the following proposition.

Proposition 5 *The stability domain of the consensus dynamics (5) in the parameter space* (k_p, k_r) *is equivalent to the stability domain of its subsystem associated with the largest Laplacian eigenvalue in (11).* □

Now, the problem is to find the setting for the parameters h, k_p and k_r as a function of the Laplacian eigenvalues such that stability of (5) is guaranteed through (35).

4 Tuning of the PR Protocol

The approach presented below is summarized from [28] for completeness. Readers are referred to the cited study for all relevant proofs. Here, we wish to show how PR protocol can be tuned for the network system in (5). As concluded above, the stability of the subsystem related to the largest eigenvalue of the Laplacian implies the stability of (5). This is now connected to the results in [28] where the objective is to place the spectral abscissa of the consensus dynamics at a desired position γ_d.

First, choose an arbitrary eigenvalue $\bar{\lambda}$ of L. Associated with $\bar{\lambda}$ we have that

$$\bar{\gamma}^* = \mathbb{R}\Big(h^{-1} W_0\left(\bar{\lambda} k_r h e^{\bar{\lambda} k_p h}\right) - \bar{\lambda} k_p\Big). \tag{36}$$

Using the delay and the gains

$$(h, k_p, k_r) = (1/\bar{\lambda}, W_0(1) - \gamma_d/\bar{\lambda}, e^{-k_p}), \tag{37}$$

into (36) reduces $\bar{\gamma}^*$ to

$$\bar{\gamma}^* = \gamma_d, \tag{38}$$

where γ_d is a free parameter introduced to arbitrarily place $\bar{\gamma}^*$.

Second, study the impact of h, k_p and k_r in the rest of the subsystems. To this end, consider a generic spectral abscissa $\hat{\gamma}^*$ associated with the eigenvalue $\hat{\lambda} > \bar{\lambda}$. Employing (37) along with $\hat{\lambda}$ into (15) we have the spectral abscissa

$$\hat{\gamma}^* = \bar{\lambda}\mathbb{R}\Big(W_0\left(\hat{\lambda}\bar{\lambda}^{-1} e^{k_p\left(\hat{\lambda}\bar{\lambda}^{-1}-1\right)}\right) - \hat{\lambda}\bar{\lambda}^{-1} k_p\Big). \tag{39}$$

Define $\delta = \hat{\lambda}\bar{\lambda}^{-1}$, since $\hat{\lambda} > \bar{\lambda} > 0$, then $\delta > 1$. Moreover, whenever k_p remains positive, W_0 is real and positive [31]. Hence, (39) is equivalent to

$$\hat{\gamma}^* = \bar{\lambda}\Big[W_0\Big(\delta e^{k_p(\delta-1)}\Big) - \delta k_p\Big]. \tag{40}$$

Third, introduce the identity

$$F(\delta) = \bar{\gamma}^* - \hat{\gamma}^*, \tag{41}$$

relating the spectral abscissas. Here, if $F(\delta) > 0$ for all $\delta \in (1, \infty)$, this would imply that the systems associated with $\hat{\lambda}$ and with $\bar{\lambda}$ are both stable provided that γ_d in (38) is strictly negative. As per proofs in [28], indeed $F(\delta) > 0$ holds so long as $k_p > e^{-1}$.

Fourth, let $\bar{\lambda} = \lambda_{\min} = \min\{\lambda_m\}_{m=1}^n \neq 0$. Since $F(\delta) > 0$ holds for $k_p > e^{-1}$, it follows that $\bar{\gamma}^* > \hat{\gamma}^*$, where $\bar{\gamma}^*$ is the spectral abscissa associated with $\lambda_{\min}$, and $\hat{\gamma}^*$ is the spectral abscissa associated with any of the remaining eigenvalues of L. Here, $\lambda_1 = 0$ is once again ignored as explained above. We conclude that, under parameters (h, k_p, k_r) in (37), the spectral abscissa of the overall network is a function of $\lambda_{\min}$ and can be placed at any desired position; i.e., $\gamma^* = \bar{\gamma}^* = \gamma_d$. Moreover, if $\hat{\lambda} = \lambda_{\max} = \max\{\lambda_m\}_{m=1}^n$, choosing $\gamma_d < 0$ such that $k_p > e^{-1}$ implies $\hat{\gamma}^* < 0$. In other words, the subsystem associated with $\lambda_{\max}$ is stable and hence, system (5) is stable as per our result in the previous section.

Finally, we have the following proposition by which the γ-stability of system (5) is ensured by means of the tuning of the parameters of the PR protocol.

Proposition 6 ([28]) Let $\lambda_{\min} = \min\{\lambda_m\}_{m=1}^n \neq 0$ be the smallest eigenvalue of L, and let $\gamma_d < 0$ be a desired locus for the spectral abscissa γ^* of system (5), then a dominant root at γ_d is placed by the following tuning of the PR protocol gains

$$(h, k_p, k_r) = \left(\frac{1}{\lambda_{\min}}, \frac{\lambda_{\min}\Omega - \gamma_d}{\lambda_{\min}}, e^{-k_p} \right), \tag{42}$$

where $\Omega = W_0(1) = 0.5671$ is the omega constant.

To summarize, on the network system at hand controlled by PR protocol, we have two key messages: **(a)** The maximum of the eigenvalue $\lambda_{\max}$ of the graph Laplacian L dictates the ultimate stability characteristics in terms of PR protocol gains, and **(b)** the minimum of the eigenvalue $\lambda_{\min}$ of the graph Laplacian L dictates how the PR protocol gains must be designed to place the dominant root of the dynamics at a user-defined spectral abscissa γ_d.

5 Numerical Examples

In this section, we present numerical results for the consensus dynamics in (5) where the parameters of the PR protocol are tuned using Proposition 6.

We investigate a fully connected topology of 5 agents with heterogeneous coupling strengths. Here, {0, 25.74, 41.84, 58.08, 70.76} is the set of eigenvalues of the Laplacian matrix L given by

$$L = \begin{pmatrix} 45.7273 & -14.9896 & -9.7127 & -17.2060 & -3.8190 \\ * & 49.0701 & -1.9947 & -13.1760 & -18.9099 \\ * & * & 26.8209 & -0.8211 & -14.2924 \\ * & * & * & 34.5030 & -3.2999 \\ * & * & * & * & 40.3212 \end{pmatrix}. \tag{43}$$

With the eigenvalues λ_m at hand, the quasipolynomial $f(s, k_p, k_r) = \det(sI - A_0 - A_1 e^{-sh})$, where $\{A_0, A_1\} = \{-k_p, k_r\} \cdot L$, is factorized with Proposition 1 as

$$\begin{aligned} f(s, k_p, k_r) = s \times \big[s + 25.74k_p \underset{}{\overset{\overset{f(\lambda_{\min})}{\uparrow}}{-}} 25.74k_r e^{-sh}\big] \times \big[s + 41.84k_p - 41.84k_r e^{-sh}\big] \\ \times \big[s + 58.08k_p - 58.08k_r e^{-sh}\big] \times \big[s + 70.76k_p \underset{\underset{f(\lambda_{\max})}{\downarrow}}{-} 70.76k_r e^{-sh}\big]. \end{aligned} \tag{44}$$

Using Proposition 6, one can now place the spectral abscissa of $f(\lambda_{\min})$ in (44) at a stable locus, which in turn implies that $F(\delta)$ in (41) remains strictly positive for any $\delta > 1$, and therefore the spectral abscissa of $f(\lambda_{\max})$ must also be placed at a stable locus.[3] We can conclude that the subsystem associated with $\lambda_{\max}$ is stable and hence, as per Proposition 5, the complete consensus network is stable.

Figure 2 shows the time simulations for the 5-agent network and considering $\gamma_d \in \{-10, -20, -30\}$. The initial states of the agents in the time interval $t \in [-h, 0]$ are $[-0.19, 0.19, -0.59, 0.05, 0.89]^\top$. In addition, we have injected uniformly distributed random signals into the network's communication channels to mimic high-frequency noise measurements of the states with a flat power spectral density and infinite total energy. Two observations are in order: i) agents' dynamics are only minimally affected by the simulated high-frequency noise in the measurements as opposed to using a controller with pure derivatives (plots suppressed due to lack of space) and ii) pushing the spectral abscissa deeper into the left-hand side of the complex plane increases the velocity of response of the system. Following these observations, we can say that the PR protocol can process the noisy measurements without any need for further filtering. Moreover, the convergence rate of the consensus network is dictated by the tuning rules in Proposition 6, hence, faster consensus can be achieved by choosing smaller negative γ_d values.

6 Conclusions

This chapter studies the stability of a LTI consensus dynamics under a PR protocol that utilizes delays as tuning parameters. We present how the PR protocol gains influence the stability of the dynamics and how the maximum eigenvalue of the underlying graph Laplacian alone ultimately determines the stability of the overall

[3] Note that Proposition 6 uses $\lambda_{\min}$ to guarantee the placement of γ^* at a desired location γ_d through a stabilizing pair (k_p, k_r). Since $\gamma_d < 0$ is a necessary and sufficient condition for the stability of the consensus network, it can be conjectured that the stabilizing pair (k_p, k_r) must lie within the stability domain associated with $\lambda_{\max}$. Therefore, one may consider Proposition 6 as a link between two important Laplacian eigenvalues, namely, $\lambda_{\min}$ and $\lambda_{\max}$.

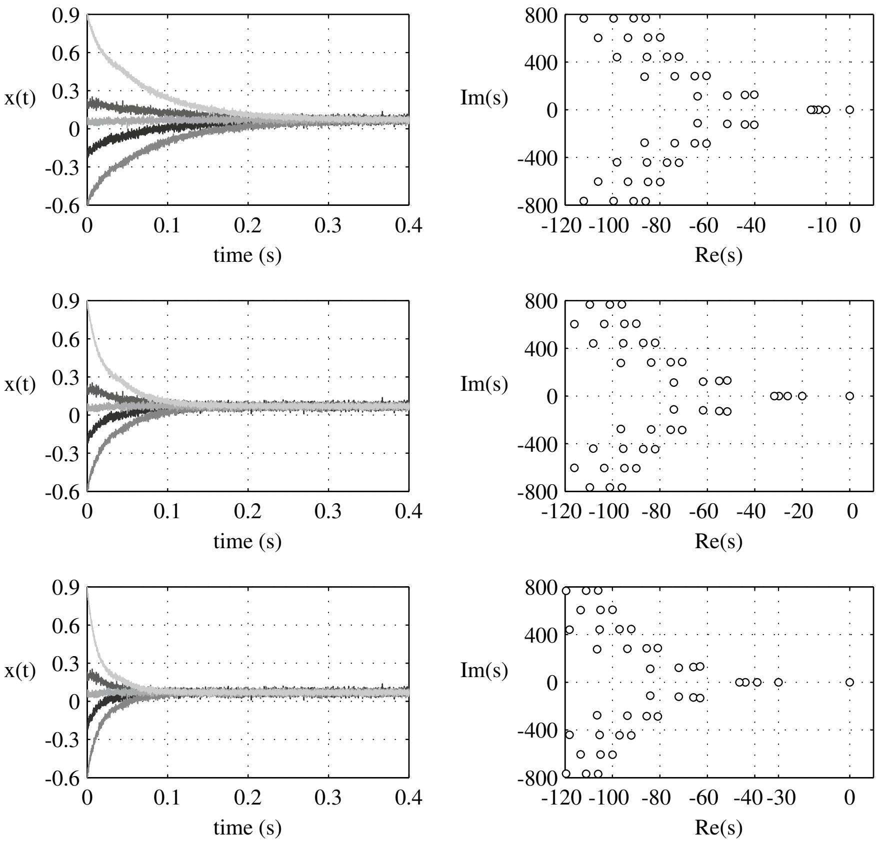

Fig. 2 5-agent network subject to high-frequency noise measurements. (Left panels) Agents' states with respect to time. (Right panels) Spectrum distribution of the consensus dynamics computed with QPMR [14]. (Top) $\gamma_d = -10$. (Center) $\gamma_d = -20$. (Bottom) $\gamma_d = -30$

network dynamics. Recent results from [28] are then summarized showing how to tune the PR protocol to achieve fast consensus. Further research directions include the analysis of different consensus protocols and the use of multiple delays.

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