Singular Perturbation Approach for Linear Coupled ODE-PDE Systems

Ying Tang, Christophe Prieur and Antoine Girard

1 Introduction

Systems modeled by coupled ordinary differential equations (ODEs)-partial differential equations (PDEs) have been studied in many research works [2, 5, 11]. It is interesting to analyze such kind of systems due to their significant physical applications. For instance, elastic beams linked to rigid bodies in [12], power converters connected to transmission lines in [4] etc.

Singular perturbation theory has been widely used in control engineering from late 1960s. It is a powerful tool for analysis and design of control systems thanks to the reduction of the system's order by neglecting the fast transitions [7, 9, 10]. This theory is effective in many applications, such as semiconductors, electrical chains and so on.

Tikhonov approximation, which describes the limiting behavior of system's solutions, is an important method for analysis of singularly perturbed systems. Tikhonov approach for finite dimensional systems modeled by ODEs has been considered in [6]. In [14], a Tikhonov theorem for infinite dimensional systems governed by hyperbolic

Y. Tang
Université de Lorraine, CRAN UMR 7039 and CNRS, 2 avenue de la Forêt de Haye, 54516 Vandoeuvre-lès-Nancy, France
e-mail: ying.tang@univ-lorraine.fr

C. Prieur
GIPSA-lab, Grenoble Campus, 11 rue des Mathématiques, BP 46, 38402 Saint Martin d'Hères Cedex, France
e-mail: christophe.prieur@gipsa-lab.fr

A. Girard (✉)
Laboratoire des signaux et systèmes (L2S), CNRS, CentraleSupélec, Université Paris-Sud, Université Paris-Saclay, 3 rue Joliot-Curie, 91192 Gif-sur-Yvette Cedex, France
e-mail: Antoine.Girard@l2s.centralesupelec.fr; antoine.girard@imag.fr

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PDEs has been established by means of a L^2 Lyapunov function. The approximation for linear hyperbolic systems has been improved by using a H^2 Lyapunov function in [15].

The present work is concerned with a class of coupled singularly perturbed linear ODE and linear hyperbolic PDE systems. Firstly, a sufficient stability condition is proposed for both coupled ODE-fast PDE and PDE-fast ODE systems. The stability of the full system implies the stability of both subsystems. Secondly, the Tikhonov approximation for such systems is achieved by Lyapunov method. Under the stability conditions of both subsystems, the coupled ODE-fast PDE system is approximated by the two subsystems for ε sufficiently small. However, for PDE-fast ODE system, the approximation is valid if the full system is stable. The error between the full system and the subsystems is estimated as the order of the perturbation parameter ε.

The paper is organized as follows. The coupled ODE-PDE systems under consideration are given in Sect. 2. The reduced and the boundary-layer subsystems are formally computed in the same section. Section 3 provides sufficient stability conditions for the full system and both subsystems. The Tikhonov approximation for such systems is stated in Sect. 4. Numerical simulations on academic examples are shown in Sect. 5. Concluding remarks end of this paper.

Notation. Given a matrix G in $\mathbb{R}^{n\times n}$, G^{-1} and $G^\top$ represent the inverse and the transpose matrix of G respectively. The minimum and maximum eigenvalues of a symmetric matrix G are denoted by $\underline{\lambda}(G)$ and $\overline{\lambda}(G)$. The symbol $\star$ in partitioned symmetric matrix stands for the symmetric block. For a positive integer n, I_n is the identity matrix in $\mathbb{R}^{n\times n}$. $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ is associated with the usual 2-norm of matrices in $\mathbb{R}^{n\times n}$. $\|\cdot\|_{L^2}$ denotes the associated norm in $L^2(0,1)$ space, defined by $\|f\|_{L^2} = \sqrt{\left(\int_0^1 |f(x)|^2 dx\right)}$ for all functions $f \in L^2(0,1)$. Similarly, the associated norm in $H^2(0,1)$ space is denoted by $\|\cdot\|_{H^2}$, defined for all functions $f \in H^2(0,1)$, by $\|f\|_{H^2} = \sqrt{\left(\int_0^1 |f(x)|^2 + |f'(x)|^2 + |f''(x)|^2 dx\right)}$. Given a real interval I and a normed space J, $C^0(I,J)$ denotes the set of continuous functions from I to J.

2 Singularly Perturbed Linear Coupled ODE-PDE Systems

In this section, the coupled ODE-fast PDE and PDE-fast ODE systems under consideration are given respectively.

2.1 Coupled ODE-Fast PDE System

We consider the following *linear ODE-fast hyperbolic PDE system*

$$\begin{cases} \dot{Z}(t) = AZ(t) + By(1,t), & \text{(1a)} \\ \varepsilon y_t(x,t) + \Lambda y_x(x,t) = 0, & \text{(1b)} \\ y(0,t) = K_1 y(1,t) + K_2 Z(t), & \text{(1c)} \\ Z(0) = Z_0, & \text{(1d)} \\ y(x,0) = y_0(x), & \text{(1e)} \end{cases}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $Z : [0,+\infty) \to \mathbb{R}^n$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}^m$. The perturbation parameter $\varepsilon > 0$ is a small constant and Λ is a diagonal positive matrix in $\mathbb{R}^{m\times m}$. The matrices A and B are of appropriate dimensions. The boundary condition matrices K_1 and K_2 are constant matrices of appropriate dimensions.

Adopting the computations of the subsystems for singularly perturbed ODEs [8], the reduced and the boundary-layer subsystems for system (1) are formally computed as follows. By setting $\varepsilon = 0$ in Eq. (1b), we obtain

$$y_x(x,t) = 0. \tag{2}$$

It implies $y(.,t) = y(1,t)$. Assuming $(I_m - K_1)$ invertible, the boundary condition (1c) becomes

$$y(.,t) = K_r Z(t), \tag{3}$$

where $K_r = (I_m - K_1)^{-1} K_2$. Using the right-hand side of (3) to replace $y(1,t)$ in (1a), the *reduced subsystem* is computed as

$$\begin{cases} \dot{\bar{Z}}(t) = A_r \bar{Z}(t), & \text{(4a)} \\ \bar{Z}_0 = Z_0. & \text{(4b)} \end{cases}$$

where $A_r = A + BK_r$. The bar indicates that the variables belong to the system with $\varepsilon = 0$. Using the following change of variable $\bar{y} = y - K_r Z$ and a new time scale $\tau = t/\varepsilon$, we have

$$\begin{cases} \bar{y}_\tau(x,\tau) + \Lambda \bar{y}_x(x,\tau) = -\varepsilon K_r (AZ(\tau) + By(1,\tau)), \\ \bar{y}(0,\tau) = K_1 \bar{y}(1,\tau). \end{cases}$$

The *boundary-layer subsystem* is formally computed with $\varepsilon = 0$

$$\begin{cases} \bar{y}_\tau(x,\tau) + \Lambda \bar{y}_x(x,\tau) = 0, & \text{(5a)} \\ \bar{y}(0,\tau) = K_1 \bar{y}(1,\tau), & \text{(5b)} \\ \bar{y}_0(x) = y_0(x) - K_r Z_0. & \text{(5c)} \end{cases}$$

2.2 Coupled PDE-Fast ODE System

Similar to system (1), we consider the following *linear hyperbolic PDE-fast ODE system*

$$\begin{cases} \varepsilon\dot{Z}(t) = AZ(t) + By(1,t), & \text{(6a)} \\ y_t(x,t) + \Lambda y_x(x,t) = 0, & \text{(6b)} \\ y(0,t) = K_1 y(1,t) + K_2 Z(t), & \text{(6c)} \\ Z(0) = Z_0, & \text{(6d)} \\ y(x,0) = y_0(x). & \text{(6e)} \end{cases}$$

The two subsystems are computed as follows. By formally setting $\varepsilon = 0$ in (6a) and assuming A invertible, we have

$$Z = -A^{-1}By(1). \tag{7}$$

Substituting (7) into (6c), the *reduced subsystem* is

$$\begin{cases} \bar{y}_t(x,t) + \Lambda\bar{y}_x(x,t) = 0, & \text{(8a)} \\ \bar{y}(0,t) = K_r\bar{y}(1,t), & \text{(8b)} \\ \bar{y}(x,0) = \bar{y}_0(x) = y_0(x), & \text{(8c)} \end{cases}$$

where $K_r = K_1 - K_2A^{-1}B$. Performing a change of variable $\bar{Z} = Z + A^{-1}By\,(1)$ and using a new time scale $\tau = t/\varepsilon$, we get

$$\frac{d\bar{Z}(\tau)}{d\tau} = A\bar{Z}(\tau) - \varepsilon A^{-1}B\Lambda y_x(1,\tau).$$

The *boundary-layer subsystem* is formally computed with $\varepsilon = 0$

$$\begin{cases} \dfrac{d\bar{Z}(\tau)}{d\tau} = A\bar{Z}(\tau), & \text{(9a)} \\ \bar{Z}(0) = \bar{Z}_0 = Z_0 + A^{-1}By_0(1). & \text{(9b)} \end{cases}$$

Remark 1 Due to [1, Theorem A.6.], the Cauchy problems (1) and (6) are well-posed, that is, for every $Z_0 \in \mathbb{R}^n$, for every $y_0 \in L^2(0,1)$, systems (1) and (6) have a unique solution $Z \in C^0([0,+\infty),\mathbb{R}^n)$, $y \in C^0([0,+\infty), L^2((0,1),\mathbb{R}^m))$.

3 Stability Condition of Coupled ODE-PDE Systems

We first provide a sufficient stability condition for both coupled ODE-PDE systems (1) and (6). Then, we study the link of the stability between the full system and both subsystems.

Proposition 1 *Systems (1) and (6) are exponentially stable for all* $\varepsilon > 0$ *if there exist diagonal positive matrix Q, symmetric positive matrix P and positive constant* μ *such that the following holds*

$$\begin{pmatrix} e^{-\mu} Q\Lambda - K_1^\top Q\Lambda K_1 & -(K_1^\top Q\Lambda K_2 + B^\top P) \\ \star & -(A^\top P + PA) - K_2^\top Q\Lambda K_2 \end{pmatrix} > 0. \tag{10}$$

The next two propositions show that condition (10) implies the stability of the reduced and the boundary-layer subsystems.

Proposition 2 *Condition (10) implies*

$$A_r^\top P + PA_r < 0, \tag{11}$$

which is equivalent to the stability of the reduced subsystem (4), and

$$e^{-\mu} Q\Lambda - K_1^\top Q\Lambda K_1 > 0, \tag{12}$$

which implies the stability of the boundary-layer subsystem (5).

Proposition 3 *Condition (10) implies*

$$e^{-\mu} Q\Lambda - K_r^\top Q\Lambda K_r > 0, \tag{13}$$

which implies the stability of the reduced subsystem (8), and

$$A^\top P + PA < 0, \tag{14}$$

which is equivalent to the stability of the boundary-layer subsystem (9).

In view of Proposition 1, the stability of the full systems (1) and (6) is guaranteed for all positive ε under condition (10). For ε sufficiently small, the stability of both subsystems (4) and (5) implies the stability of the coupled ODE-fast PDE system (1), even though condition (10) is not satisfied. However, this result is not valid in the context for PDE-fast ODE system (6). That is, system (6) could be unstable even though the two subsystems (8) and (9) are stable. We refer the readers to [13] for more details.

4 Tikhonov Approximation of Coupled ODE-PDE Systems

We deal with the Tikhonov approximation of the coupled systems when ε is sufficiently small as follows. If the two subsystems are stable, the coupled ODE-fast PDE system can be approximated by the subsystems. The approach for coupled PDE-fast ODE system is valid if the full system is stable.

4.1 Tikhonov Theorem for Linear ODE-Fast PDE System

Let us state Tikhonov theorem for system (1) in the next theorem.

Theorem 1 *Consider system (1). If (11) and (12) are satisfied, there exist positive values C_1, C_2, θ, ε^*, such that for all $0 < \varepsilon < \varepsilon^*$, and for any initial conditions $Z_0 \in \mathbb{R}^n$, $y_0 = K_r Z_0$, it holds for $t \geqslant 0$*

$$|Z(t) - \bar{Z}(t)|^2 \leqslant \varepsilon C_1 e^{-\theta t} |\bar{Z}_0|^2, \tag{15}$$

$$\|y(.,t) - K_r \bar{Z}(t)\|^2_{L^2(0,1)} \leqslant \varepsilon C_2 e^{-\theta t} |\bar{Z}_0|^2. \tag{16}$$

Before proving this theorem, we first write the error system of (1). Let us perform the following change of variables

$$\eta = Z - \bar{Z}, \tag{17a}$$

$$\delta = y - K_r \bar{Z}, \tag{17b}$$

where η represents the error between the slow dynamics of the full system and the reduced subsystem while δ is the error between the fast dynamics of the full system and its equilibrium point. Due to (1a) and (4), we write

$$\begin{aligned}\dot{\eta} &= \dot{Z} - \dot{\bar{Z}} = AZ + By(1) - A_r \bar{Z} \\ &= A(Z - \bar{Z}) + B(y(1) - K_r \bar{Z}).\end{aligned}$$

Due to (1b) and (4), we compute

$$\begin{aligned}&\delta_t = y_t - K_r \dot{\bar{Z}} = y_t - K_r (A_r \bar{Z}), \\ &\delta_x = y_x.\end{aligned}$$

Due to (1c), we have

$$\begin{aligned}\delta(0) &= y(0) - K_r \bar{Z} = K_1 y(1) + K_2 Z - K_r \bar{Z} \\ &= K_1 \Big(y(1) - K_r \bar{Z} \Big) + K_2 Z - (I_m - K_1) K_r \bar{Z}\end{aligned}$$

$$= K_1\Big(y(1) - K_r\bar{Z}\Big) + K_2 Z - (I_m - K_1)(I_m - K_1)^{-1}K_2\bar{Z}$$
$$= K_1\Big(y(1) - K_r\bar{Z}\Big) + K_2(Z - \bar{Z}).$$

Thus, the error system is written as follows

$$\begin{cases} \dot{\eta} = A\eta + B\delta(1), & \text{(18a)} \\ \varepsilon\delta_t + \Lambda\delta_x = -\varepsilon K_r A_r \bar{Z}, & \text{(18b)} \\ \delta(0) = K_1\delta(1) + K_2\eta, & \text{(18c)} \\ \eta_0 = Z_0 - \bar{Z}_0 = 0, & \text{(18d)} \\ \delta_0 = y_0 - K_r Z_0. & \text{(18e)} \end{cases}$$

Based on the above error system we are ready to prove Theorem 1.

Proof Let us consider the following candidate Lyapunov function for system (18)

$$W(\eta, \delta) = \eta^\top P\eta + \int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q(\delta - K_r\eta)\, dx, \tag{19}$$

with $\mu > 0$, matrices P and Q are specified later.
We rewrite $W(\eta, \delta) = W_1 + W_2$, with $W_1 = \eta^\top P\eta$ and $W_2 = \int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q(\delta - K_r\eta)\, dx$. The time derivative of W_1 along the solution to system (18a) is computed as

$$\begin{aligned} \dot{W}_1 &= 2\eta^\top P\dot{\eta} \\ &= 2\eta^\top P(A\eta + B\delta(1)) \\ &= \eta^\top\Big(PA_r + A_r^\top P\Big)\eta + 2\eta^\top PB\Big(\delta(1) - K_r\eta\Big). \end{aligned}$$

According to (11), there exists a symmetric positive matrix P such that

$$PA_r + A_r^\top P < -I_n. \tag{20}$$

Due to Cauchy Schwarz inequality, it holds

$$\dot{W}_1 \leqslant -|\eta|^2 + 2\|PB\|\,|\delta(1) - K_r\eta|\,|\eta|. \tag{21}$$

The time derivative of W_2 along the solution to system (18b) is

$$\begin{aligned}\dot{W}_2 &= 2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q(\delta_t - K_r\dot{\eta})\,dx\\ &= -\frac{2}{\varepsilon}\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q\Lambda\delta_x\,dx - 2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_r(A\eta + B\delta(1))\,dx\\ &\quad -2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_rA_r\bar{Z}\,dx. \qquad (22)\end{aligned}$$

Performing an integration by parts on the first integral in the right-hand side of (22), $\dot{W}_2$ follows

$$\begin{aligned}\dot{W}_2 &= -\frac{1}{\varepsilon}\Big[e^{-\mu x}(\delta - K_r\eta)^\top Q\Lambda(\delta - K_r\eta)\Big]_{x=0}^{x=1} - \frac{\mu}{\varepsilon}\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q\Lambda(\delta - K_r\eta)\,dx\\ &\quad -2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_r(A\eta + B\delta(1))\,dx - 2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_rA_r\bar{Z}\,dx. \qquad (23)\end{aligned}$$

Let the first term in (23) be W_{21}. Under the boundary condition (18c), we have

$$\begin{aligned}W_{21} &= -\frac{1}{\varepsilon}\Big[e^{-\mu}(\delta(1) - K_r\eta)^\top Q\Lambda(\delta(1) - K_r\eta) - (\delta(0) - K_r\eta)^\top Q\Lambda(\delta(0) - K_r\eta)\Big]\\ &= -\frac{1}{\varepsilon}\Big[e^{-\mu}(\delta(1) - K_r\eta)^\top Q\Lambda(\delta(1) - K_r\eta)\\ &\qquad -(K_1\delta(1) + K_2\eta - K_r\eta)^\top Q\Lambda(K_1\delta(1) + K_2\eta - K_r\eta)\Big]. \qquad (24)\end{aligned}$$

We write

$$\begin{aligned}K_2 - K_r &= K_2 - (I_m - K_1)^{-1}K_2 = (I_m - K_1)(I_m - K_1)^{-1}K_2 - (I_m - K_1)^{-1}K_2\\ &= (I_m - K_1 - I_m)\,(I_m - K_1)^{-1}K_2 = -K_1K_r. \qquad (25)\end{aligned}$$

Using the right-hand side of (25) to replace $K_2 - K_r$ in (24), we obtain

$$W_{21} = -\frac{1}{\varepsilon}\Big[(\delta(1) - K_r\eta)^\top(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)(\delta(1) - K_r\eta)\Big].$$

By using (12), there exists a diagonal positive matrix Q such that

$$e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1 > \underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1) > 0. \qquad (26)$$

Thus

$$W_{21} \leqslant -\frac{\underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)}{\varepsilon}|\delta(1) - K_r\eta|^2. \qquad (27)$$

Let W_{22} denote the second term in (23), it follows

$$W_{22} \leqslant -\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\|\delta - K_r\eta\|^2_{L^2(0,1)}. \tag{28}$$

Let the third term in (23) be W_{23}, it follows

$$W_{23} = -2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q K_r\Big(A_r\eta + B(\delta(1) - K_r\eta)\Big)\,dx.$$

Due to Cauchy Schwarz inequality, W_{23} follows

$$W_{23} \leqslant 2\|QK_rA_r\|\;|\eta|\;\|\delta - K_r\eta\|_{L^2(0,1)} + 2\|QK_rB\|\;|\delta(1) - K_r\eta|\;\|\delta - K_r\eta\|_{L^2(0,1)}. \tag{29}$$

We denote the last term in (23) as W_{24}. It holds

$$W_{24} \leqslant 2\|QK_rA_r\|\;\|\delta - K_r\eta\|_{L^2(0,1)}\;|\bar{Z}|. \tag{30}$$

Combining (27), (28), (29) and (30), the following hold for all $\kappa > 0$,

$$\begin{aligned}\dot{W}_2 \leqslant & -\frac{\underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)}{\varepsilon}|\delta(1) - K_r\eta|^2 - \frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\|\delta - K_r\eta\|^2_{L^2(0,1)}\\ & +2\|QK_rA_r\|\;|\eta|\;\|\delta - K_r\eta\|_{L^2(0,1)} + 2\|QK_rB\|\;|\delta(1) - K_r\eta|\;\|\delta - K_r\eta\|_{L^2(0,1)}\\ & +\kappa\|QK_rA_r\|\;|\bar{Z}|^2 + \frac{\|QK_rA_r\|}{\kappa}\|\delta - K_r\eta\|^2_{L^2(0,1)}.\end{aligned} \tag{31}$$

Combining (21) and (31), $\dot{W}$ follows

$$\dot{W} \leqslant -\begin{pmatrix}|\delta(1) - K_r\eta|\\ |\eta|\\ \|\delta - K_r\eta\|_{L^2(0,1)}\end{pmatrix}^\top M\begin{pmatrix}|\delta(1) - K_r\eta|\\ |\eta|\\ \|\delta - K_r\eta\|_{L^2(0,1)}\end{pmatrix} + \kappa\|QK_rA_r\|\;|\bar{Z}|^2,$$

where $M = \begin{pmatrix}M_1 & M_2\\ \star & M_4\end{pmatrix}$, with $M_1 = \begin{pmatrix}M_{11} & M_{12}\\ \star & M_{14}\end{pmatrix} = \begin{pmatrix}\frac{\underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)}{\varepsilon} & -\|PB\|\\ \star & 1\end{pmatrix}$, $M_2 = \begin{pmatrix}-\|QK_rB\|\\ -\|QK_rA_r\|\end{pmatrix}$, $M_4 = \left(\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon} - \frac{\|QK_rA_r\|}{\kappa}\right)$.
Since $M_{14} > 0$, there exists $\varepsilon_1^* > 0$ such that for $\varepsilon \in (0, \varepsilon_1^*)$, $M_{11} - M_{12}M_{14}^{-1}M_{12}^\top > 0$. Due to the Schur complement, it holds $M_1 > 0$. There exists $\sigma > 0$ sufficiently large such that $M_4 > 0$ with $\kappa = \sigma\varepsilon$. Then, there exists $\varepsilon_2^* > 0$, such that for all $0 < \varepsilon < \min(\varepsilon_1^*, \varepsilon_2^*)$, we have $M_1 - M_2M_4^{-1}M_2^\top > 0$. Using again the Schur complement, it holds $M > 0$. Hence, the following holds

$$\dot{W} \leqslant -\theta W + \sigma\varepsilon\|QK_rA_r\|\;|\bar{Z}|^2,$$

for any $0 < \theta \leqslant \min\left\{\frac{\underline{\lambda}(M)}{\overline{\lambda}(P)}, \frac{\underline{\lambda}(M)}{\overline{\lambda}(Q)}\right\}$. Condition (11) implies the exponential stability of the reduced subsystem (4a), that is, there exist positive constants $\bar{C}$ and r, such that for all $t \geqslant 0$,

$$|\bar{Z}(t)|^2 \leqslant \bar{C}e^{-rt}|\bar{Z}_0|^2.$$

Thus we get

$$\dot{W} \leqslant -\theta W + \bar{C}\sigma\varepsilon e^{-rt}\|QK_rA_r\|\ |\bar{Z}_0|^2.$$

It holds

$$\begin{aligned} W &\leqslant e^{-\theta t}W(\eta_0,\delta_0) + \bar{C}\sigma\varepsilon\|QK_rA_r\|\ |\bar{Z}_0|^2 \int_0^t e^{-\theta(t-s)}e^{-rs}ds \\ &\leqslant e^{-\theta t}W(\eta_0,\delta_0) + \frac{\bar{C}\sigma\varepsilon\|QK_rA_r\|e^{-\theta t}\left(1-e^{(\theta-r)t}\right)}{r-\theta}|\bar{Z}_0|^2. \end{aligned}$$

We may assume that $r > \theta$, the above inequality becomes

$$W \leqslant e^{-\theta t}\left(W(\eta_0,\delta_0) + \frac{\bar{C}\sigma\varepsilon\|QK_rA_r\|}{r-\theta}|\bar{Z}_0|^2\right).$$

The function W is lower and upper bounded by

$$\underline{\lambda}(P)\ |\eta|^2 + e^{-\mu}\underline{\lambda}(Q)\ \|\delta - K_r\eta\|^2_{L^2(0,1)} \leqslant W \leqslant \|P\|\ |\eta|^2 + \|Q\|\ \|\delta - K_r\eta\|^2_{L^2(0,1)}.$$

Since the initial conditions are $\eta_0 = \delta_0 = 0$, we obtain

$$|\eta|^2 \leqslant \varepsilon C_1 e^{-\theta t}|\bar{Z}_0|^2,$$

where $C_1 > 0$. Moreover, it holds

$$\|\delta\|_{L^2(0,1)} = \|\delta - K_r\eta + K_r\eta\|_{L^2(0,1)} \leqslant \|\delta - K_r\eta\|_{L^2(0,1)} + K_r|\eta|.$$

Hence, we obtain

$$\|\delta\|^2_{L^2(0,1)} \leqslant \varepsilon C_2 e^{-\theta t}|\bar{Z}_0|^2,$$

where $C_2 > 0$. This concludes the proof of Theorem 1. ■

4.2 Tikhonov Theorem for Linear Hyperbolic PDE-Fast ODE System

Following the similar computation in Sect. 4.1, the error system of (6) is written as follows

$$\begin{cases} \varepsilon\dot{\tilde{\delta}}(t) = A\tilde{\delta} + B\tilde{\eta}(1) - \varepsilon A^{-1}B\Lambda\bar{y}_x(1), & \text{(32a)}\\ \tilde{\eta}_t + \Lambda\tilde{\eta}_x = 0, & \text{(32b)}\\ \tilde{\eta}(0) = K_1\tilde{\eta}(1) + K_2\tilde{\delta}, & \text{(32c)}\\ \tilde{\delta}_0 = Z_0 + A^{-1}B\bar{y}_0(1), & \text{(32d)}\\ \tilde{\eta}_0 = y_0 - \bar{y}_0, & \text{(32e)} \end{cases}$$

where $\tilde{\eta} = y - \bar{y}$, $\tilde{\delta} = Z + A^{-1}B\bar{y}(1)$.

Theorem 2 *Consider system (6). If (10) is satisfied, there exist positive values C_1, γ, ε^*, such that for all $0 < \varepsilon < \varepsilon^*$, for any initial condition $y_0 \in H^2(0,1)$ satisfying the compatibility conditions $y_0(0) = K_r y_0(1)$ and $y_{0x}(0) = \Lambda^{-1}K_r\Lambda y_{0x}(1)$, with $\bar{y}_0 = y_0$, and for $Z_0 \in \mathbb{R}^n$, it holds for $t \geqslant 0$*

$$\|y(.,t) - \bar{y}(.,t)\|^2_{L^2(0,1)} \leqslant \varepsilon C_1 e^{-\gamma t}\left(\|\bar{y}_0\|^2_{H^2(0,1)} + |Z_0 + A^{-1}B\bar{y}_0(1)|^2\right). \quad (33)$$

Proof We consider the following candidate Lyapunov function for system (32).

$$L_\varepsilon(\tilde{\eta},\tilde{\delta}) = \varepsilon\tilde{\delta}^\top P\tilde{\delta} + \int_0^1 e^{-\mu x}\,\tilde{\eta}^\top Q\tilde{\eta}\,dx.$$

Adopting the similar computations in the proof of Theorem 1, the time derivative of $L_\varepsilon(\tilde{\eta},\tilde{\delta})$ along the solution to system (32) is

$$\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) = -\begin{pmatrix}\tilde{\eta}(1)\\ \tilde{\delta}\end{pmatrix}^\top T\begin{pmatrix}\tilde{\eta}(1)\\ \tilde{\delta}\end{pmatrix} - \mu\int_0^1 e^{-\mu x}\tilde{\eta}^\top Q\Lambda\tilde{\eta}\,dx + 2\varepsilon\tilde{\delta}^\top P(A^{-1}B\Lambda)\bar{y}_x(1),$$

where $T = \begin{pmatrix} e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1 & -(K_1^\top Q\Lambda K_2 + B^\top P)\\ \star & -(A^\top P + PA) - K_2^\top Q\Lambda K_2\end{pmatrix}$.
According to (10), using Cauchy Schwarz inequality and Young's inequality, the above inequality holds for all $\kappa > 0$

$$\begin{aligned}\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant & -\underline{\lambda}(T)|\tilde{\delta}|^2 - \mu e^{-\mu}\underline{\lambda}(Q\Lambda)\|\tilde{\eta}\|^2_{L^2(0,1)}\\ & + \kappa\varepsilon\|P(A^{-1}B\Lambda)\||\tilde{\delta}|^2 + \frac{\varepsilon\|P(A^{-1}B\Lambda)\|}{\kappa}|\bar{y}_x(1)|^2. \end{aligned} \quad (34)$$

The function $L_\varepsilon(\tilde{\eta},\tilde{\delta})$ is upper and lower bounded as

$$e^{-\mu}\underline{\lambda}(Q)\ \|\tilde{\eta}\|^2_{L^2(0,1)} + \varepsilon\underline{\lambda}(P)\ |\tilde{\delta}|^2 \leqslant L_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant \|Q\|\ \|\tilde{\eta}\|^2_{L^2(0,1)} + \varepsilon\|P\|\ |\tilde{\delta}|^2. \quad (35)$$

By choosing $\kappa = 1$, there exist $\varepsilon^*, \gamma > 0$, such that for all $\varepsilon \in (0, \varepsilon^*)$, the following holds from (34)

$$\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant -\gamma L_\varepsilon(\tilde{\eta},\tilde{\delta}) + \varepsilon\|P(A^{-1}B\Lambda)\||\bar{y}_x(1)|^2. \quad (36)$$

Condition (10) implies that $e^{-\mu}Q\Lambda - K_r^\top Q\Lambda K_r > 0$. Let Δ be a diagonal positive matrix and $Q = \Delta^2\Lambda^{-1}$. It holds $\Delta^2 - K_r^\top\Delta^2 K_r > 0$, which is equivalent to $\|\Delta K_r\Delta^{-1}\| < 1$. Then according to [3, Theorem 2.3], the reduced subsystem (8) is exponentially stable in H^2-norm. Thus, we deduce from (36)

$$\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant -\gamma L_\varepsilon(\tilde{\eta},\tilde{\delta}) + C_r\varepsilon e^{-ct}\|P(A^{-1}B\Lambda)\|\ \|\bar{y}_0\|^2_{H^2(0,1)}, \quad (37)$$

where C_r and c are positive values.
The following holds

$$L_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant e^{-\gamma t}L_\varepsilon(\tilde{\eta}_0,\tilde{\delta}_0) + C_r\varepsilon\|P(A^{-1}B\Lambda)\|\ \|\bar{y}_0\|^2_{H^2(0,1)}\int_0^t e^{-\gamma(t-s)}e^{-cs}ds.$$

We may assume that $\gamma < c$, the above inequality becomes

$$L_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant e^{-\gamma t}L_\varepsilon(\tilde{\eta}_0,\tilde{\delta}_0) + e^{-\gamma t}\frac{C_r\varepsilon}{c-\gamma}\|P(A^{-1}B\Lambda)\|\ \|\bar{y}_0\|^2_{H^2(0,1)}.$$

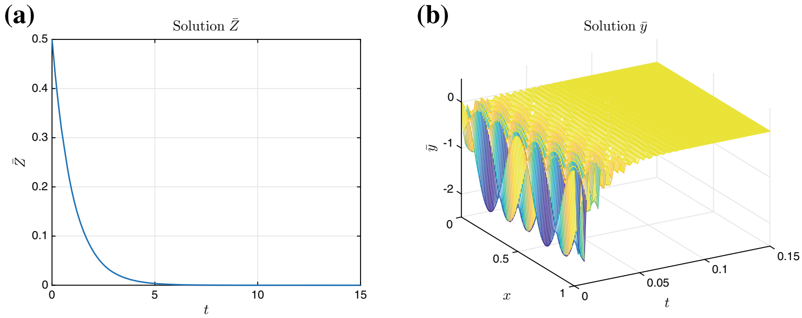
Using (35), we get

$$\|\tilde{\eta}\|^2_{L^2(0,1)} \leqslant C_1 e^{-\gamma t}\left(\|\tilde{\eta}_0\|^2_{L^2(0,1)} + \varepsilon|\tilde{\delta}_0|^2 + \varepsilon\|\bar{y}_0\|^2_{H^2(0,1)}\right),$$

where C_1 is positive constant. Since $\tilde{\eta}_0 = 0$, the inequality (33) holds for all $t \geqslant 0$. This concludes the proof of Theorem 2. ■

5 Numerical Simulations

Let us first show the numerical simulations on an ODE-fast PDE system (1), in which condition (10) is not satisfied. The full system is approximated by both subsystems under the stability conditions of the subsystems for ε sufficiently small. We next provide the numerical simulations on an academic example of PDE-fast ODE system (6). The Tikhonov approximation is achieved under the full system's stability condition (10).



Reduced subsystem Boundary-layer subsystem

Fig. 1 Time evolutions of subsystems (4) and (5)

Table 1 Estimates of the error for different values of ε with the initial condition $y_0 = K_r Z_0$

ε	0.03	0.02	0.01
$\lvert Z(t) - \bar{Z}(t)\rvert^2$	4.8×10^{-2}	2.1×10^{-2}	5.0×10^{-3}
$\lVert y(.,t) - K_r\bar{Z}(t)\rVert^2_{L^2(0,1)}$	2.2×10^{-4}	1.0×10^{-4}	2.6×10^{-5}

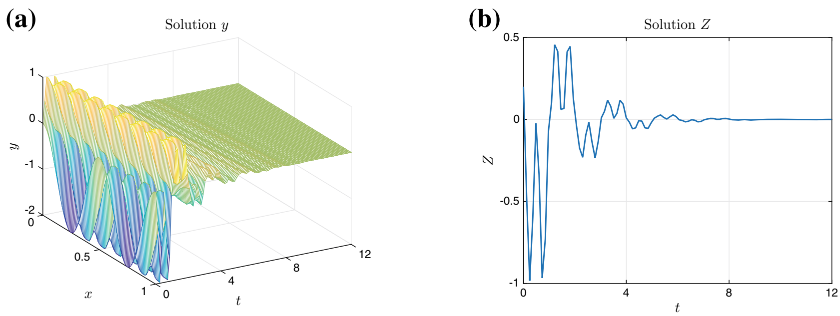
5.1 Numerical Simulations on an ODE-Fast PDE System

Let us consider system (1) with $A = 1$, $B = -1$, $\Lambda = 1$, $K_1 = \frac{1}{2}$, $K_2 = 1$. The initial conditions (1d)–(1e) are selected as $Z_0 = 0.5$ and $y_0(x) = \cos(4\pi x) - 1$. The perturbation parameter ε is chosen as $\varepsilon = 0.01$. It is computed $A_r = -1$ for the reduced subsystem (4). The initial condition (4b) is chosen as the same as for the full system $\bar{Z}_0 = Z_0 = 0.5$. The boundary condition for the boundary-layer subsystem is $K_1 = \frac{1}{2}$. The initial condition is chosen as $\bar{y}_0 = \cos(4\pi x) - \frac{3}{2}$. In view of A_r, condition (11) is satisfied for any $P > 0$. By choosing $Q = 1$, condition (12) holds. Figure 1 shows that the reduced and the boundary-layer subsystems converge to the origin as time increasing.

Let us choose $\varepsilon = \{0.03, 0.02, 0.01\}$, the initial condition y_0 is selected as the equilibrium point $y_0 = K_r Z_0$. Table 1 shows that the errors between the full system and the reduced subsystem decrease as ε decreasing, as expected from Theorem 1.

5.2 Numerical Simulations on a PDE-Fast ODE System

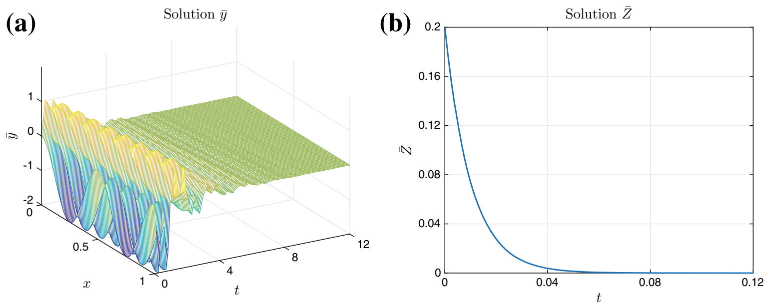
We consider system (6) with $A = -1$, $B = \frac{1}{2}$ and $\Lambda = 1$. The boundary condition (6c) is given by $K_1 = -\frac{1}{4}$, $K_2 = -\frac{1}{2}$. The initial conditions (6d)–(6e) are selected as $Z_0 = 0.2$ and $y_0(x) = \cos(4\pi x) - 1$. The perturbation parameter is $\varepsilon = 0.01$. By



Slow dynamics of the full system

Fast dynamics of the full system

Fig. 2 Time evolutions of the full system (6)



Reduced subsystem

Boundary-layer subsystem

Fig. 3 Time evolutions of subsystems (8) and (9)

Table 2 Estimates of the error for different values of ε

ε	0.03	0.02	0.01
$\|y(.,t) - \bar{y}(.,t)\|^2_{L^2}$	1.9×10^{-3}	5.7×10^{-4}	1.4×10^{-5}

choosing $P = Q = 1$, the stability condition (10) is satisfied. Therefore, Proposition 1 applies. In Fig. 2, the solutions of the slow and the fast dynamics of the full system tend to zero when time increases, as expected from Proposition 1.

Moreover, we compute the boundary condition matrix for the reduced subsystem as $K_r = -\frac{1}{2}$. The initial condition (8c) is chosen as $\bar{y}_0 = y_0(x) = \cos(4\pi x) - 1$. It is observed in Fig. 3 that the solutions of both subsystems converge to the origin as time increasing.

Let us choose $\varepsilon = \{0.03, 0.02, 0.01\}$. Table 2 shows that the errors between the slow dynamics of the full system and the reduced subsystem decrease as ε decreasing, as expected from Theorem 2.

6 Conclusions

A class of singularly perturbed linear ODE coupled with linear hyperbolic PDE systems has been considered in this work. The two subsystems have been formally computed based on the singular perturbation method. A general sufficient stability condition has been provided, which guarantees the stability of the full coupled ODE-PDE system and both subsystems. The Tikhonov approximation for full systems has been established by Lyapunov method. More precisely, based on the stability of both subsystems, the full ODE-fast PDE system is approximated by the subsystems. The estimate error is the order of the perturbation parameter ε. However, for PDE-fast ODE system, the approximation is valid if the general sufficient stability condition is satisfied.

References

1. Bastin, G., Coron, J.-M.: Stability and Boundary Stabilization of 1-D Hyperbolic Systems. PNLDE Subseries in Control, Birkhäuser (2016)
2. Bastin, G., Coron, J.-M., Tamasoiu, S.: Stability of linear density-flow hyperbolic systems under PI boundary control. Automatica **53**, 37–42 (2015)
3. Coron, J.-M., Bastin, G., d'Andréa-Novel, B.: Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. SIAM J. Control Optim. **47**(3), 1460–1498 (2008)
4. Daafouz, J., Tucsnak, M., Valein, J.: Nonlinear control of a coupled PDE/ODE system modeling a switched power converter with a transmission line. Syst. Control Lett. **70**, 92–99 (2014)
5. Dos Santos, V., Bastin, G., Coron, J.-M., d'Andréa-Novel, B.: Boundary control with integral action for hyperbolic systems of conservation laws: stability and expriments. Automatica **44**, 1013–1318 (2008)
6. Khalil, H.K.: Nonlinear Systems. Prentice-Hall (1996)
7. Kokotović, P., Haddad, A.: Singular perturbations of a class of time optimal controls. IEEE Trans. Autom. Control **20**, 163–164 (1975)
8. Kokotović, P., Khalil, H., O'Reilly, J.: Singular Perturbation Methods in Control: Analysis and Design. Academic Press (1986)
9. Kokotović, P., Sannuti, P.: Singular perturbation method for reducing the model order in optimal control design. IEEE Trans. Autom. Control **13**, 377–384 (1968)
10. Kokotović, P., Yackel, R.: Singular perturbation of linear regulators: basic theorems. IEEE Trans. Autom. Control **17**, 29–37 (1972)
11. Krstic, M., Smyshlyaev, A.: Backtepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. Syst. Control Lett. **57**, 750–758 (2008)
12. Littman, W., Markus, L.: Exact boundary controllability of a hybrid system of elasticity. Arch. Ration. Mech. Anal. **103**(3), 193–236 (1988)
13. Tang, Y., Mazanti, G.: Stability analysis of coupled linear ODE-hyperbolic PDE systems with two time scales. Automatica (accepted)
14. Tang, Y., Prieur, C., Girard, A.: Tikhonov theorem for linear hyperbolic systems. Automatica **57**, 1–10 (2015)
15. Tang, Y., Prieur, C., Girard, A.: Singular perturbation approximation by means of a H^2 Lyapunov function for linear hyperbolic systems. Syst. Control Lett. **88**, 24–31 (2016)