Advances in Delays and Dynamics 10

Giorgio Valmorbida
Alexandre Seuret
Islam Boussaada
Rifat Sipahi *Editors*

Delays and Interconnections: Methodology, Algorithms and Applications

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Silviu-Iulian Niculescu, Laboratory of Signals and Systems (L2S),
CNRS-CentraleSupèlec-Université Paris-Saclay, Gif sur Yvette, France

Delay systems are largely encountered in modeling propagation and transportation phenomena, population dynamics and representing interactions between interconnected dynamics through material, energy and communication flows. Thought as an open library on delays and dynamics, this series is devoted to publish basic and advanced textbooks, explorative research monographs as well as proceedings volumes focusing on delays from modeling to analysis, optimization, control with a particular emphasis on applications spanning biology, ecology, economy and engineering. Topics covering interactions between delays and modeling (from engineering to biology and economic sciences), control strategies (including also control structure and robustness issues), optimization and computation (including also numerical approaches and related algorithms) by creating links and bridges between fields and areas in a delay setting are particularly encouraged.

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Giorgio Valmorbida · Alexandre Seuret ·
Islam Boussaada · Rifat Sipahi
Editors

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Editors
Giorgio Valmorbida
Laboratoire des Signaux et Systèmes
CNRS-CentraleSupèlec-Université
Paris-Saclay
Gif-sur-Yvette, France

Islam Boussaada
Laboratoire des Signaux et Systèmes
CNRS-CentraleSupèlec-Université
Paris-Saclay
Gif-sur-Yvette, France

Alexandre Seuret
Laboratoire d'Analyse et d'Architecture
de Systèmes-CNRS
Toulouse, France

Rifat Sipahi
Mechanical and Industrial Engineering
Northeastern University
Boston, MA, USA

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Preface

The purpose of this volume in the established series Advances in Delays and Dynamics (ADD@S) is to provide a collection of recent results on the design and analysis of *Delays and Networked Control Systems*.

Many of current control systems operate in networks across several domains such as Energy Systems, Robotics, and Biology. The possibilities are many as resources can be shared, thanks to technological advances related to the use of communication protocols of the networks. In these systems, the individual elements such as sensor, actuators, and the computational elements might be spatially distributed, and an analysis of the individual components of the network may predict the behavior of the interconnections. Therefore, these emerging applications of controlled systems impose the study of interconnections and their effects on the system performance. The development of methods and algorithms for this class of systems is a challenge of nowadays' applications in which not only the system elements but also the interconnections are heterogeneous.

In this context, properties of the interconnections have to be assessed in terms of properties of individual systems and require specific tools whenever delays, a common phenomenon in communication networks, are introduced. Mathematical models presenting delays may arise from communication or matter transportation, from sampling phenomena or provide alternative representations of infinite-dimensional systems such as the wave equation.

For interconnected systems with delays, the uncertainties and disturbances can be related to the limitation of the communications and are induced by limited bandwidth and packet losses. It is therefore important to evaluate the impact of these disturbances in closed loop. A common strategy is to look at system properties such as the Input-to-State Stability of the system. Even for standard linear control strategies, it is important to evaluate the delays margins as a measure of the expected performance when a control loop is implemented in a network. Also, in some cases, the delays induced by the sampling or the communication can be beneficial for the system robustness. Other phenomena related to interconnected systems and communication protocols may lead to oscillations and strategies to understand and suppress undesired behavior required taking into account the nonlinear elements in the control loops.

This volume proposes different methods and algorithms for the analysis of interconnected systems, in particular systems presenting delays in interconnected systems. The majority of the methods employed to study these classes of systems are either Lyapunov methods, based on a time-domain representation of the system, or spectral methods, based on the representation of linear systems in the frequency domain. These are the main tools used by the contributions presented in the volume, which reflects the contents of the talks presented in the 4th DelSys Workshop that took place on November 25–27, 2015, in the Laboratoire des Signaux et Systèmes, Gif-sur-Yvette, France. The workshop was the last of a series of meetings of the International Scientific Coordination Network on Delay Systems—DelSys, supported by the French Center for Scientific Research (CNRS). The DelSys Network was established to promote research on delay systems and to enable scientific exchanges among experts during the period of 2012–2015.

The contents of the book are organized in three parts:

Interconnected Systems Analysis

Part one of the volume exposes several methods and strategies to cope with the analysis, control of interconnected systems. The seven chapters in this part cover continuous-time dynamical systems characterized by linear and nonlinear dynamics subject to state input or neutral delays as well as systems governed by a coupling between ordinary and partial differential equations.

This part starts with a chapter "Singular Perturbation Approach for Linear Coupled ODE-PDE Systems" authored by *Ying Tang, Christophe Prieur and Antoine Girard* where the problem of the singular perturbation approach is employed for the stability analysis and robustness of linear systems described by the interconnection of ordinary and partial differential equations.

The second chapter "On Some Neutral Functional Differential Equations Occurring in Synchronization" in this part, authored by *Vladimir Răsvan, Daniela Danciu and Dan Popescu*, deals with the relevance of neutral functional differential equations in the synchronization problem. The problem presented here arisen from the original setting of Huygens oscillators, where it appears that the dynamics of the coupled systems can be interpreted as neutral type of dynamics.

The third chapter "Dynamic Dissipativity Theory for Stability of Time-Delay Systems" provided by *Vijaysekhar Chellaboina and Wassim M. Haddad* aims at studying the dynamic dissipation theory for the stability analysis of time-delay systems. The originality of this contribution is to provide a method for systems subject to feedback interconnection that include delays. The main problem here is to create and enhance the links between the Lyapunov theory, more particularly the Lyapunov-Krasovskii method, and the dissipativity theory.

The next chapter "Stability of Interconnected Uncertain Delay Systems: A Converse Lyapunov Approach" in part one was contributed by *Ihab Haidar, Paolo Mason and Mario Sigalotti*. The objective of this chapter is to study the stability of interconnected uncertain systems subject to time delays. The originality of the present chapter relies on the characterization of a converse Lyapunov theorem for this class of systems.

A chapter "ISS-Stabilization of Delayed Neural Fields by Small-Gain Arguments" prepared by *Antoine Chaillet, Georgios Is. Detorakis, Stéphane Palfi and Suhan Senova* deals with the characterization of ISS stabilization for delayed neural fields by small-gain arguments. The main motivations of this chapter are related to the fact that integrodifferential equations describing the spatiotemporal activity of cerebral structures include delayed interconnection. The theoretical contributions therein are then to provide ISS properties of stabilization for this class of systems.

The following chapter "Robustness of Delayed Multistable Systems" copes with the robustness of delayed multistable systems and has been authored by *Denis Efimov, Johannes Schiffer and Romeo Ortega*. Sufficient conditions for input-to-state stability of delayed systems are provided based on the application of the Lyapunov-Razumikhin theorem. It is notably shown therein that ISS multistable systems are robust with respect to delays in the feedback path.

Part one ends with a chapter "A Small-Gain Method for the Design of Decentralized Stabilizing Controllers for Interconnected Systems with Delays" on the application of the small-gain theorem for the design of decentralized controllers for interconnected systems with delays, whose authors are *Pierdomenico Pepe, Hiroshi Ito and Zhong-Ping Jiang*. More particularly, a decentralized and practical input-to-state stabilization method is provided for a class of interconnected systems, affected by time delays in both the internal variables and in the communication channels, is considered.

Delay Systems: Modeling and Analysis

Part two of the volume exposes new trends in numeric as well as symbolic developments in the qualitative analysis of infinite-dimensional dynamical systems. As a matter of fact, both time-delay dynamical systems (continuous-time and discrete-time) and wave propagation equations are considered. The latter is a typical example of partial differential equations reducible to time-delay systems of neutral type. This part contains seven chapters covering theoretical contributions as well as applications in the control of infinite-dimensional systems.

This part opens with a chapter "Stability Analysis of Uniformly Distributed Delay Systems: A Frequency-Sweeping Approach" authored by *Xu-Guang Li, Silviu-Iulian Niculescu, Arben Çela and Lu Zhang* in which the stability of a class of systems including uniformly distributed delays is addressed, and a frequency-sweeping curve framework is proposed.

The second chapter "Asymptotic Analysis of Multiple Characteristics Roots for Quasi-polynomials of Retarded-Type" of this part is authored by

A. Martínez-González, S.-I. Niculescu, J. Chen, C.F. Méndez-Barrios, J.G. Romero and G. Mejía-Rodríguez where the asymptotic behavior of multiple critical roots of quasi-polynomials is addressed. The proposed analysis is based on the construction Weierstrass polynomial.

The third chapter "Scanning the Space of Parameters for Stability Regions of a Class of Time-Delay Systems; A Lyapunov Matrix Approach" authored by *Carlos Cuvas, Adrián Ramírez, Luis Juárez and Sabine Mondié* proposes a numerical stability test based on delay Lyapunov matrices allowing to detect stability regions in the space of parameters.

The fourth chapter "A Symbolic Computation Approach Towards the Asymptotic Stability Analysis of Differential Systems with Commensurate Delays" of the third part authored by *Yacine Bouzidi, Adrien Poteaux and Alban Quadrat* addresses the problem of delay perturbation effect on the stability of dynamical systems. It proposes certified symbolic-numerical algorithms for computing the set of critical pairs of a given quasipolynomial and for computing a Newton-Puiseux series at a critical pair.

The fifth chapter "Delay-Dependent Reciprocally Convex Combination Lemma for the Stability Analysis of Systems with a Fast-Varying Delay" is authored by *Alexandre Seuret and Frédéric Gouaisbaut*. It deals with the stability analysis of linear systems subject to fast-varying delays. The authors propose an improved version of the reciprocally convex lemma and use the Wirtinger-based integral inequality allowing to new stability conditions.

The sixth chapter "Wave Equation Modelling and Freeness Properties for Wind Power Systems" of this part is authored by *Hugues Mounier and Luca Greco*; it deals with some structural properties of partial differential equations. In particular, it considers two-wave equation model for strings of generators connected to a wind farm and investigates the differential flatness of the system.

This part closes by a chapter "A Delayed Mass-Action Model for the Transcriptional Control of Hmp, an NO Detoxifying Enzyme, by the Iron-Sulfur Protein FNR" on delayed mass-action model describing a biochemical reaction authored by *Marc R. Roussel*. It consists of a qualitative/quantitative study of the transcriptional control of an enzyme known to be a key contributor to the detoxification of nitric oxide.

Delay Systems: Stabilization and Control Strategies

Part three of the volume focuses on various strategies used to stabilize and control systems with delays. The five chapters in this part cover linear and nonlinear dynamics, continuous- and discrete-time systems, as well as single-input single-output (SISO) and multi-input multi-output (MIMO) systems.

This part opens with a chapter "A Comparison of Shaper-Based and Shaper-Free Architectures for Feedforward Compensation of Flexible Modes" authored by *Dan Pilbauer, Wim Michiels and Tomáš Vyhlídal* where the authors present a detailed study to compare the influence of shaper-based and shaper-free architectures for feedforward compensation of flexible modes. One highlight of the chapter is that shaper-free control design can be effective to achieve proper filtering; however, certain limitations in such controllers further motivate the use of shaper-based compensation schemes, which are designed using time delays.

The second chapter "Proportional-Retarded (PR) Protocol for a Large Scale Multi-agent Network with Noisy Measurements; Stability and Performance" in this part is authored by *Adrián Ramírez and Rifat Sipahi* on the stability analysis and performance assessment of proportional-retarded (PR) controllers in a multi-agent system. The authors identify the stable operating regions of the PR parameters with respect to the eigenvalues of the graph Laplacian of the multi-agent system, and present case studies on how PR controllers can help achieve fast consensus while satisfactorily rejecting noise in measurements.

The third chapter "Inversion of Separable Kernel Operator and Its Application in Control Synthesis", authored by *Guoying Miao, Matthew M. Peet and Keqin Gu*, demonstrates an operator-theoretic framework by which control synthesis problem can be posed in a convex optimization form. The proposed framework makes use of parameterization into finite-dimensional vectors, positive matrices, and certain inversion using algebraic manipulation, to reach a control synthesis approach amenable for computation of state-feedback controllers for differential-difference equations, which include the class of delay differential equations with discrete delays.

The next chapter "Delay Margin for Robust Stabilization of LTI Delay Systems" in part three is authored by *Tian Qi, Jing Zhu and Jie Chen*, on the investigation of delay margin for robust stabilization of LTI SISO closed-loop systems affected by a delay. Here, the authors focus on open-loop plants with unstable poles and non-minimum zeros, to first show a practical approach to obtain a lower bound on the largest delay margin achievable in the closed-loop, and next show how the results extend in the case of time-varying delays.

Part three closes with a chapter "Nonlinear Sampled-Data Stabilization with Delays" on nonlinear sampled-data stabilization with delays, authored by *Salvatore Monaco, Dorothée Normand-Cyrot and Mattia Mattioni*. In this chapter, the authors demonstrate how the effects of sampling stabilize nonlinear dynamics with delays. In particular, the authors demonstrate how to design sampled-data controllers and how sampling could actually help in the stabilization effort. The presentation includes comparisons of several pertaining approaches on this topic, and results are illustrated over academic examples.

Gif-sur-Yvette, France	Giorgio Valmorbida
Toulouse, France	Alexandre Seuret
Boston, USA	Rifat Sipahi
Gif-sur-Yvette, France	Islam Boussaada
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Acronyms

AC	Alternating Current
AG	Asymptotic Gain
CD	Critical delay
CIR	Critical imaginary root
CLF	Control Lyapunov Function
CLKF	Control Lyapunov-Krasovskii Functional
CLRF	Control Lyapunov-Razumikhin Function
CRS	Completely Regular Splitting
CTCR	Cluster Treatment of Characteristic Roots
DDE	Delay Differential Equations
DG	Distributed Generation
FSC	Frequency-Sweeping Curve
GAS	Globally Asymptotically Stable
GPe	External globus pallidus
HANSO	Hybrid Algorithm for Non-Smooth Optimization
I&I	Immersion and Invariance
ILM	Input-Lyapunov Matching
IR	Integral-Retarded
ISS	Input-to-State Stability
LHP	Left-Half Plane
LIM	Limit Property
LMI	Linear Matrix Inequality
LQR	Linear Quadratic Regulator
LR	Lyapunov-Razumikhin
LTI	Linear-Time-Invariant
MIMO	Multi-Input Multi-Output
mRNA	Messenger RNA
NO	Nitric Oxide
ODE	Ordinary Differential Equation
pAG	Practical Asymptotic Gain

PDE	Partial Differential Equation
pGS	Practical global stability
PID	Proportional Integral Derivative
PIR	Proportional-Integral-Retarded
PR	Proportional-Retarded
RFDE	Retarded Functional Differential Equation
RHP	Right-Half Plane
RNAP	RNA Polymerase
RS	Regular Splitting
RUR	Rational Univariate Representation
S-GAS	Sampled-data Globally Asymptotically Stable
SISO	Single-Input Single-Output
SOS	Sum-Of-Squares
STN	Subthalamic Nucleus
TDS	Time-Delay Systems
UDDS	Uniformly Distributed Delay System

Interconnected Systems Analysis

Singular Perturbation Approach for Linear Coupled ODE-PDE Systems

Ying Tang, Christophe Prieur and Antoine Girard

1 Introduction

Systems modeled by coupled ordinary differential equations (ODEs)-partial differential equations (PDEs) have been studied in many research works [2, 5, 11]. It is interesting to analyze such kind of systems due to their significant physical applications. For instance, elastic beams linked to rigid bodies in [12], power converters connected to transmission lines in [4] etc.

Singular perturbation theory has been widely used in control engineering from late 1960s. It is a powerful tool for analysis and design of control systems thanks to the reduction of the system's order by neglecting the fast transitions [7, 9, 10]. This theory is effective in many applications, such as semiconductors, electrical chains and so on.

Tikhonov approximation, which describes the limiting behavior of system's solutions, is an important method for analysis of singularly perturbed systems. Tikhonov approach for finite dimensional systems modeled by ODEs has been considered in [6]. In [14], a Tikhonov theorem for infinite dimensional systems governed by hyperbolic

Y. Tang
Université de Lorraine, CRAN UMR 7039 and CNRS, 2 avenue de la Forêt de Haye,
54516 Vandoeuvre-lès-Nancy, France
e-mail: ying.tang@univ-lorraine.fr

C. Prieur
GIPSA-lab, Grenoble Campus, 11 rue des Mathématiques, BP 46,
38402 Saint Martin d'Hères Cedex, France
e-mail: christophe.prieur@gipsa-lab.fr

A. Girard (✉)
Laboratoire des signaux et systèmes (L2S), CNRS, CentraleSupélec,
Université Paris-Sud, Université Paris-Saclay, 3 rue Joliot-Curie,
91192 Gif-sur-Yvette Cedex, France
e-mail: Antoine.Girard@l2s.centralesupelec.fr; antoine.girard@imag.fr

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PDEs has been established by means of a L^2 Lyapunov function. The approximation for linear hyperbolic systems has been improved by using a H^2 Lyapunov function in [15].

The present work is concerned with a class of coupled singularly perturbed linear ODE and linear hyperbolic PDE systems. Firstly, a sufficient stability condition is proposed for both coupled ODE-fast PDE and PDE-fast ODE systems. The stability of the full system implies the stability of both subsystems. Secondly, the Tikhonov approximation for such systems is achieved by Lyapunov method. Under the stability conditions of both subsystems, the coupled ODE-fast PDE system is approximated by the two subsystems for ε sufficiently small. However, for PDE-fast ODE system, the approximation is valid if the full system is stable. The error between the full system and the subsystems is estimated as the order of the perturbation parameter ε.

The paper is organized as follows. The coupled ODE-PDE systems under consideration are given in Sect. 2. The reduced and the boundary-layer subsystems are formally computed in the same section. Section 3 provides sufficient stability conditions for the full system and both subsystems. The Tikhonov approximation for such systems is stated in Sect. 4. Numerical simulations on academic examples are shown in Sect. 5. Concluding remarks end of this paper.

Notation. Given a matrix G in $\mathbb{R}^{n\times n}$, G^{-1} and $G^\top$ represent the inverse and the transpose matrix of G respectively. The minimum and maximum eigenvalues of a symmetric matrix G are denoted by $\underline{\lambda}(G)$ and $\overline{\lambda}(G)$. The symbol $\star$ in partitioned symmetric matrix stands for the symmetric block. For a positive integer n, I_n is the identity matrix in $\mathbb{R}^{n\times n}$. $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ is associated with the usual 2-norm of matrices in $\mathbb{R}^{n\times n}$. $\|\cdot\|_{L^2}$ denotes the associated norm in $L^2(0,1)$ space, defined by $\|f\|_{L^2} = \sqrt{\left(\int_0^1 |f(x)|^2 dx\right)}$ for all functions $f \in L^2(0,1)$. Similarly, the associated norm in $H^2(0,1)$ space is denoted by $\|\cdot\|_{H^2}$, defined for all functions $f \in H^2(0,1)$, by $\|f\|_{H^2} = \sqrt{\left(\int_0^1 |f(x)|^2 + |f'(x)|^2 + |f''(x)|^2 dx\right)}$. Given a real interval I and a normed space J, $C^0(I,J)$ denotes the set of continuous functions from I to J.

2 Singularly Perturbed Linear Coupled ODE-PDE Systems

In this section, the coupled ODE-fast PDE and PDE-fast ODE systems under consideration are given respectively.

2.1 Coupled ODE-Fast PDE System

We consider the following *linear ODE-fast hyperbolic PDE system*

$$\begin{cases} \dot{Z}(t) = AZ(t) + By(1,t), & \text{(1a)} \\ \varepsilon y_t(x,t) + \Lambda y_x(x,t) = 0, & \text{(1b)} \\ y(0,t) = K_1 y(1,t) + K_2 Z(t), & \text{(1c)} \\ Z(0) = Z_0, & \text{(1d)} \\ y(x,0) = y_0(x), & \text{(1e)} \end{cases}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $Z : [0,+\infty) \to \mathbb{R}^n$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}^m$. The perturbation parameter $\varepsilon > 0$ is a small constant and Λ is a diagonal positive matrix in $\mathbb{R}^{m \times m}$. The matrices A and B are of appropriate dimensions. The boundary condition matrices K_1 and K_2 are constant matrices of appropriate dimensions.

Adopting the computations of the subsystems for singularly perturbed ODEs [8], the reduced and the boundary-layer subsystems for system (1) are formally computed as follows. By setting $\varepsilon = 0$ in Eq. (1b), we obtain

$$y_x(x,t) = 0. \tag{2}$$

It implies $y(.,t) = y(1,t)$. Assuming $(I_m - K_1)$ invertible, the boundary condition (1c) becomes

$$y(.,t) = K_r Z(t), \tag{3}$$

where $K_r = (I_m - K_1)^{-1} K_2$. Using the right-hand side of (3) to replace $y(1,t)$ in (1a), the *reduced subsystem* is computed as

$$\begin{cases} \dot{\bar{Z}}(t) = A_r \bar{Z}(t), & \text{(4a)} \\ \bar{Z}_0 = Z_0. & \text{(4b)} \end{cases}$$

where $A_r = A + BK_r$. The bar indicates that the variables belong to the system with $\varepsilon = 0$. Using the following change of variable $\bar{y} = y - K_r Z$ and a new time scale $\tau = t/\varepsilon$, we have

$$\begin{cases} \bar{y}_\tau(x,\tau) + \Lambda \bar{y}_x(x,\tau) = -\varepsilon K_r (AZ(\tau) + By(1,\tau)), \\ \bar{y}(0,\tau) = K_1 \bar{y}(1,\tau). \end{cases}$$

The *boundary-layer subsystem* is formally computed with $\varepsilon = 0$

$$\begin{cases} \bar{y}_\tau(x,\tau) + \Lambda \bar{y}_x(x,\tau) = 0, & \text{(5a)} \\ \bar{y}(0,\tau) = K_1 \bar{y}(1,\tau), & \text{(5b)} \\ \bar{y}_0(x) = y_0(x) - K_r Z_0. & \text{(5c)} \end{cases}$$

2.2 Coupled PDE-Fast ODE System

Similar to system (1), we consider the following *linear hyperbolic PDE-fast ODE system*

$$\begin{cases} \varepsilon \dot{Z}(t) = AZ(t) + By(1,t), & \text{(6a)} \\ y_t(x,t) + \Lambda y_x(x,t) = 0, & \text{(6b)} \\ y(0,t) = K_1 y(1,t) + K_2 Z(t), & \text{(6c)} \\ Z(0) = Z_0, & \text{(6d)} \\ y(x,0) = y_0(x). & \text{(6e)} \end{cases}$$

The two subsystems are computed as follows. By formally setting $\varepsilon = 0$ in (6a) and assuming A invertible, we have

$$Z = -A^{-1}By(1). \tag{7}$$

Substituting (7) into (6c), the *reduced subsystem* is

$$\begin{cases} \bar{y}_t(x,t) + \Lambda \bar{y}_x(x,t) = 0, & \text{(8a)} \\ \bar{y}(0,t) = K_r \bar{y}(1,t), & \text{(8b)} \\ \bar{y}(x,0) = \bar{y}_0(x) = y_0(x), & \text{(8c)} \end{cases}$$

where $K_r = K_1 - K_2 A^{-1} B$. Performing a change of variable $\bar{Z} = Z + A^{-1}By$ (1) and using a new time scale $\tau = t/\varepsilon$, we get

$$\frac{d\bar{Z}(\tau)}{d\tau} = A\bar{Z}(\tau) - \varepsilon A^{-1} B \Lambda y_x(1,\tau).$$

The *boundary-layer subsystem* is formally computed with $\varepsilon = 0$

$$\begin{cases} \dfrac{d\bar{Z}(\tau)}{d\tau} = A\bar{Z}(\tau), & \text{(9a)} \\ \bar{Z}(0) = \bar{Z}_0 = Z_0 + A^{-1}By_0(1). & \text{(9b)} \end{cases}$$

Remark 1 Due to [1, Theorem A.6.], the Cauchy problems (1) and (6) are well-posed, that is, for every $Z_0 \in \mathbb{R}^n$, for every $y_0 \in L^2(0,1)$, systems (1) and (6) have a unique solution $Z \in C^0([0,+\infty), \mathbb{R}^n)$, $y \in C^0([0,+\infty), L^2((0,1), \mathbb{R}^m))$.

3 Stability Condition of Coupled ODE-PDE Systems

We first provide a sufficient stability condition for both coupled ODE-PDE systems (1) and (6). Then, we study the link of the stability between the full system and both subsystems.

Proposition 1 *Systems (1) and (6) are exponentially stable for all* $\varepsilon > 0$ *if there exist diagonal positive matrix* Q*, symmetric positive matrix* P *and positive constant* μ *such that the following holds*

$$\begin{pmatrix} e^{-\mu} Q\Lambda - K_1^\top Q\Lambda K_1 & -(K_1^\top Q\Lambda K_2 + B^\top P) \\ \star & -(A^\top P + PA) - K_2^\top Q\Lambda K_2 \end{pmatrix} > 0. \tag{10}$$

The next two propositions show that condition (10) implies the stability of the reduced and the boundary-layer subsystems.

Proposition 2 *Condition (10) implies*

$$A_r^\top P + PA_r < 0, \tag{11}$$

which is equivalent to the stability of the reduced subsystem (4), and

$$e^{-\mu} Q\Lambda - K_1^\top Q\Lambda K_1 > 0, \tag{12}$$

which implies the stability of the boundary-layer subsystem (5).

Proposition 3 *Condition (10) implies*

$$e^{-\mu} Q\Lambda - K_r^\top Q\Lambda K_r > 0, \tag{13}$$

which implies the stability of the reduced subsystem (8), and

$$A^\top P + PA < 0, \tag{14}$$

which is equivalent to the stability of the boundary-layer subsystem (9).

In view of Proposition 1, the stability of the full systems (1) and (6) is guaranteed for all positive ε under condition (10). For ε sufficiently small, the stability of both subsystems (4) and (5) implies the stability of the coupled ODE-fast PDE system (1), even though condition (10) is not satisfied. However, this result is not valid in the context for PDE-fast ODE system (6). That is, system (6) could be unstable even though the two subsystems (8) and (9) are stable. We refer the readers to [13] for more details.

4 Tikhonov Approximation of Coupled ODE-PDE Systems

We deal with the Tikhonov approximation of the coupled systems when ε is sufficiently small as follows. If the two subsystems are stable, the coupled ODE-fast PDE system can be approximated by the subsystems. The approach for coupled PDE-fast ODE system is valid if the full system is stable.

4.1 Tikhonov Theorem for Linear ODE-Fast PDE System

Let us state Tikhonov theorem for system (1) in the next theorem.

Theorem 1 *Consider system (1). If (11) and (12) are satisfied, there exist positive values C_1, C_2, θ, ε^*, such that for all $0 < \varepsilon < \varepsilon^*$, and for any initial conditions $Z_0 \in \mathbb{R}^n$, $y_0 = K_r Z_0$, it holds for $t \geqslant 0$*

$$|Z(t) - \bar{Z}(t)|^2 \leqslant \varepsilon C_1 e^{-\theta t} |\bar{Z}_0|^2, \tag{15}$$

$$\|y(.,t) - K_r \bar{Z}(t)\|^2_{L^2(0,1)} \leqslant \varepsilon C_2 e^{-\theta t} |\bar{Z}_0|^2. \tag{16}$$

Before proving this theorem, we first write the error system of (1). Let us perform the following change of variables

$$\eta = Z - \bar{Z}, \tag{17a}$$

$$\delta = y - K_r \bar{Z}, \tag{17b}$$

where η represents the error between the slow dynamics of the full system and the reduced subsystem while δ is the error between the fast dynamics of the full system and its equilibrium point. Due to (1a) and (4), we write

$$\begin{aligned} \dot{\eta} = \dot{Z} - \dot{\bar{Z}} &= AZ + By(1) - A_r \bar{Z} \\ &= A(Z - \bar{Z}) + B(y(1) - K_r \bar{Z}). \end{aligned}$$

Due to (1b) and (4), we compute

$$\begin{aligned} &\delta_t = y_t - K_r \dot{\bar{Z}} = y_t - K_r (A_r \bar{Z}), \\ &\delta_x = y_x. \end{aligned}$$

Due to (1c), we have

$$\begin{aligned} \delta(0) = y(0) - K_r \bar{Z} &= K_1 y(1) + K_2 Z - K_r \bar{Z} \\ &= K_1 \Big(y(1) - K_r \bar{Z} \Big) + K_2 Z - (I_m - K_1) K_r \bar{Z} \end{aligned}$$

$$= K_1\Big(y(1) - K_r\bar{Z}\Big) + K_2 Z - (I_m - K_1)(I_m - K_1)^{-1}K_2\bar{Z}$$
$$= K_1\Big(y(1) - K_r\bar{Z}\Big) + K_2(Z - \bar{Z}).$$

Thus, the error system is written as follows

$$\begin{cases} \dot{\eta} = A\eta + B\delta(1), & \text{(18a)} \\ \varepsilon\delta_t + \Lambda\delta_x = -\varepsilon K_r A_r \bar{Z}, & \text{(18b)} \\ \delta(0) = K_1\delta(1) + K_2\eta, & \text{(18c)} \\ \eta_0 = Z_0 - \bar{Z}_0 = 0, & \text{(18d)} \\ \delta_0 = y_0 - K_r Z_0. & \text{(18e)} \end{cases}$$

Based on the above error system we are ready to prove Theorem 1.

Proof Let us consider the following candidate Lyapunov function for system (18)

$$W(\eta,\delta) = \eta^\top P\eta + \int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q(\delta - K_r\eta)\,dx, \tag{19}$$

with $\mu > 0$, matrices P and Q are specified later.
We rewrite $W(\eta,\delta) = W_1 + W_2$, with $W_1 = \eta^\top P\eta$ and $W_2 = \int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q(\delta - K_r\eta)\,dx$. The time derivative of W_1 along the solution to system (18a) is computed as

$$\begin{aligned}\dot{W}_1 &= 2\eta^\top P\dot{\eta} \\ &= 2\eta^\top P(A\eta + B\delta(1)) \\ &= \eta^\top\Big(PA_r + A_r^\top P\Big)\eta + 2\eta^\top PB\Big(\delta(1) - K_r\eta\Big).\end{aligned}$$

According to (11), there exists a symmetric positive matrix P such that

$$PA_r + A_r^\top P < -I_n. \tag{20}$$

Due to Cauchy Schwarz inequality, it holds

$$\dot{W}_1 \leqslant -|\eta|^2 + 2\|PB\|\,|\delta(1) - K_r\eta|\,|\eta|. \tag{21}$$

The time derivative of W_2 along the solution to system (18b) is

$$\begin{aligned}\dot{W}_2 &= 2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q(\delta_t - K_r\dot{\eta})\,dx\\ &= -\frac{2}{\varepsilon}\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q\Lambda\delta_x\,dx - 2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_r(A\eta + B\delta(1))\,dx\\ &\quad -2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_rA_r\bar{Z}\,dx.\end{aligned}\tag{22}$$

Performing an integration by parts on the first integral in the right-hand side of (22), $\dot{W}_2$ follows

$$\begin{aligned}\dot{W}_2 &= -\frac{1}{\varepsilon}\Big[e^{-\mu x}(\delta - K_r\eta)^\top Q\Lambda(\delta - K_r\eta)\Big]_{x=0}^{x=1} - \frac{\mu}{\varepsilon}\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top Q\Lambda(\delta - K_r\eta)\,dx\\ &\quad -2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_r(A\eta + B\delta(1))\,dx - 2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_rA_r\bar{Z}\,dx.\end{aligned}\tag{23}$$

Let the first term in (23) be W_{21}. Under the boundary condition (18c), we have

$$\begin{aligned}W_{21} &= -\frac{1}{\varepsilon}\Big[e^{-\mu}(\delta(1) - K_r\eta)^\top Q\Lambda(\delta(1) - K_r\eta) - (\delta(0) - K_r\eta)^\top Q\Lambda(\delta(0) - K_r\eta)\Big]\\ &= -\frac{1}{\varepsilon}\Big[e^{-\mu}(\delta(1) - K_r\eta)^\top Q\Lambda(\delta(1) - K_r\eta)\\ &\qquad -(K_1\delta(1) + K_2\eta - K_r\eta)^\top Q\Lambda(K_1\delta(1) + K_2\eta - K_r\eta)\Big].\end{aligned}\tag{24}$$

We write

$$\begin{aligned}K_2 - K_r &= K_2 - (I_m - K_1)^{-1}K_2 = (I_m - K_1)(I_m - K_1)^{-1}K_2 - (I_m - K_1)^{-1}K_2\\ &= (I_m - K_1 - I_m)\,(I_m - K_1)^{-1}K_2 = -K_1K_r.\end{aligned}\tag{25}$$

Using the right-hand side of (25) to replace $K_2 - K_r$ in (24), we obtain

$$W_{21} = -\frac{1}{\varepsilon}\Big[(\delta(1) - K_r\eta)^\top(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)(\delta(1) - K_r\eta)\Big].$$

By using (12), there exists a diagonal positive matrix Q such that

$$e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1 > \underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1) > 0.\tag{26}$$

Thus

$$W_{21} \leqslant -\frac{\underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)}{\varepsilon}|\delta(1) - K_r\eta|^2.\tag{27}$$

Let W_{22} denote the second term in (23), it follows

$$W_{22} \leqslant -\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\|\delta - K_r\eta\|^2_{L^2(0,1)}. \tag{28}$$

Let the third term in (23) be W_{23}, it follows

$$W_{23} = -2\int_0^1 e^{-\mu x}(\delta - K_r\eta)^\top QK_r\Big(A_r\eta + B(\delta(1) - K_r\eta)\Big)\,dx.$$

Due to Cauchy Schwarz inequality, W_{23} follows

$$W_{23} \leqslant 2\|QK_rA_r\|\ |\eta|\ \|\delta - K_r\eta\|_{L^2(0,1)} + 2\|QK_rB\|\ |\delta(1) - K_r\eta|\ \|\delta - K_r\eta\|_{L^2(0,1)}. \tag{29}$$

We denote the last term in (23) as W_{24}. It holds

$$W_{24} \leqslant 2\|QK_rA_r\|\ \|\delta - K_r\eta\|_{L^2(0,1)}\ |\bar{Z}|. \tag{30}$$

Combining (27), (28), (29) and (30), the following hold for all $\kappa > 0$,

$$\begin{aligned}\dot{W}_2 \leqslant & -\frac{\underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)}{\varepsilon}|\delta(1) - K_r\eta|^2 - \frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\|\delta - K_r\eta\|^2_{L^2(0,1)}\\ & +2\|QK_rA_r\|\ |\eta|\ \|\delta - K_r\eta\|_{L^2(0,1)} + 2\|QK_rB\|\ |\delta(1) - K_r\eta|\ \|\delta - K_r\eta\|_{L^2(0,1)}\\ & +\kappa\|QK_rA_r\|\ |\bar{Z}|^2 + \frac{\|QK_rA_r\|}{\kappa}\|\delta - K_r\eta\|^2_{L^2(0,1)}.\end{aligned} \tag{31}$$

Combining (21) and (31), $\dot{W}$ follows

$$\dot{W} \leqslant -\begin{pmatrix}|\delta(1) - K_r\eta|\\ |\eta|\\ \|\delta - K_r\eta\|_{L^2(0,1)}\end{pmatrix}^\top M \begin{pmatrix}|\delta(1) - K_r\eta|\\ |\eta|\\ \|\delta - K_r\eta\|_{L^2(0,1)}\end{pmatrix} + \kappa\|QK_rA_r\|\ |\bar{Z}|^2,$$

where $M = \begin{pmatrix}M_1 & M_2\\ \star & M_4\end{pmatrix}$, with $M_1 = \begin{pmatrix}M_{11} & M_{12}\\ \star & M_{14}\end{pmatrix} = \begin{pmatrix}\frac{\underline{\lambda}(e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1)}{\varepsilon} & -\|PB\|\\ \star & 1\end{pmatrix}$, $M_2 = \begin{pmatrix}-\|QK_rB\|\\ -\|QK_rA_r\|\end{pmatrix}$, $M_4 = \left(\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon} - \frac{\|QK_rA_r\|}{\kappa}\right)$.
Since $M_{14} > 0$, there exists $\varepsilon_1^* > 0$ such that for $\varepsilon \in (0, \varepsilon_1^*)$, $M_{11} - M_{12}M_{14}^{-1}M_{12}^\top > 0$. Due to the Schur complement, it holds $M_1 > 0$. There exists $\sigma > 0$ sufficiently large such that $M_4 > 0$ with $\kappa = \sigma\varepsilon$. Then, there exists $\varepsilon_2^* > 0$, such that for all $0 < \varepsilon < \min(\varepsilon_1^*, \varepsilon_2^*)$, we have $M_1 - M_2M_4^{-1}M_2^\top > 0$. Using again the Schur complement, it holds $M > 0$. Hence, the following holds

$$\dot{W} \leqslant -\theta W + \sigma\varepsilon\|QK_rA_r\|\ |\bar{Z}|^2,$$

for any $0 < \theta \leqslant \min\left\{\frac{\underline{\lambda}(M)}{\overline{\lambda}(P)}, \frac{\underline{\lambda}(M)}{\overline{\lambda}(Q)}\right\}$. Condition (11) implies the exponential stability of the reduced subsystem (4a), that is, there exist positive constants $\bar{C}$ and r, such that for all $t \geqslant 0$,

$$|\bar{Z}(t)|^2 \leqslant \bar{C}e^{-rt}|\bar{Z}_0|^2.$$

Thus we get

$$\dot{W} \leqslant -\theta W + \bar{C}\sigma\varepsilon e^{-rt}\|QK_rA_r\|\ |\bar{Z}_0|^2.$$

It holds

$$\begin{aligned} W &\leqslant e^{-\theta t}W(\eta_0, \delta_0) + \bar{C}\sigma\varepsilon\|QK_rA_r\|\ |\bar{Z}_0|^2 \int_0^t e^{-\theta(t-s)}e^{-rs}ds \\ &\leqslant e^{-\theta t}W(\eta_0, \delta_0) + \frac{\bar{C}\sigma\varepsilon\|QK_rA_r\|e^{-\theta t}\left(1 - e^{(\theta-r)t}\right)}{r-\theta}|\bar{Z}_0|^2. \end{aligned}$$

We may assume that $r > \theta$, the above inequality becomes

$$W \leqslant e^{-\theta t}\left(W(\eta_0, \delta_0) + \frac{\bar{C}\sigma\varepsilon\|QK_rA_r\|}{r-\theta}|\bar{Z}_0|^2\right).$$

The function W is lower and upper bounded by

$$\underline{\lambda}(P)\ |\eta|^2 + e^{-\mu}\underline{\lambda}(Q)\ \|\delta - K_r\eta\|^2_{L^2(0,1)} \leqslant W \leqslant \|P\|\ |\eta|^2 + \|Q\|\ \|\delta - K_r\eta\|^2_{L^2(0,1)}.$$

Since the initial conditions are $\eta_0 = \delta_0 = 0$, we obtain

$$|\eta|^2 \leqslant \varepsilon C_1 e^{-\theta t}|\bar{Z}_0|^2,$$

where $C_1 > 0$. Moreover, it holds

$$\|\delta\|_{L^2(0,1)} = \|\delta - K_r\eta + K_r\eta\|_{L^2(0,1)} \leqslant \|\delta - K_r\eta\|_{L^2(0,1)} + K_r|\eta|.$$

Hence, we obtain

$$\|\delta\|^2_{L^2(0,1)} \leqslant \varepsilon C_2 e^{-\theta t}|\bar{Z}_0|^2,$$

where $C_2 > 0$. This concludes the proof of Theorem 1. ■

4.2 Tikhonov Theorem for Linear Hyperbolic PDE-Fast ODE System

Following the similar computation in Sect. 4.1, the error system of (6) is written as follows

$$
\begin{cases}
\varepsilon\dot{\tilde{\delta}}(t) = A\tilde{\delta} + B\tilde{\eta}(1) - \varepsilon A^{-1}B\Lambda\bar{y}_x(1), & \text{(32a)}\\
\tilde{\eta}_t + \Lambda\tilde{\eta}_x = 0, & \text{(32b)}\\
\tilde{\eta}(0) = K_1\tilde{\eta}(1) + K_2\tilde{\delta}, & \text{(32c)}\\
\tilde{\delta}_0 = Z_0 + A^{-1}B\bar{y}_0(1), & \text{(32d)}\\
\tilde{\eta}_0 = y_0 - \bar{y}_0, & \text{(32e)}
\end{cases}
$$

where $\tilde{\eta} = y - \bar{y}$, $\tilde{\delta} = Z + A^{-1}B\bar{y}(1)$.

Theorem 2 *Consider system (6). If (10) is satisfied, there exist positive values* C_1*,* γ*,* ε^**, such that for all* $0 < \varepsilon < \varepsilon^*$*, for any initial condition* $y_0 \in H^2(0,1)$ *satisfying the compatibility conditions* $y_0(0) = K_r y_0(1)$ *and* $y_{0x}(0) = \Lambda^{-1}K_r\Lambda y_{0x}(1)$*, with* $\bar{y}_0 = y_0$*, and for* $Z_0 \in \mathbb{R}^n$*, it holds for* $t \geqslant 0$

$$
\|y(.,t) - \bar{y}(.,t)\|^2_{L^2(0,1)} \leqslant \varepsilon C_1 e^{-\gamma t}\left(\|\bar{y}_0\|^2_{H^2(0,1)} + |Z_0 + A^{-1}B\bar{y}_0(1)|^2\right). \tag{33}
$$

Proof We consider the following candidate Lyapunov function for system (32).

$$
L_\varepsilon(\tilde{\eta},\tilde{\delta}) = \varepsilon\tilde{\delta}^\top P\tilde{\delta} + \int_0^1 e^{-\mu x}\,\tilde{\eta}^\top Q\tilde{\eta}\,dx.
$$

Adopting the similar computations in the proof of Theorem 1, the time derivative of $L_\varepsilon(\tilde{\eta},\tilde{\delta})$ along the solution to system (32) is

$$
\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) = -\begin{pmatrix}\tilde{\eta}(1)\\ \tilde{\delta}\end{pmatrix}^\top T\begin{pmatrix}\tilde{\eta}(1)\\ \tilde{\delta}\end{pmatrix} - \mu\int_0^1 e^{-\mu x}\tilde{\eta}^\top Q\Lambda\tilde{\eta}\,dx + 2\varepsilon\tilde{\delta}^\top P(A^{-1}B\Lambda)\bar{y}_x(1),
$$

where $T = \begin{pmatrix} e^{-\mu}Q\Lambda - K_1^\top Q\Lambda K_1 & -(K_1^\top Q\Lambda K_2 + B^\top P)\\ \star & -(A^\top P + PA) - K_2^\top Q\Lambda K_2\end{pmatrix}$.
According to (10), using Cauchy Schwarz inequality and Young's inequality, the above inequality holds for all $\kappa > 0$

$$
\begin{aligned}
\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant{}& -\underline{\lambda}(T)|\tilde{\delta}|^2 - \mu e^{-\mu}\underline{\lambda}(Q\Lambda)\|\tilde{\eta}\|^2_{L^2(0,1)}\\
&+\kappa\varepsilon\|P(A^{-1}B\Lambda)\||\tilde{\delta}|^2 + \frac{\varepsilon\|P(A^{-1}B\Lambda)\|}{\kappa}|\bar{y}_x(1)|^2.
\end{aligned} \tag{34}
$$

The function $L_\varepsilon(\tilde{\eta},\tilde{\delta})$ is upper and lower bounded as

$$e^{-\mu}\underline{\lambda}(Q)\ \|\tilde{\eta}\|^2_{L^2(0,1)} + \varepsilon\underline{\lambda}(P)\ |\tilde{\delta}|^2 \leqslant L_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant \|Q\|\ \|\tilde{\eta}\|^2_{L^2(0,1)} + \varepsilon\|P\|\ |\tilde{\delta}|^2. \quad (35)$$

By choosing $\kappa = 1$, there exist $\varepsilon^*, \gamma > 0$, such that for all $\varepsilon \in (0, \varepsilon^*)$, the following holds from (34)

$$\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant -\gamma L_\varepsilon(\tilde{\eta},\tilde{\delta}) + \varepsilon\|P(A^{-1}B\Lambda)\||\bar{y}_x(1)|^2. \quad (36)$$

Condition (10) implies that $e^{-\mu}Q\Lambda - K_r^\top Q\Lambda K_r > 0$. Let Δ be a diagonal positive matrix and $Q = \Delta^2\Lambda^{-1}$. It holds $\Delta^2 - K_r^\top\Delta^2 K_r > 0$, which is equivalent to $\|\Delta K_r\Delta^{-1}\| < 1$. Then according to [3, Theorem 2.3], the reduced subsystem (8) is exponentially stable in H^2-norm. Thus, we deduce from (36)

$$\dot{L}_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant -\gamma L_\varepsilon(\tilde{\eta},\tilde{\delta}) + C_r\varepsilon e^{-ct}\|P(A^{-1}B\Lambda)\|\ \|\bar{y}_0\|^2_{H^2(0,1)}, \quad (37)$$

where C_r and c are positive values.
The following holds

$$L_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant e^{-\gamma t}L_\varepsilon(\tilde{\eta}_0,\tilde{\delta}_0) + C_r\varepsilon\|P(A^{-1}B\Lambda)\|\ \|\bar{y}_0\|^2_{H^2(0,1)}\int_0^t e^{-\gamma(t-s)}e^{-cs}ds.$$

We may assume that $\gamma < c$, the above inequality becomes

$$L_\varepsilon(\tilde{\eta},\tilde{\delta}) \leqslant e^{-\gamma t}L_\varepsilon(\tilde{\eta}_0,\tilde{\delta}_0) + e^{-\gamma t}\frac{C_r\varepsilon}{c-\gamma}\|P(A^{-1}B\Lambda)\|\ \|\bar{y}_0\|^2_{H^2(0,1)}.$$

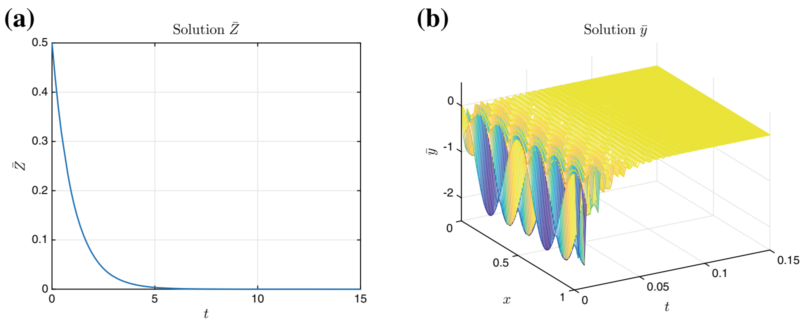
Using (35), we get

$$\|\tilde{\eta}\|^2_{L^2(0,1)} \leqslant C_1 e^{-\gamma t}\left(\|\tilde{\eta}_0\|^2_{L^2(0,1)} + \varepsilon|\tilde{\delta}_0|^2 + \varepsilon\|\bar{y}_0\|^2_{H^2(0,1)}\right),$$

where C_1 is positive constant. Since $\tilde{\eta}_0 = 0$, the inequality (33) holds for all $t \geqslant 0$. This concludes the proof of Theorem 2. ■

5 Numerical Simulations

Let us first show the numerical simulations on an ODE-fast PDE system (1), in which condition (10) is not satisfied. The full system is approximated by both subsystems under the stability conditions of the subsystems for ε sufficiently small. We next provide the numerical simulations on an academic example of PDE-fast ODE system (6). The Tikhonov approximation is achieved under the full system's stability condition (10).



Reduced subsystem Boundary-layer subsystem

Fig. 1 Time evolutions of subsystems (4) and (5)

Table 1 Estimates of the error for different values of ε with the initial condition $y_0 = K_r Z_0$

ε	0.03	0.02	0.01
$\lvert Z(t) - \bar{Z}(t)\rvert^2$	4.8×10^{-2}	2.1×10^{-2}	5.0×10^{-3}
$\lVert y(.,t) - K_r \bar{Z}(t)\rVert^2_{L^2(0,1)}$	2.2×10^{-4}	1.0×10^{-4}	2.6×10^{-5}

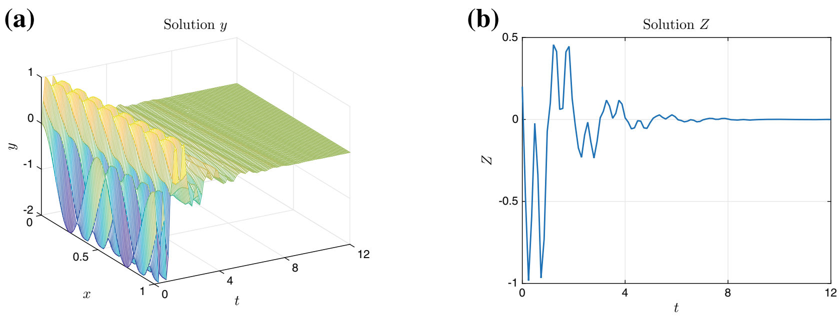
5.1 Numerical Simulations on an ODE-Fast PDE System

Let us consider system (1) with $A = 1$, $B = -1$, $\Lambda = 1$, $K_1 = \frac{1}{2}$, $K_2 = 1$. The initial conditions (1d)–(1e) are selected as $Z_0 = 0.5$ and $y_0(x) = \cos(4\pi x) - 1$. The perturbation parameter ε is chosen as $\varepsilon = 0.01$. It is computed $A_r = -1$ for the reduced subsystem (4). The initial condition (4b) is chosen as the same as for the full system $\bar{Z}_0 = Z_0 = 0.5$. The boundary condition for the boundary-layer subsystem is $K_1 = \frac{1}{2}$. The initial condition is chosen as $\bar{y}_0 = \cos(4\pi x) - \frac{3}{2}$. In view of A_r, condition (11) is satisfied for any $P > 0$. By choosing $Q = 1$, condition (12) holds. Figure 1 shows that the reduced and the boundary-layer subsystems converge to the origin as time increasing.

Let us choose $\varepsilon = \{0.03, 0.02, 0.01\}$, the initial condition y_0 is selected as the equilibrium point $y_0 = K_r Z_0$. Table 1 shows that the errors between the full system and the reduced subsystem decrease as ε decreasing, as expected from Theorem 1.

5.2 Numerical Simulations on a PDE-Fast ODE System

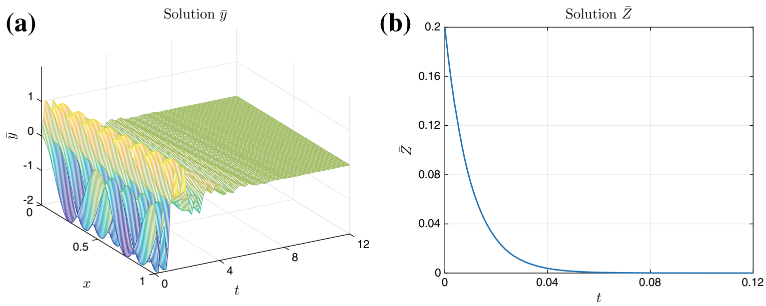
We consider system (6) with $A = -1$, $B = \frac{1}{2}$ and $\Lambda = 1$. The boundary condition (6c) is given by $K_1 = -\frac{1}{4}$, $K_2 = -\frac{1}{2}$. The initial conditions (6d)–(6e) are selected as $Z_0 = 0.2$ and $y_0(x) = \cos(4\pi x) - 1$. The perturbation parameter is $\varepsilon = 0.01$. By



Slow dynamics of the full system

Fast dynamics of the full system

Fig. 2 Time evolutions of the full system (6)



Reduced subsystem

Boundary-layer subsystem

Fig. 3 Time evolutions of subsystems (8) and (9)

Table 2 Estimates of the error for different values of ε

ε	0.03	0.02	0.01
$\|y(.,t)-\bar{y}(.,t)\|^2_{L^2}$	1.9×10^{-3}	5.7×10^{-4}	1.4×10^{-5}

choosing $P = Q = 1$, the stability condition (10) is satisfied. Therefore, Proposition 1 applies. In Fig. 2, the solutions of the slow and the fast dynamics of the full system tend to zero when time increases, as expected from Proposition 1.

Moreover, we compute the boundary condition matrix for the reduced subsystem as $K_r = -\frac{1}{2}$. The initial condition (8c) is chosen as $\bar{y}_0 = y_0(x) = \cos(4\pi x) - 1$. It is observed in Fig. 3 that the solutions of both subsystems converge to the origin as time increasing.

Let us choose $\varepsilon = \{0.03, 0.02, 0.01\}$. Table 2 shows that the errors between the slow dynamics of the full system and the reduced subsystem decrease as ε decreasing, as expected from Theorem 2.

6 Conclusions

A class of singularly perturbed linear ODE coupled with linear hyperbolic PDE systems has been considered in this work. The two subsystems have been formally computed based on the singular perturbation method. A general sufficient stability condition has been provided, which guarantees the stability of the full coupled ODE-PDE system and both subsystems. The Tikhonov approximation for full systems has been established by Lyapunov method. More precisely, based on the stability of both subsystems, the full ODE-fast PDE system is approximated by the subsystems. The estimate error is the order of the perturbation parameter ε. However, for PDE-fast ODE system, the approximation is valid if the general sufficient stability condition is satisfied.

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On Some Neutral Functional Differential Equations Occurring in Synchronization

Vladimir Răsvan, Daniela Danciu and Dan Popescu

1 Introduction. The Elementary (Single Oscillator) and Other Models

Synchronization is considered by now "a universal concept in nonlinear sciences" [12]. It is thus not surprising to find that there exist in fact "several synchronization types" i.e. several ways of conceiving (representing) synchronization. In this paper we shall leave aside the control engineering approach to synchronization e.g. [4] and focus on synchronization models as arising from physics. Here the basic model is the one described by Huygens, consisting of two pendula coupled by a distributed/lumped common support Fig. 1.

The straightforward generalization of this structure is represented by the *network of oscillators*. We refer to the paper due to Hale [8] where it is stated that "there have been many studies in recent years devoted to the dynamics induced from the ordinary differential equations (ODEs) obtained by coupling large number of oscillators on periodic lattices". Observe first that the term *oscillator* is used in the sense of Lagrange—a model of some physical device displaying "oscillations", in fact transients.

Here several synchronization types can be considered. For self excited (self sustained) periodic motions a common (unique) period is established and a phase shift is displayed (locking-in phenomenon). When the oscillations are subject of external

V. Răsvan (✉) · D. Danciu · D. Popescu
Department of Automatic Control and Electronics, University of Craiova,
A.I. Cuza str. no. 13, 200585 Craiova, Romania
e-mail: vrasvan@automation.ucv.ro; Vladimir.Rasvan@automation.ucv.ro

D. Danciu
e-mail: ddanciu@automation.ucv.ro

D. Popescu
e-mail: dpopescu@automation.ucv.ro

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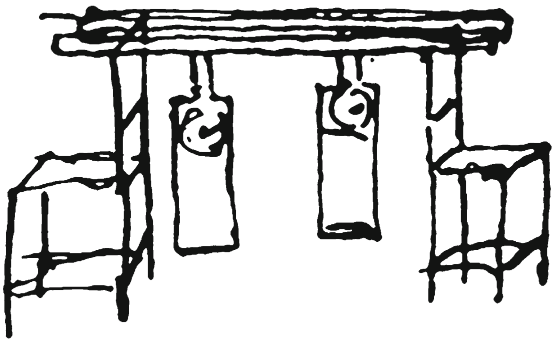


Fig. 1 The original setting of Huygens

excitation (which might be not only periodic but also almost periodic), there are suggested mainly patterns involving spatial and temporal chaos [8].

We shall consider throughout this paper some models of the aforementioned cases, starting from the basic case of Huygens. Our reference will be the papers [3, 10, 11]: one of them is concerned with self sustained oscillations, the other two with forced oscillations. Before writing the basic equations we shall complete the physical description: there are considered nonlinear mechanical oscillators (pendulum, van der Pol type) connected through distributed support (hanging on a rope/string or united by an elastic rod). In [10, 11] the support is an infinite length spring while in [3] the support is a finite length elastic rod replacing the spring that is usually connecting such oscillators.

The model of [11] might be viewed as an elementary ("toy") application: it deals (Fig. 2a) with a single oscillator—a nonlinear undamped pendulum—hanging on a infinite string. Its equations are as follows [11]

$$\begin{aligned} & y_{tt} - c^2 y_{xx} = 0\ ,\ c^2 = T/\rho,\quad -\infty < x < \infty \\ & m\ddot{\eta} + V(\eta) = T\left(y_x^r(0,t) - y_x^l(0,t)\right) \\ & \eta(t) = y_x^l(0,t) = y_x^r(0,t) \end{aligned} \tag{1}$$

with the standard notations: $y(x,t)$ is the local displacement of the string along its length at the position x and moment t while $y^l(x,t)$ and $y^r(x,t)$ are the displacements of the string for $x < 0$ and $x > 0$ respectively. The string parameters are the elastic tension T and the material density ρ; c is the propagation speed through the string. It was denoted by η the displacement of the local oscillator having the mass m; by $V(\eta)$ there was denoted the nonlinear characteristic of the spring. The second model, of [10] considers two oscillators of the aforementioned type, hanging on an infinite string (Fig. 2b). Its equations are as follows

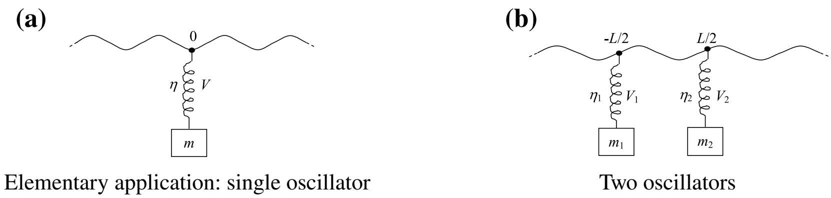
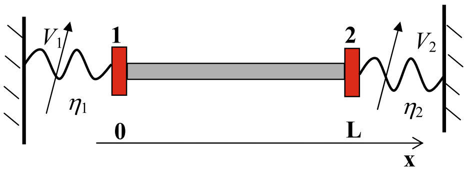


Fig. 2 Mechanical oscillators on a string

Fig. 3 Two Van der Pol oscillators on the elastic rod



$$
\begin{aligned}
& y_{tt} - c^2 y_{xx} = 0, \quad -\infty < x < \infty, \quad c^2 = T/\rho \\
& m_1 \ddot{\eta}_1 + V_1(\eta_1) = T\left(y_x^c\left(\tfrac{-L}{2}, t\right) - y_x^l\left(\tfrac{-L}{2}, t\right)\right) \\
& m_2 \ddot{\eta}_2 + V_2(\eta_2) = T\left(y_x^r\left(\tfrac{L}{2}, t\right) - y_x^c\left(\tfrac{L}{2}, t\right)\right) \\
& \eta_1(t) = y^l\left(\tfrac{-L}{2}, t\right) = y^c\left(\tfrac{-L}{2}, t\right) \\
& \eta_2(t) = y^c\left(\tfrac{L}{2}, t\right) = y^r\left(\tfrac{L}{2}, t\right)
\end{aligned}
\tag{2}
$$

The notations are as in the case of a single oscillator. One has to distinguish, however, among the string displacements at the left of the local oscillators, between them and at their right. Therefore $y^l(x,t)$, $y^c(x,t)$, $y^r(x,t)$ are the displacement of the string for $x < -L/2$, $-L/2 < x < L/2$ and $x > L/2$ respectively. About this model it has been mentioned that it led to functional differential equations. It is but only natural to obtain such equations when propagation is present.

The third model [3] is very much alike to (2) but the two van der Pol oscillators are coupled through a finite length elastic rod (Fig. 3).

Therefore the equations are as follows

$$
\begin{aligned}
& y_{tt} - c^2 y_{xx} = 0, \quad 0 < x < L, \quad c^2 = T/\rho \\
& m_1 \ddot{\eta}_1 + V_1(\eta_1, \dot{\eta}_1) = T y_x(0,t) \; ; \; m_2 \ddot{\eta}_2 + V_2(\eta_2, \dot{\eta}_2) = T y_x(L,t) \\
& y(0,t) = \eta_1(t), \quad y(L,t) = \eta_2(t)
\end{aligned}
\tag{3}
$$

By $V_i(\eta_i, \dot{\eta}_i)$, $i = 1, 2$, we denoted, in brief, the nonlinear dependencies that characterize a standard van der Pol oscillator. These models will be discussed from the point of view of the so-called augmented validation [13] which incorporates standard well posedness in the sense of Hadamard (existence, uniqueness, continuous data dependence) and stability analysis of the steady states (equilibria and oscillations) i.e. checking of the Stability Postulate of Četaev.

The approach will be the same as in many previous papers of us—association of some functional differential equations (in most of the cases—of neutral type) by integration along the characteristics. Based on the one-to-one correspondence between the solutions of the two mathematical objects, all results obtained for one object are projected back on the other.

2 The Analysis of the Single Oscillator Application

We shall consider here the system of Fig. 1 described by (1). The equations define in fact two boundary value problems, coupled at the common boundary $x = 0$. Without giving details, following [13, 16] the following representations are obtained. Denoting first

$$v(x, t) := y_t(x, t), \quad w(x, t) := y_x(x, t) \tag{4}$$

hence assuming enough smoothness to deal with classical solutions only, the following are true

$$v_r(x,t) = \begin{cases} \frac{1}{2}[y_1(x+ct) + cy_0'(x+ct) + y_1(x-ct) - cy_0'(x-ct)], & 0 < t < x/c \\ \frac{1}{2}[y_1(x+ct) + cy_0'(x+ct) + \dot{\eta}(t-x/c) - y_1(ct-x) \\ \qquad -cy_0'(ct-x)], & t > x/c \end{cases} \tag{5}$$

$$w_r(x,t) = \begin{cases} \frac{1}{2c}[y_1(x+ct) + cy_0'(x+ct) - y_1(x-ct) - cy_0'(x-ct)], & 0 < t < x/c \\ \frac{1}{2c}[y_1(x+ct) + cy_0'(x+ct) + \dot{\eta}(t-x/c) + y_1(ct-x) \\ \qquad +cy_0'(ct-x)], & t > x/c \end{cases} \tag{6}$$

and for $x > 0$ (the boundary value problem "at right"). Also, for $x < 0$ (the boundary value problem "at left") it follows that

$$v_l(x,t) = \begin{cases} \frac{1}{2}[y_1(x+ct) + cy_0'(x+ct) + y_1(x-ct) - cy_0'(x-ct)], & t < -x/c \\ \frac{1}{2}[\dot{\eta}(t+x/c) - y_1(ct-x) + cy_0'(-ct-x) + y_1(x-ct) \\ \qquad -cy_0'(x-ct)], & t > -x/c \end{cases} \tag{7}$$

$$w_l(x,t) = \begin{cases} \frac{1}{2c}[y_1(x+ct) + cy_0'(x+ct) - y_1(x-ct) + cy_0'(x-ct)], & t < -x/c \\ \frac{1}{2c}[\dot{\eta}(t+x/c) - y_1(-ct-x) + cy_0'(-ct-x) - y_1(x-ct) \\ \qquad +cy_0'(x-ct)], & t > -x/c \end{cases} \tag{8}$$

In (5)–(8) the function $\eta(t)$ is the solution of the equations of the local oscillator

$$m\ddot{\eta} + V(\eta) = T[w_r(0, t) - w_l(0, t)] \tag{9}$$

where $w_r(0, t)$ and $w_l(0, t)$ are taken from the aforementioned equations leading to

$$m\ddot{\eta} + 2(T/c)\dot{\eta} + V(\eta) = \frac{1}{c}[y_1(ct) + y_1(-ct) + y_0'(ct) - y_0'(-ct)] \quad (10)$$

In all the equations of this section we made use of the initial conditions

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad -\infty < x < \infty \quad (11)$$

with the functions $y_0(x)$ and $y_1(x)$ smooth enough.

We shall comment now on the representation formulae. The basic equation appears to be (10)—an ordinary nonlinear differential equation forced by the initial conditions (11) of the boundary problem (1). The initial conditions being defined on the entire real axis, a "steady state"—a "global (defined on $\mathbb{R}$) solution" of (10) might exist and, in order to observe it physically, be at least stable. Note also that, if $y_0(\cdot)$ and $y_1(\cdot)$ are periodic (with rationally dependent periods) then a global periodic solution might exist. Also, if $y_0(\cdot)$ and $y_1(\cdot)$ are almost periodic then a global almost periodic solution might exist.

The application defined by (1) is considered elementary (a "toy") because no propagation appears with respect to the "boundary"—the local oscillator which is synchronized with respect to the initial wave defined by the initial conditions of the string.

This application is however interesting because it displays which type of oscillator problems is involved. For instance, if $V(\cdot)$ is a sector restricted and globally Lipschitz nonlinearity, the theorem of Yakubovich [17] on the forced oscillations is valid.

Suppose $V(\eta)$ is subject to

$$0 < \delta_0 \leq \frac{V(\eta_1) - V(\eta_2)}{\eta_1 - \eta_2} \leq L. \quad (12)$$

We check the frequency domain inequality

$$\frac{1}{L} + \mathrm{Re}\frac{1}{-m\omega^2 + 2(T/c)j\omega + \delta_0} > 0, \quad 0 \leq \omega \leq +\infty. \quad (13)$$

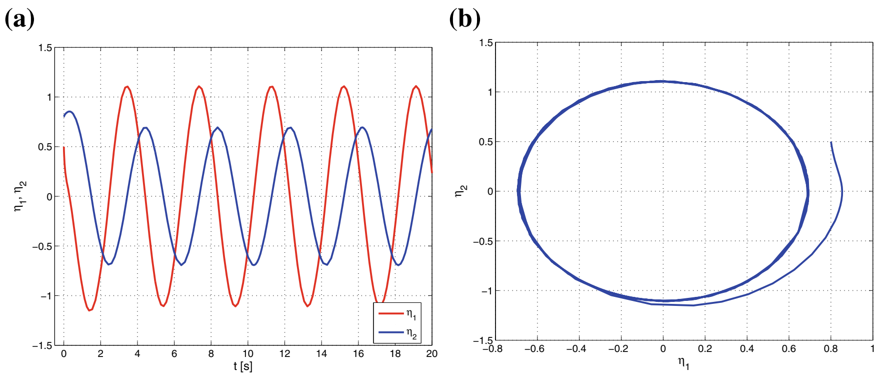
After some manipulation the following necessary and sufficient condition for (13) is found:

$$L < 4(T/c)(\sqrt{\delta_0/m} + T/(mc)) \quad (14)$$

If we take into account that for the second order system (10) whose linear part has the frequency domain characteristic of (13) we have $\omega_n^2 = \delta_0/m, \zeta\omega_n = T/(mc)$ condition (14) becomes

$$L/m < 4\zeta(1+\zeta)\omega_n^2 \quad (15)$$

According to the aforementioned theorem of Yakubovich, if (12) and (14) hold, then (10) has an unique solution defined on $\mathbb{R}$, exponentially stable, which is periodic or almost periodic provided $y_0(\cdot)$ and $y_1(\cdot)$ are periodic with rationally dependent



Time representation

Phase plane representation

Fig. 4 Simulation results for the single oscillator

periods or almost periodic (possibly periodic with rationally independent periods)—see [14]. We may say that the local oscillator is synchronized with the string.

In order to illustrate the aforementioned results, a simulation has been performed for the dynamics of system (10) with the following numerical data: $m = 0.1$, $T = 0.32$, $c = 3.2$, $y_0(x) = \sin\ 0.5x$, $y_1(x) = \sin\ 0.25x$. The nonlinear function $V(\cdot)$ is defined by $V(\eta) = \delta_0\eta + (1/2)\tanh(\eta/2)$ with $\delta_0 = 0.1$. The aforementioned data will give $\omega_n = 1$, $\zeta = 1$, $L \leq 1/2$. One can check that $L/m = 2 < 8$ hence (15) or, equivalently, (14) is fulfilled. The results of the simulations are given in Fig. 4.

We can see the standard stable limit cycle corresponding to the stable forced oscillation and the time domain representations of the oscillations of the two phase variables. Even the filtering properties are to be distinguished for the linear part whose bandwidth is $\sqrt{\sqrt{2} - 1}$. Since the nonlinearity is odd, only odd higher harmonics are present; since the frequency of the 3d harmonic is 0.75, clearly only the fundamental harmonic 0.25 "passes" as it can be seen from Fig. 4a.

3 The Case of Two Electrical Oscillators Coupled to a Lossless Transmission Line

We shall consider here the structure of Fig. 5.

As it can be seen, the local oscillators are composed of a RLC oscillating circuits and a tunnel diode displaying a S outer characteristic. The infinite transmission line is lossless—a LC transmission line described by the telegraph equations which in this case reduce to the wave equation

$$Li_t + v_x = 0, \quad Cv_t + i_x = 0 \tag{16}$$

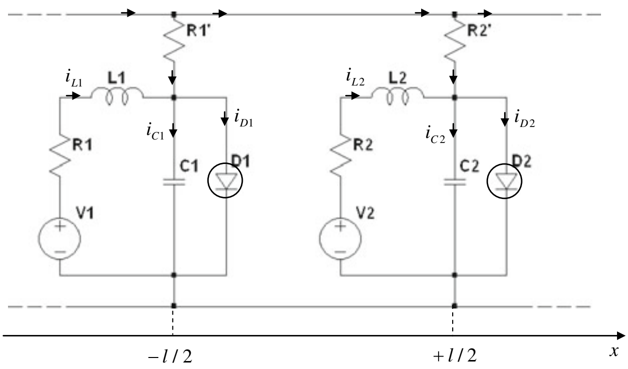


Fig. 5 Oscillators on the LC line

with the boundary conditions given at the coupling points $x = \pm\ell/2$ of the two local oscillators

$$
\begin{aligned}
& v_l(-\ell/2,t) - v_{D1} = v_c(-L/2,t) - v_{D1} = R_1'[i_l(-\ell/2,t) - i_c(-\ell/2,t)] \\
& i_{L1} + i_l(-\ell/2,t) - i_c(-\ell/2,t) = i_{c1} + i_{D1} \\
& L_1\frac{\mathrm{d}i_{L1}}{\mathrm{d}t} = -v_{D1} + E_1 - R_1 i_{L1} \\
& C_1\frac{\mathrm{d}v_{D1}}{\mathrm{d}t} = i_{C1}\ ,\ i_{D1} = f_1(v_{D1})
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
& v_c(\ell/2,t) - v_{D2} = v_r(\ell/2,t) - v_{D2} = R_2'[i_c(\ell/2,t) - i_r(\ell/2,t)] \\
& i_{L2} + i_c(\ell/2,t) - i_r(\ell/2,t) = i_{c2} + i_{D2} \\
& L_2\frac{\mathrm{d}i_{L2}}{\mathrm{d}t} = -v_{D2} + E_2 - R_2 i_{L2} \\
& C_2\frac{\mathrm{d}v_{D2}}{\mathrm{d}t} = i_{C2}\ ,\ i_{D2} = f_2(v_{D2})
\end{aligned}
\tag{18}
$$

We have to observe, in the spirit of [16], that (16)–(18) define two single point boundary value problems on semi-infinite intervals $(-\infty, -\ell/2)$ and $(\ell/2, \infty)$ and two point boundary value problem on the finite interval $(-\ell/2, \ell/2)$. As previously, the subscript l accounts for the "left hand side problem" on $(-\infty, -\ell/2)$, the subscript c—for the central problem on $(-\ell/2, \ell/2)$ etc.

We follow the same approach as in the single oscillator application suggested by [16], integrating along the characteristics—as in [13] and using the boundary conditions to couple the equations arising from the problems. We reproduce the differential equations after [2]. We send the reader to this paper for the rather long and tedious formulae describing the distributed voltages and currents along the transmission lines, name $(v_l(x,t), i_l(x,t))$, $(v_r(x,t), i_r(x,t))$ and focus on the two point boundary value problem on $(-\ell/2, \ell/2)$. After the elimination of the superfluous terms in (16)–(18) we obtain

$$L\frac{\partial i_c}{\partial t} + \frac{\partial v_c}{\partial x} = 0, \quad C\frac{\partial u_c}{\partial t} + \frac{\partial i_c}{\partial x} = 0 \tag{19}$$

with the non-standard boundary conditions at $x = -\ell/2$

$$\begin{aligned} &v_c(-\ell/2, t) + R_1' i_c(-\ell/2, t) = v_{D1}(t) \\ &L_1\frac{\mathrm{d}i_{L1}}{\mathrm{d}t} = -R_1 i_{L1} - v_{D1} + E_1 \\ &C_1\frac{\mathrm{d}v_{D1}}{\mathrm{d}t} = i_{L1} - f_1(v_{D1}) - i_c(-\ell/2, t) + i_l(-\ell/2, t) \end{aligned} \tag{20}$$

and at $x = \ell/2$

$$\begin{aligned} &v_c(\ell/2, t) + R_2' i_c(\ell/2, t) = v_{D2}(t) \\ &L_2\frac{\mathrm{d}i_{L2}}{\mathrm{d}t} = -R_2 i_{L2} - v_{D2} + E_2 \\ &C_2\frac{\mathrm{d}v_{D2}}{\mathrm{d}t} = i_{L2} - f_2(v_{D2}) + i_c(\ell/2, t) - i_r(\ell/2, t) \end{aligned} \tag{21}$$

We call the aforementioned boundary conditions non-standard since they are in fact a feedback connection between the standard ones and a system of ordinary differential equations. Observe also that the boundary conditions (20)–(21) are forced by the solution of the side boundary value problems $i_\ell(x, t)$, $x < -\ell/2$ and $i_r(x, t)$, $x > \ell/2$, computed at $x = \pm\ell/2$ (respectively).

Making use of the expressions of $i_\ell(-\ell/2, t)$ and $i_r(\ell/2, t)$ [2], the corresponding equations of the boundary conditions become

$$\begin{aligned} C_1\frac{\mathrm{d}v_{D1}}{\mathrm{d}t} &= i_{L1} - \left[\frac{1+\rho_1}{2}\sqrt{C/L}\, v_{D1} + f_1(v_{D1})\right] - i_c(-\ell/2, t) \\ &+ \frac{1+\rho_1}{2}\sqrt{C/L}\left[v_0(-\ell/2 - t/\sqrt{LC}) + \sqrt{L/C}\, i_0(-\ell/2 - t/\sqrt{LC})\right] \end{aligned} \tag{22}$$

$$\begin{aligned} C_2\frac{\mathrm{d}v_{D2}}{\mathrm{d}t} &= i_{L2} - \left[\frac{1+\rho_2}{2}\sqrt{C/L}\, v_{D2} + f_2(v_{D2})\right] + i_c(\ell/2, t) \\ &+ \frac{1+\rho_2}{2}\sqrt{C/L}\left[-v_0(\ell/2 + t/\sqrt{LC}) + \sqrt{L/C}\, i_0(\ell/2 + t/\sqrt{LC})\right] \end{aligned} \tag{23}$$

where we denoted

$$\rho_i = \frac{1 - R_i'\sqrt{C/L}}{1 + R_i'\sqrt{C/L}}, \quad i = \overline{1, 2} \tag{24}$$

while $i_0(x)$ and $v_0(x)$ are the initial current and voltage on the line; as in the previous section, these functions can be periodic or almost periodic.

The forcing terms E_1 and E_2 are constant: their role was to ensure a bias for the equilibria of the local oscillators. Here they can be integrated in the forcing terms since a constant is periodic of any period hence also almost periodic.

We associate now to (19)–(21)—with the modifications contained in (22)–(23)—a system of functional differential equations by integrating the Riemann invariants along the characteristics. We skip the rather standard (by now) computations [2, 13] and write down the system of functional differential equations. We have two (apparently) independent systems of ordinary differential equations

$$\begin{aligned} L_1\frac{\mathrm{d}i_{L1}}{\mathrm{d}t} &= -R_1 i_{L1} - v_{D1} + E_1 \\ C_1\frac{\mathrm{d}v_{D1}}{\mathrm{d}t} &= i_{L1} - \left[(1+\rho_1)\sqrt{C/L}\,v_{D1} + f_1(v_{D1})\right] \\ &\quad + \frac{1+\rho_1}{2}\sqrt{C/L}\,\eta_c^+(t-\ell\sqrt{LC}) + \phi_1(t) \end{aligned} \tag{25}$$

and

$$\begin{aligned} L_2\frac{\mathrm{d}i_{L2}}{\mathrm{d}t} &= -R_2 i_{L2} - v_{D2} + E_2 \\ C_2\frac{\mathrm{d}v_{D2}}{\mathrm{d}t} &= i_{L2} - \left[(1+\rho_2)\sqrt{C/L}\,v_{D2} + f_2(v_{D2})\right] \\ &\quad + \frac{1+\rho_2}{2}\sqrt{C/L}\,\eta_c^-(t-\ell\sqrt{LC}) + \phi_2(t) \end{aligned} \tag{26}$$

where we denoted

$$\begin{aligned} \phi_1(t) &= \frac{1+\rho_1}{2}\sqrt{C/L}\left[v_0(-\ell/2 - t/\sqrt{LC}) + \sqrt{L/C}\,i_0(-\ell/2 - t/\sqrt{LC})\right] \\ \phi_2(t) &= \frac{1+\rho_2}{2}\sqrt{C/L}\left[-v_0(\ell/2 + t/\sqrt{LC}) + \sqrt{L/C}\,i_0(\ell/2 + t/\sqrt{LC})\right] \end{aligned} \tag{27}$$

The functions $\eta_c^{\pm}(t)$ occur from the integration of the Riemann invariants along the characteristics.

$$\begin{aligned} \eta_c^+(t) &= v_c(-\ell/2, t) + \sqrt{L/C}\,i_c(-\ell/2, t) \\ \eta_c^-(t) &= v_c(\ell/2, t) - \sqrt{L/C}\,i_c(\ell/2, t) \end{aligned} \tag{28}$$

and

$$\begin{aligned} v_c(\ell/2, t) + \sqrt{L/C}\,i_c(\ell/2, t) &= \eta_c^+(t-\ell\sqrt{LC}) \\ v_c(-\ell/2, t) - \sqrt{L/C}\,i_c(-\ell/2, t) &= \eta_c^-(t-\ell\sqrt{LC}) \end{aligned} \tag{29}$$

The following representation formulae are also useful

$$\begin{aligned} v_c(x,t) &= \frac{1}{2}\left[\eta_c^+(t-\sqrt{LC}(\ell/2+x)) + \eta_c^-(t-\sqrt{LC}(\ell/2-x))\right] \\ i_c(x,t) &= \frac{1}{2}\sqrt{C/L}\left[\eta_c^+(t-\sqrt{LC}(\ell/2+x)) - \eta_c^-(t-\sqrt{LC}(\ell/2-x))\right] \end{aligned} \tag{30}$$

Systems (25)–(26) are coupled throughout the boundary conditions which now are expressed as difference equations

$$\begin{aligned} \eta_c^+(t) &= -\rho_1\eta_c^-(t-\ell\sqrt{LC}) + (1+\rho_1)v_{D1} \\ \eta_c^-(t) &= -\rho_2\eta_c^+(t-\ell\sqrt{LC}) + (1+\rho_2)v_{D2} \end{aligned} \tag{31}$$

Consider now the Eqs. (25), (26), (31): together they define a system of ordinary differential equations coupled to a system of difference equations of the form

$$\begin{aligned} \dot{x} &= A_0x(t) + A_1y(t-\tau) - \sum_{k=1}^{m} b_{1k}\phi_k(c_k^*x(t)) + f(t) \\ y(t) &= A_2x(t) + A_3y(t-\tau) - \sum_{k=1}^{m} b_{2k}\phi_k(c_k^*x(t)) + g(t) \end{aligned} \tag{32}$$

where $\phi_k(\sigma)$ are sector restricted nonlinear functions, also globally Lipschitz, subject to

$$0 \le \frac{\phi_k(\sigma_1) - \phi_k(\sigma_2)}{\sigma_1 - \sigma_2} \le L_k, \quad k = \overline{1,m} \tag{33}$$

and $f(t)$, $g(t)$ are bounded on $\mathbb{R}$, possibly periodic almost periodic sector functions. For such systems existence and exponential stability of global (periodic/almost periodic solutions have been analyzed either by frequency domain methods [7] or by using a quadratic Lyapunov functional [15]. Worth mentioning that application of either of the aforementioned methods can give only some estimates of the solution. The required properties of the oscillating behavior follow by applying a theorem of Halanay [5] on invariant manifolds for systems with time lag.

We have to recall here other two assumptions on the general system (32) in order to ensure existence and stability of the forced oscillations. First, the difference operator of the second equation of (32) namely

$$\mathcal{D}\phi(\cdot) := \phi(0) - A_3\phi(-\tau) \tag{34}$$

must be strongly stable: in this case this means that the eigenvalues of A_3 must lie inside the unit disk. Second, the linear part of (32) i.e. the linear system of ordinary differential and difference equations

$$\begin{aligned} \dot{x} &= A_0x(t) + A_1y(t-\tau) \\ y(t) &= A_2x(t) + A_3y(t-\tau) \end{aligned} \tag{35}$$

must be exponentially stable. For our application, defined by (25), (26) and (31) the aforementioned assumptions are valid. Indeed, it follows from (31) that the eigenvalues of the corresponding A_3 are $\pm\sqrt{\rho_1\rho_2}$ with $\rho_i < 1$, $i = \overline{1,2}$. Next, we have to prove that the linear part of the system of our application is exponentially stable. The characteristic equation is quite complicated but we can adopt here the energy function as a quadratic Lyapunov functional. Worth mentioning that if we consider as nonlinear functions the new ones of (25)–(26) i.e. those obtained by rotation, there exists some risk of getting an unstable linear part. Therefore it is useful to separate a subsector in order to ensure a possibly stable linear system. This will send to the following equations of the linear part

$$
\begin{aligned}
&L_1\frac{\mathrm{d}i_{L1}}{\mathrm{d}t} = -R_1 i_{L1} - v_{D1}\\
&C_1\frac{\mathrm{d}v_{D1}}{\mathrm{d}t} = i_{L1} - a_1(1+\rho_1)\sqrt{C/L}v_{D1} + \frac{1+\rho_1}{2}\sqrt{C/L}\,\eta_c^+(t-\ell\sqrt{LC})\\
&L_2\frac{\mathrm{d}i_{L2}}{\mathrm{d}t} = -R_2 i_{L2} - v_{D2}\\
&C_2\frac{\mathrm{d}v_{D2}}{\mathrm{d}t} = i_{L2} - a_2(1+\rho_2)\sqrt{C/L}v_{D2} + \frac{1+\rho_2}{2}\sqrt{C/L}\,\eta_c^-(t-\ell\sqrt{LC})\\
&\eta_c^+(t) = -\rho_1\eta_c^-(t-\ell\sqrt{LC}) + (1+\rho_1)v_{D1}\\
&\eta_c^-(t) = -\rho_2\eta_c^+(t-\ell\sqrt{LC}) + (1+\rho_2)v_{D2}
\end{aligned}
\tag{36}
$$

with $0 < a_i < 1$, $i = 1, 2$. It will appear from the following development how the Lyapunov functional is structured in this case. We shall apply the method of [15] by associating a Lyapunov functional suggested by the energy identity for (19)–(21)

$$
\begin{aligned}
&\mathcal{E}(v_{D1}, v_{D2}, i_{L1}, i_{L2}, i_c(\cdot,t), v_c(\cdot,t))\\
&\quad = \frac{1}{2}\left[C_1v_{D1}^2 + C_2v_{D2}^2 + L_1i_{L1}^2 + L_2i_{L2}^2 + \int_{-\ell/2}^{\ell/2}(Li_c^2(x,t) + Cv_c^2(x,t))\,\mathrm{d}x\right]
\end{aligned}
\tag{37}
$$

If we make use of the representation formulae (30) and introduce some free parameters to be chosen afterwards, the following Lyapunov functional associated to (25), (26), (31) is obtained

$$
\begin{aligned}
&\mathcal{V}(v_{D1}, v_{D2}, i_{L1}, i_{L2}, \phi_c^+(\cdot), \phi_c^-(\cdot)) = \tfrac{1}{2}\left[C_1v_{D1}^2 + \gamma_1C_2v_{D2}^2 + \gamma_2L_1i_{L1}^2 + \gamma_3L_2i_{L2}^2\right.\\
&\quad\left. +\gamma_4\int_{-\ell\sqrt{LC}}^{0}\phi_c^+(\theta)^2\,\mathrm{d}\theta + \gamma_5\int_{-\ell\sqrt{LC}}^{0}\phi_c^-(\theta)^2\,\mathrm{d}\theta\right]
\end{aligned}
\tag{38}
$$

From now on one can follow the approach displayed in [15] to obtain existence, uniqueness and exponential stability of the global solution (possibly periodic or almost periodic) for (25), (26), (31). If the one-to-one correspondence [13] is used, this solution exists for (19)–(21).

Some properties of this solution are interesting to be pointed out:

(i) The solution is imposed by the forcing term resulting from the solutions of the side boundary value problems (on $x < -\ell/2$ and $x > \ell/2$: if the initial conditions on the line are periodic/almost periodic, the steady states of the local oscillators are periodic/almost periodic regardless their steady state when isolated.

(ii) While the steady states of the oscillators are periodic/almost periodic in the classical sense, the steady state of the line on $(-\ell/2, \ell/2)$ are such only "in the average"—in the metrics of (28)—being thus Stepanov functions [1].
(iii) The coupling of the oscillators to the line introduces an additional damping which rotates the nonlinear characteristics of the tunnel diodes which become sector restricted—see (33)—instead of S-functions. Consequently the possible local limit cycles are "quenched" and synchronization with the external forcing term occurs.

4 The Case of the Two Mechanical Oscillators

This case (Fig. 2b), described by (2) has been briefly discussed in [14]. Here also, due to the infinite string, the analysis concerns three boundary value problems—two side, single boundary value problems and a central two point boundary value problem. When integrating along the characteristics, the solutions of the side boundary value problems become, as in the previous section, forcing terms for the system of functional differential equations associated to the central boundary value problem. This system is as follows

$$\begin{aligned}
&m_1\ddot{z}_1 + 2(T/c)\dot{z}_1 + V_1(z_1) - (T/c)\eta_c^-(t - L/c) = (T/c)f^-(t) \\
&m_2\ddot{z}_2 + 2(T/c)\dot{z}_1 + V_2(z_2) - (T/c)\eta_c^+(t - L/c) = (T/c)f^+(t) \\
&\eta_c^+(t) = -\eta_c^-(t - L/c) + 2\dot{z}_1 \\
&\eta_c^-(t) = -\eta_c^+(t - L/c) + 2\dot{z}_2
\end{aligned} \quad (39)$$

where $f^{\pm}(t)$ are the forcing terms induced by the side boundary value problems. If (39) are examined in comparison to (2), one can see that in (39) the local oscillators have an additional damping "induced" by the coupling to the string. The phenomenon is called *radiation dissipation* [11].

System (39) is also of the type (32) but here we do not have nonlinearity rotation—$V_i(\cdot)$ remain as they were. The main difficulty here is that A_3 has now the eigenvalues ± 1 i.e. *on* the unit disk. From the experience of the authors of this paper it follows that all applications arising from mechanical engineering have exactly this property—the critical case of the matrix A_3 i.e. of the difference operator. The stable case may be obtained in those applications where the "through" variables (electric currents, fluid or thermal flows) mix with the "across" variables (electric voltages, pressures) within at least one boundary condition. Unfortunately this does not happen, generally speaking, in the case of mechanical systems where one deals with forces and displacements. Even in the application of Sect. 3 where the eigenvalues of A_3 are $\pm\sqrt{\rho_1\rho_2}$, formulae (24) show that $0 < \rho_i < 1$ provided the boundary dissipation resistors $R_i' \neq 0$ otherwise the critical case will occur here also.

In the critical case the theorem on forced oscillations from [7, 15] cannot be applied. To obtain a more refined form of it would require not only more refined estimates of the solutions in the new case, but also relaxation of the 40 years old basic results of [5, 9].

For these reasons and other, the case of Fig. 3 with two oscillators coupled by an elastic rod—Eq. (3)—will be discussed elsewhere.

5 Some Conclusions

We have discussed throughout the paper several aspects concerning synchronization viewed as an existence and stability problem for forced oscillations. The oscillating structures at their turn were inspired by the old and classical problem of Huygens about synchronization throughout the coupling media e.g. a wall, a rope, a string, a rod. We constructed also an electric analogue of the aforementioned mechanical oscillating systems—two electronic oscillators coupled to a lossless (LC) transmission line. The paper focused on those systems coupled to an infinite string or electrical line. This concept—arising from Physics—allowed introduction of the forcing, synchronizing oscillatory signal from the initial conditions of the string (line).

The mathematical approach has been based again on the association of some functional equations (almost always of neutral type) by integrating the Riemann invariants along the characteristics. Due to the one-to-one correspondence between the two mathematical objects—the starting system of hyperbolic partial differential equations with non-standard boundary conditions and the associated system of functional equations—all results obtained for one object are projected back on the other. If basic theory (well posedness in the sense of Hadamard) is concerned, then mainly classical solutions (at most discontinuous) are "discovered". On the other hand, when discussing global (on $\mathbb{R}$) solutions, such as equilibria or periodic/almost periodic solutions, existence should be accompanied by their stability analysis: we view the problem in the perspective of the Četaev Stability Postulate—only those solutions are observable (in the sense of noticeable, measurable, computable) which are stable.

Referring to existence and stability of forced oscillations, this analysis strongly relies on some mathematical results concerning invariant manifolds for flows in Banach spaces [5, 9] and for specific estimates along the solutions of the dynamical systems. Here also one has to consider two competing approaches: the method of the Lyapunov function(al) and the method of Popov type frequency domain inequalities [6]. It is felt that for the cases considered in this paper—mechanical or electrical systems for which natural energy-based Lyapunov function(al)s can be easily associated, the Lyapunov approach is more feasible. On the other hand the energy functions are "weak" Lyapunov functions, only non-increasing, what requires additional mathematical instruments. In the case of neutral functional differential equations, an almost "compulsory" assumption appears to be the strong stability of the difference operator. But none of the mechanical systems discussed here has this property—their difference operator being just stable (marginally)—unlike the electrical system which has this property.

Analyzing systems with marginally stable difference operator appears to be a challenge which nevertheless might answer to the question concerning the “complex” a.k.a. “chaotic” behavior of such systems [3, 10]. But this would lead to relaxation and refinement of some old date instruments of the oscillation theory.

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Dynamic Dissipativity Theory for Stability of Time-Delay Systems

Vijaysekhar Chellaboina and Wassim M. Haddad

1 Introduction

Time delays can severely degrade system performance and in many cases drive the system to instability and hence stability analysis of time delay dynamical systems remains an important area of research (see [7, 10, 12, 14, 15] and numerous references therein). A key method for analyzing stability of linear time delay dynamical systems is Lyapunov's second method [8] as applied to functional differential equations. Specifically, stability analysis of a given linear time delay dynamical system is typically shown using a Lyapunov-Krasovskii functional [9, 11]. Standard Lyapunov-Krasovskii functionals involve a fixed quadratic function and an integral functional explicitly dependent on the system time delay. As in classical absolute stability theory [13], the fixed quadratic part of the Lyapunov-Krasovskii functional is associated with the stability of the forward delay-independent part of the retarded dynamical system. However, the system-theoretic foundation of the integral part of the Lyapunov-Krasovskii functional is less understood. See [3, 17] for a dissipativity based justification for the integral part. An alternative method involves frequency domain approaches and have been very successful in arriving at multiple necessary and sufficient conditions (typically in terms of linear matrix inequalities) for checking stability of linear time-delay systems (see [5, 6, 17] and references within).

As is evident, much of the literature on stability of time-delay systems is restricted to linear time-delay systems. In this paper, by representing a time delay dynamical system as a negative feedback interconnection of a (linear or nonlinear) finite-dimensional dynamical system and an infinite-dimensional time delay operator, we

V. Chellaboina (✉)
Mahindra École Centrale, Hyderabad, India
e-mail: vijay@mechyd.ac.in

W. M. Haddad
Georgia Institute of Technology, Atlanta, GA, USA
e-mail: wassim.haddad@aerospace.gatech.edu

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derive new sufficient conditions for stability analysis of asymptotically stable *linear and nonlinear* time delay dynamical systems by showing that the corresponding time-delay operator is dissipative [2, 3]. Specifically, based on [4], we present extensions to the notions of dissipativity [16] and exponential dissipativity [8], namely *dynamic dissipativity*, that is, $(\Sigma, \hat{Q})$-dissipativity, where Σ is a dynamical system and $\hat{Q}$ is a symmetric matrix. By choosing a certain dynamical system Σ and a symmetric matrix $\hat{Q}$ it can be shown that a system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if $\mathcal{G}$ is dissipative with respect to a quadratic supply rate. Thus, $(\Sigma, \hat{Q})$-dissipativity provides a nontrivial extension of dissipativity theory with respect to a quadratic supply rate. Based on $(\Sigma, \hat{Q})$-dissipativity theory, we also provide a result on stability of negative feedback interconnection of $(\Sigma, \hat{Q})$-dissipative systems which can then be used to establish sufficient conditions for stability of time-delay systems. Thus the overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable linear and nonlinear time delay dynamical systems based on the dissipativity properties of the time delay operator.

2 Mathematical Preliminaries

2.1 Notation and Definitions

In this section we introduce notation, several definitions, and some key results concerning dynamical systems that are necessary for developing the main results of this paper. Specifically, $\mathbb{R}$ denotes the reals and $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with Euclidean norm $\|\cdot\|$. Let $C([a,b],\mathbb{R}^n)$ denote a Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For a given real number $\tau \geq 0$ if $[a,b] = [-\tau, 0]$ we let $C = C([-\tau,0],\mathbb{R}^n)$ and designate the norm of an element ϕ in C by $|||\phi||| = \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\|$. If $\alpha, \beta \in \mathbb{R}$ and $x \in C([\alpha-\tau, \alpha+\beta],\mathbb{R}^n)$, then for every $t \in [\alpha, \alpha+\beta]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. Furthermore, for $M \in \mathbb{R}^{m\times n}$, we write M^{T} to denote the transpose of M and $M \geq 0$ (resp., $M > 0$) to denote the fact that the symmetric matrix M is nonnegative (resp., positive) definite. Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denote a state space realization of a transfer function $G(s)$; that is, $G(s) = C(sI-A)^{-1}B + D$. The notation "$\overset{\min}{\sim}$" is used to denote a minimal realization. Finally, we write I_n to denote the $n \times n$ identity matrix and C^0 to denote continuous functions.

In this paper we represent dynamical systems $\mathcal{G}$ defined on the semi-infinite interval $[0,\infty)$ as a mapping between function spaces satisfying an appropriate set of axioms. For the following definition $\mathcal{U}$ is an input space and consists of bounded continuous U-valued functions on $[0,\infty)$. The set $U \subseteq \mathbb{R}^m$ contains the set of input values; that is, at any time t, $u(t) \in U$. The space $\mathcal{U}$ is assumed to be closed under the shift operator; that is, if $u \in \mathcal{U}$, then the function u_T defined by $u_T(t) = u(t+T)$ is contained in $\mathcal{U}$ for all $T \geq 0$. Furthermore, $\mathcal{Y}$ is an output space and consists of continuous Y-valued functions on $[0, \infty)$. The set $Y \subseteq \mathbb{R}^l$ contains the set of output values; that is, each value of $y(t) \in Y$, $t \geq 0$. The space $\mathcal{Y}$ is assumed to be closed under the shift operator; that is, if $y \in \mathcal{Y}$, then the function y_T defined by $y_T(t) = y(t+T)$ is contained in $\mathcal{Y}$ for all $T \geq 0$. Finally, $\mathcal{D}$ is a metric space with topology of uniform convergence and metric $\rho : \mathcal{D} \times \mathcal{D} \to [0, \infty)$. Hence, the notions of openness, convergence, continuity, and compactness that we use in the paper refer to the topology generated on $\mathcal{D}$ by the metric $\rho(\cdot, \cdot)$.

Definition 1 ([16]) A *stationary dynamical system* on $\mathcal{D}$ is the octuple $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, [0, \infty), s, q)$, where $s : [0, \infty) \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$ and $q : \mathcal{D} \times U \to Y$ are such that the following axioms hold:

(i) (Continuity): $s(\cdot, \cdot, u)$ is jointly continuous for all $u \in \mathcal{U}$.
(ii) (Consistency): $s(0, x_0, u) = x_0$ for all $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$.
(iii) (Determinism): $s(t, x_0, u_1) = s(t, x_0, u_2)$ for all $t \in [0, \infty)$, $x_0 \in \mathcal{D}$, and $u_1, u_2 \in \mathcal{U}$ satisfying $u_1(\tau) = u_2(\tau)$, $\tau \leq t$.
(iv) (Semi-group property): $s(\tau, s(t, x_0, u), u) = s(t + \tau, x_0, u)$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $\tau, t \in [0, \infty)$.
(v) (Read-out map): There exists $y \in \mathcal{Y}$ such that $y(t) = q(s(t, x_0, u), u(t))$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $t \geq 0$.

Henceforth, we denote the dynamical system $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, [0, \infty), s, q)$ by $\mathcal{G}$. Furthermore, we refer to $s(t, x_0, u)$, $t \geq 0$, as the *trajectory* or *state transition operator* of $\mathcal{G}$ corresponding to $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$. For a given trajectory $s(t, x_0, u)$, $t \geq 0$, we refer to $x_0 \in \mathcal{D}$ as the *initial condition* of $\mathcal{G}$. For the dynamical system $\mathcal{G}$ given by Definition 1, a function $r : U \times Y \to \mathbb{R}$ is called a *supply rate* [16] if it is locally integrable; that is, for all input-output pairs $u \in U$ and $y \in Y$ satisfying the dynamical system $\mathcal{G}$, $r(\cdot, \cdot)$ satisfies $\int_{t_1}^{t_2} |r(u(s), y(s))| \mathrm{d}s < \infty$, $t_1, t_2 \geq 0$.

Definition 2 ([8, 16]) A dynamical system $\mathcal{G}$ is *exponentially dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$, called a *storage function*, and a scalar $\varepsilon > 0$ such that the *dissipation inequality*

$$e^{\varepsilon t} V_\mathrm{s}(x(t)) \leq e^{\varepsilon t_1} V_\mathrm{s}(x(t_1)) + \int_{t_1}^{t} e^{\varepsilon s} r(u(s), y(s)) \mathrm{d}s \tag{1}$$

is satisfied for all $t_1, t \geq 0$ and where $x(t) = s(t, x_0, u(t))$, $t \geq t_1$, with $x_0 \in \mathcal{D}$ and $u(t) \in U$. A dynamical system $\mathcal{G}$ is *dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$ such that (1) is satisfied with $\varepsilon = 0$.

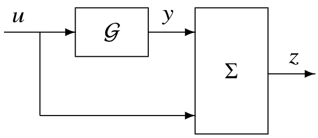

Fig. 1 Interconnection of $\mathcal{G}$ and Σ

2.2 Dynamic Dissipative Systems

Consider a dynamical system Σ given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}, U \times Y, \mathcal{Z}, Z, [0, \infty), \hat{s}, \hat{q})$, where $Z \subseteq \mathbb{R}^p$, $\mathcal{Z}$ is an output space which consists of continuous Z-valued functions on $[0, \infty)$, and consider the cascade interconnection of $\mathcal{G}$ and Σ as shown in Fig. 1. We denote the interconnected dynamical system $(\mathcal{D} \times \hat{\mathcal{D}}, \mathcal{U}, U, \mathcal{Z}, Z, [0, \infty), [s^{\mathrm{T}}, \hat{s}^{\mathrm{T}}]^{\mathrm{T}}, \hat{q})$ by $\tilde{\mathcal{G}}$. For the following definition, let $\hat{Q} \in \mathbb{R}^{p \times p}$ and $\hat{Q} = \hat{Q}^{\mathrm{T}}$.

Definition 3 ([2, 4]) A dynamical system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*exponentially dissipative* if there exists a C^0 nonnegative-definite function $\hat{V}_{\mathrm{s}} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$, called a $(\Sigma, \hat{Q})$-*storage function* and a scalar $\varepsilon > 0$, such that the $(\Sigma, \hat{Q})$-*dissipation inequality*

$$e^{\varepsilon t} \hat{V}_{\mathrm{s}}(x(t), \hat{x}(t)) \leq e^{\varepsilon t} \hat{V}_{\mathrm{s}}(x(t_1), \hat{x}(t_1)) + \int_{t_1}^{t} e^{\varepsilon s} z^{\mathrm{T}}(s) \hat{Q} z(s) \mathrm{d}s \tag{2}$$

is satisfied for all $t, t_1 \geq 0$ and where $x(t) = s(t, x_0, u(t)), \hat{x}(t) = \hat{s}(t, \hat{x}_0, u(t), y(t))$, $t \geq t_1$, with $x_0 \in \mathcal{D}$, $\hat{x}_0 \in \hat{\mathcal{D}}$, $\hat{x}_0 = 0$, $u(t) \in U$, and $y(t) = q(x(t), u(t))$. A dynamical system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*dissipative* if there exists a C^0 nonnegative-definite function $\hat{V}_{\mathrm{s}} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$ such that (2) is satisfied with $\varepsilon = 0$.

The following result provides a sufficient condition for $(\Sigma, \hat{Q})$-dissipativity of $\mathcal{G}$ in the case where $\mathcal{G}$ and Σ are linear dynamical systems. Specifically, let $\mathcal{G}$ and Σ be given by transfer functions $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$, respectively, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, $D \in \mathbb{R}^{l \times m}$, $\hat{A} \in \mathbb{R}^{\hat{n} \times \hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n} \times (l+m)}$, $\hat{C} \in \mathbb{R}^{p \times \hat{n}}$ and $\hat{D} \in \mathbb{R}^{p \times (l+m)}$. In this case, the interconnection of $\mathcal{G}$ and Σ as shown in Fig. 1 is given by the transfer function $\tilde{G}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ \hat{B}_y C & \hat{A} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ \hat{B}_y D + \hat{B}_u \end{bmatrix}, \tag{3}$$

$$\tilde{C} = [\hat{D}_y C \ \ \hat{C}], \qquad \tilde{D} = \hat{D}_u + \hat{D}_y D, \tag{4}$$

where $\hat{B}_u \in \mathbb{R}^{\hat{n} \times m}$, $\hat{B}_y \in \mathbb{R}^{\hat{n} \times l}$, $\hat{D}_u \in \mathbb{R}^{p \times m}$, and $\hat{D}_y \in \mathbb{R}^{p \times l}$ are such that $\hat{B} = [\hat{B}_u \ \hat{B}_y]$ and $\hat{D} = [\hat{D}_u \ \hat{D}_y]$.

Proposition 1 ([2, 4]) *Consider the dynamical system $\mathcal{G}$ given by the transfer function $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, let $\hat{Q} \in \mathbb{R}^{p\times p}$, $\hat{Q} = \hat{Q}^{\mathrm{T}}$, and let Σ be a linear dynamical system given by the transfer function $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$. Then, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ and a scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \varepsilon\tilde{P} & \tilde{P}\tilde{B} \\ \tilde{B}^{\mathrm{T}}\tilde{P} & 0 \end{bmatrix} \leq \begin{bmatrix} \tilde{C}^{\mathrm{T}} \\ \tilde{D}^{\mathrm{T}} \end{bmatrix} \hat{Q} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix}. \tag{5}$$

Furthermore, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ such that (5) holds with $\varepsilon = 0$.

Proof The proof is a direct consequence of the generalized Kalman-Yakubovich-Popov lemma [8]. □

Remark 1 Note that it follows from Propositon 1 that if $\tilde{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, then $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a positive-definite matrix $\tilde{P}$ such that (5) holds.

Next, we extend the proposition above for nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \ \ x(t_0) = x_0, \ \ t \geq t_0, \tag{6}$$
$$y(t) = h(x(t)) + J(x(t))u(t), \tag{7}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^l$, and $J : \mathbb{R}^n \to \mathbb{R}^{l\times m}$. We assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are continuously differentiable mappings and $f(\cdot)$ has at least one equilibrium so that, without loss of generality, $f(0) = 0$ and $h(0) = 0$. Furthermore, for the nonlinear dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $u(\cdot)$ satisfies sufficient regularity conditions such that the system (6) has a unique solution forward in time.

Here, we consider a cascade interconnection of the dynamical system $\mathcal{G}$ given by (6), (7) and a system Σ (see Fig. 1) given by

$$\dot{\hat{x}}(t) = f_\Sigma(\hat{x}(t)) + G_{\Sigma_u}(\hat{x}(t))u(t) + G_{\Sigma_y}(\hat{x}(t))y(t), \quad \hat{x}(0) = 0, \quad t \geq 0, \tag{8}$$
$$z(t) = h_\Sigma(\hat{x}(t)) + J_{\Sigma_u}(\hat{x}(t))u(t) + J_{\Sigma_y}(\hat{x}(t))y(t), \tag{9}$$

where $\hat{x} \in \mathbb{R}^{\hat{n}}$, $y \in \mathbb{R}^l$, $z \in \mathbb{R}^{\hat{l}}$, $u \in \mathbb{R}^m$, $f_\Sigma : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$, $G_{\Sigma_u} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}\times m}$, $G_{\Sigma_y} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}\times l}$, $h_\Sigma : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{l}}$, $J_{\Sigma_u} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{l}\times m}$, and $J_{\Sigma_y} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{l}\times l}$. We assume that $f_\Sigma(\cdot)$, $G_\Sigma(\cdot)$, $h_\Sigma(\cdot)$, and $J_\Sigma(\cdot)$ are continuously differentiable mappings and

$f_\Sigma(0) = 0$ and $h_\Sigma(0) = 0$. The overall cascade system consisting of $\mathcal{G}$ and Σ is given by

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t)) + \tilde{G}(\tilde{x}(t))u(t), \ \tilde{x}(0) = \tilde{x}_0, \ t \geq 0, \tag{10}$$

$$z(t) = \tilde{h}(\tilde{x}(t)) + \tilde{J}(\tilde{x}(t))u(t), \tag{11}$$

where $\tilde{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ and

$$\tilde{f}(\tilde{x}(t)) = \begin{bmatrix} f(x(t)) \\ f_\Sigma(\hat{x}(t)) + G_{\Sigma_y}(\hat{x}(t))h(x(t)) \end{bmatrix},$$

$$\tilde{G}(\tilde{x}(t)) = \begin{bmatrix} G(x) \\ G_{\Sigma_u}(\hat{x}(t)) + G_{\Sigma_y}(\hat{x}(t))J(x(t)) \end{bmatrix},$$

$$\tilde{h}(\tilde{x}(t)) = h_\Sigma(\hat{x}(t)) + J_{\Sigma_y}(\hat{x}(t))h(x(t)),$$

$$\tilde{J}(\tilde{x}(t)) = J_{\Sigma_u}(\hat{x}(t)) + J_{\Sigma_y}(\hat{x}(t))J(x(t)).$$

Proposition 2 ([4]) *Consider the nonlinear dynamical systems $\mathcal{G}$ and Σ given by (6), (7) and (8), (9) respectively. $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative (resp., $(\Sigma, \hat{Q})$-dissipative) if and only if there exist functions $\tilde{V}_s : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$, $\tilde{\ell} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{p}}$, and $\tilde{\mathcal{W}} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{p}\times m}$ and a scalar $\tilde{\varepsilon} > 0$ (resp., $\tilde{\varepsilon} = 0$), such that $\tilde{V}_s(\cdot)$ is continuously differentiable and positive definite, $\tilde{V}_s(0) = 0$, and, for all $\tilde{x} \in \mathbb{R}^{\tilde{n}}$, where $\tilde{n} = n + \hat{n}$,*

$$0 = \tilde{V}_s'(\tilde{x})\tilde{f}(\tilde{x}) + \tilde{\varepsilon}\tilde{V}_s(\tilde{x}) - \tilde{h}^{\mathrm{T}}(\tilde{x})\hat{Q}\tilde{h}(\tilde{x}) + \tilde{\ell}^{\mathrm{T}}(\tilde{x})\tilde{\ell}(\tilde{x}), \tag{12}$$

$$0 = \frac{1}{2}\tilde{V}_s'(\tilde{x})\tilde{G}(\tilde{x}) - \tilde{h}^{\mathrm{T}}(\tilde{x})\hat{Q}\tilde{J}(\tilde{x}) + \tilde{\ell}^{\mathrm{T}}(\tilde{x})\tilde{\mathcal{W}}(\tilde{x}), \tag{13}$$

$$0 = \tilde{J}^{\mathrm{T}}(\tilde{x})\hat{Q}\tilde{J}(\tilde{x}) - \tilde{\mathcal{W}}^{\mathrm{T}}(\tilde{x})\tilde{\mathcal{W}}(\tilde{x}). \tag{14}$$

Proof The proof is a direct consequence of the generalized Kalman-Yakubovich-Popov lemma [8]. □

2.3 Feedback Interconnections of Dynamic Dissipative Systems

In this section, we present a result on stability of feedback interconnection of dissipative dynamical systems. Specifically, consider the negative feedback interconnection of dynamical system $\mathcal{G}$ with a feedback system $\mathcal{G}_{\mathrm{d}}$ given by the octuple $(\mathcal{D}_{\mathrm{d}}, \mathcal{U}_{\mathrm{d}}, U_{\mathrm{d}}, \mathcal{Y}_{\mathrm{d}}, Y_{\mathrm{d}}, [0, \infty), s_{\mathrm{d}}, q_{\mathrm{d}})$. Note that with the feedback interconnection given in Fig. 2, $u = -y_{\mathrm{d}}$ and $u_{\mathrm{d}} = y$. Hence, $U = Y_{\mathrm{d}}$ and $Y = U_{\mathrm{d}}$. Furthermore, consider a dynamical system Σ_{d} given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}_{\mathrm{d}}, U_{\mathrm{d}} \times Y_{\mathrm{d}}, Z_{\mathrm{d}}, Z, [0, \infty), \hat{s}_{\mathrm{d}}, \hat{q}_{\mathrm{d}})$, where $\hat{s}_{\mathrm{d}}(t, \hat{x}, u_{\mathrm{d}}, y_{\mathrm{d}}) = \hat{s}(t, \hat{x}_0, -y_{\mathrm{d}}, u_{\mathrm{d}})$ and $\hat{q}_{\mathrm{d}}(\hat{x}, u_{\mathrm{d}}, y_{\mathrm{d}}) =$

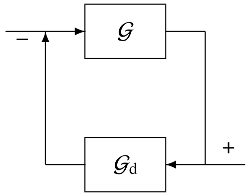

Fig. 2 Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$

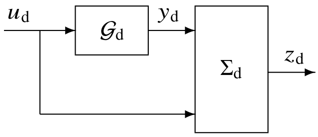

Fig. 3 Interconnection of $\mathcal{G}_\mathrm{d}$ and Σ_d

$\hat{q}(\hat{x}, -y_\mathrm{d}, u_\mathrm{d})$. In addition, consider the interconnected dynamical system $\tilde{\mathcal{G}}_\mathrm{d}$ given by the octuple $(\mathcal{D}_\mathrm{d} \times \hat{\mathcal{D}}, \mathcal{U}_\mathrm{d}, U_\mathrm{d}, \mathcal{Z}_\mathrm{d}, Z, [0, \infty), [s_\mathrm{d}^\mathrm{T} \;\; \hat{s}_\mathrm{d}^\mathrm{T}], \hat{q}_\mathrm{d})$ (see Fig. 3). The following definition is needed for the statement of the next result.

Definition 4 A dynamical system $\mathcal{G}$ with input-output pair (u, y) is *zero-state observable* if $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $s(t, x_0, u) \equiv 0$.

For the statement of the next result let $|||\cdot|||_\sigma$ and $|||\cdot|||_\mu$ denote operator norms on $\mathcal{D}$ and $\mathcal{D}_\mathrm{d}$, respectively, and let $\gamma^+(x_0, x_{\mathrm{d}0}) = \cup_{t\geq 0}\{(s(t, x_0, u), s_\mathrm{d}(t, x_{\mathrm{d}0}, u_\mathrm{d}))\}$, with $u = -y_\mathrm{d}$ and $u_\mathrm{d} = y$, denote the positive orbit of the feedback system $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$. Furthermore, recall that $\gamma^+(x_0, x_{\mathrm{d}0})$ is *precompact* if $\gamma^+(x_0, x_{\mathrm{d}0})$ can be enclosed in the union of a finite number of ε-balls around elements of $\gamma^+(x_0, x_{\mathrm{d}0})$.

Theorem 1 ([4]) *Let $\hat{Q}, \hat{Q}_\mathrm{d} \in \mathbb{R}^{p\times p}$ be such that $\hat{Q} = \hat{Q}^\mathrm{T}$ and $\hat{Q}_\mathrm{d} = \hat{Q}_\mathrm{d}^\mathrm{T}$. Consider the feedback system consisting of the stationary dynamical systems $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ with input-output pairs (u, y) and $(u_\mathrm{d}, y_\mathrm{d})$, respectively, and with $u_\mathrm{d} = y$ and $u = -y_\mathrm{d}$. Assume that $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ are $(\Sigma, \hat{Q})$-dissipative and $(\Sigma_\mathrm{d}, \hat{Q}_\mathrm{d})$-dissipative with C^0 storage functions $V_\mathrm{s} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$ and $V_\mathrm{sd} : \mathcal{D}_\mathrm{d} \times \hat{\mathcal{D}}_\mathrm{d} \to \mathbb{R}$, respectively, such that $V_\mathrm{s}(0, 0) = 0$, $V_\mathrm{sd}(0, 0) = 0$, and*

$$\alpha(|||x|||_\sigma) \leq V_\mathrm{s}(x, \hat{x}), \;\; (x, \hat{x}) \in \mathcal{D} \times \hat{\mathcal{D}}, \tag{15}$$

$$\alpha_\mathrm{d}(|||x_\mathrm{d}|||_\mu) \leq V_\mathrm{sd}(x_\mathrm{d}, \hat{x}_\mathrm{d}), \;\; (x_\mathrm{d}, \hat{x}_\mathrm{d}) \in \mathcal{D}_\mathrm{d} \times \hat{\mathcal{D}}_\mathrm{d}, \tag{16}$$

where $\alpha, \alpha_\mathrm{d} : [0, \infty) \to [0, \infty)$ are class $\mathcal{K}_\infty$ functions. Furthermore, assume that for each initial condition $(x_0, x_{\mathrm{d}0}) \in \mathcal{D} \times \mathcal{D}_\mathrm{d}$, the positive orbit $\gamma^+(x_0, x_{\mathrm{d}0})$ of the feedback system $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is precompact. Finally, assume there exists a scalar $\sigma > 0$ such that $\hat{Q} + \sigma\hat{Q}_\mathrm{d} \leq 0$. Then the following statements hold:

(i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is Lyapunov stable.
(ii) If $\mathcal{G}$ is additionally $(\Sigma, \hat{Q})$-exponentially dissipative, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is Lyapunov stable and for every $x(0) \in \mathcal{D}$, $|||x(t)|||_\sigma \to 0$ as $t \to \infty$.

Proof The proof follows from standard Lyapunov theory and invariant set arguments as applied to infinite-dimensional dynamical systems [1, 9]. □

3 Stability Theory for Time-Delay Dynamical Systems Using Dissipativity Theory

3.1 Linear Time-Delay Systems

In this section we present sufficient conditions for stability of linear time-delay systems [2]. Specifically, consider linear time delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau), \quad x(\theta) = \phi(\theta), -\tau \le \theta \le 0, \quad t \ge 0, \tag{17}$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, $A \in \mathbb{R}^{n\times n}$, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$, $\tau \ge 0$, and $\phi(\cdot) \in C = C([-\tau, 0], \mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system. Note that the state of (17) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\tau, 0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in C([-\tau, 0], \mathbb{R}^n)$. Hence, $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau, 0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definitions of Lyapunov and asymptotic stability of (17). For further details see [9, 11].

Next, we rewrite (17) as a feedback system so that

$$\dot{x}(t) = Ax(t) - A_{\mathrm{d}}u(t), \ x(0) = \phi(0), \ t \ge 0, \tag{18}$$

$$y(t) = x(t), \tag{19}$$

$$y_{\mathrm{d}}(t) = \mathcal{G}_{\mathrm{d}}(u_{\mathrm{d}}(t)), \tag{20}$$

where $u(t) = -y_{\mathrm{d}}(t)$, $u_{\mathrm{d}}(t) = y(t)$, and $\mathcal{G}_{\mathrm{d}} : C([-\tau, \infty), \mathbb{R}^n) \to C([0, \infty), \mathbb{R}^n)$ denotes a delay operator defined by $\mathcal{G}_{\mathrm{d}}(u_{\mathrm{d}}(t)) \overset{\triangle}{=} u_{\mathrm{d}}(t-\tau)$. Note that (18)–(20) is a negative feedback interconnection of a linear finite-dimensional system $\mathcal{G}$ with transfer function $G(s) \sim \left[\begin{array}{c|c} A & -A_{\mathrm{d}} \\ \hline I_n & 0 \end{array}\right]$ and the infinite-dimensional delay operator $\mathcal{G}_{\mathrm{d}}$. Hence, stability of (17) is equivalent to stability of the negative feedback interconnection of $G(s)$ and $\mathcal{G}_{\mathrm{d}}$. Next, we present a key result that shows that the delay operator $\mathcal{G}_{\mathrm{d}}$ is dissipative with respect to a quadratic supply rate. First, however, we show that the *input-output* operator $\mathcal{G}_{\mathrm{d}}$ can be characterized as a stationary dynamical system on C. Specifically, let $\mathcal{U}_{\mathrm{d}} = C([-\tau, \infty), \mathbb{R}^n)$, $\mathcal{Y}_{\mathrm{d}} = C([0, \infty), \mathbb{R}^n)$, and $U_{\mathrm{d}} = Y_{\mathrm{d}} = \mathbb{R}^n$. Now, for every $\phi \in C$, define $s_\theta : [0, \infty) \times C \times \mathcal{U}_{\mathrm{d}} \to C$ by

$$s_\theta(t, \phi, u_\mathrm{d}) = u_\mathrm{d}(t+\theta), \quad \theta \in [-\tau, 0], \quad t \geq 0, \tag{21}$$

where $u_\mathrm{d}(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$. Finally, define $q_\mathrm{d} : C \times U_\mathrm{d} \to Y_\mathrm{d}$ by

$$q_\mathrm{d}(s_\theta(t, \phi, u_\mathrm{d}), u_\mathrm{d}(t)) = s_{-\tau}(t, \phi, u_\mathrm{d}) = u_\mathrm{d}(t-\tau) = \mathcal{G}_\mathrm{d}(u_\mathrm{d}(t)). \tag{22}$$

Note that the octuple $(C, \mathcal{U}_\mathrm{d}, U_\mathrm{d}, \mathcal{Y}_\mathrm{d}, Y_\mathrm{d}, [0, \infty), s_\theta, q_\mathrm{d})$ satisfies Axioms (i)–(v) of Definition 1 which implies that the octuple $(C, \mathcal{U}_\mathrm{d}, U_\mathrm{d}, \mathcal{Y}_\mathrm{d}, Y_\mathrm{d}, [0, \infty), s_\theta, q_\mathrm{d})$ is a stationary dynamical system on C. For notational convenience we refer to this dynamical system as $\mathcal{G}_\mathrm{d}$.

To show that $\mathcal{G}_\mathrm{d}$ is $(\Sigma_\mathrm{d}, \hat{Q}_\mathrm{d})$-dissipative, let Σ denote a linear dynamical system given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}, \mathbb{R}^n \times \mathbb{R}^n, \mathcal{Z}, \mathbb{R}^{2n}, [0, \infty), \hat{s}, \hat{q})$, where $\hat{\mathcal{D}} \subset \mathbb{R}^{2\hat{n}}$ and with transfer function $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$, where

$$\hat{A} = \text{block–diag}[A_1, A_1], \; \hat{B} = \text{block–diag}[B_1, B_1], \; \hat{C} = \text{block–diag}[C_1, C_1], \tag{23}$$

$\hat{D} = I_{2n}$, and where $A_1 \in \mathbb{R}^{\hat{n}\times\hat{n}}$ is Hurwitz, $B_1 \in \mathbb{R}^{\hat{n}\times n}$, and $C_1 \in \mathbb{R}^{n\times\hat{n}}$. In this case, the dynamical system Σ_d is given by the transfer function $\hat{G}_\mathrm{d}(s) \sim \left[\begin{array}{c|c} \hat{A}_\mathrm{d} & \hat{B}_\mathrm{d} \\ \hline \hat{C}_\mathrm{d} & \hat{D}_\mathrm{d} \end{array}\right]$, where

$$\hat{A}_\mathrm{d} = \hat{A}, \quad \hat{B}_\mathrm{d} = \begin{bmatrix} 0 & -B_1 \\ B_1 & 0 \end{bmatrix}, \quad \hat{C}_\mathrm{d} = \hat{C}, \quad \hat{D}_\mathrm{d} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}. \tag{24}$$

Hence, the state space representation of the interconnection shown in Fig. 3 is given by

$$y_\mathrm{d}(t) = \mathcal{G}_\mathrm{d}(u_\mathrm{d}(t)), \quad s_\theta(0, \phi, u_\mathrm{d}) = \phi(\theta), \theta \in [-\tau, 0], \; t \geq 0, \tag{25}$$

$$\dot{x}_{\mathrm{d}_1}(t) = A_1 x_{\mathrm{d}_1}(t) - B_1 y_\mathrm{d}(t), \quad x_{\mathrm{d}_1}(0) = 0, \tag{26}$$

$$\dot{x}_{\mathrm{d}_2}(t) = A_1 x_{\mathrm{d}_2}(t) + B_1 u_\mathrm{d}(t), \quad x_{\mathrm{d}_2}(0) = 0, \tag{27}$$

$$\hat{z}_{\mathrm{d}_1}(t) = C_1 x_{\mathrm{d}_1}(t) - D_1 y_\mathrm{d}(t), \tag{28}$$

$$\hat{z}_{\mathrm{d}_2}(t) = C_1 x_{\mathrm{d}_2}(t) + D_1 u_\mathrm{d}(t). \tag{29}$$

Lemma 1 *Let* $\hat{Q}_\mathrm{d} = \textit{block-diag}[-Q, Q]$, *where* $Q \in \mathbb{R}^{n\times n}$. *If* $\phi(\theta) = 0$, $\theta \in [-\tau, 0]$, *then for every* $u_\mathrm{d}(\cdot) \in \mathcal{U}_\mathrm{d}$, *then*

$$\int_0^T \hat{z}_\mathrm{d}^\mathrm{T}(t)\hat{Q}_\mathrm{d}\hat{z}_\mathrm{d}(t)\mathrm{d}t = \int_\gamma^T \hat{z}_{\mathrm{d}_2}^\mathrm{T}(t)\hat{Q}_\mathrm{d}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t \geq 0, \; T > 0, \tag{30}$$

where $\gamma = 0$ *if* $T \in [0, \tau]$, *and* $\gamma = T - \tau$ *if* $T > \tau$.

Proof Note that

$$x_{d_1}(t) = -\int_0^t e^{A_1(t-s)} B_1 y_d(s)ds, \quad x_{d_2}(t) = \int_0^t e^{A_1(t-s)} B_1 u_d(s)ds, \quad t \geq 0.$$

Since $y_d(t) = u_d(t-\tau)$, $t \geq 0$ and $u_d(\theta) = \phi(\theta) = 0$, $\theta \in [-\tau, 0]$, it follows that $x_{d_1}(t) = 0$, $t \in [0, \tau]$, and for all $t \geq \tau$,

$$x_{d_1}(t) = -\int_\tau^t e^{A_1(t-s)} B_1 u_d(s-\tau)ds = -x_{d_2}(t-\tau).$$

Hence, $\hat{z}_{d_1}(t) = 0$, $t \in [0, \tau]$, and $\hat{z}_{d_1}(t) = -\hat{z}_{d_2}(t-\tau)$, $t > \tau$, which implies that

$$\begin{aligned}\int_0^T \hat{z}_d^T(t)\hat{Q}_d\hat{z}_d(t)dt &= \int_0^T [\hat{z}_{d_2}^T(t)Q\hat{z}_{d_2}(t) - \hat{z}_{d_1}^T(t)Q\hat{z}_{d_1}(t)]dt \\ &= \int_{T-\tau}^T \hat{z}_{d_2}^T(t)Q\hat{z}_{d_2}(t)dt \geq 0, \qquad T \geq \tau.\end{aligned}$$

The case where $T \in [0, \tau]$ follows in a similar manner. □

Theorem 2 *Consider the dynamical system* $\mathcal{G}_d$ *defined by the octuple* $(\mathcal{C}, \mathcal{U}_d, U_d, \mathcal{Y}_d, Y_d, [0, \infty), s_\theta, q_d)$, *where* s_θ *and* q_d *are given by (21) and (22), respectively. Next, let* Σ_d *be a linear dynamical system with transfer function* $\hat{G}_d(s) \sim \left[\begin{array}{c|c} \hat{A}_d & \hat{B}_d \\ \hline \hat{C}_d & \hat{D}_d \end{array}\right]$, *where* $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$ *and* $\hat{D}_d$ *are given by (24), and let* $\hat{Q}_d = \text{block-diag}[-Q, Q]$, *where* $Q \in \mathbb{R}^{n\times n}$, $Q > 0$. *Then,* $\mathcal{G}_d$ *is* $(\Sigma_d, \hat{Q}_d)$*-dissipative. Furthermore,*

$$V_{sd}(\psi, \hat{x}_{d_1}, \hat{x}_{d_2}) = -\inf_{u_d(\cdot)\in\mathcal{U}_d, T\geq 0} \int_0^T \hat{z}_d^T(t)\hat{Q}_d\hat{z}_d(t)dt \tag{31}$$

is a $(\Sigma_d, \hat{Q}_d)$*-storage function for* $\mathcal{G}_d$ *where the infimum in (31) is performed over all trajectories of* $\tilde{\mathcal{G}}_d$ *with initial conditions* $\phi(\cdot) = \psi(\cdot)$, $x_{d_1}(0) = \hat{x}_{d_1}$, *and* $x_{d_2}(0) = \hat{x}_{d_2}$.

Proof It follows from (31) that

$$\begin{aligned}V_{sd}(\psi, \hat{x}_{d_1}, \hat{x}_{d_2}) &= -\inf_{u_d(\cdot)\in\mathcal{U}_d, T\geq 0} \int_0^T \hat{z}_d^T(t)\hat{Q}_d\hat{z}_d(t)dt \\ &= \sup_{u_d(\cdot)\in\mathcal{U}_d, T\geq 0} \int_0^T [\hat{z}_{d_1}^T(t)\hat{Q}_d\hat{z}_{d_1}(t) - \hat{z}_{d_2}^T(t)\hat{Q}_d\hat{z}_{d_2}(t)]dt.\end{aligned} \tag{32}$$

Hence, $V_{sd}(\psi, \hat{x}_{d_1}, \hat{x}_{d_2}) \geq 0$, $\psi(\cdot) \in \mathcal{C}$, $\hat{x}_{d_1}, \hat{x}_{d_2} \in \mathbb{R}^n$. If $\psi(\theta) \equiv 0$, $\theta \in [-\tau, 0]$, $\hat{x}_{d_1} = 0$, $\hat{x}_{d_2} = 0$, then it follows from Lemma 1 that

$$\int_0^T \hat{z}_{\mathrm{d}}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}}(t)\mathrm{d}t = \int_0^T \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t, \quad T \geq 0,$$

which implies that

$$V_{\mathrm{sd}}(0,0,0) = \sup_{u_{\mathrm{d}}(\cdot)\in\mathcal{U}_{\mathrm{d}}, T\geq 0} -\int_0^T \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t \leq 0.$$

Hence, since $V_{\mathrm{sd}}(0,0,0) \geq 0$, $V_{\mathrm{sd}}(0,0,0) = 0$. Next, note that for every $u_{\mathrm{d}}(t)$, $t \in [t_1, t_{\mathrm{f}}]$, and $T \in [t_1, t_{\mathrm{f}}]$,

$$\begin{aligned} -V_{\mathrm{sd}}(s_\theta(t_1,\psi,u_{\mathrm{d}}), x_{\mathrm{d}_1}(t_1), x_{\mathrm{d}_2}(t_2)) &\leq \int_{t_1}^{t_{\mathrm{f}}} \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t \\ &= \int_{t_1}^{T} \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t + \int_{T}^{t_{\mathrm{f}}} \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t. \end{aligned}$$

Hence,

$$-V_{\mathrm{sd}}(s_\theta(t_1,\psi,u_{\mathrm{d}}), x_{\mathrm{d}_1}(t_1), x_{\mathrm{d}_2}(t_1)) - \int_{t_1}^{T} \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t \leq \int_{T}^{t_{\mathrm{f}}} \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t,$$

which implies that

$$\begin{aligned} &-V_{\mathrm{sd}}(s_\theta(t_1,\psi,u_{\mathrm{d}}), x_{\mathrm{d}_1}(t_1), x_{\mathrm{d}_2}(t_2)) - \int_{t_1}^{T} \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t \\ &\leq \inf_{u_{\mathrm{d}}(\cdot)\in\mathcal{U}_{\mathrm{d}}, t_{\mathrm{f}}\geq T} \int_{T}^{t_{\mathrm{f}}} \hat{z}_{\mathrm{d}}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}}(t)\mathrm{d}t = -V_{\mathrm{sd}}(s_\theta(T,\psi,u_{\mathrm{d}}), x_{\mathrm{d}_1}(T), x_{\mathrm{d}_2}(T)), \end{aligned}$$

establishing the $(\Sigma_{\mathrm{d}}, \hat{Q}_{\mathrm{d}})$-dissipativity of $\mathcal{G}_{\mathrm{d}}$. □

Remark 2 In the case where $A_1 = 0$, $B_1 = 0$, and $C_1 = 0$, it can be shown that

$$V_{\mathrm{sd}}(\psi, x_{\mathrm{d}1}, x_{\mathrm{d}2}) = V_{\mathrm{sd}}(\psi) = \int_{-\tau}^{0} \psi^{\mathrm{T}}(\theta)Q\psi(\theta)\mathrm{d}\theta. \tag{33}$$

Next, using Theorem 2, we present a sufficient condition on $G(s)$ that guarantees asymptotic stability of the negative feedback interconnection of the time delay dynamical system given by (17). For the following result we assume that $V_{\mathrm{sd}}(\cdot,\cdot,\cdot)$ given by (31) is continuously differentiable.

Theorem 3 *Consider the linear time delay dynamical system given by (17). Let* $\hat{Q} =$ *block-diag*$[Q, -Q]$, *where* $Q \in \mathbb{R}^{n\times n}$, $Q > 0$. *Assume there exists a nonnegative definite matrix* $\tilde{P} \in \mathbb{R}^{(n+2\hat{n})\times(n+2\hat{n})}$ *and scalars* $\varepsilon, \eta > 0$ *such that*

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \varepsilon\tilde{P} & \tilde{P}\tilde{B} \\ \tilde{B}^{\mathrm{T}}\tilde{P} & 0 \end{bmatrix} \leq \begin{bmatrix} \tilde{C}^{\mathrm{T}} \\ \tilde{D}^{\mathrm{T}} \end{bmatrix} \hat{Q} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix}, \tag{34}$$

holds and $\tilde{P} \geq \text{block–diag}[\eta I_n,\ 0_{\hat{n}\times\hat{n}},\ 0_{\hat{n}\times\hat{n}}]$, *where*

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A_1 & 0 \\ B_1 & 0 & A_1 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} -A_{\mathrm{d}} \\ B_1 \\ 0 \end{bmatrix},\ \tilde{C} = \begin{bmatrix} 0 & C_1 & 0 \\ I_n & 0 & C_1 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \tag{35}$$

Then the linear time delay dynamical system given by (17) is asymptotically stable for every $\tau \in [0, \infty)$.

Proof It follows from Theorem 2 that $\mathcal{G}_{\mathrm{d}}$ is $(\Sigma_{\mathrm{d}}, \hat{Q}_{\mathrm{d}})$-dissipative with $(\Sigma_{\mathrm{d}}, \hat{Q}_{\mathrm{d}})$-storage function $V_{\mathrm{sd}}(\psi, x_{\mathrm{d}_1}, x_{\mathrm{d}_2})$, $\psi \in C, x_{\mathrm{d}_1}, x_{\mathrm{d}_2} \in \mathbb{R}^{\hat{n}}$, given by (31). Next, it follows from Proposition 1 that $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative with $(\Sigma, \hat{Q})$-storage function $V_{\mathrm{s}}(\tilde{x}) = \tilde{x}^{\mathrm{T}}\tilde{P}\tilde{x}$, where $\tilde{x} = [x^{\mathrm{T}}, x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$. Furthermore, note that $x = \psi(0)$ and as in the proof of Theorem 1, it can be shown that $\hat{x}_1(t) = x_{\mathrm{d}_1}(t)$, $\hat{x}_2(t) = x_{\mathrm{d}_2}(t)$, $t \geq 0$, and hence the state of the overall interconnection of $\mathcal{G}$, $\mathcal{G}_{\mathrm{d}}$, and Σ (see Fig. 4) is given by $[\psi^{\mathrm{T}},\ \hat{x}^{\mathrm{T}}]^{\mathrm{T}}$ where $\hat{x} = [\hat{x}_1^{\mathrm{T}},\ \hat{x}_2^{\mathrm{T}}]^{\mathrm{T}}$. Next, using the Lyapunov-Krasovskii functional candidate $V(\psi, \hat{x}_1, \hat{x}_2) = V_{\mathrm{s}}(\psi(0), \hat{x}_1, \hat{x}_2) + V_{\mathrm{sd}}(\psi, \hat{x}_1, \hat{x}_2)$, it follows that

$$\dot{V}(x_t, \hat{x}_1(t), \hat{x}_1(t)) \leq -\varepsilon\tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{x}(t) \leq -\varepsilon\eta x^{\mathrm{T}}(t)x(t). \tag{36}$$

Now, Lyapunov stability follows from standard arguments as applied to time delay systems (see Theorem 2.1 of [9, p. 132] for a similar proof). The proof of asymptotic stability is similar to that of Theorem 1 and hence is omitted. □

Remark 3 Note that if $V_{\mathrm{s}}(\tilde{x})$ and $V_{\mathrm{sd}}(\psi, x_{\mathrm{d}_1}, x_{\mathrm{d}_2})$ satisfy (15) and (16), then Theorem 3 follows from Theorem 1. However, in the case of time delay dynamical systems (15) and (16) can be replaced by a weaker condition

$$\eta\psi^{\mathrm{T}}(0)\psi(0) \leq V(\psi, \hat{x}_1, \hat{x}_2), \quad \psi \in C, \quad \hat{x}_1, \hat{x}_2 \in \mathbb{R}^{\hat{n}}. \tag{37}$$

In this case, Lyapunov and asymptotic stability can be shown using the fact that $\|x(t)\| \leq \varepsilon$, $t \geq 0$, if and only if $|||x_t||| \leq \varepsilon$, $t \geq 0$.

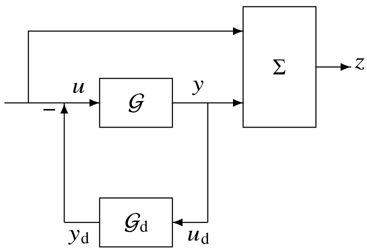

Fig. 4 Interconnection of $\mathcal{G}$, $\mathcal{G}_{\mathrm{d}}$, and Σ

Remark 4 In the case where $A_1 = 0$, $B_1 = 0$, and $C_1 = 0$, it follows from Theorem 3 that if there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + \varepsilon P + Q & -PA_{\mathrm{d}} \\ -A_{\mathrm{d}}^{\mathrm{T}}P & -Q \end{bmatrix} \leq 0, \tag{38}$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is asymptotically stable. Furthermore, it follows from Remark 2 that $V_{\mathrm{sd}}(\psi) = \int_{-\tau}^{0} \psi^{\mathrm{T}}(\theta)Q\psi(\theta)\mathrm{d}\theta$ and hence $V(\psi) = \psi^{\mathrm{T}}(0)P\psi(0) + \int_{-\tau}^{0} \psi^{\mathrm{T}}(\theta)Q\psi(\theta)\mathrm{d}\theta$ is a Lyapunov-Krasovskii functional for the linear time delay dynamical system (17).

3.2 Nonlinear Time-Delay Systems

In this section, we consider nonlinear time delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + f_{\mathrm{d}}(x(t-\tau)), \quad x(\theta) = \phi(\theta), -\tau \leq \theta \leq 0, \quad t \geq 0, \tag{39}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f_{\mathrm{d}} : \mathbb{R}^n \to \mathbb{R}^n$, $\tau \geq 0$, and $\phi(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system.

Next, we rewrite (39) as a feedback system so that

$$\dot{x}(t) = f(x(t)) - u(t), \; x(0) = \phi(0), \; t \geq 0, \tag{40}$$

$$y(t) = f_{\mathrm{d}}(x(t)), \tag{41}$$

$$y_{\mathrm{d}}(t) = \mathcal{G}_{\mathrm{d}}(u_{\mathrm{d}}(t)), \tag{42}$$

where $u(t) = -y_{\mathrm{d}}(t)$, $u_{\mathrm{d}}(t) = y(t)$, and $\mathcal{G}_{\mathrm{d}} : \mathcal{C}([-\tau, \infty), \mathbb{R}^n) \to \mathcal{C}([0, \infty), \mathbb{R}^n)$ denotes the delay operator defined by $\mathcal{G}_{\mathrm{d}}(u_{\mathrm{d}}(t)) \triangleq u_{\mathrm{d}}(t-\tau)$. Note that (40)–(42) is a negative feedback interconnection of a nonlinear finite-dimensional system $\mathcal{G}$ given by (40), (41) and the infinite-dimensional delay operator $\mathcal{G}_{\mathrm{d}}$.

Theorem 4 *Consider the nonlinear time delay dynamical system given by (39). Let $\hat{Q} = \text{block-diag}[Q, -Q]$, where $Q \in \mathbb{R}^{n\times n}$, $Q > 0$. Assume there exist $\tilde{V}_{\mathrm{s}} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$, $\tilde{\ell} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{p}}$, and $\tilde{\mathcal{W}} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{p}\times n}$ and a scalar $\tilde{\varepsilon} > 0$, such that $\tilde{V}_{\mathrm{s}}(\cdot)$ is continuously differentiable and nonnegative definite, $\tilde{V}_{\mathrm{s}}(0) = 0$, and, for all $\tilde{x} \in \mathbb{R}^{\tilde{n}}$, where $\tilde{n} = n + 2\hat{n}$, such that (12)–(14) hold and $\tilde{V}_{\mathrm{s}}(\tilde{x}) \geq \eta x^{\mathrm{T}}x$,*

$$\tilde{f}(\tilde{x}) = \begin{bmatrix} f(x) \\ A_1 x_1 \\ B_1 f_{\mathrm{d}}(x) + A_1 x_2 \end{bmatrix}, \; \tilde{G}(\tilde{x}) = \begin{bmatrix} -I_n \\ B_1 \\ 0 \end{bmatrix}, \tag{43}$$

$$\tilde{h}(\tilde{x}) = \begin{bmatrix} C_1 x_1 \\ f_{\mathrm{d}}(x) + C_1 x_2 \end{bmatrix}, \; \tilde{J}(\tilde{x}) = \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \tag{44}$$

Then the nonlinear time delay dynamical system given by (39) is asymptotically stable for every $\tau \in [0, \infty)$.

Proof The proof is similar to that of Theorem 3 and hence is omitted. □

4 Conclusion

In this chapter, by representing a time delay dynamical system as a negative feedback interconnection of a finite-dimensional dynamical system and an infinite-dimensional time delay operator, we derived new sufficient conditions for asymptotic stability of linear and nonlinear time delay dynamical systems. The overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable time delay dynamical systems based on the dissipativity properties of the time delay operator. The results of this paper are restricted to delay-independent sufficient conditions for time-delay systems and future work will focus on extending these results to delay-dependent conditions.

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Stability of Interconnected Uncertain Delay Systems: A Converse Lyapunov Approach

Ihab Haidar, Paolo Mason and Mario Sigalotti

1 Introduction

In the context of *time-varying delay systems*, many important problems concern their stability. Two principal approaches in the stability analysis are the Lyapunov–Krasovskii method and Lyapunov–Razumikhin method [6]. Based on these two approaches, a variety of stability criteria has been developed (see e.g. [1, 3, 4, 14, 16, 18] and references therein). These criteria are often formulated as linear matrix inequalities (LMIs) which yield sufficient conditions for stability. Switched systems theory offers a complementary insight in this context. In [8], Hetel, Daafouz and Iung establish a theoretical link between the Lyapunov–Krasovskii approach and the switched system representation, in the context of discrete-time systems with time-varying delays. They prove that, looking for a delay-dependent Lyapunov–Krasovskii functional for the initial system, is equivalent to applying the multiple Lyapunov functions approach to the delay-free switched system representation. The paper [5] shares the same spirit as [8] with the significant difference that it considers the case of continuous-time systems with time-varying delays. An important feature of this setting is that the switched system formulation, obtained by standard functional representation, describes an evolution in an infinite-dimensional Banach

I. Haidar
Quartz EA 7393, ENSEA, 6 Avenue du Ponceau, 95014 Cergy-Pontoise Cedex, France
e-mail: ihab.haidar@ensea.fr

P. Mason (✉)
Laboratoire des Signaux et Systèmes (L2S), CNRS, CentraleSupélec, Université Paris-Sud, Université Paris-Saclay, Gif-sur-Yvette, France
e-mail: paolo.mason@l2s.centralesupelec.fr

M. Sigalotti
Laboratoire Jacques-Louis Lions, Inria, Sorbonne Université, CNRS, Université Paris-Diderot, Paris, France
e-mail: mario.sigalotti@inria.fr

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space. Switched systems in Banach and Hilbert spaces have been studied for instance in [7, 19], where some converse Lyapunov theorems have been obtained.

The main result of [5] provides a collection of converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations. The term *uncertain* refers here to the fact that delays may vary arbitrarily in a given interval, or, more generally, that the operator describing the retarded dynamics varies arbitrarily in a given set of modes.

In this work, we review the main result of [5] and we derive from it some robustness properties of interconnected uncertain time-varying delay systems.

Consider linear retarded functional differential equations (RFDE) of the type

$$\dot{x}(t) = \Gamma(t)x_t, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $x_t : [-r, 0] \to \mathbb{R}^n$ is the standard notation for the history function defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$. For every time t, the operator $L = \Gamma(t)$ associates with x_t a vector in $\mathbb{R}^n$: a typical example is $Lx_t = A_0x(t) + A_1x(t - \tau)$ for some $n \times n$ matrices A_0 and A_1, and some delay $\tau \in [-r, 0]$. Two phase spaces are considered: $C([-r, 0], \mathbb{R}^n)$ and the Sobolev space $H^1([-r, 0], \mathbb{R}^n)$. The time-varying operator $\Gamma(\cdot)$ is assumed to be piecewise constant with values in a given set of operators Q, which is not necessarily finite nor countable. We are interested in properties that are uniform with respect to $\Gamma(\cdot)$, which plays the role of a uncertain retarded dynamics. It is proven in [5] that system (1) is *uniformly exponentially stable* in $C([-r, 0], \mathbb{R}^n)$ if and only if its restriction to $H^1([-r, 0], \mathbb{R}^n)$ is also uniformly exponentially stable (with respect to the H^1-norm). These properties are also shown to be equivalent to the existence of a Lyapunov–Krasovskii functional. One of the novelties of such results is that this functional may be *weakly-degenerate*, i.e., it may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature. On the other hand the Lyapunov–Krasovskii functional may be assumed to satisfy stronger regularity assumptions and in particular to be directionally Frechet differentiable.

This chapter is organized as follows. Section 2 is devoted to the problem framework and the switched system representation of system (1). The statement of our converse Lyapunov–Krasovskii theorem is presented in Sect. 3. Section 4 is devoted to the comparison of the latter with some previously known converse Lyapunov–Krasovskii theorems. An example that describes the applicability of our result is given in Sect. 5. The effects of small perturbations of the dynamics on the stability of system (1) is discussed in Sect. 6. Finally, sufficient conditions for the stability of interconnected uncertain linear delay systems are given in Sect. 7.

2 Problem Framework

Let $r > 0$ be a real number and let $X = X([-r, 0], \mathbb{R}^n)$ be a linear real vector space of functions mapping $[-r, 0]$ into $\mathbb{R}^n$. Assume that a norm $\|\cdot\|_X$ is given in X, and assume that $(X, \|\cdot\|_X)$ is a Banach space. We denote by $\mathcal{L}(X)$ (respectively, $\mathcal{L}(X, \mathbb{R}^n)$) the space of continuous linear operators from X into itself (respectively, $\mathbb{R}^n$) endowed with the usual operator norm $\|\cdot\|_{\mathcal{L}(X)}$ (respectively, $\|\cdot\|_{\mathcal{L}(X,\mathbb{R}^n)}$).

Two different choices of Banach spaces are considered: $X = (C([-r, 0], \mathbb{R}^n), \|\cdot\|_C)$ and $X = \left(H^1([-r, 0], \mathbb{R}^n), \|\cdot\|_{H^1}\right)$, where

$$\|\psi\|_C = \max_{\theta\in[-r,0]} |\psi(\theta)|,$$

$$\|\psi\|_{H^1} = \left(\int_{-r}^{0} \left(|\psi(\theta)|^2 + \left|\frac{d}{d\theta}\psi(\theta)\right|^2\right) d\theta\right)^{\frac{1}{2}}.$$

Let Q be any subset of the space of linear operators from $C([-r, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$. A crucial hypothesis in what follows is that Q is bounded in $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$, i.e., there exists a positive constant m such that

$$|L\psi| \le m\|\psi\|_C \qquad \forall\psi \in C([-r, 0], \mathbb{R}^n), L \in Q. \tag{2}$$

Let us associate with Q the linear Retarded Functional Differential Equation (RFDE)

$$\dot{x}(t) = \Gamma(t)x_t, \qquad t \ge t_0, \tag{3}$$

where $x : [t_0 - r, +\infty) \to \mathbb{R}^n$, $x_t : [-r, 0] \to \mathbb{R}^n$ denotes the history function defined by

$$x_t : \theta \mapsto x(t + \theta), \quad \theta \in [-r, 0], \quad t \ge t_0,$$

$\Gamma : [t_0, \infty) \to Q$ is piecewise constant (hence, with finitely many discontinuities on each bounded interval). We denote by PC($[t_0, +\infty)$, Q) (or simply PC) the class of piecewise constant functions with values in Q.

Denote by $\phi \in X$ the initial condition for (3) at time t_0, i.e.,

$$x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-r, 0]. \tag{4}$$

Example 1 A particular case of system (3)–(4) is the following type of linear time-varying delay systems

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{p} A_{\sigma(t),i}\, x\Big(t - \tau_i\big(\sigma(t)\big)\Big), \quad t \ge t_0 , \\ x_{t_0} &= \phi \in X, \end{aligned} \tag{5}$$

where $\tau_i(\sigma) \in [0, r]$ and $A_{\sigma,i}$ is a $n \times n$ matrix for every σ in a given set Σ and every $i \in \{1, \ldots, p\}$. If the set of matrices $\{A_{\sigma,i} \mid \sigma \in \Sigma, i \in \{1, \ldots, p\}\}$ is bounded, then the set Q of the operators $L_\sigma \psi = \sum_{i=1}^{p} A_{\sigma,i} \psi(-\tau_i(\sigma))$, $\sigma \in \Sigma$, is bounded in $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$.

Example 2 Another particular case of system (3)–(4) is the following integro-differential equation

$$\begin{aligned} \dot{x}(t) &= \int_0^r A_{\sigma(t),\theta}\, x(t-\theta)d\theta, \quad t \geq t_0\,, \\ x_{t_0} &= \phi \in X, \end{aligned}$$

where $A_{\sigma,\theta}$ is a $n \times n$ matrix uniformly bounded with respect to $\theta \in [0, r]$ and to σ in a given set Σ, and measurable with respect to θ. In this case the set Q of the operators $L_\sigma \psi = \displaystyle\int_0^r A_{\sigma,\theta}\, \psi(-\theta)d\theta$, $\sigma \in \Sigma$, is bounded in $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$.

2.1 Switched System Representation

With any $\Gamma(\cdot) \in \mathrm{PC}([t_0, +\infty), Q)$, we can associate an evolution operator obtained by concatenating the flows of (3) corresponding to constant values of $\Gamma(\cdot)$. It is well-known (see e.g. [6, Lemma 1.2, p. 194]) that with any $L \in Q$ one can associate a C_0-semigroup $T_L(t) : X \to X$, $t \geq 0$, with infinitesimal generator $\mathcal{A}_L$ given by

$$\begin{aligned} D(\mathcal{A}_L) &= \left\{ \psi \in X : \frac{d\psi}{d\theta} \in X, \frac{d\psi}{d\theta}(0) = L\psi \right\}, \\ \mathcal{A}_L \psi &= \frac{d\psi}{d\theta}. \end{aligned} \tag{6}$$

By definition, $T_L(t)$ is the flow at time t of Eq. (3) with $t_0 = 0$ and $\Gamma(\cdot)$ constantly equal to L.

The evolution operator corresponding to a piecewise constant $t \mapsto \Gamma(t)$ defined by $\Gamma(t) = L_k$, $t \in [t_k, t_{k+1})$, where $t_k < t_{k+1}$ for every $k \geq 0$, is given by

$$T_{\Gamma(\cdot)}(t, t_0) = T_{L_k}(t - t_k) T_{L_{k-1}}(t_k - t_{k-1}) \ldots T_{L_0}(t_1 - t_0)$$

for each $t \in [t_k, t_{k+1})$. This is exactly the notion of switched system

$$\begin{aligned} x_t &= T_{\Gamma(\cdot)}(t, t_0) x_{t_0}, \\ x_{t_0} &= \phi \in X, \end{aligned} \tag{7}$$

considered in [7] for general Banach spaces.

3 Converse Lyapunov–Krasovskii Results

The notion of uniform exponential stability is recalled in the following definition.

Definition 1 We say that system (3) is uniformly exponentially stable in X if there exist two constants $\alpha \geq 1$ and $\beta > 0$ such that for every initial condition $\phi \in X$ and every $\Gamma(\cdot) \in \mathrm{PC}([t_0, +\infty), Q)$ the solution $x(t, \phi)$ of (3)–(4) satisfies

$$\|x_t\|_X \leq \alpha e^{-\beta(t-t_0)}\|\phi\|_X, \; t \geq t_0.$$

Since the stability properties of (3)–(4) are preserved by time shifting of $\Gamma(\cdot)$, we assume from now on that $t_0 = 0$ and we set $T_{\Gamma(\cdot)}(t) = T_{\Gamma(\cdot)}(t, 0)$. In order to state our converse Lyapunov–Krasovskii result we introduce the generalized Dini derivatives of a function $V : X \to [0, \infty)$ as follows

$$\overline{D}_L V(\psi) = \lim_{t \to 0^+} \sup \frac{V(T_L(t)\psi) - V(\psi)}{t},$$
$$\underline{D}_L V(\psi) = \lim_{t \to 0^+} \inf \frac{V(T_L(t)\psi) - V(\psi)}{t},$$

noting that possibly $\underline{D}_L V(\psi), \overline{D}_L V(\psi) = \infty$ for some $\psi \in X$ and $L \in Q$. Recall that $V(\cdot)$ is said to be *directionally differentiable in the sense of Fréchet* at $\psi \in X$ if there exists a positively one-homogeneous function $V'(\psi, \cdot) : X \to \mathbb{R}$ (i.e., $V'(\psi, \lambda\xi) = \lambda V'(\psi, \xi)$ for $\xi \in X$ and $\lambda > 0$) such that

$$\frac{V(\psi + \xi) - V(\psi) - V'(\psi, \xi)}{\|\xi\|_X} \longrightarrow 0 \qquad \text{as } \xi \to 0.$$

The proof of the following theorem may be found in [5]. Item (iv) is expressed slightly differently here, rightening some loosely worded statement in [5].

Theorem 1 *Let $Q \subset \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$ be bounded. The following statements are equivalent:*

(i) System (3) is uniformly exponentially stable in $C([-r, 0], \mathbb{R}^n)$.
(ii) System (3) is uniformly exponentially stable in $H^1([-r, 0], \mathbb{R}^n)$.
(iii) There exists a function $V : C([-r, 0], \mathbb{R}^n) \to [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on $C([-r, 0], \mathbb{R}^n)$,

$$\underline{c}\|\psi\|_C^2 \leq V(\psi) \leq \overline{c}\|\psi\|_C^2 \tag{8}$$

for some constants $\underline{c}, \overline{c} > 0$ and

$$\overline{D}_L V(\psi) \leq -\|\psi\|_C^2, \quad L \in Q, \psi \in C([-r, 0], \mathbb{R}^n).$$

(iv) There exists a directionally Fréchet differentiable function $V : H^1([-r, 0], \mathbb{R}^n) \to [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on $H^1([-r, 0], \mathbb{R}^n)$,

$$\underline{c}\|\psi\|_{H^1}^2 \leq V(\psi) \leq \overline{c}\|\psi\|_{H^1}^2,$$
$$|V'(\psi,\xi)| \leq \overline{c}\|\psi\|_{H^1}\|\xi\|_{H^1},$$
$$V'(\psi,\xi_1+\xi_2) \leq V'(\psi,\xi_1) + V'(\psi,\xi_2),$$

for some constants $\underline{c}, \overline{c} > 0$ *and*

$$\overline{D}_L V(\psi) \leq -\|\psi\|_{H^1}^2, \quad L \in Q, \psi \in H^1([-r,0],\mathbb{R}^n).$$

If $\psi \in D(\mathcal{A}_L)$ *then* $\overline{D}_L V(\psi) = V'(\psi, \mathcal{A}_L\psi)$.

(v) There exists a continuous function

$$V : C([-r,0],\mathbb{R}^n) \to [0,\infty)$$

such that

$$V(\psi) \leq c\|\psi\|_C^2$$

for some constant $c > 0$ *and*

$$\underline{D}_L V(\psi) \leq -|\psi(0)|^2, \quad L \in Q, \psi \in C([-r,0],\mathbb{R}^n).$$

(vi) There exists a continuous function

$$V : H^1([-r,0],\mathbb{R}^n) \to [0,\infty)$$

such that

$$V(\psi) \leq c\|\psi\|_{H^1}^2$$

for some constant $c > 0$ *and*

$$\underline{D}_L V(\psi) \leq -|\psi(0)|^2, \quad L \in Q, \psi \in H^1([-r,0],\mathbb{R}^n).$$

Clearly, a functional $V(\cdot)$ satisfying condition (v) (respectively (vi)) does not necessarily satisfy the stronger conditions appearing in condition (iii) (respectively (iv)). Hence, condition (v) (respectively (vi)) is better suited for proving the global uniform exponential stability of a RFDE, while condition (iii) (respectively (iv)) provides more information on a linear uncertain time-delay system that is known to be globally uniformly exponentially stable, by tightening the properties satisfied by $V(\cdot)$.

Remark 1 Concerning the regularity of the Lyapunov–Krasovskii functionals satisfying conditions (iii) and (iv), recall that being a squared norm is equivalent to be positive definite, homogeneous of degree 2, continuous and convex. Note that in (iv) we do not require the function V to be Fréchet differentiable in the usual sense, which is a stronger regularity assumption.

4 Discussion

We compare here the results stated in the previous section with the Lyapunov–Krasovskii theorems given in [6, 10]. We start by recalling a Lyapunov–Krasovskii theorem given in [6, Theorem V.2.1] for general retarded functional differential equation of the form

$$\dot{x} = f(t, x_t). \tag{9}$$

Theorem 2 *Suppose that $f : \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous function. Suppose that for any bounded set $B \subset C([-r, 0], \mathbb{R}^n)$, f maps $\mathbb{R} \times B$ into a bounded set of $\mathbb{R}^n$, and $u, v, w : [0, +\infty) \to [0, +\infty)$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exists a continuous function $V : \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ such that*

$$u(|\psi(0)|) \leq V(t, \psi) \leq v(\|\psi\|_C)$$
$$\overline{D}V(t, \psi) \leq -w(|\psi(0)|)$$

then the solution $x = 0$ of Eq. (9) is uniformly stable. If $u(s) \to +\infty$ as $s \to +\infty$, the solutions of Eq. (2) are uniformly bounded. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

Theorem 2 considers systems without switching and uniformity is meant with respect to the initial condition. Its proof can however be straightforwardly adapted to linear systems of the type (3), leading to the following result.

Theorem 3 *Assume that $\mathcal{Q}$ is bounded in $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$. If there exists a continuous function $V : C([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ such that*

$$\underline{c}|\psi(0)|^2 \leq V(\psi) \leq \overline{c}\|\psi\|_C^2 \tag{10}$$

for constants $\underline{c}, \overline{c} > 0$ and

$$\overline{D}_L V(\psi) \leq -|\psi(0)|^2, \quad L \in \mathcal{Q}, \psi \in C([-r, 0], \mathbb{R}^n) \tag{11}$$

then system (3) is uniformly exponentially stable in $C([-r, 0], \mathbb{R}^n)$.

Notice that the lower bound on the Lyapunov function appearing in (10) is less restrictive than the one in (8), making it an easier condition to fulfill when looking for quadratic Lyapunov–Krasovskii functionals (see e.g. [12]).

Thanks to Theorem 1, we can give a converse version for Theorem 3. More precisely, if system (3) is uniformly exponentially stable in $C([-r, 0], \mathbb{R}^n)$, then there exists a continuous function $V : C([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ such that (10)–(11) hold.

Let us conclude this section by commenting on [10, 11, 17]. In [17] the authors show that the existence of a Lyapunov–Krasovskii functional is a necessary and

sufficient condition for the uniform global asymptotic stability and the global exponential stability of nonlinear autonomous systems described by neutral functional differential equations in Hale's form.
When restricted to linear systems of the form (3), [10, Theorem 2.10] establishes the equivalence between the uniform exponential stability in $C([-r, 0], \mathbb{R}^n)$ and the existence of a Lyapunov–Krasovskii functional either as in the statement (v) of Theorem 1 or with a lower bound as the one in (10). In the latter case, however, the Lyapunov–Krasovskii functional is defined on a space of the type $C([-r-\tau, 0], \mathbb{R}^n)$, for some $\tau \geq 0$, on which system (3) is lifted. Similar remarks apply to the results in [11], which extend the approach of [10] towards Lyapunov–Krasovskii characterizations of output stability.

5 Example of Weakly-Degenerate Lyapunov–Krasovskii Functional

In this section we discuss through an example the absence of strictly positive norm-dependent lower bounds on V in statements (v) and (vi) of Theorem 1. The example is an evidence of the fact that the relaxed condition on V might turn into an advantage while looking for Lyapunov–Krasovskii functionals.

Consider the system

$$\begin{aligned} \dot{x}(t) &= -x(t-r), \quad t \geq 0\,, \\ x_0 &= \phi, \end{aligned} \tag{12}$$

where $r \geq 0$ and $\phi \in C([-r, 0], \mathbb{R})$. Let $\mathcal{A}$ be the infinitesimal generator of the C_0-semigroup associated with (12). The spectrum of $\mathcal{A}$ is discrete and is given by (see e.g. [2])

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} : \Delta(\lambda) = 0\} = (\lambda_k)_{k\in\mathbb{N}},$$

where

$$\Delta(\lambda) = \lambda + e^{-\lambda r}. \tag{13}$$

A useful property of $\sigma_p(\mathcal{A})$ is that

$$\mathbb{R}(\lambda_k) \to -\infty \quad \text{as} \quad k \to \infty \tag{14}$$

(see, e.g. [6, Lemma 4.1, p. 18]). System (12) is exponentially stable if and only if $r < \pi/2$ (see [6]), and in such case we can define a Lyapunov–Krasovskii functional V as

$$V(\psi) = \int_0^{+\infty} \|T(t)\psi\|_C^2 dt,$$

where $T(\cdot)$ is the C_0-semigroup generated by $\mathcal{A}$. Notice that this classical choice of Lyapunov-Krasovskii functional can be generalized for exponentially stable switched systems of the type introduced in Section 2.1, taking

$$V(\psi) = \sup_{\Gamma(\cdot)\in \mathrm{PC}} \int_0^{+\infty} \|T_{\Gamma(\cdot)}\psi\|_X^2 dt,$$

as shown in [7].

For each $k \in \mathbb{N}$, let $v_k(\theta) = e^{\lambda_k(r+\theta)}$, $\theta \in [-r, 0]$ be a (complex) eigenvector of $\mathcal{A}$ corresponding to λ_k.

It follows from Eq. (13), that $\lambda \in \sigma_p(\mathcal{A})$ if and only if $\bar{\lambda} \in \sigma_p(\mathcal{A})$. Associate with $\bar{\lambda}_k$ its eigenvector $\bar{v}_k(\theta) = e^{\bar{\lambda}_k(r+\theta)}$, $\theta \in [-r, 0]$, and let $\nu_k = (v_k + \bar{v}_k)/2 = \mathbb{R}(v_k) \in C([-r, 0], \mathbb{R})$. Then

$$T(t)\nu_k(\theta) = \frac{e^{\lambda_k(t+r+\theta)} + e^{\bar{\lambda}_k(t+r+\theta)}}{2}. \tag{15}$$

Hence,

$$\begin{aligned} V(\nu_k) &= \int_0^{+\infty} \frac{\|e^{\lambda_k(t+r+\theta)} + e^{\bar{\lambda}_k(t+r+\theta)}\|_C^2}{4} dt \leq \int_0^{+\infty} \|e^{\mathbb{R}(\lambda_k)(t+r+\theta)}\|_C^2 dt \\ &= \int_0^{+\infty} e^{2\mathbb{R}(\lambda_k)t} dt = -\frac{1}{2\mathbb{R}(\lambda_k)}. \end{aligned}$$

Then $V(\nu_k) \to 0$ as $k \to +\infty$, as it follows from (14). On the other hand, we have $\|\nu_k\|_C \geq \nu_k(-r) = 1$. Then the function V does not have a strictly positive $\|\cdot\|_C$-norm-dependent lower bound as in Eq. (8). Notice however that $|\nu_k(0)| \to 0$ as $k \to +\infty$ faster then $V(\nu_k)$, hence V admits a $|\cdot|$-norm-dependent lower bound as in Eq. (10).

Another choice of Lyapunov–Krasovskii functional satisfying condition (v) of Theorem 1 is

$$\hat{V}(\psi) = \int_0^{+\infty} |(T(t)\psi)(0)|^2 dt.$$

In addition, by choosing $\hat{\nu}_k(\theta) = \mathbb{R}(e^{\lambda_k\theta})$, we can easily verify that $\hat{V}(\hat{\nu}_k) \to 0$ as $k \to +\infty$ while $\hat{\nu}_k(0) = 1$ for every k. Hence $\hat{V}$ does not have a strictly positive $|\cdot|$-norm-dependent lower bound.

6 Robustness of the Lyapunov–Krasovskii Functional with Respect to Perturbations

In this section we are interested in understanding the effects of small perturbations of the dynamics on the stability of the system. The idea is to exploit the existence of a common Lyapunov–Krasovskii functional. We start by a simple and classical result whose proof is provided by completeness.

Lemma 1 *Let* $V : X \to \mathbb{R}$ *be the square of a norm such that* $V(x) \leq C\|x\|_X^2$ *for some* $C > 0$ *and for every* $x \in X$. *Then* V *is locally Lipschitz continuous and satisfies*

$$|V(x) - V(y)| \leq C(\|x\|_X + \|y\|_X)\|x - y\|_X.$$

Proof Since $\sqrt{V}$ is a norm, we can apply the triangular inequality so that

$$|\sqrt{V(x)} - \sqrt{V(y)}| \leq |\sqrt{V(x-y)}| \leq \sqrt{C}\|x - y\|_X.$$

Thus

$$|V(x) - V(y)| = (\sqrt{V(x)} + \sqrt{V(y)})|\sqrt{V(x)} - \sqrt{V(y)}| \leq C(\|x\|_X + \|y\|_X)\|x - y\|_X.$$

□

Let Q, P be a bounded subsets of $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$. Here P has to be regarded as a set of bounded perturbations of the operators in Q. We are going to consider the case in which the system

$$\Sigma : \quad \dot{x}(t) = \Gamma(t)x_t \quad \Gamma(t) \in Q \tag{16}$$

is uniformly exponentially stable and to investigate the stability of the perturbed system

$$\Sigma_p : \quad \dot{x}(t) = (\Gamma(t) + \Lambda(t))x_t \quad \Gamma(t) \in Q, \ \Lambda(t) \in P. \tag{17}$$

Let $c = \sup_{\Lambda \in P} \|\Lambda\|_{\mathcal{L}(C([-r,0],\mathbb{R}^n),\mathbb{R}^n)}$ and let m as in (2). Before proceedings let us recall an exponential boundedness property from [5].

Lemma 2 *Let* $Q \subset \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$ *be bounded by a constant* λ. *Then the solutions of (3) with initial condition* $\psi \in C([-r, 0], \mathbb{R}^n)$ *satisfy the estimate* $\|x_t\|_C \leq e^{\lambda t}\|\psi\|_C$.

We then have the following result.

Lemma 3 *For every* $t > 0$ *we have*

$$\|T_{\Gamma(\cdot)}(t) - T_{\Gamma(\cdot)+\Lambda(\cdot)}(t)\|_{\mathcal{L}(X)} \leq C(t),$$

where $C(t) = c\dfrac{e^{(2m+c)t} - e^{mt}}{m+c}$.

Proof Let $\psi \in C([-r,0],\mathbb{R}^n)$ and let $y(\cdot) = x^{\Gamma}(\cdot) - x^{\Gamma+\Lambda}(\cdot)$, where x^{Γ} and $x^{\Gamma+\Lambda}$ are, respectively, the solutions of Σ and Σ_p with initial condition ψ. For $t \geq 0$ and $\theta \in [-r,0]$ we have

$$\begin{aligned}
|y(t+\theta)| &= \int_0^{\max\{0,t+\theta\}} |\Gamma(s)x_s^{\Gamma} - (\Gamma(s)+\Lambda(s))x_s^{\Gamma+\Lambda}|ds \\
&\leq \int_0^{\max\{0,t+\theta\}} |\Gamma(s)y_s|ds + \int_0^{\max\{0,t+\theta\}} |\Lambda(s)x_s^{\Gamma+\Lambda}|ds \\
&\leq m\int_0^{\max\{0,t+\theta\}} \|y_s\|_C ds + c\int_0^{\max\{0,t+\theta\}} \|x_s^{\Gamma+\Lambda}\|_C ds \\
&\leq m\int_0^{\max\{0,t+\theta\}} \|y_s\|_C ds + c\int_0^{\max\{0,t+\theta\}} \|\psi\|_C e^{(m+c)s} ds \\
&\leq m\int_0^{\max\{0,t+\theta\}} \|y_s\|_C ds + c\frac{e^{(m+c)t}-1}{m+c}\|\psi\|_C.
\end{aligned}$$

This leads to

$$\|y_t\|_C \leq m\int_0^t \|y_s\|_C ds + c\frac{e^{(m+c)t}-1}{m+c}\|\psi\|_C,$$

from which we obtain, thanks to Gronwall's Lemma

$$\|y_t\|_C \leq c\frac{e^{(2m+c)t}-e^{mt}}{m+c}\|\psi\|_C.$$

Thus, we get the thesis. □

Corollary 1 *Let* $V : C([-r,0],\mathbb{R}^n) \to \mathbb{R}$ *be the square of a norm such that* $V(\psi) \leq \bar{c}\|\psi\|_X^2$ *for some* $\bar{c} > 0$ *and for every* $\psi \in C([-r,0],\mathbb{R}^n)$. *Then*

$$\overline{D}_{L+\Lambda}V(\psi) \leq \overline{D}_L V(\psi) + 2\bar{c}c\|\psi\|_C^2 \quad \forall L \in Q,\ \forall \Lambda \in P,\ \forall \psi \in C([-r,0],\mathbb{R}^n).$$

Proof Thanks to the previous lemmas we have

$$\begin{aligned}
|V(T_{L+\Lambda}(t)\psi) - V(T_L(t)\psi)| &\leq \bar{c}(\|T_{L+\Lambda}(t)\psi\|_C + \|T_L(t)\psi\|_C)\|T_{L+\Lambda}(t)\psi - T_L(t)\psi\|_C \\
&\leq \bar{c}e^{mt}(1+e^{ct})\|\psi\|_C\|T_{L+\Lambda}(t)\psi - T_L(t)\psi\|_C \\
&\leq 2\bar{c}C(t)e^{(m+c)t}\|\psi\|_C^2.
\end{aligned}$$

Since $C(0) = 0$ and $C'(0) = c$ we have

$$\overline{D}_{L+\Lambda}V(\psi) \leq \limsup_{t\to 0} \frac{|V(T_{L+\Lambda}(t)\psi) - V(T_L(t)\psi)| + |V(T_L(t)\psi) - V(\psi)|}{t}$$
$$\leq 2\bar{c}c\|\psi\|_C^2 + \overline{D}_L V(\psi).$$

□

6.1 Example

We provide here an example of application of the previous result. This example may be found in [5] although, in that paper, the stability proof contains some imprecision that we fix below.

Consider the scalar time-varying delay system

$$\dot{x}(t) = -x(t - \tau(t)), \tag{18}$$

where $\tau(\cdot)$ is piecewise constant and takes values in $[0, r]$. System (18) is of the type (3) with $Q = \{L_\tau \mid \tau \in [0, r]\}$ and $L_\tau\varphi = -\varphi(-\tau)$. It is known that (18) is uniformly exponentially stable in $C([-r, 0], \mathbb{R})$ if and only if $r < 3/2$ [13, 15]. Fix $r < 3/2$ and consider the perturbed system

$$\dot{x}(t) = -x(t - \tau(t)) + \int_{-\bar{r}}^{0} a(s)x(t+s)ds, \tag{19}$$

where $a \in L^1([-\bar{r}, 0], \mathbb{R})$ and $\bar{r} \geq r$ (notice that $\bar{r}$ may be larger than $3/2$). We claim that, as a straightforward consequence of our converse Lyapunov–Krasovskii theorem and of Corollary 1, system (19) is uniformly exponentially stable in $C([-\bar{r}, 0], \mathbb{R})$ if $\|a\|_{L^1}$ is small enough. Let $\Lambda : C([-\bar{r}, 0], \mathbb{R}) \to \mathbb{R}$ be defined by

$$\Lambda\psi = \int_{-\bar{r}}^{0} a(s)\psi(s)ds$$

and notice that $|\Lambda\psi| \leq \|a\|_{L^1}\|\psi\|_C$. Hence,

$$\overline{D}_{L+\Lambda}V(\psi) \leq \overline{D}_L V(\psi) + 2\bar{c}\|a\|_{L^1}\|\psi\|_C^2 \leq (-1 + 2\bar{c}\|a\|_{L^1})\|\psi\|_C^2,$$

where V is as in Theorem 1, Item (iii). If $\|a\|_{L^1} < 1/(2\bar{c})$ then (19) is uniformly exponentially stable in $C([-\bar{r}, 0], \mathbb{R})$.

7 Stability of Interconnected Uncertain Time-Varying Delay Systems

In this section we show the stability of interconnected uncertain time-varying delay systems under a small-gain condition by taking advantage of the previous results. For non-uncertain interconnected delay systems, and in presence of external inputs, a thorough study of the (input-to-state) stability properties under small-gain conditions may be found in [9]. For $i \in \{1, 2\}$, let Q_i be a bounded subset of $\mathcal{L}(C([-r, 0], \mathbb{R}^{n_1}) \times C([-r, 0], \mathbb{R}^{n_2}), \mathbb{R}^{n_i})$ (with bound m_i in the sense of condition (2)) and consider the linear uncertain time-varying delay systems

$$\Sigma_1 : \begin{cases} \dot{x}(t) = \Gamma_1(t)(x_t, 0), \ \Gamma_1(t) \in Q_1, \\ x_0 \quad = \varphi_1 \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{y}(t) = \Gamma_2(t)(0, y_t), \ \Gamma_2(t) \in Q_2, \\ y_0 \quad = \varphi_2, \end{cases}$$

and the interconnected system

$$\Sigma : \begin{cases} \dot{z}(t) = \Gamma(t) z_t, \ \Gamma(t) \in Q_1 \times Q_2, \\ z_0 \quad = \varphi, \end{cases}$$

where $z = (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $\varphi = (\varphi_1, \varphi_2) \in C([-r, 0], \mathbb{R}^{n_1}) \times C([-r, 0], \mathbb{R}^{n_2})$.

Theorem 4 *Let $X = X_1 \times X_2$ with $X_1 = C([-r, 0], \mathbb{R}^{n_1})$ and $X_2 = C([-r, 0], \mathbb{R}^{n_2})$. Assume that Σ_1 and Σ_2 are uniformly exponentially stable in X_1 and X_2, respectively. Let $V_i : X_i \to [0, \infty)$, $i \in \{1, 2\}$, be the Lyapunov–Krasovskii functional whose existence is guaranteed by Item (iii) of Theorem 1. Let $\bar{c}_1, \bar{c}_2$ be the upper-bound constants for V_1 and V_2, respectively. Let*

$$\mu = \sup\{\|(L_1(0, \psi_2), L_2(\psi_1, 0))\| / \|\psi\|_X \mid L_1 \in Q_1, \ L_2 \in Q_2, \ \psi \in X, \ \psi \neq 0\}.$$

If $2 \max(\bar{c}_1, \bar{c}_2)\mu < 1$ then the interconnected system Σ is uniformly exponentially stable in X.

Proof Let us introduce the function $V : X \to [0, \infty)$, defined by

$$V(\psi) = V_1(\psi_1) + V_2(\psi_2), \qquad \psi = (\psi_1, \psi_2) \in X.$$

For every $L = (L_1, L_2) \in Q_1 \times Q_2$, let us set $L_{\text{diag}}(\psi) = (L_1(\psi_1, 0), L_2(0, \psi_2))$ and notice that, by definition of V_1 and V_2,

$$\overline{D}_{L_{\text{diag}}} V(\psi) \leq -\|\psi\|_X^2.$$

Moreover, V is a squared norm on X and $V(\psi) \leq \max(\bar{c}_1, \bar{c}_2)\|\psi\|_X^2$.

Since μ is an upper bound for the norm of $L - L_{\text{diag}}$, $L \in Q_1 \times Q_2$, then, thanks to Corollary 1, the upper Dini derivative of V along Σ can be bounded by

$$\overline{D}_L V(\psi) \leq \overline{D}_{L_{\mathrm{diag}}} V(\psi) + 2\max(\bar{c}_1, \bar{c}_2)\mu \|\psi\|_X^2 \leq (-1 + 2\max(\bar{c}_1, \bar{c}_2)\mu)\|\psi\|_X^2.$$

The conclusion then follows from Theorem 1. □

In the case of systems in cascade form, there is no need of imposing conditions on the non-diagonal block of the operator Γ, as stated below.

Corollary 2 *Assume that system Σ is in cascade form, namely $L(\phi_1, \phi_2) = L(0, \phi_2)$ for any $L \in Q_2$ and $(\phi_1, \phi_2) \in X$. Then Σ is uniformly exponentially stable in X if and only if Σ_1, Σ_2 are uniformly exponentially stable in X_1 and X_2, respectively.*

Proof Rewriting system Σ with respect to new coordinates $(\hat{x}, \hat{y}) = (x, \eta y)$ in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ leads to a further uncertain system in cascade form where the sets of operators corresponding to the diagonal blocks of Γ remain unchanged, while the operators corresponding to the upper-right block are multiplied by a factor $1/\eta$ with respect to the original system. Thus, if Σ_1, Σ_2 are uniformly exponentially stable in X_1 and X_2, in order to guarantee that the functional V constructed in the proof of Theorem 4 is a common Lyapunov–Krasovskii functional for Σ it is enough to take $\eta > 2\max(\bar{c}_1, \bar{c}_2)\mu$. Thus Σ is also uniformly exponentially stable in X. The opposite implication is trivial. □

8 Conclusion

Using the switched systems approach, we give a collection of converse Lyapunov–Krasovskii theorems for uncertain linear time-varying delay systems. These results are summarized by Theorem 1. One of the novelties of our results is that they use Lyapunov–Krasovskii functionals which may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature. The effect of small perturbations in the dynamics on the existence of common Lyapunov–Krasovskii functionals guaranteeing the stability of system (1) is studied, in the case when $X = C([-r, 0], \mathbb{R}^n)$. This is given by Corollary 1. Thanks to Theorem 1 and Corollary 1, sufficient conditions for the stability of interconnected uncertain linear time-varying delay systems are given by Theorem 4 and Corollary 2.

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ISS-Stabilization of Delayed Neural Fields by Small-Gain Arguments

Antoine Chaillet, Georgios Is. Detorakis, Stéphane Palfi and Suhan Senova

1 Introduction

Recent technological advances open the way for real-time control of neuronal populations. Progresses have been made both in the spatio-temporal measurement of neuronal activity, with miniaturization of sensing electrodes, and in the actuation of neurons, for instance with the advent of opogenetics which allows stimulation of targeted neurons by light impulses using a gene transfer technology [5, 30].

Real-time control of neuronal populations is of significant interest not only to decipher brain functioning, but also to develop innovative treatments. In the particular case of Parkinson's disease, abnormal oscillations in deep brain structures are known to be related to motor symptoms [17]. Although the precise mechanisms by which these pathological oscillations are generated are still debated, a possible explanation is the increase of synaptic strength between two brain structures, the subthalamic nucleus (STN) and the external globus pallidus (GPe). This disproportioned synaptic gains, combined with axonal propagation delays, could result in some instability, leading to sustained oscillations in these structures [28, 29, 34].

A. Chaillet (✉) · G. I. Detorakis
L2S - CentraleSupélec - Université Paris Sud - CNRS, University of Paris Saclay, Gif sur Yvette, France
e-mail: antoine.chaillet@l2s.centralesupelec.fr

G. I. Detorakis
e-mail: gdetor@gmail.com

S. Palfi · S. Senova
APHP H. Mondor Hospital, INSERM U955 équipe 14, University Paris Est - Faculté de Médcine, Crétil, France
e-mail: stephane.palfi@aphp.fr

S. Senova
e-mail: yannsuhansenova@gmail.com

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The possibility to disrupt pathological oscillations by relying on real-time measurements of brain activity, has been the subject of an intense research lately: see [7] for a survey. Considered approaches include adaptive and on-demand stimulation [15, 22, 24, 36, 37], delayed and multi-site stimulation [3, 23, 32, 40], optimal control strategies [13], and activity regulation [14, 16, 44].

In particular, a filtered proportional strategy was proposed in [16] in a model of the STN-GPe network by relying on linearization around an equilibrium. Nonetheless, the model employed in that reference abstracts the activity in these structures by a single averaged signal, that does no take into account spatial heterogeneity. In order to benefit from real-time spatiotemporal information in the targeted structures, and to model the dynamics involved more finely, this model has been extended in [10] by relying on delayed neural fields.

Neural fields are nonlinear integro-differential equations designed to model the spatio-temporal evolution of neuronal populations. They offer a good compromise between physiological plausibility, richness of behaviors, and analytical tractability. They have been the subject of an intense research, with a wide range of applications to neuroscience: see [6] for a detailed survey. From an analytical point of view, several works have been devoted to the existence and estimation of equilibrium patterns, local and global stability analysis, and bifurcation analysis of neural fields: see e.g. [4, 11, 21, 33, 42]. When taking into account non-instantaneous communication between neurons, the resulting model is known as delayed neural fields [1, 2, 43].

The purpose of this chapter is to provide conditions under which the activity of delayed neural fields can be robustly stabilized by proportional feedback. To that aim, we consider two subpopulations. The first one, denoted as the "controlled population" receives direct influence from the stimulation device and its activity is assumed to be measured in real time. The second one, the "uncontrolled population", is not accessible to measurements and receives no direct input from the stimulation device. The main result of this chapter is to show that oscillations of such delayed neural fields can always been disrupted by proportional feedback provided that the spatial L_2-norm of the inner synaptic weights within the uncontrolled population is below an explicit threshold.

In order to establish this result, we rely on input-to-state stability (ISS) arguments. ISS, introduced in [38], not only ensures global asymptotic stability in the absence of perturbations, but also guarantees robustness to exogenous disturbances: see [39] for a survey. Some recent works have contributed to extend ISS to infinite-dimensional systems [9, 19, 27, 35], including retarded functional differential equations [20, 25, 31, 41]. In line with these works, we provide an extension of ISS especially suited to the study of delayed neural fields, together with small-gain results that will be instrumental for Sect. 3, where we provide a condition for robust stabilizability of neural fields by a proportional feedback that relies on measurements of the controlled population only. Proofs are provided in Sect. 4. We conclude by listing some lines of future works in Sect. 5. An extended version of this work was published in [8].

Notation. Given $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm: $|x| := \sqrt{x_1^2 + \dots + x_n^2}$. Given two sets Ω_1 and Ω_2, $C(\Omega_1, \Omega_2)$ denotes the set of all continuous functions from Ω_1 to Ω_2. $L_2(\Omega_1, \Omega_2)$ denotes the set of all square integrable functions from Ω_1 to Ω_2, meaning all functions $f : \Omega_1 \to \Omega_2$ such that $\int_{\Omega_1} |f(s)|^2 ds < \infty$. Given a set $\Omega \subset \mathbb{R}^q$, $\#_\Omega$ denotes its Lebesgue measure. Given $f : \Omega \to \mathbb{R}^n$, we write $\int_\Omega f(r)dr$ to denote the multiple integral $\int_\Omega \dots \int_\Omega f(r)dr_1 \dots dr_q$, with $r =: (r_1, \dots, r_q)^T$. We define $\mathcal{F}^n := L_2(\Omega, \mathbb{R}^n)$ and $C^n := C([-\bar{d}; 0], \mathcal{F}^n)$ for some constant $\bar{d} > 0$. $\mathcal{F}^n$ is a Banach space for the L_2-norm $\|\cdot\|_{\mathcal{F}^n}$ defined as $\|x\|_{\mathcal{F}^n} := \sqrt{\int_\Omega |x(r)|^2 dr}$ for each $x \in \mathcal{F}^n$. Similarly, C^n is a Banach space for the norm $\|\cdot\|_{C^n}$ defined as $\|x\|_{C^n} := \sup_{t\in[-\bar{d};0]} \|x(t)\|_{\mathcal{F}^n}$ for all $x \in C^n$. We also denote by $\mathcal{U}^n$ the set of all measurable locally bounded functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{F}^n$. When the context is sufficiently clear, we simply refer to $\mathcal{F}^n$, C^n and $\mathcal{U}^n$ as $\mathcal{F}$, C and $\mathcal{U}$ respectively. Given $x \in C$ and $t \in [-\bar{d}; +\infty)$, we indicate by $[x(t)](r)$ the value taken by the function $x(t) \in \mathcal{F}$ at position $r \in \Omega$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if its is continuous, zero at zero and increasing. It is said to be of class $\mathcal{K}_\infty$ if it satisfies additionally $\lim_{s\to+\infty} \alpha(s) = +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $s \mapsto \beta(s, t)$ is of class $\mathcal{K}$ for any fixed $t \in \mathbb{R}_{\geq 0}$ and, for any fixed $s \in \mathbb{R}_{\geq 0}$, $t \mapsto \beta(s, t)$ is continuous and non-increasing and tends to zero as its argument tends to $+\infty$.

2 ISS for Spatio-Temporal Dynamics

We start by considering generic delayed spatio-temporal dynamics ruled by a functional differential equation of the form

$$\dot{x}(t) = f(x_t, p(t)), \tag{1}$$

where $f : C^n \times \mathcal{F}^m \to \mathcal{F}^n$ and $p \in \mathcal{U}^m$. $x(t) \in \mathcal{F}^n$ represents the state of this system: at each time instant t, it is a *function* of the space variable rather than a single point of $\mathbb{R}^n$. $x_t \in C^n$ represents the history of this function over the latest time interval of length $\bar{d}$; in other words, for each fixed $\theta \in [-\bar{d}; 0]$, $x_t(\theta) := x(t + \theta)$ is a function (of the space variable) belonging to $\mathcal{F}^n$. A natural extension of input-to-state stability for this class of systems is as follows.

Definition 1 (*ISS*) The system (1) is *input-to-state stable (ISS)* if there exist $\beta \in \mathcal{KL}$ and $\nu \in \mathcal{K}_\infty$ such that, for any initial condition $x_0 \in C$ and any input $p \in \mathcal{U}$, the system (1) admits a unique solution defined over $[-\bar{d}; +\infty)$ and it satisfies

$$\|x(t)\|_{\mathcal{F}} \leq \beta(\|x_0\|_C, t) + \nu\left(\sup_{\tau\geq 0} \|p(\tau)\|_{\mathcal{F}}\right), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

ν is then called an *ISS gain* for (1).

This definition of ISS is a particular case of the very general input-to-output stability notion introduced in [19], which applies to a wide class of systems, satisfying a weak semigroup property and encompassing in particular retarded functional differential equations, difference equations, and hybrid dynamics.

Definition 1 is also consistent with the one introduced for delayed systems in [25, 31]. The main difference stands in the fact that, in those papers, no spatial evolution is considered. As a consequence, the left hand side of the above estimate involves the L_2-norm of the solution over the spatial domain (i.e. the $\mathcal{F}$-norm), rather than a Euclidean norm of the state as in the original finite-dimensional delay-free case [38, 39]. For the same reason, the magnitude of the initial condition is here estimated through the C-norm rather than a supremum norm.

The above property is also similar to the ISS property for infinite dimensional systems in Banach spaces introduced in [9] and extensively studied in [26]. The main difference stands in the fact that the state estimate in the above definition involves the $\mathcal{F}$-norm of the state as a function of the C-norm of the initial condition, whereas the ISS notion used in [9] would involve the C-norm on both sides of this estimate. It was shown in [26, Proposition 1.4.2] that these two estimates are qualitatively equivalent. Nevertheless, we have preferred this two-norm estimate as the corresponding Lyapunov–Krasovskii results presented below will rely on each of these two norms.

Similarly to all those related notions of ISS, the above property ensures global asymptotic stability for the disturbance-free system, namely:

$$p \equiv 0 \quad \Rightarrow \quad \|x(t)\|_{\mathcal{F}} \leq \beta(\|x_0\|_C, t), \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{2}$$

In particular, when no exogenous input applies, the $\mathcal{F}$-norm of the solution asymptotically converges to zero (attractiveness), and its amplitude is arbitrarily small at all times provided that $\|x_0\|_C$ is small enough (stability). ISS also ensures the following *asymptotic gain property*:

$$\limsup_{t\to+\infty} \|x(t)\|_{\mathcal{F}} \leq \nu\left(\limsup_{t\to+\infty} \|p(t)\|_{\mathcal{F}}\right). \tag{3}$$

This means that the steady-state error in the state, measured through its $\mathcal{F}$-norm, results solely from the asymptotic value of the applied input's $\mathcal{F}$-norm and is "proportional" to it, up to the scaling factor ν. This in turn shows that the original "converging input-converging state" and "bounded input-bounded state" properties [39] are also preserved.

Lyapunov–Krasovskii Condition for ISS

We now present an extension to the class of systems (1) of the Lyapunov–Krasovskii condition for ISS. In line with [12], it relies on the upper right Dini derivative of a functional $V \in C(C, \mathbb{R}_{\geq 0})$, Lipschitz on each bounded set of C, along the solutions of (1), as defined by $\dot{V}|_{(5.1)} := \limsup_{h\to 0^+} \frac{V(x_{t+h})-V(x_t)}{h}$, where $x(\cdot)$ denotes any solution of (1).

Theorem 1 (Sufficient condition for ISS) *Let $\underline{\alpha}, \overline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ and let $V \in C(C, \mathbb{R}_{\geq 0})$ be Lipschitz on any bounded set of C. Assume that, given any initial condition $x_0 \in C$ and any input $p \in \mathcal{U}$, the system (1) admits a unique solution defined over $[-\bar{d}; +\infty)$ and satisfying, for almost all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}(\|x(t)\|_{\mathcal{F}}) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|_C) \tag{4}$$

$$V(x_t) \geq \gamma(\|p(t)\|_{\mathcal{F}}) \quad \Rightarrow \quad \dot{V}|_{(5.1)} \leq -\alpha(V(x_t)). \tag{5}$$

Then the system (1) is ISS with gain $\underline{\alpha}^{-1} \circ \gamma$.

This result is a spatiotemporal extension of [31, Theorem 3.1]. Its proof is omitted as it follows along the same lines as the ISS Lyapunov sufficient condition for finite-dimensional systems originally proposed in [38]. A similar sufficient condition for ISS of infinite-dimensional systems was proposed in [9], and further extended to integral ISS in [27]. The main difference with Theorem 1 stands in the fact that, in those works, the Lyapunov functional V is assumed to be upper and lower-bounded by a function of $\|x_t\|_C$ (whereas (4) requires only a lower-bound involving $\|x(t)\|_{\mathcal{F}}$). Such a lower bound turns out to be handier for the analysis of delayed neural fields, as will be illustrated in Sect. 3.

ISS Small-Gain Theorem

We next address the robust stability of a feedback interconnection by extending the ISS small-gain theorem to systems as in (1). More precisely, we consider feedback systems of the form

$$\dot{x}_1(t) = f_1(x_{1t}, x_{2t}, p_1(t)) \tag{6a}$$

$$\dot{x}_2(t) = f_2(x_{2t}, x_{1t}, p_2(t)), \tag{6b}$$

where $f_1 : C^{n_1} \times C^{n_2} \times \mathcal{F}^{m_1} \to \mathcal{F}^{n_1}$, $f_2 : C^{n_2} \times C^{n_1} \times \mathcal{F}^{m_2} \to \mathcal{F}^{n_2}$, $p_1 \in \mathcal{U}^{m_1}$ and $p_2 \in \mathcal{U}^{m_2}$. For these systems, we have the following result.

Theorem 2 (ISS small gain) *For each $i \in \{1, 2\}$, let $\underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \gamma_i, \chi_i \in \mathcal{K}_\infty$ and $V_i \in C(C^{n_i}, \mathbb{R}_{\geq 0})$ be Lipschitz on each bounded set of C^{n_i}. Assume that, given any $x_{i0} \in C^{n_i}$ and any $p_i \in \mathcal{U}^{m_i}$, the system (6) admits a unique solution defined over $[-\bar{d}; +\infty)$ and satisfying, for almost all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_i(\|x_i(t)\|_{\mathcal{F}}) \leq V_i(x_{it}) \leq \overline{\alpha}_i(\|x_{it}\|_C) \tag{7}$$

$$V_1 \geq \max\{\chi_1(V_2), \gamma_1(\|p_1(t)\|_{\mathcal{F}})\} \;\Rightarrow\; \dot{V}_1^{(6a)} \leq -\alpha_1(V_1)$$

$$V_2 \geq \max\{\chi_2(V_1), \gamma_2(\|p_2(t)\|_{\mathcal{F}})\} \;\Rightarrow\; \dot{V}_2^{(6b)} \leq -\alpha_2(V_2).$$

Then, under the small-gain condition $\chi_1 \circ \chi_2(s) < s$ for all $s > 0$, the feedback interconnection (6) is ISS.

This result, whose proof is omitted, is a natural extension of the main result in [31] to spatiotemporal dynamics, which was already a generalization to delayed dynamics of the ISS small-gain theorem provided in [18]. It shares strong similarities with the small-gain result in [9], which goes beyond Theorem 2 by allowing for the interconnection of more than two subsystems. As stressed before, the main difference with that work stands in the fact that the Lyapunov function associated to each subsystem is here required to be lower-bounded by a function of $\|x(t)\|_{\mathcal{F}}$ only (rather than $\|x_t\|_C$). A very general small-gain theorem for systems defined in Banach spaces was also provided in [19]. Nonetheless, the small-gain condition presented there relies on nonlinear gains involved in the estimate of the evolution of the Lyapunov functionals V_i rather than those involved in their dissipation inequalities. Estimating these nonlinear gains would thus require to first integrate the dissipation inequalities assumed in the statement of Theorem 2

In some situations, the positive terms appearing in the dissipation inequalities cannot be easily expressed in terms of the Lyapunov functional of the other subsystem. The following result, proved in Sect. 4, can help tackling this situation.

Corollary 1 *For each $i \in \{1, 2\}$, let $\underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \gamma_i, \chi_i \in \mathcal{K}_\infty$, $V_i, W_i \in C(C^{n_i}, \mathbb{R}_{\geq 0})$ be Lipschitz on any bounded set of C^{n_i}, and $k_{12}, k_{21} \geq 0$. Assume that, given any initial condition $x_{i0} \in C^{n_i}$ and any input $p_i \in \mathcal{U}^{m_i}$, the system (6) admits a unique solution defined over $[-\bar{d}; +\infty)$ and satisfying, for almost all $t \in \mathbb{R}_{\geq 0}$, both (7) and*

$$\dot{V}_1^{(6a)} \leq -\alpha_1(V_1) - W_1 + k_{12}W_2 + \gamma_1(\|p_1(t)\|_{\mathcal{F}})$$
$$\dot{V}_2^{(6b)} \leq -\alpha_2(V_2) - W_2 + k_{21}W_1 + \gamma_2(\|p_2(t)\|_{\mathcal{F}}).$$

Then, under the condition $k_{12}k_{21} \leq 1$, the feedback interconnection (6) is ISS.

3 Stabilization of Delayed Neural Fields

Based on the above ISS framework for the study of robust stability of interconnected delayed spatiotemporal dynamics, we now address stabilization of delayed neural fields with limited actuation and measurement.

Closed-Loop System

We consider a delayed neural field made of two interconnected subpopulations. The controlled population, denoted with index 1, directly receives the stimulation signal:

$$\begin{aligned}\tau_1 \frac{\partial z_1}{\partial t}(r,t) = &- z_1(r,t) \\ &+ \mathcal{S}_1\left(\sum_{j=1}^{2}\int_\Omega w_{ij}(r,r')z_j(r',t-d_j(r,r'))dr' + I_1(r,t) + \alpha(r)u(r,t)\right).\end{aligned} \tag{8}$$

On the other hand, the uncontrolled population, indexed by 2, is not directly influenced by the stimulation signal u:

$$\tau_2 \frac{\partial z_2}{\partial t}(r,t) = -z_2(r,t) + \mathcal{S}_2\left(\sum_{j=1}^{2}\int_\Omega w_{2j}(r,r')z_j(r',t-d_j(r,r'))dr' + I_2(r,t)\right). \quad (9)$$

For each each $i \in \{1, 2\}$, $z_i(r,t) \in \mathbb{R}$ represents the neuronal activity of population i, at position $r \in \Omega$ and at time $t \geq 0$. Ω denotes a set of $\mathbb{R}^p$, $p \in \{1, 2, 3\}$, representing the physical support of the populations; throughout this chapter, we will assume that Ω is compact. $\tau_i > 0$ is the time decay constant of the activity of population i. For each $j \in \{1, 2\}$, the kernel $w_{ij} : \Omega \times \Omega \to \mathbb{R}$ is a bounded function describing the synaptic strength between location r' in population j and location r in population i. $I_i : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a bounded function describing the external input of population i, arising from the influence of exogenous cerebral structures. The function $d_i : \Omega \times \Omega \to [0; \bar{d}]$, $\bar{d} \geq 0$, is a continuous function representing the axonal, dendritic and synaptic delays between a pre-synaptic neuron at position r' in population i and a post-synaptic neuron at position r. $\mathcal{S}_i : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz function, known as the activation function of the neural population i. Finally, $\alpha : \Omega \to \mathbb{R}_{\geq 0}$ is a function describing the impact of the stimulation signal u at each point $r \in \Omega$. We refer the reader to [6] for further details on neural fields dynamics, and to [10] for specific comments on this model.

As seen in [10] through simulations, (8)–(9) can produce sustained oscillations of the kernel w_{ij} are two intense. Probably the simplest closed-loop strategy one could think of to stabilize it by relying on z_1-measurements only is proportional control:

$$u(r,t) = -k\left(z_1(r,t) - z_{ref}(r)\right), \quad (10)$$

where k is a positive control gain and $z_{ref} : \Omega \to \mathbb{R}$ represents a prescribed rate for the controlled population. A similar strategy was adopted in [16] on an averaged model (i.e. neglecting spatial dependency) by relying on linearization techniques.

Up to a change of variables aiming at placing the considered equilibrium at the origin, the dynamics of the overall population (8)–(9) under the influence of the closed-loop stimulation signal (10) reads

$$\tau_1[\dot{x}_1(t)](r) = -[x_1(t)](r) + S_1\Big(r, -k\alpha(r)[x_1(t)](r) + \sum_{j=1}^{2}\int_\Omega w_{1j}(r,r')[x_j(t-d_j)](r')dr' + [p_1(t)](r)\Big) \quad (11a)$$

$$\tau_2[\dot{x}_2(t)](r) = -[x_2(t)](r) + S_2\Big(r, \sum_{j=1}^{2}\int_\Omega w_{2j}(r,r')[x_j(t-d_j)](r')dr' + [p_2(t)](r)\Big), \quad (11b)$$

for some functions $S_i : \Omega \times \mathbb{R} \to \mathbb{R}$. With this notation, (12) is in the form of (1) with state $x = (x_1, x_2) : [-\bar{d}, +\infty) \to \mathcal{F}^2$ and input $p := (p_1, p_2) \in \mathcal{U}^2$. Letting $\ell_i > 0$ denote the Lipschitz constant of $\mathcal{S}_i$, it can be seen that

$$|S_i(r,a) - S_i(r,b)| \leq \ell_i |a-b|, \quad \forall r \in \Omega, \forall a, b \in \mathbb{R}. \tag{12}$$

From now on, such functions S_i are called *activation functions with slope* ℓ_i. Since $S_i(r,0) = 0$ for all $r \in \Omega$ and $\mathcal{S}_i$ is non-decreasing, it can also be shown that

$$|S_i(r,a)| \leq \ell_i |a|, \quad \forall a \in \mathbb{R}. \tag{13}$$

$$S_i(r, a+b) \leq S_i(r, 2a) + S_i(r, 2b), \quad \forall a, b \in \mathbb{R}, \tag{14}$$

for each $r \in \Omega$. We stress that, reasoning as in the proof of [12, Theorem 3.2.1], it can be seen that the solutions of (11b) exist at all times $t \geq 0$, are unique, and are continuously differentiable over $[0; +\infty)$, provided that the kernels w_{ij} and initial states are in L_2, delays d_j are continuous, and inputs $[p_j(t)](\cdot)$ are continuous. This observation simplifies the analysis provided below by replacing the Dini derivative of Lyapunov–Krasovskii functionals by their classical time derivative.

ISS of the Uncontrolled Population

We start by providing conditions under which the uncontrolled population (11b) is ISS with respect to inputs arising from the controlled population (11a) and possibly exogenous structures. To that aim, we analyze the neural fields:

$$\tau[\dot{x}(t)](r) = -[x(t)](r) + S\left(r, \int_\Omega w(r,r')[x(t-d(r,r'))](r')dr' + [\rho(t)](r)\right). \tag{15}$$

This class of uncontrolled delayed neural fields was extensively studied in [12, 43]. A bifurcation analysis was also conducted in [1], under the additional requirement that the kernel w depends only on the distance $|r - r'|$, but allowing for higher order dynamics. From all these works, it is a known fact that the product of the Lipschitz constant ℓ of the activation function S by the square of the L_2-norm of the kernel w regulates the stability of the origin of (15) in the absence of inputs ρ: if this product is smaller than 1, then the origin is globally asymptotically stable: see [1, Theorem 2.1], [12, Theorem 4.2.3] or [43, Proposition 3.15]. Here we show that, under the same condition, the delayed neural fields (15) is actually ISS, and thus ensures robustness with respect to the input ρ. The proof is provided in Sect. 4.

Proposition 1 (ISS of the uncontrolled population) *Let S be any activation function with slope ℓ, let $\tau > 0$, $d : \Omega \times \Omega \to [0; \bar{d}]$, and $w : \Omega \times \Omega \to \mathbb{R}$ be any bounded functions satisfying*

$$\ell \int_\Omega \int_\Omega w(r,r')^2 dr' dr < 1. \tag{16}$$

Then (15) is ISS. Moreover, there exists $\beta : \Omega \to \mathbb{R}_{>0}$ and, for each $c > 0$ small enough, there exists $c_1, c_2 > 0$ such that the functional $V : C \to \mathbb{R}_{\geq 0}$ defined as

$$V(x_t) := \frac{\tau}{2}\int_\Omega [x(t)](r)^2 dr + \int_\Omega \beta(r)\int_\Omega \int_{-d(r,r')}^0 e^{c\theta}[x(t+\theta)](r')^2 d\theta dr' dr \quad (17)$$

satisfies, for all $x_0 \in C$, *all* $\rho \in \mathcal{U}$, *and all* $t \geq 0$,

$$\dot{V}|_{(5.15)} \leq -c_1 V - c_2 \int_\Omega \int_\Omega [x(t - d(r,r'))](r')^2 dr' dr + c_2 \|\rho(t)\|_{\mathcal{F}}^2.$$

We stress that the Krasovskii-Lyapunov functional in (17) cannot be lower-bounded by a $\mathcal{K}_\infty$ function of $\|x_t\|_C$, thus making the results in [9] inapplicable with this particular functional. The same observation holds for the functionals considered in Proposition 2 and in the proof of Proposition 3 below.

ISS of the Controlled Population

We now proceed to studying the controlled dynamics (11a). To this aim we consider the delayed neural fields:

$$\tau[\dot{x}(t)](r) = -[x(t)](r) + S\Big(r, \int_\Omega w(r,r')[x(t-d(r,r'))](r')dr' - k\alpha(r)[x(t)](r) + [\rho(t)](r)\Big). \quad (18)$$

The next proposition states that this system can be made ISS regardless of the strength of w by picking a sufficiently large gain k.

Proposition 2 (ISS of the controlled population) *Let* S *be any activation function and* $\tau > 0$, *and let* $\alpha : \Omega \to \mathbb{R}_{>0}$, $d : \Omega \to [0; \bar{d}]$, *and* $w : \Omega \times \Omega \to \mathbb{R}$ *be any bounded functions. Then there exist* $k^* > 0$ *such that, for all* $k \geq k^*$, *the delayed neural fields (18) is ISS and there exist* $c_1, c_2, c_3 > 0$, *independent of* k, *such that the derivative of the functional* $V : C \to \mathbb{R}_{\geq 0}$ *defined as*

$$V(x_t) := \frac{\tau}{2}\int_\Omega [x(t)](r)^2 dr + \frac{\tau}{2\#_\Omega}\int_\Omega \int_\Omega \int_{-d(r,r')}^0 e^{\theta}[x(t+\theta)](r')^2 d\theta dr' dr \quad (19)$$

satisfies, for all $x_0 \in C$, *all* $\rho \in \mathcal{U}$, *and all* $t \geq 0$,

$$\dot{V}|_{(5.18)} \leq -c_1 V - c_2 \int_\Omega \int_\Omega [x(t - d(r,r'))](r')^2 dr' dr + \frac{c_3}{k}\|\rho(t)\|_{\mathcal{F}}^2.$$

This result, proved in Sect. 4, not only states that the proportional feedback (10) allows to robustly stabilize the controlled population, but also that its ISS gain can be made arbitrarily small by picking k large enough. As we will see below, this feature is key in the stabilization of the overall interconnected neural fields.

Robust Stabilization of Coupled Neural Fields

Proposition 1 provides conditions under which the uncontrolled population is ISS. Proposition 2 shows that the controlled population can be made ISS by the proportional feedback (10) and that an arbitrarily small ISS gain can be assigned. Based on these two results, and invoking small-gain arguments, the following results provides conditions under which the overall closed-loop system (12) can be made ISS.

Theorem 3 (ISS of the whole population) *For each $i, j \in \{1, 2\}$, let S_i be any activation function, $\tau_i > 0$, and consider any bounded functions $w_{ij} : \Omega \times \Omega \to \mathbb{R}$, $d_i : \Omega \times \Omega \to [0, \bar{d}]$, and $\alpha : \Omega \to \mathbb{R}_{>0}$. Let ℓ_2 denote the slope of S_2 and assume that the internal kernel w_{22} of the uncontrolled population satisfies*

$$\ell_2 \int_\Omega \int_\Omega w_{22}(r, r')^2 dr' dr < 1. \tag{20}$$

Then there exists $k^* > 0$ such that, for any $k \geq k^*$, the coupled neural fields (12) is ISS.

This result establishes that robust stabilization of coupled delayed neural fields can be achieved by proportional feedback provided that the internal synaptic weights w_{22} within the uncontrolled population are reasonably low. Noticing that condition (20) ensures global asymptotic stability of the uncontrolled population in the absence of exogenous inputs (see Proposition 1), the above result states that proportional feedback may tackle any sustained oscillation resulting either from the interconnection between the two subpopulations or from a too strong coupling within the controlled population, but may fail at attenuating oscillations that endogenously take place within the uncontrolled population.

4 Proofs

***Proof of Corollary* 1**

Let $\sigma := \frac{k_{21}}{2} + \frac{1}{2k_{12}}$. Then, under the small-gain condition $k_{12}k_{21} \leq 1$, it holds that $k_{21} \leq \sigma \leq \frac{1}{k_{12}}$. Consider the functional defined as $V(x_t) := \sigma V_1(x_{1t}) + V_2(x_{2t})$, where $x_t := (x_{1t}, x_{2t})$. Then V satisfies (7) with $\underline{\alpha}(s) := \min\{\sigma \underline{\alpha}_1(s/\sqrt{2}); \underline{\alpha}_2(s/\sqrt{2})\}$ for all $s \in \mathbb{R}_{\geq 0}$ (as can be seen by considering separately the cases $\|x_1(t)\|_\mathcal{F} \geq \|x_2(t)\|_\mathcal{F}$ and $\|x_1(t)\|_\mathcal{F} \leq \|x_2(t)\|_\mathcal{F}$) and $\overline{\alpha} := \sigma\overline{\alpha}_1 + \overline{\alpha}_2$. Moreover, its derivative reads

$$\begin{aligned} \dot{V}^{(6)} =& \sigma \dot{V}_1^{(6a)} + \dot{V}_2^{(6b)} \\ \leq& -\sigma\alpha_1(V_1) - \alpha_2(V_2) - (\sigma - k_{21})W_1 - (1 - \sigma k_{12})W_2 + \sigma\gamma_1(\|p_1(t)\|_\mathcal{F}) + \gamma_2(\|p_2(t)\|_\mathcal{F}) \\ \leq& -\sigma\alpha_1(V_1) - \alpha_2(V_2) + \gamma(\|p(t)\|_\mathcal{F}), \end{aligned}$$

where $\gamma := \sigma\gamma_1 + \gamma_2$ is clearly a $\mathcal{K}_\infty$ function. Let $\alpha(s) := \min\{\sigma\alpha_1(s/2\sigma)\,;\, \alpha_2(s/2)\}$ for all $s \in \mathbb{R}_{\geq 0}$, then α is also a $\mathcal{K}_\infty$ function and it holds that $\alpha(V) = \alpha(\sigma V_1 + V_2) \leq \alpha(2\sigma V_1) + \alpha(2V_2) \leq \sigma\alpha_1(V_1) + \alpha_2(V_2)$. It follows that $\dot{V}^{(6)} \leq -\alpha(V) + \gamma(\|p(t)\|_\mathcal{F})$. In particular, the following implication holds:

$$V(x_t) \geq \alpha^{-1} \circ 2\gamma(\|p(t)\|_{\mathcal{F}}) \quad \Rightarrow \quad \dot{V}^{(6)} \leq -\frac{1}{2}\alpha(V(x_t)),$$

which concludes the proof in view of Theorem 1.

***Proof of Proposition* 1**

In order to avoid cumbersome notation, we will omit to write the arguments of some functions in the proof. In particular, it should be kept in mind that the delay d and the kernel w depend on both r and r'. Also, unless explicitly specified, we will denote $[x(t)](r)$ (resp. $[\rho(t)](r)$) as simply x (resp. ρ). In the same way, $S(r, x)$ will simply be denoted as $S(x)$. We decompose the functional V given in (17) as $V(x_t) = V_a(x(t)) + V_b(x_t)$ where

$$V_a(x(t)) := \frac{\tau}{2}\int_\Omega [x(t)](r)^2 dr \tag{21}$$

$$V_b(x_t) := \int_\Omega \beta(r) \int_\Omega \int_{-d(r,r')}^{0} e^{c\theta}[x(t+\theta)](r')^2 d\theta dr' dr. \tag{22}$$

With the variable change $\theta \leftarrow t + \theta$, the time derivative of V_b reads

$$\begin{aligned}\dot{V}_b|_{(5.15)} &= \int_\Omega \beta(r) \int_\Omega \left([x(t)](r')^2 - e^{-cd}[x(t-d)](r')^2\right) dr' dr - cV_b \\ &\leq \int_\Omega \beta(r) dr \int_\Omega [x(t)](r)^2 dr - e^{-c\bar{d}} \int_\Omega \int_\Omega [x(t-d(r,r'))](r')^2 dr' dr - cV_b.\end{aligned} \tag{23}$$

Furthermore, the derivative of V_a can be computed as follows. First, exploiting the fact that $|S(r,a)| \leq \ell|a|$ for all $a \in \mathbb{R}$ and all $r \in \Omega$, as recalled in (13), it holds that

$$\begin{aligned}\dot{V}_a|_{(5.15)} &\leq -\int_\Omega x^2 dr + \ell \int_\Omega |x| \left| \int_\Omega w x(t-d) dr' + \rho \right| dr \\ &\leq -\int_\Omega x^2 dr + \ell \int_\Omega \int_\Omega |x||w||x(t-d)| dr' dr + \ell \int_\Omega |x||\rho| dr.\end{aligned}$$

Using the fact $ab \leq \frac{1}{2}(\lambda a^2 + \frac{b^2}{\lambda})$ for all $a, b \geq 0$ and all $\lambda > 0$, it follows that for any functions $\lambda_1, \lambda_2 : \Omega \to \mathbb{R}_{>0}$,

$$\dot{V}_a|_{(5.15)} \leq -\int_\Omega \left(1 - \frac{\bar{w}}{2\lambda_1} - \frac{\ell}{2\lambda_2}\right) x^2 dr + \int_\Omega \frac{\ell\lambda_1}{2} \int_\Omega x(t-d)^2 dr' dr + \int_\Omega \frac{\ell\lambda_2}{2}\rho^2 dr, \tag{24}$$

where $\bar{w} : \Omega \to \mathbb{R}_{>0}$ denotes any function satisfying $\bar{w}(r) \geq \ell \int_\Omega w(r,r')^2 dr'$, for all $r \in \Omega$. Under condition (16), $\bar{w}$ can be picked in such a way that $\int_\Omega \bar{w}(r) dr < 1$. This can be seen by picking for instance $\bar{w}(r) = \frac{\varepsilon}{2\#_\Omega} + \ell \int_\Omega w(r,r')^2 dr'$, where $\varepsilon := 1 - \ell \int_\Omega \int_\Omega w(r,r')^2 dr' dr$. We claim that the proposition holds with $\beta(r) = \bar{w}(r)/2$.

With this choice, recalling that $V = V_a + V_b$, the combination of (23) and (24) yields

$$\dot{V}|_{(5.15)} \leq -\int_\Omega \Big(1 - \frac{\bar{w}}{2\lambda_1} - \frac{\ell}{2\lambda_2} - \int_\Omega \frac{\bar{w}}{2} dr'\Big) x^2 dr \\ -\int_\Omega \Big(e^{-c\bar{d}} - \frac{\ell\lambda_1}{2}\Big) \int_\Omega x(t-d)^2 dr' dr - cV_b + \int_\Omega \frac{\ell\lambda_2}{2} \rho^2 dr. \tag{25}$$

Thus, the result is proved if we can find positive functions λ_1, λ_2 and a constant $c > 0$ such that

$$\frac{\bar{w}}{2\lambda_1(r)} + \frac{\ell}{2\lambda_2(r)} + \int_\Omega \frac{\bar{w}(r')}{2} dr' < 1, \qquad e^{-c\bar{d}} > \frac{\ell\lambda_1(r)}{2}.$$

It can be checked that these conditions are fulfilled with any $c \leq \ln\left(\frac{1}{\ell\lambda_1(r)}\right)/\bar{d}$ if $\lambda_1(r) = \frac{\bar{w}(r)(3-\int_\Omega w(r')dr')}{2(2-\int_\Omega \bar{w}(r')dr')}$ and $\lambda_2(r) = \lambda_2^* > \frac{\ell}{2-\bar{w}(r)\lambda_1(r)-\int_\Omega \bar{w}(r')dr'}$. With these choices, it follows from (25) that there exist $c_1' > 0$ such that

$$\dot{V}|_{(5.15)} \leq -c_1' \int_\Omega x^2 dr - cV_b - \frac{e^{-c\bar{d}}}{2} \int_\Omega \int_\Omega x(t-d)^2 dr' dr + \frac{\ell\lambda_2}{2} \|\rho\|_{\mathcal{F}}^2 \\ \leq -\frac{2c_1'}{\tau} V_a - cV_b - \frac{e^{-c\bar{d}}}{2} \int_\Omega \int_\Omega x(t-d)^2 dr' dr + \frac{\ell\lambda_2}{2} \|\rho\|_{\mathcal{F}}^2.$$

The conclusion follows by letting $c_1 = \min\{2c_1'/\tau; c\}$, $c_2 = e^{-c\bar{d}}/2$, and $c_3 = \ell\lambda_2^*/2$.

***Proof of Proposition* 2**

For the sake of readability, we rely on the same notation simplifications as the ones used in the proof of Proposition 1 everywhere the context is explicit enough. We decompose the functional V given in (19) as $V(x_t) = V_a(x(t)) + V_b(x_t)$ where

$$V_a(x(t)) := \frac{\tau}{2} \int_\Omega [x(t)](r)^2 dr \tag{26}$$

$$V_b(x_t) := \frac{1}{2\#_\Omega} \int_\Omega \int_\Omega \int_{-d(r,r')}^0 e^\theta [x(t+\theta)](r')^2 d\theta dr' dr. \tag{27}$$

The derivative of V_a along the solutions of (18) reads

$$\dot{V}_a|_{(5.18)} = -\int_\Omega x^2 dr + \int_\Omega x\, S\left(\int_\Omega w[x(t-d)](r')dr' - k\alpha x + \rho\right) dr. \tag{28}$$

Let $\underline{\alpha} := \min_{r\in\Omega} \alpha(r) > 0$. We consider two cases. First, consider the case when $k\underline{\alpha}|x| \geq \left|\int_\Omega w[x(t-d)](r')dr' + \rho\right|$. Recalling that that $S(r,\cdot)$ is nondecreasing and has the same sign as its argument for all $r \in \Omega$, it holds that $S(a+b)$ has the same sign as a for all $a, b \in \mathbb{R}$ satisfying $|a| \geq |b|$. Consequently,

$$xS\left(\int_\Omega w[x(t-d)](r')dr' - k\alpha x + p\right) \leq 0. \tag{29}$$

On the other hand, when $k\underline{\alpha}|x| < \left|\int_\Omega w[x(t-d)](r')dr' + \rho\right|$, the fact that $|S(x)| \leq \ell|x|$, where ℓ is the slope of S (see (13)), ensures that

$$\begin{aligned}
xS\Big(\int_\Omega w[x(t-d)](r')dr' - kx + \rho\Big) &\leq \ell|x|\left(\left|\int_\Omega w[x(t-d)](r')dr'\right| + k\underline{\alpha}|x| + |\rho|\right)\\
&\leq \frac{\ell}{\underline{\alpha}k}\left|\int_\Omega w[x(t-d)](r')dr' + \rho\right|\left(2\left|\int_\Omega w[x(t-d)](r')dr'\right| + |\rho|\right)\\
&\leq \frac{2\ell}{\underline{\alpha}k}\left(\left|\int_\Omega w[x(t-d)](r')dr'\right| + |\rho|\right)^2\\
&\leq \frac{4\ell}{\underline{\alpha}k}\left(\int_\Omega w^2dr'\int_\Omega [x(t-d)](r')^2dr' + \rho^2\right),
\end{aligned} \tag{30}$$

where we used Cauchy-Schwarz inequality for the latter step. In view of (29), we conclude that this bound (30) holds in any case. Plugging it into (28), we get that

$$\dot{V}_a|_{(5.18)} \leq -\int_\Omega x^2dr + \frac{4\ell}{\underline{\alpha}k}\int_\Omega\left(\int_\Omega w^2dr'\int_\Omega [x(t-d)](r')^2dr' + \rho^2\right)dr. \tag{31}$$

Let us now move to the computation of the functional V_b introduced in (27). Proceeding as in the proof of Proposition 1 (see from (27) on), we get that

$$\dot{V}_b|_{(5.18)} \leq \frac{1}{2}\int_\Omega x^2dr - \frac{e^{-\bar{d}}}{2\#\Omega}\int_\Omega\int_\Omega [x(t-d)](r')^2dr'dr - V_b.$$

Combining this with (31), we get that the derivative of $V = V_a + V_b$ satisfies

$$\dot{V}|_{(5.18)} \leq -\frac{1}{\tau}V_a - V_b + \frac{4\ell}{\underline{\alpha}k}\|\rho(t)\|^2_{\mathcal{F}} - \int_\Omega\Big(\frac{e^{-\bar{d}}}{2\#\Omega} - \frac{4\ell}{\underline{\alpha}k}\int_\Omega w^2dr'\Big)[x(t-d)](r')^2dr'dr.$$

Thus, by picking $k \geq k^* := \frac{16\ell\#_\Omega e^{\bar{d}}}{\underline{\alpha}}\max_{r\in\Omega}\int_\Omega w(r,r')dr'$ and by letting $c_1 := \min\{1/\tau; 1\}, c_2 := \frac{e^{\bar{d}}}{4\#\Omega}$ and $c_3 = 4\ell/\underline{\alpha}$, we obtain $\dot{V}|_{(5.18)} \leq -c_1V - c_2\int_\Omega\int_\Omega [x(t-d)](r')^2dr'dr + \frac{c_3}{k}\|\rho(t)\|^2_{\mathcal{F}}$.

***Proof of Proposition* 3**

First note that the subsystems (11a) and (11b) can be respectively written in the form of the systems addressed by Propositions 2 and 1, namely (18) and (15), by considering the inputs $\rho_1, \rho_2 \in \mathcal{U}$ defined, for all $t \in \mathbb{R}_{\geq 0}$, as $\rho_1(t) := \int_\Omega w_{12}(\cdot, r')[x_2(t - d_2(\cdot, r'))](r')dr' + p_1(t)$ and $\rho_2(t) := \int_\Omega w_{21}(\cdot, r')[x_1(t - d_1(\cdot, r'))](r')dr' + p_2(t)$. Under the assumptions of Proposition 3, those of Propositions 2 and 1 are satisfied for each relevant subsystem, and we get that there exist Lyapunov functionals $V_1, V_2 : C \to \mathbb{R}_{\geq 0}$ (respectively defined as in (19) and (17)) and positive constants

c_{ij}, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$, such that, for all $k > 0$ large enough,

$$\dot{V}_1|_{(11a)} \leq -c_{11}V_1 - c_{12}\int_\Omega\int_\Omega [x_1(t-d_1)](r')^2 dr'dr + \frac{c_{13}}{k}\|\rho_1(t)\|_{\mathcal{F}}^2 \tag{32}$$

$$\dot{V}_2|_{(11b)} \leq -c_{21}V_2 - c_{22}\int_\Omega\int_\Omega [x_2(t-d_2)](r')^2 dr'dr + c_{23}\|\rho_2(t)\|_{\mathcal{F}}^2. \tag{33}$$

Using Cauchy-Schwarz inequality, it holds that

$$\begin{aligned}\|\rho_1(t)\|_{\mathcal{F}}^2 &\leq 2\int_\Omega\left(\int_\Omega w_{12}(r,r')[x_2(t-d_2(r,r'))](r')dr'\right)^2 dr + 2\int_\Omega [p_1(t)](r)^2 dr\\ &\leq 2\bar{\bar{w}}_{12}\int_\Omega\int_\Omega [x_2(t-d_2(r,r'))](r')^2 dr'dr + 2\|p_1(t)\|_{\mathcal{F}}^2,\end{aligned}$$

where $\bar{\bar{w}}_{12} := \max_{r\in\Omega}\int_\Omega w_{12}(r,r')^2 dr'$. In the same way, defining $\bar{\bar{w}}_{21} := \max_{r\in\Omega}$ $\int_\Omega w_{21}(r,r')^2 dr'$, we get that $\|\rho_2(t)\|_{\mathcal{F}}^2 \leq 2\bar{\bar{w}}_{21}\int_\Omega\int_\Omega [x_1(t-d_1(r,r'))](r')^2 dr'dr +$ $2\|p_2(t)\|_{\mathcal{F}}^2$. Plugging these bounds into (32)–(33) yields

$$\begin{aligned}\dot{V}_1|_{(11a)} \leq & -c_{11}V_1 - c_{12}\int_\Omega\int_\Omega [x_1(t-d_1)](r')^2 dr'dr\\ & + \frac{2c_{13}\bar{\bar{w}}_{12}}{k}\int_\Omega\int_\Omega [x_2(t-d_2)](r')^2 dr'dr + \frac{2c_{13}}{k}\|p_1(t)\|_{\mathcal{F}}^2\\ \dot{V}_2|_{(11b)} \leq & -c_{21}V_2 - c_{22}\int_\Omega\int_\Omega [x_2(t-d_2)](r')^2 dr'dr\\ & + 2c_{23}\bar{\bar{w}}_{21}\int_\Omega\int_\Omega [x_1(t-d_1)](r')^2 dr'dr + 2c_{23}\|p_2(t)\|_{\mathcal{F}}^2.\end{aligned}$$

Letting $W_1(x_{1t}) := c_{12}\int_\Omega\int_\Omega [x_1(t-d_1(r,r'))](r')^2 dr'dr$ and $W_2(x_{2t}) := c_{22}\int_\Omega$ $\int_\Omega [x_2(t-d_2(r,r'))](r')^2 dr'dr$, we finally get that

$$\begin{aligned}\dot{V}_1|_{(11a)} &\leq -c_{11}V_1 - W_1 + \frac{2c_{13}\bar{\bar{w}}_{12}}{kc_{22}}W_2 + \frac{2c_{13}}{k}\|p_1(t)\|_{\mathcal{F}}^2\\ \dot{V}_2|_{(11b)} &\leq -c_{21}V_2 - W_2 + \frac{2c_{23}\bar{\bar{w}}_{21}}{c_{12}}W_1 + 2c_{23}\|p_2(t)\|_{\mathcal{F}}^2.\end{aligned}$$

Invoking Corollary 1, the system is thus ISS provided that $4c_{13}c_{23}\bar{\bar{w}}_{12}\bar{\bar{w}}_{21}/kc_{12}c_{22} \leq$ 1. The conclusion follows as this can be satisfied by picking k sufficiently large.

5 Conclusions and Future Works

We have shown that a simple proportional feedback on the controlled subpopulation achieves ISS of the overall neural field provided that the inner synaptic weight of the uncontrolled subpopulation does not exceed some explicit threshold. This strategy is appealing from an applicative viewpoint since no precise knowledge of the parameters involved is required. The robustness induced by ISS is likely to be exploited to analyze stability of more advanced stimulation policies as well as delays in the feedback stimulation [8].

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Robustness of Delayed Multistable Systems

Denis Efimov, Johannes Schiffer and Romeo Ortega

1 Introduction

The increasing penetration of renewable distributed generation (DG) units at the low and medium voltage levels has a strong impact on the power system structure [13, 14, 45]. This fact requires new control and operation strategies to ensure a reliable and efficient electrical power supply [14, 17]. An emerging concept to address these challenges is the microgrid [14, 20, 23]. A microgrid is a locally controllable subset of a larger electrical network. It is composed of several DG units, storage devices and loads.

Typically, most DG units in an AC microgrid are connected to the network via AC inverters [17]. Under ideal conditions, an inverter-based DG unit can be modeled as an ideal controllable voltage source [24, 33]. Furthermore, a popular control scheme to operate inverter-based DG units with the purpose to achieve frequency synchronization and power sharing in the network is droop control [6, 19]. Conditions for stability in droop-controlled microgrids with inverters modeled as ideal controllable voltage sources have been derived, e.g., in [27, 35, 37].

D. Efimov (✉)
Inria, 40 avenue Halley, 59650 Villeneuve d'Ascq, France
e-mail: denis.efimov@inria.fr

D. Efimov
CRIStAL (UMR-CNRS 9189), Ecole Centrale de Lille, Avenue Paul Langevin, 59651 Villeneuve d'Ascq, France

D. Efimov
Department of Control Systems and Informatics, University ITMO, 49 avenue Kronverkskiy, 197101 Saint Petersburg, Russia

J. Schiffer
Control Systems and Network Control Technology, Brandenburg University of Technology (BTU), Cottbus 03046, Germany
e-mail: schiffer@b-tu.de

R. Ortega
Laboratoire des Signaux et Systémes, École Supérieure d'Electricité (SUPELEC), Gif-sur-Yvette 91192, France
e-mail: ortega@lss.supelec.fr

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However, in a practical setup, the droop control scheme is applied to an inverter by means of digital discrete time control. Besides clock drifts, see, e.g. [36], digital control usually introduces time delays [22, 25, 29]. According to [29], the main reasons for this are (1) sampling of control variables, (2) calculation time of the digital controller and (3) generation of the pulse-width modulation. We refer the reader to, e.g. [29] for further details. Hence, it is important to consider time delays in the stability analysis of microgrids.

In general, inverter-based microgrids operated with droop control have several equilibria [35, 37]. Thus they are multistable systems. Stability analysis [4, 10, 26, 28, 31, 32, 34, 38, 44] and robust stability analysis [1, 3, 5, 7, 42] for this class of systems is rather complicated. Recently, the ISS theory [39] has been extended to multistable systems in [2, 3] (see also [21] for discussion on ISS property with respect to an unbounded set).

Motivated by the abovementioned phenomenon, the papers [11, 12] have extended the ISS framework for multistable systems [2, 3] to multistable systems with delay. In particular, sufficient conditions for ISS of multistable systems in the presence of delays are given in terms of a Lyapunov-Razumikhin function. It is also shown that ISS multistable systems are robust with respect to feedback delays (a simple but important illustration is via the example of a nonlinear pendulum). We would like to point out that related works on ISS of time-delay systems by employing Lyapunov functions [9, 16, 30] are limited to systems with a single equilibrium point or a compact attracting set. Based on the results in [11, 12] (their detailed presentation is given below), in this chapter, a condition for asymptotic phase-locking in a microgrid composed of two droop-controlled inverters with delay is developed. The analysis is conducted for a simplified inverter model derived under the assumptions of constant voltage amplitudes and ideal clocks, as well as negligible dynamics of the internal inverter LC filter and controllers. In that scenario, the delay merely affects the phase angle of the inverter output voltage. The stability results are illustrated by simulations.

2 Preliminaries

For an n-dimensional C^2 connected and orientable Riemannian manifold M without a boundary, let the map $f(x, d) : M \times \mathbb{R}^m \to T_x M$ be of class C^1, and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t), d(t)), \tag{1}$$

where the state $x \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X(t, x_0; d)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x_0; d) = x_0$. Together with (1) we will analyze its unperturbed version:

$$\dot{x}(t) = f(x(t), 0). \tag{2}$$

A set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$. Define the distance from a point $x \in M$ to the set $S \subset M$ as $|x|_S = \min_{a \in S} \delta(x, a)$, where the symbol $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M, $|x| = |x|_{\{0\}}$ for $x \in M$ (in this case 0 represents a designated element of M) or a usual Euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \operatorname{ess\,sup}_{t \geq 0} |d(t)|$.

2.1 Decomposable Sets

Let $\Lambda \subset M$ be a compact invariant set for (2).

Definition 1 ([28]) A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$.

Definition 2 ([28]) Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1. An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \cdots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2. A 1-cycle is an index i such that $[\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)] - \Lambda_i \neq \emptyset$.
3. A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 2, existence of an r-cycle with $r \geq 2$ is equivalent to existence of a heteroclinic cycle for (2) [18]. Furthermore, existence of a 1-cycle implies existence of a homoclinic cycle for (2) [18].

Definition 3 The set $\mathcal{W}$ is called decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which form a filtration ordering of $\mathcal{W}$, as detailed in Definitions 1 and 2.

2.2 Robustness Notions

The following robustness notions for systems represented by (1) have been introduced in [2, 3] (see also [8] for a survey on the ISS framework).

Definition 4 We say that the system (1) has the practical asymptotic gain (pAG) property if there exist $\eta \in \mathcal{K}_\infty$ and a non-negative real q such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \to +\infty} |X(t, x; d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty) + q.$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 5 We say that the system (1) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |X(t, x; d)|_{\mathcal{W}} \leq \mu(\|d\|_\infty).$$

Definition 6 We say that the system (1) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and measurable essentially bounded inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t, x; d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}).$$

It has been shown in [2, 3] that to characterize pAG property in terms of Lyapunov functions the following notion is appropriate.

Definition 7 We say that a C^1 function $V : M \to \mathbb{R}$ is a practical ISS-Lyapunov function for (1) if there exists $\mathcal{K}_\infty$ functions α_1, $[\alpha_2]$, α_3 and γ, and scalar $q \geq 0$ [and $c \geq 0$] such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq [\alpha_2(|x|_{\mathcal{W}} + c)],$$

the function V is constant on each $\mathcal{W}_i$ and the following dissipation holds:

$$DV(x)f(x, d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If the latter inequality holds for $q = 0$, then V is said to be an ISS-Lyapunov function.

Notice that α_2 and c are in square brackets as their existence follows (without any additional assumptions) by standard continuity arguments.

The main result of [2, 3] connecting these robust stability properties is stated below. It extends the results of [40, 41] obtained for connected sets.

Theorem 1 *Consider a nonlinear system as in (1) and let a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 3). Then the following facts are equivalent.*

1. *The system admits an ISS Lyapunov function;*
2. *The system enjoys the AG property;*
3. *The system admits a practical ISS Lyapunov function;*
4. *The system enjoys the pAG property;*
5. *The system enjoys the LIM property and the pGS.*

Definition 8 ([3]) Suppose that a nonlinear system as in (1) satisfies the assumptions and the list of equivalent properties of Theorem 1. Then this system is called ISS with respect to the set $\mathcal{W}$.

3 Multistable Systems with Delays

Let $\tau > 0$, for a function $d : [-\tau, +\infty) \to \mathbb{R}^m$ and $t \geq 0$ denote a function $d_t(\cdot) : [-\tau, 0] \to \mathbb{R}^m$ defined by $d_t(\theta) = d(t+\theta)$ for $\theta \in [-\tau, 0]$. Denote by $\mathcal{D}$ a set of bounded and piecewise continuous functions $d_t(\cdot) : [-\tau, 0] \to \mathbb{R}^m$. Consider a functional differential equation on an n-dimensional C^2 connected and orientable Riemannian manifold M without a boundary:

$$\dot{x}(t) = F(x_t, d_t), \;\; x_0 \in C_\tau, \tag{3}$$

where the map $F : C_\tau \times \mathcal{D} \to T_x M$ is of class C^1 (we will denote a set of continuous functions $\xi : [-\tau, 0] \to M$ by C_τ), $x(t) \in M$ is the state, $x_t \in C_\tau$ and $d_t \in \mathcal{D}$ for all $t \geq 0$. We denote by $X(t, x_0; d)$ the uniquely defined solution of (3) at time t fulfilling $X(\theta, x_0; d) = x_0(\theta)$ for all $\theta \in [-\tau, 0]$; $X_t^{x_0,d}(\theta) = X(t+\theta, x_0; d)$ for $\theta \in [-\tau, 0]$. Define as in [43]

$$|x_t| = \max_{\theta\in[-\tau,0]} |x(t+\theta)|, \;\; ||x||_{t_0} = \sup_{t\geq t_0} |x_t| = \sup_{t\geq t_0-\tau} |x(t)|.$$

Again, together with (3), we will analyze its unperturbed version:

$$\dot{x}(t) = F(x_t, 0). \tag{4}$$

A set $\mathcal{S} \subset C_\tau$ is invariant for the unperturbed system (4) if $X_t^{x_0,0} \in \mathcal{S}$ for all $t \in \mathbb{R}_+$ and for all $x_0 \in \mathcal{S}$. Define the distance from a function $\xi \in C_\tau$ to a set $\mathcal{S} \subset C_\tau$ as $||\xi||_\mathcal{S} = \inf_{\alpha\in\mathcal{S}} |\xi - \alpha|$.

Let $\mathcal{W} \subset M$ be a set, denote by $\overbrace{\mathcal{W}}$ a subset of $\overline{\mathcal{W}} = \{\xi \in C_\tau : \xi(t) \in \mathcal{W} \; \forall t \in [-\tau, 0]\}$ such that if $\zeta \in \overbrace{\mathcal{W}}$ then $\zeta = X_\tau^{\xi,0}$ for $\xi \in \overline{\mathcal{W}}$. For stability analysis of time-delay systems it is necessary to define a distance to invariant sets in two spaces: in $\mathbb{R}^n$ with respect to the set $\mathcal{W}$ and in C_τ with respect to corresponding invariant set $\overbrace{\mathcal{W}}$ (functions from C_τ taking values in $\mathcal{W}$ and solutions of (3)). The following stability notions for (3) are considered in this work [12] (for a recent survey on stability tools for time-delay systems see [16]).

Definition 9 The system (3) has the pAG property with respect to the set $\mathcal{W}$ if there exist $\eta \in \mathcal{K}_\infty$ and a non-negative real q such that for all $x_0 \in C_\tau$ and all bounded piecewise continuous inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \to +\infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \eta(\|d_t\|_0) + q.$$

If $q = 0$, then we say that the AG property holds.

This property can be equivalently stated as

$$\limsup_{t \to +\infty} ||X_t^{x_0,d}||_{\widetilde{\mathcal{W}}} \leq \eta(\|d_t\|_0) + q$$

and it implies that (a subset of) $\widetilde{\mathcal{W}}$ is invariant for (4) if $q = 0$.

Definition 10 The system (3) has the pGS property with respect to the set $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x_0 \in C_\tau$ and all bounded piecewise continuous inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t, x_0; d)|_{\mathcal{W}} \leq q + \beta(\max\{||x_0||_{\widetilde{\mathcal{W}}}, \|d_t\|_0\}).$$

To characterize pAG and pGS properties for a time-delay system (3) the Lyapunov-Razumikhin approach is used in this work [9, 30]. Given a continuous function $x : [-\tau, +\infty) \to M$ with a C^1 function $U : M \to \mathbb{R}$ denote $U(t) = U(x(t))$, if $x(t) = X(t, x_0; d)$ is a solution to (3) for some piecewise continuous $d : [-\tau, +\infty) \to \mathbb{R}^m$ and initial condition $x_0 \in C_\tau$, then the upper right-hand side derivative of U along this solution is

$$D^+U(t) = \limsup_{h \to 0^+} \frac{U(t + h) - U(t)}{h}.$$

Definition 11 A C^1 function $U : M \to \mathbb{R}$ is a practical ISS-Lyapunov-Razumikhin (ISS-LR) function for (3) if there exist $\mathcal{K}_\infty$ functions α_1, $[\alpha_2]$, α_4, γ and γ_U, $\gamma_U(s) < s$ for all $s > 0$, and scalar $q \geq 0$ [and $c \geq 0$] such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq U(x) \leq [\alpha_2(|x|_{\mathcal{W}} + c)],$$
$$U(t) \geq \max\{\gamma_U(|U_t|), \gamma(|d_t|), q\} \Rightarrow D^+U(t) \leq -\alpha_4[U(t)].$$

If the latter inequality holds for $q = 0$, then U is said to be an ISS-LR function.

Definition 12 The system in (3) is said to be ISS with respect to the set $\mathcal{W}$ if it admits pAG and pGS properties with respect to the set $\mathcal{W}$.

Note that Definitions 8 and 12 introduce the same property, but for different classes of systems, (1) and (3), respectively.

Theorem 2 ([12]) *Consider the system (3). Suppose there exists an ISS-LR function $U : M \to \mathbb{R}$ as in Definition 11. Then the system (3) admits the pAG property from Definition 9 with $\eta(s) = \alpha_1^{-1} \circ \gamma(s)$ and the pGS property from Definition 10.*

4 ISS of Multistable Systems with Delayed Perturbations

In this section we consider the robustness of the system (1) with respect to a disturbance d, which is dependent on a delayed state. The analysis is conducted under the assumption that the system (1) is ISS with respect to a set $\mathcal{W}$. The proposed approach is illustrated via example of a nonlinear pendulum with delay.

4.1 Robustness Analysis

If (1) is ISS with respect to the set $\mathcal{W}$, then by Theorem 1 there exists an ISS Lyapunov function V as in Definition 7. From the inequalities $\alpha_3[0.5\alpha_2^{-1} \circ V(x)] \leq \alpha_3(0.5[|x|_{\mathcal{W}} + c]) \leq \alpha_3(|x|_{\mathcal{W}}) + \alpha_3(c)$ we obtain

$$DV(x)f(x,d) \leq -\alpha_4[V(x)] + \gamma(|d|) + \tilde{q},$$

where $\alpha_4(s) = \alpha_3[0.5\alpha_2^{-1}(s)]$ and $\tilde{q} = q + \alpha_3(c)$.

Assume that the input d has two terms d_1 and d_2, and d_2 is a function of $x_t \in C_\tau$ for some $\tau > 0$, i.e.:

$$d = d_1 + d_2, \;\; d_2 = g(x_t), \tag{5}$$

where g is a continuous function, $|g(x_t)| \leq \upsilon(|V_t|) + \upsilon_0$ for $\upsilon \in \mathcal{K}_\infty$ and $\upsilon_0 \geq 0$ (here V_t denotes a function $V_t(\cdot) : [-\tau, 0] \to \mathbb{R}_+$ defined by $V_t(\theta) = V(t+\theta)$ for $\theta \in [-\tau, 0]$). Denote further for simplicity of notation $d = d_1$, then (1) is transformed to (3) with

$$F(x_t, d_t) = f(x(t), d + g(x_t)),$$
$$D^+V(t) \leq -\alpha_4(V(t)) + \gamma(2\upsilon(|V_t|) + 2\upsilon_0) + \gamma(2|d_t|) + \tilde{q}.$$

This estimate can be rewritten as follows:

$$V(t) \geq \max\{\hat{\gamma}_V(|V_t|), \hat{\gamma}(|d_t|), \hat{q}\} \;\Rightarrow\; D^+V(t) \leq -0.5\alpha_4(V(t)),$$
$$\hat{\gamma}_V(s) = \alpha_4^{-1}[6\gamma(4\upsilon(s))], \;\; \hat{\gamma}(s) = \alpha_4^{-1}[6\gamma(2s)], \;\; \hat{q} = \alpha_4^{-1}[6\tilde{q} + 6\gamma(4\upsilon_0)].$$

It is straightforward to see that if $\hat{\gamma}_V(s) < s$ for all $s > 0$, then V is an ISS-LR function for (1) with (5), and by Theorem 2 this system possesses pAG and pGS properties.

4.2 Illustration for a Nonlinear Pendulum

Now, the procedure for a robust ISS analysis of a multistable system with delays outlined in Sect. 4.1 is illustrated via the example of a nonlinear pendulum. First, we prove the assumption made in Sect. 4.1 that the pendulum is ISS with respect to a set $\mathcal{W}$. Second, a condition for ISS of a pendulum with delay is derived. During our analysis, we also establish almost global attractivity of an equilibrium of a nonlinear pendulum with constant nonzero input. To the best of our knowledge, such result is not available in the literature thus far.

4.2.1 Delay-Free Case

Consider a nonlinear pendulum:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\Omega^2 \sin(x_1) - \kappa x_2 + d, \end{aligned} \tag{6}$$

where the state $x = [x_1, x_2]$ takes values on the cylinder $M := \mathbb{S} \times \mathbb{R}$, $d(t) \in \mathbb{R}$ is an exogenous disturbance, and Ω, κ are constant positive parameters. The total energy of (6) is $H(x) = 0.5x_2^2 + \Omega^2(1 - \cos(x_1))$ and $\dot{H} = x_2 d - \kappa x_2^2$. The unperturbed system (6) has two equilibria $[0, 0]$ and $[\pi, 0]$ (the former is attractive and the latter one is a saddle-point). Thus, $\mathcal{W} = \{[0, 0] \cup [\pi, 0]\}$ is a compact set containing all α- and ω-limit sets of (6) for $d = 0$. In addition, it is straightforward to check that $\mathcal{W}$ is decomposable in the sense of Definition 3.

Lemma 1 ([12]) *The system (6) is ISS with respect to the set* $\mathcal{W}$.

By using this result it is possible to prove that for a constant input d (with $d < \Omega^2$) the pendulum still has two steady-state points with similar stability properties.

Lemma 2 ([12]) *Let* $d < \min\{\Omega^2, \sqrt{\frac{\varrho\lambda_{\min}(Y)}{2}}\frac{\pi}{\zeta}, 0.5\sqrt{\epsilon}\Omega\frac{\pi}{\zeta}, \xi\}$ *be a constant input in (6), where*

$$\varrho = \min\left[\frac{\kappa - \epsilon}{1 + \epsilon}, \frac{1}{\sqrt{2\pi}}\frac{\epsilon\Omega^2}{(\Omega^2 + (\kappa + 1)\epsilon)}\right],$$

$$\zeta = \sqrt{\sqrt{2\pi}\left[\epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}\right]},\ \xi = \frac{2\sqrt{\Omega^2 + \kappa\epsilon}}{\zeta}\left(\frac{\Omega^2 + (\kappa + 1)\epsilon}{\epsilon\Omega^2} + \frac{1}{\sqrt{2\pi}\varrho}\right)$$

and $0 < \epsilon < \min\{1, \kappa\}$ *is a parameter. Then the system has two equilibria,* $[\arcsin(d\Omega^{-2}), 0]$ *and* $[\pi - \arcsin(d\Omega^{-2}), 0]$. *The former one is almost globally attractive.*

4.2.2 A Delayed Case Study

Now consider a time-delay modification of (6):

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\Omega^2 \sin[x_1(t-\tau)] - \kappa x_2(t) + d(t), \end{aligned} \tag{7}$$

where $\tau > 0$ is a fixed delay. The unperturbed system (7) with $d(t) = 0$ has the same equilibria as (6), i.e. $[0, 0]$ and $[\pi, 0]$. The system (7) can be represented as follows:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\Omega^2 \sin[x_1(t)] - \kappa x_2(t) + d(t) + \Omega^2\{\sin[x_1(t)] - \sin[x_1(t-\tau)]\}. \end{aligned}$$

By the mean value theorem

$$|\sin[x_1(t)] - \sin[x_1(t-\tau)]| = |\cos[x_1(\phi)]x_2(\phi)\tau| \le |x_2(\phi)|\tau$$

for some $\phi \in [t-\tau, t]$. Thus, the system (7) can be analyzed as a perturbed nonlinear pendulum with part of the input d dependent on the delay. Using V we obtain for $\mu = \epsilon\Omega^{-2} + \frac{1}{\kappa-\epsilon}$:

$$D^+V(t) \le -0.5[\kappa-\epsilon]x_2^2 - 0.5\epsilon\Omega^2\sin^2(x_1) + \mu\Omega^4 x_2^2(\phi)\tau^2 + \mu d^2.$$

It is straightforward to check that

$$\begin{aligned} V(x) &\le 0.5[1+\epsilon]x_2^2 + 0.5\epsilon\sin^2(x_1) + 2[\Omega^2 + \kappa\epsilon], \\ x_2^2 &\le \frac{2}{1-\epsilon}V(x) + \frac{\epsilon}{1-\epsilon} \end{aligned}$$

for $0 < \epsilon < \min\{1, \kappa\}$, then for $\rho = \min\{\frac{\kappa-\epsilon}{1+\epsilon}, \Omega^2\}$

$$\begin{aligned} D^+V(t) &\le -\rho\{V(t) - 2[\Omega^2 + \kappa\epsilon]\} + \mu\Omega^4 x_2^2(\phi)\tau^2 + \mu d^2 \\ &\le -\rho\{V(t) - 2[\Omega^2 + \kappa\epsilon]\} + \frac{\mu\Omega^4}{1-\epsilon}\tau^2[2V(\phi) + \epsilon] + \mu d^2. \end{aligned}$$

Therefore,

$$V(t) \ge \frac{6}{\rho}\max\left\{2\frac{\mu\Omega^4}{1-\epsilon}\tau^2|V_t|, 2\rho[\Omega^2+\kappa\epsilon] + \frac{\mu\Omega^4}{1-\epsilon}\tau^2\epsilon, \mu d^2\right\} \Rightarrow D^+V(t) \le -0.5\rho V(t) \tag{8}$$

and V is an ISS-LR function for (7) provided that

$$\frac{12}{\rho}\frac{\mu\Omega^4}{1-\epsilon}\tau^2 < 1. \tag{9}$$

The inequality (9) is a delay-dependent stability condition for (7), which is always satisfied for a sufficiently small delay τ. The set of asymptotic attraction for (7) can be evaluated from (8).

Remark 1 If we assume that $\max\{0, \frac{\kappa-\Omega^2}{1+\Omega^2}\} < \epsilon < \min\{1, \kappa\}$, then $\min\{\frac{\kappa-\epsilon}{1+\epsilon}, \Omega^2\} = \frac{\kappa-\epsilon}{1+\epsilon}$ and the condition (9) can be rewritten as follows:

$$\tau^2 < \frac{12}{\Omega^2}\frac{1-\epsilon}{1+\epsilon}\frac{(\kappa-\epsilon)^2}{\epsilon(\kappa-\epsilon)+\Omega^2}.$$

Since the functions $\frac{1-\epsilon}{1+\epsilon}$ and $\frac{(\kappa-\epsilon)^2}{\epsilon(\kappa-\epsilon)+\Omega^2}$ are decreasing for $\epsilon \in (\max\{0, \frac{\kappa-\Omega^2}{1+\Omega^2}\}, \min\{1, \kappa\})$, selecting $\epsilon = \max\{0, \frac{\kappa-\Omega^2}{1+\Omega^2}\} + \varepsilon$ for a sufficiently small $\varepsilon > 0$ (if $\kappa > \Omega^2$ then the optimal choice is $\epsilon = \frac{\kappa-\Omega^2}{1+\Omega^2}$) optimizes the value of the admissible delay τ to

$$\tau^* = \frac{\kappa-\epsilon}{\Omega}\sqrt{\frac{1-\epsilon}{1+\epsilon}\frac{12}{\epsilon(\kappa-\epsilon)+\Omega^2}},$$

i.e. for any $\tau < \tau^*$ the system (7) admits V as an ISS-LR function.

5 Application to a Microgrid Composed of Two Droop-Controlled Inverters with Delay

In this section the theoretical results of Sect. 3 are applied to our main motivating application: a droop-controlled microgrid with delays. In particular, we are interested in conditions for ISS of such systems. In order to tackle this problem, we proceed along the lines detailed in Sect. 4. The analysis is conducted under a reasonable assumption of constant voltage amplitudes. Then, a lossless droop-controlled microgrid formed by two inverters with delay can be modeled as [35]:

$$\begin{aligned} \dot\theta(t) &= \omega_1(t) - \omega_2(t), \\ \tau_{P_1}\dot\omega_1(t) &= -\omega_1(t) - k_{P_1}a_{12}\sin[\theta(t-\tau_{d_1})] + c_1 + \delta_1(t), \\ \tau_{P_2}\dot\omega_2(t) &= -\omega_2(t) + k_{P_2}a_{12}\sin[\theta(t-\tau_{d_2})] + c_2 + \delta_2(t), \end{aligned} \tag{10}$$

where $\theta(t) \in [0, 2\pi)$ is the phase difference between the inverters, $\omega_1(t), \omega_2(t) \in \mathbb{R}$ are the time-varying frequencies of the inverters; $\tau_{d_1} > 0$ and $\tau_{d_2} > 0$ are delays caused by the digital controls required to implement the droop controls; $\tau_{P_1} > 0$,

$\tau_{P_2} > 0$, $k_{P_1} > 0$, $k_{P_2} > 0$, $a_{12} > 0$, c_1 and $c_2 = -\frac{k_{P_2}}{k_{P_1}} c_1$ are constant parameters; the disturbances $\delta_1(t)$ and $\delta_2(t)$ represent additional model uncertainties. We say that a solution of (10) is phase-locked if $\theta(t) = \theta_0$ is constant $\forall t \in \mathbb{R}_+$ for some $\theta_0 \in [0, 2\pi)$ [15]. If this property holds asymptotically, i.e., for $t \to +\infty$, we speak about an asymptotic phase-locking.

`Assumption.` $\tau_{d_1} = \tau_{d_2} = \tau > 0$.

Under this assumption, define the new coordinates:

$$x_1 = \theta,\ x_2 = \omega_1 - \omega_2,\ x_3 = \frac{k_{P_2}\tau_{P_1}}{k_{P_1}\tau_{P_2}}\omega_1 + \omega_2.$$

Then the system (10) can be rewritten as follows:

$$\dot{x}_1(t) = x_2(t), \tag{11}$$

$$\dot{x}_2(t) = -b_1 x_2(t) + b_2 x_3(t) - a^2 \sin[x_1(t-\tau)] + d_1(t), \tag{12}$$

$$\dot{x}_3(t) = -b_3 x_3(t) + b_4 x_2(t) + d_2(t), \tag{13}$$

where

$$b_1 = \tau_{P_1}^{-1} + (\tau_{P_2}^{-1} - \tau_{P_1}^{-1})\left(1 + \frac{k_{P_2}\tau_{P_1}}{k_{P_1}\tau_{P_2}}\right)^{-1} \frac{k_{P_2}\tau_{P_1}}{k_{P_1}\tau_{P_2}},\ b_2 = (\tau_{P_2}^{-1} - \tau_{P_1}^{-1})\left(1 + \frac{k_{P_2}\tau_{P_1}}{k_{P_1}\tau_{P_2}}\right)^{-1},$$

$$b_3 = \tau_{P_1}^{-1} + (\tau_{P_2}^{-1} - \tau_{P_1}^{-1})\left(1 + \frac{k_{P_2}\tau_{P_1}}{k_{P_1}\tau_{P_2}}\right)^{-1},\ b_4 = (\tau_{P_2}^{-1} - \tau_{P_1}^{-1})\left(1 + \frac{k_{P_2}\tau_{P_1}}{k_{P_1}\tau_{P_2}}\right)^{-1} \frac{k_{P_2}\tau_{P_1}}{k_{P_1}\tau_{P_2}},$$

$$a = \sqrt{(k_{P_1}\tau_{P_1}^{-1} + k_{P_2}\tau_{P_2}^{-1})a_{12}},$$

$$d_1(t) = (\tau_{P_1}^{-1} + \tau_{P_2}^{-1}\frac{k_{P_2}}{k_{P_1}})c_1 + \tau_{P_1}^{-1}\delta_1(t) - \tau_{P_2}^{-1}\delta_2(t),\ d_2(t) = \frac{k_{P_2}}{k_{P_1}\tau_{P_2}}\delta_1(t) + \tau_{P_2}^{-1}\delta_2(t).$$

In [12] it has been also assumed that $\tau_{P_1} = \tau_{P_2} = \tau_P > 0$. The stability analysis above is based on the assumption that $b_1 > 0$ and $b_3 > 0$, which we imposed without loss of generality. If $\tau_{P_2} > \tau_{P_1}$ and the coefficients $b_1 < 0$ or $b_3 < 0$ for the given values of k_{P_1} and k_{P_2}, then the above equations can be rewritten to have the term $\tau_{P_1}^{-1} - \tau_{P_2}^{-1}$ instead of $\tau_{P_2}^{-1} - \tau_{P_1}^{-1}$ by flipping the indices. Thus, the system (10) is decomposed into two interrelated subsystems: (11)–(13). The variable x_3 converges asymptotically to zero if $b_4 x_2 + d_2 = 0$ (then asymptotically the frequencies ω_1 and ω_2 are locked), moreover this subsystem is ISS with respect to the input $b_4 x_2 + d_2$, with the ISS asymptotic gain $|b_4 x_2 + d_2| \to |x_3|$ being equal to b_3^{-1} (this simple result can be obtained using an ISS Lyapunov function $V_3(x_3) = 0.5x_3^2$).

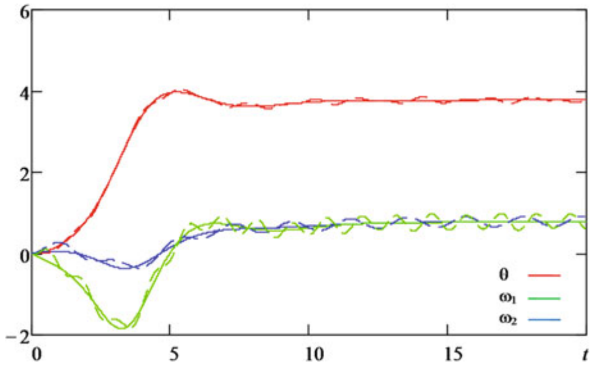

Fig. 1 Simulation results for the system (10). The solid lines show the state trajectories for the case $d_1(t) = d_2(t) = 0$. The dashed lines correspond to the case $d_1(t) = 0.8\sin(3t)$, $d_2(t) = 0.9\sin(5t)$

The dynamics (11), (12) have the form of (7) for $d = d_1 + b_2 x_3$ and, as it has been established above, have pAG and pGS properties from Definitions 9 and 10 respectively if condition (9) is satisfied, which for (11), (12) takes the form:

$$\tau^2 < \frac{12}{a^2}\frac{1-\epsilon}{1+\epsilon}\frac{(b_1-\epsilon)^2}{\epsilon(b_1-\epsilon)+a^2} \tag{14}$$

for $0 < \epsilon < \min\{1, b_1\}$, and the asymptotic gain $d^2 \to V$ is $\epsilon a^{-2} + \frac{1}{b_1-\epsilon}$. Therefore, under the small-gain condition

$$b_3^2 b_4^2 b_2^2 \left(\epsilon a^{-2} + \frac{1}{b_1-\epsilon}\right) < 1 \tag{15}$$

and for a sufficiently small delay τ verifying (14) the system (11)–(13) is ISS with respect to inputs d_1 and d_2. In that case the inverters will demonstrate a phase-locking behavior. According to [29], a good estimate of the overall delay introduced by the digital control is $\tau = 1.75T_S$,[1] where $T_S = 1/f_S$ and $f_S \in \mathbb{R}_{>0}$ is the switching frequency of the inverter. Since usually $f_S \in [5, 20]$ kHz [17], τ is reasonably small in most practical applications. Hence, the condition (14) may be satisfied for most practical choices of parameters in (10).

The analysis is illustrated in a simulation example with the following set of parameters for the system (10): $\tau_{P_1} = 2$, $\tau_{P_2} = 1$, $k_{P_1} = 10$, $k_{P_2} = 20$, $a_{12} = 0.1$, $c_1 = 0.2$ and $\tau = 0.05$. Conditions (14) and (15) are satisfied for $\epsilon = 0.5\min\{1, b_1\}$. The simulation results are shown in Fig. 1. The solid lines represent the state $(\theta, \omega_1, \omega_2)^T$ trajectories for the case $d_1(t) = d_2(t) = 0$, and the dashed lines correspond to $d_1(t) = 0.8\sin(3t)$, $d_2(t) = 0.9\sin(5t)$. The phase-locking phenomenon is observed in these simulation results.

[1]The overall delay reduces to $\tau = 1.5T_S$ if no moving average function for the measurement is used [29].

6 Conclusions

Sufficient conditions for ISS of multistable systems with delay have been presented. The conditions are formulated using Lyapunov-Razumikhin functions. The potential of the approach has been illustrated by demonstrating several robustness properties for a nonlinear pendulum with delay. Furthermore, it has been shown that asymptotic phase-locking in a lossless droop-controlled microgrid formed by two inverters with delays can be analyzed based on a perturbed pendulum model. By exploiting this fact, a delay-dependent condition for ISS of such a microgrid has been presented.

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A Small-Gain Method for the Design of Decentralized Stabilizing Controllers for Interconnected Systems with Delays

Pierdomenico Pepe, Hiroshi Ito and Zhong-Ping Jiang

1 Introduction

Feedback stabilization by means of static state-feedback, dynamic output-feedback and input-output feedback linearization for nonlinear retarded systems has been extensively studied in the literature (see, for instance, [1–5, 9–11, 16, 26–29, 43]). Nevertheless, though many approaches are available, the stabilization problem for general nonlinear systems, with an arbitrary number of discrete and distributed time-delays, is still far from being fully solved. The technique of control Lyapunov functions has been exploited to practically or asymptotically stabilize a large class of time-invariant retarded systems in affine form in [16], using Lyapunov Razumikhin functions. The domination redesign control methodology is employed (see [39]). Results concerning the use of control Lyapunov-Krasovskii functionals (instead of control Lyapunov-Razumikhin functions) for the design of stabilizing control laws for retarded systems can be found in [7, 15, 21, 33, 34, 38]. In [15] a fixed type of control Lyapunov-Krasovskii functionals is exploited. For this type of control Lyapunov-Krasovskii functionals, remarkable results are achieved for a broad class of retarded systems. For instance, it is shown that both Sontag's (see [41]) and Free-

P. Pepe (✉)
Department of Information Engineering, Computer Science, and Mathematics, University of L'Aquila, Via Vetoio (Coppito I), 67100 L'Aquila, Italy
e-mail: pierdomenico.pepe@univaq.it

H. Ito
Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka 820-8502, Japan
e-mail: hiroshi@ces.kyutech.ac.jp

Z.-P. Jiang
Department of Electrical and Computer Engineering, Tandon School of Engineering, New York University, Six Metrotech Center, Brooklyn, NY 11201, USA
e-mail: zjiang@nyu.edu

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man's (see [8]) formulas can be successfully used for global stabilization purposes. If the small control property holds (see [15, 41]), the proposed feedback control laws are, at least, locally Lipschitz outside the origin and continuous at the origin. In [7] the authors propose a predictive control scheme with guaranteed closed-loop stability for nonlinear retarded systems, utilizing the same fixed type of control Lyapunov-Krasovskii functionals as in [15]. In [21] the authors prove the equivalence of the existence of a completely locally Lipschitz control Lyapunov-Krasovskii functional satisfying the small control property (see [41] and references therein), and the stabilizability property by means of completely locally Lipschitz control laws, for fully nonlinear, retarded systems. Moreover, stabilizability is intended robust with respect to vanishing disturbances. In [38] the inverse optimality approach for delay-free nonlinear systems is extended to time-delay systems, by the use of complete quadratic control Lyapunov-Krasvoskii functionals. In the paper [33] it is shown how invariantly differentiable functionals (see [24, 25]) can play an important role for the input-to-state practical stabilization (see Definition 2.1 in [17]) of retarded systems, by using the Sontag's universal formula with a slight modification. The hypotheses introduced in [33] do not guarantee that the state feedback law obtained by the Sontag's formula, as proposed in [15], is locally Lipschitz. Therefore, the Sontag's formula extended to retarded systems is modified in the critical subsets of the infinite dimensional state space where the Lipschitz property of the related feedback control law may be lost. By this modification, the problem of non Lipschitz feedback control law is solved. Then, an input-to-state stabilizing term (see [32, 35, 40]) is added to the control law, thus achieving the twofold result of attenuation of the actuator disturbance and attenuation of the bounded error due to the above modification of the Sontag's formula. Sontag's stabilizer is studied in [34] also for neutral systems in Hale's form, which include retarded systems as a special case. Sufficient conditions for both the global asymptotic stabilization and for the global practical stabilization, by Sontag's and modified Sontag's formula, are provided. A robustifying controller for retarded interconnected systems is studied in [12]. It is shown that, under a suitable small-gain condition (see [14]), decentralized controllers can be found in order to achieve actuator disturbance attenuation, in the sense of input-to-state stability, for interconnected systems stabilizable by means of decentralized state feedback control laws. The interested reader can refer to the recent monograph [20] for an extensive presentation of Lyapunov-based stabilization methods for nonlinear systems, in both the finite dimensional and the retarded cases, in the continuous-time as well as in the discrete-time. It is well known that finding control Lyapunov-Krasovskii functionals is in general a not easy task, as well as that small-gain methods (see [18]) can provide an important tool in order to simplify the construction of suitable Lyapunov functions and Lyapunov-Krasovskii functionals (see [13, 14, 17]). For this reason, we propose here a constructive methodology for the design of controllers for a class of interconnected systems, which makes use of control Lyapunov-Krasovskii functionals for each subsystem, aimed at simplifying the search of these functionals. Then, for each subsystem, the locally Lipschitz control law proposed in [33] is found. By means of a small-gain condition for retarded systems developed in [14], it is proved that the resulting overall closed-loop system is input-to-state practically stable with respect to actuator disturbances. If the disturbances are bounded, it is proved that, by suitably tuning a control parameter, the system state can be driven to an arbitrarily small neighborhood of the origin. The easier search of control Lyapunov-Krasovskii functionals is evident in the special case when delays appear on the communication channels only. In this case, it is possible to look for just control Lyapunov functions in Euclidean spaces, and transfer to the small-gain condition the problem of dealing with interconnection delays (see (17) in [14]). Moreover, the provided conditions do not include the small control property, which may well be not satisfied, as shown in [33]. Actually, the small control property may be an additional property hard to be satisfied for retarded systems (see examples in [33] and, in the neutral case, in [34]). The proposed control law is locally Lipschitz in the infinite dimensional state space of the systems described by retarded functional differential equations here considered. Moreover, the control law is decentralized, that is, each control law depends only on the state of each sub-system, which may be an interesting property for the practical situation where subsystems are located far away each other, and each subsystem is provided with a controller. A numerical example is studied in details, in order to show the effectiveness of the proposed methodology.

A preliminary version of this chapter has been published in [36].

Notation $\mathcal{R}$ denotes the set of real numbers, $\mathcal{R}^{\star}$ denotes the extended real line $[-\infty, +\infty]$, $\mathcal{R}^{+}$ denotes the set of non-negative reals $[0, +\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_{\infty}$. A function $v : \mathcal{R}^{+} \to \mathcal{R}^{m}$, m positive integer, is said to be *essentially bounded* if $ess\ \sup_{t\geq 0} |v(t)| < +\infty$. For given times $0 \leq T_1 < T_2$, with $v_{[T_1,T_2)} : \mathcal{R}^{+} \to \mathcal{R}^{m}$ we mean the function given by $v_{[T_1,T_2)}(t) = v(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input v is said to be *locally essentially bounded* if, for any $T > 0$, $v_{[0,T)}$ is essentially bounded. For a positive integer n, for a positive real Δ (maximum involved time delay), C_n and Q_n denote the space of the continuous functions mapping $[-\Delta, 0]$ into R^n and the space of the bounded, continuous except at a finite number of points with jump discontinuities, and right-continuous functions mapping $[-\Delta, 0)$ into $\mathcal{R}^n$, respectively. For $\phi \in C_n$, $\phi_{[-\Delta,0)}$ is the function in Q_n defined as $\phi_{[-\Delta,0)}(\tau) = \phi(\tau)$, $\tau \in [-\Delta, 0)$. For a continuous function $x : [-\Delta, c) \to \mathcal{R}^n$, with $0 < c \leq +\infty$, for any real $t \in [0, c)$, x_t is the function in C_n defined as $x_t(\tau) = x(t + \tau)$, $\tau \in [-\Delta, 0]$. For a positive real δ, $\phi \in C_n$, $I_{\delta}(\phi) = \{\psi \in C_n : \|\psi - \phi\|_{\infty} \leq \delta\}$. For given positive integers n, m, a map $f : C_n \to \mathcal{R}^{n\times m}$ is said to be: completely continuous if it is continuous and takes bounded subsets of C_n into bounded subsets of $\mathcal{R}^{n\times m}$; locally Lipschitz in C_n if, for any $\phi \in C_n$, there exist positive reals δ, η such that, for any $\phi_1, \phi_2 \in I_{\delta}(\phi)$, the inequality $|f(\phi_1) - f(\phi_2)| \leq \eta\|\phi_1 - \phi_2\|_{\infty}$ holds. Let us here recall that a function $\gamma : \mathcal{R}^{+} \to \mathcal{R}^{+}$ is: of class $\mathcal{P}$ if it is continuous, zero at zero, and positive at any positive real; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_{\infty}$ if it is of class $\mathcal{K}$ and it is unbounded; of class $\mathcal{L}$ if it is continuous and it monotonically decreases to zero as its argument tends to $+\infty$. A function $\beta : \mathcal{R}^{+} \times \mathcal{R}^{+} \to \mathcal{R}^{+}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \geq 0$. The symbol $\circ$ denotes composition of functions. With the symbol M_a is indicated any functional mapping C_n into $\mathcal{R}^+$ (see [32]), such that, for some $\mathcal{K}_\infty$ functions $\gamma_a, \overline{\gamma}_a$, the inequalities $\gamma_a(|\phi(0)|) \leq M_a(\phi) \leq \overline{\gamma}_a(\|\phi\|_\infty)$ hold for any $\phi \in C_n$. Throughout the chapter, RFDE stands for retarded functional differential equation, ISS stands for input-to-state stability or input-to-state stable, ISpS stands for input-to-state practical stability or input-to-state practically stable, GAS stands for global asymptotic stability or globally asymptotically stable. A system with an equilibrium at zero is said to be 0-GAS if the zero solution is GAS. CLF stands for control Lyapunov function, CLRF stands for control Lyapunov-Razumikhin function, CLKF stands for control Lyapunov-Krasovskii functional.

2 Preliminaries

Let us consider the system described by the following RFDE

$$\begin{aligned} &\dot{x}(t) = f(x_t) + g(x_t)u(t), \qquad t \geq 0, \qquad a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \qquad \xi_0 \in C_n, \end{aligned} \tag{1}$$

where: $x(t) \in \mathcal{R}^n$, n is a positive integer; $\Delta > 0$ is the maximum involved time delay; the maps $f : C_n \to \mathcal{R}^n$ and $g : C_n \to R^{n\times m}$ are completely continuous and locally Lipschitz in C_n, $f(0) = 0$; m is a positive integer; $u(t) \in \mathcal{R}^m$ is the input signal, assumed to be Lebesgue measurable and locally essentially bounded.

Given a locally Lipschitz continuous functional $V : C_n \to \mathcal{R}^+$, the upper right-hand derivative $D^+V : C_n \times \mathcal{R}^m \to \mathcal{R}^\star$ of the functional V, in the Driver's form (see [6, 19, 37]), is defined, for $\phi \in C_n$, $v \in \mathcal{R}^m$, as

$$D^+V(\phi, v) = \limsup_{h\to 0^+} \frac{1}{h} \left(V(\phi_h) - V(\phi)\right), \tag{2}$$

where $\phi_h \in C_n$ is given, for $h \in [0, \Delta)$, by

$$\phi_h(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h), \\ \phi(0) + (f(\phi) + g(\phi)v)(\theta + h), & \theta \in [-h, 0] \end{cases} \tag{3}$$

Remark 1 It is proved in [30] that, for locally Lipschitz continuous functionals V, the following equality holds

$$\limsup_{h\to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h} = D^+V(x_t, u(t)), \qquad t \in [0, b), \qquad a.e., \tag{4}$$

where x_t is the solution of (1) in a maximal time interval $[0, b)$, $0 < b \leq +\infty$. It is proved in [31] that, for locally Lipschitz functionals V, the problem of the local absolute continuity of the function $t \to V(x_t)$ is overcome.

The following definition of invariant differentiable functionals is taken from [25], see Definitions 2.2.1, 2.5.2 in Chap. 2. The formalism used in [25] is here slightly modified for the purpose of formalism uniformity over the chapter. For any given $x \in \mathcal{R}^n$, $\phi \in \mathcal{Q}_n$ and for any given continuous function $\mathcal{Y} : [0, \Delta] \to \mathcal{R}^n$ with $\mathcal{Y}(0) = x$, let $\psi_h^{(x,\phi,\mathcal{Y})} \in \mathcal{Q}_n$, $h \in [0, \Delta)$, be defined as

$$\psi_h^{(x,\phi,\mathcal{Y})}(s) = \begin{cases} \phi(s), \quad s \in [-\Delta, 0), \quad h = 0, \\ \left\{ \begin{array}{ll} \phi(s+h), & s \in [-\Delta, -h), \\ \mathcal{Y}(s+h), & s \in [-h, 0), \end{array} \right\}, \quad h \in (0, \Delta) \end{cases} \tag{5}$$

For $\phi \in C_n$, $h \in [0, \Delta)$, let $\phi^h \in C_n$ be defined as follows

$$\phi^h(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h) \\ \phi(0), & s \in [-h, 0] \end{cases} \tag{6}$$

Definition 1 (see [25]) A functional $V : \mathcal{R}^n \times \mathcal{Q}_n \to \mathcal{R}^+$ is said to be invariantly differentiable if, at any point $(x, \phi) \in R^n \times \mathcal{Q}_n$, the following conditions hold:

(i) for any continuous function $\mathcal{Y} : [0, \Delta] \to R^n$ with $\mathcal{Y}(0) = x$, the right-hand derivative $\left.\frac{\partial V\left(x, \psi_h^{(x,\phi,\mathcal{Y})}\right)}{\partial h}\right|_{h=0}$ exists and such derivative is invariant with respect to the function $\mathcal{Y}$;

(ii) the derivative $\frac{\partial V(x,\phi)}{\partial x}$ exists;

(iii) for any given continuous function $\mathcal{Y} : [0, \Delta] \to R^n$ with $\mathcal{Y}(0) = x$, the following limit holds (involved $z \in R^n$ and $h \in [0, \Delta)$),

$$\lim_{z \to 0,\, h \to 0^+} \frac{1}{\sqrt{|z|^2 + h^2}} \cdot \left(V\left(x + z, \psi_h^{(x,\phi,\mathcal{Y})}\right) - V(x, \phi) - \frac{\partial V(x, \phi)}{\partial x} z - \left.\frac{\partial V\left(x, \psi_\ell^{(x,\phi,\mathcal{Y})}\right)}{\partial \ell}\right|_{\ell=0} h \right) = 0 \tag{7}$$

For a given locally Lipschitz and invariantly differentiable functional $V : \mathcal{R}^n \times \mathcal{Q}_n \to \mathcal{R}^+$, let $V_0 : C_n \to \mathcal{R}^+$ be the locally Lipschitz continuous functional defined, for $\phi \in C_n$, as $V_0(\phi) = V(\phi(0), \phi_{[-\Delta,0)})$. Then, the following result holds, for any $\phi \in C_n$ and any $v \in \mathcal{R}^m$,

$$D^+V_0(\phi, v) = \left.\frac{\partial V(x, \phi_{[-\Delta,0)})}{\partial x}\right|_{x=\phi(0)} (f(\phi) + g(\phi)v) + \left.\frac{\partial V\left(\phi(0), \phi^h_{[-\Delta,0)}\right)}{\partial h}\right|_{h=0}, \tag{8}$$

where the second term of the right-hand side of (8) is a right-hand derivative (see point (i) in Definition 1 and (6)).

In the following, for given positive integer n, $\mathcal{V}_n$ is the class of functionals $V : \mathcal{R}^n \times \mathcal{Q}_n \to \mathcal{R}^+$ which have the following properties: i) V is locally Lipschitz in $\mathcal{R}^n \times \mathcal{Q}_n$ and invariantly differentiable; ii) the maps ($\phi \in C_n$, involved $x \in \mathcal{R}^n$, $h \in [0, \Delta)$)

$$\phi \to \left.\frac{\partial V\left(\phi(0), \phi^h_{[-\Delta,0)}\right)}{\partial h}\right|_{h=0}, \qquad \phi \to \left.\frac{\partial V(x, \phi_{[-\Delta,0)})}{\partial x}\right|_{x=\phi(0)} \tag{9}$$

are completely continuous and locally Lipschitz in C_n.

3 Interconnected Retarded Systems

Consider an interconnected system Σ described by the following RFDEs

$$\Sigma \begin{cases} \Sigma_1 : \dot{x}_1(t) = f_1(x_{1,t}) + H_1(x_{1,t}, x_{2,t}) + g_1(x_{1,t})(u_1(t) + d_1(t)) \\ \Sigma_2 : \dot{x}_2(t) = f_2(x_{2,t}) + H_2(x_{2,t}, x_{1,t}) + g_2(x_{2,t})(u_2(t) + d_2(t)) \end{cases} \tag{10}$$
$$x_{1,0} = \xi_{1,0}, \quad x_{2,0} = \xi_{2,0},$$

where, for $i = 1, 2$: $x_i(t) \in \mathcal{R}^{n_i}$; $d_i(t) \in \mathcal{R}^{m_i}$ is a disturbance adding to the control input (measurable, locally essentially bounded); n_i and m_i are positive integers; for $t \in \mathcal{R}^+$, $x_{i,t} : [-\Delta, 0] \to \mathcal{R}^{n_i}$ denotes (see Notation section) the function $x_{i,t}(\tau) = x_i(t + \tau), \tau \in [-\Delta, 0]$, where $\Delta > 0$ is the maximum involved delay; $\xi_{i,0} \in C_{n_i}$. The maps $f_i : C_{n_i} \to \mathcal{R}^{n_i}$, $H_i : C_{n_i} \times C_{n_{3-i}} \to \mathcal{R}^{n_i}$, $g_i : C_{n_i} \to \mathcal{R}^{n_i \times m_i}$ are locally Lipschitz and completely continuous. We combine vectors as $x(t) = [x_1(t)^T, x_2(t)^T]^T \in \mathcal{R}^n$, $n = n_1 + n_2$, $u(t) = [u_1(t)^T, u_2(t)^T]^T \in \mathcal{R}^m$, $d(t) = [d_1(t)^T, d_2(t)^T]^T \in \mathcal{R}^m$, $m = m_1 + m_2$, $\xi_0 = [\xi^T_{1,0}, \xi^T_{2,0}]^T \in C_n$, $f(\cdot) = [f_1(\cdot)^T, f_2(\cdot)^T]^T$, $H(\cdot) = [H_1(\cdot)^T, H_2(\cdot)^T]^T$ and $g(\cdot) = [g_1(\cdot)^T, g_2(\cdot)^T]^T$. The element $x_t \in C_n$ is defined as for its i-th component $x_{i,t}$ (see Notations section). It is assumed that $f_i(0) = H_i(0, 0) = 0$, $i = 1, 2$. We use functionals $M_{a,i} : C_{n_i} \to \mathcal{R}^+$ for which there exist class $\mathcal{K}_\infty$ functions $\underline{\gamma}_{a,i}$, $\overline{\gamma}_{a,i}$, such that

$$\underline{\gamma}_{a,i}(|\phi_i(0)|) \le M_{a,i}(\phi_i) \le \overline{\gamma}_{a,i}(\|\phi_i\|_\infty), \quad \forall \phi_i \in C_{n_i} \tag{11}$$

For functionals $V_i : \mathcal{R}^{n_i} \times \mathcal{Q}_{n_i} \to \mathcal{R}^+$ in the class $\mathcal{V}_{n_i}$, $i = 1, 2$, let the maps $a_i : C_{n_i} \to \mathcal{R}$, $b_i : C_{n_i} \to \mathcal{R}^{m_i}$ (row vectors), $c_i : C_{n_i} \times C_{n_{3-i}} \to \mathcal{R}$, and $\rho_i : C_{n_i} \times C_{n_{3-i}} \to \mathcal{R}$ be defined, for $\phi_i \in C_{n_i}$, as follows:

$$
\begin{aligned}
a_i(\phi_i) &= \left.\frac{\partial V_i(x_i, \phi_{i_{[-\Delta,0)}})}{\partial x_i}\right|_{x_i=\phi_i(0)} f_i(\phi_i) + \left.\frac{\partial V_i\left(\phi_i(0), \phi^{\ell}_{i_{[-\Delta,0)}}\right)}{\partial \ell}\right|_{\ell=0}, \\
b_i(\phi_i) &= \left.\frac{\partial V_i(x_i, \phi_{i_{[-\Delta,0)}})}{\partial x_i}\right|_{x_i=\phi_i(0)} g_i(\phi_i), \\
c_i(\phi_i, \phi_{3-i}) &= \left.\frac{\partial V_i(x_i, \phi_{i_{[-\Delta,0)}})}{\partial x_i}\right|_{x_i=\phi_i(0)} H_i(\phi_i, \phi_{3-i}), \\
\rho_i(\phi_i, \phi_{3-i}) &= -\sqrt{a_i^2(\phi_i) + |b_i(\phi_i)|^4} + c_i(\phi_i, \phi_{3-i})
\end{aligned} \tag{12}
$$

Moreover, for a positive real r, let $k_{i,r} : C_{n_i} \to R^{m_i}$ be defined as follows, for $\phi_i \in C_{n_i}$,

$$
k_{i,r}(\phi_i) = \begin{cases} -\frac{a_i(\phi_i)+\sqrt{a_i^2(\phi_i)+|b_i(\phi_i)|^4}}{|b_i(\phi_i)|^2} b_i^T(\phi_i), & |b_i(\phi_i)| > r, \\ -\frac{a_i(\phi_i)+\sqrt{a_i^2(\phi_i)+|b_i(\phi_i)|^4}}{r^2} b_i^T(\phi_i), & |b_i(\phi_i)| \le r \end{cases} \tag{13}
$$

The following assumption will be used in the forthcoming theorem (see Assumption 6 in [14], Hypothesis 4 in [33], Hypothesis 18 in [34]).

Assumption 1 There exist functionals $V_i : \mathcal{R}^{n_i} \times \mathcal{Q}_{n_i} \to \mathcal{R}^+$, $i = 1, 2$, in the class $\mathcal{V}_{n_i}$, with corresponding maps a_i, b_i, c_i, ρ_i, positive reals r, p, non-negative integers h, h_d, functions $\underline{\alpha}_i, \overline{\alpha}_i, \alpha_i$ of class $\mathcal{K}_\infty$, integers $S_{i,j} \in \{0, 1\}$, functions $\sigma_{i,j}$ of class $\mathcal{K}$ and positive reals $\Delta_j \in (0, \Delta]$, $j = 0, 1, \ldots, h + h_d$, such that, $\forall\ \phi_i \in C_{n_i}$, the following conditions hold, for $i = 1, 2$:

(i) $\underline{\alpha}_i(M_{a,i}(\phi_i)) \le V_i(\phi_i(0), \phi_{i_{[-\Delta,0)}}) \le \overline{\alpha}_i(M_{a,i}(\phi_i))$;
(ii) $(b_i(\phi_i) = 0) \ \Rightarrow\ (a_i(\phi_i) \le 0)$;
(iii)

$$
\begin{aligned}
&\rho_i(\phi_i, \phi_{3-i}) \le -\alpha_i(M_{a,i}(\phi_i)) + S_{i,0}\sigma_{i,0}(M_{a,3-i}(\phi_{3-i})) \\
&+ \sum_{j=1}^{h} S_{i,j}\sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right) + \sum_{j=h+1}^{h+h_d} S_{i,j}\int_{-\Delta_j}^{0} \sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau;
\end{aligned} \tag{14}
$$

(iv) $\sup_{\{\psi_i \in C_i,\ 0<|b_i(\psi_i)|\le r\}} \frac{a_i(\psi_i)}{|b_i(\psi_i)|} \le p$.

Remark 2 By $h = 0$ (resp., $h_d = 0$), it is meant that the first (resp., second) sum in (14) vanishes. As well, when $h_d = 0$, the maximum term involving h_d in forthcoming equality (15) is meant to be zero.

Theorem 1 *Let Assumption 1 hold. Let $\sigma_i : \mathcal{R}^+ \to \mathcal{R}^+$, $i = 1, 2$, be the functions defined, for $s \in \mathcal{R}^+$, as*

$$\sigma_i(s) = \left(\sum_{k=0}^{h+h_d} S_{i,k} \right) \max \left\{ \max_{j=0,1,\ldots,h} S_{i,j}\sigma_{i,j}(s), \quad \max_{j=h+1,\ldots,h+h_d} S_{i,j}\Delta_j\sigma_{i,j}(s) \right\} \tag{15}$$

Assume also there exist reals $c_i > 1$, $i = 1, 2$, such that, $\forall s \in \mathcal{R}^+$, the small-gain inequality holds (see (17) in [14])

$$c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \le \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s) \tag{16}$$

Then:

(1) the maps $k_{i,r} : C_{n_i} \to \mathcal{R}^{m_i}$, $i = 1, 2$, are completely continuous and locally Lipschitz in C_{n_i};
(2) there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, chosen any positive real q, for any initial state ξ_0 and any measurable, locally essentially bounded disturbance $d(t)$, the corresponding solution $x(t)$ of the closed loop system (10) with decentralized control laws

$$u_i(t) = k_{i,r}(x_{i,t}) - qb_i^T(x_{i,t}), \qquad i = 1, 2, \tag{17}$$

exists for all $t \ge 0$ and, furthermore, satisfies the following inequality

$$|x(t)| \le \beta(\|x_0\|_\infty, t) + \gamma\left(\frac{\|d_{[0,t)}\|_\infty + 2p + r}{\sqrt{q}} \right) \tag{18}$$

Proof Let the map $k_i : C_{n_i} \to R^{m_i}$, $i = 1, 2$, be defined, for $\phi_i \in C_{n_i}$, as (Sontag's universal stabilizer, see [33, 34, 41])

$$k_i(\phi_i) = \begin{cases} -\frac{a_i(\phi_i)+\sqrt{a_i^2(\phi_i)+|b_i(\phi_i)|^4}}{|b_i(\phi_i)|^2} b_i^T(\phi_i), & b_i(\phi_i) \neq 0 \\ 0, & b_i(\phi_i) = 0 \end{cases} \tag{19}$$

Under Assumption 1, it is proved in [33] that the maps $k_{i,r}$ are completely continuous and locally Lipschitz in C_{n_i} and satisfy the inequality

$$|k_{i,r}(\phi_i) - k_i(\phi_i)| \le 2p + r, \ \forall \phi_i \in C_{n_i} \tag{20}$$

Let $W_i : C_{n_i} \to \mathcal{R}^+$ be defined, for $\phi_i \in C_{n_i}$ as $W_i(\phi_i) = V_i(\phi_i(0), \phi_{i_{[-\Delta,0)}}), i = 1, 2$. Then, the following inequalities hold for the functional $D^+W_i : C_{n_i} \times C_{n_{3-i}} \times \mathcal{R}^{m_i} \to \mathcal{R}^\star$, for any $\phi_i \in C_{n_i}$, $d_i \in \mathcal{R}^{m_i}$, $i = 1, 2$,

$$\begin{aligned}&D^+W_i(\phi_i, \phi_{3-i}, d_i) =\\&a_i(\phi_i) + b_i(\phi_i)k_{i,r}(\phi_i) - q b_i(\phi_i) b_i^T(\phi_i) + c_i(\phi_i, \phi_{3-i}) + b_i(\phi_i)d_i =\\&a_i(\phi_i) + b_i(\phi_i)k_i(\phi_i) + b_i(\phi_i)(k_{i,r}(\phi_i) - k_i(\phi_i)) - q|b_i(\phi_i)|^2 + c_i(\phi_i, \phi_{3-i}) + b_i(\phi_i)d_i\end{aligned} \tag{21}$$

By definition of the maps k_i, it is obtained

$$a_i(\phi_i) + b_i(\phi_i)k_i(\phi_i) = \begin{cases} -\sqrt{a_i^2(\phi_i) + |b_i(\phi_i)|^4}, & b_i(\phi_i) \neq 0, \\ a_i(\phi_i), & b_i(\phi_i) = 0 \end{cases} \tag{22}$$

Therefore, by the point (ii) in Assumption 1, taking into account of the definition of the map ρ_i in (12), it follows that

$$a_i(\phi_i) + b_i(\phi_i)k_i(\phi_i) + c_i(\phi_i, \phi_{3-i}) = \rho_i(\phi_i, \phi_{3-i}) \tag{23}$$

From (20), (21), (23), by Young's inequality, it is obtained

$$\begin{aligned}&D^+W_i(\phi_i, \phi_{3-i}, d_i) \leq |b_i(\phi_i)|(2p + r) - q|b_i(\phi_i)|^2 + b_i(\phi_i)d_i + \rho_i(\phi_i, \phi_{3-i}) \leq\\&-q|b_i(\phi_i)|^2 + |b_i(\phi_i)|(|d_i| + 2p + r) + \rho_i(\phi_i, \phi_{3-i}) \leq\\&-q|b_i(\phi_i)|^2 + q|b_i(\phi_i)|^2 + \frac{(|d_i| + 2p + r)^2}{4q} + \rho_i(\phi_i, \phi_{3-i}) =\\&\rho_i(\phi_i, \phi_{3-i}) + \sigma_{Ri} \circ \eta(|d_i|),\end{aligned} \tag{24}$$

where $\eta : \mathcal{R}^+ \to \mathcal{R}^+$ is the function defined, for $s \in \mathcal{R}^+$, as $\eta(s) = \frac{s+2p+r}{\sqrt{q}}$, and σ_{Ri}, $i = 1, 2$, is the function of class $\mathcal{K}_\infty$ defined, for $s \in \mathcal{R}^+$, as $\sigma_{Ri}(s) = \frac{1}{4}s^2$. We now remark that Lemmas 21, 23 in [14] hold as well if the argument $s \in \mathcal{R}^+$ of the functions σ_{Ri}, $i = 1, 2$, as defined in (14) in [14], is replaced by a continuous, increasing function mapping $\mathcal{R}^+$ to $\mathcal{R}^+$, as, for instance, η. Then, from point (iii) in Assumption 1, from Theorem 8 in [14] (taking into account of the above remark), it follows that there exist a locally Lipschitz functional $W_{cl} : C_n \to \mathcal{R}^+$ (see (18) in [14]), a functional $M_a : C_n \to \mathcal{R}^+$, functions $\underline{\gamma}_a$, $\overline{\gamma}_a$, $\underline{\alpha}$, $\overline{\alpha}$ and α_{cl} of class $\mathcal{K}_\infty$, and a function σ_{cl} of class $\mathcal{K}$, such that, for any $\phi \in C_n$, $d \in \mathcal{R}^m$, the inequalities hold (see (18) and D.4 in [14]),

$$\underline{\alpha}(M_a(\phi)) \leq W_{cl}(\phi) \leq \overline{\alpha}(M_a(\phi)), \qquad \underline{\gamma}_a(|\phi(0)|) \leq M_a(\phi) \leq \overline{\gamma}_a(\|\phi\|_\infty), \tag{25}$$

$$D^+W_{cl}(\phi, d) \leq -\alpha_{cl}(M_a(\phi)) + \sigma_{cl} \circ \eta(|d|) \tag{26}$$

From (25), (26), it follows that the solution exists $\forall t \geq 0$ and that the inequality (18) holds. The same reasoning used in the proof of Theorem 3.1 in [37] can be used here in order to obtain the ISpS result (see Definition 2.1 in [17]) described by the inequality (18).

Remark 3 We provide here a discussion on Assumption 1. The point (i) in Assumption 1 is standard in the 0-GAS, ISS theory for systems described by RFDEs (see [14, 37, 44], see in particular Lemma 4 in [14] as far as the lower bound is concerned). The point (ii) is the standard key property for a function V to be referred as a CLF (see [16, 20, 41]) and, for a functional V, to be referred as a CLKF (see, for instance, [15, 20, 21]). Notice that, in this case, it is allowed, for non-zero $\phi_i \in C_i$ satisfying $b_i(\phi_i) = 0$, that $a_i(\phi) = 0$ (see related discussions in [33, 34]). The point (iii) allows that, if the control input were equal to the state feedback obtained with Sontag's universal formula, then the derivative in Driver's form of the functionals V_i would satisfy a very general dissipative inequality with supply rates which may cope with both discrete and distributed time delays, in both subsystems and interconnections (see (13), (14) and Remark 10 in [14]). Notice that, in each dissipative inequality, only the map describing the dynamics of the related lower dimension subsystem is involved. This lower dimension may significantly simplify the analysis (namely, the computation of involved functions of class $\mathcal{K}$ and $\mathcal{K}_\infty$). The condition (14) incorporates the fact that, if the interconnection terms were zero (i.e., $H_i(\phi_i, \phi_{3-i}) = 0$, $i = 1, 2$, $\forall \phi_j \in C_{n,j}$, $j = 1, 2$, and $S_{i,j} = 0$, $i = 1, 2$, $j = 0, 1, \ldots, h + h_d$), each resulting subsystem would satisfy a standard inequality for 0-GAS, ISS concerns (see [14, 20, 22, 23, 37, 44]). The point (iv) is a key condition by which, using the methodology presented in [33], the problems related to non locally Lipschitz maps k_i, $i = 1, 2$, in (19) (i.e., the Sontag's universal stabilizers for subsystems), can be overcome. A similar condition was introduced in [16] in the framework of CLRFs (see Assumption 1 in [16]), and in [15] in the framework of CLKFs, for exploiting the domination redesign formula (see [39]).

Remark 4 Because of the inequality (18), the closed-loop system (10), (17) is ISpS (see Definition 2.1 in [17]) with respect to the disturbance $d(t)$. Notice in (18) that, if the disturbance is bounded, the solution can achieve an arbitrarily small neighborhood of the origin by increasing the control tuning parameter q.

4 Illustrative Numerical Example

Consider the interconnected system described by the following RFDE

$$\begin{aligned}
\dot{x}_1(t) &= x_1^3(t) + \omega_1 x_1(t) x_2(t-\Delta) + u_1(t) + d_1(t),\\
\dot{x}_2(t) &= x_2(t) + \omega_2 x_1^2(t-\Delta) + x_2(t)\left(u_2(t) + d_2(t)\right),
\end{aligned} \tag{27}$$

where $x_i, u_i, d_i \in \mathcal{R}$, $i = 1, 2$, Δ is a positive unknown real, $\omega_i \in (-2, 2)$, $i = 1, 2$. Let $V_i : \mathcal{R} \times Q_1 \to \mathcal{R}^+$, $i = 1, 2$, be defined, for $x_i \in \mathcal{R}$, $\psi_i \in Q_1$, as $V_i(x_i, \psi_i) =$

$\frac{1}{2}x_i^2$. Let $M_{a,i} : C_1 \to \mathcal{R}^+$, $i = 1, 2$, be defined, for $\phi_i \in C_1$, as $M_{a,i}(\phi_i) = |\phi_i(0)|$. As far as (11) and point (i) in Assumption 1 are concerned, they are satisfied by the functions $\underline{\gamma}_{a,i}$, $\overline{\gamma}_{a,i}$, $\underline{\alpha}_{a,i}$, $\overline{\alpha}_{a,i}$ of class $\mathcal{K}_\infty$, $i = 1, 2$, defined, for $s \in R^+$, as

$$\underline{\gamma}_{a,i}(s) = \overline{\gamma}_{a,i}(s) = s; \qquad \underline{\alpha}_i(s) = \overline{\alpha}_i(s) = \frac{1}{2}s^2. \tag{28}$$

As far as the functions a_i, b_i, ρ_i, $i = 1, 2$, defined in (12) are concerned, we have, for $\phi_i \in C_1$, $i = 1, 2$,

$$\begin{aligned}
&a_1(\phi_1) = \phi_1^4(0); \qquad b_1(\phi_1) = \phi_1(0); \qquad c_1(\phi_1, \phi_2) = \omega_1\phi_1^2(0)\phi_2(-\Delta);\\
&\rho_1(\phi_1, \phi_2) = -\phi_1^2(0)\sqrt{1 + \phi_1^4(0)} + \omega_1\phi_1^2(0)\phi_2(-\Delta);\\
&a_2(\phi_2) = \phi_2^2(0); \qquad b_2(\phi_2) = \phi_2^2(0); \qquad c_2(\phi_2, \phi_1) = \omega_2\phi_2(0)\phi_1^2(-\Delta);\\
&\rho_2(\phi_2, \phi_1) = -\phi_2^2(0)\sqrt{1 + \phi_2^4(0)} + \omega_2\phi_2(0)\phi_1^2(-\Delta)
\end{aligned} \tag{29}$$

Point (ii) in Assumption 1 is satisfied. As far as the point (iv) in Assumption 1 is concerned, it is satisfied and, in particular, for any positive real r, we obtain $p = \max\{r^3, 1\}$. As far as the point (iii) in Assumption 1 is concerned, the following inequalities hold, for any $\phi_i \in C_1$, $i = 1, 2$,

$$\begin{aligned}
\rho_1(\phi_1, \phi_2) &\le -\left(M_{a,1}(\phi_1)\right)^4 + \frac{1}{2}|\omega_1|\left(M_{a,1}(\phi_1)\right)^4 + \frac{1}{2}|\omega_1|\phi_2^2(-\Delta);\\
\rho_2(\phi_2, \phi_1) &\le -\left(M_{a,2}(\phi_2)\right)^2 + \frac{1}{2}|\omega_2|\left(M_{a,2}(\phi_2)\right)^2 + \frac{1}{2}|\omega_2|\phi_1^4(-\Delta)
\end{aligned} \tag{30}$$

Therefore, the point (iii) in Assumption 1 is satisfied with $h = 1$, $h_d = 0$ $S_{1,0} = S_{2,0} = 0$, $S_{1,1} = S_{2,1} = 1$, and the functions α_i, $\sigma_{i,1}$, $i = 1, 2$, of class $\mathcal{K}_\infty$ defined, for $s \in R^+$, as

$$\begin{aligned}
\alpha_1(s) &= \left(1 - \frac{1}{2}|\omega_1|\right)s^4; \qquad \sigma_{1,1}(s) = \frac{1}{2}|\omega_1|s^2;\\
\alpha_2(s) &= \left(1 - \frac{1}{2}|\omega_2|\right)s^2; \qquad \sigma_{2,1}(s) = \frac{1}{2}|\omega_2|s^4
\end{aligned} \tag{31}$$

Let σ_i, $i = 1, 2$, be the functions of class $\mathcal{K}_\infty$ defined, for $s \ge 0$, as $\sigma_i(s) = \sigma_{i,1}(s)$, according to (15). If the inequality holds

$$|\omega_1| + |\omega_2| < 2, \tag{32}$$

then the small-gain inequality (16) is satisfied for this example. All the hypotheses of Theorem 1 are satisfied for this example, provided that (32) holds. By Theorem 1, the feedback control law, for any chosen positive real q (see (13), (17)),

$$u_1(t) = \begin{cases} -x_1(t)\left(x_1^2(t) + \sqrt{1 + x_1^4(t)}\right) - qx_1(t), & |x_1(t)| > r \\ -\frac{1}{r^2}x_1^3(t)\left(x_1^2(t) + \sqrt{1 + x_1^4(t)}\right) - qx_1(t), & |x_1(t)| \le r, \end{cases}$$

$$u_2(t) = \begin{cases} -1 - \sqrt{1 + x_2^4(t)} - qx_2^2(t), & x_2^2(t) > r, \\ -\frac{1}{r^2}x_2^4(t)\left(1 + \sqrt{1 + x_2^4(t)}\right) - qx_2^2(t), & x_2^2(t) \le r, \end{cases} \tag{33}$$

is such that the closed-loop system (27), (33) satisfies the inequality (18), provided that the inequality (32) is satisfied. The control law (33) is memoryless, decentralized. As can be seen, since the system (27) involves time delays only in interconnections, the CLKFs, used for each subsystem, are actually CLFs defined in $\mathcal{R}$.

Finding controllers, by Sontag's formula, for (27) directly with an overall CLKF would be, at least, much more complicated than the methodology here presented, since this overall CLKF should involve integral terms to cope with the time-delays. One could try, for instance, with the candidate CLKF $V : \mathcal{R}^2 \times \mathcal{Q}_2 \to \mathcal{R}^+$ defined, for $x \in \mathcal{R}^2, \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in \mathcal{Q}_2$, as $V(x,\phi) = x^T P x + \int_{-\Delta}^{0} e^{\mu\theta}(\phi_1^4(\theta) + g\phi_2^2(\theta))d\theta$, with μ, g suitable positive reals and P a suitable positive definite symmetric matrix to be chosen. Then, one should prove that the points $(i) - (iv)$, Hypothesis 4, in [33], are satisfied. The analytical proof is not easy and, anyway, the resulting controller would be neither memoryless nor decentralized. Notice also that the small control property is not satisfied by subsystem 2 and V_2. If one applied the Sontag's universal formula for the controller in the subsystem 2, that controller would be not continuous whenever $x_2(t) = 0$. This would mean discontinuity of the overall feedback control law in the infinite dimensional subspace $\left\{\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C_2,\ \phi_i \in C_1,\ i = 1, 2,\ \phi_2(0) = 0\right\}$. By the use of the results in [33] this discontinuity problem is overcome (the feedback control law (33) is locally Lipschitz in C_2). If the disturbances $d_i(t)$, $i = 1, 2$, are bounded, then an arbitrarily small neighborhood of the origin can be reached, by increasing the control parameter q (see the inequality (18)). Simulations have been performed with $r = 1, q = 10, \omega_1 = \omega_2 = 0.9, d_1 = sin(2t), d_2(t) = cos(2t)$, $t \ge 0$, $x_0(\tau) = [\,1\ -1\,]^T$, $\tau \in [-\Delta, 0]$, $\Delta = 1.2$. The state variables are reported in Fig. 1. As can be seen, the state variables are kept suitably bounded by the control law (33), thus validating the theoretical results. In Fig. 2 the control signals are reported. If $u_1(t) = u_2(t) = 0$, $t \ge 0$, then simulations show divergence of the magnitude of the state variables to ∞. As well, simulations show that, increasing the tuning parameter q, smaller neighborhoods of the origin are asymptotically reached. In Fig. 3, the state variables are reported with the tuning parameter choice $q = 100$. The better performance with the increased value of the parameter q is achieved at the price of

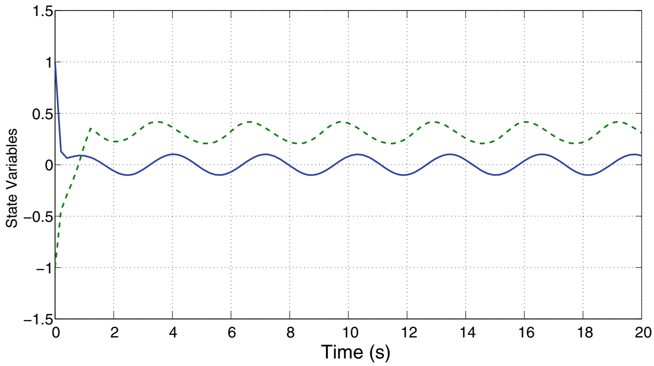

Fig. 1 State variables of (27), (33), with $\Delta = 1.2$, $q = 10$, $r = 1$, $\omega_1 = \omega_2 = 0.9$

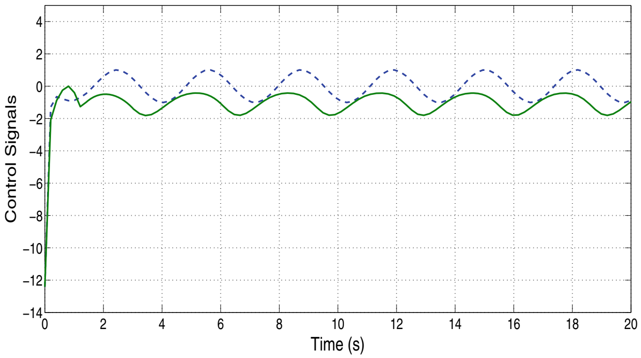

Fig. 2 Control signals (33), with $\Delta = 1.2$, $q = 10$, $r = 1$, $\omega_1 = \omega_2 = 0.9$

an increased control effort (the control signals reach in this case a maximum absolute value close to 120).

Remark 5 In general, when delays appear in the subsystems, CLKFs are involved for subsystems and checking Assumption 1 becomes more difficult, even in the linear case (the proposed control law is nonlinear also in this case), because of involved integral terms (see [14]). Numerical software tools may often be used to provide a sufficient confidence about satisfaction of inequalities involved in Assumption 1. Alternative conditions, which however require the satisfaction of the small control property, may be used in the disturbance-free case (see Hypothesis 8 in [34]). These alternative conditions avoid the use of the M_a functionals and may return to be easier to be checked, than the ones in Assumption 1, provided that the small control property holds. These alternative conditions will be topic of forthcoming investigation, as far as the control design by small-gain arguments is concerned.

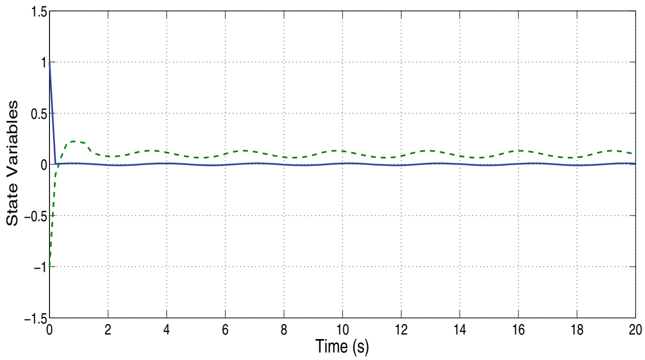

Fig. 3 State variables of (27), (33), with $\Delta = 1.2$, $q = 100$, $r = 1$, $\omega_1 = \omega_2 = 0.9$

5 Conclusions

Since finding a CLKF, and a related controller, for retarded systems, is in general not an easy task, this chapter has the aim to provide a constructive methodology for the design of controllers for a class of systems which are in the interconnection form. The approach makes use of a CLKF for each subsystem and of a suitably modified Sontag's universal formula. By suitable hypotheses, the resulting closed-loop system is proved to be ISpS with respect to actuator disturbances, exploiting a small-gain condition which copes with time delays. Here time delays in both subsystems and interconnections are dealt with, and a significant simplification of the control design is achieved, with respect to the case of control design by an overall CLKF. For instance, when the delays affect only the interconnections, the CLKF for each subsystem becomes a CLF, defined on finite dimensional Euclidean spaces rather than on infinite dimensional Banach spaces, with evident improvement towards simplification. On the other hand, a small gain condition is required to be satisfied. A major goal of this work is to make it possible to exploit, in a unified framework, some existing results in the past literature, at the aim of the controller design. We still assume the strict matching condition (i.e., disturbance belonging to the input space and adding to the control law) in the chapter. This restrictive assumption may be removed through systematic use of backstepping and small-gain techniques. An interesting research topic may be the application of the methodology here shown to networked systems, for instance by the use of the small-gain results provided in [13, 42]. This would further simplify the design of decentralized, input-to-state practically stabilizing controllers, provided that a suitable network small-gain condition can be satisfied.

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Delay Systems: Modeling and Analysis

Stability Analysis of Uniformly Distributed Delay Systems: A Frequency-Sweeping Approach

Xu-Guang Li, Silviu-Iulian Niculescu, Arben Çela and Lu Zhang

1 Introduction

Consider the following general distributed delay system

$$\dot{x}(t) = Ax(t) + B\int_{-\infty}^{t} \kappa(t-\theta)x(\theta)d\theta, \tag{1}$$

under some appropriate initial conditions, where A and B are constance matrices and $\kappa(\theta) : [0, \infty) \mapsto [0, \infty)$ is a scalar kernel function. The model (1) *includes* the scenario of point-wise delay systems. For instance, if $\kappa(\theta) = \delta(\theta - \tau)$ ($\delta(\theta)$ is the Dirac delta function), system (1) reduces to: $\dot{x}(t) = Ax(t) + Bx(t-\tau)$.

In the literature on distributed delay systems, there are two common kernel functions: gamma distribution and uniform distribution. One may refer to e.g., [13] for

X.-G. Li (✉) · L. Zhang
School of Information Science and Engineering, Northeastern University, Shenyang 110819, Liaoning, China

State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang 110819, Liaoning, China
e-mail: masdanlee@163.com

L. Zhang
e-mail: zhanglu880226@163.com

S.-I. Niculescu
Laboratoire des Signaux et Systèmes (L2S UMR CNRS 8506), CNRS-CentraleSupélec-Université Paris-Sud, 91192 Gif-Sur-Yvette, France
e-mail: Silviu.Niculescu@l2s.centralesupelec.fr

A. Çela
UPE, ESIEE Paris, Laboratoire Images, Signaux et Systèmes Intelligents, 93162 Noisy-Le-Grand, France
e-mail: a.cela@esiee.fr

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a detailed introduction to these distributions from the perspective of probability and statistics.

For system (1) with gamma distribution, the characteristic function is equivalent to a quasipolynomial (as for a point-wise delay system), see the analysis in [12]. Therefore, the existing results for systems with point-wise delays can be applied, directly or with some slight modifications, to such distributed delay systems.

In this chapter, we consider $\kappa(\theta)$ of uniform distribution:

$$\kappa(\theta) = \begin{cases} \frac{1}{d_1+d_2}, & \text{if } \tau - d_1 < \theta < \tau + d_2, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $\tau \geq d_1 \geq 0$, $d_2 \geq 0$, and $d_1 + d_2 > 0$.

The system described by (1)–(2) is called a uniformly distributed delay system (UDDS). The application of the UDDS model can be found in e.g., [1, 2, 10, 14].

The objective of this chapter is to analyze the stability of the UDDS w.r.t. τ along the whole interval $[d_1, \infty)$, given d_1 and d_2. Compared to the studies for retarded and neutral systems (for which the stability in the whole τ domain now can be solved), some additional issues need to be considered for the UDDS.

Here, Let us have a quick look at a scalar UDDS (leaving the general UDDS to be studied in later sections), i.e., when A and B are scalars a and b, which corresponds to a simple characteristic function $f(\lambda, \tau) : \mathbb{C} \times [d_1, \infty) \mapsto \mathbb{C} = \lambda - a - b\frac{e^{-(\tau-d_1)\lambda} - e^{-(\tau+d_2)\lambda}}{(d_1+d_2)\lambda}$. To study the stability for $\tau \in [d_1, \infty)$, three technical issues arise.

(i) First, $f(\lambda, \tau)$ is not defined at $\lambda = 0$. However, $\lambda = 0$ may be a potential characteristic root. By L'Hôpital's rule, $\lim_{\lambda \to 0} f(\lambda, \tau) = -a - b$. Hence, $\lambda = 0$ is a characteristic root if and only if $a + b = 0$. One can see that if $\lambda = 0$ is a characteristic root, it is independent of τ. The criterion for the general case will be given in this chapter.

(ii) Second, we need to analyze the spectrum at the minimum value of τ, i.e., $\tau = d_1$. This is a necessary step required by the τ-decomposition idea, which will be explained later in this chapter. For a retarded system with its characteristic function, say, $\lambda - a - be^{-\tau\lambda}$, it is easy to study at the minimum value $\tau = 0$: $\lambda = a$. However, the UDDS is still infinitely dimensional at the minimum value $\tau = d_1$.

(iii) Third, we need to analyze the asymptotic behavior of the critical imaginary roots (CIRs) at the corresponding critical delays (CDs). This is a key step of the stability analysis for most types of delay systems (see more details in Sect. 2.2). Suppose $\lambda = j\omega^*$ is a CIR for the UDDS at a CD $\tau = \tau^*$ (i.e., $f(j\omega^*, \tau^*) = 0$, $\omega^* \in \mathbb{R}_+$, $\tau^* \in \mathbb{R}_+ \cup \{0\}$). Then, $\lambda = j\omega^*$ is a CIR for all $\tau = \tau^* + \frac{2k\pi}{\omega^*} \geq 0$, $k \in \mathbb{Z}$. That is, a CIR has *infinitely many* CDs and hence it is impossible to analyze the asymptotic behavior at all the infinitely many CDs one by one.

The stability of scalar UDDSs has been extensively investigated, see e.g., [1, 2]. In this chapter, we will address the general form of the UDDS (i.e., the coefficients are allowed to be matrices) and study the stability along the whole semi-infinite interval $[d_1, \infty)$.

Towards this end, we need to solve all the above technical issues. For technical issue (ii), we will adopt a method based on the argument principle, while for technical issues (i) and (iii) the solutions can be obtained by extending the recently-proposed frequency-sweeping framework [8].

Then, we will obtain a procedure for studying the stability of UDDS. This procedure, mainly based on the frequency-sweeping approach, is simple to implement.

This chapter is organized as follows. Preliminaries and prerequisites are given in Sect. 2. The main results are presented in Sect. 3. Illustrative examples are given in Sect. 4. Finally, some concluding remarks end the chapter in Sect. 5.

Notations: $\mathbb{R}$ ($\mathbb{R}_+$) denotes the set of (positive) real numbers and $\mathbb{C}$ is the set of complex numbers. $\mathbb{C}_-$ and $\mathbb{C}_+$, denote respectively the left half-plane and right half-plane in $\mathbb{C}$. $\mathbb{C}_0$ is the imaginary axis and $\partial\mathbb{D}$ is the unit circle, in $\mathbb{C}$. $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$ are the sets of integers, non-negative integers, and positive integers, respectively. ε is a sufficiently small positive real number, mainly used to describe the infinitesimal change of λ ($\Delta\lambda = \pm\varepsilon j$) and τ ($\Delta\tau = \pm\varepsilon$). I is the identity matrix of appropriate dimensions. For $\gamma \in \mathbb{R}$, $\lceil\gamma\rceil$ denotes the smallest integer greater than or equal to γ. Finally, $\det(\cdot)$ denotes the determinant of its argument.

2 Preliminaries and Prerequisites

In this section, preliminaries and prerequisites regarding the stability problem of UDDSs are given.

2.1 Characteristic Function

The characteristic function of the general form of the UDDS is

$$f(\lambda, \tau) = \det(\lambda I - A - B\tfrac{e^{-(\tau-d_1)\lambda} - e^{-(\tau+d_2)\lambda}}{(d_1+d_2)\lambda}), \lambda \neq 0. \tag{3}$$

Clearly, the characteristic function $f(\lambda, \tau)$ (3) is not defined at $\lambda = 0$. As earlier mentioned, $\lambda = 0$ may be a potential characteristic root. The related analysis will be given in Sect. 3.1.

The asymptotic stability of UDDS is determined by its characteristic roots (i.e., the roots λ for the characteristic equation $f(\lambda, \tau) = 0$): The UDDS is asymptotically stable if and only if all the characteristic roots are located in $\mathbb{C}_-$.

For *any* $\tau \in [d_1, \infty)$, the UDDS has *infinitely many* characteristic roots and hence we need to follow the τ-decomposition idea for studying the stability problem in this chapter, see the next subsection.

2.2 Stability Problem and τ-Decomposition Idea

Naturally, we want to determine the stability property (asymptotically stable or not) for the UDDS along the whole interval $\tau \in [d_1, \infty)$. This stability problem is the objective of the current chapter.

As commonly adopted in the literature (see e.g., [7]), the notation $NU(\tau) \in \mathbb{N}$ denotes the number of characteristic roots located in $\mathbb{C}_+$, in the presence of delay τ. In order to study the stability, we will inspect $NU(\tau)$ as τ increases from the minimum point $\tau = d_1$. Clearly, for a given τ, the UDDS is asymptotically stable if and only if there are no critical imaginary roots (CIRs), i.e., characteristic roots located on the imaginary axis $\mathbb{C}_0$, and $NU(\tau) = 0$.

As for retarded and neutral delay systems, in this chapter we adopt the τ-decomposition idea (see e.g., [7]), which is based on the continuity property of the spectra. We now briefly introduce this idea.

As τ increases from d_1, $NU(\tau)$ changes only when the system has CIRs. The values of τ at which the system has CIRs are called the critical delays (CDs). It is trivial to conclude that if the system has no CDs for any $\tau \in [d_1, \infty)$, then $NU(\tau)$ is a constant for all $\tau \in [d_1, \infty)$. Thus, in the sequel we mainly consider the case with CDs. All the CDs divide the interval $[d_1, \infty)$ into subintervals and within each subinterval $NU(\tau)$ is a constant. If we know the change of $NU(\tau)$ at each CD (corresponding to a boundary point of two adjacent subintervals), we are able to inspect $NU(\tau)$ along the whole interval $[d_1, \infty)$.

The above is the so-called τ-decomposition idea, along which the stability analysis requires to solve the following Problems 1 and 2.

Problem 1 How to exhaustively detect the critical imaginary roots (CIRs).

As a straightforward application of the frequency-sweeping framework [8], Problem 1 can be easily solved from the FSCs.

Letting $z = e^{-\tau\lambda}$ and $\mu(\lambda) = \frac{e^{d_1\lambda} - e^{-d_2\lambda}}{(d_1+d_2)\lambda}$, we can rewrite the characteristic function $f(\lambda, \tau)$ (3) as

$$p(\lambda, z) = \det(\lambda I - A - B\mu(\lambda)z), \lambda \neq 0. \tag{4}$$

Furthermore, we express $p(\lambda, z)$ as a polynomial of z:

$$p(\lambda, z) = a_0(\lambda) + a_1(\lambda)\mu(\lambda)z + \cdots + a_q(\lambda)\mu^q(\lambda)z^q, \tag{5}$$

where $a_i(\lambda)$ are polynomials of λ such that

$$\deg(a_0(\lambda)) > \max\{\deg(a_1(\lambda)), \ldots, \deg(a_q(\lambda))\}.$$

Remark 1 We rule out a trivial case that $a_0(\lambda), \ldots, a_q(\lambda)\mu^q(\lambda)$ have common zeros in $\mathbb{C}_+ \cup \mathbb{C}_0$ (otherwise, the UDDS is not asymptotically stable for any $\tau \in [d_1, \infty)$).

The detection of the CIRs and CDs for $f(\lambda, \tau) = 0$ amounts to detecting the critical pairs (λ, z) ($\lambda \in \mathbb{C}_0$ and $z \in \partial\mathbb{D}$) for $p(\lambda, z) = 0$. Due to the conjugate symmetry of the spectrum, it suffices to consider only the CIRs with non-negative imaginary parts.

Without loss of generality, suppose there are u such critical pairs denoted by $(\lambda_0 = j\omega_0, z_0)$, …, $(\lambda_{u-1} = j\omega_{u-1}, z_{u-1})$ with $0 < \omega_0 \leq \cdots \leq \omega_{u-1}$. Once all the critical pairs $(\lambda_\alpha, z_\alpha)$ are found, all the critical pairs (λ, τ) for $f(\lambda, \tau) = 0$ can be obtained: For each CIR λ_α, the corresponding (infinitely many) CDs are given by $\tau_{\alpha,k} \triangleq \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}$, $k \in \mathbb{N}$, $\tau_{\alpha,0} \triangleq \min\{\tau \geq d_1 : e^{-\tau\lambda_\alpha} = z_\alpha\}$ (recall that $\tau \geq d_1$ for the UDDS). The pairs $(\lambda_\alpha, \tau_{\alpha,k})$, $k \in \mathbb{N}$, define a set of critical pairs associated with $(\lambda_\alpha, z_\alpha)$.

All the critical pairs may be detected from the frequency-sweeping curves (FSCs), which are generated by the procedure to be introduced in Sect. 2.3.

Problem 2 How to analyze the asymptotic behavior of the CIRs w.r.t. the infinitely many CDs.

For a CIR λ_α, its asymptotic behavior at a CD $\tau_{\alpha,k} > d_1$, from the stability perspective, can be described by a notation $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$. Recall that the notation $\Delta NU_\alpha(\beta)$, where (α, β) is a critical pair, stands for the number change of the unstable roots caused by the variation of the CIR $\lambda = \alpha$ as τ increases from $\beta - \varepsilon$ to $\beta + \varepsilon$.

The value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ at a $\tau_{\alpha,k}$ can be precisely calculated by invoking the Puiseux series for the critical pair $(\lambda_\alpha, \tau_{\alpha,k})$. The general method for invoking the Puiseux series can be found in Chap. 4 of [8]. However, since a CIR has infinitely many CDs, such a method can not be applied to all the infinitely many CDs one by one. That is why we need an in-depth understanding of the CIRs' asymptotic behavior for the UDDS.

In this chapter, we will prove that for a CIR λ_α, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant for all $\tau_{\alpha,k} > d_1$. With this crucial property, called the *invariance property*, we can solve Problem 2.

Finally, we will obtain the explicit expression of $NU(\tau)$ and hence we can analyze the stability for the UDDS in the whole τ domain.

2.3 Frequency-Sweeping Framework

For the UDDS, the frequency-sweeping curves (FSCs) can be generated by the following procedure.

Frequency-Sweeping Curves (FSCs): Sweep $\omega > 0$ and for each $\lambda = j\omega$ we have q solutions of z such that $p(j\omega, z) = 0$ (denoted by $z_1(j\omega), \ldots, z_q(j\omega)$). In this way, we obtain q FSCs $\Gamma_i(\omega)$: $|z_i(j\omega)|$ vs. ω, $i = 1, \ldots, q$. For simplicity, we denote by $\mathfrak{I}_1$ the line parallel to the abscissa axis with ordinate equal to 1. If $(\lambda_\alpha, \tau_{\alpha,k})$ is a critical pair, then some FSCs intersect $\mathfrak{I}_1$ at $\omega = \omega_\alpha$.

It is easy to see that all the CIRs and CDs can be detected from the FSCs (i.e., Problem 1 may be solved without much difficulty).

A new idea of the frequency-sweeping framework established in [8] is that the asymptotic behavior of the FSCs is taken into account. For a set of critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$, there must exist some FSCs such that $z_i(j\omega_\alpha) = z_\alpha = e^{-\tau_{\alpha,0}\lambda_\alpha}$ intersecting $\mathfrak{I}_1$ when $\omega = \omega_\alpha$. Among these FSCs, we denote the number of those when $\omega = \omega_\alpha + \varepsilon$ $(\omega = \omega_\alpha - \varepsilon)$ above $\mathfrak{I}_1$ by $NF_{z_\alpha}(\omega_\alpha + \varepsilon)$ $(NF_{z_\alpha}(\omega_\alpha - \varepsilon))$. We introduce a notation $\Delta NF_{z_\alpha}(\omega_\alpha)$ to describe the asymptotic behavior of the FSCs, as

$$\Delta NF_{z_\alpha}(\omega_\alpha) = NF_{z_\alpha}(\omega_\alpha + \varepsilon) - NF_{z_\alpha}(\omega_\alpha - \varepsilon). \tag{6}$$

Remark 2 It is a useful property that for a set of critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$, $k \in \mathbb{N}$, the corresponding $\Delta NF_{z_\alpha}(\omega_\alpha)$ is a constant, independent of k. For retarded- and neutral-type delay systems, the invariance property was confirmed through proving that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$ (the mathematical development is from an analytic curve perspective, see [8]). This line will be used as well in this chapter.

3 Main Results

In this section, the three technical issues mentioned earlier will be solved separately and then a procedure for the stability analysis along the whole interval $\tau \in [d_1, \infty)$ will be presented.

3.1 Detecting Characteristic Roots $\lambda = 0$

As mentioned, $f(\lambda, \tau)$ (3) is not defined at $\lambda = 0$. In this chapter, we study the case $\lambda \to 0$ by using L'Hôpital's rule and have:

Property 1 *For the uniformly distributed delay system described by (1) and (2), $\lambda = 0$ is a characteristic root for all $\tau \in [d_1, \infty)$ if and only if* $\det(A + B) = 0$.

Proof In view of the expression (5), Property 1 can be proved if the two conditions "$z = e^{-\tau \times 0} = 1$ is a characteristic root for $p(\lambda, z) = 0$ as $\lambda \to 0$" and "$\det(A + B) = 0$" are equivalent.

By L'Hôpital's rule, $\lim_{\lambda \to 0} \mu(\lambda) = 1$ and hence

$$\lim_{\lambda \to 0} p(\lambda, z) = a_0(0) + a_1(0)z + \cdots + a_q(0)z^q. \tag{7}$$

It is not hard to find that the limit (7) is exactly the expression of $\det(-A - Bz) = \det(-(A + Bz))$. The equivalence can be seen and thus the proof is complete. □ ■

Furthermore, we may directly check the condition in Property 1 from the FSCs (without calculating $\det(A + B)$), as stated below.

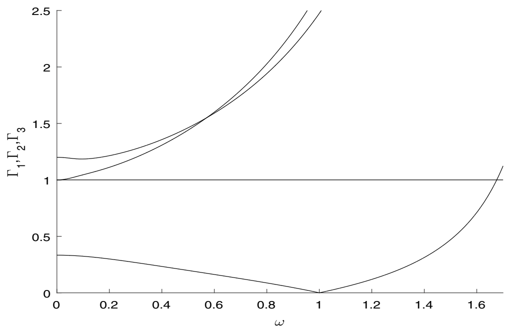


Fig. 1 FSCs for Example 1

Property 2 *For the uniformly distributed delay system described by (1) and (2), $\lambda = 0$ is a characteristic root for all $\tau \in [d_1, \infty)$ if and only if there exists a $z_i(j\omega)$ ($i \in \{1, \dots, q\}$) such that $\lim_{\omega \to 0} z_i(j\omega) = 1$.*

(Recall that $z_i(j\omega)$, $i = 1, \dots, q$, denote the q solutions of $p(j\omega, z) = 0$, see the procedure for generating the FSCs introduced in Sect. 2)

Proof The FSCs are generated according to the equation $p(j\omega, z) = 0$. It follows from (7) that

$$\lim_{\omega \to 0} p(j\omega, z) = a_0(0) + a_1(0)z + \cdots + a_q(0)z^q.$$

Then, following the line of the proof of Property 1, we may prove Property 2. □ ■

That is, if $\lambda = 0$ is a characteristic root, one of the FSCs must approach $\Im_1$ as $\omega \to 0$.

Obviously, the UDDS can not be asymptotically stable for any $\tau \in [d_1, \infty)$ if $\lambda = 0$ is a characteristic root.

Example 1 Consider the UDDS with

$$A = \begin{pmatrix} -1 & 0.5 & 2.5 \\ 1 & 3 & 2 \\ -1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0.5 & -1.5 \\ 3 & 2 & 4 \\ 3 & 2 & 1 \end{pmatrix}.$$

We may know that $\lambda = 0$ is a characteristic root for all $\tau \in [d_1, \infty)$ either by Property 1 or by Property 2 (the FSCs are shown in Fig. 1). □

Although the UDDS in Example 1 can not be asymptotically stable, we may use the procedure to be developed to check if the UDDS may be marginally stable, if needed.

Remark 3 For other types of distributed delay systems, $\lambda = 0$ may be a characteristic root *only at finitely many values of* τ, see [16].

3.2 Some Spectral Properties at Minimum Value of τ

For most types of time-delay systems, the minimum value of τ is 0. For instance, a retarded system $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ reduces to $\dot{x}(t) = (A+B)x(t)$ at the minimum point $\tau = 0$ whose (finite-dimensional) spectrum is simply composed of the eigenvalues of $A + B$.

However, it is not as straightforward to study the UDDS at the minimum value of τ (i.e., d_1), since the UDDS retains infinitely dimensional at $\tau = d_1$.

For this reason, we will adopt an argument principle-based method to compute $NU(d_1 + \varepsilon)$ (the value of $NU(d_1 + \varepsilon)$ is always needed for studying the stability problem in this chapter, see Theorem 3 given later). Similar applications of the argument principle can be found in e.g., [4, 6, 15].

First, it is easy to see that any nonzero characteristic root for the UDDS must be a characteristic root for the following characteristic equation

$$\det(\lambda^2 I - \lambda A - B\frac{e^{-(\tau-d_1)\lambda} - e^{-(\tau+d_2)\lambda}}{d_1 + d_2}) = 0. \tag{8}$$

As the characteristic function in (8) is a quasipolynomial of retarded type (i.e., the highest-order term of λ does not involve a transcendental term), we have the following properties (from Proposition 1.8 and Corollary 1.9 of [11]).

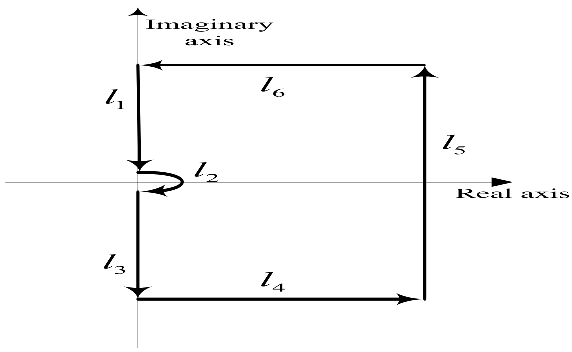
Property 3 *For a finitely large $\tau \geq d_1$, $NU(\tau)$ for the uniformly distributed delay system described by (1) and (2) is finite.*

Property 4 *If the uniformly distributed delay system described by (1) and (2) has unstable roots, their real parts and imaginary parts must be bounded.*

Therefore, if the UDDS has unstable roots, they must lie in the interior of a positively oriented Jordan curve l, where $l = l_1 \cup l_2 \cup l_3 \cup l_4 \cup l_5 \cup l_6$ is depicted in Fig. 2. The construction of l is explained below:

First, since the characteristic function (3) is not defined at $\lambda = 0$, we choose a semicircle sufficiently close to the origin, l_2, to link l_1 and l_3. In this way, the Jordan

Fig. 2 Jordan curve



curve l does not pass through the origin. Second, for simplicity, the Jordan curve l is constructed in a symmetric structure w.r.t. the real axis. Finally, to assure that all unstable roots (if any!) are contained in the interior of l, we may simply let the lengths of l_1, l_3, l_4, l_5, and l_6 be sufficiently large.

Then, according to the argument principle (see e.g., Chap. 4 of [5]), we have the following theorem.

Theorem 1 *The value of $NU(d_1 + \varepsilon)$ equals the winding number of $f(\lambda, d_*)$ w.r.t. the origin as λ varies along the positively oriented Jordan curve l, where*

$$d_* = \begin{cases} d_1, & \text{if } d_1 \text{ is not a critical delay,} \\ d_1 + \varepsilon, & \text{otherwise.} \end{cases}$$

Remark 4 If d_1 is a CD, then the image of $f(\lambda, d_1)$ passes through the origin. Thus, for a practical application of Theorem 1, it is easy to examine if d_1 is a CD.

Remark 5 As will be illustrated by Examples 2 and 3, Theorem 1 mainly requires an argument test. The computational load for such a graphical method is not high.

Example 2 Consider the UDDS with $d_1 = d_2 = \frac{\pi}{2}$ and

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{\pi^4+3\pi^2-4}{\pi^2(\pi^2+1)} & \frac{2\pi}{\pi^2+1} \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ \frac{\pi^2+4}{\pi(\pi^2+1)} & \frac{-1}{\pi^2+1} \end{pmatrix}.$$

We use Theorem 1 to analyze the spectrum at the minimum value of τ. As d_1 is not a CD, we analyze the argument of $f(\lambda, d_1)$ as λ varies along the Jordan curve. The image of $f(\lambda, d_1)$ is given in Fig. 3a, where we see that the winding number w.r.t. the origin is 2. According to Theorem 1, $NU(d_1 + \varepsilon) = 2$. □

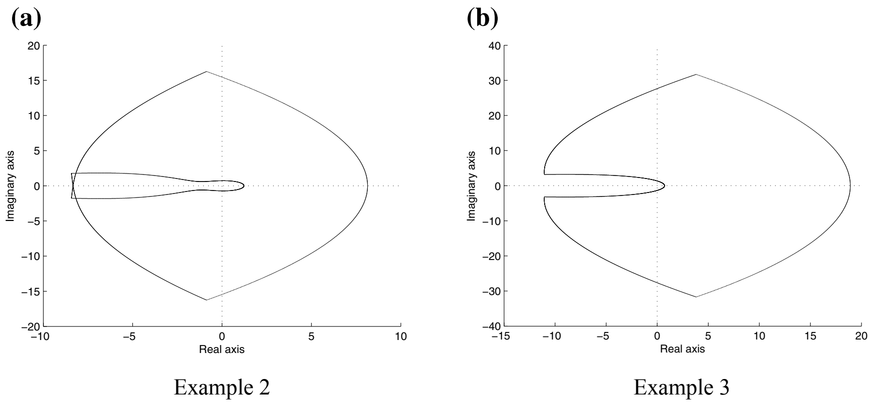


Fig. 3 Image of $f(\lambda, d_1)$ for Examples 2 and 3

Example 3 Consider the UDDS with $d_1 = d_2 = 0.2$ and

$$A = \begin{pmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.$$

As the system has no CIRs when $\tau = d_1$, we analyze the argument of $f(\lambda, d_1)$ according to Theorem 1. The image of $f(\lambda, d_1)$ as λ varies along the Jordan curve is shown in Fig. 3b. As the winding number w.r.t. the origin is 0, $NU(d_1 + \varepsilon) = 0$ in the light of Theorem 1. □

3.3 Invariance Property

It was seen in Sect. 3.1 that if $\lambda = 0$ is a characteristic root then the UDDS is not asymptotically stable for any $\tau \in [d_1, \infty)$. In this context, when considering the asymptotic behavior of CIRs and the related invariance property, we refer to the *nonzero* CIRs.

The characteristic functions for the retarded- and neutral-type delay systems are *quasipolynomials* of the form

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}, \tag{9}$$

where $a_0(\lambda), \ldots, a_q(\lambda)$ are polynomials of λ with real coefficients.

It was confirmed in [8] that for the CIRs of $f(\lambda, \tau) = 0$ (with $f(\lambda, \tau)$ in the form (9)), the invariance property holds. The main result is given by Theorem 8.5 therein (the idea of the proof is introduced in Remark 2 of this chapter).

We now analyze if the above invariance property holds for the UDDS.

First, in view of (5), the characteristic function $f(\lambda, \tau)$ (3) can be expressed as:

$$f(\lambda, \tau) = \sum_{i=0}^{q} \widetilde{a}_i(\lambda)e^{-q\tau\lambda}, \tag{10}$$

where

$$\widetilde{a}_i(\lambda) = a_i(\lambda)\mu^i(\lambda).$$

The characteristic functions (9) and (10) have two common points: (i) They are both polynomials of $e^{-\tau\lambda}$ and the corresponding coefficient functions (i.e., $a_i(\lambda)$ for (9) and $\widetilde{a}_i(\lambda)$ for (10)) are all independent of τ. (ii) The coefficient functions for (9) and (10) are all analytic near the CIRs (it is easy to see that $a_i(\lambda)$ are analytic in $\mathbb{C}$ and that $\widetilde{a}_i(\lambda)$ are analytic in $\mathbb{C}/\{0\}$).

Then, based on the above common points and following the line of the proof for Theorem 8.5 in [8], we have:

Theorem 2 *For a critical imaginary root λ_a of the uniformly distributed delay system described by (1) and (2), $\Delta NU_{\lambda_a}(\tau_{a,k})$ is a constant $\Delta NF_{z_a}(\omega_a)$ for all $\tau_{a,k} > d_1$.*

The contribution of Theorem 2 is twofold: First, the invariance property is confirmed for the UDDS, with which we will be able to systematically study the stability (see Sect. 3.4). Second, a graphical criterion is obtained to determine $\Delta NU_{\lambda_a}(\tau_{a,k})$ (since the constant value of $\Delta NF_{z_a}(\omega_a)$ can be easily observed from the FSCs).

Remark 6 The invariance property for the UDDS (Theorem 2) may also be proved from the perspective of general quasipolynomials [9]. From this perspective, the invariance property for a broader class of time-delay systems, such as retarded-type, neutral-type, distributed-type, fractional-order time-delay systems, and systems with incommensurate delays, can be proved.

Example 4 Consider the UDDS in Example 2.

At $\tau = (2k+1)\pi$, $\lambda = j$ is a CIR: $\lambda = j$ is a double CIR at $\tau = \pi$ while $\lambda = j$ is simple at all $\tau = (2k+1)\pi$, $k \in \mathbb{N}_+$.

The FSCs are given in Fig. 4a, where we see that $\Delta NF_{-1}(1) = 0$. Then, by Theorem 2 , $\Delta NU_j((2k+1)\pi) = \Delta NF_{-1}(1) = 0$ for all $k \in \mathbb{N}$.

Next, we verify the above result through the series analysis. The Puiseux series for the critical pair (j, π) is

$$\Delta\lambda = (0.2290 + 0.2930j)(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}}).$$

The Taylor series for the critical pairs $(j, 3\pi)$ and $(j, 5\pi)$ are respectively:

$$\Delta\lambda = -0.1592j\Delta\tau + (-0.0283 + 0.0324j)(\Delta\tau)^2 + o((\Delta\tau)^2),$$

and

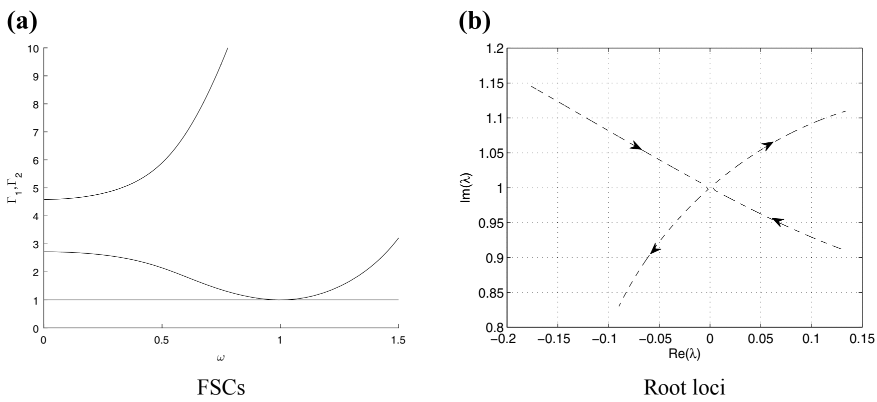


Fig. 4 FSCs and root loci for Example 4

$$\Delta\lambda = -0.0796j\Delta\tau + (-0.0035 + 0.0072j)(\Delta\tau)^2 + o((\Delta\tau)^2).$$

We also numerically generate the root loci of the double CIR j near the critical pair (λ, π) (using the MATLAB-based package DDE-BIFTOOL [3]), as shown in Fig. 4b.

Both the series analysis and the root loci are consistent with the result derived from Theorem 2. □

3.4 Stability Analysis Procedure

Combining the results proposed in the previous subsections, we are now able to study the stability of the UDDS along the whole delay interval $\tau \in [d_1, \infty)$. The procedure is as follows:

Step 1: Generate the frequency-sweeping curves (FSCs).

Step 2: Check if $\lambda = 0$ is a characteristic root for all $\tau \in [d_1, \infty)$ by Property 2. If so, the UDDS can not be asymptotically stable for any $\tau \in [d_1, \infty)$.

Step 3: Determine all the critical imaginary roots (CIRs) and the corresponding critical delays (CDs) according to the FSCs.

Step 4: Calculate $NU(d_1 + \varepsilon)$ by using Theorem 1.

Step 5: For each CIR λ_α, we may choose any CD $\tau_{\alpha,k} > d_1$ to compute $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ (the value is denoted by U_{λ_α}). Alternatively, we may directly have from the FSCs that $U_{\lambda_\alpha} = \Delta NF_{z_\alpha}(\omega_\alpha)$, according to Theorem 2.

With the steps above, we obtain the explicit expression of $NU(\tau)$ for the UDDS, as stated in the following theorem.

Theorem 3 *For any $\tau > d_1$ which is not a critical delay, $NU(\tau)$ for the uniformly distributed delay system described by (1) and (2) can be explicitly expressed as*

$$NU(\tau) = NU(d_1 + \varepsilon) + \sum_{\alpha=0}^{u-1} NU_\alpha(\tau), \tag{11}$$

where

$$NU_\alpha(\tau) = \begin{cases} 0, \ \tau < \tau_{\alpha,0}, \\ 2U_{\lambda_\alpha}\left\lceil \frac{\tau-\tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, \ \tau > \tau_{\alpha,0}, \end{cases} \text{if } \tau_{\alpha,0} \neq d_1,$$

$$NU_\alpha(\tau) = \begin{cases} 0, \ \tau < \tau_{\alpha,1}, \\ 2U_{\lambda_\alpha}\left\lceil \frac{\tau-\tau_{\alpha,1}}{2\pi/\omega_\alpha} \right\rceil, \ \tau > \tau_{\alpha,1}, \end{cases} \text{if } \tau_{\alpha,0} = d_1,$$

The UDDS is asymptotically stable if and only if τ lies in the interval(s) with $NU(\tau) = 0$ excluding the CDs.

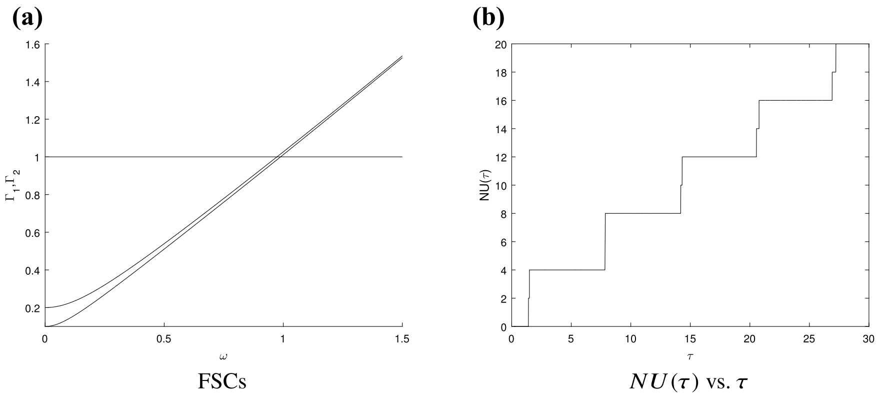


Fig. 5 FSCs and $NU(\tau)$ plot for Example 6

4 Illustrative Examples

Some examples are given to illustrate the proposed procedure for stability analysis.

Example 5 Study the stability of the UDDS in Examples 2 and 4.

The FSCs were already given in Fig. 4a (Step 1). From the FSCs, we know that $\lambda = 0$ is not a characteristic root (Step 2) and this system has only one set of critical pairs (Step 3). Next, in Example 2 we have that $NU(d_1 + \varepsilon) = 2$ (Step 4). The invariance property was illustrated in Example 4 (Step 5).

Finally, according to Theorem 3, for all $\tau \in [d_1, \infty)$ other than the CDs, $NU(\tau) = 2$. □

Example 6 Consider the UDDS in Example 3.

The FSCs are shown in Fig. 5a (Step 1). From the FSCs, we know that $\lambda = 0$ is not a characteristic root (Step 2) and that the system has two sets of critical pairs: ($\lambda_0 = 0.9734j$, $\tau_{0,k} = 1.4056 + \frac{2k\pi}{0.9734}$) and ($\lambda_1 = 0.9885j$, $\tau_{1,k} = 1.4871 + \frac{2k\pi}{0.9885}$), $k \in \mathbb{N}$ (Step 3). In Example 3, we have that $NU(d_1 + \varepsilon) = 0$ (Step 4). We have from the FSCs that, for all $k \in \mathbb{N}$, $\Delta NU_{\lambda_0}(\tau_{0,k}) = +1$ and $\Delta NU_{\lambda_1}(\tau_{1,k}) = +1$ (Step 5).

Finally, we have the explicit expression of $NU(\tau)$ (by Theorem 3) as plotted in Fig. 5b. The UDDS is asymptotically stable if and only if $\tau \in [0.2, 1.4056)$. □

5 Conclusion

We studied the stability of uniformly distributed delay systems (UDDSs). For such systems, three new technical issues need to be specifically addressed, compared to the existing results for retarded- and neutral-type delay systems. For one of the technical issues, we adopt an argument principle-based method and the other two technical issues can be covered by the frequency-sweeping framework, which was recently established for solving the stability problems of retarded and neutral delay systems. As a consequence, the stability of UDDSs in the whole domain of delay can be systematically studied.

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Asymptotic Analysis of Multiple Characteristics Roots for Quasi-polynomials of Retarded-Type

A. Martínez-González, S.-I. Niculescu, J. Chen, C. F. Méndez-Barrios, J. G. Romero and G. Mejía-Rodríguez

1 Introduction

It is commonly accepted in the literature that the presence of *delays* in dynamical systems is accompanied with “undesired” behaviors, (as, for example, instabilities, oscillations, bandwidth sensitivity), as pointed out by [11, 22] and the references therein. However, there exist some situations in which the delay may *induce* stability; a classical example is presented in the work of Abdallah et al. (1993) (see, for instance, [1]), where a simple oscillator is controlled by one delay “block” (gain, delay), with positive gains and extremely small delay values. As discussed in [22], such a property opens an interesting perspective in using *delays* as *control parameters*

A. Martínez-González · C. F. Méndez-Barrios (✉) · G. Mejía-Rodríguez
Universidad Autónoma de San Luis Potosí (UASLP), Facultad de Ingeniería,
Dr. Manuel Nava No. 8, Zona Universitaria, 78290 San Luis Potosí, S.L.P., Mexico
e-mail: cerfranfer@gmail.com

A. Martínez-González
e-mail: jano.mtz.glz@gmail.com

G. Mejía-Rodríguez
e-mail: gilberto.mejia@uaslp.mx

S.-I. Niculescu
Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CNRS-Supélec 3,
rue Joliot Curie, 91192 Gif-sur-Yvette, France
e-mail: Silviu.Niculescu@lss.supelec.fr

J. Chen
Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China
e-mail: jichen@cityu.edu.hk

J. G. Romero
Departamento Académico de Sistemas Digitales ITAM, Río Hondo 1,
01080 Ciudad de México, Mexico
e-mail: jose.romerovelazquez@itam.mx

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in some situations, as, for example in, [24] (stabilizing chains of integrators by using delays), [16] (multiple delay blocks), and [20] (bounded input, single delay). Nonetheless the approach appears *conservative* in other cases; see, for instance, [23, 29].

The above observations have been deeply explored in [8], where, for a general *retarded* linear time-invariant delay system with commensurate delays, the authors have first, fully characterized the stability properties of such a systems by finding a set of critical delay values, at which the system's characteristic quasi-polynomial has critical zeros on the imaginary axis. Secondly, considering the delay as a *variable parameter* and by adopting an operator based-approach they have expanded the solutions of the quasi-polynomial in terms of a Taylor (or Puiseux) series, allowing analyzing the solutions behavior as the delay varies around a critical delay value.

As discussed in [7], even in the case of a fixed delay, the testing of stability for a time-delay system is not a simple task. The difficulty arises from the fact that delay systems (and in consequence, quasi-polynomials) have always infinitely many solutions (see, for instance, [12] and the references therein). However, in general, we will only be interested in analyzing the behavior of a critical zero of finite multiplicity (for the purpose of this work, we consider only the case of multiplicity $m > 1$). In fact, from one hand the analysis of multiple characteristic roots it is more challenging than the case of single roots, where the roots tendency can be performed by applying the implicit function theorem. On the other hand, in the recent works of [25–27] it has been shown that multiple dominant spectral values can create an *optimal asymptotic stable* behavior by means of a delay-based controller, which ensures the optimal exponential decay rate. Hence, the study of multiple roots can be applied to explore such properties. Further discussions on computing bounds for the eigenvalue with the largest multiplicity can be found in [2, 3]. In this vein, it will be interesting if instead of analyzing the stability behaviour of a time-delay through their corresponding quasi-polynomial, such an analysis is performed through a polynomial (with degree equal to the multiplicity of the critical zero) that preserve the full information concerning the stability behaviour. Such an approach has been adopted by [4], where by means of the *calculus of residues* (see, for instance, [28]) they have proposed an analytical method to construct such a polynomial (known as Weierstrass polynomial).

This chapter focuses on the analysis of multiple characteristic roots of time-delayed systems when the delay is subject to small parameter variations. More precisely, by means of the *Weierstrass Preparation Theorem*, we propose an algorithm to construct the Weierstrass polynomial that will capture all stability information corresponding to the multiple critical zero. Finally, by adopting similar ideas to those developed by [7–9] we obtain the crossing directions of the critical zeros, which is a method to determine the asymptotic behaviour of the critical zeros when the delay varies in a neighborhood of the critical delay. Some preliminary ideas of this chapter can be found in [21].

The remaining chapter is organized as follows: Sect. 2 introduces some preliminary results and the problem formulation. Section 3 is devoted to the main results. More precisely, first, an algorithm is proposed to compute the Weierstrass Polynomial associated with the critical zero under study; second, by means of the Newton

Diagram method, the corresponding Puiseux series are computed; finally, the crossing directions are explicitly determined. Section 4 includes some numerical examples illustrating the proposed algorithms. The chapter ends with some concluding remarks.

Notations: In the sequel, the following notations will be adopted: $\mathbb{C}$ (RHP, LHP) is the set of complex numbers (with strictly positive, and strictly negative real parts), $\boldsymbol{i} := \sqrt{-1}$. For $z \in \mathbb{C}$, $\angle z \in [0, 2\pi)$, $\Re(z)$ ($\Im(z)$): argument, real (imaginary) part of z. Next, $\mathbb{R}_+$ denotes the set of positive real values. The unit open (closed) disk will be denoted by $\mathbb{D}$ ($\overline{\mathbb{D}}$). For a matrix $A \in \mathbb{C}^{n\times n}$, denote its spectrum by $\sigma(A)$, and the kth eigenvalue by $\lambda_k(A)$. For a matrix pair (A, B) denote the set of all generalized eigenvalues by $\sigma(A, B)$, i.e., $\sigma(A, B) := \{\lambda \in \mathbb{C} : \det(A - \lambda B) = 0\}$. The *order* of a power series $f(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ will be denoted by ord (f) and defined as the smallest number $n = i + j$ such that $a_{i,j} \neq 0$. Finally, given two polynomials $f(z) = \sum_{j=0}^{n} a_{n-j} z^j$ and $g(z) = \sum_{j=0}^{m} b_{m-j} z^j$, the *resultant* of f and g is defined as

$$\mathcal{R}(f, g) := \det \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & \\ & a_0 & a_1 & \cdots & \cdots & a_n & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & b_m & & & \\ & b_0 & b_1 & \cdots & \cdots & b_m & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \cdots & \cdots & b_m \end{bmatrix}.$$

Remark 1 It is well known that one of the properties of this resultant (see, for instance [5, 31], and references therein) is that $\mathcal{R}(f, g) \equiv 0$ if and only if f and g have common non-constant factors.

2 Prerequisites and Problem Formulation

2.1 Preliminary Results

Consider a retarded linear time-invariant delay system described by:

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t - k\tau), \qquad \tau \geq 0, \tag{1}$$

or by the differential-difference equation

$$y^{(n)}(t) + \sum_{\ell=0}^{n-1}\sum_{k=0}^{q} a_{k\ell} y^{(\ell)}(t - k\tau) = 0, \qquad \tau \geq 0. \tag{2}$$

Let $f : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ be the corresponding quasi-polynomial given by:

$$f(s,\tau) = \det\left(sI - \sum_{k=0}^{q} A_k e^{-sk\tau}\right), \tag{3a}$$

$$= \sum_{k=0}^{q} p_k(s) e^{-k\tau s}, \qquad \tau \geq 0, \tag{3b}$$

where

$$p_0(s) = s^n + \sum_{\ell=0}^{n-1} a_{0\ell} s^\ell, \quad p_k(s) = \sum_{\ell=0}^{n-1} a_{k\ell} s^\ell, \quad k = 1, \ldots, q.$$

The corresponding critical delay values can be computed as follows:

Lemma 1 ([6, 8]) *Define* $H_n := 0$*,* $J_n := I$*, and for* $j = 0, \ , 1, \ \cdots, \ n-1$*,*

$$H_j := \begin{bmatrix} a_{q,j} & a_{q-1,j} & \cdots & a_{1,j} \\ 0 & a_{q,j} & \cdots & a_{2,j} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{q,j} \end{bmatrix}, J_j := \begin{bmatrix} a_{0,j} & 0 & \cdots & 0 \\ a_{1,j} & a_{0,j} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{q-1,j} & a_{q-2,j} & \cdots & a_{0,j} \end{bmatrix}.$$

For $j = 0, \ 1, \ \cdots, \ n$*, define further* $G(s) := diag\left(1, \ \cdots \ 1, \ p_q(s)\right)$*,*

$$F(s) := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -p_0(s) & -p_1(s) & \cdots & -p_{q-1}(s) \end{bmatrix}, \ T_j := \begin{bmatrix} (\boldsymbol{i})^j J_j & (\boldsymbol{i})^j H_j \\ (-\boldsymbol{i})^j H_j^T & (-\boldsymbol{i})^j J_j^T \end{bmatrix},$$

$$T := \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -T_n^{-1}T_0 & -T_n^{-1}T_1 & \cdots & -T_n^{-1}T_{n-1} \end{bmatrix}.$$

The quasi-polynomial $f(s,\tau)$ *has a critical zero on the imaginary axis if and only if the following conditions are satisfied:*

(i) $\sigma(T) \cap \mathbb{R}_+ \neq \emptyset$*;*
(ii) For some $\omega_j \in \sigma(T) \cap \mathbb{R}_+$*,* $\sigma(F(\boldsymbol{i}\omega_j), \ G(\boldsymbol{i}\omega_j)) \cap \partial\mathbb{D} = \emptyset$*.*

The imaginary number $\boldsymbol{i}\omega_j$*, where* $\omega_j \in \mathbb{R}_+$ *is a critical zero. The set*

$$\mathcal{T}(\omega_j) = \left\{ \frac{\theta_j + 2\pi\ell}{\omega_j} > 0,\ \ell = 1, 2, \ldots \right\},$$

where $e^{-\boldsymbol{i}\theta_j} \in \sigma(F(\boldsymbol{i}\omega_j),\ G(\boldsymbol{i}\omega_j))$, $\theta_j \in [0,\ 2\pi]$, *contains all the critical delay values corresponding to the critical zero* $\boldsymbol{i}\omega_j$.

Now, since f is an analytic function, the following result holds:

Theorem 1 (Weierstrass Preparation Theorem [10]) *Suppose that* $f(z, \boldsymbol{p})$ *is an analytic function vanishing at the point* $z_0 \in \mathbb{C}$, $\boldsymbol{p}_0 \in \mathbb{C}^n$, *where* $z = z_0$ *is an* $m-$*multiple root of the equation* $f(z, \boldsymbol{p}) = 0$, *i.e.,*

$$f(z, \boldsymbol{p}) = \frac{\partial f}{\partial z} = \cdots = \frac{\partial^{m-1} f}{\partial z^{m-1}} = 0, \quad \frac{\partial^m f}{\partial z^m} \neq 0,$$

where the derivatives are taken at the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$. *Then, there exist a neighborhood* $U_0 \subset \mathbb{C}^{n+1}$ *of the point* $(z_0, \boldsymbol{p}_0) \in \mathbb{C}^{n+1}$ *in which the function* $f(z, \boldsymbol{p})$ *can be expressed as*

$$f(z, \boldsymbol{p}) = W(z, \boldsymbol{p})\, b(z, \boldsymbol{p}), \tag{4}$$

where

$$W(z, \boldsymbol{p}) = (z - z_0)^m + a_{m-1}(\boldsymbol{p})(z - z_0)^{m-1} + \cdots + a_0(\boldsymbol{p}),$$

and $a_0(\boldsymbol{p})$,…, $a_{m-1}(\boldsymbol{p})$, $b(z, \boldsymbol{p})$ *are analytic functions uniquely defined by the function* $f(z, \boldsymbol{p})$ *and* $a_i(\boldsymbol{p}_0) = 0$, $b(z_0, \boldsymbol{p}_0) \neq 0$.

Remark 2 The function $W(z, \boldsymbol{p})$ is known as the Weierstrass polynomial (for further details on Weierstrass polynomials, see, for instance, [13, 28]).

Given a known solution $(z_0, \boldsymbol{p}_0)$ of $f(z, \boldsymbol{p})$, the local behaviour of the solution $z(\boldsymbol{p})$ in the neighborhood $\mathbb{C}^n$ of $\boldsymbol{p}$ can be obtained by means of the Newton-diagram method. Thus, in order to introduce such a procedure, let us consider the following notation (for more details, see, for instance, [30]). Let $f(x, y)$ be a *pseudo-polynomial* in y, i.e.,

$$f(x, y) = \sum_{k=0}^{n} a_k(x) y^k, \tag{5}$$

where the corresponding coefficients are given by,

$$a_k(x) = x^{\rho_k} \sum_{r=0}^{\infty} a_{rk} x^{r/q}, \tag{6}$$

a_{rk} are complex numbers, x and y are complex variables, ρ_k are non-negative rational numbers, q is an arbitrary natural number, $a_n(x) \not\equiv 0$, and $a_0(x) \not\equiv 0$.

Then, a solution of (5) can be written in the form of a series as

$$y = y_0 + \alpha_1 (x - x_0)^{\epsilon_1} + \alpha_2 (x - x_0)^{\epsilon_2} + \cdots ,$$

where ϵ_1, ϵ_2, ..., is an increasing sequence of rational numbers. To determine the possible values of $\epsilon_1, \alpha_1, \epsilon_2, \alpha_2, \ldots$, it is necessary to consider the *Newton's diagram*. Since by simple translation, any point on a curve can be moved to the origin, we will only consider expansions of the solution of $f(x, y) = 0$ around the origin. In this vein, we will consider solution of (5) in the form of the following series:

$$y(x) = y_{\epsilon_1} x^{\epsilon_1} + y_{\epsilon_2} x^{\epsilon_2} + y_{\epsilon_3} x^{\epsilon_3} + \cdots , \tag{7}$$

where $\epsilon_1 < \epsilon_2 < \epsilon_3 < \cdots$, $y_{\epsilon_1} \neq 0$, or, in its compact form,

$$y(x) = y_{\epsilon_1} x^{\epsilon_1} + o(x^{\epsilon_1}) . \tag{8}$$

Definition 1 (*Newton Diagram*) Given a pseudo- polynomial equation of the form (5) with coefficients given by (6), plot ρ_k versus k for k for $k = 0, 1, \ldots, n$ (if $a_k(\cdot) \equiv 0$, the corresponding point is disregarded). Denote each of these points by $\pi_k = (k, \rho_k)$ and let

$$\Pi = \{\pi_k \, : \, a_k(\cdot) \neq 0\}$$

be the set of all plotted points. Then, the Newton diagram associated with $f(x, y)$ is the lower boundary of the convex hull of the set Π.

For a given pseudo-polynomial $f(x, y)$, Fig. 1 simply illustrates Definition 1.

Theorem 2 (Puiseux Theorem, [5, 31]) *The equation $f(x, y) = 0$, with f given in formal power series such that $f(0, 0) = 0$, posses at least one solution in power series of the form:*

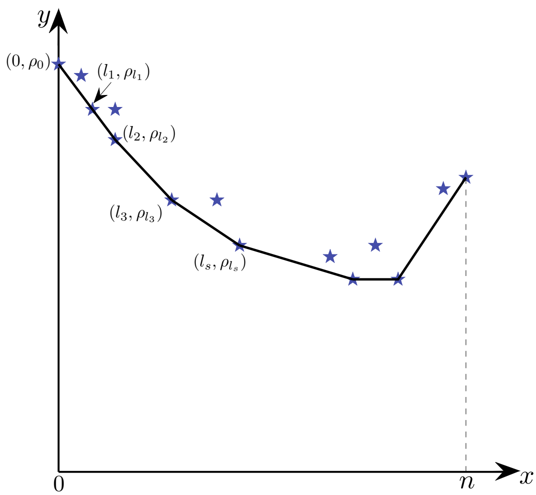

Fig. 1 The Newton diagram for the pseudo-polynomial $f(x, y)$ given in (5)

$$x = t^q, \quad s = \sum_{i=1}^{\infty} c_i t^i, \quad q \in \mathbb{N}.$$

2.2 Problem Formulation

As mentioned in the Introduction, in this chapter we will focus on the following problems:

(i) for a given quasi-polynomial $f(s, \tau)$ and a known $m-$multiple solution $(\boldsymbol{i}\omega^*, \tau^*) \in \mathbb{C} \times \mathbb{R}_+$, find the corresponding Weierstrass polynomial $W(s, \tau)$, such that:

$$f(s, \tau) = W(s, \tau)\, b(s, \tau),$$

where $W(s, \tau) = (s - \boldsymbol{i}\omega^*)^m + a_{m-1}(\tau)(s - \boldsymbol{i}\omega^*)^{m-1} + \cdots + a_0(\tau)$, and $W(s, \tau)$, $b(s, \tau)$ analytic functions, such that $b(\boldsymbol{i}\omega^*, \tau^*) \neq 0$;

(ii) for the solution $s(\tau)$, find the first coefficients of its Puiseux series expansion, i.e., compute γ_1 such that

$$s(\tau) = \boldsymbol{i}\omega^* + \gamma_1 \left(\tau - \tau^*\right)^{\frac{p}{q}} + o\left(\left|\tau - \tau^*\right|^{\frac{p}{q}}\right), \quad \text{with } q \leq m, \ p \in \mathbb{Z};$$

(iii) give conditions on $f(s, \tau)$ which describes the splitting properties of its solutions $s(\tau)$: Regular Splitting (RS) and Completely Regular Splitting (CRS); and,

(iv) find the stability crossing directions, that is, determine whether the solution $s(\tau)$ enter to the right half-plane (or to the left half-plane) for $\tau > \tau^*$.

3 Main Results

3.1 The Weierstrass Polynomial

In order to simplify the presentation, we will assume in the sequel that $s^* = 0$ is a zero of multiplicity $m > 1$ of $f(s^*, \tau^*)$ for $\tau^* = 0$, making appropriate shifts $s \mapsto s - s^*$, $\tau \mapsto \tau - \tau^*$ if necessary. Now, since $f(s, \tau)$ is an analytic function, it is not difficult to see that $f(s, \tau)$ can be expressed as

$$f(s, \tau) = \sum_{i=0}^{\infty} f_i(\tau)\, s^i, \tag{9}$$

with f_i given as

$$f_i(\tau) := \frac{1}{i!} \sum_{j=i}^{\infty} \frac{1}{(j-i)!} \frac{\partial^j f}{\partial s^i \partial \tau^{j-i}} \bigg|_{(0,0)} \tau^{j-i}.$$

As suggested by [13], let us express f as follows:

$$f(s,\tau) = l(s,\tau) + s^m h(s,\tau), \tag{10}$$

where $l(s,\tau)$ and $h(s,\tau)$ retain the lower and higher order terms of f, respectively, i.e.,

$$l(s,\tau) := \sum_{j=0}^{m-1} f_j(\tau) s^j, \quad h(s,\tau) := \sum_{j=m}^{\infty} f_j(\tau) s^{j-m}.$$

Clearly, both l and h are holomorphic functions. Furthermore, h is nonvanishing and analytic in a neighborhood of $0 \in \mathbb{C} \times \mathbb{R}$. From the Weierstrass Preparation Theorem we have:

$$W(s,\tau) b(s,\tau) = l(s,\tau) + s^m h(s,\tau), \tag{11}$$

where, according to this result, b is holomorphic and $b(0,0) \neq 0$, implying that b^{-1} is analytic in a neighborhood of the origin. Thus, (11) can be rewritten as:

$$W(s,\tau) = \left(s^m + \frac{l}{h}\right) hb^{-1}. \tag{12}$$

Since W is monic, then it can be written as $W(s,\tau) := s^m - \widehat{W}(s,\tau)$, and defining $\varphi := hb^{-1}$, (12) can be expressed as:

$$s^m - \widehat{W} = \left(s^m + \frac{l}{h}\right) \varphi, \tag{13}$$

$$\Rightarrow s^m - \frac{l}{h}\varphi = s^m \varphi + \widehat{W}. \tag{14}$$

Equation (14) can be solved by successive approximations, to this end it can be written as:

$$s^m - \frac{l}{h}\varphi_{k-1} = s^m \varphi_k + \widehat{W}_k, \tag{15}$$

where $\widehat{W}_k$ is a polynomial in s of degree $< m$. For $k = 1$ set $\varphi_0 := 0$ getting

$$s^m = s^m \varphi_1 + \widehat{W}_1,$$

that clearly imposes the following initial conditions $\varphi_1 = 1$ and $\widehat{W}_1 = 0$, or equivalently $\varphi_1 = 1$ and $\widehat{W}_0 = 0$. For $k = 1, 2, \ldots,$ $\widehat{W}_k$ can be obtained as the remainder after left-hand side (15) is divided by s^m.

Hence, the following result is a slight modification of the Weierstrass Preparation Theorem presented in [13, 17].

Proposition 1 *Let $f(s, \tau)$ be a quasi-polynomial written as (10), such that $s = 0$ is a zero of multiplicity m, with $m > 1$. Assume that $f(0, \tau)/s^m$ is holomorphic in a neighborhood of $0 \in \mathbb{C}$ and for $k = 1, 2, \ldots,$ let $\widehat{W}_k(s, \tau)$ be obtained by the procedure (15). Then, Weierstrass polynomial is given by*

$$W(s, \tau) = s^m - \widehat{W}(s, \tau),$$

where

$$\widehat{W}(s, \tau) = \lim_{k\to\infty} \widehat{W}_k(s, \tau).$$

Remark 3 Even though the above result seems to be a powerful tool to derive the Weierstrass polynomial, it has the inconvenience that it requires an infinite number of iterations to obtain the exact Weierstrass polynomial. However, since we are just interested in analyzing the stability behavior for the solutions around the critical pair, then, according to the Newton procedure, we just need to know the leading terms of each $a_j(\tau)$.

In the light of Remark 3, let us adopt the following notation for $\widehat{W}$:

$$\widehat{W}(s, \tau) = w_{m-1}(\tau)s^{m-1} + w_{m-2}(\tau)s^{m-2} + \cdots + w_0(\tau),$$

with

$$w_i(\tau) := \tau^{\rho_i} \sum_{j=0}^{\infty} w_{i,j}\tau^j. \tag{16}$$

While for its kth approximation, $\widehat{W}_k$ will be expressed as:

$$\widehat{W}_k(s, \tau) = w_{m-1}^{(k)}(\tau)s^{m-1} + w_{m-2}^{(k)}(\tau)s^{m-2} + \cdots + w_0^{(k)}(\tau),$$

where

$$w_i^{(k)}(\tau) := \sum_{j=1}^{\infty} w_{i,j}^{(k)}\tau^j. \tag{17}$$

Bearing in mind the previous notations, the following remark holds:

Remark 4 Let $\mathrm{ord}_\tau(a_0) = \rho_0$. Then according to Theorem 1, it is clear to see that $\rho_0 \in \mathbb{N}$. Moreover, ρ_0 satisfies the following relations:

$$f(s, \tau) = \frac{\partial f}{\partial \tau} = \cdots = \frac{\partial^{\rho_0 - 1} f}{\partial \tau^{\rho_0 - 1}} = 0, \quad \frac{\partial^{\rho_0} f}{\partial \tau^{\rho_0}} \neq 0, \tag{18}$$

where the partial derivatives are taken at (0, 0). In fact, since f is analytic at (0, 0), according to the Weierstrass Preparation Theorem, in a neighborhood of (0, 0), we have that $f(s,\tau) = W(s,\tau)b(s,\tau)$ where $b(0,0) \neq 0$. Then, by (4) we have that $f(0,\tau) = a_0(\tau)b(0,\tau)$ with $b(0,\tau) = b_0 + \sum_{i=1}^{\infty} b_i\tau^i$. Thus, for $i < \rho_0$

$$\left.\frac{\partial^i f(0,\tau)}{\partial\tau^i}\right|_{\tau=0} = \frac{\partial^i}{\partial\tau^i}\left[a_0(\tau)\left(b_0 + \sum_{i=1}^{\infty} b_i\tau^i\right)\right]\Bigg|_{\tau=0},$$

the right hand side of the above expression must be equal zero.

Remark 5 Observe that there may be cases at which there exist a $\kappa < m$ satisfying

$$\left.\frac{\partial^j f}{\partial s^j}\right|_{(0,0)} = \left.\frac{\partial^{j+1} f}{\partial\tau\partial s^j}\right|_{(0,0)} = \cdots = \left.\frac{\partial^{j+\rho_0-1} f}{\partial\tau^{\rho_0-1}\partial s^j}\right|_{(0,0)} = 0, \quad \forall\rho_0 \in \mathbb{N},\ 0 \le j < \kappa, \tag{19a}$$

$$\left.\frac{\partial^{j+\rho_0} f}{\partial\tau^{\rho_0}\partial s^j}\right|_{(0,0)} \neq 0, \quad \rho_0 \in \mathbb{N},\ j = \kappa. \tag{19b}$$

in such a situation, since according to the Weierstrass preparation theorem all w_j are analytic functions, this simply implies that W can be written as

$$W(s,\tau) = s^{\kappa}\left(s^{m-\kappa} + w_{m-1}(\tau)s^{m-\kappa-1} + \cdots + w_{\kappa}(\tau)\right), \tag{20}$$

$$\Rightarrow \widehat{W}(s,\tau) = s^{m-\kappa} + w_{m-1}(\tau)s^{m-\kappa-1} + \cdots + w_{\kappa}(\tau). \tag{21}$$

Thus, the analysis will be focused in the remaining $(m-\kappa)-$roots. In this vein, in order to keep the notation as simple as possible, when $\kappa > 0$ we will consider the shifts $m \mapsto m-\kappa$, $w_j \mapsto w_{j+\kappa}$ for $j = 0,\ldots,m-\kappa-1$ and ρ_0 will be defined as satisfying (18).

We have the following result:

Proposition 2 *Let* $s^* = 0$ *be a* $m-$*multiple zero of* $f(s,\tau)$ *at* $\tau^* = 0$ *and assume* $\kappa = 0$ (18). *Then, the full stability information for the* $m-$*zeros* $s_{\ell}(\tau)$*, with* $\ell = 1,\ldots,m$ *is completely determined by* $W(s,\tau) \approx s^m - \widehat{W}_k(s,\tau)$ *for* $k = \rho_0$.

Proof First, let us denote l/h by:

$$\frac{l(s,\tau)}{h(s,\tau)} = \sum_{i=0}^{\infty} q_i(\tau)s^i, \tag{22}$$

where simple computations reveal that the coefficients q_i can be recursively expressed as:

$$f_m(\tau)\sum_{i=0}^{\infty} q_i(\tau)s^i = \sum_{i=0}^{m-1}\left(f_i - \sum_{k=0}^{i-1} q_k f_{m+i-k}\right)s^i - \left(\sum_{k=0}^{m-1} q_k f_{m-1-k}\right)s^m - \sum_{i=m+1}^{\infty}\left(\sum_{k=0}^{i-2} q_k f_{m+i-k} + q_{i-1}f_{m+1}\right)s^i, \tag{23}$$

where $q_0 = f_0/f_m$. Now, since each q_i depends solely on the f_j's, it will be useful to determine the order of each f_j. From (9), it is evident that ord $\left(f_j\right) \geq 1\ \forall j \in \mathbb{N}\cup\{0\}$. Since $\kappa = 0$, from Remark 4 we know that $\mathrm{ord}_\tau\,(f_0) = \rho_0$.

Then, the result will be proved if we are able to show that after ρ_0-iterations the coefficients of the approximation $W(s,\tau) \approx s^m - \widehat{W}_k(s,\tau)$ that fall in the convex hull of the Newton diagram remain unchanged for all $k \geq \rho_0$. Bearing in mind the above observations, let us introduce the following:

$$\mathrm{ord}_\tau\,(q_0) \equiv \rho_0; \quad \mathrm{ord}_\tau\,(q_i) := \rho_{q_i} \geq 1,\ \forall i \neq m; \quad \mathrm{ord}_\tau\,(q_m) := \rho_{q_m} \geq 2.$$

Next, from (15), we consider $\widehat{W}_k$ for $k = 1, 2, 3$. Thus, we obtain:

$$\widehat{W}_1(s,\tau) = -\sum_{i=0}^{m-1} q_i s^i, \quad \widehat{W}_2(s,\tau) = \widehat{W}_1 + \sum_{i=0}^{m-1}\left(\sum_{j=0}^{i} q_{i-j}q_{m+j}\right)s^i,$$

$$\widehat{W}_3(s,\tau) = \widehat{W}_2 - \sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}\sum_{k=0}^{m+j} q_{i-j}q_{m+j-k}q_{m+k}\right)s^i.$$

From the above expressions, we have that $w_i^{(k)}$ is given as:

$$w_i^{(1)} = -q_i;\ w_i^{(2)} = w_i^{(1)} + \tilde{w}_i^{(2)};\ w_i^{(3)} = w_i^{(2)} - \tilde{w}_i^{(3)},$$

where

$$\tilde{w}_i^{(2)} := \sum_{j=0}^{i} q_{i-j}q_{m+j}; \qquad \tilde{w}_i^{(3)} := \sum_{j=0}^{i}\sum_{k=0}^{m+j} q_{i-j}q_{m+j-k}q_{m+k}.$$

From the previous expressions it is clear that $\mathrm{ord}_\tau\left(w_i^{(1)}\right) \equiv \rho_{q_i}$, whereas $\mathrm{ord}_\tau\left(w_i^{(2)}\right)$ and $\mathrm{ord}_\tau\left(w_i^{(3)}\right)$ depends on $\mathrm{ord}_\tau\left(\tilde{w}_i^{(2)}\right)$ and $\mathrm{ord}_\tau\left(\tilde{w}_i^{(3)}\right)$, respectively. Hence, the order of these expressions are given as follows:

$$\mathrm{ord}_\tau\left(\tilde{w}_i^{(2)}\right) = \min\left\{\rho_{q_i}+\rho_{q_m},\ \rho_{q_{i-1}}+\rho_{q_{m+1}},\ldots,\rho_{q_1}+\rho_{q_{m+i-1}},\ \rho_{q_0}+\rho_{q_{m+i}}\right\},$$
$$\mathrm{ord}_\tau\left(\tilde{w}_i^{(3)}\right) = \min\left\{\rho_{q_i}+2\rho_{q_m},\ \rho_{q_i}+2\rho_{q_{m+1}},\ldots,\rho_{q_0}+\rho_{q_1}+\rho_{q_{2m+i-1}},\ 2\rho_{q_0}+\rho_{q_{2m+i}}\right\}.$$

Thus, the following inequalities hold:

$$\mathrm{ord}_\tau\left(\tilde{w}_i^{(2)}\right) \le \mathrm{ord}_\tau\left(w_i^{(1)}\right) \quad \text{and} \quad \mathrm{ord}_\tau\left(\tilde{w}_i^{(2)}\right) < \mathrm{ord}_\tau\left(\tilde{w}_i^{(3)}\right),$$

implying that

$$\mathrm{ord}_\tau\left(w_i^{(2)}\right) = \min\left\{\mathrm{ord}_\tau\left(w_i^{(1)}\right), \mathrm{ord}_\tau\left(\tilde{w}_i^{(2)}\right)\right\} \quad \text{and} \quad \mathrm{ord}_\tau\left(w_i^{(3)}\right) = \mathrm{ord}_\tau\left(w_i^{(2)}\right).$$

For the next steps ($k > 3$), from the iterative construction (15) it is clear that $w_i^{(k)}$ is given as

$$w_i^{(k)} = w_i^{(k-1)} + (-1)^k \tilde{w}_i^{(k)}. \tag{24}$$

Moreover, since q_j (for some $j \in \mathbb{N} \cup \{0\}$) is always a factor of $\tilde{w}_i^{(k)}$ we will have the following relations:

$$\mathrm{ord}_\tau\left(\tilde{w}_i^{(k)}\right) < \mathrm{ord}_\tau\left(\tilde{w}_i^{(k+1)}\right),$$
$$\Rightarrow \mathrm{ord}_\tau\left(w_i^{(k)}\right) = \mathrm{ord}_\tau\left(w_i^{(k+1)}\right).$$

Then, from the above discussion, we can conclude that all coefficients $w_{i,j}$ belonging to the convex hull of $W(s,\tau)$ will stop updating at most at the ρ_0-th step, leading thus to the expected result. ∎

Remark 6 Generally, we have $\kappa < m$ and ρ_κ. Now, following the above proof, it can be seen that ρ_κ represents an upper bound that ensures the computation of the exact leading coefficient of $W(s,\tau)$ that belongs to the convex hull of the Newton diagram. However, it is worth mentioning that such coefficients can be obtained in less number of steps.

3.2 Splitting Properties

The main goal of this subsection is to explore some qualitative properties of the solutions $s(\tau)$ of the quasi-polynomial $f(s,\tau)$ around the $m-$multiple critical pair $(0,0)$. In this vein, as discussed by Wall (2004) (for further details, see, [31]) it is possible to characterize the root locus of f by its branches. In fact, the equation $f(s,\tau) = 0$ defines a solution curve $\mathcal{C} \in \mathbb{C}^2$ which is composed by the finite union of $r-$branches $s_\ell\left(\tau^{1/m_\ell}\right)$, each of these branches can be expressed as a Puiseux series:

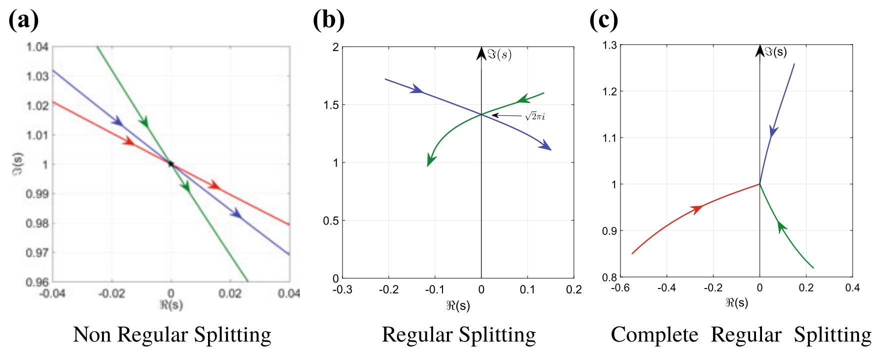

Fig. 2 Splitting properties

$$s_{\ell}(\tau) = \gamma_{\ell,j}\tau^{\frac{1}{m_{\ell}}} + o\left(|\tau|^{\frac{1}{m_{\ell}}}\right), \quad \ell = 1, \ldots, r, \quad j = 1, \ldots, m_{\ell}, \tag{25}$$

where each branch has multiplicity m_{ℓ}, such that $m = m_1 + m_2 + \cdots + m_r$. We have the following definition.

Definition 2 We say that there is a *Complete Regular Splitting* (CRS) property of the solution $s^* = 0$ at $\tau^* = 0$ if $\gamma_{\ell,j} \neq 0, \ \forall \ell$. For the *Regular Splitting* (RS) property some of the coefficients $\gamma_{\ell,j}$ for which $m_j = 1$ may be equal zero. Otherwise, we say that the solution possess the *Non Regular Splitting* (NRS) property.

The properties introduced in Definition 2 are illustrated in Fig. 2.

Remark 7 The above definition was inspired by the matrix case introduced by Langer et al. (1992) in [18] (see also [14]).

The proposed approach to deal with the splitting properties is based on the Newton diagram applied in conjunction with the Weierstrass polynomial. To this end, it can be observed from the Theorem 1, that since $b(s, \tau)$ is an holomorphic non vanishing function at (0, 0), then, there must exist some neighborhood $\Omega(0, 0) \subset \mathbb{C}^2$ on which $b(s, \tau)$ preserves the same property. Hence, based on this observation, we can ensure that the root loci of the quasi-polynomial f and of $W(s, \tau)$ in the neighborhood Ω are the same. Along these lines, based on the Newton procedure introduced in Sect. 2, we propose the following algorithm:

The above algorithm is useful to derive the following:

Proposition 3 *Let $s^* = \boldsymbol{i}\omega^*$ be a m-multiple critical root of the quasi-polynomial $f(s, \tau)$ at $\tau = \tau^*$. Let $W(s, \tau)$ be the Weierstrass polynomial of $f(s + \boldsymbol{i}\omega^*, \tau + \tau^*)$ given by (16), and assume that $\widetilde{r}$, β_j, (i_j, ℓ_j) and $\Pi^{(j)}$ are given by the Algorithm 1. Define $\widetilde{m}_j := i_j - i_{j-1}$, then the following properties hold:*

(i) if $\widetilde{m}_j \cdot \beta_j \equiv 1$ then the solution $(\boldsymbol{i}\omega^*, \tau^*)$ of $f(s, \tau)$ has the completely regular splitting property;

Algorithm 1: Auxiliary Puiseux Series Expansion

1 Let $W(s,\tau)$ denote the Weierstrass polynomial associated to the critical pair $(\boldsymbol{i}\omega,\tau^*)$ of multiplicity m. Consider the initial values as $\widetilde{r} := 0, j := 0$ and $k := \rho_0$.

Step 1) Set $\mathcal{E}_{\widetilde{r}} := \left\{\frac{\ell-k}{j-i} : (i,\ell)\in\Pi,\ \text{and } i>j\right\}$;

Step 2) Let $\beta_{\widetilde{r}} := \max\mathcal{E}_j$ and $\Pi^{(r)} := \left\{(i,\ell)\in\Pi : \beta_r \equiv \frac{k-\ell}{j-i}\right\}$;

Step 3) Set $(i_{\widetilde{r}},\ell_{\widetilde{r}})\in\Pi^{(\widetilde{r})}$ such that $i_{\widetilde{r}} \geq i,\ \forall(i,\ell)\in\Pi^{(\widetilde{r})}$;

Step 4) Set $j := i_{\widetilde{r}}, m_{\widetilde{r}} = i_{\widetilde{r}} - i_{\widetilde{r}-1}$ and $\widetilde{r} = \widetilde{r}+1$;

Step 5) If $j < m$ go to step 1. Otherwise the algorithm ends.

(ii) *if some β_j satisfies $\widetilde{m}_j\cdot\beta_j > 1$ for $\widetilde{m}_j > 1$, then non regular splitting property for the solution $(\boldsymbol{i}\omega^*,\tau^*)$ is possible;*

(iii) *if the pairs $(\widetilde{m}_k,\beta_k)$ that do not fulfill (i), satisfy the inequality $\beta_k \geq \widetilde{m}_k \equiv 1$, then the solution $(\boldsymbol{i}\omega^*,\tau^*)$ of $f(s,\tau)$ has the regular splitting property;*

(iv) *let Ω_0 be a neighborhood of $(0,0)\in\mathbb{C}^2$, and assume that $\mathcal{R}\left(W,\frac{\partial}{\partial s}W\right)\neq 0, \forall s\in\Omega_0\setminus\{(0,0)\}$. Then, there are $m-$different Puiseux series solutions $g_i\left(\tau^{\frac{1}{n_i}}\right)$ such that,*

$$f(s,\tau) = \prod_{i=1}^{m}\left(s - g_i\left(\tau^{\frac{1}{n_i}}\right)\right)b(s,\tau),$$

where n_i is arranged in terms of m_j as

$$\underbrace{n_1 = n_2 = \cdots = n_{m_1}}_{n_{i_1}=m_1},\ \underbrace{n_{m_1+1} = \cdots = n_{m_2}}_{n_{i_2}=m_2},\ \cdots,\ \underbrace{n_{m_1+\cdots+m_{r-1}+1} = \cdots = n_{m_1+\cdots+m_r}}_{n_{i_r}=m_r}$$

with $\sum m_i = m$.

Proof First of all, observe that $\widetilde{r}$ in the Algorithm 1 corresponds to the number of branches for the solution $(\boldsymbol{i}\omega^*,\tau^*)$ and $\widetilde{m}_j$ the multiplicity of each branch. Then, in terms of the notations introduced in Sect. 3.2 we consider in the following $m_\ell = m_j$ and $r = \widetilde{r}$.

(i) In this case, we have that $\beta_j = \frac{1}{m_i}$, then from the Newton procedure we know that the rational numbers β_j are associated to the first exponents in the solutions, since we have r branches, thus the root locus of $f(s,\tau)$, is given by

$$s(\tau)_\ell = \gamma_{\ell,j}\tau^{\frac{1}{m_i}} + o\left(\tau^{\frac{1}{m_i}}\right) \quad \ell = 1,\ldots,r \quad j = 1,\ldots,m_j.$$

Since $\gamma_{\ell,j}$ are related to the nonzero solution of a polynomial formed with the coefficients of the convex hull, clearly $\gamma_{\ell,j}\neq 0$. Then, the solution $(\boldsymbol{i}\omega^*,\tau^*)$ has the CRS property.

(ii)-(iii) These follows similar lines than those presented in (i).

(iv) This case can be stated by induction. To this end, from the Puiseux Theorem we know that there exist some Puiseux series $g_1\left(\tau^{1/n_i}\right)$ such that $f(g_1, \tau) = 0$, then the factor s_1 can be taken out, such that $f = g_1 f_1$. Assume now that the above factorization is valid for some $k \in \mathbb{N}$, i.e., the following relation holds:
$$f(s, \tau) = g_1(\tau) \cdots g_k(\tau) f_k(s, \tau).$$
Then, the quotient f_k has order $m - k$ in s. Applying the induction hypothesis to f_k, leads to m different factors g_i
$$f(s, \tau) = g_1(\tau) \cdots g_m(\tau) f_m(s, \tau),$$
where $f_m(s, \tau)$ has order $\text{ord}_\tau(f_m) = 0$.

■

Corollary 1 *Consider the hypothesis proposed in Proposition 3. Assume that $\rho_0 = 1$. Then at $\tau = \tau^*$, the $m-$roots of $f(s, \tau)$ have the CRS property, i.e. these roots can be expanded as:*

$$s_\ell(\tau) = \boldsymbol{i}\omega^* + \gamma_\ell \left(\tau - \tau^*\right)^{\frac{1}{m}} + o\left(\left|\tau - \tau^*\right|^{\frac{1}{m}}\right), \quad \textit{for } \ell = 1, 2, \ldots, m. \tag{26}$$

Moreover, the following properties hold:

(i) if $m = 2$ and $\Re(\gamma_\ell) \neq 0$ with $\ell \in \{1, 2\}$. Then for $\tau > \tau^*$ sufficiently close to τ^*, one of the zeros $s_\ell(\tau)$ will enter the RHP, whereas the other one will enter the LHP;

(ii) if $m > 2$, then at least one of the zeros $s_\ell(\tau)$ will enter the RHP.

3.3 Crossing Directions Characterization

As mentioned in the Introduction, the Weierstrass polynomial will be our main tool in analyzing the stability behavior of the critical characteristic roots. In the lines of [8], we have the following:

Proposition 4 *Let $s^* = \boldsymbol{i}\omega^*$ be a $m-$multiple root of $f(s, \tau)$ at $\tau = \tau^*$. Let $W(s, \tau)$ be the Weierstrass polynomial of $f(s + \boldsymbol{i}\omega^*, \tau + \tau^*)$, and assume that r, β_j, $\left(i_j, \ell_j\right)$ and $\Pi^{(j)}$ are given by the Algorithm 1. Define $m_\ell := i_\ell - i_{\ell-1}$ and $i_{-1} := \rho_0$, then at $\tau = \tau^*$, the $m-$zeros of $f(s, \tau)$ can be expanded as*

$$s_{\ell,k}(\tau) = \boldsymbol{i}\omega^* + \gamma_{\ell,k}\left(\tau - \tau^*\right)^{\beta_\ell} + o\left(\left|\tau - \tau^*\right|^{\beta_\ell}\right), \tag{27}$$

for $\ell = 0, 1, \ldots, r-1, k = 1, \ldots, m_\ell$ *and* $m = m_1 + \cdots + m_r$. *Where* $\gamma_{l,k}$ *are roots of the polynomial* $\mathcal{P}_\ell : \mathbb{C} \mapsto \mathbb{C}$,

$$\mathcal{P}_\ell(z) := \sum_{\nu=i_{\ell-1}}^{i_\ell} w_{\nu,0} z^{\nu - i_{\ell-1}}, \quad s.t. \ (\nu, \rho_\nu) \in \Pi^{(\ell)}. \tag{28}$$

For $\tau > \tau^*$ *sufficiently close to* τ^*, *the zeros* $s_{\ell,k}(\tau)$ *will enter the right half-plane (or to the left half-plane) if*

$$\Re\{\gamma_{\ell,k}\} > 0 (< 0). \tag{29}$$

Proof By hypothesis, we have that W corresponds to the Weierstrass polynomial associated to f. Then, by taking into account the Newton procedure, the proof follows by observing that after applying the Algorithm 1 to W, r corresponds to the number of branches of the solution $s^* = \boldsymbol{i}\omega^*$ at $\tau = \tau^*$, where β_ℓ is the main exponent of the Puiseux series for the $\ell-$branch, whereas the coefficients $\gamma_{\ell,k}$ can be computed with the polynomial with coefficients given by the leading terms of $w_j(\tau)$ that falls in the convex hull $\Pi^{(\ell)}$, which is nothing else than the polynomial $\mathcal{P}$. Finally, the direction of crossing follows straightforwardly by condition (29). ■

4 Illustrative Examples

In order to illustrate the effectiveness of the proposed approach, consider in the sequel several numerical examples.

Example 1 Let us consider the following quasi-polynomial, borrowed from [15],

$$f(s,\tau) = \left(s^4 + 2s^2 + 2\right) + 2e^{-\tau s} + e^{-2\tau s}. \tag{30}$$

In this case, the critical pair is given by $(s^*, \tau^*) = (\boldsymbol{i}, \pi)$, with a double multiplicity, i.e., $m = 2$. Then, in order to apply the above results let us consider the changes of variables $s \mapsto s + \boldsymbol{i}$ and $\tau \mapsto \tau + \pi$, leading to:

$$\widetilde{f}(s,\tau) = \left(s^4 + 4\boldsymbol{i}s^3 - 4s^2 + 1\right) + \left(2e^{-(\pi+\tau)(\boldsymbol{i}+s)} + e^{-2(\pi+\tau)(\boldsymbol{i}+s)}\right).$$

By the Weierstrass Preparation Theorem, we know that $\widetilde{f}$ can be described locally by $W(s,\tau)b(s,\tau)$ where:

$$W(s,\tau) = s^2 - w_1(\tau)s - w_0(\tau),$$

which can be considered as a polynomial in s. This polynomial has the same root locus as f in some neighborhood of $(\boldsymbol{i}, \pi)$. Now, let $\widetilde{f}(s,\tau) = l(s,\tau) + s^2 h(s,\tau)$ and compute the approximation $W \approx s^2 - \widehat{W}_k$. This is done by making two successive

Table 1 Results summary for the quasi-polynomial (30)

Initial data	Algorithm output	$\mathcal{Z} := \{z \in \mathbb{C} : \mathcal{P}(z) = 0\}$
$m = 2, \rho_0 = 2$ $\Pi = \{(0, 2),\ (1, 1),\ (2, 0)\}$	$r = 1, \ell = 0$ $\Pi^{(0)} = \{(0, 2),\ (1, 1),\ (2, 0)\}$ $m_0 = m = 2$	$\mathcal{P}(\tilde{z}) := z^2 + \frac{2i\pi}{\pi^2-4}z - \frac{1}{\pi^2-4}$ $\left\{z_{0,1} = -\frac{i}{\pi+2},\ z_{0,2} - \frac{i}{\pi-2}\right\}$

approximations of (15), i.e.,

$$s^2 - \frac{l}{h}\varphi_{k-1} = s^2\varphi_k + \widehat{W}_k \quad k = 1, 2.$$

The fact that only two iterations are sufficient for a good approximation of $W(s, \tau) = w_1(\tau)s + w_0(\tau)$ is due to Theorem 2, since $\rho_0 = 2$. Thus, he have that

$$w_1(\tau) = w_{11}\tau + O(\tau) \quad w_0(\tau) = w_{02}\tau^2 + O\left(\tau^2\right),$$

where the leading coefficients of w_1 and w_0 are given by:

$$w_{11} = -\frac{2i\pi}{-4+\pi^2} \quad \text{and} \quad w_{02} = \frac{1}{-4+\pi^2}.$$

After applying Algorithm 1 along with Proposition 3, Table 1 summarizes the results (Fig. 3).

Thus, according to the previous results, the solutions of (30) around $(s^*, \tau^*) = (\boldsymbol{i}, \pi)$ can be expressed as:

$$s_{1,2}(\tau) = \boldsymbol{i} - \frac{\boldsymbol{i}}{\pi \pm 2}(\tau - \pi) + o(|\tau - \pi|).$$

Finally, since $m = 2$ and $\beta_0 = 1 > 1/m_0$, thus the solution $s^* = \boldsymbol{i}$ possess the NRS property. Such a behavior is illustrated in Fig. 1.

Example 2 Let us consider now the following example borrowed from [4],

$$f(s, \tau) = -\left(\tfrac{\pi}{2}s^5 + \tfrac{\pi}{2}s^3 + s^2\right) + \left(\tfrac{\pi}{2}s^3 - s^2 + \tfrac{\pi}{2}s + 1\right)e^{-\tau s} + e^{-2\tau s}. \tag{31}$$

The critical pair is given by $(s^*, \tau^*) = (\boldsymbol{i}, \pi)$, and has multiplicity $m = 3$. Now, in order to apply the previous results let us consider the changes of variables $s \mapsto s + \boldsymbol{i}$ and $\tau \mapsto \tau + \pi$, leading to:

$$\widetilde{f}(\tilde{z}, \tau) = \frac{1}{2}e^{-2\tilde{z}}\left(1 + e^{\pi\tau + (\boldsymbol{i}+\tau)\tau}(\boldsymbol{i} + \tau)^2\right)\left(2 + e^{\tilde{z}}(2 + \pi\tau(\boldsymbol{i} + \tau)(2\boldsymbol{i} + \tau))\right),$$

with $\tilde{z} := (\boldsymbol{i} + s)(\pi + \tau)$. From $\widetilde{f}(s, \tau)$, let us compute the polynomial $\widehat{W}^{(k)}(s, \tau) = s^3 + w_2^{(k)}(\tau)s^2 + w_1^{(k)}(\tau)s + w_0^{(k)}(\tau)$. Next, according to Proposition 2, we have that:

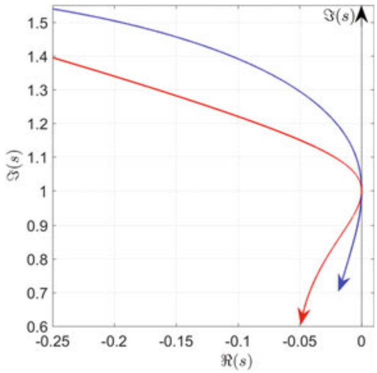

Fig. 3 Root loci for Example 1: $\Re(s)$ versus $\Im(s)$

Table 2 Results summary for the quasi-polynomial (31)

Initial data	Algorithm output	$\mathcal{Z} := \{z \in \mathbb{C}:\ \mathcal{P}(z) = 0\}$
$m = 3$, $\rho_0 = 2$ $\Pi = \{(0,2),\ (1,1),\ (2,1),\ (3,0)\}$	$r = 1, m_0 = 1, \beta_0 = 1\ \ell = 0$ $\Pi^{(0)} = \{(0,2),\ (1,1)\}$	$\mathcal{P}_0(\tilde{z}) := \frac{-2\boldsymbol{i}}{\pi(\pi-3\boldsymbol{i})} z + \frac{2}{\pi\left(\pi^2-5\boldsymbol{i}\pi-6\right)}$ $\left\{z_{0,1} = -\frac{2-\boldsymbol{i}\pi}{\pi^2+4}\right\}$
	$r = 2, m_1 = 2, \beta_1 = \frac{1}{2}\ \ell = 1$ $\Pi^{(1)} = \{(1,1),\ (3,0)\}$	$\mathcal{P}_1(\tilde{z}) := z^2 - \frac{2\boldsymbol{i}}{\pi(-3\boldsymbol{i}+\pi)}^{1}$ $\left\{z_{1,k} = (-1)^k \frac{1+\boldsymbol{i}}{\sqrt{\pi(\pi-3\boldsymbol{i})}}\right\}$

$$w_2(\tau) = w_{21}\tau + O(\tau) \quad w_1(\tau) = w_{11}\tau + O(\tau) \quad w_0(\tau) = w_{02}\tau^2 + O\left(\tau^2\right),$$

with leading coefficients given by:

$$w_{02} = \frac{2}{\pi\left(6+5\boldsymbol{i}\pi-\pi^2\right)}, \quad w_{11} = \frac{2\boldsymbol{i}}{\pi(-3\boldsymbol{i}+\pi)}, \quad w_{21} = \frac{\boldsymbol{i}\left(24\boldsymbol{i}+39\pi+38\boldsymbol{i}\pi^2-7\pi^3\right)}{3\pi(-2\boldsymbol{i}+\pi)(-3\boldsymbol{i}+\pi)^2}.$$

After applying Algorithm 1 along with Proposition 3, Table 2 summarizes the results.

Thus, according to the previous results, the solutions of (30) around $(s^*, \tau^*) = (\boldsymbol{i}, \pi)$ can be expressed as:

$$s_1(\tau) = \boldsymbol{i} + \frac{2-\boldsymbol{i}\pi}{4+\pi^2}(\tau-\pi) + o(|\tau-\pi|),$$
$$s_{2,3}(\tau) = \boldsymbol{i} \pm \frac{1+\boldsymbol{i}}{\sqrt{\pi(-3\boldsymbol{i}+\pi)}}(\tau-\pi)^{\frac{1}{2}} + o\left(|\tau-\pi|^{\frac{1}{2}}\right).$$

Then, the solution $s^* = \boldsymbol{i}$ at $\tau^* = \pi$ with $\beta_0 = 1/m_0$ and $\beta_1 = 1/m_1$ has the CRS property. However, it is worth mentioning that since the solution has two-branches with multiplicities $\beta_0 = 1$ and $\beta_1 = 1/2$, the CRS property implies that one root will behave as a Taylor series (s_1 in Fig. 4), whereas the other branch will behave as a

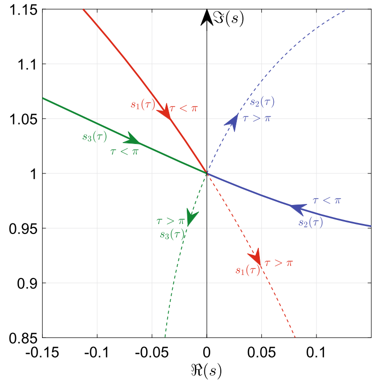

Fig. 4 Root loci for Example 2: $\Re(s)$ versus $\Im(s)$

fractional series of order $1/2$ (s_2 and s_3 in Fig. 4). Such a behavior is illustrated in Fig. 4.

Example 3 As a final, let us consider the the following quasi-polynomial [19]:

$$f(s,\tau) = \frac{1}{8}\left(8 - \pi^2 + 7\pi s - 2\pi^2 s^2 + 10\pi s^3 - \pi^2 s^4 + 3\pi s^5\right) + e^{-\tau s}.$$

In this case, the critical pair $(\boldsymbol{i}, \pi)$ has multiplicity $m = 3$. Now, based on Remark 4 we compute $\rho_0 = 1$. Then, in this case we have $m = 3$ and $\rho_0 = 1$, allowing us to use the Corollary 1. Thus we can immediately conclude that the solution $s^* = \boldsymbol{i}$ has the Completely Regular Splitting property. In the light of this corollary, we know that at least one of the solutions $s_\ell(\tau)$ will enter to *RHP* (Fig. 5).

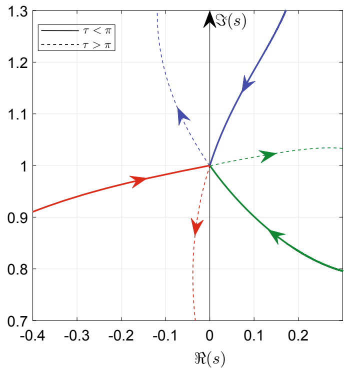

Fig. 5 Root loci for Example 3: $\Re(s)$ versus $\Im(s)$

5 Conclusions

In this chapter we have presented an alternative approach to the study of asymptotic behaviour of multiple imaginary roots for quasi-polynomials of retarded-type. By means of the Weierstrass Preparation Theorem, the structure of the root locus of $f(s, \tau) = 0$ at (s^*, τ^*) was described as a finite union of branches $s_j\left(\tau^{1/m_i}\right)$ with multiplicity m_j. In addition, the behavior of the solution s^* for τ sufficiently close to τ^* have been analyzed by means of leading coefficient of its branches. Finally, some splitting properties of the solutions s_j, such as CRS or RS were fully characterized.

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Scanning the Space of Parameters for Stability Regions of a Class of Time-Delay Systems; A Lyapunov Matrix Approach

Carlos Cuvas, Adrián Ramírez, Luis Juárez and Sabine Mondié

1 Introduction

In this chapter, we study the robust stability of linear time-invariant multiple time-delay systems with respect to delays $0 = h_0 < h_1 < \ldots < h_m = H$. The system is expressed in state-space form as

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - h_j), \quad t \geq 0, \tag{1}$$

where $A_0, \ldots, A_m$ are constant real $n \times n$ matrices, and $x(t) \in \mathbb{R}^n$ is the state vector. When investigating the stability of (1), one key challenge is to determine the set of all possible delay combinations under which the system remains stable. One opportunity in this endeavor is the characterization of stable regions in the delay-parameter space. Moreover, for systems of the form (1), stability charts can help reveal such regions and hence, all delay values favoring stability [25, 26].

C. Cuvas
Cuerpo Académico de Electrónica y Control, CITIS, Universidad Autónoma del Estado de Hidalgo, Carretera Pachuca-Tulancingo km. 4.5, C. U., 42084 Pachuca, Hidalgo, Mexico
e-mail: ccuvas@ctrl.cinvestav.mx

A. Ramírez · L. Juárez · S. Mondié (✉)
Departamento de Control Automático, CINVESTAV-IPN,
Av. IPN 2508, 07360 Ciudad de México, Mexico
e-mail: smondie@ctrl.cinvestav.mx

A. Ramírez
e-mail: aramirez@ctrl.cinvestav.mx

L. Juárez
e-mail: ljuarez@ctrl.cinvestav.mx

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Characterization of stability charts relies on the $\mathcal{D}$-decomposition principle [13], which states that boundaries in the delay-parameter space divide the space into regions. Their production evolves in a two-steps fashion: first; one must find the so-called stability crossing boundaries which generate a partition of the delay-parameter space, and second; one must determine the number of roots with positive real part in each region, also known as the instability degree, with which stability regions can be finally declared. Hereafter, the collection of regions with no right-hand side roots is referred to as the stability domain.

While the production of stability charts with fixed delays is well established, the case of two or more delays as variable parameters remains challenging. Recent results in [22] indicate that the use of resultant theory in combination with Rekasius transformations can facilitate the process. The method, called advanced clustering with frequency sweeping, allows a successive reduction of the problem, shortening the computational burden of NP-hard complexity arising from the presence of multiple delays, and ultimately revealing the stability outlook in a two-dimensional delay-parameter space, see also [27]. The reader is referred to [23] where the Cluster Treatment of Characteristic Roots (CTCR) paradigm is introduced and to [8] for an elegant geometrical approach in the case of systems with two delays. Having found the partition of the delay-parameter space, determining the instability degree in each region can be handled by deploying, for instance, semi-discretization methods [9], pseudospectral techniques [1], or computational approaches [20, 28]. Alternatively, the instability degree may be obtained based on root tendency properties of the imaginary roots, as reported in [14].

Considering that the main objective is to determine the stability domain associated with system (1), and not the instability degree in each region, our proposal is to obviate the tedious and prone to error unstable roots counting task by using instead the delay Lyapunov matrix to assess the stability regions.[1] This shall be done by efficiently scanning the delay-parameter space[2] equipped with a set of necessary stability conditions that depends, exclusively, in the delay Lyapunov matrix [6]. Moreover, we state that the stability conditions presented here and summarized from our previous contributions [3, 4, 6, 12] turn out to be a stability criterion under some special conditions [5].

The contribution is organized as follows. Section 2 revisits the main concepts and definitions on Lyapunov–Krasovskii functionals of complete type and summarizes the stability test in terms of the delay Lyapunov matrix. Section 3 presents the scanning process using the delay Lyapunov matrix. Sections 4 and 5 verify our method using various challenging examples found in the literature. Finally, some concluding remarks are given in Sect. 6.

[1] Local stability analysis of non-linear systems follows from linearization, meaning that the technique presented here may be used as a first approximation to the stability analysis of non-linear dynamics.

[2] It is worthy of mention that the delay Lyapunov matrix has exact solution only in the commensurate delays case, in the rest of the chapter, we take advantage of this characteristic to optimize the scanning process such that efficiency is improved.

Notation: The Euclidean norm for vectors is denoted by $\|.\|$. *The norm* $\|\varphi\|_{\mathcal{H}} = \sup_{\theta\in[-H,0]} \|\varphi(\theta)\|$ *is used for functions.* $Q > 0$ *means that* $Q = Q^\top$ *is positive definite. The square block matrix with* i*th row and* j*th column element* A_{ij} *is denoted* $\{A_{ij}\}_{i,j=1}^{r}$. *The symbol * indicates transposed terms in symmetric block matrices.*

2 Preliminaries: Stability Conditions Based on the Delay Lyapunov Matrix

The initial function φ of system (1) is taken from the set of piecewise functions defined on the interval $[H, 0]$, $PC([-H, 0], \mathbb{R}^n)$. The restriction of the solution $x(t, \varphi)$ of system (1) on the interval $[t - H, t]$ is denoted by $x_t(\varphi) : \theta \to x(t + \theta, \varphi), \theta \in [-H, 0]$. The delay Lyapunov matrix $U(\tau)$, $\tau \in \mathbb{R}$ of (1) associated with a positive definite matrix W, is the solution of the boundary value problem [10]

$$U'(\tau) = \sum_{j=0}^{m} U(\tau - h_j)A_j, \quad \tau \geq 0, \tag{2}$$

$$U(\tau) = U^T(-\tau), \qquad \tau \geq 0, \tag{3}$$

$$\sum_{j=0}^{m} \left[U(-h_j)A_j + A_j^T U(h_j)\right] = -W. \tag{4}$$

The above equations are also known as the dynamic, symmetric and algebraic properties of $U(\tau)$. They admit a unique solution whenever the system satisfies the Lyapunov condition.[3] In the case of commensurate delays; i.e. $h_j = jh$ with h as the *basic delay*, the semi-analytic method [10] provides an exact solution for (2), (3), and (4), up to computation of exponential matrices. The main results concerning this construction are summarized below.

Lemma 1 ([10]) *Let* $U(\tau)$ *be a Lyapunov matrix associated with* W. *Then for* $\xi \in [0, h]$, *the auxiliary matrices*

$$X_i(\xi) = U(\xi + hi), \quad i = -m, \ldots, 0, \ldots, m - 1, \tag{5}$$

satisfy the system of linear delay-free matrix differential equations

[3] That is, the characteristic equation of (1), namely $\det(sI - \sum_{j=0}^{m} A_j e^{-h_j s}) = 0$, has no symmetric eigenvalues with respect to the imaginary axis in the complex plane.

$$
\begin{aligned}
X_i'(\xi) &= \sum_{j=0}^{m} X_{i-j}(\xi) A_j,\ i \geq 0, \\
X_i'(\xi) &= -\sum_{j=0}^{m} A_j^T X_{i+j}(\xi),\ i < 0,
\end{aligned} \tag{6}
$$

with boundary conditions

$$
\begin{aligned}
X_{i+1}(0) &= X_i(h),\ i = -m, \ldots, 0, \ldots, m-2, \\
-W &= \sum_{j=0}^{m} \left[X_j(0) A_j + A_j^T X_j^T(0) \right].
\end{aligned} \tag{7}
$$

Corollary 1 ([10]) *If the boundary value problem (6)-(7) admits a unique solution (5), then there exists a unique Lyapunov matrix $U(\tau)$ associated with W and defined in $[0, mh]$ by $U(ih + \xi) = X_i(\xi),\ \xi \in [0, h],\ i = 0, 1, \ldots, m-1$.*

Using vectorization techniques based on Kronecker products properties, the delay-free system (6) and the boundary condition (7) can be rewritten in the form

$$
\begin{aligned}
z'(\xi) &= Lz(\xi), \\
Mz(0) + Nz(h) &= -W_v.
\end{aligned}
$$

where z is the vectorization of the auxiliary variables, and L, M and N are appropriate arrangements of matrices $A_j,\ j = 0, \ldots, m$.

The solution of the dynamic system of auxiliary matrices is readily computed as

$$
z(\xi) = e^{L\xi} z(0), \quad \xi \in [0, h] \tag{8}
$$

subject to

$$
z(0) = (M + Ne^{Lh})^{-1} W_v. \tag{9}
$$

Then, the Lyapunov matrix $U(\tau)$ for $\tau \in (0, mh]$ can be retrieved from z.

In [6], necessary stability conditions formulated exclusively in terms of the delay Lyapunov matrix were introduced. They are summarized next for completeness.

Theorem 1 *If system (1) is exponentially stable, then*

$$
K_r(\tau_1, \ldots, \tau_r) = \{U(-\tau_i + \tau_j)\}_{i,j=1}^{r} > 0, \tag{10}
$$

where $\tau_k \in [0, H],\ k = \overline{1, r},\ \tau_i \neq \tau_j$ if $i \neq j$, and r is a natural number.

The sufficiency of these conditions was established for a special choice of the delays and a large enough parameter r in [5].

Theorem 2 *System (1) is exponentially stable if and only if the Lyapunov condition holds and for every natural number $r \geq 2$,*

$$\left\{U\left(\frac{j-i}{r-1}H\right)\right\}_{i,j=1}^{r} > 0. \tag{11}$$

Moreover, if the Lyapunov condition holds and system (1) is unstable, there exists r such that

$$\left\{U\left(\frac{j-i}{r-1}H\right)\right\}_{i,j=1}^{r} \ngtr 0.$$

Remark 1 A family of conditions of increasing complexity can be obtained for example as

$$r=2:\begin{pmatrix} U(0) & U(H)\\ * & U(0)\end{pmatrix} > 0,$$
$$r=3:\begin{pmatrix} U(0) & U(H/2) & U(H)\\ * & U(0) & U(H/2)\\ * & * & U(0)\end{pmatrix} > 0,$$
$$r=4:\begin{pmatrix} U(0) & U(H/3) & U(2H/3) & U(H)\\ * & U(0) & U(H/3) & U(2H/3)\\ * & * & U(0) & U(H/3)\\ * & * & * & U(0)\end{pmatrix} > 0.$$

Remark 2 In [3, 4, 7] these results are extended to cover the case of distributed parameter systems. This class of delay systems arises naturally in a number of control problems, namely, optimal control of delay systems [21], control of systems with input delays [11], and regulation of spectrally controllable delay systems where the underlying distributed parameters ring is a Bézout ring [2].

3 Scanning Methodology

To test the stability condition (10) for equally spaced points in the space of parameters, one must compute the delay Lyapunov matrix, and then test condition (11). If more than one delay is involved, the complexity of the task can become cumbersome due to multiple-dimensionality issues. In the following, we explain how we can use some underlying properties of (11) to alleviate the computational burden in the two-delay case.

3.1 Minimizing the Dimension of the Delay-Free System

A simple computation shows that the dimension of the delay-free system is given by $2n^2m$. Testing for instance an arbitrary point $h_1 = 80$ and $h_2 = 120$, and taking the basic delay as 1, the dimension of the associated delay-free system becomes $2 \cdot 2^2 \cdot 120 = 960$. Noticing, however, that the largest possible basic delay might be taken as the greatest common divisor (gcd) of h_1 and h_2; e.g. 40, the dimension of the delay-free system reduces to $2 \cdot 2^2 \cdot 3 = 24$. Hence, to minimize the delay-free system dimension, we propose the following scanning organization.

(i) Take $N + 1$ equidistant points with separation Δ on each axis.

(ii) Set a counter for every axis; that is,

$$h_1^k = k\Delta,\ h_2^l = l\Delta, \quad k, l = 0,\ \ldots,\ N.$$

(iii) Set the basic delay as

$$h = gcd(k, l)\Delta.$$

(iv) Define

$$q_1 = \frac{k}{gcd(k, l)} \quad \text{and} \quad q_2 = \frac{l}{gcd(k, l)},$$

and compute the maximum delay qh with

$$q = \max(q_1,\ q_2).$$

The dimension of the delay-free system is $2 \cdot q \cdot n^2$.

(v) Using Corollary 1, obtain the Lyapunov matrix

$$U(\xi + jh) = X_j(\xi), \quad j = -q, -q + 1, \ldots, 0, 1, \ldots, q - 1,$$

in the interval $\xi \in [0, h]$ by solving the set of equations

$$X_j'(\xi) = X_j(\xi)A_0 + X_{j-q_1}(\xi)A_1 + X_{j-q_2}(\xi)A_2, \quad j = 0, 1, \ldots, q - 1,$$
$$X_j'(\xi) = -A_0^T X_{-j}(\xi) - A_1^T X_{j+q_1}(\xi) - A_2^T X_{j+q_2}(\xi), \quad j = -q, -q + 1, \ldots, -1.$$

The boundary conditions are

$$\begin{aligned} X_{j+1}(0) &= X_j(h), \quad j = -q, -q + 1, \ldots, 0, 1, \ldots, q - 2, \\ -W &= A_0^T X_0(0) + A_1^T X_{q_1-1}(h) + A_2^T X_{q_2-1}(h) \\ &\quad + X_0(0)A_0 + X_{q_1-1}^T(h)A_1 + X_{q_2-1}^T(h)A_2. \end{aligned}$$

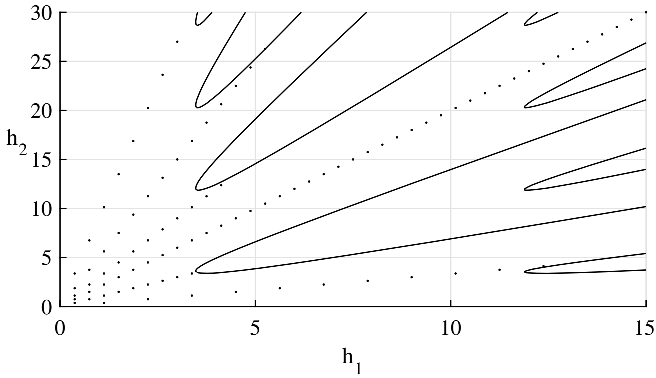

Fig. 1 Example 1. Highlights of the scanning methodology (search along rays). The stability crossing boundaries in solid lines are computed using CTCR [24]. The pairs satisfying condition (11) are depicted with isolated points forming various rays emanating from the origin.

3.2 Organizing the Search Along Rays

Conventionally, the scanning of the grid of the space parameters is realized through nesting loops. In the outer loop the delay h_1 is fixed and in the inner loop, the delay h_2 ranges from zero to its upper bound. In this way, scanning of the grid is usually implemented either in a vertical or horizontal fashion.

Observe that scanning the delay-parameter space along rays significantly reduces computational burden. Indeed, for pairs $ih_1,\ ih_2,\ i = 1, 2, \ldots$ located on a ray emerging from zero, the matrices L, M and N remain the same. Moreover, the exponential matrix in (8) can be computed as

$$e^{Lih} = (e^{Lh})^i = (e^{Lh})^{i-1}e^{Lh}, \qquad i = 1,\ 2,\ \ldots.$$

Therefore, preallocating the term $(e^{Lh})^{i-1}$ after each scanning, allows reducing the computational effort in the calculation of the next points of the ray.

To illustrate the above methodology, we next use an academic example studied in [25]. Figure 1 depicts the verification of condition (11) along a few number of rays. This example will be completely analyzed later in Sect. 4.

3.3 Avoiding Redundant Verification of the Conditions

Once the matrix $U(\tau)$, with $\tau \in [0, H]$, is constructed at a given fixed point, one can construct $K_r(\tau_1, \ldots, \tau_r) = \{U(-\tau_i + \tau_j)\}_{i,j=1}^{r}$ matrix and test condition (11). Clearly, the complexity of the test increases with r. On the other hand, notice from Remark 1, that $U(0) > 0$ represents the simplest necessary condition as $U(0)$ can

always be found in the diagonal of $K_r(\tau_1, \ldots, \tau_r)$. Therefore, using a simpler condition to discard points may speed up the process. To alleviate the computational burden, let us propose the following algorithm.
ALGORITHM 1.

(i) Scan the space of parameters and test each point for the condition $U(0) > 0$. Flag the points where $U(0) > 0$ does not hold true.
(ii) Initialize $r = 2$ and scan the space of parameters using (11) while skipping flagged points. Flag the points where (11) does not hold true.
(iii) If there are new flagged points, increase r by one and repeat step 2.
(iv) If there are no new flagged points, stop the process.

Since the construction of $U(\theta)$ and the stability test are independent from each other at every point in the delay-parameter space, parallel computing might be used. Consequently, we shall classify each variable into one of several categories, namely, loop variables, sliced variables, broadcast variables, among others. Notice also that at points where the ratio between the largest and smaller delay is large, the delay-free system is sparse, hence appropriate numerical techniques could be applied.

4 Illustrative Examples

Next, two challenging examples analyzed in [25] illustrate the interest of the above results. The first one is a two-delay scalar system, and the second one has a cross-talking delay. In the presented figures, the continuous lines describing imaginary axis crossings of the roots are generated by using the CTCR technique [24], and the isolated dots indicate points of the space of parameters where the necessary stability conditions hold.

Example 1 Consider the scalar equation with two delays taken from [25]

$$\dot{x}(t) = -1.3x(t) - x(t - h_1) - 0.5x(t - h_2).$$

For this system, the exact stability domain in (h_1, h_2) delay-parameter space is detected for $r = 4$ and $r = 8$, see Fig. 2a, b. Since the delay-free system is stable, the region connected to the origin, is stable. Choosing $r = 4$, we can observe from Fig. 2a the existence of regions where the stability conditions hold, but are actually unstable regions. As expected, an improvement is achieved increasing $r = 8$ as demonstrated on Fig. 2b where the exact stability domain is found.

Example 2 Consider the characteristic equation given by

$$f(s, h_1, h_2) = s^2 + s + 20 + (2s + 3)e^{-h_1 s} + (s + 4)e^{-h_2 s} + e^{-(h_1+h_2)s}.$$

A representation in the time domain is

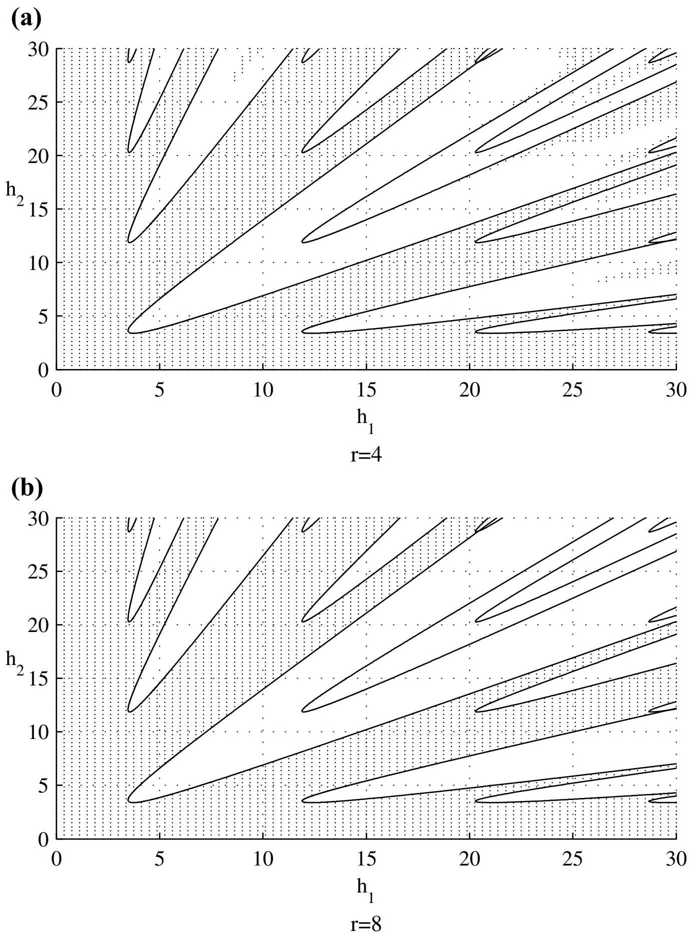

Fig. 2 Candidate stability domain for Example 1. The stability crossing boundaries in solid lines are computed using CTCR [24]. The pairs satisfying the necessary stability condition (11) are depicted with isolated points for a fixed value r.

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + A_2 x(t - h_2) + A_3 x(t - h_1 - h_2),$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -20 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -3 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -4 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The stability conditions are tested for $r = 2$ and $r = 4$. The results are shown in Fig. 3a, b respectively. From the figures, we can conclude that the exact stability domain is found when r increases.

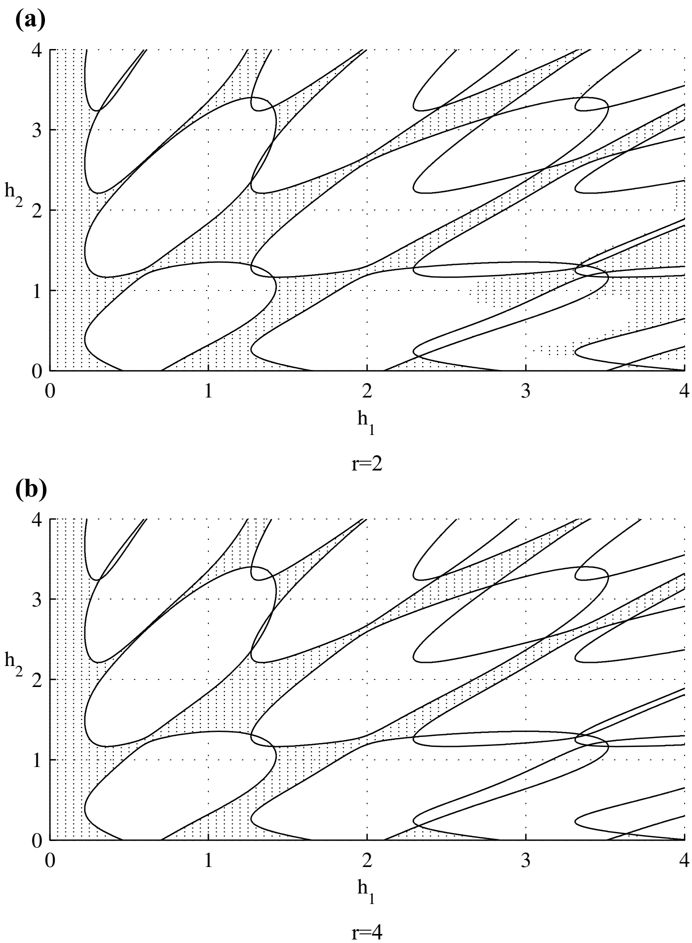

Fig. 3 Candidate stability domain for Example 2. The stability crossing boundaries in solid lines are computed using CTCR [24]. The pairs satisfying the necessary stability condition (11) are depicted with isolated points for a fixed value r.

5 Case Study: Effect of Input Delay on IR and PIR Controllers

In practical applications, the uncertainty induced by unavoidable measurement noise is a significant problem, hence filtering is essential if the control energy is to be constrained. The deliberate introduction of a delay h_1 in the feedback loop has proved to be a reasonable filtering option in the Integral-Retarded (IR) and the Proportional-Integral-Retarded (PIR) controllers proposed in [15–19].

As an auxiliary design tool to address the presence of an input delay, we present stability charts depicting the non-induced input delay h_2 against the controller delay parameter h_1. For each control scheme, the delay-parameter space is decomposed deploying the CTCR technique [24] and further scanned via the necessary stability conditions.

Example 3 IR control of first-order linear systems. Let us consider first the IR controller operating on a first-order linear system. The existence of a delay h_2 in the input is assumed, resulting in the closed-loop system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ bk_{ir} & 0 \end{pmatrix} x(t - h_1 - h_2) + \begin{pmatrix} 0 & 0 \\ -bk_i & 0 \end{pmatrix} x(t - h_2),$$

where (a, b) are the system parameters, and (k_i, k_{ir}) are the controller gains. Following [17], in the absence of input delay the operational parameters $(a, b) = (6, 42)$ lead to the IR tuning $(k_i, k_{ir}) = (0.9086, 0.2931)$, and $h_1 = 0.1851$. For this parameter setting, the exact stability domain in (h_1, h_2) delay-parameter space is detected for $r = 4$, see in Fig. 4. For this choice, the exact h_2-stability interval of the system is $(0, 0.2981)$. Observe also that by reducing the intentional delay the upper bound on the h_2-stability interval can be increased.

Example 4 PIR control of second-order systems. We now investigate the stability of the PIR controller regulating the behavior of a second-order linear system subject to a non-intentional input delay. The closed-loop state-space representation is:

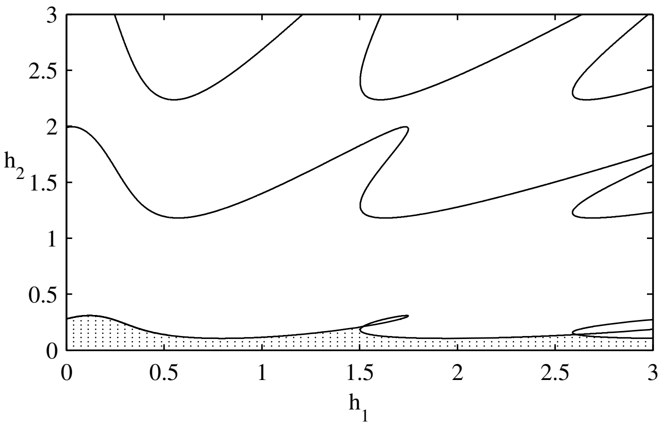

Fig. 4 Candidate stability domain for Example 3. The stability crossing boundaries in solid lines are computed using CTCR [24]. The pairs satisfying the necessary stability condition (11) are depicted with isolated points with $r = 4$.

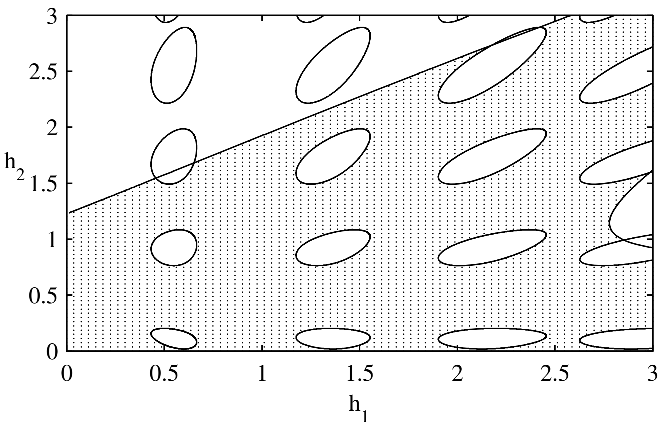

Fig. 5 Candidate stability domain for Example 4. The stability crossing boundaries in solid lines are computed using CTCR [24]. The pairs satisfying the necessary stability condition (11) are depicted with isolated points with $r = 4$.

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b & -a \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & ck_r & 0 \end{pmatrix} x(t - h_1 - h_2)$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -ck_i & -ck_p & 0 \end{pmatrix} x(t - h_2).$$

Here, (a, b, c) are the system parameters, and (k_p, k_i, k_r) are the controller gains. The gain values $(k_p, k_i, k_r) = (0.0789, 1.3415, 1.0858)$ and the delay design parameter $h_1 = 0.0789$ are obtained using the tuning formulae in [19] corresponding to the operational parameters $(a, b, c) = (6, 83, 42)$ which ensure the stable operation of the closed-loop system in the absence of input delay. The stability chart in Fig. 5 obtained with $r = 4$, shows the exact stability domain in (h_1, h_2) delay-parameter space. It is now clear that the choice for h_1 has a great impact on the h_2-stability margins, and that some choices are dangerous as their reduce it to zero.

Remark 3 A noteworthy advantage of the proposed approach is that at points of the space where the test is conclusive and stability is established, the delay Lyapunov matrix of system (1) associated with W, defines a Lyapunov–Krasovskii functional of complete type introduced in [10]. This functional has a prescribed quadratic negative derivative that depends on the complete state of the system and, if the system is stable, it admits a quadratic lower bound. This functional satisfies the conditions of the Krasovskii Theorem and hence, it can be used to find exponential estimates of the response or robust stability bounds with respect to parameter uncertainty or delay uncertainty, among other applications. Study of robustness properties with respect to delays is of special significance, as it allows to conclude on the stability of points in parameter space in a neighborhood of the tested commensurate points.

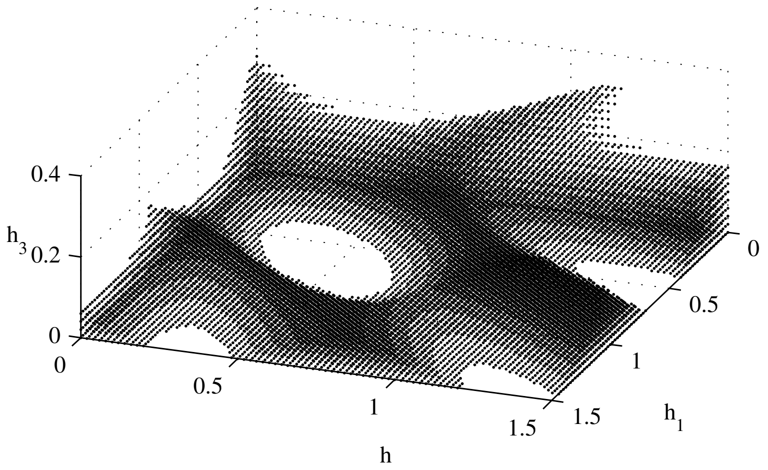

Fig. 6 Candidate stability domain for Example 5. The pairs satisfying the necessary stability condition (11) are depicted with isolated points with $r = 8$.

5.1 Extension to the Three Delays Case

The introduced methodology may be further extended to produce stability maps in a three-delay space of parameters as presented next.

Example 5 Consider the following characteristic equation taken from [24]

$$f(s, h_1, h_2, h_3) =(s^2 + s + 3) + (3s + 2)e^{-h_1 s} + (s + 8)e^{-h_2 s} + (3s + 28)e^{-h_3 s},$$

and associated with the system

$$\dot{x}(t) =A_0 x(t) + A_1 x(t - h_1) + A_2 x(t - h_2) + A_3 x(t - h_3),$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -3 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -2 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -8 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ -28 & -3 \end{pmatrix}.$$

Here, each delay is consider as a variable parameter. Deploying the necessary stability conditions with $r = 8$ we obtain the candidate stability domain shown on Fig. 6. In the particular case $h_1 = h_2 = 0$, we know that the system is stable in the interval $h_3 \in [0, 0.28)$, which agrees with our results based on the Lyapunov matrix.

6 Concluding Remarks

In this contribution, we introduce a methodology for finding the stability domain of a class of time-delay systems in the delay-parameter space. The approach is based on the positiveness of a set of necessary stability conditions written exclusively in terms of the delay Lyapunov matrix. An appealing by-product of our method is the immediate availability of a Lyapunov–Krasovskii functional at every detected stable points.

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A Symbolic Computation Approach Towards the Asymptotic Stability Analysis of Differential Systems with Commensurate Delays

Yacine Bouzidi, Adrien Poteaux and Alban Quadrat

1 Introduction

This paper aims at studying the asymptotic stability of *linear time-invariant differential systems with commensurate time-delays* by means of computer algebra methods and implementations recently developed by the symbolic computation community.

An example of such a system is defined by the state-space representation

$$\dot{x}(t) = \sum_{k=0}^{m} A_k \, x(t - k\,\tau), \tag{1}$$

where $\tau \in \mathbb{R}_+ := \{\tau \in \mathbb{R} \mid \tau \geq 0\}$, $m \in \mathbb{Z}_{\geq 0} := \{0, 1, \ldots\}$ and $A_0, \ldots, A_m \in \mathbb{K}^{n \times n}$ with $\mathbb{K}$ a field (e.g., $\mathbb{K} = \mathbb{Q}, \mathbb{R}$). The *characteristic function* of (1) is then a *quasi-polynomial* of the form $f(s, \tau) = \det\left(s\, I_n - \sum_{k=0}^{m} A_k\, e^{-k\,\tau\,s}\right) = \sum_{j=0}^{l} p_j(s)\, e^{-j\,\tau\,s}$, where the p_j's are polynomials in the complex variable s with coefficients in $\mathbb{K}$.

In this paper, we consider *retarded type* linear time-invariant differential commensurate time-delay systems, namely systems whose characteristic functions are

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Y. Bouzidi
Inria Lille - Nord Europe, Non-A project, Parc Scientifique de la Haute Borne,
40 Avenue Halley, Bat. A - Park Plaza, 59650 Villeneuve d'Ascq, France
e-mail: yacine.bouzidi@inria.fr

A. Poteaux
University of Lille, CRIStAL - UMR CNRS 9189, CFHP, Batiment M3,
59655 Villeneuve d'Ascq, France
e-mail: adrien.poteaux@univ-lille.fr

A. Quadrat (✉)
Inria Paris, Ouragan project, IMJ – PRG, Sorbonne University, Paris, France
e-mail: alban.quadrat@inria.fr

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defined by quasipolynomials of the form of $f(s, \tau) = \sum_{j=0}^{l} p_j(s)\, e^{-j\,\tau\,s}$, where $\deg p_0 \geq \max\{\deg p_1, \ldots, \deg p_l\}$ and $\deg p_j$ stands for the degree of the polynomial p_j in s (e.g., (1)). We investigate the *asymptotic stability* of this class of systems by means of the location of the complex solutions s of $f(s, \tau) = 0$. Recall that a retarded type differential commensurate time-delay system is *asymptotically stable* [2, 12] if and only if all the complex solutions of its characteristic function $f(s, \tau) = 0$ have negative real parts, i.e., $f(s, \tau) \neq 0$ for all s in the closed right half-plane $\overline{\mathbb{C}}_+ := \{s \in \mathbb{C} \mid \mathbb{R}(s) \geq 0\}$ (see, e.g., [2, 12, 16]). While considering the asymptotic stability of a quasipolynomial $f(s, \tau)$, three different problems can be studied: the first one consists in checking the stability for a fixed value of τ, the second one is the study of the so-called *delay independent stability* property which guarantees the stability for all values of τ in $\mathbb{R}_{\geq 0}$, and the last one, *the stability analysis depending on the time-delay* τ, which considers τ as a parameter of the system and aims at computing the values of τ for which the system is asymptotically stable. In this paper, we mainly focus on the last problem. Our approach for analyzing the asymptotic stability of this class of systems is based on the computation of the so-called *critical pairs* of $f(s, \tau)$, that is the pairs $(\omega, \tau) \in \mathbb{R} \times \mathbb{R}_+$ satisfying $f(i\,\omega, \tau) = 0$. For more details, see [11, 16, 18–20] and the references therein. If such critical pairs exist, the asymptotic stability can then be derived from the way the first component of these pairs, called *critical imaginary roots* of $f(s, \tau)$, behaves under a small variation $\Delta\tau$ of the time-delay τ with respect to $\overline{\mathbb{C}}_+$. Thus, the asymptotic stability analysis is divided into two distinct problems. First, the critical pairs of $f(s, \tau)$ must be computed. Then, for each critical pair (ω_0, τ_0), the behavior of the critical imaginary root ω_0, particularly its real part, must be studied with respect to a small variation of τ_0. For more details, see [11, 16, 18–20].

There exist several methods for computing the critical pairs of a general quasipolynomial. The study of the asymptotic behavior of critical imaginary roots has been addressed for the case of simple imaginary roots (see [11, 16, 18–20] and the references therein). For *multiple imaginary roots*, a recent method based on the so-called *Puiseux series* [29] was developed in [15, 16].

The contributions of the paper are twofold. We first present a new approach for the efficient computation of the critical pairs of a general quasipolynomial. After a *Möbius transformation*, the problem reduces to the computation of the real solutions of a system formed by two polynomial equations in two variables. An efficient method to solve this last problem is to compute the so-called *Rational Univariate Representation* (RUR) [25, 27] of the polynomial system which is a one-to-one mapping between the solutions of the polynomial system and of the roots of a univariate polynomial. The complex/real solutions of the polynomial system can then be obtained in a certified manner by numerical isolation of the complex/real roots of the univariate polynomial [13, 26]. The RUR of a polynomial system admitting a finite number of complex solutions can be obtained by means of the command `RationalUnivariateRepresentation` of the `Maple` package `Groebner`. Motivated by applications in *computational geometry* (computation of the topology of algebraic curves), an extremely efficient algorithm for the RUR computation was recently obtained in [4, 5] for systems formed by two polynomial equations in two variables. This algorithm avoids a *Gröbner basis computation* [8].

Moreover, based on recent advances in the direction of Newton–Puiseux series for algebraic curves developed in [21–23], the second contribution of the papier is to present an efficient algorithm which efficiently computes terms of Newton–Puiseux series of a quasipolynomial $f(s,\tau)$ at a critical pair (ω_0, τ_0). These Newton–Puiseux series can be used to study the variation Δs in terms of $\Delta \tau$ in a neighborhood of the critical pair (ω_0, τ_0), which allows one to study the asymptotic stability of the corresponding retarded type differential time-delay system as explained in [16].

This paper is based on the congress paper [6].

The paper is organized as follows. In Sect. 2, we present an efficient algorithm which computes, via a Rational Univariate Representation, certified numerical approximations of the critical pairs of a quasipolynomial. Then, an algorithm which computes Newton–Puiseux series on these critical pairs is presented in Sect. 3.

2 An Efficient Algorithm for the Computation of the Critical Pairs

In this section, we focus on the computation of the *critical pairs* of a quasipolynomial $f(s,\tau) = \sum_{j=0}^{l} p_j(s)\, e^{-j\,\tau\, s}$, namely the set $\{(\omega, \tau) \in \mathbb{R} \times \mathbb{R}_+ \mid f(i\,\omega, \tau) = 0\}$. Due to the presence of transcendental terms $e^{-j\,\tau\,s}$ in $f(s,\tau)$, $f(i\,\omega,\tau) = 0$ usually admits an infinite number of zeros $(i\,\omega, \tau)$. Using the so-called *Möbius transformation* (or the *Rekasius transformation* [19, 20]) introduced below, we show that this problem is reduced to the computation of the real solutions of a system defined by two polynomial equations in two variables. An efficient computer algebra method [5] and its implementation in the library RS [28] can be used to solve the latter problem, and thus to compute critical pairs of quasipolynomials in a certified manner.

When the delay τ (resp., ω) varies in $\mathbb{R}_{\geq 0}$ (resp., $\mathbb{R}$), $e^{-\tau\, i\,\omega}$ covers the complex torus $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. The problem of studying the zeros of $f(i\,\omega, \tau)$ then amounts to studying the zeros of the bivariate polynomial $f(i\,\omega, z) = \sum_{j=0}^{l} p_j(i\,\omega)\, z^j$, where $\omega \in \mathbb{R}$ and $z \in \mathbb{T}$.

A first approach consists in considering $z = u + i\,v$, where $u,\ v \in \mathbb{R}$, which yields $f(i\,\omega,\ u + i\,v) = \mathcal{R}(\omega,\ u,\ v) + i\,\mathcal{I}(\omega,\ u,\ v)$, where $\mathcal{R}$ and $\mathcal{I}$ are two polynomials with real coefficients. Thus, we have to compute the real solutions $(\omega,\ u,\ v)$ of:

$$\mathcal{R}(\omega,\ u,\ v) = 0, \quad \mathcal{I}(\omega,\ u,\ v) = 0, \quad u^2 + v^2 - 1 = 0. \tag{2}$$

Generically, the above system is zero-dimensional, i.e., it only admits a finite number of complex solutions. Standard computer algebra methods, particularly the so-called *Rational Univariate Representation* (RUR) [25], can then be used to obtain certified numerical approximations of the real solutions of (2).

For an efficiency issue, it is important to reduce the number of the indeterminates (ω, u and v) of (2). To do that, we use the so-called *Möbius transform* defined by:

$$\begin{aligned}\mathcal{M} : \mathbb{R} \cup \{\infty\} &\mapsto \mathbb{T}\\ x &\mapsto z = \frac{x-i}{x+i} = \frac{x^2-1}{x^2+1} - i\,\frac{2\,x}{x^2+1},\\ \infty &\mapsto 1.\end{aligned}$$

Substituting $e^{-\tau\, i\,\omega}$ by $\frac{x-i}{x+i}$ into $f(i\,\omega, \tau)$ and cleaning the denominators, we obtain a polynomial of the form $\mathcal{R}(\omega, x) + i\,\mathcal{I}(\omega, x)$, where $\mathcal{R}$ and $\mathcal{I}$ are two polynomials in ω and x with real coefficients. The critical pairs can then be computed by first solving the following system formed by two polynomial equations in two variables:

$$\mathcal{R}(\omega, x) = 0, \quad \mathcal{I}(\omega, x) = 0. \tag{3}$$

Accordingly, the critical delays can easily be obtained as follows:

$$\tau_k = \frac{1}{\omega}\left(\arctan\left(\frac{2\,x}{x^2-1}\right) + k\,\pi\right), \quad k \in \mathbb{Z}. \tag{4}$$

Thus, the computation of the critical pairs is reduced to searching for the real solutions of the polynomial system (3).

Remark 1 Since $z = 1$ is sent to ∞ by the above Möbius transformation, this case, which corresponds to $\tau_k\,\omega = 2\,k\,\pi$, where $k \in \mathbb{Z}$, can be independently studied. The quasipolynomial f then becomes the pure polynomial $f(i\,\omega, 1) = \sum_{j=0}^{l} p_j(i\,\omega)$ in ω whose real roots can be isolated using, e.g., classical *bisection algorithms* [7, 26].

Remark 2 The *Rekasius transformation* is defined by mapping $e^{-\tau\, i\,\omega}$ to $\frac{1-i\,T\,\omega}{1+i\,T\,\omega}$, where $T \in \mathbb{R}$ (see [19, 20] and the references therein). Note that the Rekasius transformation can be seen as a particular Möbius transform.

In what follows, we consider the case of the Möbius transformation (the approach being similar for the Rekasius transformation). Let us now consider the polynomial system (3). Without loss of generality, we can assume that (3) admits only a finite number of complex solutions, which means that the polynomials $\mathcal{R}$ and $\mathcal{I}$ do not have a non-trivial common factor (see Theorem 1 below for the computation of $\gcd(\mathcal{R}, \mathcal{I})$). Our objective is to develop an efficient approach for the certified computation of the real solutions of (3) by means of modern computer algebra methods.

We note that the ω-coordinate of the solutions of $f(i\,\omega, \tau) = 0$ (called the *imaginary roots* of f) is a root of the *resultant* $\mathrm{Res}_x(\mathcal{R}, \mathcal{I})$ of $\mathcal{R}$ and $\mathcal{I}$ with respect to the variable x. Let us recall the definition of a (sub)resultant. Let $A = \mathbb{K}[\omega]$, $\mathcal{R} = \sum_{i=0}^{p} a_i(\omega)\,x^i \in A[x]$ and $\mathcal{I} = \sum_{j=0}^{q} b_j(\omega)\,x^j \in A[x]$, i.e., the a_i's and b_j's belong to $\mathcal{A}$. Let us suppose that $a_p \neq 0$ and $b_q \neq 0$ so that $\deg_x \mathcal{R} = p$ and $\deg_x \mathcal{I} = q$, and $p \geq q$. Let $A[x]_n = \{P \in \mathcal{A}[x] \mid \deg_x P \leq n\}$ be the set of polynomials with degree at most n and $\{x^i\}_{i=0,\ldots,n}$ the standard basis of the free $\mathcal{A}$-module

$A[x]_n$ of rank $n+1$. We set $A[x]_n = 0$ for negative integer n. For $0 \leq k \leq q$, we can consider the following homomorphism of free $\mathcal{A}$-modules:

$$\begin{aligned} \varphi_k : A[x]_{q-k-1} \times A[x]_{p-k-1} &\longrightarrow A[x]_{p+q-k-1} \\ (\mathcal{U}, \mathcal{V}) &\mapsto \mathcal{U}\mathcal{R} + \mathcal{V}\mathcal{I}. \end{aligned}$$

Using the standard basis of $A[x]_{q-k-1}$ (resp., $A[x]_{p-k-1}$, $A[x]_{p+q-k-1}$) and identifying $\sum_{i=0}^{q-k-1} u_i\, x^i \in A[x]_{q-k-1}$ with the row vector $(u_0, \ldots, u_{q-k-1}) \in A^{1\times(p-k)}$, we obtain that

$$\varphi_k(u_0, \ldots, u_{q-k-1}, v_0, \ldots, v_{p-k-1}) = (u_0, \ldots, u_{q-k-1}, v_0, \ldots, v_{p-k-1})\, S_k,$$

where the matrix S_k is the matrix defined by:

$$S_k = \begin{pmatrix} U_k \\ V_k \end{pmatrix} \in A^{(q-k+p-k)\times(p+q-k)},$$

$$U_k = \begin{pmatrix} a_0 & a_1 & \ldots & a_{q-k} & \ldots & a_p & 0 & \ldots & 0 \\ 0 & a_0 & \ldots & a_{q-k-1} & \ldots & a_{p-1} & a_p & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & a_0 & \ldots & \ldots & \ldots & \ldots & a_p \end{pmatrix} \in A^{(q-k)\times(p+q-k)},$$

$$V_k = \begin{pmatrix} b_0 & b_1 & \ldots & b_{p-k} & \ldots & b_q & 0 & \ldots & 0 \\ 0 & b_0 & \ldots & b_{q-k-1} & \ldots & b_{q-1} & b_q & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & b_0 & \ldots & \ldots & \ldots & \ldots & b_q \end{pmatrix} \in A^{(p-k)\times(p+q-k)}.$$

To be coherent with the degree of polynomials, we attach index $i-1$ to the ith column of S_k so that the index of the columns goes from 0 to $p+q-k-1$.

Definition 1 For $0 \leq j \leq p+q-k-1$ and $0 \leq k \leq q$, let $\mathrm{sr}_{k,j}$ be the determinant of the submatrix of S_k formed by the last $p+q-2k-1$ columns, the column of index j and all the $p+q-2k$ rows. The polynomial $\mathrm{Sres}_k(\mathcal{R}, \mathcal{I}) = \mathrm{sr}_{k,k}\, x^k + \cdots + \mathrm{sr}_{k,0}$ is the kth*subresultant* of $\mathcal{R}$ and $\mathcal{I}$, and its leading term $\mathrm{sr}_{k,k}$ is the kth*principal subresultant* of $\mathcal{R}$ and $\mathcal{I}$. The matrix $S_0 \in A^{p+q)\times(p+q)}$ is the *Sylvester matrix* associated with $\mathcal{R}$ and $\mathcal{I}$ and $\mathrm{Res}_x(\mathcal{R}, \mathcal{I}) = \det S_0$ is the *resultant* of $\mathcal{R}$ and $\mathcal{I}$.

Remark 3 For $k < j \leq p+q-2k-1$, we note that $\mathrm{sr}_{k,j} = 0$ since $\mathrm{sr}_{k,j}$ is the determinant of a matrix having twice the same column. Moreover, we can check that we have $\mathrm{sr}_{q,q} = b_q^{p-q}$ and $\mathrm{Sres}_q(\mathcal{R}, \mathcal{I}) = b_q^{p-q-1}\, \mathcal{I}$ for $q < p$.

Since $A = \mathbb{K}[\omega]$ is an *integral domain*, we can consider its *field of fractions* $Q(A) = \mathbb{K}(\omega)$ and the *Euclidean domain* $B = Q(A)[x]$. Since $\mathcal{R}, \mathcal{I} \in B$, we can define the *greatest common factor* $\gcd(\mathcal{R}, \mathcal{I})$, which is defined up to a non-zero element of $Q(A)$, so that we can suppose that $\gcd(\mathcal{R}, \mathcal{I}) \in A$.

Theorem 1 ([1]) *The first* $\mathrm{Sres}_k(\mathcal{R}, \mathcal{I})$ *such that* $\mathrm{sr}_{k,k} \neq 0$ *is equal to* $\gcd(\mathcal{R}, \mathcal{I})$.

Theorem 2 ([1]) *The roots of* $\mathrm{Res}_x(\mathcal{R}, \mathcal{I})$ *are the projection onto the* ω*-axis of the common solutions of* $\mathcal{R}$ *and* $\mathcal{I}$ *and the common roots of* a_p *and* b_q.

If we want to check whether or not (3) admits imaginary roots, we can first compute $\mathrm{Res}_x(\mathcal{R}, \mathcal{I})$ and then check whether or not $\mathrm{Res}_x(\mathcal{R}, \mathcal{I})$ admits real roots. A method to do that is, for instance, to use the *Descartes rule of sign* [26]. If $f(i\,\omega, \tau)$ does not admit solutions with real ω-coordinates, then it means that the stability of the system is independent of τ. We can then set, for instance, $\tau = 0$ in $f(s, \tau)$ and then apply the *Routh–Hurwitz criterion* to the univariate polynomial $\sum_{j=0}^{l} p_j(s)$.

Example 1 Let us consider $f(s, \tau) = s + 2 + e^{-\tau s}$. Applying the Möbius transformation to $f(i\,\omega, \tau) = i\,\omega + 2 + e^{-i\,\tau\,\omega}$, we obtain $\mathcal{R} = -\omega + 3\,x$ and $\mathcal{I} = \omega\,x + 1$. Then, we get $\mathrm{Res}_x(\mathcal{R}, \mathcal{I}) = \omega^2 + 3$ whose solutions are complex. Considering $\tau = 0$, $f(s, 0) = s + 3$ is stable which proves that f is stable independently of the delay.

If $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, then a convenient way to express the solutions

$$V(\langle \mathcal{R}, \mathcal{I} \rangle) := \left\{(\omega, x) \in \mathbb{K}^2 \mid \mathcal{R}(\omega, x) = \mathcal{I}(\omega, x) = 0\right\}$$

of the polynomial system (3) is to use the so-called *Rational Univariate Representation* (RUR) [25, 27], that is the following one-to-one mapping

$$\begin{array}{rcl} V(\langle \mathcal{R}, \mathcal{I} \rangle) & \longrightarrow & V(\langle f_a \rangle) \\ (\omega, x) & \mapsto & \xi, \\ \left(\dfrac{g_{a,\omega}(\xi)}{g_a(\xi)}, \dfrac{g_{a,x}(\xi)}{g_a(\xi)}\right) & \leftarrow\!\shortmid & \xi, \end{array} \tag{5}$$

between the solutions $V(\langle \mathcal{R}, \mathcal{I} \rangle)$ of (3) and the roots $V(\langle f_a \rangle) := \{t \in \mathbb{K} \mid f_a(t) = 0\}$ of a certain univariate polynomial f_a. In order to achieve the one-to-one condition, the representation (5) is computed with respect to a so-called *separating linear form* $t = a_1\,\omega + a_2\,x \in \mathbb{Q}[\omega, x]$ for certain $a = (a_1, a_2) \in \mathbb{Q}^2$, that takes different values when evaluated at the different points of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$. Using (5), the solutions of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$ are then defined by the following rational parametrization

$$f_a(t) = 0, \quad \omega = \frac{g_{a,\omega}(t)}{g_a(t)}, \quad x = \frac{g_{a,x}(t)}{g_a(t)}, \tag{6}$$

where $f_a, g_a, g_{a,\omega}, g_{a,x} \in \mathbb{Q}[t]$ and f_a and g_a satisfy $\gcd(f_a, g_a) = 1$.

Computing a RUR requires solving the following two problems:

- Find a separating linear form $t = a_1\,\omega + a_2\,x$ for $V(\langle \mathcal{R}, \mathcal{I} \rangle)$.
- Given a linear form $t = a_1\,\omega + a_2\,x \in \mathbb{Q}[\omega, x]$, compute a RUR-candidate, that is to say the polynomials $f_a, g_a, g_{a,\omega}$ and $g_{a,x}$.

Computation of the RUR-candidate. Given a polynomial h in $\mathbb{Q}[\omega, x]$ (not necessarily separating), a RUR-candidate with respect to h can be computed using an algorithm given in [25, 27]. This algorithm requires the knowledge of a $\mathbb{K}$-basis of the finite $\mathbb{K}$-vector space $\mathcal{A} := \mathbb{K}[\omega, x]/\langle \mathcal{R}, \mathcal{I} \rangle$ and a *reduction algorithm* which computes *normal forms* modulo the ideal $\langle \mathcal{R}, \mathcal{I} \rangle$ [8]. In order to explicitly characterize the polynomials appearing in the RUR-candidate, we first define the $\mathbb{Q}$-endomorphism defined by the multiplication by $h \in \mathbb{Q}[\omega, x]$ in $\mathcal{A}$, i.e., we consider $m_h : \mathcal{A} \longrightarrow \mathcal{A}$ defined by $m_h(\overline{p}) = \overline{h\, p}$, where $\overline{p}$ denotes the residue class of $p \in \mathbb{Q}[\omega, x]$ in $\mathcal{A}$ (i.e., modulo $\langle \mathcal{R}, \mathcal{I} \rangle$). A representative of $\overline{p}$ is the *normal form* of p with respect to the reduction algorithm (e.g., based on a *Gröbner basis* computation) of $\langle \mathcal{R}, \mathcal{I} \rangle$. For more details, see [8].

Given a $\mathbb{K}$-basis $\{e_1, \ldots, e_n\}$ of $\mathcal{A}$—which can be deduced from, e.g., a Gröbner basis computation of $\langle \mathcal{R}, \mathcal{I} \rangle$ for the *graded reverse lexicographic order* [8]—we can compute the $(n \times n)$-matrix $M_{a_1\, \omega + a_2\, x}$ associated to $m_{a_1\, \omega + a_2\, x}$. The polynomial f_a of the RUR-candidate is defined as the characteristic polynomial of the matrix $M_{a_1\, \omega + a_2\, x}$. If $\overline{f_a} := \frac{f_a}{\gcd\left(f_a, \frac{d f_a}{dt}\right)} = \sum_{i=0}^{d} v_i\, t^{d-i} \in \mathbb{Q}[t]$ is the *square-free part* of f_a of degree d, and if we note $H_j = \sum_{i=0}^{j} v_i\, t^{j-i} \in \mathbb{Q}[t]$ for $j = 0, \ldots, d-1$, then we have:

$$\begin{cases} g_a = \displaystyle\sum_{i=0}^{d-1} \mathrm{Trace}(M_{a_1\, \omega + a_2\, x}^{i})\, H_{d-i-1}, \\ g_{a,\omega} = \displaystyle\sum_{i=0}^{d-1} \mathrm{Trace}(M_\omega\, M_{a_1\, \omega + a_2\, x}^{i})\, H_{d-i-1}, \\ g_{a,x} = \displaystyle\sum_{i=0}^{d-1} \mathrm{Trace}(M_x\, M_{a_1\, \omega + a_2\, x}^{i})\, H_{d-i-1}. \end{cases}$$

Finding of a separating linear form. For the computation of a separating polynomial of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$, a critical remark is that the number of non separating elements is bounded by $m = n\,(n-1)/2$, where n denotes the cardinal of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$, that is the number of lines passing by two distinct points of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$. Thus, a separating form can always be found among the set $\{t = \omega + a\, x \mid a = 0, \ldots, m\}$. On the other hand, the *Bezout theorem* [1, 8] states that for polynomials $\mathcal{R}$ and $\mathcal{I}$ of total degree respectively d_1 and d_2, the cardinal of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$ is bounded by $d_1\, d_2$. Hence, a strategy for computing a separating element for $V(\langle \mathcal{R}, \mathcal{I} \rangle)$ is to consider $m := d_1\, d_2\, (d_1\, d_2 - 1)/2 + 1$ distinct integers a and for each a, compute the number of distinct roots of the polynomial f_a (see above), i.e., the degree of its squarefree part $\overline{f_a}$, and finally select an a for which this number is maximal. This ensures that the degree of $\overline{f_a}$ is equal to the cardinal of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$, and thus that the roots of f_a are in bijection with the points of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$. This strategy is actually time-consuming since it requires the computation of m characteristic polynomials as well as their squarefree parts. In practice, noticing that an arbitrary chosen linear form is separating with high probability, one prefers a strategy that consists in choosing randomly a linear form and testing that it separates the points of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$ a posteriori of the computation of a RUR-candidate. This test is based on the fact that a linear form t is separating for $V(\langle \mathcal{R}, \mathcal{I} \rangle)$ if and only if the polynomials $u_\omega := g_a\,\omega - g_{a,\omega}$ and $u_x := g_a\,x - g_{a,x}$ vanish on all the points of $V(\langle \mathcal{R}, \mathcal{I} \rangle)$, i.e., belong to the *radical* of $\langle \mathcal{R}, \mathcal{I} \rangle$ [8]. In computational terms, this condition can be translated into $(\text{Trace}(M_{u_\omega e_i}))_{i=1,\ldots,n} = (0, \ldots, 0)$ and $(\text{Trace}(M_{u_x e_i}))_{i=1,\ldots,n} = (0, \ldots, 0)$ [25, 27].

Efficient RUR algorithm. The above algorithm for the computation of the RUR is designed for general zero-dimensional systems and is thus not optimized for the specific system (3) formed by two polynomials in two variables (i.e., ω, x). The computation of a Gröbner basis of $\langle \mathcal{R}, C \rangle$, required for the computation of the RUR-candidate as well as for the separating form, is usually time-consuming.

Alternatively, we propose below an efficient method, developed in [4], that computes a decomposition of the solutions of (3) into *univariate representations*. This method avoids the computation of a Gröbner basis. This method first consists in applying a change of variables on the variables (ω, x) and then using resultants and subresultants polynomials to compute parameterizations of the solutions of (3). These parameterizations then encode the solutions of the system in a one-to-one correspondence with the roots of a univariate polynomial provided that the change of variables shears the system into *generic position*. Let us introduce this last definition.

Definition 2 Let $\mathcal{R}(\omega, x)$, $\mathcal{I}(\omega, x) \in \mathbb{Q}[\omega, x]$. If $\sharp S$ denotes the cardinality of a finite set S, then the system $\{\mathcal{R}, \mathcal{I}\}$ is said to be in *generic position* if we have:

$$\forall\, \alpha \in \mathbb{C}, \quad \sharp\,\{\beta \in \mathbb{C} \mid \mathcal{R}(\alpha, \beta) = \mathcal{I}(\alpha, \beta) = 0\} \leq 1.$$

Let us start with the following theorem which shows that there exists $t = \omega + a\,x$ such that, up to a factor in $\mathbb{Q}$, the polynomial f_a is equal to the resultant of two polynomials obtained from $\mathcal{R}$ and $\mathcal{I}$ after a change of variables.

Theorem 3 ([4]) *Let $\mathcal{R}(\omega, x)$, $\mathcal{I}(\omega, x) \in \mathbb{Q}[\omega, x]$. Define $\mathcal{R}'(t, x) = \mathcal{R}(t - a\,x, x)$ and $\mathcal{I}'(t, x) = \mathcal{I}(t - a\,x, x)$, where $a \in \mathbb{Z}$ is such that the leading coefficient of $\mathcal{R}'$ and $\mathcal{I}'$ with respect to x are coprime. Then, the resultant of $\mathcal{R}'$ and $\mathcal{I}'$ with respect to x is then equal to*

$$\text{Res}_x(\mathcal{R}', \mathcal{I}') = c \prod_{(\alpha_1,\, \alpha_2) \in V(\langle \mathcal{R}, \mathcal{I} \rangle)} (t - \alpha_1 - a\,\alpha_2)^{\mu(\alpha_1, \alpha_2)} = c\, f_a(t),$$

where $c \in \mathbb{Q}$ and $\mu(\alpha_1, \alpha_2)$ denotes the multiplicity *of $(\alpha_1,\, \alpha_2) \in V(\langle \mathcal{R}, \mathcal{I} \rangle)$ [8].*

Given a linear form $t = \omega + a\,x$, it can be shown that it is separating for $V(\langle \mathcal{R}, \mathcal{I} \rangle)$ if and only if the system $\{\mathcal{R}', \mathcal{I}'\}$ is in generic position (see Definition 2). Algebraically, this means that for each root α of $\text{Res}_x(\mathcal{R}', \mathcal{I}')$ (where $\mathcal{R}'$ and $\mathcal{I}'$ are defined as in Theorem 3), the gcd of $\mathcal{R}'(\alpha, x)$ and $\mathcal{I}'(\alpha, x)$, denoted $\mathcal{G}(\alpha, x)$, has exactly one distinct root.

To check the above genericity condition, we consider a *triangular description* of the solutions of $\{\mathcal{R}', \mathcal{I}'\}$ given by a finite union of *triangular systems*:

$$V(\langle \mathcal{R}',\ \mathcal{I}' \rangle) = \bigcup_{k=1}^{l} \left\{ (\alpha,\ \beta) \in \mathbb{C}^2 \mid r_k(\alpha) = \mathcal{G}_k(\alpha, \beta) = 0 \right\}.$$

Such a triangular description can be obtained via a *triangular decomposition algorithm* based on the resultant and subresultant polynomials (see Algorithm 1 of [4] for more details). Such a triangular decomposition yields a set of triangular systems of the form $\{r_k(t), \mathrm{Sres}_k(t, x)\}_{k=1,\dots,l}$, where $l = \min\{\deg_x \mathcal{R}',\ \deg_x \mathcal{I}'\}$, $\mathrm{Res}_x(\mathcal{R}', \mathcal{I}') = \prod_{k=1}^{l} r_k(t)$, $r_k \in \mathbb{K}[t]$ is the factor of $\mathrm{Res}_x(\mathcal{R}', \mathcal{I}')$ (possibly equal to 1) whose roots α's satisfy the property that the degree of $\mathcal{G}(\alpha, x)$ (i.e., the gcd of $\mathcal{R}'(\alpha, x)$ and $\mathcal{I}'(\alpha, x)$) in x is equal to k and $\mathrm{Sres}_k(t, x) = \sum_{i=0}^{k} \mathrm{sr}_{k,i}(t)\, x^i$ is the kth subresultant of $\mathcal{R}'$ and $\mathcal{I}'$. Once a triangular decomposition $\{r_k(t), \mathrm{Sres}_k(t, x)\}_{k=1,\dots,l}$ of $\{\mathcal{R}', \mathcal{I}'\}$ is computed, the genericity condition is equivalent to the fact that $\mathrm{Sres}_k(t, x)$ can be written as $(a_k(t)\, x - b_k(t))^k$ modulo $r_k(t)$, with $\gcd(a_k, b_k) = 1$. The next theorem checks this last condition.

Theorem 4 ([9]) *Let $\mathcal{R}(\omega, x),\ \mathcal{I}(\omega, x) \in \mathbb{Q}[\omega, x]$. Define the polynomials $\mathcal{R}'(t, x)$, $\mathcal{I}'(t, x)$ as in Theorem 3, and let $\{r_k(t), \mathrm{Sres}_k(t, x)\}_{k=1,\dots,l}$ be the triangular decomposition of $\{\mathcal{R}', \mathcal{I}'\}$. Then, $\{\mathcal{R}', \mathcal{I}'\}$ is in generic position if and only if we have*

$$k\,(k-i)\,\mathrm{sr}_{k,i}\,\mathrm{sr}_{k,k} - (i+1)\,\mathrm{sr}_{k,k-1}\,\mathrm{sr}_{k,i+1} = 0 \mod r_k, \tag{7}$$

for all $k \in \{1, \dots, l\}$ and for all $i \in \{0, \dots, k-1\}$.

If $\{\mathcal{R}', \mathcal{I}'\}$ is in generic position, i.e., if (7) is satisfied, then we obtain that $\mathrm{Sres}_k(t, x) = \sum_{i=0}^{k} \mathrm{sr}_{k,i}(t)\, x^i = (a_k(t)\, x + b_k(t))^k$ modulo $r_k(t)$, with $\gcd(a_k, b_k) = 1$. An explicit expression for x can be obtained by differentiating $(k-1)$-times $\mathrm{Sres}_k(t, x)$ with respect to x, which yields $x = -\frac{b_k(t)}{a_k(t)} = -\frac{\mathrm{sr}_{k,k-1}(t)}{k\,\mathrm{sr}_{k,k}(t)}$. Finally, the solutions of the system $\{\mathcal{R}, \mathcal{I}\}$ can be parametrized as follows:

$$r_k(t) = 0, \quad x = -\frac{\mathrm{sr}_{k,k-1}(t)}{k\,\mathrm{sr}_{k,k}(t)}, \quad \omega = t - a\,\frac{\mathrm{sr}_{k,k-1}(t)}{k\,\mathrm{sr}_{k,k}(t)}, \quad k = 1, \dots, l. \tag{8}$$

Numerical approximations. Once a RUR of the solutions of (3) is computed (respectively a parametrization by means of the subresultants (8)), we can obtain certified numerical approximations of these solutions by first isolating the real roots of f_a (resp., r_k) by means of intervals using, for instance, the algorithm in [26], and then substituting these intervals into the rational functions $g_{a,\omega}/g_a$ and $g_{a,x}/g_a$ (resp., $t - a\,\mathrm{sr}_{k,k-1}(t)/k\,\mathrm{sr}_{k,k}(t)$ and $-\mathrm{sr}_{k,k-1}(t)/k\,\mathrm{sr}_{k,k}(t)$) in order to get isolating intervals for the coordinates ω and x of the solutions of (3). Moreover, substituting these intervals into (4) yields intervals for the delays corresponding to each solution.

An efficient algorithm for the RUR computation is implemented in the `Maple` command `RationalUnivariateRepresentation` of the `Groebner` package. The computation of the parametrization (8) is done in the external library `RS` [28]. Moreover, the numerical approximations of the solutions, obtained by means of a RUR, can be computed using the `Maple` command `Isolate` of the package `RootFinding` with the option `Method="RS"` and `Output="interval"`.

Example 2 We consider $f(s,\tau) = e^{-3\tau s} - 3\,e^{-2\tau s} + 3\,e^{-\tau s} + s^4 + 2\,s^2$ studied in [16]. To compute the critical pairs of f, we consider the polynomial system

$$\begin{cases} \mathcal{R}(\omega,x) = \left(x^3 - 3\,x\right)\omega^4 + \left(-2\,x^3 + 6\,x\right)\omega^2 + x^3 - 3\,x = 0,\\ \mathcal{I}(\omega,x) = \left(3\,x^2 - 1\right)\omega^4 + \left(-6\,x^2 + 2\right)\omega^2 + 3\,x^2 + 7 = 0, \end{cases} \tag{9}$$

obtained by substituting $s = i\,\omega$ and $e^{-\tau\,i\,\omega} = \frac{x-i}{x+i}$ into $f(s,\tau)$ and separating the real and imaginary parts of the numerator of the result, as well as the univariate polynomial $f(i\,\omega, 2\,k\,\pi) = (\omega^2 - 1)^2$ (see Remark 1), whose solutions are $\{-1,\ -1,\ 1,\ 1\}$. The latter polynomial yields the critical pairs $(i, 2\,k\,\pi)$ and $(-i, 2\,k\,\pi)$ where $k \in \mathbb{Z}$. Computing the RUR of (9) with respect to the separating form $h = \omega + x$, we obtain:

$$\begin{cases} f_a = \left(t^4 - 2\,t^2 - 7\right)\left(t^8 - 16\,t^6 + 74\,t^4 - 32\,t^2 + 25\right),\\ g_a = t\left(t^{10} - 15\,t^8 + 66\,t^6 - 34\,t^4 - 143\,t^2 + 29\right),\\ g_{a,\omega} = t^{10} - 9\,t^8 - 26\,t^6 + 234\,t^4 + 53\,t^2 + 35,\\ g_{a,x} = 2\,\left(t^4 - 2\,t^2 - 7\right)\left(t^6 - 10\,t^4 + 17\,t^2 - 10\right). \end{cases}$$

Then, isolating the real roots of f_a, we obtain the two real roots

$$\begin{aligned} t_1 &\in \left[-\tfrac{268918098581}{137438953472}, -\tfrac{67229524645}{34359738368}\right], & t_1 &\approx -1.956636687,\\ t_2 &\in \left[\tfrac{67229524645}{34359738368}, \tfrac{268918098581}{137438953472}\right], & t_2 &\approx 1.956636687, \end{aligned}$$

which, after substitution into the rational functions of the RUR, yields $x_1 = x_2 = 0$ and thus $\omega_j = t_j$ for $j = 1,\ 2$.

Note that the two ω-coordinates are roots of the polynomial $t^4 - 2\,t^2 - 7$ which can be solved symbolically. We then get $\omega_j = (-1)^j\sqrt{1 + 2\sqrt{2}}$ for $j = 1,\ 2$.

Finally, the critical delays $\tau_{j,k}$ can then be obtained by (4). In our case, the solutions $(\omega_1,\ x_1)$ and $(\omega_2,\ x_2)$ yields to $\tau_{j,k} = \frac{k\,\pi}{\omega_j}$, where $k \in \mathbb{Z}$ and $j = 1,\ 2$.

Alternatively, we can compute the parametrization (8) of (9). Since all the solutions of (9) are simple (i.e., their multiplicities are equal to 1), this yields the following single parametrization:

$$\begin{cases} r_1 = \left(t^4 - 2\,t^2 - 7\right)\left(t^8 - 16\,t^6 + 74\,t^4 - 32\,t^2 + 25\right),\\ \mathrm{sr}_{1,0} = 65536\,t\left(3\,t^6 - 18\,t^4 + 7\,t^2 + 18\right),\\ \mathrm{sr}_{1,1} = -327680\,t^6 + 1179648\,t^4 + 983040\,t^2 + 655360. \end{cases}$$

The two algorithms, based respectively on the computation of (6) and of (8), are implemented in `Maple`. We report below their running times (on a laptop with an Intel i-7 processor and L8 cache) for randomly generated quasipolynomials of total degree deg (i.e., the maximum of the total degree of the monomials in s and $z = e^{-\tau s}$ in the quasipolynomial $f(s, \tau)$) and density dens (sparse/dense).

deg	dens	numsol	Timing for (6)	Timing for (8)
10	0.5	13	1.17	0.45
10	1	15	1.62	0.6
15	0.5	28	26.58	2.8
15	1	28	24.32	3.7
20	0.5	32	166.40	9.5
20	1	37	182.37	14.3
40	0.5	92	/	300

In the table, the timings are the running times in CPU seconds and numsol stands for the average number of critical pairs (ω, x).

3 Computing Puiseux Series and Asymptotic Stability Analysis

In this section, we first provide basic definitions and algorithms for the computation of *Puiseux series* in the case of polynomials. This problem is well-studied in the computer algebra literature [10, 22, 23]. We then explain how these results can be adapted to the case of quasipolynomials. As shown in [15, 16], Puiseux series can be used to effectively decide the stability of a linear time-invariant differential system with commensurate delays (multiples of $\tau \in \mathbb{R}_{>0}$) when τ changes.

Let $F \in \mathbb{K}[X, Y]$ be a bivariate polynomial, where $\mathbb{K}$ is a field of characteristic 0 (e.g. $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). In what follows, we shall assume that the polynomial F is *square-free* and *monic* in Y, i.e., as a polynomial in Y with coefficients in $\mathbb{K}[X]$, F has no factor of multiplicity 2 and its leading coefficient is 1. Let $\overline{\mathbb{K}}$ denote an *algebraic closure* of $\mathbb{K}$. With these notations, we have the following *Puiseux theorem*.

Theorem 5 *Let $d = \deg_Y F$ be the degree of $F \in \mathbb{K}[X, Y]$ in Y and ζ_e the eth root of unity where $e \in \mathbb{Z}_{>0}$. Given $x_0 \in \overline{\mathbb{K}}$, there exist positive integers $e_1, \ldots, e_s$ with $\sum_{i=1}^{s} e_i = d$ and d distinct series $S_{ij}(X) = \sum_{k=0}^{\infty} \alpha_{ik}\, \zeta_{e_i}^{j\,k}\, (X - x_0)^{\frac{k}{e_i}}$ such that $F(X, S_{ij}(X)) = 0$ for $1 \leq i \leq s$ and $0 \leq j \leq e_i - 1$.*

Over most points x_0, the polynomial $F(x_0, Y)$ has d distinct roots. In this case, the *Implicit Function Theorem* ensures that the series solutions are Taylor series. Such series can be quickly computed using, e.g., *quadratic Newton iterations* [14]. When dealing with multiple roots over a point x_0, this may not be the case anymore.

In what follows, we shall compute the Puiseux series using a variant of the well-known *Newton–Puiseux algorithm* [3, 29], namely the *rational Newton–Puiseux algorithm* [10]. Let us explain the main idea of this algorithm by means of an example.

Example 3 Let $H = Y^6 + X\, Y^5 + 5\, X^3\, Y^4 - 2\, X\, Y^4 + 4\, X^2\, Y^2 + X^5 - 3\, X^4$ and let us compute the Puiseux series at $x_0 = 0$. Note that we can always be in this situation via a change of the variable $X \leftarrow X + x_0$. From Theorem 5, we know that the first term of any series $Y(X)$ is of the form $\alpha\, X^{\frac{m}{q}}$, $\alpha \in \mathbb{K}$ and $(m,\ q) \in \mathbb{N}^2$. We then get:

$$H\left(X, \alpha\, X^{\frac{m}{q}} + \cdots\right) = \alpha^6\, X^{\frac{6m}{q}} + \alpha^5\, X^{\frac{5m}{q}+1} + 5\,\alpha^4\, X^{\frac{4m}{q}+3} - 2\,\alpha^4\, X^{\frac{4m}{q}+1}$$
$$+ 4\,\alpha^2\, X^{\frac{2m}{q}+2} + X^5 - 3\,X^4 + \cdots$$

To get $H(X, Y(X)) = 0$, at least two terms of the above sum must cancel, i.e. (m, q) must be chosen so that two of the above exponents coincide.

Definition 3 The *support* of a polynomial $F(X, Y) = \sum_{i,j} a_{ij} X^j\, Y^i$ is defined by:

$$\mathrm{Supp}(F) = \left\{(i,\ j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\right\}.$$

Example 3 (continued) $\mathrm{Supp}(H) = \{(6, 0), (5, 1), (4, 3), (4, 1), (2, 2), (0, 5), (0, 4)\}$.

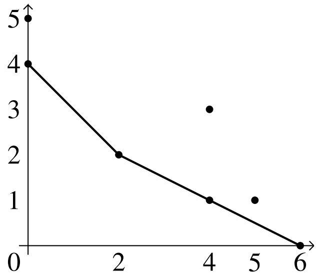
The condition on the pairs $(m,\ q)$ can be translated as follows: two points of $\mathrm{Supp}(F)$ must lie on the same line $m\, i + q\, j = l$ for a certain $l \in \mathbb{Z}$. To increase the X-order of the *valuation*, there must be no point under this line. This leads us to the following definition.

Definition 4 The *Newton polygon* $\mathcal{N}(F)$ of $F \in \mathbb{K}[X, Y]$ is the lower part of the *convex hull* of its support $\mathrm{Supp}(F)$.

Example 3 (continued) As illustrated in the figure below, $\mathcal{N}(H)$ is given by two edges: $\Delta_1 = ((6, 0), (2, 2))$ – corresponding to the line $i + 2\, j = 6$ (i.e., $m = 1$ and $q = 2$ – and $\Delta_2 = ((2, 2), (0, 4))$. The points of $\mathrm{Supp}(H)$ belonging to Δ_1 are $(6, 0)$, $(2, 2)$ and $(4, 1)$. We have

$$H(T^q, \alpha T^m) = H(T^2, \alpha T) = (\alpha^6 - 2\alpha^4 + 4\alpha^2)T^6 + \alpha^5 T^7 + (5\alpha^4 + 1)T^{10} - 3T^8 + \cdots$$

Since we want to increase the T-order of $H(T^2, \alpha T)$, we must set α equal to a root of $P = Z^6 - 2\, Z^4 + 4\, Z^2$. We first note that $P \in \mathbb{K}[Z^2]$. Indeed, by construction, the polynomial P always belongs to $\mathbb{K}[Z^q]$. Note also that 0 is a root of P and we are not interested in this root (we shall get the first non-zero term of the associated Puiseux series by considering the edge Δ_2). For these reasons, only the roots of the polynomial $\phi = T^2 - 2\, T + 4$ are interesting. This polynomial is called the *characteristic polynomial* of Δ_1.



The Newton polygon $\mathcal{N}(H)$

Definition 5 If $F = \sum a_{ij} X^j Y^i \in \mathbb{K}[X, Y]$, then the *characteristic polynomial* ϕ_Δ of $\Delta \in \mathcal{N}(F)$ is defined by $\phi_\Delta(T) = \sum_{(i,\, j)\in\Delta} a_{ij} T^{\frac{i-i_0}{q}} \in \mathbb{K}[T]$, where i_0 is the smallest value such that $(i_0,\ j_0)$ belongs to Δ for some j_0.

Computing more terms. Let us sum up the computation of the first non zero term of the Puiseux series of F at 0: we first compute the Newton polygon $\mathcal{N}(F)$ of F, then the characteristic polynomial ϕ_Δ for all the edges Δ of $\mathcal{N}(F)$, and finally compute the roots of these polynomials. Note that we assume that we can decide whether or not a coefficient of F is 0 to be able to certify the correctness of our computations. At the end of this section, we shall briefly discuss a symbolic-numeric strategy developed in [21, 22] to overcome possibly costly symbolic computations.

Assuming the generalization of the above results for quasipolynomials (see the next paragraph), if one of the roots of the characteristic polynomial leads to a purely imaginary coefficient for one of the Puiseux series (in the sequel, such a root is denoted by ξ), we then have to compute its next term to decide on the stability of the differential time-delay system. This can actually be done by applying the above strategy to $F\left(X, Y + \xi^{\frac{1}{q}} X^{\frac{m}{q}}\right) \in \mathbb{K}\left(\xi^{\frac{1}{q}}\right)\left[X^{\frac{1}{q}}, Y\right]$, or similarly to the polynomial $F\left(X^q, Y + \xi^{\frac{1}{q}} X^m\right) \in \mathbb{K}\left(\xi^{\frac{1}{q}}\right)[X, Y]$. To avoid taking a qth root of ξ, following [10, Sect. 4], we can consider the following polynomial

$$F_{\Delta,\xi}(X, Y) = \frac{F(\xi^v X^q, X^m (Y + \xi^u))}{X^l} \in \mathbb{K}(\xi)[X, Y],$$

where $u, v \in \mathbb{Z}$ are such that $u\, q - m\, v = 1$ and $m\, i + q\, j = l$ with $(i,\ j) \in \Delta$. Figure 1 illustrates such a transformation.

Let us now introduce the concept of a *rational Puiseux expansion*.

Definition 6 A *rational Puiseux expansion* over $\mathbb{K}$ of F above 0 is a pair of non-constant formal power series $(X(T), Y(T)) = \left(\lambda\, T^e,\ \sum_{j=0}^{\infty} \beta_j\, T^j\right)$ such that:

(i) $e \geq 1$ and $\lambda \neq 0$.
(ii) $(X(T), Y(T))$ is a *parametrization* of F, i.e., $F(X(T), Y(T)) = 0$.

(iii) The parametrization $(X(T), Y(T))$ of F is *irreducible*, namely e is minimal: for any $u > 1$, $Y(T) \notin \mathbb{K}[[T^u]]$.

From a rational Puiseux expansion of F, we can deduce e Puiseux series:

$$Y_k(X) = Y\left(\zeta_e^k\,(\lambda^{-1}\,X)^{\frac{1}{e}}\right) = \sum_{j=0}^{\infty} \beta_j\,\lambda^{-j}\,\zeta_e^{k\,j}\,X^{\frac{j}{e}}, \quad 0 \le k \le e-1.$$

The *rational Newton–Puiseux algorithm* [10, 22], which computes truncated rational Puiseux expansions of $F(X, Y) = 0$ at $x_0 = 0$, is given below. It uses the subroutine `Factor`$(\mathbb{K}, \phi)$, that given a field $\mathbb{K}$ and $\phi \in \mathbb{K}[T]$, returns a finite set $\{(\phi_i, M_i)\}_{i \in I}$ of its irreducible factors and the associated multiplicities, i.e. $\phi_i \in \mathbb{K}[T]$ is a monic and irreducible polynomial and $\phi = c \prod_{i\in I} \phi_i^{M_i}$, where $c \in \mathbb{K}$ and $M_i \in \mathbb{Z}_{>0}$ for $i \in I$.

Algorithm RNPuiseux(F,K_i,π):

Input: $F \in \mathbb{K}[X, Y]$, monic and squarefree with $\mathbb{K}$ a field
  π the result of previous computations ($\pi = (X, Y)$ for the initial call)
Output: Truncated rational Puiseux expansions of F

$\mathcal{R} \leftarrow \{\}$; `// results of the algorithm will be grouped in` $\mathcal{R}$
foreach $\Delta \in \mathcal{N}(F)$ **do**
  Compute $m, q, l \in \mathbb{N}$ s.t. q and m are coprime and $m\,i + q\,j = l$ for all $(i, j) \in \Delta$;
  Compute $(u, v) \in \mathbb{Z}^2$ such that $u\,q - m\,v = 1$ and ϕ_Δ associated to Δ (see Definition 5);
  foreach (ϕ, M) **in** `Factor`$(\mathbb{K}, \phi_\Delta)$ **do**
    Consider a new symbol ξ satisfying $\phi(\xi) = 0$;
    $\pi_1 = \pi(\xi^v X^q, X^m (Y + \xi^u))$;
    if $M = 1$ **then** $\mathcal{R} \leftarrow \mathcal{R} \cup \{(\pi_1(T, 0), \mathbb{K}(\xi))\}$;
    else
      $G(X, Y) \leftarrow F(\xi^v X^q, X^m (Y + \xi^u))/X^l$; `// Puiseux transformation`
      $\mathcal{R} \leftarrow \mathcal{R} \cup$ `RNPuiseux`$(G, \mathbb{K}(\xi), \pi_1)$;
return $\mathcal{R}$;

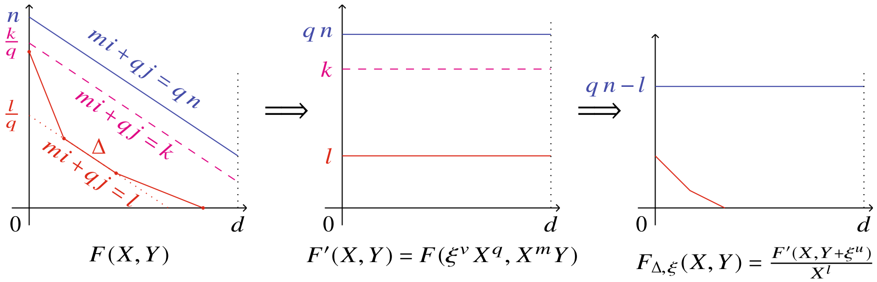


Fig. 1 Geometry of a change of variables: each diagonal becomes a horizontal line

The quasipolynomial case. As mentioned in [15], we can approximate the quasipolynomial by truncated power series. In [22, 23], it is shown that truncation bounds exist to ensure the exactness of the Puiseux series. A first answer is therefore to compute such an approximation and then apply the standard algorithm explained above.

For the computation of the first term, we can replace any coefficient access in the polynomial case by an evaluation of some derivatives of the quasipolynomial [15]. For instance, if $f(s, \tau)$ is a quasi-polynomial, then $\left(\frac{1}{2!\,1!}\,\frac{\partial^3 f}{\partial^2 s\,\partial\tau}\right)(\tau_0, s_0)$ corresponds to the coefficient of the term $(s - s_0)^2\,(\tau - \tau_0)$.

For higher order terms, instead of evaluating the derivatives of f at a point, we can evaluate f at the truncated Puiseux series already computed (e.g., via the parameter π of Algorithm `RNPuiseux`). It would also be interesting to develop such a strategy and compare the two approaches.

Finally, the condition on line 8 in Algorithm `RNPuiseux` has to be changed. Indeed, the aim of this algorithm is to *desingularize* the curve $F(X, Y) = 0$, i.e., to stop when the multiplicity of the root ξ is 1. In our context, we stop when the computed coefficient has a non-zero real part. This can happen before or after the condition $M = 1$ is reached.

A symbolic-numeric strategy. In [21, 22], a symbolic-numeric strategy was developed to compute a numerical approximation of the Puiseux series coefficients with *correct* exponents. Roughly speaking, the idea is to first compute the Puiseux series modulo a well-chosen prime number, which gives the *structure* of the series (exponents, etc.), and then to use it to conduct purely numeric operations. Adapting this strategy to quasi-polynomials is an interesting challenge which will be studied.

Examples. We implemented a `Maple` prototype to compute the Puiseux series of a quasipolynomial. Note that this is a symbolic algorithm. We illustrate the results with examples and we show the logs of some computations (including the timing).

Example 4 We consider again the quasipolynomial defined in Example 2 and compute its Puiseux series at $(\tau_0, s_0) = (2\pi, i)$. We get the following:

```
Newton polygon is   [[0, 3], [2, 0]]
xi is   1/4*I
Real part of the coefficient is 1/4*2^(1/2)
It took .88e-1 seconds
```

Thus, we obtain $\Delta s = \dfrac{\sqrt{2}}{4}\,(1 + i)\,(\Delta\tau)^{\frac{3}{2}} \approx (.3535 + .3535\,i)\,(\Delta\tau)^{\frac{3}{2}}$. Thus, for $\Delta\tau > 0$ (resp., $\Delta\tau < 0$), i.e., for an increasing (resp., decreasing) delay $\tau > 2\pi$ (resp., $\tau < 2\pi$), $(\Delta\tau)^{\frac{3}{2}} = \pm\,(\Delta\tau)\,\sqrt{\Delta\tau}$ (resp., $(\Delta\tau)^{\frac{3}{2}} = \pm i\,(-\Delta\tau)\,\sqrt{-\Delta\tau}$), which yields $\mathbb{R}(\Delta s) = \pm\,\dfrac{\sqrt{2}}{4}\,(\Delta\tau)\,\sqrt{\Delta\tau}$ (resp., $\mathbb{R}(\Delta s) = \pm\,\dfrac{\sqrt{2}}{4}\,(-\Delta\tau)\,\sqrt{-\Delta\tau}$). Therefore, the double root i splits into two branches toward $\mathbb{C}_+$ and $\mathbb{C}_- = \{s \in \mathbb{C} \mid \mathbb{R}(s) < 0\}$, which shows that the quasipolynomial is unstable for small variation around $\tau = 2\pi$.

Example 5 Let us consider the quasipolynomial $F(\tau, s) = \sum_{k=0}^{4} a_k(s)\, e^{-k\, s\, \tau}$, where

$$\begin{cases} a_4(s) = s^3 + 2\,s^2 + 2\,s + 1, \\ a_3(s) = 3\,s^3 + 9\,s^2 + 9\,s + 4, \\ a_2(s) = \frac{5}{4}\,\pi\, s^5 + \frac{11}{4}\,\pi\, s^4 + 3\,s^3 - \pi\, s^3 + \frac{1}{2}\, s^2\pi + 13\,s^2 + 15\,s - \frac{9}{4}\,s\,\pi + 6 - \frac{9}{4}\,\pi, \\ a_1(s) = \frac{5}{4}\,\pi\, s^5 + \frac{11}{2}\,\pi\, s^4 + s^3 + \frac{7}{2}\,\pi\, s^3 + s^2\,\pi + 7\,s^2 + 11\,s + \frac{9}{4}\,s\,\pi + 4 - \frac{9}{2}\,\pi, \\ a_0(s) = 1 + 3\,s + \frac{9}{2}\,\pi\, s^3 + s^2 - \frac{45}{8}\,\pi^2 + \frac{9}{2}\,s\,\pi - \frac{15}{8}\,s^4\,\pi^2 + \frac{1}{2}\,s^2\,\pi - \frac{75}{8}\,s^2\,\pi^2 \\ \qquad\quad -\frac{9}{4}\,\pi + \frac{11}{4}\,\pi\, s^4 + \frac{15}{8}\,\pi^2\, s^6, \end{cases}$$

considered in [17]. Computing the Puiseux series at the point $(5\,\pi, i)$, we obtain:

```
Newton polygon is  [[0 2] [1 1] [4 0]]
Edge 1, xi is  -((1/2)*I)/Pi
Real part is 0; Going for a recursive call
   Newton polygon is [[0 1] [1 0]]
   xi is (-3/208+(1/104)*I)*(117*Pi+2-3*I)/Pi^2
   Real part is -(27/16)/Pi,  We're done.
Edge 2, xi is ((6/5)*I)/(50*Pi^3-30*Pi^2-3*Pi)
Real part is 0; Going for a recursive call
   Newton polygon is [[0 1] [1 0]]
   xi is (see below)
   real part is non zero. We're done.
 It took 3.596 seconds
```

We obtain the two Puiseux series $\Delta s = -0.1591\, i\, \Delta\tau - (0.5371 - 0.3644\, i)$ and $\Delta s = -0.0987\, i\, (\Delta\tau)^{1/3} - (0.03557 - 0.0028\, i)\, (\Delta\tau)^{2/3}$. The coefficient growth is important in this example. For instance, the root ξ that is not written above is:

$$\frac{3}{25\pi^2}\, \frac{200\, i\, \pi^2 - 180\, i\, \pi - 750\,\pi^3 - 21\, i + 600\,\pi^2}{125000\,\pi^6 - 225000\,\pi^5 + 112500\,\pi^4 - 6750\,\pi^2 - 810\,\pi - 27}.$$

This explains that the running time is longer than for Example 4. This fact advocates for the development of an efficient symbolic-numeric strategy.

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Delay-Dependent Reciprocally Convex Combination Lemma for the Stability Analysis of Systems with a Fast-Varying Delay

Alexandre Seuret and Frédéric Gouaisbaut

1 Introduction

This paper aims at providing less conservatism and computationally efficient stability conditions for linear systems subject to fast-varying delays. This class of delays corresponds to the situation where the function representing the delay varies with time and lies in a bounded interval of $(0, \infty)$. No assumption on the derivative of the delay function are imposed. This topic of research has attracted many researchers over the past decades (see for instance [1, 7, 14, 15] and the reference therein). The main difficulties for the study of such a class of systems rely on two technical steps that are the derivation of efficient integral and matrix inequalities. Indeed, the differentiation of usual Lyapunov-Krasovskii functional candidates leads to integral quadratic terms that cannot be included straightforwardly in a linear matrix inequality (LMI) setup. Including these terms requires the use of integral inequalities such as Jensen [3], Wirtinger-based [9], auxiliary-based [5, 8] or Bessel inequalities [10]. Although these inequalities have shown a great interest for constant delay systems, their application to time- or fast-varying delays reveals additional difficulties related to the non convexity of the resulting terms. Therefore, some matrix inequalities are employed to address this last problem and to derive convex conditions. A huge number of papers (see e.g. [7, 15] and the references therein) have studied the ways to combine efficiently integral and matrix inequalities. Hence, a first method corresponds to the application of Young's or Moon's inequalities [6], after the application of an integral inequality, which basically results from the positivity of a square posi-

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A. Seuret (✉) · F. Gouaisbaut
LAAS-CNRS, Université de Toulouse, CNRS, UPS, Toulouse, France
e-mail: alexandre.seuret@laas.fr

F. Gouaisbaut
e-mail: gouaisb@laas.fr

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tive definite term. It can also be noted that the recent free-matrix inequality [16] can be interpreted as the merge of the Wirtinger-based inequality and Moon's inequality. Recently, the reciprocally convex lemma was proposed in [7]. The novelty of this method consists in merging the non convex terms into a single expression to derive an accurate convex inequality. It was notably shown that the conservatism of the reciprocally convex combination lemma [7] and the Moon's inequality are similar when considering Jensen-based stability criteria, with a lower computational burden. In the present paper, the objective is to refine the reciprocally convex lemma by introducing delay dependent terms. The resulting lemma includes the initial reciprocally convex lemma as a particular case. Examples show a clear reduction of conservatism at a reasonable increase of the computational cost.

Notations: Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ are the set of $n \times m$ real matrices and of $n \times n$ real symmetric matrices, respectively. Moreover the notation $P \in \mathbb{S}^n_+$, means that $P \in \mathbb{S}^n$ and $P \succ 0$, which means that P is symmetric positive definite. For any matrices A, B of appropriate dimension, the matrix $diag(A, B)$ stands for $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. The matrices I_n and $0_{n,m}$ represent the identity and null matrices of appropriate dimension and, when no confusion is possible, the subscript will be omitted. For any $h > 0$ and any function $x : [-h, +\infty) \to \mathbb{R}^n$, the notation $x_t(\theta)$ stands for $x(t+\theta)$, for all $t \geq 0$ and all $\theta \in [-h, 0]$. Finally, for given positive scalars $h_1 \leq h_2$, we use the notation $h_{21} = h_2 - h_1$.

2 Problem Formulation and Preliminaries

Consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h_2, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, ϕ is the initial condition and $A, A_d \in \mathbb{R}^{n\times n}$ are constant matrices. The delay $h(t)$ is assumed to be a continuous time-varying function for which there exist positive scalars $h_1 \leq h_2$ such that

$$h(t) \in [h_1,\ h_2], \quad \forall t \geq 0. \tag{2}$$

No assumption on $\dot{h}$ is included to represent fast-varying delays. When possible, the time argument of the delay function h will be omitted.

Objectives

Looking at the literature, providing efficient stability conditions for time-varying or fast-varying delay systems relies on the accuracy of matrix and integral inequalities. There are mainly two directions for deriving less conservative stability conditions.

First, a lot of attention has been paid recently to integral inequalities such as Jensen's [3], Wirtinger-based [9] or Bessel's inequalities [10], auxiliary functions [8] or on free-weighting matrix [16]. They indeed represent tools to efficiently reduce the conservatism of the conditions, at least in the constant delay case. In this paper, we will only concentrate on the Wirtinger-based inequality stated in the next lemma taken from [9], noting that the main result of this paper can be adapted to the other integral inequalities.

Lemma 1 (Wirtinger-Based inequality) *Let $R \in \mathbb{S}_+^n$ and x be a continuously differentiable function from $[-h_2,\ -h_1]$ to $\mathbb{R}^n$. The following inequality holds*

$$h_{21} \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) \mathrm{d}s \geq \omega_0^T R \omega_0 + 3\omega_1^T R \omega_1,$$

where $\omega_0 = x(-h_1) - x(-h_2)$, and $\omega_1 = x(-h_1) + x(-h_2) - 2/h_{21} \int_{-h_2}^{-h_1} x(s)\mathrm{d}s$.

Second, when considering time-varying delays, the problem often relies on finding lower bound Θ_m of the parameter dependent matrix given by [7–9]

$$\forall \alpha \in (0, 1), \qquad \begin{bmatrix} \frac{1}{\alpha} R & 0 \\ 0 & \frac{1}{1-\alpha} R \end{bmatrix} \succeq \Theta_m.$$

There are two main methods to find lower bounds Θ_m. The first one is based on the Moon's inequality (see, for instance, the survey paper [15]). The second method is the so-called reciprocally convex combination lemma developed in [7]. The conservatism induced by these two inequalities are independent. While, in some cases, such as stability conditions resulting from the application of the Jensen inequality, the two methods lead to equivalent results on examples, the reciprocally convex combination lemma is in general more conservative than Moon's inequality (see for instance [16]). In this paper, we present an extended version of the reciprocally convex combination lemma, which reduces notably the conservatism of the resulting stability conditions.

3 Extended Reciprocally Convex Inequality

This section is devoted to the derivation of a new matrix inequality which refines the reciprocally convex combination lemma from [7]. It is presented in the next lemma.

Lemma 2 *Let n be a positive integer, and R in $\mathbb{S}^n$. If there exist X_1, X_2 in $\mathbb{S}^n$ and Y_1, Y_2 in $\mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} - \alpha \begin{bmatrix} X_1 & Y_1 \\ Y_1^T & 0 \end{bmatrix} - (1-\alpha) \begin{bmatrix} 0 & Y_2 \\ Y_2^T & X_2 \end{bmatrix} \succeq 0 \tag{3}$$

for $\alpha = 0, 1$, the next inequality holds, for all $\alpha \in (0, 1)$

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \succeq \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} + (1-\alpha)\begin{bmatrix} X_1 & Y_2 \\ Y_2^T & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & Y_1 \\ Y_1^T & X_2 \end{bmatrix} \tag{4}$$

Proof Following [7], the proof consists in noting that

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} \frac{1-\alpha}{\alpha}R & 0 \\ 0 & \frac{\alpha}{1-\alpha}R \end{bmatrix}. \tag{5}$$

The objective is to find a lower bound of the second term of the right-hand-side of (5). Using a convexity argument, if (3) holds for $\alpha = 0, 1$, it also holds for any α in $[0, 1]$. Then, by pre- and post-multiplying inequality (3) by $\begin{bmatrix} \sqrt{\frac{1-\alpha}{\alpha}}I & 0 \\ 0 & \sqrt{\frac{\alpha}{1-\alpha}}I \end{bmatrix}$, for all α in $(0,\ 1)$, we obtain

$$\begin{bmatrix} \frac{1-\alpha}{\alpha}R & 0 \\ * & \frac{\alpha}{1-\alpha}R \end{bmatrix} \succeq \alpha \begin{bmatrix} \frac{1-\alpha}{\alpha}X_1 & Y_1 \\ Y_1^T & 0 \end{bmatrix} + (1-\alpha)\begin{bmatrix} 0 & Y_2 \\ Y_2^T & \frac{\alpha}{1-\alpha}X_2 \end{bmatrix} = (1-\alpha)\begin{bmatrix} X_1 & Y_2 \\ Y_2^T & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & Y_1 \\ Y_1^T & X_2 \end{bmatrix}. \tag{6}$$

for all α in $(0,\ 1)$. Re-injecting (6) into (5) concludes the proof. ■

It is worth noting that (3) is affine with respect to α, therefore it suffices to verify the inequality at the boundary of the interval $[0,\ 1]$. The second inequality of the previous lemma provides a lower bound which is also affine, and consequently convex in α. Note moreover that, selecting $X_1 = X_2 = 0$ and $Y_1 = Y_2 = Y \in \mathbb{R}^{n\times n}$, inequalities (3) and (4) recover the reciprocally convex combination lemma [7]. Therefore, Lemma 2 brings additional degrees of freedom and is potentially less conservative.

The conditions of Lemma 2 can be straightforwardly extended to find a lower bound of the matrix $\begin{bmatrix} \frac{1}{\alpha}R_1 & 0 \\ 0 & \frac{1}{1-\alpha}R_2 \end{bmatrix}$, where $\alpha \in (0, 1)$, R_1 and R_2 are in $\mathbb{S}_+^n$ and $\mathbb{S}_+^m$, respectively, where n and m in $\mathbb{N}$ are not necessarily equal.

4 Stability Conditions and Potentialities

The following stability theorem is provided.

Theorem 1 *Assume that there exist matrices P in $\mathbb{S}_+^{3n}$, S_1, S_2, R_1, R_2 in $\mathbb{S}_+^n$, X_1, X_2 in $\mathbb{S}_+^{2n}$ and two matrices Y_1, Y_2 in $\mathbb{R}^{2n\times 2n}$, such that the conditions*

$$\begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} - \begin{bmatrix} X_1 & Y_1 \\ Y_1^T & 0 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} - \begin{bmatrix} 0 & Y_2 \\ Y_2^T & X_2 \end{bmatrix} \succeq 0, \tag{7}$$

$$\Phi(h_i) = \Phi_0(h_i) - \Gamma^T \Psi(h_i)\Gamma \prec 0, \tag{8}$$

are satisfied, for $i = 1, 2$*, where*

$$\begin{aligned} \Phi_0(\theta) &= G_1^T(\theta)PG_0 + G_0^T PG_1(\theta) + \hat{S} + g_0^T(h_1^2 R_1 + h_{12}^2 R_2)g_0, \\ \hat{S} &= diag(S_1, -S_1 + S_2, 0_n, -S_2, 0_{3n}), \\ \tilde{R}_i &= diag(R_i, 3R_i), \quad \forall i = 1, 2, \end{aligned} \tag{9}$$

$$\begin{aligned} \Psi(h_1) &= diag\left(\tilde{R}_1, \begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} + \begin{bmatrix} X_1 & Y_2 \\ Y_2^T & 0 \end{bmatrix}\right), \\ \Psi(h_2) &= diag\left(\tilde{R}_1, \begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} + \begin{bmatrix} 0 & Y_1 \\ Y_1^T & X_2 \end{bmatrix}\right), \end{aligned} \tag{10}$$

and where the matrices g_0, Γ *and* G_i*, for* $i = 0, 1, \ldots 4$ *are given by*

$$\begin{aligned} g_0 &= \begin{bmatrix} A \; 0 \; A_d \; 0 \; 0 \; 0 \; 0 \end{bmatrix}, \\ G_0 &= \begin{bmatrix} A & 0 & A_d & 0 & 0 & 0 & 0 \\ I & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}, \\ G_1(\theta) &= \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\theta - h_1)I & (h_2 - \theta)I \end{bmatrix}, \\ G_2 &= \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 \\ I & I & 0 & 0 & -2I & 0 & 0 \end{bmatrix}, \\ G_3 &= \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 \\ 0 & I & I & 0 & 0 & -2I & 0 \end{bmatrix}, \\ G_4 &= \begin{bmatrix} 0 & 0 & I & -I & 0 & 0 & 0 \\ 0 & 0 & I & I & 0 & 0 & -2I \end{bmatrix}, \quad \Gamma = \begin{bmatrix} G_2 \\ G_3 \\ G_4 \end{bmatrix}. \end{aligned} \tag{11}$$

Then system (1) is asymptotically stable for all time-varying delay h satisfying (2).

Proof Consider the Lyapunov-Krasovskii functional introduced in [12], given by

$$\begin{aligned} V(x_t, \dot{x}_t) &= \begin{bmatrix} x(t) \\ \int_{t-h_1}^{t} x(s)\mathrm{d}s \\ \int_{t-h_2}^{t-h_1} x(s)\mathrm{d}s \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \int_{t-h_1}^{t} x(s)\mathrm{d}s \\ \int_{t-h_2}^{t-h_1} x(s)\mathrm{d}s \end{bmatrix} \\ &+ \int_{t-h_1}^{t} x^T(s)S_1 x(s)\mathrm{d}s + \int_{t-h_2}^{t-h_1} x^T(s)S_2 x(s)\mathrm{d}s, \\ &+ h_1 \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)\mathrm{d}s + h_{12} \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)\mathrm{d}s, \end{aligned} \tag{12}$$

where we recall that $h_{12} = h_2 - h_1$. Note that the positive definiteness of P, S_1, S_2, R_1 and R_2 implies the positive definiteness of the functional V. Following the same procedure as in [12], the differentiation of the functional V along the trajectories of system (1) leads to

$$\begin{aligned}\dot{V}(x_t,\dot{x}_t) = \zeta^T(t)\Phi_0(h)\zeta(t) - h_1\int_{t-h_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)\mathrm{d}s \\ - h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)\mathrm{d}s,\end{aligned} \tag{13}$$

with $\Phi_0(h)$ given in (9) and $\zeta(t) = [\zeta_1^T(t), \zeta_2^T(t)]^T$ with

$$\zeta_1(t) = \begin{bmatrix} x(t) \\ x(t-h_1) \\ x(t-h) \\ x(t-h_2) \end{bmatrix}, \ \zeta_2(t) = \begin{bmatrix} \frac{1}{h_1}\int_{t-h_1}^{t} x^T(s)\mathrm{d}s \\ \frac{1}{h-h_1}\int_{t-h}^{t-h_1} x^T(s)\mathrm{d}s \\ \frac{1}{h_2-h}\int_{t-h_2}^{t-h} x^T(s)\mathrm{d}s \end{bmatrix}.$$

Applying Lemma 1 to the two integral terms, after splitting the second integral into two parts, leads to

$$\dot{V}(x_t,\dot{x}_t) \le \zeta^T(t)\left(\Phi_0(h) - \Gamma^T\Psi(h)\Gamma\right)\zeta(t), \tag{14}$$

where Γ is given in (11) and

$$\Psi(h) = diag\left(\tilde{R}_1, \begin{bmatrix} \frac{h_{12}}{h-h_1}\tilde{R}_2 & 0 \\ * & \frac{h_{12}}{h_2-h}\tilde{R}_2 \end{bmatrix}\right).$$

Define $\alpha = (h-h_1)/h_{12}$. Then Lemma 2 ensures that, if there exist matrices X_1, X_2 in $\mathbb{S}^{2n}$ and Y_1, Y_2 in $\mathbb{R}^{2n\times 2n}$ such that conditions (7) hold, then we have

$$\Psi(h) \succeq (1-\alpha)\Psi(h_1) + \alpha\Psi(h_2).$$

Noting that the matrix $\Phi_0(h)$ is affine in h, we have $\Phi_0(h) = (1-\alpha)\Phi_0(h_1) + \alpha\Phi_0(h_2)$ and it holds

$$\dot{V}(x_t,\dot{x}_t) \le \zeta^T(t)\left[(1-\alpha)\,\Phi(h_1) + \alpha\Phi(h_2)\right]\zeta(t).$$

Therefore if the two LMIs $\Phi(h_1) \prec 0$ and $\Phi(h_2) \prec 0$ are satisified, any linear combination of these two matrices is also definite negative and we can conclude that the system is asymptotically stable for all time-varying delay in the interval $[h_1\ h_2]$. ■

It is worth noting that the proof of Theorem 1 is very similar to the one provided in [12]. The only difference relies on the use of Lemma 2. The impact in terms of reduction of the conservatism will be exposed in the example section. It is also possible to limit the number of additional decision variables in Theorem 1 with respect to [12]. The following corollary is provided where only a symmetric matrix X and a matrix Y are introduced.

Corollary 1 *Assume that there exist matrices P in $\mathbb{S}_+^{3n}$, S_1, S_2, R_1, R_2 in $\mathbb{S}_+^n$, X in $\mathbb{S}_+^{2n}$ and Y in $\mathbb{R}^{2n\times 2n}$, such that the conditions (7) and (8) hold with $X_1 = X_2 = X$ and*

$Y_1 = Y_2 = Y$. *Then system (1) is asymptotically stable for any time-varying delay h satisfying (2).*

The only remaining difference with respect to [12] is the introduction of a symmetric matrix X. We will show in the example section, that the sole introduction of this matrix leads to a notable reduction of the conservatism. We have presented here an application of Lemma 2, which copes with the case of systems subject to a fast-varying delay. It is worth mentioning that this lemma is also relevant for systems subject to a slow-varying delay, which usually includes the constraints $\dot{h} < 1$. We have only presented how Lemma 2 can be combined with the Wirtinger-based integral inequality. Other recent integral inequalities such that the auxiliary function-based inequality [8] or the Bessel-Legendre inequality [10] can be associated to Lemma 2 in order to obtain new efficient stability conditions.

5 Illustrative Examples

Three numerical examples from the literature illustrate the efficiency of the proposed conditions in Theorems 1 and its corollary. Before entering into the numerical results, a discussion on the numerical complexity of the various stability conditions for system system (1)–(2) is deserved. Table 1 points out the number of decision variables involved in Theorem 1 and Corollary 1 compared with the ones from existing results from the literature. For the next three examples, we expose in Tables 2, 3 and 4, the maximal upper-bound, h_2 of the delay functions for various values of h_1 obtained by solving by Theorem 1, its corollary and several recent stability conditions from literature.

There exists a large number of papers dealing with the stability analysis of such a class of systems. Because of space limitations, we consider only few representative conditions from the literature. On a first side, we present conditions derived using Jensen's inequality [2, 7], Wirtinger-based inequality [12], auxiliary-based inequality [8] and the recent free-matrix-based inequality [16]. On the other hand, we also discriminate conditions issued from Moon's inequalities [2, 16], or on the reciprocally convex combination lemma [7, 8, 12]. At last, the theorem proposed in [8]

Table 1 Number of decision variables involved in several conditions from the literature and in Theorem 1 and Corollary 1

Th.	No. of variables	Th.	No. of variables
[7]	$3.5n^2 + 2.5n$	[2]	$11.5n^2 + 3.5n$
[8]	$21n^2 + 6n$	[12]	$10.5n^2 + 3.5n$
[16]	$54.5n^2 + 9.5n$		
Th. 1	$18.5n^2 + 5.5n$	Cor. 1	$12.5n^2 + 4.5n$

Table 2 Example 1: Admissible upper bounds of h_2 for different h_1

h_1	0.0	0.4	0.7	1.0	2.0	3.0
[7]	1.86	1.88	1.95	2.06	2.61	3.31
[2]	1.86	1.89	1.98	2.12	2.72	3.45
[12]	2.11	2.17	2.23	2.31	2.79	3.49
[8]	2.14	2.19	2.24	2.31	2.80	3.50
[16]	2.18	2.21	2.25	2.32	2.79	3.49
Th. 1	2.21	2.25	2.28	2.34	2.80	3.49
Cor. 1	2.19	2.24	2.28	2.34	2.80	3.49

Table 3 Example 2: Admissible upper bounds of h_2 for different h_1

h_1	0.0	0.3	0.5	0.8	1.0	2.0
[4]	0.77	0.94	1.09	1.34	1.51	2.40
[13]	0.87	1.07	1.21	1.45	1.61	2.47
[7]	1.06	1.24	1.38	1.60	1.75	2.58
[12]	1.19	1.35	1.47	1.67	1.82	2.63
[8]	1.19	1.35	1.47	1.67	1.82	2.63
[16]	1.20	1.35	1.47	1.67	1.82	2.63
Th. 1	1.20	1.35	1.47	1.67	1.82	2.63
Cor. 1	1.20	1.35	1.47	1.67	1.82	2.63

Table 4 Example 3: Admissible upper bound of h_2 for different h_1

h_1	0.0	0.3	0.7	1.0	2.0
[12]	1.59	2.01	2.41	2.62	3.59
[8]	1.64	2.13	2.70	2.96	3.63
[16]	1.80	2.19	2.58	2.79	3.68
Th. 1	1.76	2.18	2.59	2.79	3.75
Cor. 1	1.69	2.11	2.50	2.71	3.67

are based on an integral inequality, which is proven to be less conservative than the Wirtinger-based inequality.

Example 1 Consider the following much-studied linear time-delay system (1)–(2) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \; A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

The results obtained by solving Theorem 1 and its corollary show a clear reduction of the conservatism. Moreover, the improvements due to the use of Lemma 2 and its corollary can be seen when comparing the results obtained with [12] and the stability conditions provided in the present paper. Indeed the only difference between these two papers is the use of the delay-dependent reciprocally convex lemma. Moreover, it is worth noting that Theorem 1 and its corollaries provide less conservative results, on this example, than other conditions from the literature except for [8] with $h_1 = 3$. This improvement of [8] can be explained by the use of the auxiliary function integral inequality, which is less conservative than the Wirtinger-based inequality. As noted at the end of Sect. 4, it is possible to reduce the conservatism of the stability conditions by considering less conservative integral inequalities, but this will not be presented in this note, because it is out of the scope of this paper.

It is also worth noting that Theorem 1 and its corollary leads in general to the same results except for small lower bounds $h_1 = 0$ even if the computational complexities of the stability conditions are different. This is due to the fact that when the quantity h_{21} is large, more decision variables need to be included to enlarge the maximal allowable delay h_2.

Example 2 We consider now the linear time-delay system (1)–(2), taken from [7], with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

For this example, we present in Table 3 the maximal allowable upper-bound of the delay obtained, for different values of h_1, by application of various conditions from the literature and the ones presented in this paper. We first note that Theorem 1, its corollary and the conditions from [8, 12, 16] deliver the same results, (except when $h_1 = 0$ for [8, 12]), while their associated number of decision variables are different (see Table 1). This demonstrates again the potential of the improved reciprocally convex lemma.

Example 3 Consider system (1) borrowed from [8] with the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

Table 4 presents admissible upper bound of h_2 for different h_1. We can see that the method of Theorem 1 is competitive with the stability conditions.

Table 4 shows that the conditions from [8, 16] and from Theorem 1 provide less conservative results than [12]. Then depending on the value of the lower bound of the delay, h_1, the best numerical results are obtained by the conditions from [8, 16] or from Theorem 1. For small values of h_1, deliver [16] provides less conservative results. A possible interpretation is that the free matrix-based inequality introduces a large number of decision variables. This example proves that this additional complexity is in some case relevant. When for greater values of h_1, the best results are obtained by [8]. Again, this is due to the fact that this results relies on a less conservative integral inequality. Finally for larger values of h_1, the best numerical results are provided by Theorem 1, which demonstrates, again, the potential of the delay-dependent reciprocally convex combination lemma.

6 Conclusions

In this chapter, an improved version of the reciprocally convex lemma is provided. The novelty of this technical lemma brings a notable reduction of the conservatism of LMI stability conditions for fast-varying delay systems, on some examples, with a reasonable additional computational burden, which is still lower than the most recent and efficient conditions from the literature.

In this chapter we have introduced a new stability conditions based on a refined reciprocally convex lemma and on the application of the Wirtinger-based integral inequality. This integral inequality has been the subject of many improvements through, for example, the auxiliary function based inequality, the free-matrix-based integral inequality or the Bessel-Legendre inequality. Several recent papers have already consider similar results related to Lemma 2 and more involved integral inequality as in [11, 17] or more involved Lyapunov-Krasovskii functionals as for instance in [18].

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Wave Equation Modelling and Freeness Properties for Wind Power Systems

Hugues Mounier and Luca Greco

1 Introduction

We here model the electric power system with transmission lines, generators and loads to be a continuum. A distributed placement of the string of generators leads to a wave equation model, whose disturbances are called inter area oscillations [3, 10, 13, 16, 18]. The involved wave equations are amenable to delay systems, the study of which has a rich literature. Quite a few authors have used algebraic techniques [1, 2, 9]. We here envision the problem using module theoretic techniques, which have been used for delay systems ([4–6] as well as for distributed parameter systems [7, 12, 15, 17]). The main advantages of this approach are threefold: notions are intrinsic to the system, many controllability notions can be recast in this setting using extension of scalars, and a complete system parametrization is obtained through the freeness property (the linear analogue to differential flatness of lumped nonlinear systems).

The module properties of torsion freeness, projectivity and freeness, as well as the change of base ring through tensor product (i.e. extension of scalars) give rise to a huge number of possible controllability notions. One can then combine the choice of the base ring (the simplest, i.e. the nearest to a principal ideal domain, the better) and the module property (the strongest, i.e. the nearest to freeness, the better) to obtain a basis which can generate all the distributed system (such a system can be viewed as a collection of input/output systems parametrized by the spatial variable). The present contribution is based on a first version which appeared in [11].

H. Mounier (✉) · L. Greco
Laboratoire des Signaux et Systèmes, CentraleSupélec, 3, rue Joliot Curie, 91192 Gif sur Yvette, France
e-mail: hugues.mounier@l2s.centralesupelec.fr

L. Greco
e-mail: luca.greco@l2s.centralesupelec.fr

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The goal of this chapter is first to propose a seemingly new type of model for the inter area oscillations in wind power systems, and second to give some controllability (e.g. flatness) results for very simple configurations, as a first step.

The paper is organized as follows: in Sect. 2, the module theoretic setting is recalled. In Sect. 3, two models are reviewed, one stemming from the literature, and another new one. The control appearing at the boundary is supposed to come from a wind farm. In Sect. 4, the controllability properties of the two systems are given, wherefrom a placement of the wind farm can be deduced.

2 Module Theoretic Setting

2.1 R-Linear Systems

We shall consider in this section quite general definitions for linear systems viewed as modules over a ring R (*see,* e.g., [5]).

Definition 1 An R-*system* Λ, or a *system over* R, is an R-module. A *presentation matrix* of a finitely presented R-system Σ is a matrix P such that $\Sigma \cong [v]/[Pv]$ where $[v]$ is free with basis v. An *output* $\boldsymbol{y}$ is a subset, which may be empty, of Λ.

Remarks 1 (i) Finite dimensional systems will be modules over $R = \mathbb{R}[\frac{d}{dt}]$; systems with pointwise delays will be modules over $R = \mathbb{R}[\frac{d}{dt}, \boldsymbol{\delta}]$, where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_r)$ are the pointwise delay operators, i.e. $\delta_i f(t) = f(t - h_i)$ for some $h_i \in \mathbb{R}$, the amplitude of the delay.

(ii) Recall that a module is a linear structure admitting the very same axioms as the ones of vector spaces, the sole difference being that the scalars are taken in a ring instead of being taken in a field. In informal terms, a finite dimensional system with equations

$$a\left(\tfrac{d}{dt}\right) y = b\left(\tfrac{d}{dt}\right) u$$

where $a, b \in R = \mathbb{R}[\frac{d}{dt}]$ will be represented by a set formed of all the linear combinations of y, u and their derivatives, such that the above relation is satisfied

$$\Lambda = \left\{ p\left(\tfrac{d}{dt}\right) y + q\left(\tfrac{d}{dt}\right) u \,\middle|\, p, q \in R,\ a\left(\tfrac{d}{dt}\right) x = b\left(\tfrac{d}{dt}\right) u \right\}$$

(iii) A finitely generated module over $\mathbb{R}[\frac{d}{dt}, \boldsymbol{\delta}]$ is entirely specified by the datum of generators and of a set of relations (the latter being itself a module) verified by the latter. The module is then said to be given by generators and relations. We then have generators $\Lambda = [w_1, \ldots, w_\alpha] = [\boldsymbol{w}]$ and relations,

$$P_\Lambda\left(\tfrac{d}{dt}, \boldsymbol{\delta}\right) \boldsymbol{w} = 0$$

where $P_\Lambda \in \mathbb{R}[\frac{d}{dt}, \boldsymbol{\delta}]^{\beta\times\alpha}$ and $\mathrm{rk}_{\mathbb{R}[\frac{d}{dt},\delta]} P_\Lambda = \beta$. The matrix P_Λ is named a presentation matrix of Λ. Let us remark that this matrix is not unique.

The next definition allows, by extension of scalars, to obtain much nicer algebraic properties when needed.

Definition 2 Let A be an R-algebra and Λ be an R-system. The A-module $A \otimes_R \Lambda$ is an A-system, which *extends* Λ.

Examples 1 (i) Let Λ_1 be an $\mathbb{R}[\frac{d}{dt}, \delta]$-system (i.e. a pointwise delay system) with y, u as generators and

$$\dot{y} = \delta u, \qquad \text{or} \quad \dot{y}(t) = u(t-h)$$

as relation. In such a system, no advance is allowed. In the extended system $\mathbb{R}[\frac{d}{dt}, \delta, \delta^{-1}] \otimes_{\mathbb{R}[\frac{d}{dt},\delta]} \Lambda_1$, advances are allowed, and one may write

$$u = \delta^{-1}\dot{y}, \qquad \text{or} \quad u(t) = \dot{y}(t+h)$$

Similarly, let Λ_2 be the $\mathbb{R}[\frac{d}{dt}, \delta]$-system with y, u as generators and

$$\dot{y} = (1+\delta)u, \qquad \text{or} \quad \dot{y}(t) = u(t) + u(t-h)$$

as relation. In the extended system $\mathbb{R}[\frac{d}{dt}, \delta, (1+\delta)^{-1}] \otimes_{\mathbb{R}[\frac{d}{dt},\delta]} \Lambda_2$, one may write

$$u = (1+\delta)^{-1}\dot{y}, \qquad \text{or} \quad u(t) = \frac{1}{1+\delta}\,\dot{y}(t+h)$$

2.2 System Controllabilities

In this section we emphasize several controllability notions which are defined directly without referring to a solution space (*see* e.g., [5]). We shall explicit three algebraic properties of modules, which are directly relevant to controllability.

Torsion. Let us first explicit an important notion for controllability studies: that of torsion Consider a finite dimensional system Σ, modelled by a module over the ring $\mathbb{R}[\frac{d}{dt}]$. An element w of Σ is said *torsion* if it exists a non void polynomial p of $\mathbb{R}[\frac{d}{dt}]$ such that

$$p(\tfrac{d}{dt})\, w = 0$$

Note that this phenomenon cannot happen in a non trivial manner in a vector space; indeed, a relation of the form $pw = 0$ in a vector space implies $w = 0$, since p is invertible as element of a field. A torsion element of Σ satisfies a differential equation with coefficients in $\mathbb{R}$. A module whose elements are all torsion is said torsion. On the contrary, a module with no torsion element is said *torsion free*. The absence of torsion means that no variable satisfies an autonomous differential (resp. difference-differential,..., according to the ring R) equation, i.e. non influenced by the input.

Freeness and Projectivity. The *freeness* amounts to the existence of a basis, i.e. a maximally independent and minimally generating set of the module. A basis always exists in a vector space, which is far from being the case for modules, since non constant scalars may not be invertible.

The projectivity covers the notion of sub space of a free module. On the one hand, a sub-module M of a free module N may be free but not necessarily a term in direct sum of N; on the other hand, there may be terms in direct sum of N which are not free. It is this latter concept which furnishes a natural generalisation of the notion of sub-space. A module M over a commutative ring R is said to be *projective* if $N = M \oplus \widetilde{M}$ where N is a free module; $\widetilde{M}$ is then also projective for the same reason. A characterization of projectivity is the following: being given a module M over a commutative ring R, M is projective if a presentation matrix P_M ($P_M \in R^{\beta\times\alpha}$, $\mathrm{rk}_R P_M = \beta$) is right invertible (if there exists $Q \in R^{\alpha\times\beta}$ such that $P_M Q = I_\beta$).

Note that this characterization is directly linked to matrix Bezout equations. If we consider a dynamics Λ with equations

$$\dot{\boldsymbol{x}} = F(\delta)\boldsymbol{x} + G(\delta)\boldsymbol{u}$$

with F and G matrices with coefficients in $\mathbb{R}[\delta]$ of appropriate size, the projectivity of Λ amounts to the existence of matrices $\overline{F}$ and $\overline{G}$ with coefficients in $\mathbb{R}[\delta]$ such that

$$\left[\frac{d}{dt}I_n - F(\delta)\right]\overline{F}(\tfrac{d}{dt},\delta) + G(\delta)\overline{G}(\tfrac{d}{dt},\delta) = I_n.$$

This type of equations has be used for stabilization purposes by Vidyasagar and others.

Definition 3 Let A be an R algebra. An R-system Λ is said to be *A-torsion free controllable* (resp. *A-projective controllable*, *A-free controllable*) if the A-module $A \otimes_R \Lambda$ is torsion free (resp. projective, free). An *R-torsion free* (resp. *R-projective*, *R-free*) *controllable* R-system is simply called *torsion free* (resp. *projective*, *free*) *controllable*.

Examples 2 (i) Consider the model Σ_1 with equations

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_1 + u\end{aligned}$$

The element x_1 is torsion, hence Σ_1 is not free.

(ii) Consider the model Σ_2 with equations

$$\dot{x}_1 = x_1 - u$$
$$\dot{x}_2 = x_1 + u$$

The model Σ_2 is free, with basis $\omega = x_1 + x_2$. Indeed, by summing up the two equations, we have

$$x_1 = \frac{1}{2}\,\dot{\omega}$$

then, the definition of ω yields

$$x_2 = \omega - \frac{1}{2}\,\dot{\omega}$$

and, through the first equation

$$u = x_1 - \dot{x}_1 = \frac{1}{2}\,(\dot{\omega} - \ddot{\omega})$$

Remark 1 The presence of torsion elements is a severe obstruction to controllability. In example i above, one has $\dot{x}_1 = x_1$ whatever the value of u may be. The torsion freeness character could have been taken as the definition for a controllability notion, but it is not constructive, whereas the freeness is.

Elementary homological algebra (*see,* e.g., [14]) yields

Proposition 1 *A-free (resp. A-projective) controllability implies A-projective (resp. A-torsion free) controllability.*

Proposition 2 *R-free controllability implies A-free controllability for any R-algebra A. More generally, given any R-system Σ that is a direct sum of a torsion module* $\mathrm{t}\Sigma$ *and a free module Λ, the extended system $A \otimes_R \Sigma$ is a direct sum of the torsion module $A \otimes_R \mathrm{t}\Sigma$ and the free module $A \otimes_R \Lambda$.*

Definition 4 Take an A-free controllable R-system Λ with a finite output $\boldsymbol{y}$. This output is said to be *A-flat*, or *A-basic*, if $\boldsymbol{y}$ is a basis of $A \otimes_R \Lambda$. If $A \cong R$ then $\boldsymbol{y}$ is simply called *flat*, or *basic*.

3 Modelling

3.1 Distributed Parameter System Models

The following derivation is taken from [3, 10]. Consider a transmission line with series of generators. The generation G_i and power angle change δ_i are supposed to be continuously distributed over the spatial dimension z. The Rotor dynamics of the ith generator is taken to be (see, e.g. [8])

$$\left(\frac{2H_i}{\Omega_s}\right)\ddot{\delta}_i + \chi\dot{\delta}_i = P_i \tag{1}$$

with H_i the inertia constant, Ω_s the electrical frequency with 60Hz base, P_i the real power flowing out the ith machine, and χ a damping coefficient. Then, the real power flow from node i to node $i+1$ over a lossless line is

$$P_{i,i+1} = \frac{E_i E_{i+1}\sin(\delta_i - \delta_{i+1})}{x_i}$$

with E_i the voltage magnitude at bus i and x_i is the linearized effective impedance between machine i and $i+1$. We then make the following two standard assumptions: the change angle δ_i is small and $E_i = 1$. With these, we get

$$P_i = P_{i+1,i} - P_{i,i+1} = \frac{(\delta_{i-1} - \delta_i)(\delta_i - \delta_{i+1})}{x_i}$$

By substitution and division by $\Delta L = x_{i+1} - x_i$, one obtains

$$\frac{2}{\Omega_i}\,\frac{H_i}{\Delta L}\,\ddot{\delta}_i + \frac{\chi}{\Delta L}\,\dot{\delta}_i = \frac{\Delta L}{x_i}\,\frac{\delta_i - \delta_{i-1}}{(\Delta L)^2} - \frac{\Delta L}{x_i}\,\frac{\delta_i - \delta_{i+1}}{(\Delta L)^2}$$

Then, taking the limit $\Delta L \to 0$, and setting

$$H_T = \frac{1}{L}\int_0^L dH(z),\quad H(L) = LH_T,\quad \gamma = \frac{x(L)}{L},\quad \eta = \frac{\chi(L)}{L}$$

yields, with $\nu = \sqrt{377/2H_T\gamma}$ (see, e.g. [10])

$$\partial_t^2\delta(z,t) + \eta\,\partial_t\delta(z,t) = \nu^2\,\partial_z^2\delta(z,t) \tag{2}$$

The quantity dH/dz is the inertia density of the continuous model, whereas H_i is the density of generator i in a discrete model. The corresponding power flow is

$$P(z,t) = -\frac{1}{\gamma}\,\partial_z\delta(z,t) \tag{3}$$

This type of model has been used to take into account inter area oscillation phenomena (see, e.g. [3, 8, 10, 13, 16, 18]). Adding power injection to the previous model leads to

$$\partial_t^2\delta(z,t) + \eta\,\partial_t\delta(z,t) - \nu^2\,\partial_z^2\delta(z,t) = W(z,t) \tag{4}$$

with boundary conditions

$$P(0,t) = P(1,t) = 0, \quad \text{or} \quad \partial_z \delta(0,t) = \partial_z \delta(1,t) = 0$$

A first model, used in [10], is a point source injection

$$W(u,t) = \rho P_g(t)\Delta(z-\alpha)$$

where Δ denotes the delta Dirac distribution, α the injection location, and P_g the net power injected. Another possible model, which we introduce here, is a power flow injection

$$W(u,t) = -\gamma P_g(t)\Delta'(z-\alpha)$$

with Δ' is the Dirac's derivative, in the sense of distributions. The previous model (4) with point source injection

$$\partial_t^2 \delta_p(z,t) + \eta \partial_t \delta_p(z,t) - \nu^2 \partial_z^2 \delta_p(z,t) = \rho P_g(t)\Delta_p(z-\alpha)$$

is equivalent to the following model

$$\forall z \in [0,\alpha], \quad \partial_t^2 \delta_p^-(z,t) + \eta\, \partial_t \delta_p^-(z,t) - \nu^2\, \partial_z^2 \delta_p^-(z,t) = 0 \tag{5a}$$

$$\partial_z \delta_p^-(0,t) = 0 \tag{5b}$$

$$\delta_p^-(\alpha,t) = \rho P_g(t) \tag{5c}$$

$$\forall z \in [\alpha,L], \quad \partial_t^2 \delta_p^+(z,t) + \eta\, \partial_t \delta_p^+(z,t) - \nu^2\, \partial_z^2 \delta_p^+(z,t) = 0 \tag{5d}$$

$$\delta_p^+(\alpha,t) = \rho P_g(t) \tag{5e}$$

$$\partial_z \delta_p^+(L,t) = 0 \tag{5f}$$

Hence, $\delta_p^-(z,t) = \delta_p(z,t)$ for $z \in [0,\alpha]$ and $\delta_p^+(z,t) = \delta_p(z,t)$ for $z \in [\alpha, L]$. The other model we introduce, corresponding to the model (4) with power flow injection:

$$\partial_t^2 \delta_f(z,t) + \eta \partial_t \delta_f(z,t) - \nu^2 \partial_z^2 \delta_f(z,t) = -\gamma P_g(t)\Delta_f'(z-\alpha)$$

is equivalent to the following model

$$\forall z \in [0,\alpha], \quad \partial_t^2 \delta_f^-(z,t) + \eta\, \partial_t \delta_f^-(z,t) - \nu^2\, \partial_z^2 \delta_f^-(z,t) = 0 \tag{6a}$$

$$\partial_z \delta_f^-(0,t) = 0 \tag{6b}$$

$$\partial_z \delta_f^-(\alpha,t) = -\gamma P_g(t) \tag{6c}$$

$$\forall z \in [\alpha,L], \quad \partial_t^2 \delta_f^+(z,t) + \eta\, \partial_t \delta_f^+(z,t) - \nu^2\, \partial_z^2 \delta_f^+(z,t) = 0 \tag{6d}$$

$$\partial_z \delta_f^+(\alpha,t) = -\gamma P_g(t) \tag{6e}$$

$$\partial_z \delta_f^+(L,t) = 0 \tag{6f}$$

Hence, $\delta_f^-(z,t) = \delta_f(z,t)$ for $z \in [0,\alpha]$ and $\delta_f^+(z,t) = \delta_f(z,t)$ for $z \in [\alpha, L]$.

Remark 2 Note that a point source actuation (as in (5c)) implies that the actuator is able to directly modify the controlled variable (here the angular variable). A flow actuation (as in (6c)) includes a form of actuation model, which is consistent with power as space derivative of angle change depicted in (3). The first model implies highly powerful actuation, to be able to directly steer the angle change. Following a electromechanical analogy, consider the torsional displacements of a metallic rod of length L with wave speed c and actuation u. Two models are

$$\begin{aligned} c^2\partial_z^2\theta(z,t) - \partial_t^2\theta(z,t) = 0, &\quad z \in [0,L] \\ \partial_z\theta(0,t) = 0, &\quad \theta(L,t) = u(t) \end{aligned}$$

for point actuation and

$$\begin{aligned} c^2\partial_z^2\theta(z,t) - \partial_t^2\theta(z,t) = 0, &\quad z \in [0,L] \\ \partial_z\theta(0,t) = 0, &\quad \partial_z\theta(L,t) = u(t) \end{aligned}$$

for flow actuation. Here, in the second model, the actuation $\partial_z\theta(L,t)$ is a torque actuation model. In the first one, the actuator is supposed to be sufficiently powerfull to act directly on the displacement.

3.2 Point Source Model Solution

General Solution

Let us consider the first half point source model for $z \in [0,\alpha]$ (Eqs. (5a)–(5c)):

$$\partial_t^2\delta_p^-(z,t) + \eta\,\partial_t\delta_p^-(z,t) - \nu^2\,\partial_z^2\delta_p^-(z,t) = 0 \tag{7a}$$

$$\partial_z\delta_p^-(0,t) = 0 \tag{7b}$$

$$\delta_p^-(\alpha,t) = \rho P_g(t) \tag{7c}$$

The temporal Laplace transform of (7) yields

$$\begin{aligned} s^2\hat{\delta}_p^-(z,s) + \eta s\hat{\delta}_p^-(z,s) - \nu^2\partial_z^2\hat{\delta}_p^-(z,s) = 0 \\ \partial_z\hat{\delta}_p^-(0,s) = 0 \\ \hat{\delta}_p^-(\alpha,s) = \rho\hat{P}_g(s) \end{aligned}$$

For fixed s one obtains the ODE in space as the boundary value problem:

$$s^2\hat{\delta}_p^-(z) + \eta s\hat{\delta}_p^-(z) - \nu^2\frac{d^2\hat{\delta}_p^-}{dz^2}(z) = 0 \tag{8}$$

$$\frac{d\hat{\delta}_p^-}{dz}(0) = 0, \qquad \hat{\delta}_p^-(\alpha) = \rho\hat{P}_g(s) \tag{9}$$

where we have kept the symbol δ_p^- (since δ_p^- is a function of two variables z and s) by abuse of notation.

The general solution of the previous ODE is investigated through the characteristic equation in X:

$$s^2 + \eta s - \nu^2 X^2 = 0$$

yielding

$$X = \pm\varsigma\sqrt{s^2 + \eta s} = \pm\sigma(s), \quad \text{with} \quad \varsigma = 1/\nu$$

Thus, the general solution of (8) is

$$\hat{\delta}_p^-(z) = e^{\varsigma z\sqrt{s^2+\eta s}}\,\hat{\lambda}_1 + e^{-\varsigma z\sqrt{s^2+\eta s}}\,\hat{\lambda}_2 = e^{\sigma z}\hat{\lambda}_1 + e^{-\sigma z}\hat{\lambda}_2$$

Boundary Value Problem Solution

This general solution, and its spatial derivative can be rewritten as

$$\hat{\delta}_p^-(z,s) = \hat{C}_z(s)\hat{\mu}_{p1}^-(s) + \hat{S}_z(s)\hat{\mu}_{p2}^-(s), \quad \partial_z\hat{\delta}_p^-(z,s) = \sigma^2\hat{S}_z\hat{\mu}_{p1}^- + \hat{C}_z\hat{\mu}_{p2}^- \text{ where}$$
$$\hat{C}_z(s) = \cosh(\sigma z), \quad \hat{S}_z(s) = \frac{\sinh(\sigma z)}{\sigma}$$

Similarly, the general solution of the second half point source model for $z \in [\alpha, L]$ (Eqs. (5d)–(5f)) and its spatial derivatives are

$$\hat{\delta}_p^+(z,s) = \hat{C}_z\hat{\mu}_{p1}^+ + \hat{S}_z\hat{\mu}_{p2}^+ \quad \partial_z\hat{\delta}_p^+(z,s) = \sigma^2\hat{S}_z\hat{\mu}_{p1}^+ + \hat{C}_z\hat{\mu}_{p2}^+$$

The boundary conditions of the point source model (5)

$$\partial_z\delta_p^-(0,t) = 0, \quad \delta_p^-(\alpha,t) = \rho P_g(t), \quad \delta_p^+(\alpha,t) = \rho P_g(t), \quad \partial_z\delta_p^+(L,t) = 0$$

are then expressed as, in the Laplace domain

$$\hat{\mu}_{p2}^- = 0 \qquad \hat{C}_\alpha\hat{\mu}_{p1}^- + \hat{S}_\alpha\mu_{p2}^- = \rho\hat{P}_g(s) \tag{10a}$$
$$\hat{C}_\alpha\hat{\mu}_{p1}^+ + \hat{S}_\alpha\hat{\mu}_{p2}^+ = \rho\hat{P}_g(s) \qquad \sigma^2\hat{S}_L\hat{\mu}_{p1}^+ + \hat{C}_L\hat{\mu}_{p2}^+ = 0 \tag{10b}$$

Recalling the general solutions and/or spatial derivatives:

$$\hat{\delta}_p^-(z,s) = \hat{C}_z \hat{\mu}_{p1}^-, \quad \hat{\delta}_p^+(z,s) = \hat{C}_z \hat{\mu}_{p1}^+ + \hat{S}_z \hat{\mu}_{p2}^+, \quad \partial_z \hat{\delta}_p^+(z,s) = \sigma^2 \hat{S}_z \hat{\mu}_{p1}^+ + \hat{C}_z \hat{\mu}_{p2}^+$$

we get the following expressions for $\hat{\mu}_{p1}^-$, $\hat{\mu}_{p1}^+$ and $\hat{\mu}_{p2}^+$:

$$\hat{\mu}_{p1}^- = \hat{\delta}_p^-(0,s) = \hat{\delta}_{p0}^- \qquad \hat{\mu}_{p1}^+ = \hat{\delta}_p^+(0,s) = \hat{\delta}_{p0}^+ \tag{11a}$$

$$\hat{\mu}_{p2}^+ = \partial_z \hat{\delta}_p^+(0,s) = \hat{\delta}_{p0}^{+'} \tag{11b}$$

Thus, the preceding Eqs. (10) become

$$\hat{C}_\alpha \hat{\delta}_{p0}^+ + \hat{S}_\alpha \hat{\delta}_{p0}^{+'} = \hat{C}_\alpha \hat{\delta}_{p0}^- \tag{12a}$$

$$\sigma^2 \hat{S}_L \hat{\delta}_{p0}^+ + \hat{C}_L \hat{\delta}_{p0}^{+'} = 0 \tag{12b}$$

which form the relations of the $\mathcal{R}_\mathbb{Q}$-system $\Lambda_\mathbb{Q}^p = [\hat{\delta}_{p0}^-, \hat{\delta}_{p0}^+, \hat{\delta}_{p0}^{+'}]_{\mathcal{R}_\mathbb{Q}}$, where the ring $\mathcal{R}_\mathbb{X}$ is given by

$$\mathcal{R}_\mathbb{X} = \mathbb{C}(\partial_t)[\mathfrak{S}_\mathbb{X}] \cap \mathcal{E}^{'*} \qquad \text{with} \quad \mathfrak{S}_\mathbb{X} = \{\hat{C}_{z\alpha}, \hat{S}_{z\alpha} \mid z \in \mathbb{X}\}$$

with $\mathcal{E}^{'*}$ a ring of compactly supported Gevrey ultradistributions. The presentation of $\Lambda_\mathbb{Q}^p$ is then

$$\begin{pmatrix} -\hat{C}_\alpha & \hat{C}_\alpha & \hat{S}_\alpha \\ 0 & \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{p0}^- \\ \hat{\delta}_{p0}^+ \\ \hat{\delta}_{p0}^{+'} \end{pmatrix} = 0 \tag{13}$$

To be more specific from a physical viewpoint, the model can be written as

$$\hat{C}_\alpha \hat{\delta}_{p0}^+ + \hat{S}_\alpha \hat{\delta}_{p0}^{+'} = \hat{C}_\alpha \hat{\delta}_{p0}^- \tag{14a}$$

$$\sigma^2 \hat{S}_L \hat{\delta}_{p0}^+ + \hat{C}_L \hat{\delta}_{p0}^{+'} = 0 \tag{14b}$$

$$\rho \hat{P}_g = \hat{C}_\alpha \hat{\delta}_{p0}^- \tag{14c}$$

$$\hat{\delta}_{pz}^- = \hat{C}_z \hat{\delta}_{p0}^- \tag{14d}$$

$$\hat{\delta}_{pz}^+ = \hat{C}_z \hat{\delta}_{p0}^+ + \hat{S}_z \hat{\delta}_{p0}^{+'} \tag{14e}$$

with $\delta_{pz}^- = \delta_p^-(z,s)$, $\delta_{pz}^+ = \delta_p^+(z,s)$.

3.3 Power Flow Model Solution

General and Boundary Value Problem Solution

The general solution of power flow model (6) and its spatial derivatives are

$$\hat{\delta}_f^-(z,s) = \hat{C}_z\hat{\mu}_{f1}^- + \hat{S}_z\hat{\mu}_{f2}^- \quad \partial_z\hat{\delta}_f^-(z,s) = \sigma^2\hat{S}_z\hat{\mu}_{f1}^- + \hat{C}_z\mu_{f2}^-$$
$$\hat{\delta}_f^+(z,s) = \hat{C}_z\hat{\mu}_{f1}^+ + \hat{S}_z\hat{\mu}_{f2}^+ \quad \partial_z\hat{\delta}_f^+(z,s) = \sigma^2\hat{S}_z\hat{\mu}_{f1}^+ + \hat{C}_z\hat{\mu}_{f2}^+$$

The boundary conditions of the power flow model (6)

$$\partial_z\delta_f^-(0,t) = 0,\ \partial_z\delta_f^-(\alpha,t) = -\gamma P_g(t),\ \partial_z\delta_f^+(\alpha,t) = -\gamma P_g(t),\ \partial_z\delta_f^+(L,t) = 0$$

are then expressed as

$$\hat{\mu}_{f2}^- = 0 \tag{15a}$$
$$\sigma^2\hat{S}_\alpha\hat{\mu}_{f1}^- + \hat{C}_\alpha\mu_{f2}^- = -\gamma\hat{P}_g(s) \tag{15b}$$
$$\sigma^2\hat{S}_\alpha\hat{\mu}_{f1}^+ + \hat{C}_\alpha\mu_{f2}^+ = -\gamma\hat{P}_g(s) \tag{15c}$$
$$\sigma^2\hat{S}_L\hat{\mu}_{f1}^+ + \hat{C}_L\hat{\mu}_{f2}^+ = 0 \tag{15d}$$

The following expressions are obtained for $\hat{\mu}_{f1}^-$, $\hat{\mu}_{f1}^+$ and $\hat{\mu}_{f2}^+$:

$$\hat{\mu}_{f1}^- = \hat{\delta}_f^-(0,s) = \hat{\delta}_{f0}^- \qquad \hat{\mu}_{f1}^+ = \hat{\delta}_f^+(0,s) = \hat{\delta}_{f0}^+ \tag{16}$$
$$\hat{\mu}_{f2}^+ = \partial_z\hat{\delta}_f^+(0,s) = \hat{\delta}_{f0}^{+'} \tag{17}$$

We then get the following equations

$$\sigma^2\hat{S}_\alpha\hat{\delta}_{f\,0}^{\ +} + \hat{C}_\alpha\delta_{f0}^{+'} = \sigma^2\hat{S}_\alpha\hat{\delta}_{f0}^- \tag{18a}$$
$$\sigma^2\hat{S}_L\hat{\delta}_{f0}^+ + \hat{C}_L\delta_{f0}^{+'} = 0 \tag{18b}$$

which form the relations of the $\mathcal{R}_{\mathbb{Q}}$-system $\Lambda_{\mathbb{Q}}^f = [\hat{\delta}_{f0}^-, \hat{\delta}_{f0}^+, \hat{\delta}_{f0}^{+'}]_{\mathcal{R}_{\mathbb{Q}}}$. The presentation of $\Lambda_{\mathbb{Q}}^f$ is then

$$\begin{pmatrix} -\sigma^2\hat{S}_\alpha & \sigma^2\hat{S}_\alpha & \hat{C}_\alpha \\ 0 & \sigma^2\hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{f0}^- \\ \hat{\delta}_{f0}^+ \\ \hat{\delta}_{f0}^{+'} \end{pmatrix} = 0 \tag{19}$$

To be more specific from a physical viewpoint, the model can be written as

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{\;+} - \hat{C}_\alpha \delta_{f0}^{+'} = \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{-} \tag{20a}$$

$$\sigma^2 \hat{S}_L \hat{\delta}_{f0}^{+} - \hat{C}_L \delta_{f0}^{+'} = 0 \tag{20b}$$

$$-\gamma \hat{P}_g = \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{-} \tag{20c}$$

$$\hat{\delta}_{fz}^{-} = \hat{C}_z \hat{\delta}_{f0}^{-} \tag{20d}$$

$$\hat{\delta}_{fz}^{+} = \hat{C}_z \hat{\delta}_{f0}^{+} + \hat{S}_z \hat{\delta}_{f0}^{+'} \tag{20e}$$

with $\delta_{fz}^{-} = \delta_f^{-}(z, s)$, $\delta_{fz}^{+} = \delta_f^{+}(z, s)$.

3.4 Associated I/O Systems

Point Source I/O System

To obtain the input/output system, we should extract from (14) the relation between δ_{pz}^{-} and $\hat{P}_g$ on the one hand and between δ_{pz}^{+} and $\hat{P}_g$ on the other hand. From (14c) and (14c), we get

$$\hat{C}_\alpha \hat{\delta}_{pz}^{-} = \rho\, \hat{C}_z \hat{P}_g \tag{21}$$

Then, from (14a) and (14b):

$$\begin{pmatrix} \hat{C}_\alpha & \hat{S}_\alpha \\ \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{p0}^{+} \\ \hat{\delta}_{p0}^{+'} \end{pmatrix} = \begin{pmatrix} \hat{C}_\alpha \hat{\delta}_{p0}^{-} \\ 0 \end{pmatrix}$$

Hence, we get

$$\hat{C}_{L-\alpha} \begin{pmatrix} \hat{\delta}_{p0}^{+} \\ \hat{\delta}_{p0}^{+'} \end{pmatrix} = \begin{pmatrix} \hat{C}_L & -\hat{S}_\alpha \\ -\sigma^2 \hat{S}_L & \hat{C}_\alpha \end{pmatrix} \begin{pmatrix} \hat{C}_\alpha \hat{\delta}_{p0}^{-} \\ 0 \end{pmatrix}$$

wherefrom the expression involving $\hat{\delta}_{pz}^{+}$:

$$\begin{aligned} \hat{C}_{L-\alpha} \hat{\delta}_{pz}^{+} &= \hat{C}_z \hat{C}_{L-\alpha} \hat{\delta}_{p0}^{+} + \hat{S}_z \hat{C}_{L-\alpha} \hat{\delta}_{p0}^{+'} = \big(\hat{C}_z \hat{C}_L - \sigma^2 \hat{S}_z \hat{S}_L\big) \hat{C}_\alpha \hat{\delta}_{p0}^{-} \\ &= \rho\, \hat{C}_{L-z} \hat{P}_g \end{aligned} \tag{22}$$

And, gathering (21) and (22), we get the point source injection input/output model:

$$\hat{C}_\alpha \hat{\delta}_{pz}^{-} = \rho\, \hat{C}_z \hat{P}_g \tag{23}$$

$$\hat{C}_{L-\alpha} \hat{\delta}_{pz}^{+} = \rho\, \hat{C}_{L-z} \hat{P}_g \tag{24}$$

on which we can see that this model is a purely discrete one, i.e. from a system dynamics viewpoint, it has no dynamics.

Power Flow I/O System

Similarly to obtain the input/output system, we should extract from (20) the relation between δ_{fz}^- and $\hat{P}_g$ on the one hand and between δ_{fz}^+ and $\hat{P}_g$ on the other hand.

From (20c) and (20c), we get

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{fz}^- = -\gamma\, \hat{C}_z \hat{P}_g \tag{25}$$

Then, from (20a) and (20b):

$$\begin{pmatrix} \sigma^2 \hat{S}_\alpha & -\hat{C}_\alpha \\ \sigma^2 \hat{S}_L & -\hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{f0}^+ \\ \hat{\delta}_{f0}^{+'} \end{pmatrix} = \begin{pmatrix} \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^- \\ 0 \end{pmatrix}$$

Hence, we get

$$-\hat{S}_{L-\alpha} \begin{pmatrix} \hat{\delta}_{f0}^+ \\ \hat{\delta}_{f0}^{+'} \end{pmatrix} = \begin{pmatrix} -\hat{C}_L & \hat{C}_\alpha \\ -\sigma^2 \hat{S}_L & \sigma^2 \hat{S}_\alpha \end{pmatrix} \begin{pmatrix} \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^- \\ 0 \end{pmatrix}$$

wherefrom the expression involving $\hat{\delta}_{fz}^+$:

$$\begin{aligned} \hat{S}_{L-\alpha} \hat{\delta}_{fz}^+ &= \hat{C}_z \hat{S}_{L-\alpha} \hat{\delta}_{f0}^+ + \hat{S}_z \hat{S}_{L-\alpha} \hat{\delta}_{f0}^{+'} = \big(\hat{C}_z \hat{C}_L - \sigma^2 \hat{S}_z \hat{S}_L\big) \sigma^2 S_\alpha \hat{\delta}_{f0}^- \\ &= -\gamma\, \hat{C}_{L-z} \hat{P}_g \end{aligned} \tag{26}$$

And, gathering (25) and (26), we get the power flow injection input/output model:

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{fz}^- = -\gamma\, \hat{C}_z \hat{P}_g \tag{27}$$

$$\hat{S}_{L-\alpha} \hat{\delta}_{fz}^+ = -\gamma\, \hat{C}_{L-z} \hat{P}_g \tag{28}$$

on which we can see that this model is a second order neutral distributed delay system. In the temporal domain, the preceding equations rewrite

$$\sigma^2 \big(\delta_{fz}^-(t+\alpha) - \delta_{fz}^-(t-\alpha)\big) = -\gamma\, \big(\dot{P}_g(t+z) + \dot{P}_g(t-z)\big) \tag{29}$$

$$\delta_{fz}^+\big(t+L-\alpha\big) - \delta_{fz}^+\big(t-L+\alpha\big) = -\gamma\, \big(\dot{P}_g(t+L-z) + \dot{P}_g(t-L+z)\big) \tag{30}$$

4 Structural Properties

4.1 Point Source System

The presentation matrix associated to $\Lambda^p_{\mathbb{Q}}$ is

$$P_{\Lambda^p_{\mathbb{Q}}} = \begin{pmatrix} -\hat{C}_\alpha & \hat{C}_\alpha & \hat{S}_\alpha \\ 0 & \sigma^2\hat{S}_L & \hat{C}_L \end{pmatrix} \tag{31}$$

and the associated minors are:

$$m_1(s) = -\sigma^2\hat{C}_\alpha\hat{S}_L,\ m_2(s) = -\hat{C}_\alpha\hat{C}_L,\ m_3(s) = \hat{C}_\alpha\hat{C}_L - \sigma^2\hat{S}_L\hat{S}_\alpha = \hat{C}_{L-\alpha} \tag{32a}$$

We then have the following proposition

Theorem 1 *The $\mathcal{R}_{\mathbb{Q}}$-system $\Lambda^p_{\mathbb{Q}}$ is $\mathcal{R}_{\mathbb{Q}}$-free controllable if, and only if, $\hat{C}_{L-\alpha}$ and $\hat{C}_\alpha$ have no common zeros in $\mathbb{C}$.*

Proof Common zeros between the minors m_1 and m_2 can only be the ones of their common factor $\hat{C}_\alpha$. Thus, if $\hat{C}_{L-\alpha}$ and $\hat{C}_\alpha$ have no common zeros, $\Lambda^p_{\mathbb{Q}}$ is $\mathcal{R}_{\mathbb{Q}}$-free controllable. ■

4.2 Power Flow System

The presentation matrix associated to $\Lambda^p_{\mathbb{Q}}$ and the associated minors are:

$$P_{\Lambda^p_{\mathbb{Q}}} = \begin{pmatrix} -\sigma^2\hat{S}_\alpha & \sigma^2\hat{S}_\alpha & \hat{C}_\alpha \\ 0 & \sigma^2\hat{S}_L & \hat{C}_L \end{pmatrix} \tag{33a}$$

$$m_1(s) = -\sigma^4\hat{S}_\alpha\hat{S}_L,\ m_2(s) = -\sigma^2\hat{S}_\alpha\hat{C}_L,\ m_3(s) = \sigma^2\hat{S}_\alpha\hat{C}_L - \sigma^2\hat{S}_L\hat{C}_\alpha = \sigma^2\hat{S}_{\alpha-L} \tag{33b}$$

We then have the following proposition and theorem

Proposition 3 *The $\mathcal{R}_{\mathbb{Q}}$-system $\Lambda^f_{\mathbb{Q}}$ is not $\mathcal{R}_{\mathbb{Q}}$-free controllable.*

Proof The minors m_1 to m_3 have $\sigma = 0$ (corresponding to $s = 0$) as a common zero. Hence, $\Lambda^f_{\mathbb{Q}}$ is not $\mathcal{R}_{\mathbb{Q}}$-spectrally controllable, and hence not $\mathcal{R}_{\mathbb{Q}}$-torsion free. ■

Let $\mathcal{R}^\sigma_{\mathbb{Q}}$ be the ring $\mathcal{R}^\sigma_{\mathbb{Q}} = \mathbb{C}(\partial_t)[\sigma^{-1}, \mathfrak{S}_{\mathbb{Q}}] \cap \mathcal{E}'^*$ Then we have:

Theorem 2 *The system $\mathcal{R}^\sigma_{\mathbb{Q}} \otimes_{\mathcal{R}_{\mathbb{Q}}} \Lambda^f_{\mathbb{Q}}$ is $\mathcal{R}^\sigma_{\mathbb{Q}}$-free controllable if, and only if, $\hat{S}_\alpha$ and $\hat{S}_{L-\alpha}$ have no common zeros in $\mathbb{C}$.*

Proof Common zeros between the minors m_1 and m_2 can only be the ones of their common factor $\hat{S}_\alpha$. Thus, if $\hat{S}_{L-\alpha}$ and $\hat{S}_\alpha$ have no common zeros, $\Lambda^f_{\mathbb{Q}}$ is $\mathcal{R}_{\mathbb{Q}}$-free controllable. ■

5 Discussion

We see that the controllability properties depend on the location of the source (the control): the fact that $\hat{S}_{L-\alpha}$ and $\hat{S}_\alpha$ may have common zeros. This leads to some transcendental properties between L and α. This yields a simple rule for actuator placement.

6 Conclusion

We have established a novel model for inter area oscillations in wind power networks. We have then derived the associated delay system models and input/output models. We then have examined the flatness properties of the most simple networks, i.e. with one source. The controllability properties we have established are a first step towards the investigation of the network structural properties, which shall be the object of further studies.

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A Delayed Mass-Action Model for the Transcriptional Control of Hmp, an NO Detoxifying Enzyme, by the Iron-Sulfur Protein FNR

Marc R. Roussel

1 Introduction

Protein expression in living organisms involves, at minimum, two steps: transcription of the corresponding gene to a messenger RNA (mRNA), and translation of that mRNA to protein. Transcription and translation are among the slower processes in cells [46]. For a typical bacterial gene of 900 nt (nucleotides) [55], transcription at a rate of 24 nt s^{-1} [8] means that the mRNA is synthesized in 38 s. A 900 nt transcript encodes a 300 aa (amino acid) protein. Under optimal conditions, *Escherichia coli* ribosomes translate an mRNA at a rate of roughly 17 aa s^{-1} [56], so translation of a 300 aa protein transcript would take 18 s.

It is possible to model a gene expression network by including detailed models of transcription and translation, i.e. explicitly modeling every nucleotide addition to an mRNA, and every amino acid addition to a protein [33, 34, 43]. In most cases however, this is an undesirable level of detail, avoided by replacing the repetitive steps of transcription and translation by delays, which were perhaps first used in gene expression models in the 1970s [2, 6, 29].

As part of their immune response, mammals generate nitric oxide (NO), which inhibits a number of bacterial enzymes [31] and causes DNA damage. In many bacteria, Hmp, an enzyme that converts nitric oxide into nitrate ions, is a key part of the defenses against NO [31, 39]. Hmp is synthesized under the control of an NO-responsive repressor protein known as FNR [32]. Understanding the control system that regulates the production of Hmp could therefore prove useful in devising strategies for weakening bacterial defenses against nitrosative stress, thus helping the immune system combat pathogenic bacteria.

M. R. Roussel (✉)
Alberta RNA Research and Training Institute, Department of Chemistry and Biochemistry, University of Lethbridge, Lethbridge, Alberta T1K 3M4, Canada
e-mail: roussel@uleth.ca

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Tolla, Savageau and coworkers developed a model for FNR's role as a regulator of the aerobic/anaerobic metabolic switch in *E. coli* [50, 51], but did not consider FNR's role in the nitrosative stress response. Robinson and Brynildsen have constructed a general model for NO metabolism in *E. coli* [39, 40] that includes detoxification by Hmp. However, they use an empirical equation for the dependence of the rate of synthesis of Hmp on [NO]. In Sect. 2, a detailed biochemical model for the control of the synthesis of Hmp is developed, based on the delayed mass-action framework [42]. Parameter estimates are obtained in Sect. 3. The steady-state solutions of this model are studied in Sect. 4. The paper concludes with some perspectives for future work.

2 A Model for the Controlled Synthesis of Hmp

We consider a cell that is exposed to an external source of nitric oxide, leading to a net inflow of this dissolved gas into the cytoplasm at a constant rate:

$$\xrightarrow{k_{\text{in}}} \text{NO}. \tag{1}$$

Hmp catalyzes the conversion of NO to nitrate [39]:

$$2\text{NO} + 2\text{O}_2 + \text{NAD(P)H} \longrightarrow 2\text{NO}_3^- + \text{NAD(P)}^+ + \text{H}^+$$

For catalysis to occur, the substrates must bind in order, NAD(P)H first, then O_2, then NO [16]. We can assume that the enzyme is constantly saturated with NADH or NADPH, given that the Michaelis constant of Hmp for NADH, the concentration of NADH required for half-maximal saturation of the enzyme, is 4.8 μM [16], and that the concentration of NADH in an *E. coli* cell is at least 83 μM [5]. The relevant reactions are therefore

$$\text{Hmp} + \text{O}_2 \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \text{Hmp} \cdot \text{O}_2, \tag{2a}$$

$$\text{Hmp} \cdot \text{O}_2 + \text{NO} \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} \text{Hmp} \cdot \text{O}_2 \cdot \text{NO} \xrightarrow{k_{-3}} \text{Hmp} + \text{NO}_3^-, \tag{2b}$$

$$\text{Hmp} + \text{NO} \underset{k_{-4}}{\overset{k_4}{\rightleftharpoons}} \text{Hmp} \cdot \text{NO}. \tag{2c}$$

Equation (2c) is a substrate inhibition reaction, forming the catalytically inactive species Hmp · NO. Each of these reactions is assumed to obey the law of mass-action, which states that the rate of an elementary reaction is proportional to the product of the reactant concentrations. The products are taken to appear instantaneously. The proportionality constant is known as a rate constant, and is written over (under) the arrow corresponding to the forward (reverse) reaction. For example, the rate of the forward reaction in (2a) is $k_1[\mathrm{Hmp}][\mathrm{O}_2]$.

Because of the recycling of RNA polymerase (RNAP) following the completion of transcription, the pool of free RNAP is roughly constant. Accordingly, we can subsume the dependence of the transcription initiation rate on [RNAP] into the corresponding rate constant. The transcription process can then be modeled as follows, using the delayed mass-action notation [42]:

$$\mathrm{Pro}^{hmp}(t) \xrightarrow{k_{tc}} \mathrm{Pro}^{hmp}(t+\tau_1) + \mathrm{RBS}^{hmp}(t+\tau_2), \tag{3}$$

where RBS^{hmp} represents the ribosome binding site of the mRNA. The interpretation of this equation is that, if there is a transcription initiation event at time t, the RNAP clears the promoter τ_1 time units later, and the ribosome binding site becomes available τ_2 time units later. The initiation event itself proceeds at a rate governed by the law of mass-action, in this case at a rate $k_{tc}[\mathrm{Pro}^{hmp}]$.

Translation of the mRNA can be similarly modeled, assuming a constant pool of ribosomes:

$$\mathrm{RBS}^{hmp}(t) \xrightarrow{k_{tl}} \mathrm{RBS}^{hmp}(t+\tau_3) + \mathrm{Hmp}(t+\tau_4), \tag{4}$$

τ_3 is the time required for the ribosome to clear the RBS, and τ_4 is the time required to translate the mRNA.

We must also consider the decay of the mRNA, which typically starts at the ribosome binding site, allowing translation events already initiated to run off [8, 25]. This explains the focus on the RBS in reactions (3) and (4).

$$\mathrm{RBS}^{hmp} \xrightarrow{k_5} \emptyset, \tag{5}$$

where Ø represents decay products. Hmp also decays. We assume for simplicity that all of the enzyme-substrate complexes decay with similar kinetics:

$$\mathrm{Hmp} \xrightarrow{k_6} \emptyset, \tag{6a}$$

$$\mathrm{Hmp}\cdot\mathrm{O}_2 \xrightarrow{k_6} \mathrm{O}_2 + \emptyset, \tag{6b}$$

$$\mathrm{Hmp}\cdot\mathrm{O}_2\cdot\mathrm{NO} \xrightarrow{k_6} \mathrm{O}_2 + \mathrm{NO} + \emptyset, \tag{6c}$$

$$\mathrm{Hmp}\cdot\mathrm{NO} \xrightarrow{k_6} \mathrm{NO} + \emptyset. \tag{6d}$$

Hmp transcription is repressed by FNR binding to the *hmp* promoter [9]. The active form of FNR (the holoprotein) is a dimer, with each monomer containing a [4Fe-4S] cluster [26]. Reactions of the iron-sulfur cluster with NO cause the dimer to dissociate into its constituent monomers [9] and the affinity of FNR for the *hmp* promoter to decrease [12], allowing transcription to proceed. We focus here on the FNR monomer, for a number of reasons: First of all, the reactions at the two iron-sulfur clusters appear to be independent and to proceed with identical kinetics [9]. More importantly, dimerization of FNR or dissociation of these dimers is, under metabolic conditions, rapid and nearly quantitative [27].

Each iron-sulfur cluster reacts with NO in a 1:8 stoichiometry. This sequence of reactions is not fully understood at this time, so what follows, while based on the observations and interpretations of Crack and coworkers [9], will necessarily be somewhat speculative. Of the nine possible nitrosylation states, corresponding to the addition of 0–8 NO molecules, only five, labeled A to E ($FNR_A \equiv FNR, FNR_B, \ldots, FNR_E$), are kinetically distinguishable, and each of these is formed by a process which is kinetically of the first order in [NO]. The other four states are presumably transient species rapidly converted to kinetically more stable forms by further reactions with NO. The following is a plausible reaction sequence, based on the kinetic and spectroscopic analysis of Crack and coworkers:

$$\mathrm{FNR} + \mathrm{NO} \longrightarrow \mathrm{FNR_B}, \qquad v_{AB} = k_{AB}[\mathrm{FNR}][\mathrm{NO}]; \tag{7a}$$
$$\mathrm{FNR_B} + \mathrm{NO} \longrightarrow \mathrm{FNR_C}, \qquad v_{BC} = k_{BC}[\mathrm{FNR_B}][\mathrm{NO}]; \tag{7b}$$
$$\mathrm{FNR_C} + 2\mathrm{NO} \longrightarrow \mathrm{FNR_D}, \qquad v_{CD} = k_{CD}[\mathrm{FNR_C}][\mathrm{NO}]; \tag{7c}$$
$$\mathrm{FNR_D} + 4\mathrm{NO} \longrightarrow \mathrm{FNR_E}, \qquad v_{DE} = k_{DE}[\mathrm{FNR_D}][\mathrm{NO}]. \tag{7d}$$

Spectroscopic evidence indicates that FNR_D is the net product of the addition of four NO molecules [9], which required the allocation of an "extra" NO to one of reactions (7a)–(7c), arbitrarily added to (7c). Further assume that FNR_E is monomeric, unlike FNR A to D.

Not only does nitrosylated, monomerized FNR have a lower affinity for its DNA binding site than the holoprotein, it also has a lower specificity, i.e. it will bind to DNA sequences different from its natural binding site [12]. These two factors taken together along with the abundance of potential binding sites in the genome of a bacterial cell mean that very little of the monomerized form may bind to the promoter in the end. Therefore, assuming that FNR_E has negligible occupation of the promoter, repression of *hmp* by FNR is modeled by the following reactions:

$$\mathrm{Pro}^{hmp} + \mathrm{FNR} \underset{k_{-A}}{\overset{k_A}{\rightleftharpoons}} \mathrm{Pro}^{hmp} \cdot \mathrm{FNR}, \tag{8a}$$

$$\mathrm{Pro}^{hmp} + \mathrm{FNR_B} \underset{k_{-B}}{\overset{k_B}{\rightleftharpoons}} \mathrm{Pro}^{hmp} \cdot \mathrm{FNR_B}, \tag{8b}$$

$$\mathrm{Pro}^{hmp} + \mathrm{FNR_C} \underset{k_{-C}}{\overset{k_C}{\rightleftharpoons}} \mathrm{Pro}^{hmp} \cdot \mathrm{FNR_C}, \tag{8c}$$

$$\mathrm{Pro}^{hmp} + \mathrm{FNR_D} \underset{k_{-D}}{\overset{k_D}{\rightleftharpoons}} \mathrm{Pro}^{hmp} \cdot \mathrm{FNR_D}. \tag{8d}$$

It is not known whether NO can react with FNR bound to the promoter. However, oxygen does react with the FNR-promoter complex [10]. It is therefore likely that NO does too:

$$\mathrm{Pro}^{hmp} \cdot \mathrm{FNR} + \mathrm{NO} \longrightarrow \mathrm{Pro}^{hmp} \cdot \mathrm{FNR_B}, \quad v_{AB}^{\mathrm{Pro}} = \alpha k_{AB}[\mathrm{Pro}^{hmp} \cdot \mathrm{FNR}][\mathrm{NO}]; \tag{9a}$$

$$\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_B} + \mathrm{NO} \longrightarrow \mathrm{Pro}^{hmp} \cdot \mathrm{FNR_C}, \quad v_{BC}^{\mathrm{Pro}} = \alpha k_{BC}[\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_B}][\mathrm{NO}]; \tag{9b}$$

$$\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_C} + 2\mathrm{NO} \longrightarrow \mathrm{Pro}^{hmp} \cdot \mathrm{FNR_D}, \quad v_{CD}^{\mathrm{Pro}} = \alpha k_{CD}[\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_C}][\mathrm{NO}]; \tag{9c}$$

$$\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_D} + 4\mathrm{NO} \longrightarrow \mathrm{Pro}^{hmp} + \mathrm{FNR_E}, \quad v_{DE}^{\mathrm{Pro}} = \alpha k_{DE}[\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_D}][\mathrm{NO}]. \tag{9d}$$

Note the dissociation of the monomerized $\mathrm{FNR_E}$ from the promoter in the last step. Lacking detailed information, the rate constants in reactions (9) are assumed to be uniformly scaled by a factor of α relative to those in reactions (7).

Dibden and Green showed that FNR recycling is more important than *de novo* protein synthesis [14] to the dynamics of this system, so a constant pool of FNR should be a good approximation. Let us suppose that the FNR recycling pathway operates on $\mathrm{FNR_E}$ in a reaction with effective first-order kinetics:

$$\mathrm{FNR_E} \xrightarrow{k_r} \mathrm{FNR}. \tag{10}$$

The Hmp-FNR model thus consists of reactions (1)–(10). The assumption of a constant pool of FNR leads to the following conservation law:

$$\begin{aligned}[\mathrm{FNR}]_{\mathrm{total}} &= [\mathrm{FNR}] + [\mathrm{FNR_B}] + [\mathrm{FNR_C}] + [\mathrm{FNR_D}] + [\mathrm{FNR_E}] \\ &+ [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_B}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_C}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_D}].\end{aligned} \tag{11}$$

Assuming that the oxygen concentration is constant, the delayed mass-action framework [42] gives the following set of delay-differential equations (DDEs), using the notation $[\mathrm{A}]_{-\tau} \equiv [\mathrm{A}](t-\tau)$.

$$\begin{aligned}\frac{d[\mathrm{NO}]}{dt} &= k_{\mathrm{in}} - k_2[\mathrm{Hmp} \cdot \mathrm{O_2}][\mathrm{NO}] + k_{-2}[\mathrm{Hmp} \cdot \mathrm{O_2} \cdot \mathrm{NO}] - k_4[\mathrm{Hmp}][\mathrm{NO}] \\ &+ k_{-4}[\mathrm{Hmp} \cdot \mathrm{NO}] + k_6\left([\mathrm{Hmp} \cdot \mathrm{O_2} \cdot \mathrm{NO}] + [\mathrm{Hmp} \cdot \mathrm{NO}]\right) \\ &- v_{AB} - v_{BC} - 2v_{CD} - 4v_{DE} - v_{AB}^{\mathrm{Pro}} - v_{BC}^{\mathrm{Pro}} - 2v_{CD}^{\mathrm{Pro}} - 4v_{DE}^{\mathrm{Pro}},\end{aligned} \tag{12a}$$

$$\begin{aligned}\frac{d[\mathrm{Hmp}]}{dt} &= -k_1[\mathrm{Hmp}][\mathrm{O_2}] + k_{-1}[\mathrm{Hmp} \cdot \mathrm{O_2}] + k_{-3}[\mathrm{Hmp} \cdot \mathrm{O_2} \cdot \mathrm{NO}] \\ &- k_4[\mathrm{Hmp}][\mathrm{NO}] + k_{-4}[\mathrm{Hmp} \cdot \mathrm{NO}] + k_{tl}[\mathrm{RBS}^{hmp}]_{-\tau_4} - k_6[\mathrm{Hmp}],\end{aligned} \tag{12b}$$

$$\begin{aligned}\frac{d[\mathrm{Hmp} \cdot \mathrm{O_2}]}{dt} &= k_1[\mathrm{Hmp}][\mathrm{O_2}] - (k_{-1} + k_6)[\mathrm{Hmp} \cdot \mathrm{O_2}] \\ &- k_2[\mathrm{Hmp} \cdot \mathrm{O_2}][\mathrm{NO}] + k_{-2}[\mathrm{Hmp} \cdot \mathrm{O_2} \cdot \mathrm{NO}],\end{aligned} \tag{12c}$$

$$\frac{d[\mathrm{Hmp}\cdot\mathrm{O_2}\cdot\mathrm{NO}]}{dt} = k_2[\mathrm{Hmp}\cdot\mathrm{O_2}][\mathrm{NO}] - (k_{-2}+k_{-3}+k_6)[\mathrm{Hmp}\cdot\mathrm{O_2}\cdot\mathrm{NO}], \tag{12d}$$

$$\frac{d[\mathrm{Hmp}\cdot\mathrm{NO}]}{dt} = k_4[\mathrm{Hmp}][\mathrm{NO}] - (k_{-4}+k_6)[\mathrm{Hmp}\cdot\mathrm{NO}], \tag{12e}$$

$$\begin{aligned}\frac{d[\mathrm{Pro}^{hmp}]}{dt} &= -k_{tc}[\mathrm{Pro}^{hmp}] + k_{tc}[\mathrm{Pro}^{hmp}]_{-\tau_1} - k_A[\mathrm{Pro}^{hmp}][\mathrm{FNR}] \\ &\quad + k_{-A}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR}] - k_B[\mathrm{Pro}^{hmp}][\mathrm{FNR_B}] + k_{-B}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_B}] \\ &\quad - k_C[\mathrm{Pro}^{hmp}][\mathrm{FNR_C}] + k_{-C}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_C}] - k_D[\mathrm{Pro}^{hmp}][\mathrm{FNR_D}] \\ &\quad + k_{-D}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_D}] + v_{DE}^{\mathrm{Pro}},\end{aligned} \tag{12f}$$

$$\begin{aligned}\frac{d[\mathrm{RBS}^{hmp}]}{dt} &= k_{tc}[\mathrm{Pro}^{hmp}]_{-\tau_2} - k_{tl}[\mathrm{RBS}^{hmp}] + k_{tl}[\mathrm{RBS}^{hmp}]_{-\tau_3} \\ &\quad - k_5[\mathrm{RBS}^{hmp}],\end{aligned} \tag{12g}$$

$$\frac{d[\mathrm{FNR}]}{dt} = -v_{AB} - k_A[\mathrm{Pro}^{hmp}][\mathrm{FNR}] + k_{-A}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR}] + k_r[\mathrm{FNR_E}], \tag{12h}$$

$$\frac{d[\mathrm{FNR_B}]}{dt} = v_{AB} - v_{BC} - k_B[\mathrm{Pro}^{hmp}][\mathrm{FNR_B}] + k_{-B}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_B}], \tag{12i}$$

$$\frac{d[\mathrm{FNR_C}]}{dt} = v_{BC} - v_{CD} - k_C[\mathrm{Pro}^{hmp}][\mathrm{FNR_C}] + k_{-C}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_C}], \tag{12j}$$

$$\frac{d[\mathrm{FNR_D}]}{dt} = v_{CD} - v_{DE} - k_D[\mathrm{Pro}^{hmp}][\mathrm{FNR_D}] + k_{-D}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_D}], \tag{12k}$$

$$\frac{d[\mathrm{FNR_E}]}{dt} = v_{DE} + v_{DE}^{\mathrm{Pro}} - k_r[\mathrm{FNR_E}], \tag{12l}$$

$$\frac{d[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR}]}{dt} = k_A[\mathrm{Pro}^{hmp}][\mathrm{FNR}] - k_{-A}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR}] - v_{AB}^{\mathrm{Pro}}, \tag{12m}$$

$$\frac{d[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_B}]}{dt} = k_B[\mathrm{Pro}^{hmp}][\mathrm{FNR_B}] - k_{-B}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_B}] + v_{AB}^{\mathrm{Pro}} - v_{BC}^{\mathrm{Pro}}, \tag{12n}$$

$$\frac{d[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_C}]}{dt} = k_C[\mathrm{Pro}^{hmp}][\mathrm{FNR_C}] - k_{-C}[\mathrm{Pro}^{hmp}\cdot\mathrm{FNR_C}] + v_{BC}^{\mathrm{Pro}} - v_{CD}^{\mathrm{Pro}}. \tag{12o}$$

In these equations, all terms are evaluated instantaneously unless otherwise noted. Production delays in reactions (3) and (4) give rise to the delayed terms in Eqs. (12b), (12f) and (12g).

Chemical rate equations, including delayed mass-action systems [42], are conservation equations describing the transformation of matter from one form to another. Delayed reactions can be understood using a pipe metaphor: the delayed terms represent matter that has entered a reaction channel and that will emerge from this channel at some later time. The material "in the pipe" must be accounted for in mass balances. The promoter in this model is neither created nor destroyed, although it is occluded by the polymerase for a period of time τ_1 following initiation of transcription (Eq. 3). This gives rise to an integral conservation relationship, which can be verified by differentiation with respect to time:

$$[\mathrm{Pro}^{hmp}]_{\mathrm{total}} = [\mathrm{Pro}^{hmp}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_B}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_C}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_D}] + k_{tc} \int_{t-\tau_1}^{t} [\mathrm{Pro}^{hmp}](t')\, dt'. \tag{13}$$

3 Parameter Estimates

This study targets physiological conditions for *E. coli*, which would imply among other things a temperature of 37 °C. The parameters for reactions (2) were generally taken from the kinetic study of Gardner and coworkers [16]. When rate constants were unavailable at 37 °C, the rule of thumb that rate constants roughly double for every 10° rise in temperature [30], corresponding to an activation energy of about 50 kJ mol^{-1}, was used to estimate rate constants at 37 °C.

In the lower gastrointestinal (GI) tract, the oxygen tension is about 11 torr [21]. The concentration of oxygen there can be calculated from the Henry's law coefficient of oxygen in water at 37 °C of 1.409×10^{-6} M torr^{-1} [37].

τ_1 can be estimated to be roughly 1.9 s based on the requirement for the polymerase to move 30–60 nt in order to clear the promoter [36], and the typical rate of transcription elongation of 24 nt s^{-1} [8]. Since there are no data on initiation frequencies from the *hmp* promoter, data from the well-characterized *lac* promoter are used instead, where under optimal conditions, transcription initiates on average every 3.3 s [24]. Given that 1.9 s is spent clearing the promoter, this means that the average wait time for initiation at an empty promoter is 1.4 s, corresponding to a rate constant of $k_{tc} = 0.71\ \mathrm{s}^{-1}$. The transcriptional delay τ_2 was estimated from the gene length for Hmp (1191 nt [52]) and the elongation rate.

For translation, given an RBS clearance time of approximately 2 s [38] and a mean time between translation initiations, again for the *lac* transcript, of 3.2 s [24], we can calculate a translation initiation rate constant of $k_{tl} = 0.83\ \mathrm{s}^{-1}$. At a translation rate of 17 aa s^{-1} [56], the 396 aa Hmp protein would be translated in a time $\tau_4 = 23$ s. This estimate assumes that transcription and translation are spatially and temporally separated, in accord with recent evidence in *E. coli* [3].

Throughout this paper, we assume that an *E. coli* cell has a volume of 1.7 fL [19].

Cell growth has the effect of diluting the cellular contents. Under anaerobic conditions, *E. coli* grows at a median rate of 0.25 h^{-1} [20]. Adding this to the chemical decay constant calculated from the 66 min half-life of Hmp in *Salmonella* Typhimurium [53], we obtain an effective value of k_6 of $2.4 \times 10^{-4}\ \mathrm{s}^{-1}$.

The in vivo concentration of FNR can be calculated from the number of molecules of FNR present in an *E. coli* cell grown at 37 °C under anaerobic conditions, which is approximately 2600 [48].

For reactions (8), if we assume a typical bimolecular rate constant for all four steps of $10^7\,\mathrm{M^{-1}s^{-1}}$ (similar to the rate constant for binding of the structurally related protein CAP to the *lac* operator [17]), then we can calculate the dissociation rate constants given the dissociation equilibrium constants (K_d). For the holoprotein, $K_d = 1 \times 10^{-6}\,\mathrm{M}$ [12]. It is also known that adding NO loosens the complex. Assuming a moderate loosening (i.e. an increase in K_d) of $0.5 \times 10^{-6}\,\mathrm{M}$ per step keeps the K_d within the measured range for nitrosylated FNR [12].

All of the parameter estimates obtained are shown in Table 1. The only model parameters not estimated in this section are k_r and α, for which literature values are not available, and k_{in}, which is treated as a control parameter in this study.

Table 1 Model parameters. Values carrying the notation [T] in the reference field were adjusted for temperature as described in the text. The notation [a] indicates an assumed value, while [c] indicates that the value was calculated as described in the text. The [s] notation indicates that data from *E. coli* were unavailable and that data from another bacterial species were therefore used. For k_{-A}, the first value given was calculated in Sect. 3, and the second is proposed in Sect. 4

Parameter	Value	Ref.	Parameter	Value	Ref.
Hmp catalysis			*FNR dynamics*		
$[O_2]$	16 µM	[c]	$[\mathrm{FNR}]_{\mathrm{total}}$	2.5 µM	[c]
k_1	$8 \times 10^6\,\mathrm{M^{-1}s^{-1}}$	[16]	k_{AB}	$6.1 \times 10^5\,\mathrm{M^{-1}s^{-1}}$	[9][T]
k_{-1}	$1.4\,\mathrm{s^{-1}}$	[16][T]	k_{BC}	$4.1 \times 10^4\,\mathrm{M^{-1}s^{-1}}$	[9][T]
k_2	$2.5 \times 10^9\,\mathrm{M^{-1}s^{-1}}$	[16]	k_{CD}	$1.0 \times 10^4\,\mathrm{M^{-1}s^{-1}}$	[9][T]
k_{-2}	$6.2 \times 10^{-4}\,\mathrm{s^{-1}}$	[16][T]	k_{DE}	$1.6 \times 10^3\,\mathrm{M^{-1}s^{-1}}$	[9][T]
k_{-3}	$6.2 \times 10^2\,\mathrm{s^{-1}}$	[16][T]	α	1	[a]
k_4	$8.0 \times 10^7\,\mathrm{M^{-1}s^{-1}}$	[16][T]	k_A	$10^7\,\mathrm{M^{-1}s^{-1}}$	[a]
k_{-4}	$6.2 \times 10^{-4}\,\mathrm{s^{-1}}$	[16][T]	k_{-A}	$10\,\mathrm{s^{-1}} \to 0.11\,\mathrm{s^{-1}}$	[c]
Hmp expression			k_B	$10^7\,\mathrm{M^{-1}s^{-1}}$	[a]
k_{tc}	$0.71\,\mathrm{s^{-1}}$	[c]	k_{-B}	$15\,\mathrm{s^{-1}}$	[c]
τ_1	1.9 s	[c]	k_C	$10^7\,\mathrm{M^{-1}s^{-1}}$	[a]
τ_2	50 s	[c]	k_{-C}	$20\,\mathrm{s^{-1}}$	[c]
k_{tl}	$0.83\,\mathrm{s^{-1}}$	[c]	k_D	$10^7\,\mathrm{M^{-1}s^{-1}}$	[a]
τ_3	2 s	[38]	k_{-D}	$25\,\mathrm{s^{-1}}$	[c]
τ_4	23 s	[c]	k_r	$0.1\,\mathrm{s^{-1}}$	[a]
k_5	$0.014\,\mathrm{s^{-1}}$	[22][s]			
k_6	$2.4 \times 10^{-4}\,\mathrm{s^{-1}}$	[c]			

4 Steady-State Analysis

In the steady state, Eq. (13) becomes

$$[\mathrm{Pro}^{hmp}]_{\mathrm{total}} = [\mathrm{Pro}^{hmp}](1 + k_{tc}\tau_1) + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_B}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_C}] + [\mathrm{Pro}^{hmp} \cdot \mathrm{FNR_D}]. \quad (14)$$

The steady states satisfy Eqs. (11), (12a)–(12e), (12g)–(12o) and (14). Since, in a steady state, $x(t - \tau) = x(t)$, the values of the delays typically drop out of the steady-state analysis of models with delays. This is not necessarily the case in gene expression models, as we see here. The delay τ_1, while small, has an effect on the steady state via Eq. (14) because it accounts for the period during which access to the promoter is blocked by the polymerase. Note that $k_{tc}\tau_1 = 1.3 \sim 1$, so τ_1 is not negligible.

The steady-state equations were solved in Maple using the `fsolve()` function. Branches of steady states were followed by taking small steps along a branch and using the solution at the last step as an initial guess.

The NO-free ($k_{\mathrm{in}} = 0$) steady state using the parameters of Table 1, with $k_{-A} = 10\,\mathrm{s}^{-1}$ as calculated in the previous section, and $[\mathrm{Pro}^{hmp}]_{\mathrm{total}} = 0.98$ nM, corresponding to one promoter per cell, has $[\mathrm{Pro}^{hmp}] = 0.20$ nM and $[\mathrm{Pro}^{hmp} \cdot \mathrm{FNR}] = 0.51$ nM. The remaining 0.27 nM of the promoter is "in the pipe", i.e. occluded by a polymerase. A single promoter cannot be divided in this manner. Rather, we should think about these "concentrations" as being related to probabilities over a large ensemble of cells. In particular, only 52% of the *hmp* promoters in an ensemble of cells are repressed at any given time. The rest are either waiting for a polymerase, or have a polymerase engaged in transcription initiation. As a result, the total Hmp concentration in this calculation is very high ($[\mathrm{Hmp}] + [\mathrm{Hmp} \cdot \mathrm{O_2}] = 35\,\mu\mathrm{M}$). In the absence of NO however, we expect a low level of Hmp. A lower level of Hmp expression could be obtained by lowering k_{tc}, a parameter estimated based on data from a different gene. However, the very low level of repression of the *hmp* promoter, which depends on an experimentally measured K_d [12], seems more anomalous than does the value of k_{tc}. Thus assume, arbitrarily, that the affinity of holo-FNR for the *hmp* promoter is sufficiently high for 99% repression. Given the large size assumed for k_A, k_{-A} is adjusted to a value of $0.11\,\mathrm{s}^{-1}$, a hundredfold lower than the estimate of Sect. 3. The rate constants associated with the binding of the nitrosylated forms of FNR to DNA have not been adjusted, on the assumption that there is a significant drop-off in affinity as FNR is progressively nitrosylated.

Figure 1 shows a bifurcation diagram obtained by numerically following branches of solutions in Maple as described above. As k_{in} increases from zero, a bifurcation to bistability occurs near $k_{\mathrm{in}} = 0.084\,783\,\mu\mathrm{M\,s}^{-1}$, with two branches of stable steady states (verified by numerical integration of the DDEs (12) supplemented by the conservation relation (11) using the "stiff" integrator in XPPAUT [15]) separated by a branch of unstable steady states. In the bistable region, the lower branch corresponds to the desired behavior as the sub-nanomolar concentrations obtained here correspond

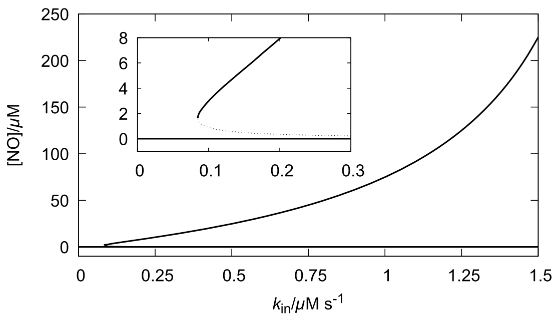

Fig. 1 Steady states of the model vs k_{in} obtained for the parameters of Table 1 with the lower value of k_{-A}. The initial conditions were chosen to correspond to one promoter per cell. The inset is a blowup of the small-k_{in} region. Solid curves: stable steady states; dotted curve: unstable steady state

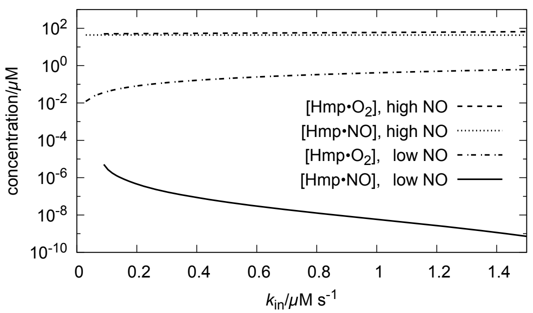

Fig. 2 Concentrations of key complexes of Hmp for the high- and low-NO steady states shown in Fig. 1

to less than one molecule of NO per cell. In this steady state, Hmp $\cdot$ O_2 is the dominant form of the Hmp enzyme (Fig. 2). Accordingly, the cell's biochemical state is primed for NO detoxification, and NO does not accumulate. In the high-NO steady state, the unproductive complex Hmp $\cdot$ NO has become dominant (Fig. 2). Because k_4 is approximately ten times larger than k_1, once an enzyme has completed a cycle of catalysis, at high NO levels, it becomes more likely to add a molecule of NO than a molecule of O_2. The small value of k_{-4} then results in very effective inhibition of Hmp's catalytic activity. The high-NO state is therefore self-reinforcing, leading to the observed bistability. Note that substrate inhibition has previously been shown to be a potential cause of bistability in biochemical systems [1, 13, 45].

It may be asked whether the ad hoc adjustment of k_{-A} had any effect on the results. A bifurcation diagram computed with the larger value of k_{-A} (not shown) is identical to the one shown in Fig. 1 to within the numerical resolution of the calculations. There are differences in some variables, particularly at lower values of k_{in}, but the NO concentration being the key variable from the point of view of the cell's health, we can conclude that the network's steady-state response is robust with respect to the value of k_{-A}. The model with a lower value of k_{-A} is much less wasteful, accumulating roughly half as much Hmp (all forms combined) at lower values of k_{in} (below about $0.1\ \mu M\ s^{-1}$; data not shown). Further calculations were carried out, again somewhat arbitrarily, with the lower value of k_{-A}.

The unknown parameters k_r and α were varied (results not shown). Neither of these parameters changes the qualitative dynamics. While there are clear quantitative

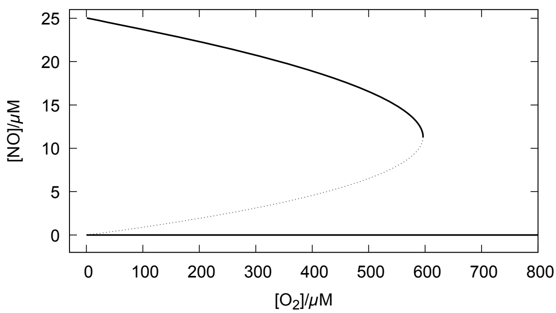

Fig. 3 Steady states as a function of $[O_2]$. All parameters are as in Fig. 1 with $k_{in} = 0.5\,\mu M\,s^{-1}$

effects associated with varying k_r, notably a steep increase in the high-NO steady state at lower values of k_r, the solutions are remarkably insensitive to α.

E. coli experiences varied oxygen concentrations during its life cycle, including its transit through the GI tract [21]. We would expect that higher O_2 concentrations would shift the balance towards the $Hmp \cdot O_2$ complex relative to the $Hmp \cdot NO$ complex, and thus increase the rate of NO removal, possibly abolishing bistability. The latter hypothesis is correct, as can be seen in Fig. 3. However, the oxygen concentrations at which bistability gives way to a monostable low-[NO] solution are unrealistically high. For comparison, consider that water in equilibrium with air at normal atmospheric pressure at 37 °C would hold 225 μM of dissolved O_2. The saddle-node bifurcation at $[O_2] \approx 596\,\mu M$ is therefore of no physiological consequence.

Robinson and Brynildsen used the following empirical equation to model the net dependence of the translation rate of Hmp on [NO] [40]:

$$v_{tl}^{RB2016} = k_{basal} + (k_{max} - k_{basal})[NO]/\,(K_{NO} + [NO])\,. \tag{15}$$

We can compute the steady-state translation rate in the Hmp-FNR model, $v_{tl} = k_{tl}[RBS^{hmp}]$, with [NO] as a parameter, by dropping Eq. (12a) and solving the remaining steady-state conditions, including the conservation equations (11) and (14). The solid curve in Fig. 4 is the rate of translation calculated from the Hmp-FNR model. Treating [NO] as a parameter eliminates the feedback that leads to bistability, and only one steady state is found for any given value of [NO]. Interestingly, this steady state is a high-$[Hmp \cdot NO]$ steady state. Coupling of the NO concentration to the catalytic degradation of NO by Hmp is thus critical to the creation of the low-[NO], low-$[Hmp \cdot NO]$ steady state that allows cells to survive nitrosative stress. The dashed curve in Fig. 4 is the best fit of Eq. (15) to the steady-state translation rate curve over the concentration range shown in the figure. The shape of the translation curve generated by the Hmp-FNR model is clearly quite different from the shape assumed by Robinson and Brynildsen. Whether this has any material effect on the results of their calculations is unknown.

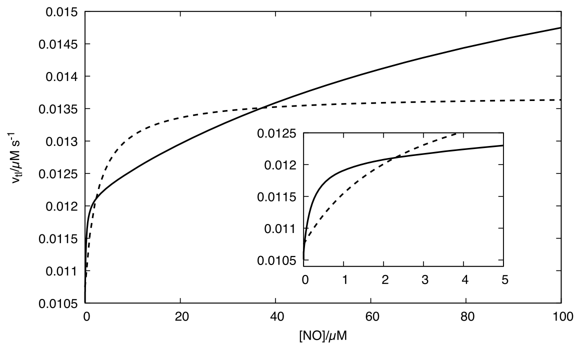

Fig. 4 Steady-state translation curve from the Hmp-FNR model (solid curve), treating [NO] as a parameter, for the parameters of Table 1 using the lower value of k_{-A}. The dashed curve is a fit of Eq. (15) to the translation rate from the Hmp-FNR model, yielding $k_{basal} = 0.010\,75\,\mu M\,s^{-1}$, $k_{max} = 0.013\,71\,\mu M\,s^{-1}$, and $K_{NO} = 2.7\,\mu M$. The inset magnifies the small-[NO] region

5 Discussion and Conclusions

The reactions in this model were, insofar as was possible, drawn from the literature. Still, the model contains some assumptions whose effects on the qualitative and quantitative behavior of the solutions are yet to be examined. Perhaps the most important modeling decision was the focus on nitrosylation states of monomers rather than explicitly considering dimers. A lumping analysis [54] to be presented elsewhere however suggests that this is a tenable approximation.

It is possible that the low-NO steady state is selected by additional regulatory or biochemical interactions. For instance, in *E. coli*, *hmp* transcription is regulated both by FNR and by NsrR [41]. NsrR is, like FNR, an iron-sulfur protein that represses *hmp* transcription while its iron-sulfur cluster is intact. The kinetics of nitrosylation of the iron-sulfur cluster in NsrR is analogous to that of FNR [11]. It would be interesting to model the joint control of *hmp* transcription by FNR and NsrR.

Figure 1 suggests that, in the bistable range, the basin of attraction of the desired low-NO steady state is relatively small with respect to the NO concentration, neglecting the dependence of this basin on the initial function [49]. In vivo, the NO inflow rate (k_{in}) will not be a fixed parameter but will have a time course dictated by the dynamics of the immune system and by the kinetics of permeation of NO through the cell membrane. Accordingly, it would be interesting to study the basin of attraction of the low-NO steady state with respect to a family of parameterized temporal programs for $k_{in}(t)$. Is there a critical rate of increase of k_{in} beyond which the system jumps to the high-NO steady state? How long and how intense a pulse of NO can the system handle and still return to the low-NO steady state?

In principle, all of the delays can be manipulated experimentally by changing the gene's sequence. In particular, the promoter clearance delay τ_1 hides a number of mechanistic details, including repeated rounds of abortive initiation, during which the polymerase "scrunches" the DNA as it adds the first few nucleotides to the new mRNA, then either releases an abortive transcript and returns to its starting position, or escapes from the promoter [23]. The mean number of cycles of abortive initiation, and thus the mean promoter clearance time, depends on the sequence of the promoter and of the initially transcribed region [47]. Whether there are any qualitative changes in the behavior of the model over a plausible range of τ_1 values remains to be seen.

Gene expression raises a modeling complication not addressed here, namely that of the low copy numbers of various species. For instance, the number of active promoters is not a continuous variable as modeled here, but a discrete one that can only take on integer values, limited by the gene copy number. Similarly, free ribosome binding site (mRNA) concentrations generated by this model correspond to one to two dozen molecules per cell, again in a range where a continuous description is problematic. A differential equation model provides population-level average behavior, but it cannot describe the dynamics in a single cell. Accordingly, we need models that incorporate both delays and stochasticity due to finite populations. Methods for simulating systems with these two features have recently been developed [4, 7, 18, 44], and software implementing some of these algorithms is available [28, 35]. In addition to finite population effects, the intrinsic unpredictability of certain processes, such as the random number of rounds of abortive initiation prior to promoter escape, are more easily incorporated into a stochastic model than into a differential equation model. Studying a stochastic counterpart of the DDE model described here is therefore an important next step for this project.

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Delay Systems: Stabilization and Control Strategies

A Comparison of Shaper-Based and Shaper-Free Architectures for Feedforward Compensation of Flexible Modes

Dan Pilbauer, Wim Michiels and Tomáš Vyhlídal

1 Introduction

Techniques modifying a reference input and filtering undesired frequency by a time delay filter (an input shaper) are broadly used. The input shaping idea was firstly proposed in [13] and named "Posi-cast", for a review on input shaping since then see [12]. The architecture using Posi-cast is shown in Fig. 1. The scheme depicts a classical feedback scheme with input shaper S connected on the input reference r and flexible structure F on the output, and P denoting the plant and C the controller. The goal of such a scheme is to compensate oscillatory modes of the flexible structure represented by a couple of oscillatory poles, as a rule, by including the input shaper that compensates the poles by dominant zeros from its infinite spectrum. Since the introduction of the input shaper technique, many modifications of the control scheme have been developed. Modifications with shapers incorporated directly in feedback loop were motivated by filtering not only the reference signal but also external disturbances. A first attempt to develop this scheme was in [13], where a rather complicated scheme with the shaper and compensator is combined. As shown in [17], such a scheme is limited by the controller and the system which both have to be biproper. Later on, the hybrid control approach proposed in [7] combined

D. Pilbauer · W. Michiels (✉)
Department of Computer Science, KU Leuven, Celestijnenlaan 200A,
3001 Heverlee, Belgium
e-mail: Wim.Michiels@cs.kuleuven.be

D. Pilbauer
e-mail: Dan.Pilbauer@cs.kuleuven.be

D. Pilbauer · T. Vyhlídal
Dept. of Instrumentation and Control Eng., Faculty of Mechanical Engineering,
and Czech Institute of Informatics, Robotics and Cybernetics - CIIRC, Czech Technical
University in Prague, Technická 4 166 07, Praha 6, Czech Republic
e-mail: Tomas.Vyhlidal@fs.cvut.cz

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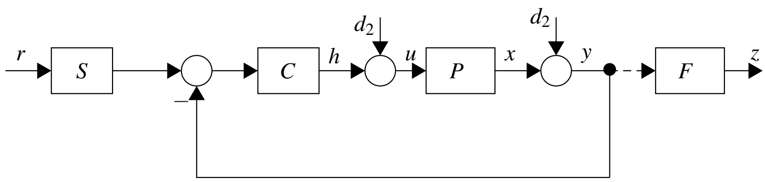

Fig. 1 Classical feedforward application of input shaper

the Posi-cast principle with a classical feedback system design followed by [5, 14], where the authors anaylse the closedloop stability of systems with shapers within the feedback via root locus plots. The scheme with the shaper within the feedback loop, where the shaper is placed in between controller and the system, is effective only when disturbances appear on the sensor but not on the actuator, see [6]. The novel architecture proposed in [16, 17] suggests to use an inverse shaper in the feedback loop.

Here we show an alternative scheme without shaper and with only one controller. The controller's parameters are designed with constraints on its zeros which results in similar properties as the scheme with shaper. However, as will be shown, this approach has limited usage.

Firstly in Sect. 2, we introduce feedback architecture for feedforward compensation. This section shows a technique without inverse shaper and describes limitations for this method. A shaper-free method is motivating shaper-based method, described in Sect. 3. Section 3 firstly introduces the classical input shaping techniques and continues with inverse shaper based technique. Both, the shaper-free and shaper-based techniques, are compared in numerical simulations in Sect. 4.

2 Architecture Without Input Shapers

Consider a system with a block scheme depicted in Fig. 2, where system P with strictly proper transfer function $P(s)$, and with input u and output x. System F, the flexible structure, has transfer function $F(s) = \frac{F_N(s)}{F_D(s)}$ with y being the input and z the output. The inputs $d_{1,2}$ are unmeasurable input and output disturbances actuating

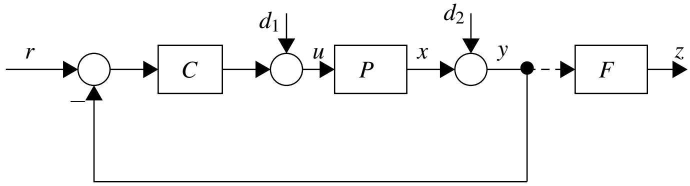

Fig. 2 Block scheme of the classical configuration with controller C, plant P and flexible structure F

on system $P(s) = \frac{P_N(s)}{P_D(s)}$. The controller $C(s) = \frac{C_N(s)}{C_D(s)}$ is assumed to be a fixed-order controller of the form

$$C \begin{cases} \dot{p}(t) = A_c p(t) + B_c(r(t) - y(t)) \\ u(t) = C_c p(t) \end{cases} \quad (1)$$

where capital letters are real-valued matrices of appropriate dimensions. We assume for the moment a single input single output (SISO) system and controller. Therefore (1) is described in the frequency domain by a single transfer function

$$C(s) = C_c(sI - A_c)^{-1} B_c. \quad (2)$$

Closing the loop with the controller and system, the following transfer functions are in the scope of interest. The first transfer function is from reference r to output z

$$T_{zr} = \frac{CP}{1 + CP} F = \frac{\frac{C_N P_N}{C_D P_D}}{\frac{C_D P_D + C_N P_N}{C_D P_D}} \frac{F_N}{F_D} = \frac{C_N P_N}{C_D P_D + C_N P_N} \frac{F_N}{F_D} \quad (3)$$

and the transfer functions from disturbances $d_{1,2}$ to output y are

$$T_{zd_1} = \frac{P}{1 + CP} F = \frac{\frac{P_N}{P_D}}{\frac{C_D P_D + C_N P_N}{C_D P_D}} \frac{F_N}{F_D} = \frac{C_D P_N}{C_D P_D + C_N P_N} \frac{F_N}{F_D} \quad (4)$$

$$T_{zd_2} = \frac{1}{1 + CP} F = \frac{1}{\frac{C_D P_D + C_N P_N}{C_D P_D}} \frac{F_N}{F_D} = \frac{C_D P_D}{C_D P_D + C_N P_N} \frac{F_N}{F_D} \quad (5)$$

in order to compensate the oscillatory pole pole of $F(s)$ by zeros, the transfer function T_{zr} requires the numerator of the controller C_N to have zeros placed on the position of poles of the flexible structure whereas transfer functions T_{zd_1} and T_{zd_2} require denominator C_D to have zeros placed there. These two requirements are contradictory because they cannot be satisfied simultaneously. When the numerator of the controller has required zeros, a change of reference does not excite the oscillatory mode of the flexible structure but the disturbance does. On the other hand, when the denominator of the controller has required zeros, the disturbance does not excite the flexible structure but the change of reference does. As only one one requirement can be satisfied the specific application decides what is the most important. Here, we show how to achieve partial zero placement for controller's numerator.

Consider system P as a linear SISO system described by sets of differential equations

$$P \begin{cases} \dot{x}(t) = A_P x(t) + B_P u(t) \\ y(t) = C_P x(t) \end{cases} \quad (6)$$

where capital letters are real-valued matrices of appropriate dimensions.

The flexible structure F is an oscillatory, low damping system with transfer function $F(s) = \frac{F_N}{F_D}$, where denominator F_D is defined by flexible mode as $s_{1,2} = f(\zeta, \omega)$, where ζ is the damping ratio and ω the natural frequency. The controller C is defined by (1)–(2). To maintain linearity in the design of the parameters the controller is considered in canonical form with matrices

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C_c = \begin{bmatrix} c_n \ c_{n-1} \ \cdots \ c_1 \end{bmatrix}, \tag{7}$$

where parameters c_k modify system's zeros. Parameters c_k together with a_k modify poles of the system.

The primary goal is to a achieve zero pole cancellation in the transfer function T_{zr}. Coefficients c_k can be tuned in a way that at least one couple of zeros is placed at position of poles to be compensated $\hat{z}_{1,2} = s_{1,2}$. To place a couple of zeros, the following set of constraints are

$$\mathbb{R}\{\hat{z}_1^n + c_1\hat{z}_1^{n-1} + \cdots + c_{n-1}\hat{z}_1 + c_n\} = 0, \tag{8}$$

$$\Im\{\hat{z}_1^n + c_1\hat{z}_1^{n-1} + \cdots + c_{n-1}\hat{z}_1 + c_n\} = 0. \tag{9}$$

When a couple of complex zeros is placed the controller exhibits filtering properties. This can be seen in the example of a magnitude frequency response of T_{zr} in Fig. 3, where the drop of the amplitude at the given frequency is shown.

Each constraint (8), (9) removes one degree of freedom of the controller, where total number of degrees is determined by the order of the controller N_c. Therefore, to place one couple of complex conjugated zeros $N_c \geq 2$ and the remaining parameters of c_k and a_k can be used for further purposes, e.g. H_∞ optimization, minimization of the spectral abscissa etc.

The set of Eqs. (8)–(9) can be rewritten into form

$$HC_c = R, \quad with \quad C_c = [c_1 \quad \cdots \quad c_n]^T, \tag{10}$$

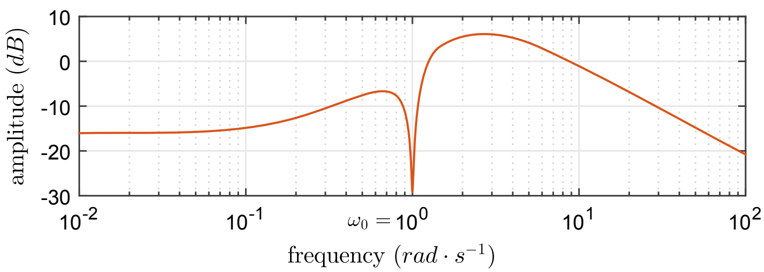

Fig. 3 Magnitude frequency response of a controller designed with partial zero placement

and by applying the singular value decomposition which gives us $H = U\Sigma G^*$, the gains can be then separated in two parts

$$C_c = C_0 + EL \tag{11}$$

where

$$C_c = \underbrace{[g_1|\cdots|g_m]\begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix}}_{C_0} + \underbrace{[g_{m+1}|\cdots|g_n]}_{E}\underbrace{\begin{bmatrix} l_{m+1} \\ \vdots \\ l_n \end{bmatrix}}_{L} \tag{12}$$

where $l_i = \frac{\bar{e}_i}{\sigma_i}$, $U^*R = [\bar{e}_1 \cdots \bar{e}_m]^T$ and $\Sigma = [diag(\sigma_1 \cdots \sigma_m)]$. A size of the first part of (12) is determined by m which is given by a number of constraints (8)–(9). The remaining degrees of freedom is defined as $n - m$, where n is given by an order of the controller (1). This separation of the controller's coefficients provides elimination of the constraints (8)–(9) which would be very difficult to incorporate into an optimization routine.

Connecting a system (6) and the controller (1), with C_c matrix defined in (12), the following system is obtained

$$\begin{cases} \dot{x}(t) = A_P x(t) + B_P (C_0 + EL)^T (r(t) - y(t)), \\ \dot{p}(t) = B_c C x(t) + A_c (r(t) - y(t)). \end{cases} \tag{13}$$

Now, the remaining parameters in L and a_c are available to modify the closed loop system. To show the functionality of the proposed method, the minimization of the spectral abscissa c is presented here. The spectral abscissa is in general a non-convex function where differentiability may not occur when more than one eigenvalue is active, i.e., an eigenvalue whose real part equals the spectral abscissa [8, 9]. Lipschitz continuity fails when an active eigenvalue is multiple and non-semisimple. On the other hand, the spectral abscissa function is differentiable at points where there is only one active eigenvalue with multiplicity one. Since this is the case with probability one when randomly sampling parameter values, the spectral abscissa is smooth almost everywhere. The above properties exclude classical methods to solve the problem Hybrid algorithm for non-smooth optimization (HANSO) software [10] where combination of BFGS with Wolfe weak line search algorithm is able to solve such problems. The software only requires objective function and its derivatives with respect to controller parameters wherever the objective function is differentiable. The objective is to minimize the spectral abscissa of the closed loop system. Defining a vector of variables $p = [L \quad a_1 \quad \cdots \quad a_n]^T$ of length N_p, the optimization can be then defined as

$$\min_p c(p), \tag{14}$$

where the spectral abscissa is defined as

$$c(p) := \sup\left\{\mathbb{R}(s) : \ s \in M(s; p)\right\}, \tag{15}$$

with

$$M(s; p) := \left\{s \in \mathbb{C} : \det\left(sI - \mathcal{A}\ \right) = 0\right\}. \tag{16}$$

where

$$\mathcal{A} = \begin{bmatrix} A_P & B_P(C_0 + EL)^T \\ -B_c C_P & A_c \end{bmatrix} \tag{17}$$

is the matrix of the closed loop system (13). The software also requires derivatives of the objective function with respect to controller parameters. If only one characteristic root with multiplicity one is active then the spectral abscissa is differentiable and expressed as

$$\frac{\partial c}{\partial p} = \begin{bmatrix} \frac{\partial c}{\partial p_1} & \dots & \frac{\partial c}{\partial p_{N_p}} \end{bmatrix}^T = \mathbb{R}\left\{ -\frac{1}{v^* \frac{\partial M}{\partial s} w} \begin{bmatrix} v^* \frac{\partial M}{\partial p_1} w & \dots & v^* \frac{\partial M}{\partial p_{N_p}} w \end{bmatrix}^T \right\}, \tag{18}$$

where v and w are the left and right eigenvectors corresponding to the rightmost eigenvalue.

It is important to note that the technique used to eliminate controller parameters requires that the polynomial, which determines its zeros, linearly depends on the controller parameters. This reduces the applicability to SISO systems. Furthermore a linear dependence also requires that throughput gain D_c is not part of the controller (1).

3 Architectures with Input Shapers

The technique utilizing the inverse shaper is based on classical reference input shaping. Hereby, this section introduces basics of input shaping with time delays followed by a method with an inverse shaper. The section points out the main advantages of inverse shapers, which will be compared with the technique from Sect. 2.

3.1 Input Shaping Background

The classical feedforward scheme with input shaper is show in Fig. 1. The main idea is to filter undesirable frequency coming from the reference signal. Filtering with time delays has main advantage that the modified input reference can be preserved as non-decreasing which is hard to achieve with a classical notch filter.

Utilizing time delays every delay-based input shaper can be described with a Stieltjes integral as

$$h(t) = \int_0^T r(t - \mu) d\lambda(\mu) \tag{19}$$

where $r, u \in R$ are the input and the output, respectively. The function $\lambda(\mu)$ is the distribution of the delay over time interval $[0, T]$, $T \in R$ and $T > 0$. The delay can be distributed in a different ways. The distribution can be made of discrete delays and the shaper then has the form

$$h(t) = A_0 r(t) + \sum_{k=1}^{N} A_k r(t - \tau_k) \tag{20}$$

where $A_k \in R$ are gains and $\tau_k > 0$ are delays. The classical Posicast (also called ZV) shaper has one only delay and has form $h(t) = A_0 r(t) + A_1 r(t - \tau)$ with first, non-delayed parameter $0 < A_0 < 1$ and the gain for the delayed part $A_1 = (1 - A_0)$. For more general discrete distributions see [1].

The distribution of the delay can also be continuous. Then the shaper is in the form

$$h(t) = A_0 r(t) + \sum_{k=1}^{N} A_k \int_0^{\tau_k} g_k(\mu) r(t - \mu) d\mu, \tag{21}$$

where the $g_k(\mu)$ functions can be linear (equally distributed delay, the shaper is then called DZV [18, 19] or even more complicated distributions as shown in [15]). The delay distributed with smooth polynomial functions is described in [11]. The shaper in form (21) has retarded spectrum whereas the shaper with discrete delays (20) has undesirable neutral spectra ([3]). Neutrality brings difficulties in dynamic analysis and requires special attention in the feedback design [2, 4], which would be crucial in method using an inverse shaper. For this reason, we focus only on shapers with retarded dynamics.

3.2 *Inverse Shaper*

The closed loop architecture with inverse shaper proposed in [17] is show in Fig. 4. The shaper S has spectrum consisting only of zeros and the inversion of the shaper S^{-1} turns its zeros into the poles of transfer function $\frac{1}{S(s)}$. Of course, this mathematical operation is only possible when the transfer function is proper. In case of shapers in form of (20) or (21) the inversion always exists if $A_0 > 0$.

The main idea of including the inverse shaper in the feedback is to project its filtering properties in the transfer functions from all possible inputs. For reference input as

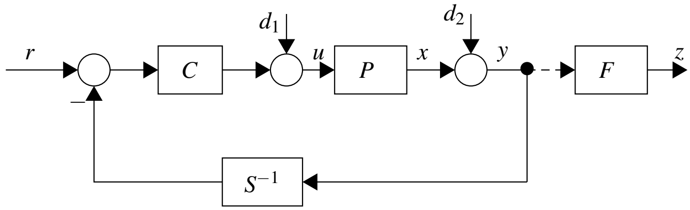

Fig. 4 Block scheme with inverse shaper in the feedback

$$T_{zr} = \frac{CP}{1 + CP\frac{1}{S}}F = \frac{\frac{C_N P_N}{C_D P_D}}{\frac{C_D P_D S + C_N P_N}{C_D P_D S}} \frac{F_N}{F_D} = \frac{C_N P_N S}{C_D P_D S + C_N P_N} \frac{F_N}{F_D} \tag{22}$$

and for the input d_1 and output d_2 disturbances

$$T_{zd_1} = \frac{P}{1 + CP\frac{1}{S}}F = \frac{\frac{P_N}{P_D}}{\frac{C_D P_D S + C_N P_N}{C_D P_D S}} \frac{F_N}{F_D} = \frac{C_D P_N S}{C_D P_D S + C_N P_N} \frac{F_N}{F_D} \tag{23}$$

$$T_{zd_2} = \frac{1}{1 + CP\frac{1}{S}}F = \frac{1}{\frac{C_D P_D S + C_N P_N}{C_D P_D S}} \frac{F_N}{F_D} = \frac{C_D P_D S}{C_D P_D S + C_N P_N} \frac{F_N}{F_D} \tag{24}$$

As can be seen in the transfer functions (22)–(24), infinitely many zeros of the shaper S appear in all numerators of the transfer functions. Then the dominant zeros can be used to compensate the oscillatory modes of $F(s)$. This means, that neither a reference change, nor an input or input disturbance excite the given frequency. On the other hand, the quasi-polynomial form of the shaper also appear in the denominator of the transfer functions and projects its zeros into poles of the system spectra. The main advantage of the shapers with retarded spectra is revealed now. If the shaper has retarded spectra also the closed loop with inverse shaper has retarded spectra and stability issues with small delay perturbations (see, [2, 4]) are omitted.

The closed loop system with the inverse shaper can be unstable or relatively slow. To stabilize the system or modify system spectra, a fixed-order controller can be used to perform the tasks. A fixed-order controller design for infinite dimensional system allows to obtain relatively simple controller and the order of the controller does not necessary need to be the same as the open-loop system. As in the previous, shaper-free, case the design of the controller can be executed for various objective functions, e.g. minimizing the spectral abscissa or H-infinity norms. For the fixed-order controller and the mentioned objectives, the optimization problem is in general non-convex, non-smooth etc. Such problems can once again be handled by recently developed non-smooth, non-convex optimization techniques that are implemented in the package HANSO.

To demonstrate the applicability of the scheme with an inverse shaper, one specific shaper is chosen and applied to the system. The system is then optimized in sense of minimizing the spectral abscissa.

Firsty, we define the input shaper which will be used. The shaper is in form

$$h(t) = ar(t) + (1 - a) \int_0^T r(t - \mu) d\lambda(\mu) \tag{25}$$

where r and h are the shaper input and output, respectively, $a \in R^+$, $a < 1$ is the gain parameter, and the distribution of the delays is prescribed by the non-decreasing function $\lambda(\mu)$. Considering that the overall delay consists of a series of lumped and equally distributed delay of the lengths T the input shaper has transfer function

$$S_{DZV}(s) = a + (1 - a) \frac{1 - e^{-sT}}{Ts} e^{-s\tau} \tag{26}$$

which consists of lumped delay τ and equally distributed delay of the length T. The interpretation of the given transfer function is shown in the time domain in Fig. 5 and in frequency domain in Fig. 6, where its filtering properties for the given nominal frequency $\omega_0 = 1$ rad/s is obvious.

The inversion of the shaper is realized by the following formula

$$h(t) = \frac{1}{a}(y(t) - (1 - a)b(t)) \tag{27}$$

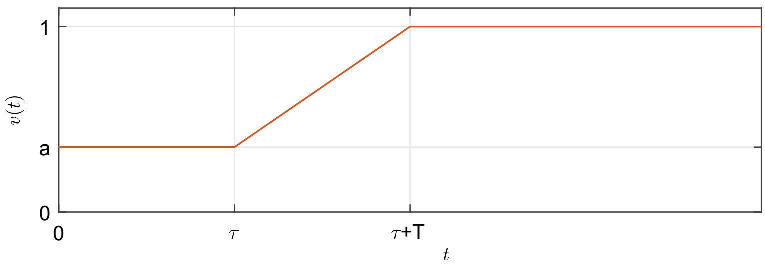

Fig. 5 Step response of the shaper

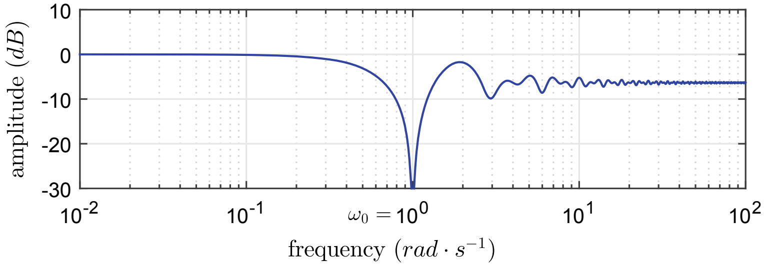

Fig. 6 Magnitude frequency response of the shaper

where

$$b(t) = \frac{1}{\tau}\int_{t-(T+\tau)}^{t-\tau} h(\mu)d\mu \tag{28}$$

which can be implemented as a dynamic equation

$$\dot{b}(t) = \frac{1}{T}\left(h(t-T) - h\left(t-(T+\tau)\right)\right). \tag{29}$$

Connecting equations for shaper (27) and (29) together with the system (6) the system can be described by a set of Delayed Differential and Algebraic Equations (DDAEs) as

$$\begin{cases} \dot{x}(t) = A_P x(t) + B_P u(t), \\ h(t) = \frac{1}{a}C_P x(t) - \frac{1-a}{a}b(t), \\ \dot{b}(t) = \frac{1}{T}h(t-T) - \frac{1}{T}h(t-(T+\tau)). \end{cases} \tag{30}$$

The next task is to design a controller that stabilizes and optimizes the system. To get a good comparison with the shaper-free method the spectral abscissa is the objective function for both cases. The controller is in form (1) with no requirements on structure as in shaper-free method. The vector of variables is here defined as $q = vec\begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}$ with length N_q. The spectral abscissa for the closed loop is defined as

$$c(q) := \sup\left\{\mathbb{R}(s) : \ s \in M_S(s,q)\right\}, \tag{31}$$

with

$$M_S(s,q) := \left\{s \in \mathbb{C} : \det\left(s\mathcal{E} - \mathcal{A}_0 - \mathcal{A}_1\left(e^{-sT} + e^{-s(T+\tau)}\right)\right) = 0\right\}. \tag{32}$$

where

$$\mathcal{A}_0 = \begin{bmatrix} A_P & -B_P D_c & 0 & B_P Cc \\ \frac{1}{a}C_P & -1 & -\frac{1-a}{a} & 0 \\ 0 & 0 & 0 & 0 \\ B_c C & 0 & 0 & A_c \end{bmatrix}, \mathcal{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{T} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

The function of spectral abscissa has the same properties as in previous shaper-free case and HANSO software can be chosen to handle the problem. Also here, derivatives are necessary for the optimization and the same rules for derivatives as in (18) applies.

4 Numerical Simulations

Consider the mechanical system depicted in Fig. 7 where the primary structure is a 2DOF system described by

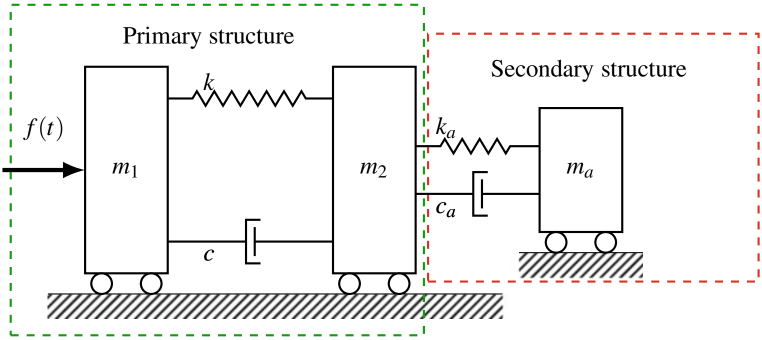

Fig. 7 Mechanical scheme of 2 degrees of freedom primary structure with attached secondary structure. Note that we neglect the coupling between primary and secondary structure as $m_2 << m_a$

Table 1 Parameters of the system

m_1	c	k	m_2	m_a	c_a	k_a	ω	ζ
1 kg	10 kg s^{-1}	1000 N m^{-1}	10 kg	1 kg	1 kg s^{-1}	1 N m^{-1}	1 rad s^{-1}	0.01

$$A_P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & -\frac{c}{m_1} & \frac{c}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} & \frac{c}{m_2} & -\frac{c}{m_2} \end{bmatrix}, B_P = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} \tag{34}$$

with the parameters given in Table 1. The secondary structure is attached to the primary structure by a spring and a damper. We assume that mass $m_a \ll m_2$ and therefore the dynamic of the primary structure may be considered as decoupled from the dynamics of the secondary structure. The oscillatory mode of the secondary structure is defined by its natural frequency $\omega = \sqrt{\frac{k_a}{m_a}}$ and damping ratio $\zeta = \frac{c_a}{2\sqrt{m_a k_a}}$ as $\bar{s}_{1,2} = -\beta \pm j\Omega$, where $\beta = \omega\zeta$, $\Omega = \omega\sqrt{1-\zeta^2}$.

The both cases, shaper-free scheme and scheme with inverse shaper, are designed for the same system. The results are compared in both the compared in frequency and time domain. The results can be seen in the Fig. 8, where zeros and poles of the system are shown. As shown, the initial system without a controller is at the stability boundary with a double pole at the origin and has no zeros. Closing the feedback with designed controller makes the system stable with required zeros in the transfer function from reference to system output (5). In case with the inverse shaper, the zeros of the shaper merge with poles of the system and introduce its dynamics into the closed loop.

Unfortunately, in case without inverse shaper the couple of assigned zeros appear only in this particular transfer function (5) but not in ones coming from the disturbances $d_{1,2}$. Zeros for different transfer functions are compared in Fig. 9. For the

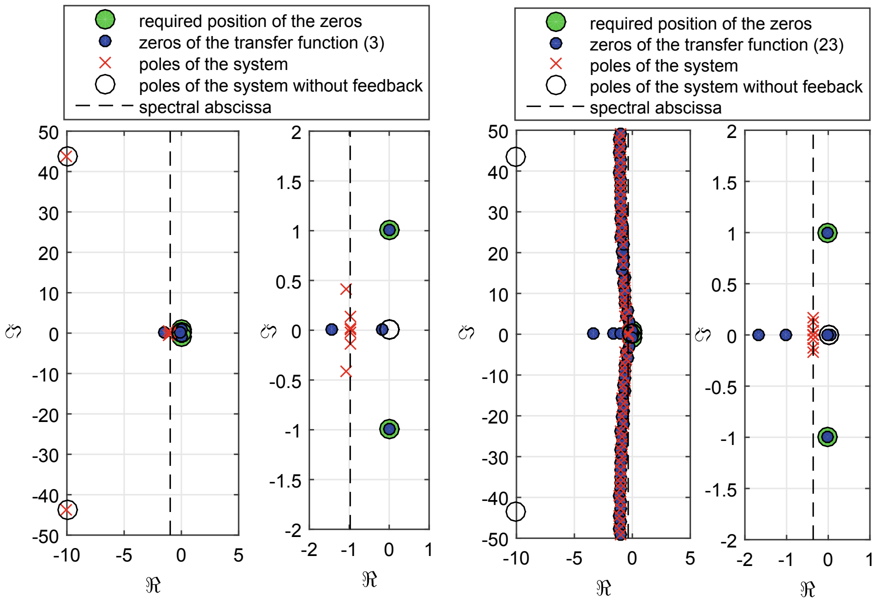

Fig. 8 Spectra of the shaper-free (left) and shaper-based (right) systems

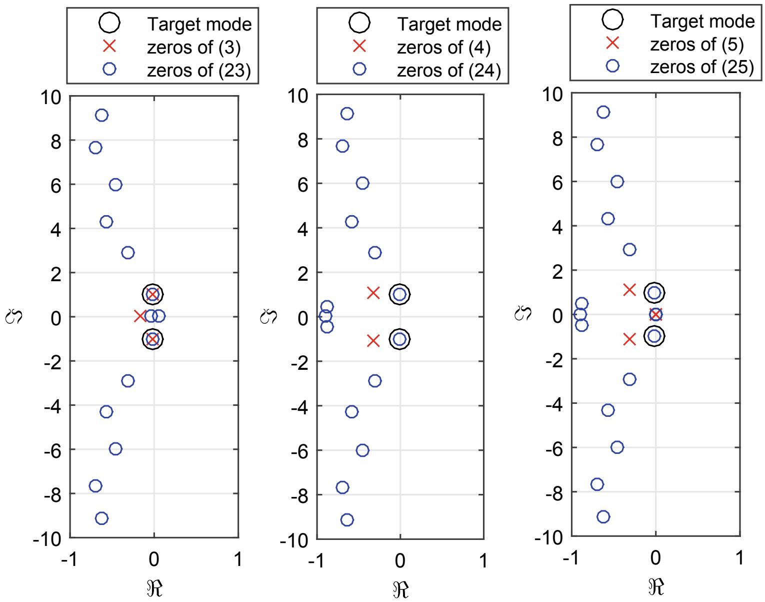

Fig. 9 Comparison of zeros of: left-from reference, middle-from input disturbance d_1, right-from output disturbance d_2

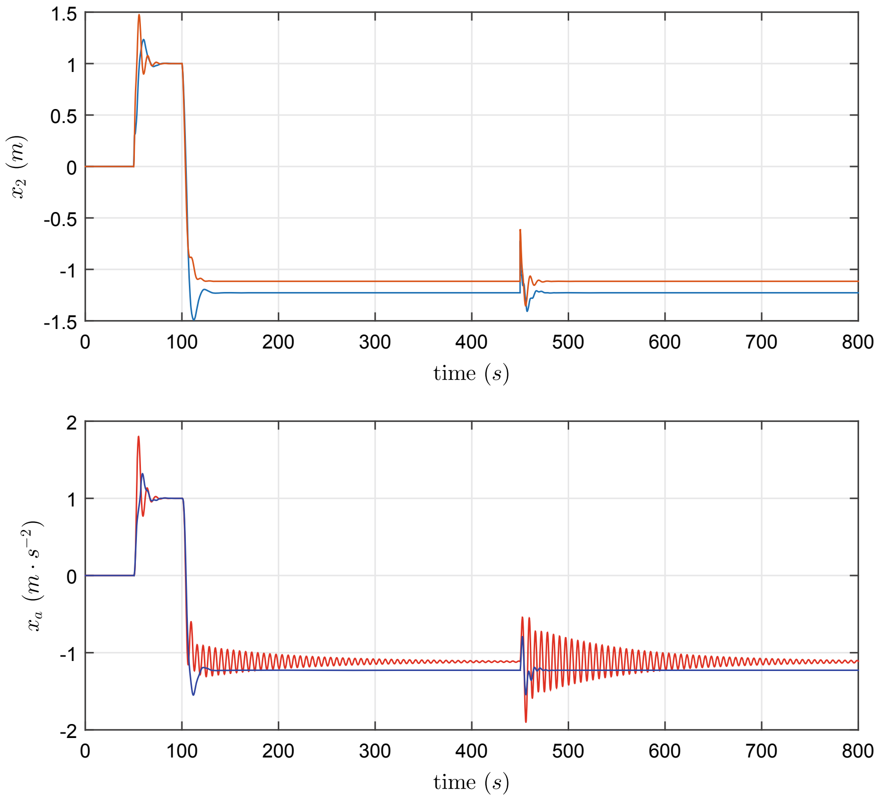

Fig. 10 Red-Response of the system without inverse shaper, designed with partial zero placement; Blue- Response of the system with inverse shaper. Response to the chane of reference r at time $t = 5$ s, input disturbance d_1 appears at time $t = 100$ s and output disturbance d_2 at time $t = 450$ s. The upper figure shows the position of the second cart x_2 and the lower figure shows the position of the flexible structure x_a

case with shapers, part of infinitely many zeros are shown. In fact, the spectrum is retarded, hence the chain of zeros have real parts moving off to minus infinity.

As shown in Fig. 10, the oscillations do not appear when the reference signal is changed but appear when one of the disturbances are present. This behaviour disappear when the inverse shaper is applied in the feedback. Note, that, the response to the disturbance has non-zero steady state error. The error could be eliminated by putting another constraints on the controller or implementing additional integrator. Another advantage of the inverse shaper scheme is the tendency to form smoother responses, even though the desirable monotonous responses have not been achieved in this particular example due to 'fast' controller setting.

5 Summary

We shown how certain constraints can help to construct controllers with properties mimicking filtering properties of the inverse shaper. This approach allows only certain channels, either a single reference input or a couple of input and output disturbances, to have filtering properties, and applies only to SISO system only. Moreover, compared to the input shaping, the responses do not tend to have desirable monotonous character.

In order to address both the set-point and disturbance cases, the controller would need to be separated into two blocks analogously as it is done in Fig. 4. Instead of the inverse shaper, one can place the filter with the flexible mode as its poles. By this option however, the monotonicity of the response from the set-point changes cannot be achieved neither.

The inverse shaper introduces additional disturbance rejections without exciting oscillatory modes of the flexible structure. On the other hand, infinitely many zeros of the shaper turn into poles of the system and make design of the controller more complicated. Stability issues of neutral systems are removed by using shapers with distributed time delay, whereas the shaper-free method does not need any special stability treatment.

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Proportional-Retarded (PR) Protocol for a Large Scale Multi-agent Network with Noisy Measurements; Stability and Performance

Adrián Ramírez and Rifat Sipahi

1 Introduction

Study of multi-agent systems has attracted tremendous attention especially in the past decade, with applications involving robotic networks [1], traffic flow dynamics [2], human-machine interactions and collaborative human-robot systems [3]. While such systems can enjoy rich information flow amongst the agents with the network interconnectivity, distributed nature of the agents and the need to utilize advanced technologies to tailor these agents inevitably bring about a number of unique challenges to the design and control of multi-agent systems. One key challenge is the presence of time delays in the network dynamics [4], which may arise due to various reasons including agents' actuation times, the need to use a communication medium to enable the agents to exchange information, and necessary computation times to process and interpret large stream of data. The presence of time delay in a dynamical system often imports undesirable characteristics, including poor performance, oscillatory response, and instability [5]. Nevertheless, if carefully engineered, time delay can also be used as a vehicle to craft the dynamic response, including fast stabilization [6–11].

In the context of multi-agent networks however, use of delays as a design parameter to achieve fast stabilization is under-explored although this is of high interest [12, 13]. One opportunity in this endeavor is to utilize reliable computational tools to approximate the rightmost eigenvalues of the dynamics [14], or to use those tools to

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A. Ramírez · R. Sipahi (✉)
Department of Mechanical and Industrial Engineering,
Northeastern University, Boston, MA 02115, USA
e-mail: rifat@coe.neu.edu

A. Ramírez
e-mail: a.ramirezlopez@northeastern.edu

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tune the controller gains via optimization schemes [15]. Other ideas to achieve fast stabilization include strategically removing certain links between some of the agents to expedite consensus reaching [16], or re-designing their coupling strengths [17].

Recent results on linear time-invariant (LTI) single-input single-output (SISO) systems [9, 10] indicate that analytical tuning rules can be developed with certain classes of controllers to optimize the spectrum of the closed-loop system. Specifically, in [18, 19], authors analytically designed Proportional-Retarded (PR) protocols that can assign a closed-loop system's spectral abscissa to a user-defined location on the complex plane. These results point out opportunities also for large scale LTI network control problems see, e.g., [20–27] for studies utilizing PR controllers in network settings.

In this chapter we seek to develop distributed PR-based protocols for a benchmark large-scale LTI consensus system. The main objective is to utilize Lambert W functions to analyze the stability of the system in terms of PR protocol parameters. For this, we first take advantage of standard decomposition tools to break down the corresponding characteristic equation into subsystems and treat each subsystem stability one by one. This result provides a transparent understanding in terms of which specific eigenvalue of the graph Laplacian underlying the network governs directly the stability of the entire consensus system. Furthermore, it connects with our recent study in [28] where, with each subsystem being in a particular form, we utilized some inherent features of Lambert W functions to tune the PR protocol without any approximation while shifting the spectrum of the subsystems all at once, thereby yielding fast stabilization. With the novelty of the results pertaining to this tuning approach left to [28], here we summarize some key findings from the cited work for the completeness of the presentation. Overall, in an undirected network, the proposed approach as we demonstrate is scalable and easy to implement, even in the presence of signals with high-frequency noise components.

The rest of the chapter is organized as follows. Section 2 describes the consensus dynamics under analysis and states the problem formulation in light of the Lambert W function. Section 3 starts with a useful factorization of the system that enables a comprehensive study of the stability of the complete network using dimensional analysis. Section 4 summarizes some results without proofs from [28] regarding the design of the spectral abscissa of the system ensuring fast consensus. Section 5 verifies the findings via the analysis of a challenging numerical example. Finally some concluding remarks and further directions on research are given in Sect. 6.

2 Preliminaries and Problem Formulation

In the following we consider a system with n identical agents whose dynamics is captured by the integrator plant

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where $x_i(t)$ is the state of the ith agent and $u_i(t)$ is the control input by which agent i communicates with the rest of the agents. The communication topology of the network is described by an undirected weighted graph $\mathcal{G} = (N, E, A)$ where $N = \{1, 2, \ldots, n\}$ is the set of nodes, $E \subset N \times N$ is the set of edges (communication channels), and $A = [a_{ij}]$ is the weighted adjacency matrix. We assume that each edge has an associated weight $a_{ij} = a_{ji}$, also known as the coupling strength, where the indices $(i, j) \in E$ indicate that agent $i \in N$ receives information either instantaneously or with delay from agent $j \in N$ whenever $a_{ij} > 0$.

Let $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ be the Laplacian matrix with

$$l_{ij} = \begin{cases} \sum_{m=1, m\neq i}^{n} a_{im} & i = j, \\ -a_{ij} & i \neq j, \end{cases} \tag{2}$$

then L is symmetric $l_{ij} = l_{ji}$ and accepts the diffusive property $\sum_{j=1}^{n} l_{ij} = 0$. Hence, from the spectral theorem for Hermitian matrices [29], its eigenvalues are real.

The control objective, as proposed in [28], is to achieve agreement of the states amongst all the agents of the network. To this end, we consider that the agents are coupled via the Proportional-Retarded (PR) protocol originally developed for SISO systems [11],

$$u_i(t) = k_p \sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)] - k_r \sum_{j=1}^{n} a_{ij}[x_j(t-h) - x_i(t-h)]. \tag{3}$$

Here, k_p and k_r determine respectively the strength of the proportional and retarded actions, and $h > 0$ is an intentional delay induced as part of the input with the aim of obtaining a delayed term by which high-frequency measurement noise is attenuated. We say that protocol (3) solves the consensus problem if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, for all $i, j \in N$.

Note that the introduction of the retarded part in the PR protocol mimics a pure derivative action, thus improving transient response but being insensitive to measurement noise. To see this, observe that (3) can be written in terms of the entries of L as: $u_i(t) = -\sum_{j=1}^{n} \ell_{ij}[k_p x_j(t) - k_r x_j(t-h)]$. Introducing the null term $\pm k_r x_j(t)$ into the above and defining $\tilde{k}_p \equiv k_p - k_r$ and $\tilde{k}_r \equiv h k_r$ we obtain the following alternative representation

$$u_i(t) = -\sum_{j=1}^{n} \ell_{ij}\Big[\tilde{k}_p x_j(t) + \tilde{k}_r \frac{1}{h}\int_{t-h}^{t} \dot{x}_j(\tau)d\tau\Big]. \tag{4}$$

Hence, the proposed protocol performs an averaged derivative action [11] distributed throughout the network by which high-frequency noise components are attenuated without relying on measurements or approximations of $\dot{x}_j(t)$.

Let $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^\top$ be the stack vector of the states at all nodes and $\{A_0, A_1\} = \{-k_p, k_r\} \cdot L$, then system (1) with (3) can be conveniently expressed in matrix form as

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-h), \tag{5}$$

whose stability properties are defined by the location of the *characteristic roots* of the function

$$f(s, k_p, k_r) = \det(sI - A_0 - A_1 e^{-sh}) = 0, \tag{6}$$

which is also known as the *characteristic equation* of system (5). Let Γ be the collection of all characteristic roots satisfying (6) and define the spectral abscissa

$$\gamma^* = \max\{\mathbb{R}(s) \mid s \in \Gamma\}. \tag{7}$$

Then, $\gamma^* < 0$ implies that the spectrum of the system, Γ, lies in the open left-half of the complex plane, thus leading to the following definition [10, 30].

Definition 1 The system (5) is exponentially stable if and only if the spectral abscissa is strictly negative.

Remark on the solution of DDEs via the Lambert *W* function: A Lambert W function is any function $W : \mathbb{C} \to \mathbb{C}$ satisfying

$$W(z)e^{W(z)} = z, \tag{8}$$

for all $z \in \mathbb{C}$. Due to the fact that W is multi-valued, it possesses infinitely-many branches [31]. For a Delay-Differential Equation (DDE), such as the one in (1) with (3), each of these branches can be associated to an element of its spectrum; i.e., to an eigenvalue. In particular, the Lambert W function is useful for the stability analysis and control of LTI-TDS represented by DDEs [32]. For example, the principal branch W_0 can be employed to find the system's dominant root s_0. Then, if $\mathbb{R}(s_0)$ is negative, we can conclude that the system is stable.[1] Computation of W_0 follows from the Lagrange inversion theorem [31] as the series expansion

$$W_0(z) = \sum_{r=1}^{\infty} \frac{(-r)^{r-1}}{r!} z^r. \tag{9}$$

Moreover, W_0 may also be defined as the only branch of W that is analytic at 0.[2] In addition, the Lambert W function can also be extended to design feedback controllers placing s_0 at a desired position by using numerical procedures [33].

[1] It is worthy of mention that the radius of convergence of the series is e^{-1}. For practical computation, the reader is referred to [31] where additional asymptotic formulae can be found considering all the branches of the Lambert W function.

[2] The scalar Lambert W function is available as embedded function in MATLAB, see the function *lambertw*.

3 Stability of the Network

Next, we present a decomposition of system (5) by which stability conditions can be derived using the Lambert W function [33] and the D-subdivision method [34].

Factorization of the Consensus Dynamics

Let us begin with a modal transformation that rotates the vector fields of system (5) aiming at obtaining a diagonal representation of it. Similar transformations have been widely utilized in the context of time delay systems, see for example [35–38]. Here, we will say that both systems are equivalent if they share the same stability properties in terms of their spectrum.

Proposition 1 *Let* $\{\lambda_1, \ldots, \lambda_n\}$ *be the set of eigenvalues of L ordered increasingly. Then, system (5) is equivalent to the diagonal system*

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{\Lambda}_0 \boldsymbol{\xi}(t) + \boldsymbol{\Lambda}_1 \boldsymbol{\xi}(t-h), \tag{10}$$

where $\{\boldsymbol{\Lambda}_0, \boldsymbol{\Lambda}_1\} = \{-k_p, k_r\} \cdot \boldsymbol{\Lambda}$, *and* $\boldsymbol{\Lambda} = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$.

Proof Since the graph is undirected the Laplacian matrix is symmetric, hence the Schur's theorem [29] guarantees the existence of a nonsingular orthogonal matrix $U \in \mathbb{R}^{n \times n}$, such that the following representation holds: $L = U \boldsymbol{\Lambda} U^{-1}$. Introducing the change of variable $\mathbf{x}(t) = U \boldsymbol{\xi}(t)$, system (5) is reduced to the diagonal form (10), which can be treated as a set of n decoupled subsystems with dynamics

$$\dot{\xi}_m(t) = -\lambda_m k_p \xi_m(t) + \lambda_m k_r \xi_m(t-h), \quad m = 1, \ldots, n. \tag{11}$$

The characteristic equation of any subsystem of the form (11) is

$$f_m(s, k_p, k_r) = s + \lambda_m k_p - \lambda_m k_r e^{-sh} = 0. \tag{12}$$

The fact that the matrix coefficients of systems (5) and (10) share the same set of eigenvalues implies that each $f_m(s, k_p, k_r)$ in (12) is a factor of $f(s, k_p, k_r)$ in (6); i.e.,

$$f(s, k_p, k_r) = \prod_{m=1}^{n} \left[s + \lambda_m k_p - \lambda_m k_r e^{-sh} \right] = 0. \tag{13}$$

Then, the spectra and thus the stability properties of (10) and (5) are equivalent. □ ■

Proposition 2 *The system (5) is exponentially stable if and only if*

$$\gamma^* = \max \left\{ \gamma_m^* \right\}_{m=1}^{n} < 0, \tag{14}$$

where

$$\gamma_m^* = \mathbb{R}\big(h^{-1} W_0 \big(\lambda_m k_r h e^{\lambda_m k_p h} \big) - \lambda_m k_p \big), \tag{15}$$

and W_0 is the principal branch of the Lambert W function.

Proof Consider the mth factor $f_m(s, k_p, k_r)$ in (12). Multiplying both sides of this equation by $he^{\lambda_m k_p h}$ yields

$$h\big(s + \lambda_m k_p\big)e^{h\left(s+\lambda_m k_p\right)} = \lambda_m k_r h e^{\lambda_m k_p h}. \tag{16}$$

Comparing (8) and (16), we can see that $h\big(s + \lambda_m k_p\big) = W(\lambda_m k_r h e^{\lambda_m k_p h})$. Solving the above equation for s leads to the solution

$$s = h^{-1} W\big(\lambda_m k_r h e^{\lambda_m k_p h}\big) - \lambda_m k_p. \tag{17}$$

Then, Eq. (15) follows from the real part of (17) using the principal branch W_0. As per Definition 1, the exponential stability of the system is equivalent to (14). □ ■

Assuming that agents are connected, matrix L has a zero eigenvalue $\lambda_1 = 0$ corresponding to the consensus state, and with $\ell_{ij} > 0$, its remaining eigenvalues $\lambda_2, \ldots, \lambda_n$ are positive [39]. Therefore, while ignoring the case of $m = 1$ since this corresponds to the consensus state $s = 0$, we have the following corollary.

Corollary 1 *Let γ_m^* be the spectral abscissa corresponding to the* mth *subsystem (11). Then, if $\gamma_m^* < 0$ for all $m = 2, \ldots, n$, system (5) is exponentially stable around the consensus state of the network.*

Proof Note that $\gamma_m^* < 0$ for all $m = 2, \ldots, n$ guarantees $\gamma^* < 0$. □ ■

Corollary 1 states that separately analyzing the stability of the individual subsystems is equivalent to analyzing the stability of the complete system.

Decomposition of the Space of Parameters

The above discussion indicates that the stability analysis of system (5) can be performed by studying a finite set of subsystems with reduced complexity. With this in mind, using the D-subdivision method, we next decompose the space of controller parameters to study the stability switches of each subsystem (11) with the aim of obtaining a complete stability picture of the system. First, we transform the characteristic Eq. (12) into a dimensionless form. To this end, let the quasipolynomial $f_m(s, k_p, k_r)$ be scaled by h and introduce the time-scaled Laplace operator $\tilde{s} = hs$, this then transforms (12) into

$$hf_m(\tilde{s}/h, k_p, k_r) = \tilde{s} + \lambda_m k_p h - \lambda_m k_r h e^{-\tilde{s}} = 0. \tag{18}$$

Note that the new quasipolynomial retains the stability properties of the original one but with a time delay transformed to unity and where h is now acting as a gain in the system. Defining the lumped gains $\rho_p = \lambda_m k_p h$ and $\rho_r = \lambda_m k_r h$ and the scaled function $\tilde{f}(\tilde{s}, \rho_p, \rho_r) = hf_m(\tilde{s}/h, k_p, k_r)$, we can recast (18) as

$$\tilde{f}(\tilde{s}, \rho_p, \rho_r) = \tilde{s} + \rho_p - \rho_r e^{-\tilde{s}} = 0. \tag{19}$$

Remark 1 Observe that under this transformation, all factors $f_m(s, k_p, k_r)$ in (12) share the uniform structure (19). Hence, for the sake of generality, we temporarily drop the index m associated with the mth eigenvalue.

Following the same logic as in Proposition 2, we multiply (19) by a factor $e^{(\tilde{s}+\rho_p)}$ and obtain $(\tilde{s}+\rho_p)e^{(\tilde{s}+\rho_p)} = \rho_r e^{\rho_p}$ with which, using the real part of the principal branch W_0 of the Lambert W function, we find the spectral abscissa

$$\tilde{\gamma}^* = \mathbb{R}(W_0\,(\rho_r e^{\rho_p}) - \rho_p). \tag{20}$$

As per Corollary 1, stability of any subsystem of the form (11) follows from (20) if and only if $\tilde{\gamma}^* < 0$. Moreover, a stability switch can only occur if some characteristic roots cross the imaginary axis. Therefore, we next search for the lumped crossing points $(\rho_p^\sharp, \rho_r^\sharp)$ and the corresponding scaled crossing frequencies $\tilde{\omega} = \omega h$ such that

$$\tilde{f}(j\tilde{\omega}, \rho_p^\sharp, \rho_r^\sharp) = 0. \tag{21}$$

Due to symmetry of the characteristic roots with respect to the real axis, we can consider only nonnegative frequencies.

Proposition 3 *For a given $\tilde{\omega} \neq k\pi$, $k \in \mathbb{N}$ the corresponding lumped crossing point $(\rho_p^\sharp, \rho_r^\sharp)$ is given by*

$$(\rho_p^\sharp, \rho_r^\sharp) = \left(-\tilde{\omega}\cos(\tilde{\omega})\big/\sin(\tilde{\omega}), -\tilde{\omega}\big/\sin(\tilde{\omega})\right). \tag{22}$$

Moreover, any point on the line

$$\rho_p^\sharp - \rho_r^\sharp = 0, \tag{23}$$

is also a lumped crossing point.

Proof Collecting real and imaginary parts of (21) and some algebraic manipulations generate (22), which are well defined for $\tilde{\omega} \neq k\pi$, $k \in \mathbb{N}$. Corresponding to a root on the origin of the complex plane, (23) satisfies (21) as $\tilde{\omega} \to 0$. □ ■

Let $(\rho_p^\sharp, \rho_r^\sharp)$ be a lumped crossing point and define

$$\tilde{C} = \left\{(\rho_p^\sharp, \rho_r^\sharp) \mid \tilde{\omega} \in [0, \infty), \tilde{\omega} \neq k\pi, k \in \mathbb{N}\right\}. \tag{24}$$

The collection of points in (24) generates almost-everywhere smooth curves [40] known as stability crossing boundaries. Then, $\tilde{C}$ decomposes the lumped space of parameters $\tilde{D} = \{(\rho_p, \rho_r) \in \mathbb{R} \times \mathbb{R}\}$ into a finite number of regions. Since the spectrum of system (5), Γ, behaves continuously with respect to small variations of the lumped parameters, each of these regions is characterized by the same number of unstable roots ν. Let us denote each region by $\tilde{D}(\nu)$, thus,

$$\tilde{D} = \bigcup_{\nu=0}^{\infty} \tilde{D}(\nu), \tag{25}$$

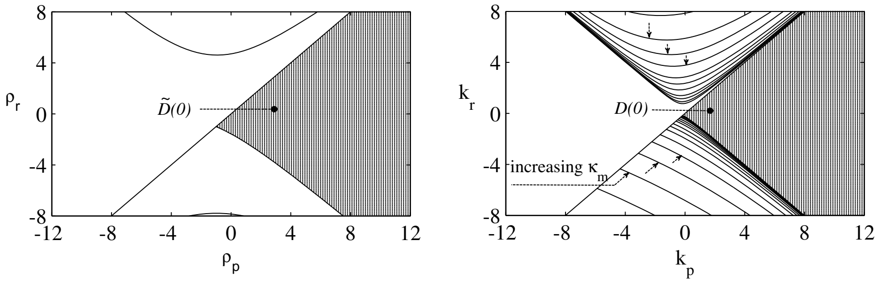

Fig. 1 (Left panel) Stability map obtained with the quasipolynomial (19), the stability crossing boundaries $\widetilde{C}$ are shown in solid lines and the (ρ_p, ρ_r) pairs satisfying $\tilde{\gamma}^* < 0$ are depicted with isolated points. (Right panel) The stability crossing boundaries C_m obtained with Proposition 4 are shown in solid lines and the (k_p, k_r) pairs satisfying $\check{\gamma}^* < 0$ are depicted with isolated points

forms a partition of the lumped space of parameters. Here, $\tilde{D}(0)$ is referred to as the lumped stability domain.

From Proposition 3, we compute the stability crossing boundaries depicted in solid line in Fig. 1 (Left panel). The stability condition $\tilde{\gamma}^* < 0$ is next tested, with $\tilde{\gamma}^*$ in (20), using the embedded function *lambertw* in MATLAB and sweeping both ρ_p and ρ_r. The isolated points in Fig. 1 (Left panel) correspond to (ρ_p, ρ_r) pairs where the stability condition holds. Here, the stability domain is given by

$$\tilde{D}(0) = \{(\rho_p, \rho_r) \mid \tilde{\gamma}^* < 0\}, \tag{26}$$

whose outlook, shaped by $\tilde{C}$, remains invariant with respect to both the eigenvalues of the Laplacian and the amount of induced delay.

To determine the impact of λ_m and h in the stability properties of the system in the original coordinates (k_p, k_r), we now consider a network with an infinite number of agents $(n \to \infty)$ represented by an undirected graph. According to the spectral theorem for Hermitian matrices [29], all the eigenvalues of L are real. Without loss of generality, let the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}^\infty$ be ordered increasingly and define the merged parameter $\kappa_m = \lambda_m h$. Once again, we ignore $\lambda_1 = 0$ as explained above, and therefore $\kappa_m > 0$. Then, the following proposition is well defined.

Proposition 4 *For a given $\kappa_m > 0$, $m = 2, 3, \ldots$, and $\tilde{\omega} \neq k\pi$, $k \in \mathbb{N}$ the corresponding crossing point $(k_p^\sharp, k_r^\sharp)$ is given by*

$$(k_p^\sharp, k_r^\sharp) = \left(-\kappa_m^{-1}\tilde{\omega}\cos(\tilde{\omega})\big/\sin(\tilde{\omega}), -\kappa_m^{-1}\tilde{\omega}\big/\sin(\tilde{\omega})\right). \tag{27}$$

Moreover, any point on the line

$$k_p^\sharp - k_r^\sharp = 0, \tag{28}$$

is also a crossing point.

Proof The result follows directly from Proposition 3. □ ■

Since $n \to \infty$ and $\ell_{ij} = \ell_{ji}$ is a free parameter, the eigenvalues of L are allowed to take any real value, then $\kappa_m \in (0, \infty)$ and $\kappa_{m+1} > \kappa_m$ with $m = 2, 3, \ldots$. Considering a fixed value of κ_m, the crossing points $(k_p^\sharp, k_r^\sharp)$ define the stability crossing boundary

$$C_m = \left\{(k_p^\sharp, k_r^\sharp) \mid \tilde{\omega} \in [0, \infty), \tilde{\omega} \neq k\pi, k \in \mathbb{N}\right\}. \quad (29)$$

Associated with κ_m, let us define the stability domain

$$D_m(0) = \left\{(k_p, k_r) \mid \check{\gamma}_m^* < 0\right\}, \quad (30)$$

where $\check{\gamma}_m^* = h\gamma_m^*$ follows from (15) and is given by

$$\check{\gamma}_m^* = \mathbb{R}\left(W_0\left(\kappa_m k_r e^{\kappa_m k_p}\right) - \kappa_m k_p\right). \quad (31)$$

Corollary 1 states that the stability of all subsystems in (12) implies the stability of the complete network. Moreover, as per Proposition 2, the stability of the complete network implies

$$\check{\gamma}^* = \max\left\{\check{\gamma}_m^*\right\}_{m=2}^n < 0. \quad (32)$$

Conversely, condition (32) implies the stability of the complete network. From Proposition 4, we compute the stability crossing boundaries depicted in solid line in Fig. 1 (Right panel) with several values of $\kappa_m \to \infty$. Condition $\check{\gamma}^* < 0$ in (32) is next tested, using the embedded function *lambertw* in MATLAB and sweeping both k_p and k_r. The isolated points in Fig. 1 (Right panel) corresponds to (k_p, k_r) pairs where the stability condition of (32) holds. Here, the stability domain is given by

$$D(0) = \bigcap_{m=2}^{n} D_m(0), \quad (33)$$

where $D(0)$ is the stability domain of the overall system (5). Note that $D_2(0) \supset D_3(0) \supset \cdots \supset D_n(0)$, therefore (33) reduces to

$$D(0) = D_n(0). \quad (34)$$

Since $D_n(0)$ is related to κ_n, and κ_n is related to λ_n, we can conclude that ensuring the stability of the subsystem that corresponds to the largest eigenvalue of the Laplacian will ensure the stability of the complete system. That is,

$$D(0) = \left\{(k_p, k_r) \mid \check{\gamma}_n^* < 0\right\}. \quad (35)$$

The above result is formalized in the following proposition.

Proposition 5 *The stability domain of the consensus dynamics (5) in the parameter space (k_p, k_r) is equivalent to the stability domain of its subsystem associated with the largest Laplacian eigenvalue in (11).* □

Now, the problem is to find the setting for the parameters h, k_p and k_r as a function of the Laplacian eigenvalues such that stability of (5) is guaranteed through (35).

4 Tuning of the PR Protocol

The approach presented below is summarized from [28] for completeness. Readers are referred to the cited study for all relevant proofs. Here, we wish to show how PR protocol can be tuned for the network system in (5). As concluded above, the stability of the subsystem related to the largest eigenvalue of the Laplacian implies the stability of (5). This is now connected to the results in [28] where the objective is to place the spectral abscissa of the consensus dynamics at a desired position γ_d.

First, choose an arbitrary eigenvalue $\bar{\lambda}$ of L. Associated with $\bar{\lambda}$ we have that

$$\bar{\gamma}^* = \mathbb{R}\Big(h^{-1} W_0\left(\bar{\lambda} k_r h e^{\bar{\lambda} k_p h}\right) - \bar{\lambda} k_p\Big). \tag{36}$$

Using the delay and the gains

$$(h, k_p, k_r) = (1/\bar{\lambda},\ W_0(1) - \gamma_d/\bar{\lambda},\ e^{-k_p}), \tag{37}$$

into (36) reduces $\bar{\gamma}^*$ to

$$\bar{\gamma}^* = \gamma_d, \tag{38}$$

where γ_d is a free parameter introduced to arbitrarily place $\bar{\gamma}^*$.

Second, study the impact of h, k_p and k_r in the rest of the subsystems. To this end, consider a generic spectral abscissa $\hat{\gamma}^*$ associated with the eigenvalue $\hat{\lambda} > \bar{\lambda}$. Employing (37) along with $\hat{\lambda}$ into (15) we have the spectral abscissa

$$\hat{\gamma}^* = \bar{\lambda}\mathbb{R}\Big(W_0\left(\hat{\lambda}\bar{\lambda}^{-1} e^{k_p\left(\hat{\lambda}\bar{\lambda}^{-1}-1\right)}\right) - \hat{\lambda}\bar{\lambda}^{-1} k_p\Big). \tag{39}$$

Define $\delta = \hat{\lambda}\bar{\lambda}^{-1}$, since $\hat{\lambda} > \bar{\lambda} > 0$, then $\delta > 1$. Moreover, whenever k_p remains positive, W_0 is real and positive [31]. Hence, (39) is equivalent to

$$\hat{\gamma}^* = \bar{\lambda}\Big[W_0\Big(\delta e^{k_p(\delta-1)}\Big) - \delta k_p\Big]. \tag{40}$$

Third, introduce the identity

$$F(\delta) = \bar{\gamma}^* - \hat{\gamma}^*, \tag{41}$$

relating the spectral abscissas. Here, if $F(\delta) > 0$ for all $\delta \in (1, \infty)$, this would imply that the systems associated with $\hat{\lambda}$ and with $\bar{\lambda}$ are both stable provided that γ_d in (38) is strictly negative. As per proofs in [28], indeed $F(\delta) > 0$ holds so long as $k_p > e^{-1}$.

Fourth, let $\bar{\lambda} = \lambda_{\min} = \min\{\lambda_m\}_{m=1}^n \neq 0$. Since $F(\delta) > 0$ holds for $k_p > e^{-1}$, it follows that $\bar{\gamma}^* > \hat{\gamma}^*$, where $\bar{\gamma}^*$ is the spectral abscissa associated with $\lambda_{\min}$, and $\hat{\gamma}^*$ is the spectral abscissa associated with any of the remaining eigenvalues of L. Here, $\lambda_1 = 0$ is once again ignored as explained above. We conclude that, under parameters (h, k_p, k_r) in (37), the spectral abscissa of the overall network is a function of $\lambda_{\min}$ and can be placed at any desired position; i.e., $\gamma^* = \bar{\gamma}^* = \gamma_d$. Moreover, if $\hat{\lambda} = \lambda_{\max} = \max\{\lambda_m\}_{m=1}^n$, choosing $\gamma_d < 0$ such that $k_p > e^{-1}$ implies $\hat{\gamma}^* < 0$. In other words, the subsystem associated with $\lambda_{\max}$ is stable and hence, system (5) is stable as per our result in the previous section.

Finally, we have the following proposition by which the γ-stability of system (5) is ensured by means of the tuning of the parameters of the PR protocol.

Proposition 6 ([28]) Let $\lambda_{\min} = \min\{\lambda_m\}_{m=1}^n \neq 0$ be the smallest eigenvalue of L, and let $\gamma_d < 0$ be a desired locus for the spectral abscissa γ^* of system (5), then a dominant root at γ_d is placed by the following tuning of the PR protocol gains

$$(h, k_p, k_r) = \left(\frac{1}{\lambda_{\min}}, \frac{\lambda_{\min}\Omega - \gamma_d}{\lambda_{\min}}, e^{-k_p} \right), \tag{42}$$

where $\Omega = W_0(1) = 0.5671$ is the omega constant.

To summarize, on the network system at hand controlled by PR protocol, we have two key messages: **(a)** The maximum of the eigenvalue $\lambda_{\max}$ of the graph Laplacian L dictates the ultimate stability characteristics in terms of PR protocol gains, and **(b)** the minimum of the eigenvalue $\lambda_{\min}$ of the graph Laplacian L dictates how the PR protocol gains must be designed to place the dominant root of the dynamics at a user-defined spectral abscissa γ_d.

5 Numerical Examples

In this section, we present numerical results for the consensus dynamics in (5) where the parameters of the PR protocol are tuned using Proposition 6.

We investigate a fully connected topology of 5 agents with heterogeneous coupling strengths. Here, {0, 25.74, 41.84, 58.08, 70.76} is the set of eigenvalues of the Laplacian matrix L given by

$$L = \begin{pmatrix} 45.7273 & -14.9896 & -9.7127 & -17.2060 & -3.8190 \\ * & 49.0701 & -1.9947 & -13.1760 & -18.9099 \\ * & * & 26.8209 & -0.8211 & -14.2924 \\ * & * & * & 34.5030 & -3.2999 \\ * & * & * & * & 40.3212 \end{pmatrix}. \tag{43}$$

With the eigenvalues λ_m at hand, the quasipolynomial $f(s, k_p, k_r) = \det(sI - A_0 - A_1 e^{-sh})$, where $\{A_0, A_1\} = \{-k_p, k_r\} \cdot L$, is factorized with Proposition 1 as

$$\begin{aligned} f(s, k_p, k_r) = s \times \big[s + 25.74k_p \underset{}{\overset{\overset{f(\lambda_{\min})}{\uparrow}}{-}} 25.74k_r e^{-sh}\big] \times \big[s + 41.84k_p - 41.84k_r e^{-sh}\big] \\ \times \big[s + 58.08k_p - 58.08k_r e^{-sh}\big] \times \big[s + 70.76k_p \underset{\underset{f(\lambda_{\max})}{\downarrow}}{-} 70.76k_r e^{-sh}\big]. \end{aligned} \tag{44}$$

Using Proposition 6, one can now place the spectral abscissa of $f(\lambda_{\min})$ in (44) at a stable locus, which in turn implies that $F(\delta)$ in (41) remains strictly positive for any $\delta > 1$, and therefore the spectral abscissa of $f(\lambda_{\max})$ must also be placed at a stable locus.[3] We can conclude that the subsystem associated with $\lambda_{\max}$ is stable and hence, as per Proposition 5, the complete consensus network is stable.

Figure 2 shows the time simulations for the 5-agent network and considering $\gamma_d \in \{-10, -20, -30\}$. The initial states of the agents in the time interval $t \in [-h, 0]$ are $[-0.19, 0.19, -0.59, 0.05, 0.89]^\top$. In addition, we have injected uniformly distributed random signals into the network's communication channels to mimic high-frequency noise measurements of the states with a flat power spectral density and infinite total energy. Two observations are in order: i) agents' dynamics are only minimally affected by the simulated high-frequency noise in the measurements as opposed to using a controller with pure derivatives (plots suppressed due to lack of space) and ii) pushing the spectral abscissa deeper into the left-hand side of the complex plane increases the velocity of response of the system. Following these observations, we can say that the PR protocol can process the noisy measurements without any need for further filtering. Moreover, the convergence rate of the consensus network is dictated by the tuning rules in Proposition 6, hence, faster consensus can be achieved by choosing smaller negative γ_d values.

6 Conclusions

This chapter studies the stability of a LTI consensus dynamics under a PR protocol that utilizes delays as tuning parameters. We present how the PR protocol gains influence the stability of the dynamics and how the maximum eigenvalue of the underlying graph Laplacian alone ultimately determines the stability of the overall

[3] Note that Proposition 6 uses $\lambda_{\min}$ to guarantee the placement of γ^* at a desired location γ_d through a stabilizing pair (k_p, k_r). Since $\gamma_d < 0$ is a necessary and sufficient condition for the stability of the consensus network, it can be conjectured that the stabilizing pair (k_p, k_r) must lie within the stability domain associated with $\lambda_{\max}$. Therefore, one may consider Proposition 6 as a link between two important Laplacian eigenvalues, namely, $\lambda_{\min}$ and $\lambda_{\max}$.

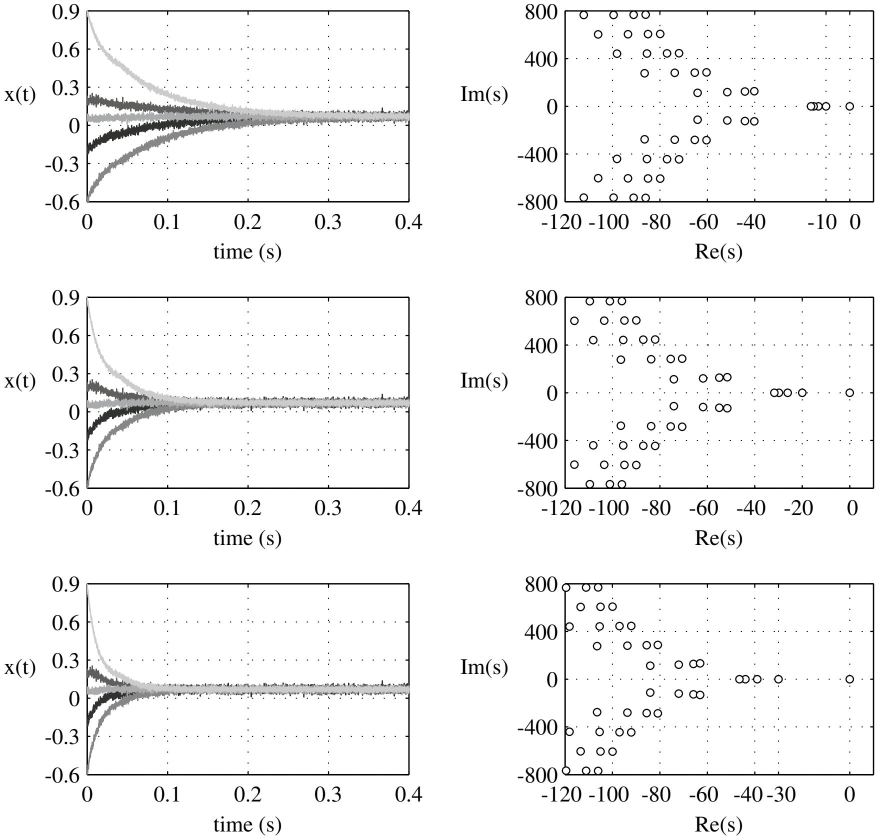

Fig. 2 5-agent network subject to high-frequency noise measurements. (Left panels) Agents' states with respect to time. (Right panels) Spectrum distribution of the consensus dynamics computed with QPMR [14]. (Top) $\gamma_d = -10$. (Center) $\gamma_d = -20$. (Bottom) $\gamma_d = -30$

network dynamics. Recent results from [28] are then summarized showing how to tune the PR protocol to achieve fast consensus. Further research directions include the analysis of different consensus protocols and the use of multiple delays.

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Inversion of Separable Kernel Operator and Its Application in Control Synthesis

Guoying Miao, Matthew M. Peet and Keqin Gu

1 Introduction

A necessary and sufficient condition for stability of coupled differential-difference equations (of which delay-differential equations is a special case) is the existence of a so-called "complete quadratic" Lyapunov-Krasovskii functional. Such functionals have the form $V = \langle x, \mathcal{P}x \rangle$ where $\mathcal{P}$ is defined by a combination of multiplier and integral operators, the inner product is defined on L_2, and the state, $x \in \mathbb{R}^n \times L_2[-\tau, 0]^m$ is a combination of the present state and memory of certain delayed channels. The first numerical algorithm to use semidefinite programming(SDP) to parameterize and optimize over the set of complete-quadratic functionals was the discretized Lyapunov-Krasovskii functional method in [3], later refined in [4], wherein the multiplier and kernel of the Lyapunov-Krasovskii functional were assumed to be piecewise linear. A more recent SDP-based approach to parameterizing and optimizing these functionals is the Sum-Of-Squares (SOS) method presented in [13]. In the SOS method, the multiplier and kernel are parameterized by polynomials.

In this chapter, we focus on the problem of full-state feedback controller synthesis for coupled differential-difference equations of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + By(t-r) + Fu(t),\\ y(t) &= Cx(t) + Dy(t-r),\end{aligned}$$

G. Miao
School of Information and Control, Nanjing University of Information Science and Technology, Nanjing 210044, People's Republic of China
e-mail: mgyss66@163.com

M. M. Peet
School of Matter, Transport and Energy, Arizona State University, Tempe, AZ 85287, USA
e-mail: mpeet@asu.edu

K. Gu (✉)
Department of Mechanical and Industrial Engineering, Southern Illinois University Edwardsville, Edwardsville, IL 62026, USA
e-mail: kgu@siue.edu

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where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $F \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{m\times m}$ and $r > 0$ is the delay. For most practical systems, the number of delayed channels is significantly smaller than the total number of state variables ($m \ll n$). For such systems, the complexity of the complete-quadratic functional associated with the coupled differential-difference form is significantly lower than that associated with the delay-differential framework. By exploiting this reduced functional, the complexity of numerical algorithms such as the discretized and SOS approaches can be significantly reduced—resulting in more efficient and accurate tests for stability. Such reductions were explored and documented for the discretized Lyapunov-Krasovskii functional approach in [5, 6, 8], and for the SOS formulation in [14].

Full-state feedback controller synthesis means that $u(t) = \mathcal{K}\left[\, x^T(t)\ y_t^T \,\right]^T$ where y_t is the history of $y(t)$ on the interval $[t-r, t]$ and $\mathcal{K} : \mathbb{R}^n \times L_2[-r, 0]^m \to \mathbb{R}^p$. That is, the feedback controller uses knowledge of the entire state in determining the input. This is in contrast to most work on controller synthesis for time-delay systems which only use more directly measurable parts of the state such as $x(t)$ to determine the control input. The use of subsets of the state for controller synthesis are best classified as output feedback. However, the use of state-estimation for delayed systems in the output-feedback framework has not hitherto been explored.

More generally, conditions for control synthesis of delayed systems based on the complete quadratic Lyapunov-Krasovskii functional is still rare. An early example is [2], in which a more limited class of Lyapunov-Krasovskii functional is used, and some parameter constraints are imposed. Recently, a synthesis condition based on the inverse of the multiplier/kernel operators associated with the complete Lyapunov-Krasovskii functional for time-delay systems of retarded type in the SOS formulation was developed in [10, 12]. This chapter extends this method to coupled differential-difference equations. The inverse operator is derived using a direct algebraic approach rather than the series expansion approach in [10, 12]. The basic idea of such synthesis is outlined as follows.

Consider the coupled differential functional equations either in closed-loop or autonomous form:

$$\dot{x}(t) = Ax(t) + By(t-r) + \int_{-r}^{0} H(\theta)y(t+\theta)d\theta, \tag{1}$$

$$y(t) = Cx(t) + Dy(t-r), \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $H(\theta) \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$ and $r > 0$. Eqs. (1)–(2) have a unique solution $(x(t), y(t))$ for any initial condition

$$x(0) = \psi \in \mathbb{R}^n, \qquad y(t) = \phi(t) \quad \forall t \in [-r, 0] \qquad \phi \in \mathcal{PC}(r, m),$$

where $\mathcal{PC}(r, m)$ represents the set of piecewise continuous functions from $[-r, 0]$ to $\mathbb{R}^m$. For any piecewise-continuous function $y(t)$ and $\tau \in \mathbb{R}^+$, we define $y_\tau \in \mathcal{PC}(r, m)$ by $y_\tau(\theta) = y(\tau + \theta)$, $-r \le \theta \le 0$. Let

$$Z := \mathbb{R}^n \times \mathcal{PC}(r, m). \tag{3}$$

Then any initial condition $(\psi, \phi) \in Z$ uniquely defines a solution, which may be represented by a strongly continuous semigroup (C_0-semigroup) $\mathcal{S} : Z \to Z$,

$$z(t) = \mathcal{S}(t - \tau)z(\tau). \tag{4}$$

Solutions of System (1)–(2) satisfy an abstract differential equation on Z,

$$\dot{z} = \mathcal{A}z, \tag{5}$$

where $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup $\mathcal{S}$.

Stability of solutions of the system defined by Eqs. (1)–(2) is equivalent to the existence of a quadratic Lyapunov-Krasovskii functional of the form

$$V(z) =< z, \mathcal{P}z >, \tag{6}$$

where $\mathcal{P}$ is a self-adjoint operator, and $\langle\cdot,\cdot\rangle$ represents the inner product on L_2 [1]. The system is stable if $\mathcal{P}$ is coercive, and its derivative along the system trajectory

$$\dot{V}(z) =< z, (\mathcal{PA} + \mathcal{A}^*\mathcal{P})z >,$$

is negative definite in the sense that $-\mathcal{PA} - \mathcal{A}^*\mathcal{P}$ is coercive, where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$. Stability analysis is therefore equivalent to the existence of an operator $\mathcal{P}$ which satisfies the above conditions. Then, by using positive matrices to parameterize a suitably rich cone of positive operators, as described in [11] or [4], and by observing that the operator $\mathcal{P}$ appears linearly in $V(z)$ and $\dot{V}(z)$, the problem of stability analysis can be reduced to a SDP problem. Specifically, the methods defined in [11] consider the case where the operators are the combination of multiplier and integral operators with polynomial multipliers and kernels. The approach in [4] uses piecewise linear multipliers and kernels.

For the problem of controller synthesis, however, we search for both a Lyapunov operator $\mathcal{P}$ and a feedback operator $\mathcal{K}$. This poses a challenge in that, while we can parameterize both operators, the resulting expression for $\dot{V}(z)$ is bilinear in $\mathcal{K}$ and $\mathcal{P}$ and is hence non-convex. To illustrate this, consider a system with input $u(t)$ described by the abstract differential equation

$$\dot{z} = \mathcal{A}z + \mathcal{F}u. \tag{7}$$

If we want to design a linear feedback control in the form of

$$u = \mathcal{K}z, \tag{8}$$

such that the closed-loop system is stable, and use the Lyapunov-Krasovskii functional given in (6), then its derivative becomes

$$\dot{V}(z) =< z, (\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{F}\mathcal{K} + (\mathcal{P}\mathcal{F}\mathcal{K})^*)z > . \tag{9}$$

Because we need to determine the feedback gain $\mathcal{K}$ in addition to the operator $\mathcal{P}$, $\dot{V}(z)$ becomes a bilinear function of the variables which define these operators. Since no reliable and efficient numerical method to solve bilinear matrix inequalities is currently available, we conclude that any solution to the controller synthesis problem must involve a reformulation or change of variables.

The approach we describe in this chapter is to use the transformation of variables

$$\mathcal{Q} = \mathcal{P}^{-1}, \tag{10}$$

$$\hat{\mathcal{K}} = \mathcal{K}\mathcal{P}^{-1}. \tag{11}$$

Clearly, given coercive $\mathcal{Q} > 0$ and $\hat{\mathcal{K}}$, and a procedure for finding the inverse $\mathcal{Q}^{-1}$, we can recover the original operators $\mathcal{P}$ and $\mathcal{K}$ as $\mathcal{P} = \mathcal{Q}^{-1}$ and $\mathcal{K} = \hat{\mathcal{K}}\mathcal{P}$. Now, by examination of V and $\dot{V}$ and by defining the transformed state $\hat{z} = \mathcal{P}z$, we obtain

$$V(z) = \langle \hat{z}, \mathcal{Q}\hat{z} \rangle, \tag{12}$$

$$\dot{V}(z) = \langle \hat{z}, (\mathcal{A}\mathcal{Q} + \mathcal{Q}\mathcal{A}^* + \mathcal{F}\hat{\mathcal{K}} + \hat{\mathcal{K}}^*\mathcal{F}^*)\hat{z} \rangle, \tag{13}$$

which are linear with respect to the new operator-variables $\mathcal{Q}$ and $\hat{\mathcal{K}}$. Therefore, if $\mathcal{P} : Z \to Z$ and we can parameterize operators which are positive on Z, then the controller synthesis problem can be represented as an SDP.

Critical to implementation of this approach, however, is to enforce the condition $\mathcal{P} : Z \to Z$ and the ability to invert the bounded linear operator $\mathcal{Q}$. While inversion of combined multiplier and integral operators is, in general, difficult, in the following sections we will show that it is possible to obtain an analytic expression for this inverse in the case where $\mathcal{Q}$ is separable, similar to the case discussed in [11]. Of course, numerical approximations to the inverse are possible through series expansion methods, as described in [10]. However, the existence of a closed-form analytic expression eliminates the approximation error due to finite truncation of the series and significantly reduces the complexity of the resulting inverse operator.

2 Preliminaries

Consider the coupled differential functional equations given in (1) and (2). If $\rho(D) < 1$, stability of this system is equivalent to the existence of a complete Lyapunov-Krasovskii functional of the following form,

$$V(\psi,\phi)=r\psi^TP\psi+2r\psi^T\int_{-r}^{0}Q(\eta)\phi(\eta)d\eta+\int_{-r}^{0}\int_{-r}^{0}\phi^T(\xi)R(\xi,\eta)\phi(\eta)d\xi d\eta$$
$$+\int_{-r}^{0}\phi^T(\eta)S(\eta)\phi(\eta)d\eta, \tag{14}$$

where

$$P=P^T\in\mathbb{R}^{n\times n},\qquad Q(\eta)\in\mathbb{R}^{n\times m}, \tag{15}$$
$$R(\xi,\eta)=R^T(\eta,\xi)\in\mathbb{R}^{m\times m},\qquad S(\eta)=S^T(\eta)\in\mathbb{S}^n, \tag{16}$$

where $\mathbb{S}^n\subset\mathbb{R}^{n\times n}$ represents the set of symmetric matrices. The use of complete Lyapunov-Krasovskii functionals of this form was described in [6, 8], and these results imply the following lemma.

Lemma 1 *System (1)–(2) with $\rho(D)<1$ is exponentially stable if and only if there exists a quadratic Lyapunov-Krasovskii functional of the Form (14)–(16), such that $\epsilon||\psi||^2\le V(\psi,\phi)$, and*

$$\dot V(\psi,\phi)\triangleq\limsup_{t\to0^+}\frac{V(x(t,\psi,\phi),y_t(\psi,\phi))-V(\psi,\phi)}{t} \tag{17}$$

satisfies $\dot V(\psi,\phi)\le-\epsilon||\psi||^2$ for some $\epsilon>0$, where $(x(t,\psi,\phi),y_t(\psi,\phi))$ is the solution of (1) and (2) with initial condition (ψ,ϕ).

Define the inner product on Z (Recall Z is defined in (3)),

$$\left\langle\begin{bmatrix}\psi_1\\\phi_1\end{bmatrix},\begin{bmatrix}\psi_2\\\phi_2\end{bmatrix}\right\rangle=r\psi_1^T\psi_2+\int_{-r}^{0}\phi_1^T(s)\phi_2(s)ds.$$

For a matrix P and matrix functions Q,R,S that satisfy (15)–(16), we define the linear operator $\mathcal{P}:Z\to Z$ as follows

$$\mathcal{P}\begin{bmatrix}\psi\\\phi\end{bmatrix}(s)=\begin{bmatrix}P\psi+\int_{-r}^{0}Q(\theta)\phi(\theta)d\theta\\ rQ^T(s)\psi+\int_{-r}^{0}R(s,\theta)\phi(\theta)d\theta+S(s)\phi(s)\end{bmatrix}. \tag{18}$$

Obviously, (15)–(16) implies that $\mathcal{P}$ is a bounded and self-adjoint linear operator. The complete Lyapunov-Krasovskii functional may now be expressed as

$$V(\psi,\phi)=\left\langle\begin{bmatrix}\psi\\\phi\end{bmatrix},\mathcal{P}\begin{bmatrix}\psi\\\phi\end{bmatrix}\right\rangle.$$

System (1)–(2) defines a C_0-semigroup $\mathcal{S}:Z\to Z$ that satisfies (4). The corresponding abstract differential equation is (5). Let the domain of definition of $\mathcal{A}$ be X. Then, $X:=\left\{\begin{bmatrix}\psi\\\phi\end{bmatrix}\in Z\,\middle|\,\dot\phi(s)\in C,\phi(0)=C\psi+D\phi(-r)\right\}$, where C represents the set of continuous functions. As is standard for abstract differential equation (5), the domain of definition X for $\mathcal{A}$ is only a subset of the solution space Z. Similar to [7], the stability in the more restricted space X implies the stability in larger space Z. For controller synthesis, we would like to restrict $\mathcal{P}$ so that X is an invariant subspace of $\mathcal{P}$,

$$\mathcal{P}X \subset X. \tag{19}$$

The conditions for $\mathcal{P}$ to satisfy (19) are as follows, which is a generalization of Theorem 3 in [9].

Lemma 2 *$\mathcal{P}$, as defined in (18), satisfies (19) if and only if the following conditions are satisfied,*

$$rQ^T(0) + S(0)C = CP + rDQ^T(-r), \tag{20}$$

$$R(0,s) = CQ(s) + DR(-r,s), \quad \forall s, \tag{21}$$

$$DS(-r) = S(0)D. \tag{22}$$

Proof Let $h(s) = rQ^T(s)\psi + \int_{-r}^{0} R(s,\theta)\phi(\theta)d\theta + S(s)\phi(s)$ and $g = P\psi + \int_{-r}^{0} Q(s)\phi(s)ds$. Then, $\mathcal{P}X \subset X$ is equivalent to

$$h(0) = Cg + Dh(-r), \tag{23}$$

for all ψ and ϕ that satisfy

$$\phi(0) = C\psi + D\phi(-r). \tag{24}$$

Using (24), we have

$$\begin{aligned} h(0) &= rQ^T(0)\psi + S(0)\phi(0) + \int_{-r}^{0} R(0,\theta)\phi(\theta)d\theta \\ &= (rQ^T(0) + S(0)C)\psi + \int_{-r}^{0} R(0,\theta)\phi(\theta)d\theta + S(0)D\phi(-r), \end{aligned} \tag{25}$$

$$\begin{aligned} Cg + Dh(-r) &= CP\psi + \int_{-r}^{0} CQ(s)\phi(s)ds + rDQ^T(-r)\psi + \int_{-r}^{0} DR(-r,s)\phi(s)ds \\ &\quad + DS(-r)\phi(-r) \\ &= (CP + rDQ^T(-r))\psi + \int_{-r}^{0} (CQ(s) + DR(-r,s))\phi(s)ds \\ &\quad + DS(-r)\phi(-r). \end{aligned} \tag{26}$$

Therefore, the right-hand sides of (25)–(26) are equal for arbitrary ψ and ϕ if and only if (20)–(22) are satisfied. ■

3 Inverting Separable Operators

In this section, we present an analytical expression for the inverse of the operator $\mathcal{P}$ when it is separable. Similar to [10], such an analytic expression for the inverse operator can be used to expedite the construction of the stabilizing controller in the controller synthesis problem.

Definition 1 An operator $\mathcal{P}$, as defined in (18), is said to be separable if

$$R(s,\theta) = Z^T(s)\Gamma Z(\theta), \qquad Q(s) = HZ(s), \tag{27}$$

for some constant matrices $\Gamma = \Gamma^T$ and H, and matrix-valued function $Z(s)$. Note that a sufficient condition for $\mathcal{P}$ to be separable is that R and Q are polynomials.

Theorem 1 *Assume $\mathcal{P}$ in (18) is separable. Then, provided that all the inverse matrices below are well defined, its inverse may be expressed as*

$$\mathcal{P}^{-1}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) = \begin{bmatrix}\hat{P}\psi + \int_{-r}^{0}\hat{Q}(\theta)\phi(\theta)d\theta\\ r\hat{Q}^T(s)\psi + \hat{S}(s)\phi(s) + \int_{-r}^{0}\hat{R}(s,\theta)\phi(\theta)d\theta\end{bmatrix}, \tag{28}$$

where $\hat{R}(s,\theta)$, $\hat{Q}(\theta)$ and $\hat{S}(s)$ are given as follows

$$\hat{R}(s,\theta) = \hat{Z}^T(s)\hat{\Gamma}\hat{Z}(\theta), \tag{29}$$

$$\hat{Q}(\theta) = \hat{H}\hat{Z}(\theta), \qquad \hat{S}(s) = S^{-1}(s), \qquad \hat{Z}(s) = Z(s)S^{-1}(s), \tag{30}$$

and $\hat{H}$, $\hat{P}$ and $\hat{\Gamma}$ are given below,

$$\hat{H} = -P^{-1}HT, \qquad \hat{P} = [I + rP^{-1}HTKH^T]P^{-1}, \tag{31}$$

$$\hat{\Gamma} = [rT^TH^TP^{-1}H - \Gamma](I + K\Gamma)^{-1}, \qquad T = (I + K\Gamma - rKH^TP^{-1}H)^{-1}, \tag{32}$$

where $K = \int_{-r}^{0} Z(s)S^{-1}(s)Z^T(s)ds$, and I denotes the identity matrix with appropriate dimension.

Proof Let the operator defined by the right hand side of (28) be denoted as $\hat{\mathcal{P}}$, then

$$\begin{aligned}
&\hat{\mathcal{P}}\mathcal{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) = \begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix},\\
\Lambda_1 &= \int_{-r}^{0}\left(\hat{P}Q(\theta) + \hat{Q}(\theta)S(\theta) + \int_{-r}^{0}\hat{Q}(\xi)R(\xi,\theta)d\xi\right)\phi(\theta)d\theta\\
&\quad + \left(\hat{P}P + \int_{-r}^{0} r\hat{Q}(\theta)Q^T(\theta)d\theta\right)\psi,\\
\Lambda_2 &= r\left(\hat{Q}^T(s)P + \hat{S}(s)Q^T(s) + \int_{-r}^{0}\hat{R}(s,\theta)Q^T(\theta)d\theta\right)\psi + \hat{S}(s)S(s)\phi(s)\\
&\quad + \int_{-r}^{0}\left(r\hat{Q}^T(s)Q(\theta) + \hat{S}(s)R(s,\theta) + \hat{R}(s,\theta)S(\theta)\right.\\
&\qquad \left. + \int_{-r}^{0}\hat{R}(s,\xi)R(\xi,\theta)d\xi\right)\phi(\theta)d\theta.
\end{aligned}$$

Using (27) and (29)–(32), we obtain

$$
\begin{aligned}
&\hat{P}Q(\theta) + \hat{Q}(\theta)S(\theta) + \int_{-r}^{0} \hat{Q}(\xi)R(\xi,\theta)d\xi \\
&\quad = \left(\hat{P}H + \hat{H} + \hat{H}K\Gamma\right)Z(\theta) \\
&\quad = \left([I + rP^{-1}HTKH^T]P^{-1}H - P^{-1}HT - P^{-1}HTK\Gamma\right)Z(\theta) \\
&\quad = \left[P^{-1}H + P^{-1}HT(rKH^TP^{-1}H - I - K\Gamma)\right]Z(\theta) \\
&\quad = \left(P^{-1}H - P^{-1}H\right)Z(\theta) = 0, \\
&\hat{P}P + r\int_{-r}^{0} \hat{Q}(\theta)Q^T(\theta)d\theta \\
&\quad = \hat{P}P + r\hat{H}KH^T \\
&\quad = [I + rP^{-1}H(I + K\Gamma - rKH^TP^{-1}H)^{-1}KH^T] - rP^{-1}HTKH^T = I, \\
&\hat{Q}^T(s)P + \hat{S}(s)Q^T(s) + \int_{-r}^{0} \hat{R}(s,\theta)Q^T(\theta)d\theta \\
&\quad = \hat{Z}^T(s)(\hat{H}^TP + H^T + \hat{\Gamma}KH^T) \\
&\quad = \hat{Z}^T(s)\left\{-T^TH^TP^{-1}P + H^T + [rT^TH^TP^{-1}H - \Gamma](I + K\Gamma)^{-1}KH^T\right\} \\
&\quad = \hat{Z}^T(s)\left\{I - T^T(I - rH^TP^{-1}H(I + K\Gamma)^{-1}K) - \Gamma(I + K\Gamma)^{-1}K\right\}H^T \\
&\quad = \hat{Z}^T(s)\left\{I - T^T(I - rH^TP^{-1}HK(I + \Gamma K)^{-1}) - \Gamma K(I + \Gamma K)^{-1}\right\}H^T \\
&\quad = \hat{Z}^T(s)\left\{I - T^T(I + \Gamma K - rH^TP^{-1}HK)(I + \Gamma K)^{-1} - \Gamma K(I + \Gamma K)^{-1}\right\}H^T \\
&\quad = \hat{Z}^T(s)[I - (I + \Gamma K)^{-1} - \Gamma K(I + \Gamma K)^{-1}]H^T = 0, \\
&r\hat{Q}^T(s)Q(\theta) + \hat{S}(s)R(s,\theta) + \hat{R}(s,\theta)S(\theta) + \int_{-r}^{0} \hat{R}(s,\xi)R(\xi,\theta)d\xi \\
&\quad = \hat{Z}^T(s)(r\hat{H}^TH + \Gamma + \hat{\Gamma} + \hat{\Gamma}K\Gamma)Z(\theta) \\
&\quad = \hat{Z}^T(s)\left\{-rT^TH^TP^{-1}H + \Gamma + [rT^TH^TP^{-1}H - \Gamma](I + K\Gamma)^{-1}(I + K\Gamma)\right\}Z(\theta) \\
&\quad = \hat{Z}^T(s)\left(-rT^TH^TP^{-1}H + \Gamma + rT^TH^TP^{-1}H - \Gamma\right)Z(\theta) = 0.
\end{aligned}
$$

Thus, we have shown

$$\hat{\mathcal{P}}\mathcal{P}\begin{bmatrix}\psi \\ \phi\end{bmatrix} = \begin{bmatrix}\psi \\ \phi\end{bmatrix}, \tag{33}$$

for all $\begin{bmatrix}\psi \\ \phi\end{bmatrix} \in Z$. Similarly, we can show

$$\mathcal{P}\hat{\mathcal{P}}\begin{bmatrix}\psi \\ \phi\end{bmatrix} = \begin{bmatrix}\psi \\ \phi\end{bmatrix}. \tag{34}$$

From (33) and (34), we conclude that $\hat{\mathcal{P}} = \mathcal{P}^{-1}$. ■

Theorem 2 *If the separable operator* $\mathcal{P}$ *satisfies* $\mathcal{P}X \subset X$ *and if its inverse is well defined, Then,* $\mathcal{P}^{-1}X \subset X$ *holds.*

Proof Let the linear operator $\mathcal{P}$ satisfy $\mathcal{P}X \subset X$. By Lemma 2, this is equivalent to (20)–(22), from which, we obtain

$$CP^{-1} = rS^{-1}(0)(DZ^T(-r) - Z^T(0))H^T P^{-1} + S^{-1}(0)C, \tag{35}$$
$$CH = (Z^T(0) - DZ^T(-r))\Gamma, \tag{36}$$
$$S^{-1}(0)D = DS^{-1}(-r). \tag{37}$$

Applying (35)–(37) to the operator $\mathcal{P}^{-1}$ defined in (28), after tedious calculations, we can obtain the following equation,

$$r\hat{Q}^T(0)\psi + \hat{S}(0)\phi(0) + \int_{-r}^{0} \hat{R}(0,\theta)\phi(\theta)d\theta = C\left(\hat{P}\psi + \int_{-r}^{0} \hat{Q}(\theta)\phi(\theta)d\theta\right)$$
$$+D\left(r\hat{Q}^T(-r)\psi + \hat{S}(-r)\phi(-r) + \int_{-r}^{0} \hat{R}(-r,\theta)\phi(\theta)d\theta\right),$$

from which, we conclude that $\mathcal{P}^{-1}X \subset X$. ■

4 Controller Synthesis

In this section, we consider a system with control input as follows

$$\dot{x}(t) = Ax(t) + By(t-r) + Fu(t), \tag{38}$$
$$y(t) = Cx(t) + Dy(t-r). \tag{39}$$

For this system, let us define the infinitesimal generator $\mathcal{A}$ as follows

$$\left(\mathcal{A}\begin{bmatrix} x \\ y_t \end{bmatrix}\right)(s) = \begin{bmatrix} Ax + By(t-r) \\ \frac{d}{ds}y_t(s) \end{bmatrix}.$$

Likewise, we define the input operator $\mathcal{F} : \mathbb{R}^q \to X$ as

$$(\mathcal{F}u)(s) := \begin{bmatrix} Fu \\ 0 \end{bmatrix}.$$

We define the controller synthesis problem as the search for matrices K_0, K_1 and matrix-valued function $K_2(s)$ such that the closed-loop system described by (38), (39) and

$$u(t) = \mathcal{K}\begin{bmatrix} x(t) \\ y_t \end{bmatrix} \tag{40}$$

is stable. In (40), $\mathcal{K} : X \to \mathbb{R}^q$ is defined as

$$\left(\mathcal{K}\begin{bmatrix} x \\ y_t \end{bmatrix}\right)(s) = K_0 x(t) + K_1 y(t-r) + \int_{-r}^{0} K_2(s) y(t+s) ds. \tag{41}$$

Before we give the main result of the section, we briefly address SOS methods for enforcing joint positivity of coupled multiplier and integral operators using positive matrices. These methods have been developed in a series of papers, a summary of which can be found in the survey paper [11]. Specifically, for matrix-valued functions $M(s)$ and $N(s,\theta)$, we say that$\{M, N\} \in \Xi$, if M and N satisfy the conditions of Theorem 8 in [11]. The constraint $\{M, N\} \in \Xi$ can be cast as an LMI using SOSTOOLS as described in [11] and this constraint ensures that the operator $\mathcal{P}$, defined as

$$\left(\mathcal{P}\begin{bmatrix} \psi \\ \phi \end{bmatrix}\right)(s) = \begin{bmatrix} M_{11}\psi + \int_{-r}^{0} M_{12}(\theta)\phi(\theta)d\theta \\ rM_{21}^T(s)\psi + \int_{-r}^{0} N(s,\theta)\phi(\theta)d\theta + M_{22}(s)\phi(s) \end{bmatrix},$$

is positive on X. Furthermore, we note that $\{M, N\} \in \Xi$ implies that $\mathcal{P}$ is separable and $P = \int M_{11}(s)ds$ and $S = M_{22}$ are invertible. We now state the main result.

Proposition 1 *Suppose there exist matrices M_0, M_1, $P = P^T$, matrix-valued functions $M_2(s)$, $Q(s)$, $R(s,\theta)$, $S(s) = S^T(s) \in \mathbb{S}^n$, and scalar $\epsilon > 0$ such that (20)–(22) are satisfied and the following conditions hold*

$$\{T, R\} \in \Xi, \tag{42}$$

$$\{-U, -W\} \in \Xi, \tag{43}$$

where

$$T(s) = \begin{bmatrix} P & rQ(s) \\ rQ^T(s) & S(s) \end{bmatrix} - \epsilon I, \tag{44}$$

$$U(s) = \begin{bmatrix} \Gamma + \epsilon I & BS(-r) + FM_1 + \frac{1}{r}C^T S(0) D & \Upsilon \\ * & \frac{-1}{r}(S(-r) - D^T S(0) D) & 0 \\ * & * & \dot{S}(s) \end{bmatrix}, \tag{45}$$

$$W(s,\theta) = \frac{\partial}{\partial s} R(s,\theta) + \frac{\partial}{\partial \theta} R(s,\theta), \tag{46}$$

where $*$ denotes entries in the matrix determined by symmetry,

$$\Gamma = AP + PA^T + r(BQ^T(-r) + Q(-r)B^T) + \frac{1}{r}C^T S(0) C + FM_0 + M_0^T F,$$
$$\Upsilon = r[\dot{Q}(s) + BR(-r,s) + AQ(s) + FM_2(s)].$$

Then System (38)–(39) is stabilizable with a controller (40). In other words, let

$$u(t) = K_0 x(t) + K_1 y(t-r) + \int_{-r}^{0} K_2(s) y(t+s) ds, \tag{47}$$

where

$$K_0 = M_0 \hat{P} + r M_1 \hat{Q}^T(-r) + r \int_{-r}^{0} M_2(s) \hat{Q}^T(s) ds, \tag{48}$$

$$K_1 = M_1 \hat{S}(-r), \tag{49}$$

$$K_2(s) = M_0 \hat{Q}(s) + M_1 \hat{R}(-r, s) + M_2(s) \hat{S}(s) + \int_{-r}^{0} M_2(\theta) \hat{R}(\theta, s) d\theta, \tag{50}$$

and $\hat{P}$, $\hat{Q}$, $\hat{R}$ and $\hat{S}$ are defined in Theorem 1. Then the closed-loop System (38)–(39) and (47) is stable.

Proof Define $\mathcal{P}$ by (18). Then $\mathcal{P}$ is bounded and self-adjoint. Per Lemma 2, $\mathcal{P}X \subset X$. (42) implies $\mathcal{P} \geq \epsilon I$. Per Theorem 1, the inverse $\mathcal{P}^{-1}$ can be expressed as in (28) and is likewise bounded and coercive with $\mathcal{P}^{-1} \geq \epsilon' I$. Furthermore, from Theorem 2, $\mathcal{P}^{-1} X \subset X$ and $\mathcal{P}^{-1} = \mathcal{P}^{-*}$. In other words, the Lyapunov-Krasovskii functional satisfies

$$V = \left\langle \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathcal{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle \geq \epsilon' \left\| \begin{matrix} \psi \\ \phi \end{matrix} \right\|^2 \tag{51}$$

for some $\epsilon' > 0$ and all $\begin{bmatrix} \psi \\ \phi \end{bmatrix} \in X$. Furthermore,

$$\begin{aligned} &\left\langle \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathcal{P}^{-1} \mathcal{A} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle + \left\langle \mathcal{A} \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathcal{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle \\ &= \left\langle \mathcal{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathcal{A}\mathcal{P}\mathcal{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle + \left\langle \mathcal{A}\mathcal{P}\mathcal{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathcal{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle. \end{aligned}$$

Next, we note that if we define $\mathcal{K}$ as in (41) and $\mathcal{M}$ as follows

$$\left(\mathcal{M} \begin{bmatrix} x \\ y_t \end{bmatrix} \right) = M_0 x(t) + M_1 y(t-r) + \int_{-r}^{0} M_2(s) y(t+s) ds.$$

Then (48)–(50) implies $\mathcal{K} := \mathcal{M}\mathcal{P}^{-1}$. Now we define a new state $\begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} = \mathcal{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix}$ $\in X$. Continuing, if $u = \mathcal{K} \begin{bmatrix} x \\ y_t \end{bmatrix} = \mathcal{K}\mathcal{P}\mathcal{P}^{-1} \begin{bmatrix} x \\ y_t \end{bmatrix} = \mathcal{M}\mathcal{P}^{-1} \begin{bmatrix} x \\ y_t \end{bmatrix}$, then the closed-loop system is stable if $\dot{V} < 0$, where

$$\dot{V} = \left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathcal{AP} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \mathcal{AP} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \mathcal{FM} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathcal{FM} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle.$$

To show that $\dot{V} < 0$, we examine $\mathcal{AP}$ and $\mathcal{FM}$ separately. First, we have

$$\mathcal{AP} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}(s) = \begin{bmatrix} \Psi \\ \Phi(s) \end{bmatrix}, \tag{52}$$

where

$$\Psi = AP\hat{\psi} + \int_{-r}^{0} AQ(s)\hat{\phi}(s)ds + BrQ^T(-r)\hat{\psi} + BS(-r)\hat{\phi}(-r) + \int_{-r}^{0} BR(-r,\theta)\hat{\phi}(\theta)d\theta,$$
$$\Phi(s) = r\dot{Q}^T(s)\hat{\psi} + \dot{S}(s)\hat{\phi}(s) + S(s)\dot{\hat{\phi}}(s) + \int_{-r}^{0} \tfrac{d}{ds}R(s,\theta)\hat{\phi}(\theta)d\theta.$$

Then,

$$\begin{aligned}
&\left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathcal{AP} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle \\
&= \int_{-r}^{0} \hat{\psi}^T \Psi ds + \int_{-r}^{0} \hat{\phi}^T(s)\Phi(s)ds \\
&= r\hat{\psi}^T AP\hat{\psi} + r\int_{-r}^{0} \hat{\psi}^T AQ(s)\hat{\phi}(s)ds + r\hat{\psi}^T BrQ^T(-r)\hat{\psi} + r\hat{\psi}^T BS(-r)\hat{\phi}(-r) \\
&\quad + r\int_{-r}^{0} \hat{\psi}^T BR(-r,\theta)\hat{\phi}(\theta)d\theta + \int_{-r}^{0} r\hat{\phi}^T(s)\dot{Q}^T(s)\hat{\psi}ds + \int_{-r}^{0} \hat{\phi}^T(s)\dot{S}(s)\hat{\phi}(s)ds \\
&\quad + \int_{-r}^{0}\int_{-r}^{0} \hat{\phi}^T(s)\tfrac{d}{ds}R(s,\theta)\hat{\phi}(\theta)dsd\theta + \int_{-r}^{0} \hat{\phi}^T(s)S(s)\dot{\hat{\phi}}(s)ds \\
&= \int_{-r}^{0} \begin{bmatrix} \hat{\psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} AP + rBQ^T(-r) & BS(-r) & \Theta \\ 0 & 0 & 0 \\ r\dot{Q}^T(s) & 0 & \dot{S}(s) \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix} ds \\
&\quad + \int_{-r}^{0}\int_{-r}^{0} \hat{\phi}^T(s)\tfrac{\partial}{\partial s}R(s,\theta)\hat{\phi}(\theta)dsd\theta + \int_{-r}^{0} \hat{\phi}^T(s)S(s)\dot{\hat{\phi}}(s)ds,
\end{aligned}$$

where $\Theta = r(AQ(s) + BR(-r,s))$. Since $\begin{bmatrix} \hat{\psi} \\ \hat{\phi}(s) \end{bmatrix} \in X$, we have $\hat{\phi}(0) = C\hat{\psi} + D\hat{\phi}(-r)$. Then,

$$\begin{aligned}
&\int_{-r}^{0} \hat{\phi}^T(s)S(s)\dot{\hat{\phi}}(s)ds \\
&= \hat{\phi}^T(0)S(0)\hat{\phi}(0) - \hat{\phi}^T(-r)S(-r)\hat{\phi}(-r) - \int_{-r}^{0} \hat{\phi}^T(s)\dot{S}(s)\hat{\phi}(s)ds - \int_{-r}^{0} \dot{\hat{\phi}}^T(s)S(s)\hat{\phi}(s)ds \\
&= \tfrac{1}{2}\left(\hat{\phi}^T(0)S(0)\hat{\phi}(0) - \hat{\phi}^T(-r)S(-r)\hat{\phi}(-r)\right) - \tfrac{1}{2}\int_{-r}^{0} \hat{\phi}^T(s)\dot{S}(s)\hat{\phi}(s)ds \\
&= \tfrac{1}{2}\int_{-r}^{0} \begin{bmatrix} \hat{\psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} \frac{1}{r}C^T S(0)C & \frac{1}{r}(C^T S(0)D) & 0 \\ \frac{1}{r}(D^T S(0)C) & -\frac{1}{r}(S(-r) - D^T S(0)D) & 0 \\ 0 & 0 & -\dot{S}(s) \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix} ds.
\end{aligned}$$

Thus,

$$\dot{V} = \left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathcal{AP} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \mathcal{AP} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle$$
$$+ \left\langle \mathcal{FM} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathcal{FM} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle.$$
$$= \int_{-r}^{0} \begin{bmatrix} \hat{\psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} \Gamma & BS(-r) + FM_1 + \frac{1}{r}(C^T S(0) D) & \Upsilon \\ * & -\frac{1}{r}(S(-r) - D^T S(0) D) & 0 \\ * & * & \dot{S}(s) \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix} ds$$
$$+ \int_{-r}^{0} \int_{-r}^{0} \hat{\phi}^T(s) \left(\tfrac{\partial}{\partial s} R(s,\theta) + \tfrac{\partial}{\partial \theta} R(s,\theta) \right) \hat{\phi}(\theta) ds d\theta.$$

From conditions (43), (45) and (46), we have $\dot{V} < 0$, which, along with (51) means that the closed-loop system defined by (38)–(39) and (47) is stable. ■

Remark 1 When $D = 0$, System (38)–(39) may be written in the standard delay-differential framework studied in [9, 10]:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - r_1) + F u(t)$$
$$x(t) = \phi(t).$$

The primary computational advantage of the differential-difference framework over control of System (38)–(39) is that we can replace $A_1 \in \mathbb{R}^{n \times n}$ with BC where $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times n}$ and m is typically strictly less than n. Because the dimension of the decision variables in the optimization problem defined in this paper scale as $n + 2m$ as opposed to $3n$ using the framework in [9, 10], the complexity of the resulting algorithm is significantly reduced.

Remark 2 Although not explicitly stated, in order to use SOS to enforce the conditions of Theorem 1 and Proposition 1, we choose our decision variables to be polynomial and use SOSTOOLS and the Positivstellensatz to enforce positivity/negativity on the interval $[-r, 0]$. This approach is described in more detail in [9, 10].

In the following, we present a numerical example to illustrate the controller obtained from the condition in Proposition 1. We consider the following system with a feedback controller as follows

$$\dot{x}(t) = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.9 \end{bmatrix} x(t) + By(t - r) + Fu(t), \quad (53)$$

$$y(t) = \begin{bmatrix} -0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t), \quad (54)$$

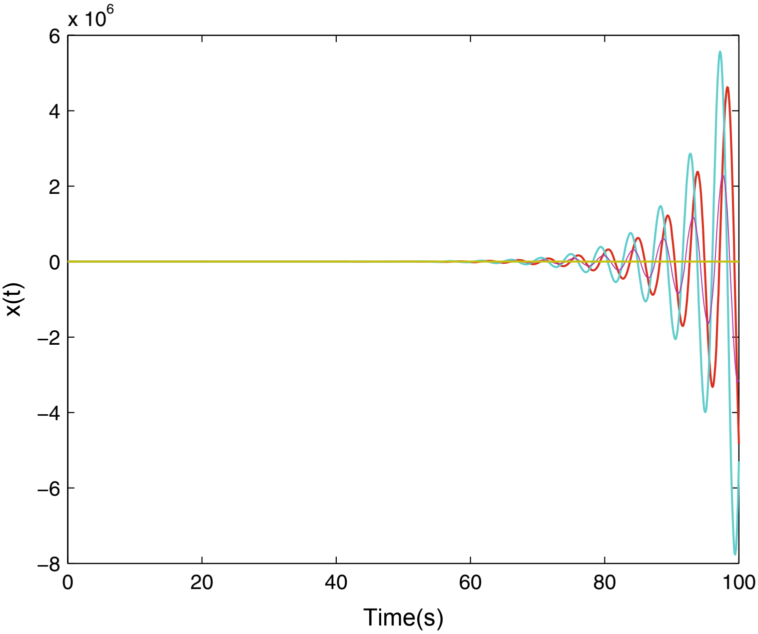

Fig. 1 System (53)–(54) is unstable in open loop

where $r = 1.6$s, $B = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$, $F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$. By using Proposition1, together with the tools of MuPad, Matlab, SOSTOOLS and polynomials with degree 2, we obtain the controller

$$u(t) = K_0 x(t) + \begin{bmatrix} -0.239 \\ -0.343 \end{bmatrix}^T y(t-r) + \int_{-1.6}^{0} K_2(s) y(t+s) ds, \qquad (55)$$

$$K_0 = \begin{bmatrix} -1.874 & 2.232 & -0.830 & 3.099 & 0.030 & -1.033 \end{bmatrix},$$

$$K_2(s) = \begin{bmatrix} -0.246 + 0.221s + 0.122s^2 - 0.012s^3 - 0.032s^4 \\ 0.238 - 0.398s + +0.007s^2 + 0.037s^3 + 0.010s^4, \end{bmatrix}^T.$$

Let $u(t) = 0$ in (53), System (53)–(54) is unstable, which is shown in Fig. 1. Using Controller (55) coupled with System (53)–(54) we simulate the closed-loop system, which is illustrated in Fig. 2.

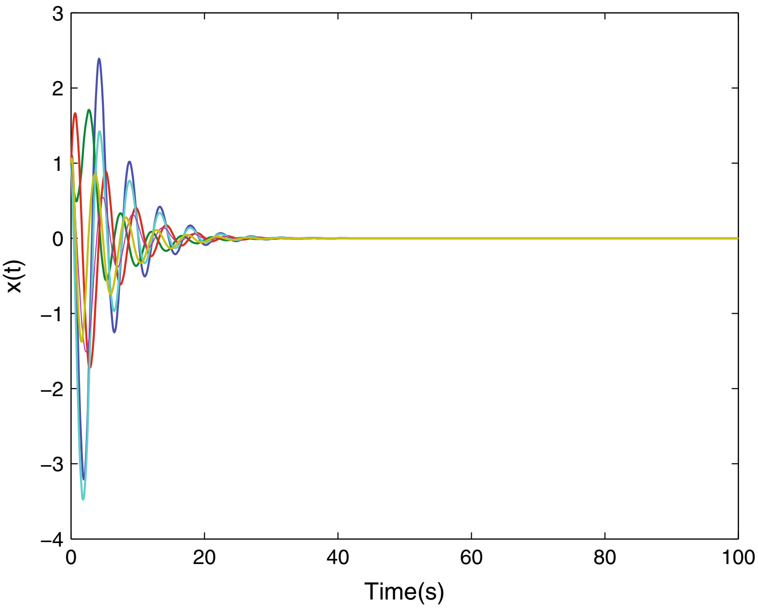

Fig. 2 States of system (53)–(54) coupled with a stabilizing controller from Proposition 1

5 Conclusions

In this chapter, we have obtained an analytic formulation for the inverse of jointly positive multiplier and integral operators as defined in [9]. This formulation has the advantage that it eliminates the need for either individual positivity of the multiplier and integral operators or the need to use a series expansion to find the inverse. This inversion formula is applied to controller synthesis of coupled differential-difference equations. The use of the coupled differential-difference formulation has the advantage that the size of the resulting decision variables is reduced, thereby allowing for control of systems with larger numbers of states. These methods are illustrated by designing a stabilizing controller for a system with 6 states and a 2-dimensional delay channels.

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Delay Margin for Robust Stabilization of LTI Delay Systems

Tian Qi, Jing Zhu and Jie Chen

1 Introduction

With a steadily growing interest, over the last two decades or so there have been significant advances in the study of time-delay systems, thanks to the development of analysis methods drawing upon robust control theory, and the development of computational methods in solving *linear matrix inequality (LMI)* problems. In particular, an extraordinary volume of the literature is in existence on stability problems, and various time- and frequency-domain stability analysis approaches have been developed (see, e.g., [8, 16, 17, 21, 25], and the references therein).

Despite the considerable advances on stability studies, stabilization of time-delay systems poses a more difficult problem. The existing work has been largely focused on synthesis problems for systems with a *fixed* delay. Feedback design for such systems can be conducted based on LQR and $\mathcal{H}_\infty$ techniques (see, e.g., [20, 30] and the references therein), via predictor feedback [14, 31], or using LMI-based solutions [6, 17]. On the other hand, fundamental robustness of stabilization in the presence of *uncertain, variable* delays has been seldom investigated. Nor is it clear how the above methods may be extended to address the robust stabilization problem.

T. Qi
School of Automation Science and Engineering, South China University of Technology, Guangzhou, China
e-mail: auqt@scut.edu.cn

J. Zhu
Collage of Automation Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, China
e-mail: drzhujing@nuaa.edu.cn

J. Chen (✉)
Department of Electronic Engineering, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong SAR, China
e-mail: jichen@cityu.edu.hk

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In this vein, particularly noteworthy is the problem of *delay margin* [18], which by nature addresses a system's robust stabilization against uncertain delays and seeks to answer the question: *What is the largest range of delay such that there exists a single feedback controller that can stabilize all the plants subject to delays within the range?* An age-old problem by itself [3, 5], this problem bears a close similarity to the gain margin and phase margin problems, which are two classical stability margin optimization problems solvable analytically by solving a finite-dimensional $\mathcal{H}_\infty$ optimal control problem [4]. Unlike the gain and phase margin however, the delay margin problem proves fundamentally more challenging, due to obstacles in solving infinite-dimensional optimization problems. Indeed, the problem has been open except in isolated cases. In [17, pp. 154], the delay margin was determined for first-order systems achievable by static feedback, while in [27], the delay margin was found for first-order systems when PID controllers are used instead. Other related results concerning stabilizability via delayed feedback can be found in, e.g., [13, 22].

In [10, 18], upper bounds on the delay margin were obtained for general SISO systems subject to an uncertain constant delay. These bounds serve to provide a limit beyond which no single LTI output feedback controller may exist to robustly stabilize the delay plant family within the margin. The results show that this fundamental limit is determined by the unstable poles and nonminimum phase zeros in the plant. In its essence, however, the work of [18] is by and large limited to systems with no more than one unstable pole and nonminimum phase zero, for which the bounds were found to be exact; otherwise, under more general circumstances, the bounds may be crude and pessimistic. Moreover, the analysis in [18] was carried out largely case by case, and for this reason, its technique does not appear readily generalizable. The same can be said of the improvement in [10].

This chapter aims at developing lower bounds on the delay margin. Unlike in [18], which addresses the question when a delay system is *not* stabilizable, we ask when it *is* stabilizable. Thus, the results provide a *guaranteed* range of delay ensuring robust stabilization. Built on small-gain stability conditions, our approach employs rational approximation of delay elements, which enables us to cast the problem as one of finite-dimensional, parameter-dependent $\mathcal{H}_\infty$ optimization; the latter may then be tackled and solved using such analytic interpolation techniques as Nevanlinna-Pick interpolation [1]. This operator-theoretic approach ensures not only that the bounds can be efficiently computed, but also that it can be cohesively extended, and indeed, in a unified manner, to more general classes of systems with more general classes of delays, e.g., systems with time-varying delays. Furthermore, since the approach amounts to solving a standard $\mathcal{H}_\infty$ control synthesis problem, it in fact yields a robustly stabilizing controller that achieves the bounds and guarantees the stabilization for all possible delay values within the bounds.

We consider LTI output feedback controllers. Our contribution is twofold. First, for a SISO system with an arbitrary number of plant unstable poles and nonminimum phase zeros, we provide an explicit bound on the delay margin, which requires computing only the largest real eigenvalue of a constant matrix. Second, we extend our analysis to systems subject to time-varying delays, which yield similar bounds. In both cases, which are unified in our interpolation approach, the results not only are computationally attractive, but shed useful conceptual insights; when specialized to more specific cases, e.g., to plants with one unstable pole and one nonminimum phase zero, they furnish analytical expressions exhibiting explicit dependence of the bounds on the pole and zero, showing how fundamentally unstable poles and nonminimum phase zeros may limit the range of delays over which a plant may be robustly stabilized by a LTI controller. It should be emphasized nonetheless that the results and conclusions presented herein address only the limitation of LTI controllers in stabilizing time-delay systems. More general controllers with varying degrees of implementation complexity, such as linear periodic controllers [19], nonlinear periodic controllers [7], and nonlinear adaptive controllers [15, 23] can be constructed to lend an infinite delay margin, allowing a LTI delay plant to be stabilized for arbitrarily long uncertain delays.

The notation used throughout this chapter is fairly standard. Let $\mathbb{R}$ be the space of real numbers, $\mathbb{R}^n$ the space of n-dimensional real vectors, and $\mathbb{R}^n_+$ the n-dimensional space of positive real numbers. For any complex number z, we denote its conjugate by $\bar{z}$. For any complex vector x, we denote its transpose by x^T and its conjugate transpose by x^H. Similarly, for any complex matrix A, A^H denotes its conjugate transpose. The largest real eigenvalue of a matrix A will be written as $\sigma_{\max}(A)$, and if A is a Hermitian matrix, its largest eigenvalue will be written as $\bar{\lambda}(A)$. We write $A \geq 0$ if A is nonnegative definite, and $A > 0$ if it is positive definite. The symbol $\otimes$ denotes the Kronecker product. Let $\mathbb{C}_- := \{s : \ \mathrm{Re}(s) < 0\}$, $\mathbb{C}_+ := \{s : \ \mathrm{Re} > 0\}$, and $\bar{\mathbb{C}}_+ := \{s : \ \mathrm{Re} \geq 0\}$ be the open left and the open right-half of the complex plane, and the closed right-half of the complex plane, respectively. For any stable transfer function matrix $G(s)$, define its $\mathcal{H}_\infty$ norm by $\|G(s)\|_\infty = \sup_\omega \bar{\sigma}\left(G(j\omega)\right)$, where $\bar{\sigma}(\cdot)$ stands for the largest singular value. For any unitary vectors $u, v \in \mathbb{C}^n$, we denote the principal angle between the directions spanned by u and v as $\cos \angle(u, v) = \left|u^H v\right|$.

We note that subsequent to this chapter, an extended version reporting the results herein has appeared in [24], which develops in full the approach with extensions to multi-input multi-output delay systems. We refer to [24] as well for all the proofs of the results in this chapter.

2 Bounds on Delay Margin of SISO Systems

In this section, we present the results for SISO delay systems, which consist of a general lower bound on the delay margin that amounts to computing an eigenvalue problem. We also present explicit bounds for more specialized cases, which exhibit the dependence of the delay margin on the plant's unstable poles and nonminimum phase zeros.

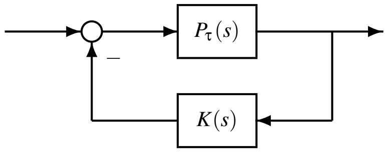

Fig. 1 Standard feedback control structure

The Delay Margin Problem

We consider the feedback control system depicted in Fig. 1, in which $P_\tau(s)$ represents a family of plants subject to an unknown delay τ, with $P_0(s)$ being the delay-free plant:

$$P_\tau(s) = e^{-\tau s} P_0(s), \quad \tau \geq 0. \tag{1}$$

Suppose that $P_0(s)$ is stabilized by a certain finite-dimensional LTI controller $K(s)$. By continuity, $K(s)$ can stabilize $P_\tau(s)$ for sufficiently small $\tau > 0$. But how large may τ be, before the system loses closed-loop stability?

The *delay margin problem* seeks to answer the above question, which amounts to computing

$$\tau^* = \sup\{\nu : \ K(s) \text{ stabilizes } P_\tau(s), \ \forall \tau \in [0, \ \nu)\}.$$

In other words, we want to determine the largest delay range within which $P_\tau(s)$ can be stabilized by a finite-dimensional LTI controller $K(s)$. Note that for $K(s)$ to stabilize $P_\tau(s)$, it is both necessary and sufficient that

$$1 + P_\tau(s)K(s) \neq 0, \quad \forall s \in \bar{\mathbb{C}}_+.$$

Under the condition that $P_0(s)$ is stabilized by $K(s)$, this condition is equivalent to

$$1 + T_0(s)\left(e^{-\tau s} - 1\right) \neq 0, \quad \forall s \in \bar{\mathbb{C}}_+, \tag{2}$$

where $T_0(s) = P_0(s)K(s)\,(1 + P_0(s)K(s))^{-1}$ is the system's complementary sensitivity function. It is clear that there exists some stabilizing $K(s)$ for all $\tau \in [0, \ \bar{\tau}]$ if

$$\sup_{\tau \in [0, \ \bar{\tau}]} \inf_{K(s)} \left\| T_0(s)(e^{-\tau s} - 1) \right\|_\infty < 1. \tag{3}$$

Define

$$\phi_{\bar{\tau}}(\omega) = \sup_{\tau \in [0, \ \bar{\tau}]} |e^{-j\omega\tau} - 1| = \begin{cases} 2\sin(\omega\bar{\tau}/2), & |\omega\bar{\tau}| \leq \pi, \\ 2, & |\omega\bar{\tau}| > \pi. \end{cases} \tag{4}$$

Evidently, the condition (3) holds whenever

$$\inf_{K(s)} |T_0(j\omega)\phi_{\bar{\tau}}(\omega)| < 1, \quad \forall \omega \in \mathbb{R}. \tag{5}$$

Unfortunately, the problems in (3), (5) and the delay margin problem itself all pose a formidable challenge, for they all require solving infinite-dimensional optimization problems due to the presence of the weighting function $\left(e^{-\tau s}-1\right)$.

One instrumental step in our approach is to construct a parameter-dependent rational approximation

$$w_\tau(s)=\frac{b_\tau(s)}{a_\tau(s)}=\frac{b_q(\tau s)^q+\cdots+b_1(\tau s)+b_0}{a_q(\tau s)^q+\cdots+a_1(\tau s)+a_0}, \tag{6}$$

such that

$$\phi_\tau(\omega)\le |w_\tau(j\omega)|\,,\quad \forall\omega\in\mathbb{R}. \tag{7}$$

We require that $w_\tau(s)$ be stable and have no nonminimum phase zero, excluding the origin where $w_\tau(s)$ might have a zero, that is $w_\tau(0)=0$. This latter condition may be imposed to ensure a close-fit of $|w_\tau(j\omega)|$ to $\phi_\tau(\omega)$ at low frequencies. Note that under this requirement, with no loss of generality, it is necessary that $a_i>0$ for $i=0,1,\ldots,q$, and $b_i>0$ for $i=1,\ \ldots,\ q$. Some of specific, low-order approximants in this spirit can be found in, e.g., [9, 24, 28]:

$$w_{1\tau}(s)=\tau s, \tag{8}$$

$$w_{2\tau}(s)=\frac{\tau s}{1+\tau s/3.465}, \tag{9}$$

$$w_{3\tau}(s)=\frac{1.216\tau s}{1+\tau s/2}, \tag{10}$$

$$w_{4\tau}(s)=\frac{\tau s(2\times 0.2152^2\tau s+1)}{(0.2152\tau s+1)^2}, \tag{11}$$

$$w_{5\tau}(s)=\frac{\tau s}{1+\tau s/2}\frac{0.1791(\tau s)^2+0.7093\tau s+1}{0.1791(\tau s)^2+0.5798\tau s+1}, \tag{12}$$

$$w_{6\tau}(s)=\frac{\tau s}{1+\tau s/2}\frac{0.02952(\tau s)^4+0.210172(\tau s)^3+0.70763(\tau s)^2+1.3188\tau s+1}{0.02952(\tau s)^4+0.191784(\tau s)^3+0.64174(\tau s)^2+1.195282\tau s+1}. \tag{13}$$

Note that $w_{6\tau}(s)$, with a highest order, betters all other $w_{i\tau}(s)$, for $i=1,\ldots,5$. Figure 2 shows the magnitude responses of these rational functions.

With the rational approximant alluded to above, we may then attempt to compute

$$\underline{\tau}=\sup\left\{\tau\ge 0:\ \inf_{K(s)}\|T_0(s)w_\tau(s)\|_\infty<1\right\}, \tag{14}$$

which, unlike in (5), amounts to solving a finite-dimensional $\mathcal{H}_\infty$ optimal control problem, nonetheless parameterized by a nonnegative parameter $\tau\ge 0$; for a different

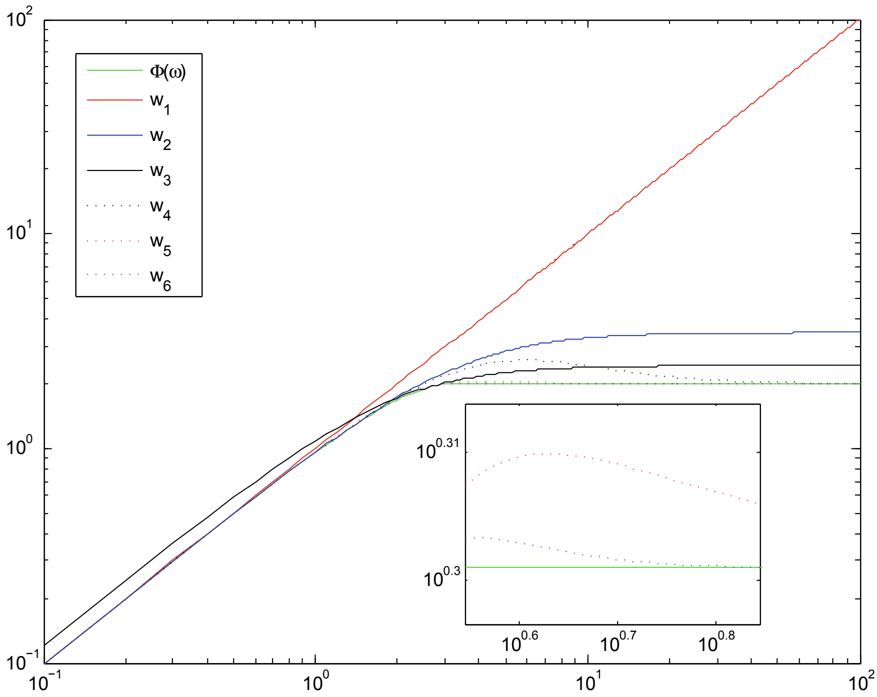

Fig. 2 Rational approximation for $\phi(\omega)$

$w_{i\tau}(s)$, a corresponding $\mathcal{H}_\infty$ problem is solved, resulting in a different $\underline{\tau}_i$. Clearly, the condition (5) holds for $\phi_{\underline{\tau}}(\omega)$ whenever

$$\inf_{K(s)} \left\| T_0(s) w_{\underline{\tau}}(s) \right\|_\infty < 1. \tag{15}$$

Note that $\phi_\tau(\omega)$ is monotonically increasing with $\tau \geq 0$ within the range of $0 \leq \omega\tau \leq \pi$. As such, $\underline{\tau}$ serves as a lower bound on the delay margin τ^*, and in turn provides a range guaranteeing the stabilizability of $P_\tau(s)$: there exists a controller $K(s)$ that can stabilize $P_\tau(s)$ for all $\tau \in [0,\ \underline{\tau})$.

A Computational Formula

We compute the lower bound $\underline{\tau}$ with a general rational approximant given in (6), by casting the problem (14) into one of the Nevanlinna-Pick interpolation [1]. The following result illustrates this point.

Theorem 1 *Let $p_i \in \mathbb{C}_+$, $i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then for any $w_\tau(s)$ in (6),*

$$\underline{\tau} = \sigma_{\max}^{-1}\left(\begin{bmatrix} -\Phi_0^{-1}\Phi_1 & \cdots & -\Phi_0^{-1}\Phi_{2q-1} & -\Phi_0^{-1}\Phi_{2q} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}\right),$$

where $\Phi_0 = \begin{bmatrix} Q_p & b_0 \\ b_0 & a_0^2 Z^{-1} \end{bmatrix}$,

$$\Phi_k = \begin{cases} \begin{bmatrix} 0 & b_k (D_p^H)^k \\ b_k D_p^k & \sum_{l=0}^{k} a_l a_{k-l} D_p^l Z^{-1} (D_p^H)^{k-l} \end{bmatrix}, & k = 1, \ \ldots, \ q, \\ \mathrm{diag}\left(0, \ \sum_{l=k-q}^{q} a_l a_{k-l} D_p^l Z^{-1} (D_p^H)^{k-l}\right), & k = q+1, \ \ldots, \ 2q, \end{cases}$$

$$Z = Q_p + Q_{zp}^H Q_z^{-1} Q_{zp}, \quad D_p = diag\,(p_1, \ \ldots, \ p_n),$$

$$Q_z = \left[\frac{1}{z_i + \bar{z}_j}\right], \quad Q_p = \left[\frac{1}{\bar{p}_i + p_j}\right], \quad Q_{zp} = \left[\frac{1}{z_i - p_j}\right].$$

We note that Theorem 1 can be extended to accommodate multiple poles and zeros in $P_0(s)$, using a more sophisticated result on the mixed Nevanlinna-Pick and Carathéodory-Fejér interpolation problem [26, 29]. Imaginary poles and zeros can also be incorporated in the analysis as boundary interpolation constraints [1, 2]. For technical simplicity, however, we choose not to address such poles and zeros herein.

In view of Theorem 1, a lower bound $\underline{\tau}$ on the delay margin can be found by solving rather efficiently an eigenvalue problem, which guarantees that $P_\tau(s)$ can be stabilized by a certain LTI controller $K(s)$ for all $\tau \in [0, \ \underline{\tau})$. Since $\underline{\tau}$ corresponds to an optimal $\mathcal{H}_\infty$ optimization problem, a robustly stabilizing controller can be synthesized accordingly. Indeed, to synthesize this robustly stabilizing controller $K(s)$, it suffices to solve the standard $\mathcal{H}_\infty$ control problem in (15), once $\underline{\tau}$ is computed according to Theorem 1. This gives rise to an optimal controller $K(s)$ depending on $\underline{\tau}$. In this vein, it is worth pointing out that a lower order $w_\tau(s)$, such as those given in (8)–(13), can be particularly desirable, since they potentially result in low-order controllers.

Special Cases

A number of special cases are further examined in this section. The first result concerns the circumstance where $P_0(s)$ has only a single unstable pole. In this case, an explicit lower bound is obtained which exhibits how the plant unstable pole may confine the delay margin.

Corollary 1 *Suppose that $P_0(s)$ has only one unstable pole $p \in \mathbb{C}_+$, and no non-minimum phase zero. Then for any $w_\tau(s)$ in (6) with $b_0 < a_0$,*

$$\underline{\tau} = \frac{\lambda_{\min}}{p}, \tag{16}$$

where

$$\lambda_{\min} = \min\left\{\lambda > 0 : \sum_{k=0}^{q}(b_k - a_k)\lambda^k = 0\right\}. \tag{17}$$

In particular, if $w_\tau(s) = w_{i\tau}(s)$ *for* $w_{i\tau}(s)$, $i = 1, \ldots, 6$ *given in (8)–(13), then we have* $\underline{\tau} = \underline{\tau}_i$, *with*

(1) $\underline{\tau}_1 = 1/p$; (2) $\underline{\tau}_2 \approx 1.406/p$; (3) $\underline{\tau}_3 \approx 1.397/p$; (4) $\underline{\tau}_4 \approx 1.5582/p$; (5) $\underline{\tau}_5 \approx 1.7008/p$; (6) $\underline{\tau}_6 \approx 1.722/p$.

In other words, for plants with a sole unstable pole, it suffices to solve the smallest positive real root of a polynomial.

While Corollary 1 shows a varying degree of conservatism in the various lower bounds resulted from their respective approximants $w_{i\tau}(s)$, it is interesting to observe that $w_{5\tau}(s)$ and $w_{6\tau}(s)$, despite being only a third-order and a fifth-order approximant respectively, provide rather accurate estimates of the true delay margin; in these cases, $\underline{\tau}_5 = 1.7008/p$, and $\underline{\tau}_6 = 1.722/p$, respectively, as opposed to the exact delay margin $\tau^* = 2/p$, obtained in [18]. Note however that the exact delay margin $\tau^* = 2/p$ may not be attainable in a realistic sense, for the robustly stabilizing controller corresponding to τ^* will result in an arbitrarily small loop bandwidth [18] and thus will be hardly of use. In practice, one must then accept to find a robustly stabilizing controller for a smaller range of delay, which will be even closer to the lower bounds obtained herein.

More generally, Corollary 1 can be extended to systems containing nonminimum phase zeros as well, as demonstrated by the following result.

Corollary 2 *Suppose that* $P_0(s)$ *has only one unstable pole* $p \in \mathbb{C}_+$ *and distinct nonminimum phase zeros* $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$. *Let*

$$M = \prod_{i=1}^{m}\left|\frac{z_i - p}{\bar{z}_i + p}\right|.$$

Then for any $w_\tau(s)$ *in (6) with* $b_0 < Ma_0$,

$$\underline{\tau} = \frac{\lambda_{\min}}{p}, \tag{18}$$

where

$$\lambda_{\min} = \min\left\{\lambda > 0 : \sum_{k=0}^{q}(b_k - Ma_k)\lambda^k = 0\right\}. \tag{19}$$

Furthermore, for $w_\tau(s) = w_{i\tau}(s)$, $i = 1,\ 2,\ 3$ *given in (8)–(10), we have* $\underline{\tau} = \underline{\tau}_i$, *with*

$$(1)\ \ \underline{\tau}_1 = \frac{M}{p}; \quad (2)\ \ \underline{\tau}_2 = \frac{M}{(1-0.289M)p}; \quad (3)\ \ \underline{\tau}_3 = \frac{M}{(1.216-0.5M)p}.$$

Evidently, Corollary 2 shows that in the presence of nonminimum phase zeros, the range of delay with guaranteed stabilizability will be further shrunk. This is consistent with the finding of [18], which shows that it is less likely to stabilize a delay plant containing nonminimum phase zeros. The explicit relations given in (1)–(3) of Corollary 2 show that $\underline{\tau}$ is a monotonically increasing function of M. In the limit when $M \to 0$, stabilization is rendered impossible. This scenario occurs when the plant has a pair of closely located unstable pole and nonminimum phase zero. Note also that for the fourth, fifth, and sixth order approximants $w_{4\tau}(s)$, $w_{5\tau}(s)$, and $w_{6\tau}(s)$, similar yet more complex expressions of $\underline{\tau}$ can be found explicitly in terms of M, by more tedious calculations.

3 Systems with Time-Varying Delays

With an added advantage, the interpolation approach can be expanded to analyze linear systems with time-varying delays. Consider the system

$$\begin{cases} \dot{x} = Ax + Bu(t-\tau(t)), \\ y = Cx. \end{cases} \tag{20}$$

It is customary to confine the time-varying delay $\tau(t)$ to a given range $[0,\ \bar{\tau})$, i.e.,

$$0 \le \tau(t) \le \bar{\tau}, \tag{21}$$

and bound the variation rate $\dot{\tau}(t)$ as,

$$|\dot{\tau}(t)| \le \delta < 1. \tag{22}$$

Let $P_0(s) = C(sI-A)^{-1}B$ be the transfer function of the delay-free system. We want to find a LTI controller $K(s)$ so as to stabilize the delay system (20) by way of the output feedback $u(s) = K(s)y(s)$ within a region defined by $(\bar{\tau},\ \delta)$.

Rate-Independent Bound

It is readily recognized that the closed loop system can be represented by Fig. 3, in which Δ is a linear time-varying operator such that

$$\Delta u(t) = u(t-\tau(t)).$$

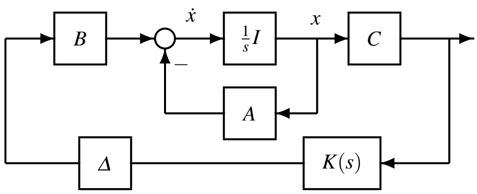

Fig. 3 Feedback system with time-varying input delay

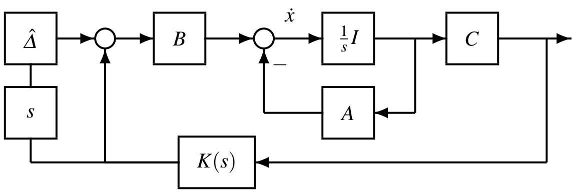

Fig. 4 Small-gain setup of systems with time-varying delay

By employing the *model transformation* [8],

$$u(t-\tau(t)) = u(t) - \int_{t-\tau(t)}^{t} \dot{u}(\sigma)\mathrm{d}\sigma,$$

the system can be transformed into the one depicted in Fig. 4, where

$$\hat{\Delta}x = -\int_{t-\tau(t)}^{t} x(\sigma)\mathrm{d}\sigma. \tag{23}$$

It is well-known [8] that the system in Fig. 3, i.e., the original system (20) with the controller $K(s)$, is stable whenever the system in Fig. 4 is stable. Thus, by applying the small-gain condition developed in [12, 32], we conclude that $K(s)$ stabilizes the system (20) if it stabilizes $P_0(s)$ and the small-gain condition

$$\|\bar{\tau} s T_0(s)\|_\infty < 1 \tag{24}$$

holds. As a result, in much the same manner, a lower bound on $\bar{\tau}$ can be found by solving the $\mathcal{H}_\infty$ optimization problem in (24), which will guarantee the existence of a controller $K(s)$ that can stabilize the system (20) for all $\tau(t) \in [0,\ \bar{\tau})$ regardless of δ. Evidently, this problem coincides with that in (14), with $w_\tau(s) = \bar{\tau}s$. The following result is thus clear.

Theorem 2 *Let $p_i \in \mathbb{C}_+, i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+, i = 1, \ldots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then the system (20) can be stabilized by some $K(s)$ for all $\tau(t) \in [0,\ \bar{\tau})$ with*

$$\bar{\tau} = \bar{\lambda}^{-\frac{1}{2}} \left(Q_{p1}^{-\frac{1}{2}} (Q_{p2} + Q_{zp}^H Q_z^{-1} Q_{zp}) Q_{p1}^{-\frac{1}{2}} \right),$$

where

$$Q_z = \left[\frac{1}{z_i + \bar{z}_j}\right], \quad Q_{p1} = \left[\frac{1}{\bar{p}_i + p_j}\right], \quad Q_{p2} = \left[\frac{\bar{p}_i p_j}{\bar{p}_i + p_j}\right], \quad Q_{zp} = \left[\frac{p_j}{z_i - p_j}\right].$$

Rate-Dependent Bound

More generally, it is possible to employ more elaborate approximations of the time-varying operator Δ. It may also be useful to incorporate the delay variation rate in the approximation. One such approximation scheme is suggested in [11], which stipulates that for any $\bar{\tau} \geq 0$ and $0 \leq \delta < 1$, $K(s)$ can stabilize the system (20) whenever

$$|T_0(j\omega)\psi_\epsilon(j\omega)| < 1, \qquad \forall \omega \in \mathbb{R}, \tag{25}$$

where $\psi_\epsilon(s)$ is a stable rational function meeting the condition

$$|\psi_\epsilon(j\omega)| \geq \sqrt{\frac{2}{2-\delta}}\,\phi_\tau(\omega) + \epsilon$$

and hence can be constructed so that

$$|\psi_\epsilon(j\omega)| = \sqrt{\frac{2}{2-\delta}}\,|w_\tau(j\omega)| + \epsilon,$$

for any $\epsilon > 0$ and any rational function $w_\tau(s)$ given in (6) and satisfying (7). Since $\epsilon > 0$ can be made arbitrarily small, the condition (25) is met whenever

$$\inf_{K(s)} \|T_0(s)w_\tau(s)\|_\infty < \sqrt{\frac{2-\delta}{2}}\,. \tag{26}$$

As a consequence, the stabilizability of the system (20) can also be ascertained using the same interpolation approach. The following result extends Theorem 1 to systems described by (20), with time-varying delays.

Theorem 3 *Let $p_i \in \mathbb{C}_+$, $i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then the system (20) can be stabilized by some $K(s)$ for all $\tau(t) \in [0, \bar{\tau})$, $|\dot{\tau}(t)| \leq \delta$ if*

$$\bar{\tau} = \sigma_{\max}^{-1}\left(\begin{bmatrix} -\Phi_0^{-1}\Phi_1 & \cdots & -\Phi_0^{-1}\Phi_{2q-1} & -\Phi_0^{-1}\Phi_{2q} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}\right),$$

where for any $w_\tau(s)$ in (6), Φ_k, $k = 1, \ldots, 2q$ are defined as in Theorem 1, and Φ_0 is given by

$$\Phi_0 = \begin{bmatrix} \frac{2-\delta}{2} Q_p & b_0 \\ b_0 & a_0^2 Z^{-1} \end{bmatrix},$$

with Q_p and Z defined in Theorem 1 as well.

Analogously, explicit bounds can be obtained for more special cases. The following corollary summarizes the time-varying counterparts to Corollaries 1 and 2.

Corollary 3 *Suppose that $P_0(s)$ is minimum phase and has only one unstable pole $p \in \mathbb{C}_+$. Define*

$$N = \sqrt{\frac{2-\delta}{2}}\,.$$

Then the system (20) can be stabilized by some $K(s)$ for all $\tau(t) \in [0, \bar{\tau})$ with $\bar{\tau} = \bar{\tau}_i$, $i = 1, \ldots, 4$, where

$$(1)\ \bar{\tau}_1 = 1/p; \quad (2)\ \bar{\tau}_2 = \frac{N}{(1-0.289N)p}; \quad (3)\ \bar{\tau}_3 = \frac{N}{(1.216-0.5N)p};$$
$$(4)\ \bar{\tau}_4 = \frac{10.81 - 4.654N - \sqrt{116.9 - 57.32N}}{(N-2)\,p}.$$

Additionally, suppose also that $P_0(s)$ has distinct nonminimum phase zeros $z_i \in \mathbb{C}_+$, $i = 1, \ldots, m$. Let M be defined in Corollary 2. Then the system (20) can be stabilized by some $K(s)$ for all $\tau(t) \in [0, \bar{\tau})$ with $\bar{\tau} = \bar{\tau}_i$, $i = 1, \ldots, 4$, where

$$(1)\ \bar{\tau}_1 = M/p; \quad (2)\ \bar{\tau}_2 = \frac{N}{(1-0.289N)p}M; \quad (3)\ \bar{\tau}_3 = \frac{N}{(1.216-0.5N)p}M;$$
$$(4)\ \bar{\tau}_4 = \frac{10.81 - 4.654N - \sqrt{116.9 - 57.32N}}{(N-2)\,p}M.$$

Theorems 2 and 3 differ from each other due to the incorporation of the variation rate δ, which results from the difference between (24) and (26). Similarly, in Corollary 3, the bound $\bar{\tau}_1$ is derived using the condition (24), while $\bar{\tau}_2$, $\bar{\tau}_3$ and $\bar{\tau}_4$ are obtained using (26), together with $w_{2\tau}(s)$, $w_{3\tau}(s)$ and $w_{4\tau}(s)$, respectively. Among the rate-dependent bounds, $\bar{\tau}_i, i = 2, 3, 4$ become progressively less conservative. Compared to the rate-independent $\bar{\tau}_1$, they may or may not be advantageous depending on the value of δ. It is also worth noting that for $\delta = 0$, the condition (26) reduces to (15), and hence the results in this section all recover the bounds for LTI systems presented in Sect. 3.

4 Examples

We now consider a number of illustrating examples. Example 1 presents a system with a constant delay, while Example 2 addresses systems with a time-varying delay. In both examples, we assume that the plant is excited by a unit step input.

Example 1 Consider the plant

$$P_0(s) = \frac{0.1(s-10)(s-0.1659)}{(s-0.1081)(s^2+0.2981s+0.06281)}. \tag{27}$$

This system has an unstable pole $p = 0.1081$ and two nonminimum phase zeros $z_1 = 10,\ z_2 = 0.1659$. Using the approximant $w_{6\tau}(s)$, we find $\underline{\tau}_6 = 2.0741$, achievable by the optimal controller $K(s)$ solving (15):

$$\begin{aligned} K(s) = & \frac{10643(s+0.9643)(s^2+0.2981s+0.06282)}{(s+1044)(s+30.13)(s+1.134)(s+0.7959)} \\ & \times \frac{(s^2+1.965s+1.109)(s^2+1.167s+1.651)}{(s+0.617)(s^2+1.27s+1.647)}. \end{aligned} \tag{28}$$

The closed-loop output response is plotted in Fig. 5 for $\tau = 0.6,\ 1,\ 1.5,\ 2$, respectively. For $\tau = 2$, Fig. 6 shows the state responses of the system (27). Clearly, the system is internally stable. Moreover, Fig. 7 shows the magnification of the output responses for $t \in [0,\ 25]$, exhibiting the typical undershoot behavior of nonminimum phase systems.

Example 2 The following system, given in state-space form, contains a time-varying delay $\tau(t)$:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -599 & 600 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-\tau(t)), \\ y(t) &= \begin{bmatrix} 1 & -3 \end{bmatrix} x(t). \end{aligned} \tag{29}$$

The delay-free part of the system has an unstable pole $p = 1$ and a nonminimum phase zeros $z = 3$. The time-varying delay under consideration is described by

$$\tau(t) = \alpha(1 - \sin(\beta t))$$

for some $\alpha > 0,\ \beta > 0$. It is evident that $0 \le \tau(t) \le 2\alpha$, and $\delta = \alpha\beta$. From Theorem 2, we assert that the system (29) is robustly stabilizable regardless of β whenever $\alpha < 0.25$. For all $\alpha \in [0,\ 0.25)$, the system (29) can be stabilized by the feedback controller $K(s)$ solving the $\mathcal{H}_\infty$ optimal control problem in (24):

$$K(s) = -1.0273 \times 10^8 \frac{(s+600)(s+100)}{(s+3.2\times10^6)(s+1640)(s+5.853)}. \tag{30}$$

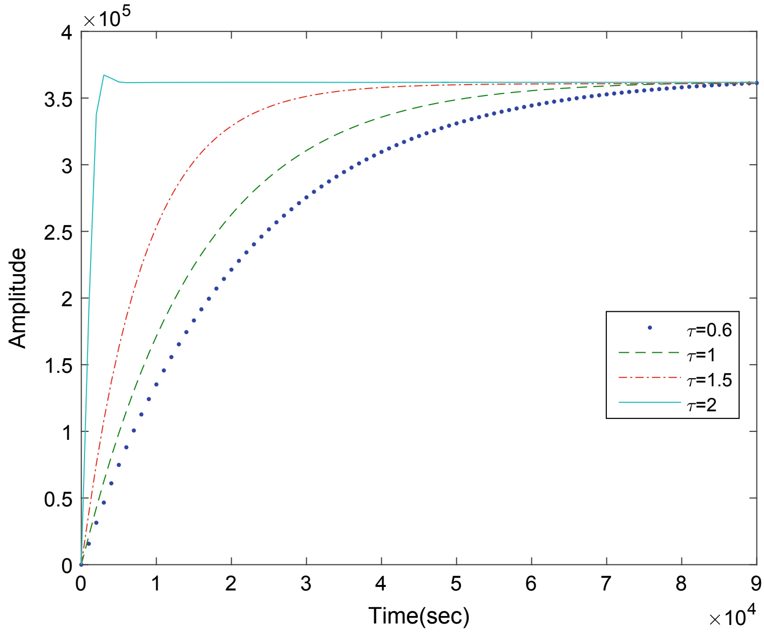

Fig. 5 Step response of the system (27) with controller (28)

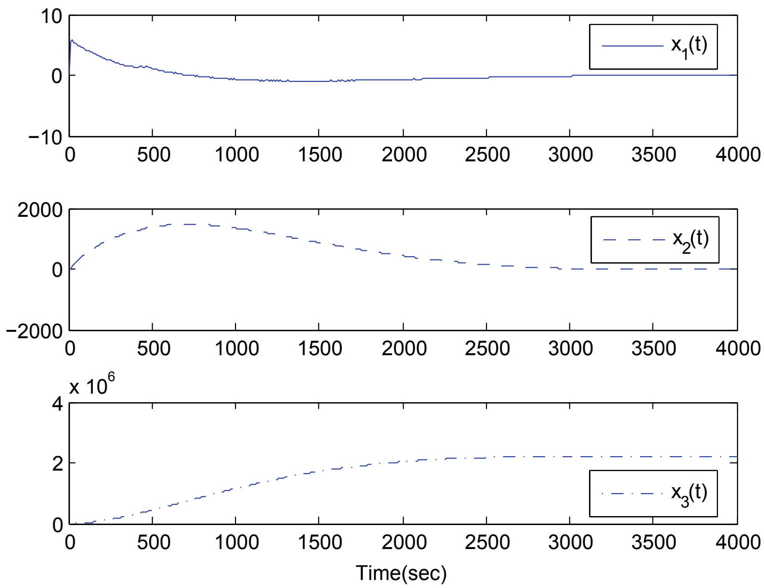

Fig. 6 State response of the system (27) with controller (28) for $\tau = 2$

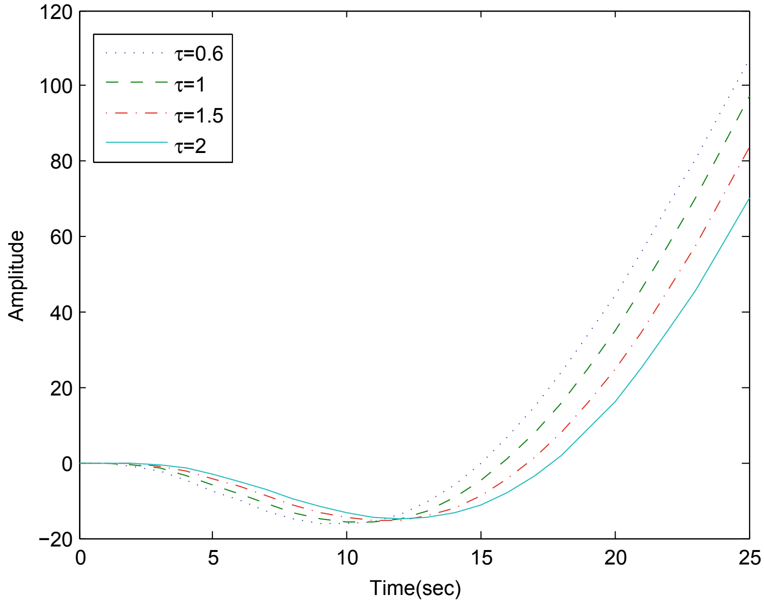

Fig. 7 Step response of the system (27) with controller (28) $t \in [0,\ 25]$

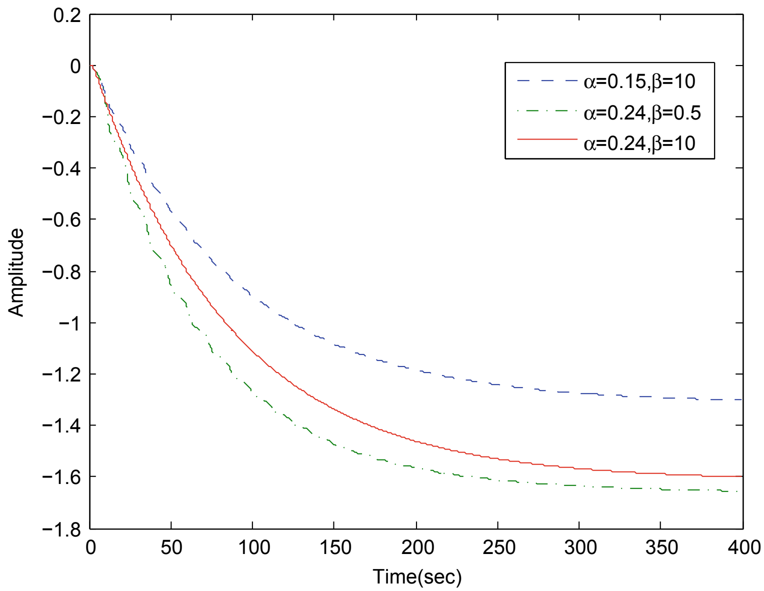

Fig. 8 Output response of the system (29) with controller (30)

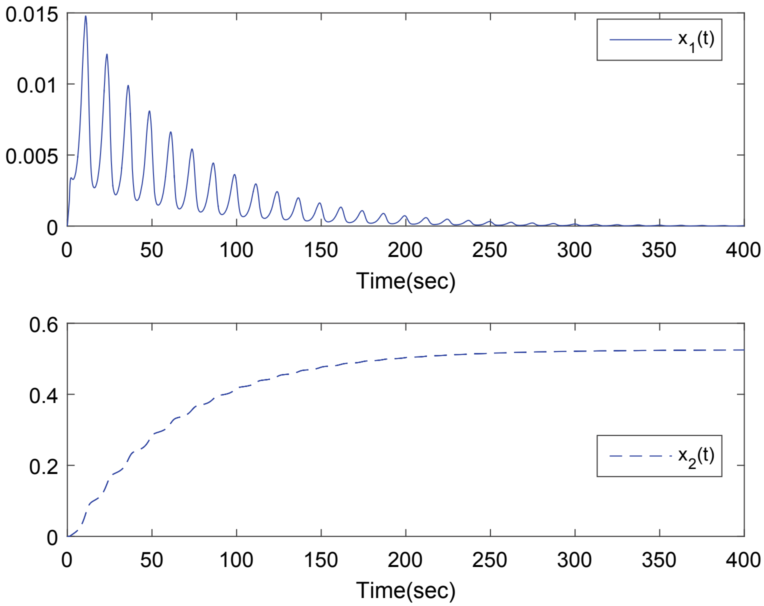

Fig. 9 State response of the system (29) with controller (30) for $\alpha = 0.24$, $\beta = 0.5$

Figure 8 shows the stable output responses of the closed-loop system for various combinations of α within the interval $[0,\ 0.25)$ and arbitrarily selected β. For $\alpha = 0.24$, $\beta = 0.5$, Fig. 9 shows the state responses.

5 Conclusion

In this chapter we have studied the delay margin and delay robust stabilization problems for linear delay systems. Our solutions seek to ascertain the existence of a finite-dimensional LTI output feedback controller that can robustly stabilize an entire family of plants subject to uncertain, possibly time-varying delays within a given range. Built on small-gain stability conditions, we employed analytic interpolation and rational approximation techniques to develop bounds on the delay margin. The development has led to a unified interpolation-based approach, applicable to SISO systems with constant and time-varying delays. The results consist of readily computable bounds on the delay margin of SISO systems, within which a delay plant is guaranteed to be stabilizable. The bounds can in general be computed by solving an eigenvalue problem. For more special plants admitting, e.g., only one unstable pole, explicit results are found which show how unstable poles and nonminimum phase zeros may fundamentally confine the range of delay allowed for robust stabilization.

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Nonlinear Sampled-Data Stabilization with Delays

Salvatore Monaco, Dorothée Normand-Cyrot and Mattia Mattioni

1 Introduction

Several recent approaches (see among them [3, 9, 12, 13, 21, 23]) discuss the compensation of delays for nonlinear continuous-time systems in terms of reduction, descriptor or predictor-based strategies. Simultaneously, mainly motivated by implementation issues, a growing interest has been addressed toward systems under sampling. In particular, robustness of sample-and-hold stabilizing controllers with respect to actuators uncertainties (see [28] and the references therein) or with respect to delays are investigated in the literature [8, 10, 11, 19, 20, 22, 27, 29]. These contributions are essentially concerned with implementation of existing feedback laws designed over the continuous plant or a priori assuming the existence of ad-hoc sampled-data controllers for the delay free model. That sampling is instrumental when dealing with time delays is even more clear when considering nonlinear dynamics affected by fixed and known input or measurements delays. In this case, sampled-data predictor based schemes (see [10, 11]) can be exactly computed for

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S. Monaco
Dipartimento di Ingegneria Informatica, Automatica e Gestionale,
DIAG Università di Roma "La Sapienza", via Ariosto 25, 00185 Roma, Italy
e-mail: salvatore.monaco@uniroma1.it

D. Normand-Cyrot (✉)
Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506)
CNRS; 3, rue Joliot Curie, 91192 Gif-sur-Yvette, France
e-mail: cyrot@l2s.centralesupelec.fr

M. Mattioni
DIAG (Università di Roma 'La Sapienza'), Rome, Italy
e-mail: mattia.mattioni@uniroma1.it

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particular classes of systems admitting closed-form (possibly *finitely* computable) expressions of the predictor map.

The purpose of this work is to go further in the direct design of sampled-data feedback laws for input-delayed dynamics and investigate some simple extensions to special classes of systems affected by state delays. The results stated by the authors for the class of input-affine dynamics in some recent works [15, 24, 25, 31] are in the sequel extended to more general classes of nonlinear time-delayed systems with several comments on some open perspectives.

Throughout the paper we address the problem of designing sampled-data controllers for a given continuous input-delayed dynamics starting from its sampled equivalent model; the resulting control laws admit parametrized expressions in the sampling period which can be used to underline the possible advantages of sampling. The design is developed in three steps starting by showing how to get global asymptotic stabilization of the equivalent sampled delay free dynamics; then a sampled-data predictor-based controller is proposed, and, finally, robustness improvement of the prediction errors is achieved through a suitable redesign.

Section 2 states the problem and specifies the classes of systems under study: nonlinear systems affected by input-delays; *strict feedforward*-like dynamics, admitting finitely computable predictor map, directly or via preliminary coordinates change and feedback; *strict feedback*-like systems, characterized by a delay in the connection state variable. Section 3 reports on the sampled-data predictor based stabilizing controller with discrete-time predictor map. Section 4 discusses a modified sampled-data predictor based controller via Immersion and Invariance (I&I) design. Section 5 deals with the specific case of a two block cascade dynamics with delay in the state connection variable. Again it is shown how to recast the stabilizing problem in the Immersion and Invariance context. Academic examples illustrate the computational aspects. No complete proofs are reported but adequately referred to from previous authors' work.

2 Sampled-Data Models of Differential Dynamics with Delays

2.1 The Class of Systems Under Study

We consider nonlinear dynamics over $\mathbb{R}^n$ with scalar valued input $u \in \mathbb{R}$

$$\dot{x}(t) = f(x(t), u(t-\tau)) \tag{1}$$

with equilibrium x_* ($f(x_*, 0) = 0$) and known constant delay $\tau \geq 0$; when $\tau = 0$ in (1) the so called *delay-free dynamics* is recovered as

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$

Throughout the paper, maps and vector fields are assumed smooth over the respective definition spaces (i.e. infinitely differentiable - C^∞) and the delay-free system corresponding to (1) when $\tau = 0$ forward complete.[1] In (1), $u \in M_U^{[-\tau,\infty)}$ where M_U^I denotes the space of measurable and locally bounded functions $u : I \subset \mathbb{R}^+ \to U$ ($u : I \to U$, with $U \subseteq \mathbb{R}$).

The following standing assumptions are set:

- measures are available only at the sampling instants $t = k\delta$; $k \geq 0$; $\delta \in]0, T[$, where δ is the constant sampling period and T is the maximum allowable sampling period;
- the control is constant over time intervals of length δ; i.e. $u \in \mathcal{U}_\delta = \{u \in M_U$ s.t.$\}$ $\{u(t) = u_k, \forall t \in [k\delta, (k+1)\delta[; k \geq 0\}$ (*sampled-data control*);
- δ is chosen so that $\tau = N\delta$ for a suitable integer N.
- Assumption **A -** The delay free dynamics associated to (1) is smoothly stabilizable; i.e. there exists a smooth continuous-time feedback $u = \gamma(x)$ with $\gamma(x_*) = 0$ and a proper Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $V(x_*) = 0$ such that $\dot{V}(x) = \mathrm{L}_{f(\cdot,\gamma(\cdot))}V(x) < 0$ and with $\frac{\partial L_{f(\cdot,u)}V(\cdot)}{\partial u}\Big|_{u=\gamma(x)} \neq 0$ for all $x \in \mathbb{R}^n / \{x_*\}$.

Remark 1 Smoothness of the feedback $\gamma(\cdot)$ is instrumental to computational purposes.

Under these assumptions, the objective of this work is to discuss direct sampled-data strategies which make the equilibrium x_* of (1) S-GAS in the sense of the following definition.

Definition 1 Consider the sampled-data system $\dot{x} = f(x, u_k)$. We say that the closed-loop equilibrium of $\dot{x} = f(x, \alpha(x_k))$ is sampled-data globally asymptotically stable (S-GAS) for some piecewise constant feedback $u_k = \alpha(x_k)$ if the equilibrium of its sampled equivalent $x_{k+1} = F^\delta(x_k, \alpha(x_k))$ is globally asymptotically stable (GAS) at the sampling instants $t = k\delta$, $k \geq 0$.

Remark 2 The scalar input case is developed for the sake of simplicity but multi-input illustrative examples are reported in [5, 31].

2.2 *Equivalent Sampled-Data State Representations*

As, well know, when setting $u(t) = u_k$ for $t \in [k\delta, (k+1)\delta[$ and assuming $\tau = N\delta$, (1) rewrites as the interconnection of a continuous-time delay free dynamics and a discrete-time finite dimensional linear dynamics; namely, one gets the so-called *equivalent hybrid dynamics*

---

[1] Assuming the delay free dynamics forward complete ensures that the delayed one (1) is too: $\forall x_0$ and $u \in M_U^{[-\tau,\infty)}$ the solution $x(t)$ of (1) with initial condition $x(0) = x_0 \in \mathbb{R}^n$ exists $\forall t \geq 0$.

$$\dot{x}(t) = f(x(t), v_k^1); \quad t \in [k\delta, (k+1)\delta[\tag{3a}$$

$$v_{k+1}^1 = v_k^2; \qquad \ldots; \qquad v_{k+1}^N = u_k \tag{3b}$$

that is now free of delays. By integrating (3a) over $[k\delta, (k+1)\delta[$ and initial condition $x_k := x(k\delta)$, one describes the *equivalent sampled-data dynamics* in the form of a map (or a difference equation) over $\mathbb{R}^n \times \mathbb{R}^N$ as

$$x_{k+1} = F^\delta(x_k, v_k^1) = e^{\delta \mathrm{L}_{f(\cdot, v_k^1)}} x \Big|_{x_k} \tag{4a}$$

$$v_{k+1}^1 = v_k^2; \quad \ldots \quad v_{k+1}^N = u_k. \tag{4b}$$

The dynamics (4a) describes the exact sampled-data equivalent of the (3a). In most cases, a closed form of (4a) cannot be computed and the map $F^\delta(\cdot, v^1)$ is described by its series expansion[2] in powers of δ

$$F^\delta(x, v^1) = e^{\delta \mathrm{L}_{f(\cdot, v^1)}} x = x + \sum_{i \geq 1} \frac{\delta^i}{i!} \mathrm{L}^i_{f(\cdot, v^1)} x.$$

Finite order approximations in $O(\delta^p)$[3] are currently used in practice though the above series might possess a finite number of terms in δ (*finite discretizability of* $f(\cdot, v^1)$) in some specific cases.

Stabilization of the input-delayed dynamics (1) can be thus reformulated in terms of stabilization of the extended dynamics (4) so clearly involving nonlinear discrete-time control strategies. However, by exploiting the cascade structure exhibited by (4) and assuming the existence of a discrete-time stabilizing controller for the delay free dynamics (i.e., when $v_k^1 = u_k$, $N = 0$ in (4)), predictor-based control with discrete-time prediction map can be worked out.

Remark 3 The case of measurement delays can be treated analogously.

Remark 4 In case of non entire delays (say $\tau = N\delta + \sigma$ with $\sigma \in]0, \delta[$), the extended sampled dynamics can be defined over $\mathbb{R}^n \times \mathbb{R}^{N+1}$ as below [4]

$$x_{k+1} = F_1^\delta(x_k, \sigma, v_k^1, v_k^2) = e^{\sigma \mathrm{L}_{f(\cdot, v_k^1)}} \circ e^{(\delta - \sigma) \mathrm{L}_{f(\cdot, v_k^2)}} x \Big|_{x_k}$$

$$v_{k+1}^1 = v_k^2; \qquad \ldots \qquad ; v_{k+1}^N = u_k.$$

$F_1^\delta(\sigma, v_k^1, v_k^2)$ is parameterized by both the fractional part of the delay σ and past values of the input variables so that the design strategies developed in the sequel

[2] L_f denotes the Lie derivative operator, $L_f = \sum_{i=1}^n f_i(\cdot) \frac{\partial}{\partial x_i}$. e^{L_f} (or e^f, when no confusion arises) denotes the associated Lie series operator, $e^{\mathrm{L}_f} := 1 + \sum_{i \geq 1} \frac{L_f^i}{i!}$.

[3] A function $R(x, \delta) = O(\delta^p)$ is said of order δ^p; $p \geq 1$ if whenever it is defined it can be written as $R(x, \delta) = \delta^{p-1} \tilde{R}(x, \delta)$ and there exist a function $\theta \in \mathcal{K}_\infty$ and $\delta^* > 0$ s.t. $\forall \delta \leq \delta^*$, $|\tilde{R}(x, \delta)| \leq \theta(\delta)$.

[4] $\circ$ denotes the composition of operators and functions.

need to be suitably modified. The extension of predictor-based techniques to this case has been proposed in [16] while an alternative solution relying on the concept of reduction has been discussed in [18].

2.3 Finite Sampling Under Coordinates Change and Feedback

As previously anticipated, the existence of a closed form (possibly finitely computable) expression of the sampled model may be useful. As well known, upper triangular (say strict-feedforward) forms as

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_2(t), \dots, x_n(t), u(t-\tau)) \\ \dot{x}_2(t) &= f_2(x_3(t), \dots, x_n(t), u(t-\tau)) \\ &\dots \\ \dot{x}_n(t) &= f_n(u(t-\tau)) \end{aligned} \tag{5}$$

where $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ can be finitely integrated through a bottom-up iterative procedure. When assuming the input piecewise constant and affected by delay $\tau = N\delta$, the associated sampled-data model still preserves finite discretizability in δ.

Several mechanical systems exhibit those triangular forms (as the chained forms or their extensions) either from direct modeling or through preliminary coordinates changes and continuous-time feedback [4, 30]. Accordingly, in [7] we examine the conditions allowing finite discretizability through coordinates change and feedback. In the present context, we underline that if such a transformation exists on the delay free dynamics associated to (1), then it can be extended to the input-delayed ones. This fact is easily verified and stated as follows.

Proposition 1 *Let the dynamics (1) and consider when $\tau = 0$ the delay-free dynamics (2). Assume the existence of coordinates change $\tilde{x} = \phi(x)$ and a state feedback $u(t) = k(x(t), v(t))$ with $v \in M_U^{[-\tau,\infty)}$ making (2) finitely discretizable in δ of order p. Then, for $\tau > 0$, the retarded feedback $u(t-\tau) = k(x(t), v(t-\tau))$ and the same $\tilde{x} = \phi(x)$ make (1) finitely discretizable at the same order p in δ.*

Basically, Proposition 1 implies that the predictor map is finitely computable once a preliminary feedback is applied. Denoting by $\tilde{f}(\tilde{x}, v)$ the continuous-time dynamics in the $(\tilde{x}, v)$ coordinates and by $\tilde{F}^\delta(\tilde{x}, v)$ its sampled model, the design is performed in the $(\tilde{x}_k, v_k)$ coordinates. Then, the original control is expressed in terms of the computable predictor map $\tilde{F}^\delta(\cdot, v_k)$ so getting the piecewise continuous control signal

$$u(k\delta + s) = k(x((k+N)\delta + s), v_k) = \tilde{k}(\tilde{x}((k+N)\delta + s), v_k)$$

for $s \in [0, \delta[$, $k \geq 0$ and $\tilde{x}(k\delta + s) = \tilde{F}^s(\tilde{x}(k\delta), v_k)$ for $N = 0$ and

$$\tilde{x}((k+N)\delta+s) = \tilde{F}^s(\cdot, v_k) \circ \tilde{F}^\delta(\cdot, v_{k-1}) \circ \cdots \circ \tilde{F}^\delta(\tilde{x}(k\delta), v_{k-N}) \quad \text{for } N \geq 1.$$

Dynamics admitting chained form-like representations generally do not admit continuous-time smooth stabilizing control laws. Thus, sampled-data multirate control strategies have been proposed in [6] to deal with trajectory planning or finite-time tracking objectives. In presence of delays, Proposition 1 suggests to resettle these control problems in the sampled-data context by taking advantage of the simplified transformed dynamics. Arguing so, Assumption **A** is naturally relaxed and sampling becomes properly an instrumental tool of the design.

2.3.1 The Unicycle: an Illustrative Example

Let the kinematics equations of a wheeled vehicle over $\mathbb{R}^3$

$$\dot{x}(t) = v(t)\cos\theta(t); \quad \dot{y}(t) = v(t)\sin\theta(t); \quad \dot{\theta}(t) = \omega(t) \tag{6}$$

where v and ω denote respectively the forward and steering velocities. As well known, the change of coordinates

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^\top = \begin{pmatrix} x\cos\theta + y\sin\theta & \sin\theta - y\cos\theta, & \theta \end{pmatrix}^\top \tag{7}$$

and state feedback

$$u_1(t) = v(t) - x_2(t)\omega(t); \quad u_2(t) = \omega(t) \tag{8}$$

transform the dynamics (6) into the nonholonomic integrator

$$\dot{x}_1(t) = u_1(t); \quad \dot{x}_2(t) = u_2(t)x_1(t); \quad \dot{x}_3(t) = u_2(t).$$

When setting $u_i(t) = u_{ik}$ for $t \in [k\delta, (k+1)\delta[$, $i = 1, 2$ and $x_{ik} = x_i(k\delta)$ for $i = 1, 2, 3$, one gets an exact sampled equivalent dynamics of finite order 2 in δ; namely,

$$x_{1k+1} = x_{1k} + \delta u_{1k}; \quad x_{2k+1} = x_{2k} + \delta u_{2k}x_{1k} + \frac{\delta^2}{2}u_{2k}u_{1k} \quad x_{3k+1} = x_{3k} + \delta u_{2k}.$$

Accordingly, introducing a delay of length $\tau > 0$ over the inputs, the coordinate change (7) and the delayed version of feedback (8) transform (6) into the retarded chained form

$$\dot{x}_1(t) = u_1(t-\tau); \ \dot{x}_2(t) = u_2(t-\tau)x_1(t); \ \dot{x}_3(t) = u_2(t-\tau)$$

which still admits a finite sampled model. Assuming now $\tau = \delta$, the sampled-data model in the extended coordinates $(x_1, x_2, x_3, w_1, w_2)$ with $w_i = u_{ik-1}$ $(i = 1, 2)$ takes the form

$$x_{1k+1} = x_{1k} + \delta w_{1k}; \quad x_{2k+1} = x_{2k} + \delta w_{2k} x_{1k} + \frac{\delta^2}{2} w_{2k} w_{1k}; \quad x_{3k+1} = x_{3k} + \delta w_{2k}$$
$$w_{1k+1} = u_{1k}; \quad w_{2k+1} = u_{2k}.$$

Once a control solution over (u_1, u_2) is computed, the piecewise continuous controller in terms of (v, ω) is described for $s \in [0, \delta[$ as

$$v(k\delta + s) = u_{1k} + x_2(k\delta + \delta + s) u_{2k}; \quad \omega(k\delta + s) = u_{2k}$$

with $x_2(k\delta + \delta + s) = x_{2k+1} + s u_{2k} x_{1k+1} + \frac{\sigma^2}{2} u_{2k} u_{1k}$, so getting after substitutions

$$v(k\delta + s) = u_1 + x_2 u_2 + s x_1 (u_2)^2 + \delta x_1 u_2 w_2 + s\delta w_1 (u_2)^2 + \frac{s^2}{2} u_1 (u_2)^2 + \frac{\delta^2}{2} u_2 w_2 w_1.$$

For the interested reader, we refer to [31] for a complete discussion while mechanical structures of this type can be worked out along these lines [5].

2.4 Cascade Dynamics With State Delays

Let us now extend the previous arguments to the case of feedback-like cascade dynamics affected by state delays. Consider the dynamics

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)); \qquad \dot{x}_2(t) = u(t) \tag{9}$$

with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}$ and $f_1(x_{1*}, x_{2*}) = 0$ and assume that a time delay τ is affecting the connection variable x_2; i.e.,

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t - \tau)); \qquad \dot{x}_2(t) = u(t). \tag{10}$$

By setting $x_2^r(t) = x_2(t - \tau)$, the delay τ is moved onto the input variable u; namely, one gets

$$\dot{x}_1(t) = f_1(x_1(t), x_2^r(t)); \qquad \dot{x}_2^r(t) = u(t - \tau) \tag{11}$$

that exhibits the form (1) with $x = (x_1^\top, x_2^r)^\top$. Analogously, one can consider a delay in the connection variable x_2 in the feedforward like extension of (9)

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t - \tau)) \\ \dot{x}_2(t) &= g_2(x_2(t), x_3(t - \tau), u(t)) \\ &\;\;\vdots \\ \dot{x}_n(t) &= g_n(x_n(t), u(t)) \end{aligned} \tag{12}$$

with $x_1 \in \mathbb{R}^{n_1}$, $x_i \in \mathbb{R}$ for $i = 2, \ldots, n$ and $f_1(x_{1*}, x_{2*}) = 0$. By setting $x_i^r(t) = x_i(t - \tau)$ for $i = 2, \ldots, n$, one moves the delay τ onto the successive interconnecting variable x_{i+1} and onto the input u until one recovers a specific case of (1) with $x = (x_1, x_2^r, \ldots, x_n^r)$. In Sect. 5, we prove sampled-data stabilizability of (10) by assuming the existence of a *fictitious feedback law* $x_2 = k(x_1)$ ensuring stabilization of the x_1-dynamics (thus replacing Assumption **A**).

3 Sampled-Data Predictor-Based Stabilization

Consider now the input delayed dynamics (1) with extended sampled equivalent model (4). The design of predictor-based controllers with discrete-time prediction map is described below. We first recall that from Assumption **A**, one directly infers the existence of a sampled-data feedback $u_k = \gamma^\delta(x_k)$ stabilizing the delay-free sampled dynamics while guaranteeing, at the sampling instants, the same Lyapunov performances as in continuous time. The following result is recalled from [26].

Theorem 1 (Input-Lyapunov Matching - ILM) *Consider the delay free dynamics associated to (1) under* **A**. *Then, there exist* $T > 0$ *and for each* δ *in* $]0, T[$, *a sampled feedback* $\gamma^\delta(x)$ *which satisfies the ILM equality*

$$V(F^\delta(x_k, u_k)) - V(x_k) = \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f(\cdot,\gamma(\cdot))} V(x(s)) ds \tag{13}$$

i.e. S-GAS is yielded by $u_k = \gamma^\delta(x_k)$.

The proof is constructive and $\gamma^\delta(\cdot)$ is described by its series expansion around $\gamma(x)$; i.e. $\gamma^\delta(x) = \gamma(x) + \sum_{i \geq 1} \frac{\delta^i}{(i+1)!} \gamma_i(x)$ with $\gamma^\delta(x_*) = 0$. This result is developed in [32] when considering $f(x, u) = f(x) + ug(x)$ with constructive algorithms for computing the feedback solution.

Stabilization of the input delayed dynamics (4) follows through state prediction; i.e. $u_k = \gamma^\delta(x_{k+N})$ with N-steps ahead prediction and suitably chosen initial conditions.

Theorem 2 *Consider (1) under the assumptions of Theorem 1. Then,* $x_k^p := F^\delta(\cdot, u_{k-N}) \circ \ldots \circ F^\delta(x_k, u_{k-1})$ *is a predictor for (1) with dynamics*

$$x_{k+1}^p = F^\delta(x_k^p, u_k) \tag{14}$$

and initial conditions $x_0^p = F^\delta(\cdot, u_{-N}) \circ \ldots \circ F^\delta(x_0, u_{-1})$. *Moreover, the feedback* $u_k = \gamma^\delta(x_k^p)$ *makes the closed-loop equilibrium of (1) S-GAS.*

The above result underlines how sampled-data stabilizability is directly ensured by smooth stabilizability of the continuous-time delay-free dynamics. Contrarily to existing works dealing with sample-and-hold solutions, the proposed compensating feedback is based on a two step redesign procedure: first, for $\tau = 0$, a sampled-data feedback ensuring GAS in closed-loop is constructed through Input Lyapunov Matching (ILM); then, the final sampled-data stabilizing predictor-based feedback for the retarded dynamics is built by defining a discrete-time predictor dynamics.

Remark 5 Contrarily to the predictor-based techniques proposed in [10, 11], we describe a discrete-time prediction map (14) that exploits the piecewise constant nature of the input. This implies that prediction of the full state at any time instant t is unnecessary for stabilizing purposes so overcoming some tough numerical issues.

4 Immersion and Invariance Stabilization with Input Delays

As well known, predictor-based techniques strongly suffer from robustness to prediction errors. In this context, stabilization through invariant sets offers interesting refinements to Lyapunov-based control by exploiting the system structure. Taking advantage of the underlying cascade of (4), in [24] we have shown that Immersion and Invariance [1, 2] provides a natural set-up for input-delayed dynamics under sampling. More in detail, the delay-free stable closed loop dynamics identifies the target systems evolving over a stable manifold. Accordingly, I&I stabilization requires to drive the off-the manifold trajectories to zero with boundedness of the full state ones. Manifold invariance guarantees that the on-the-manifold closed-loop dynamics recover the predictor evolutions defining the target. The I&I feedback modifies the predictor-based one so preventing from big control effort and improving robustness when predictor-based controllers cannot be exactly computed.

4.1 I&I Stabilization

Immersion and Invariance was firstly introduced in continuous time in [2] and then proposed for discrete-time adaptive control in [33]. It is reformulated below in the discrete-time and sampled contexts for completeness (see also [1, 14, 17]).

Theorem 3 *Consider a nonlinear discrete-time dynamics in the form of a map*

$$x_{k+1} = F(x_k, u_k) \tag{15}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and equilibrium state x_* to be stabilized. Assume that there exists a $p < n$ so that there exist mappings $\alpha(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$, $\pi(\cdot) : \mathbb{R}^p \to \mathbb{R}^n$, $c(\cdot) : \mathbb{R}^p \to \mathbb{R}$, $\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n-p}$ and $\psi(\cdot, \cdot) : \mathbb{R}^{n\times(n-p)} \to \mathbb{R}$ such that the following four conditions hold:

($\boldsymbol{I}_1$) *the target dynamics* $\xi_{k+1} = \alpha(\xi_k)$ *with* $\xi \in \mathbb{R}^p$ *has a GAS equilibrium at* ξ_* *and* $x_* = \pi(\xi_*)$*;*
($\boldsymbol{I}_2$) $F(\pi(\xi_k), c(\xi_k)) = \pi(\alpha(\xi_k))$;
($\boldsymbol{I}_3$) *the following set-identity holds:*

$$\left\{x \in \mathbb{R}^n | \phi(x) = 0\right\} = \left\{x \in \mathbb{R}^n | x = \pi(\xi) \quad for \quad \xi \in \mathbb{R}^p\right\};$$

($\boldsymbol{I}_4$) *setting* $z = \phi(x)$ *with* $z_0 = \phi(x_0)$*, the trajectories of*

$$z_{k+1} = \phi(F(x_k, \psi(x_k, z_k))); \qquad x_{k+1} = F(x_k, \psi(x_k, z_k))$$

are bounded and satisfy $\lim_{k\to\infty} z_k = 0$ *and* $\psi(\pi(\xi), 0) = c(\xi)$.

Then, x_* *is a globally asymptotically stable equilibrium of the closed loop dynamics*

$$x_{k+1} = F(x_k, \psi(x_k, \phi(x_k))). \tag{16}$$

Definition 2 Any discrete-time system (15) satisfying conditions I_1 to I_4 of Theorem 3 is said *I&I stabilizable* with target dynamics $\xi_{k+1} = \alpha(\xi_k)$.[5]

Definition 3 The continuous-time dynamics $\dot{x} = f(x, u)$ is said *sampled-data I&I stabilizable* if its sampled equivalent dynamics $x_{k+1} = F^\delta(x_k, u_k) = e^{\delta f(\cdot, u_k)} x\big|_{x_k}$ with $u \in \mathcal{U}_\delta$ is I&I stabilizable in the sense of Definition 2.

4.2 I&I Stabilization With Input Delays

Further exploiting the structure of the extended sampled model (4), we show that Assumption **A** implies its I&I stabilizability. We are now extending the result in [24] to more general systems of the form (1). Let us rewrite (4) in compact form as

$$x^e_{k+1} = F^\delta_e(x^e_k, u_k) \tag{17}$$

with $x^e = (x^\top, \bar{v}^\top)^T \in \mathbb{R}^{n+N}$, $\bar{v} = (v^1, \ldots, v^N)^\top \in \mathbb{R}^N$ and equilibrium $x^e_* = (x_*^\top, 0)^\top$. According to Theorem 3, one defines the target dynamics

$$x_{k+1} = \alpha^\delta(x_k) := F^\delta(x_k, \gamma^\delta(x_k)) \tag{18}$$

whose equilibrium x_* is GAS by construction of $\gamma^\delta(\cdot)$ in Theorem 1. Setting now

$$\bar{z} := (z^1, \cdots, z^N)^\top := \phi^\delta(x^e) = (\phi^\delta_1(x^e), \ldots, \phi^\delta_N(x^e))^\top \tag{19}$$

[5] Mappings and dynamics are parameterized by δ as indicated with superscript $(\cdot)^\delta$.

with

$$\phi_i^\delta(x_k, \bar{v}_k) := v_k^i - \gamma^\delta(x_{k+i-1}), \quad x_{k+i} = \underbrace{e^{\delta f(\cdot, v_k^1)} \circ \cdots \circ e^{\delta f(\cdot, v_k^{i-1})}}_{i \text{ times}} x\Big|_{x_k}$$

and $\pi^\delta(\cdot) := ((\cdot)^\top, \gamma^\delta(\cdot), \ldots, \gamma^\delta((\alpha^\delta)^{N-1}(\cdot)))^\top$, the following result can be stated.

Theorem 4 *Let the continuous-time input delayed dynamics (1) satisfy Assumption* **A** *and the delay-free stabilizer $\gamma^\delta(\cdot)$ be the solution of (13). Then (1) is sampled-data I&I stabilizable; i.e. its sampled equivalent dynamics (17) is I&I stabilizable with target dynamics (18).*

Consider $\gamma^\delta(\cdot)$ as defined in (13) with control Lyapunov function $V(\cdot)$. Computing now $\pi^\delta(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n+N}$ as above, it is a matter of computations to verify that conditions I_1 to I_3 of Theorem 3 are satisfied by construction with $\phi^\delta(\cdot)$ described in (19) and

$$c^\delta(x_k) = \gamma^\delta(x_{k+N}) = \gamma^\delta \circ \underbrace{\alpha^\delta \circ \cdots \circ \alpha^\delta}_{N \text{ times}}(x_k). \tag{20}$$

On these bases and from Theorem 3, one directly infers that any feedback $u = \psi^\delta(x, \bar{v}, \bar{z})$ such that $\psi^\delta(\pi^\delta(x), 0) = c^\delta(x)$ and ensuring I&I stabilization also achieves GAS of the equilibrium of the extended system

$$x_{k+1} = F^\delta(x_k, v_k^1); \quad v_{k+1}^1 = v_k^2; \quad \ldots \quad ; v_{k+1}^{N-1} = v_k^N; \qquad v_{k+1}^N = \psi^\delta(x_k, \bar{v}_k, \bar{z}_k)$$

whenever it is designed to drive $\bar{z}$ to zero with boundedness of the state trajectories of the full dynamics

$$\begin{aligned} z_{k+1}^1 &= z_k^2; \quad \ldots \quad ; z_{k+1}^{N-1} = z_k^N; \qquad z_{k+1}^N = u_k - \gamma^\delta(x_{k+N}) \\ x_{k+1} &= F^\delta(x_k, v_k^1) \\ v_{k+1}^1 &= v_k^2; \quad \ldots \quad ; v_{k+1}^{N-1} = v_k^N; \qquad v_{k+1}^N = u_k. \end{aligned} \tag{21}$$

As a consequence of the structure of (21), condition I_4 of Theorem 3 relaxes to requiring $\lim_{k\to\infty} \psi^\delta(x_k, \bar{v}_k, \bar{z}_k) - \psi^\delta(x_k, \bar{v}_k, 0) = 0$ with $\psi^\delta(x_k, \bar{v}_k, 0) = \gamma^\delta \circ (\alpha^\delta)^N(x_k)$. In [24], a multi-rate control strategy is proposed for stretching $\bar{z}$ to zero in exactly N steps. However, for implementation issues, an approximate controller of the form

$$u_k = \gamma^\delta \circ (\alpha^\delta)^N(x_k) + L^\delta(x_k)\bar{z}_k \tag{22}$$

with suitably defined gain matrix $|L^\delta(x)| < 1$ is needed. The approximate feedback (22) achieves, at least locally, asymptotic stabilization of the equilibrium with improved robustness performances with respect to uncertainties on the delay length or discarded higher order components in the predictor dynamics. As a matter of fact, I&I introduces a feedback term over the prediction error while prediction-based feedback usually work in open-loop (see [24] for a more complete discussion).

Remark 6 When the stable manifold is reached (i.e., $\bar{z} = 0$), $c^\delta(x)$ in (20) recovers the N-steps ahead predictor-based feedback $\gamma^\delta \circ (\alpha^\delta)^N(x)$ while in the delay free case ($N = 0$), $c^\delta(x)$ reduces to the original ILM-based feedback $\gamma^\delta(x)$.

Remark 7 When $N = 1$, the I&I feedback is reminescent of the first step of a back-stepping design with Lyapunov function $\bar{V}(x, z) = V(x) + \frac{1}{2}z^2$. When $N > 1$, the N-steps ahead I&I strategy recalls the N-rate backstepping design developed in [32].

4.2.1 An Example

The following two block cascade system is exploited as an illustrative example throughout the paper. Let

$$\dot{x}_1(t) = x_1(t)x_2(t); \qquad \dot{x}_2(t) = u(t) \tag{23}$$

be stabilized by the feedback $u = -2x_2 - x_1^2 e^{2x_2}$ and Lyapunov function $V(x) = \frac{1}{2}(x_1^2 e^{2x_2} + x_2^2)$. The exact sampled model of (23) takes the form

$$x_{1k+1} = e^{\delta(x_{2k} + \frac{\delta}{2}u_k)} x_{1k}; \qquad x_{2k+1} = x_{2k} + \delta u_k. \tag{24}$$

and a stabilizing sampled-data controller can be computed through ILM design as

$$u_k = \gamma^\delta(x_k) = -2x_{2k}\left(1 - \frac{\delta}{2}\right) - x_{1k}^2 e^{2x_{2k}}\left(1 - \frac{\delta}{2}(1 + x_{1k}^2 e^{2x_{2k}})\right) + O(\delta^2)$$

which is approximated in $O(\delta^2)$. Assuming now an input delay of amplitude δ (i.e. $u(t - \tau) = u(t - \delta)$) in (23), one defines the extended sampled equivalent dynamics

$$x_{1k+1} = e^{\delta(x_{2k} + \frac{\delta}{2}v_k)} x_{1k}; \quad x_{2k+1} = x_{2k} + \delta v_k; \quad v_{k+1} = u_k. \tag{25}$$

Accordingly, the *sampled predictor-based feedback* is $u_k = \gamma^\delta(x_k^p)$ with $x_k^p = x_{k+1}$:

$$\gamma^\delta(x^p) = -2a^\delta(x) - b^\delta(x) + \frac{\delta}{2}(2a^\delta(x) + b^\delta(x)(1 + b^\delta(x)) \tag{26}$$

with $a^\delta(x) = x_2 + \delta\gamma^\delta(x)$; $b^\delta(x) = e^{2(x_2 + \gamma^\delta(x))} e^{2\delta(x_2 + \frac{\delta}{2}\gamma^\delta(x))} x_1^2$. As discussed, a modified I&I feedback with sampled-data target dynamics

$$x_{1k+1} = e^{\delta(x_{2k} + \frac{\delta}{2}\gamma^\delta(x_k))} x_{1k}; \qquad x_{2k+1} = x_{2k} + \delta\gamma^\delta(x_k). \tag{27}$$

and off-the-manifold component $z = v - \gamma^\delta(x)$, can be designed to bring z to zero for the dynamics

$$x_{1k+1} = e^{\delta(x_{2k}+\frac{\delta}{2}\gamma^{\delta}(x_k)+\frac{\delta}{2}z_k)}x_{1k}; \quad x_{2k+1} = x_{2k} + \delta\gamma^{\delta}(x_k) + \delta z_k; \quad z_{k+1} = u_k - \gamma^{\delta}(x_{k+1}).$$

Accordingly, one computes (omitting the $_k$-dependency)

$$\psi^{\delta}(x) = -\,2a^{\delta}(x) - 2\delta z - b^{\delta}(x)e^{3\delta z} + \frac{\delta}{2}(2a^{\delta}(x) + 2\delta z + b^{\delta}(x)e^{3\delta z}(1 + b^{\delta}(x)e^{3\delta z})$$

so recovering $\gamma^{\delta}(x_k^p)$ when $z = 0$ with robustness improvement when $z \neq 0$.

5 Sampled-Data I&I Stabilization with State Delays

In this section, stabilization of the strict-feedback dynamics (10) with delays on the connecting state variable is addressed. Since (10) exhibits the form (1), the former arguments still apply for sampled stabilization when setting as extended state $x = (x_1^T, x_2)^T$. A different solution exploiting the interconnection structure of (9) is described below when reformulating Assumption **A** over $f_1(\cdot)$ in (9) as follows.

Assumption **B** - There exists a smooth mapping $x_2(t) = k(x_1)$ with $k(x_{1*}) = x_{2*}$ and a proper Lyapunov function $W : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ such that $\dot{W}(x_1) = \mathrm{L}_{f(x_1,k(x_1))}W(x_1) < 0$ and $\frac{\partial L_{f_1}W}{\partial x_1}\Big|_{x_2=k(x_1)} \neq 0$ for any $x_1 \neq x_{1*}$.

Remark 8 Through easy backstepping-like arguments, one can prove that Assumption **B** implies Assumption **A**.

Consider (9) under Assumption **B** (equivalently (11) when setting $x_2^r(t) = x_2(t-\tau)$) with sampled equivalent dynamics described over $\mathbb{R}^{n_1+1} \times \mathbb{R}^N$ as

$$x_{1k+1} = F_1^{\delta}(x_{1k}, x_{2k}^r, v_k^1); \quad x_{2k+1}^r = x_{2k}^r + \delta v_k^1; \quad v_{k+1}^1 = v_k^2; \quad \ldots; v_{k+1}^N = u_k \tag{28}$$

or, in a more compact form, as $x_{k+1}^e = F_e^{\delta}(x_k^e, u_k)$ with $x^e = (x_1^{\top}, x_2^r, v^1, \ldots, v^N)^{\top}$. Two preliminary results are instrumental for extending to (9) the results provided in [14, 17] for delay-free input-affine systems. More in detail, we show what follows:

- when $\tau = 0$, Assumption **B** implies I&I stabilizability of the delay free dynamics (9) which implies I&I stabilizability of its equivalent sampled model (Proposition 2);
- sampled I&I stabilizability of the delay-free (9) implies sampled-data I&I stabilizability of the retarded (10) (Theorem 5).

Proposition 2 *Let (9) satisfy Assumption* **B**. *Then, its sampled equivalent dynamics*

$$x_{1k+1} = F_1^{\delta}(x_{1k}, x_{2k}, u_k); \quad x_{2k+1} = x_{2k} + \delta u_k$$

is I&I stabilizable with target dynamics

$$x_{1k+1} = F_1^{\delta}(x_{1k}, k^{\delta}(x_{1k}), \gamma^{\delta}(x_{1k})) \tag{29}$$

where $k^\delta = k + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!} k_i$ *and* $\gamma^\delta = \dot{k} + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!}\gamma_i$ *are solutions of the equalities*

$$W(x_{1k+1}) = W(x_{1k}) + \int_{k\delta}^{(k+1)\delta} L_{f(\cdot,k(\cdot))} W(x_1(\tau))d\tau \tag{30a}$$

$$k^\delta(x_{1k+1}) = k^\delta(x_{1k}) + \delta\gamma^\delta(x_{1k}). \tag{30b}$$

Equality (30a) ensures ILM at the sampling instants of the closed loop Lyapunov function $W(x_1)$ on the target dynamics (29) and, hence, the stability of its closed-loop equilibrium. Equality (30b) guarantees invariance of the corresponding manifold that is implicitly defined by the condition $z = x_2 - k^\delta(x_1) = \phi^\delta(x_1, x_2) \equiv 0$. On these bases, the sampled I&I stabilizing delay-free feedback $u = \psi^\delta(x, z)$ is designed to drive z to zero while preserving boundedness of the complete state trajectories

$$x_{1k+1} = F_1^\delta(x_{1k}, x_{2k}, u_k);\ x_{2k+1} = x_{2k} + \delta u_k;\ z_{k+1} = z_k + \delta u_k - k^\delta(x_{1k+1}) + k^\delta(x_{1k}).$$

GAS of the closed-loop x-dynamics follows (delay free case) with

$$\psi^\delta(x_1, k^\delta(x_1), 0) = \gamma^\delta(x_1) = \frac{1}{\delta}\Big(k^\delta(F_1^\delta(x_{1k}, k^\delta(x_{1k}), \gamma^\delta(x_{1k}))) - k^\delta(x_{1k})\Big)$$

when $z = 0$ as implied by (30b).

Remark 9 The existence of solutions to equalities (30) is guaranteed by the cascade structure of (9) and $\frac{\partial L_{f_1} W}{\partial x_1}\Big|_{x_2=k(x_1)} \neq 0, \forall x_1 \neq x_{1*}$.

Remark 10 It is a matter of computations to verify that the pair $(k, \dot{k})$ satisfies equalities (30) in $O(\delta^2)$ so emphasizing the fact that the sampled-data pair $(k^\delta, \gamma^\delta)$ is computed around the continuous-time solution $(k, \dot{k})$.

The following result generalizes Theorem 3.1 in [15]

Theorem 5 *Consider the continuous-time dynamics (10) with* $\tau = N\delta$. *Suppose that when* $\tau = 0$, *(10) satisfies Assumption* **B** *with* $\frac{\partial L_{f_1} W}{\partial x_1}\Big|_{x_2=k(x_1)} \neq 0, \forall x_1 \neq x_{1*}$. *Then, the extended sampled equivalent dynamics (28) is I&I stabilizable with target dynamics (29).*

The above result follows when defining the pair $(k^\delta, \gamma^\delta)$ as in Proposition 2. Accordingly, the subsequent extended immersion mapping $\bar{\pi}^\delta : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1+1} \times \mathbb{R}^N$ takes the form

$$\bar{\pi}^\delta(x_{1k}) = (x_{1k}^\top, k^\delta(x_{1k}), \gamma^\delta(x_{1k}), \ldots, \gamma^\delta(x_{1k+N-1}))^\top \tag{31}$$

while the extended $\bar{\phi}^\delta : \mathbb{R}^{n_1+1} \times \mathbb{R}^N \to \mathbb{R}^{N+1}$ with $\bar{v} = (v^1, \ \ldots, \ v^N)^\top$ is provided by

$$z_{1k} = \bar{\phi}_1^{\delta}(x_{1k}, x_{2k}^r, v_k) = x_{2k}^r - k^{\delta}(x_{1k}); \quad z_{2k} = \bar{\phi}_2^{\delta}(x_{1k}, x_{2k}^r, v_k) = v_k^1 - \gamma^{\delta}(x_{1k})$$
$$\ldots$$
$$z_{N+1k} = \bar{\phi}_{N+1}^{\delta}(x_{1k}, x_{2k}^r, v_k) = v_k^N - \gamma^{\delta}(x_{1k+N-1}).$$

On these bases, I&I stabilization of (10) at the sampling instants is achieved by any feedback $u_k = \bar{\psi}^{\delta}(x_{1k}, x_{2k}^r, \bar{z}_k)$ satisfying $\bar{\psi}^{\delta}(x_1, k^{\delta}(x_1), 0) = \gamma^{\delta}(x_1)$ and designed to bring $\bar{z}$ to zero with boundedness of the trajectories of the $(x_1^{\top}, x_2, \bar{z})$-dynamics. We showed in [17] that multi-rate digital control strategies of order equal to the dimension of $\bar{z}$, the off-the manifold component, are suitable to ensure invariance of the manifold and drive $\bar{z}$ to zero.

5.1 Example

Consider again the system (23) but now notice that for $x_2 = -\frac{1}{2}x_1^2$ the sub-system $\dot{x}_1 = -\frac{1}{2}x_1^3$ has a GAS equilibrium at the origin with Lyapunov function $V_0(x_1) = \frac{1}{2}x_1^2$. Suppose now that a delay τ is acting on the transmission variable x_2, namely,

$$\dot{x}_1(t) = x_1(t)x_2(t-\tau), \quad \dot{x}_2(t) = u(t).$$

Under state transformation $x_2^r(t) = x_2(t-\tau)$ and dynamical extension, one gets

$$\dot{x}_1(t) = x_1(t)x_2^r(t); \quad \dot{x}_2^r(t) = v_k; \quad v_{k+1} = u_k$$

with exact (single-rate) sampled equivalent model

$$x_{1k+1} = e^{\delta(x_{2k}^r + \frac{\delta}{2}v_k)}x_{1k}; \quad x_{2k+1}^r = x_{2k}^r + \delta v_k; \quad v_{k+1} = u_k.$$

Accordingly, I&I applies to the above system by setting the target dynamics as $x_{1k+1} = e^{\delta(k^{\delta}(x_{1k}) + \frac{\delta}{2}c^{\delta}(x_{1k}))}x_{1k}$ with the solutions to (30a)–(30b) in $O(\delta^4)$ as

$$k^{\delta}(x_1) = -\frac{1}{2}x_1^2 - \frac{\delta^2}{6}\left(x_1^2 + \frac{1}{8}\right)x_1^4 + O(\delta^3) \quad c^{\delta}(x_1) = \frac{1}{2}x_1^4 + \delta\left(\frac{1}{8} - x_1^2\right)x_1^4 + O(\delta^2).$$

Setting $z_1 := x_2 - k^{\delta}(x_1)$, $z_2 := v - c^{\delta}(x_1)$ and $u_{ik} = u(k\delta + \frac{(i-1)}{2}\delta)$ for $i = 1, 2$, one computes the double-rate equivalent model as

$$x_{1k+1} = e^{\frac{\delta}{2}(2(z_1 + k^{\frac{\delta}{2}}(x_{1k})) + \frac{3\delta}{4}(z_2 + c^{\frac{\delta}{2}}(x_{1k})) + \frac{\delta}{4}u_k^1)}x_{1k}; \quad x_{2k+1}^r = x_{2k}^r + \frac{\delta}{2}(v_k + u_k^1); \quad v_{k+1} = u_k^2$$
$$z_{1k+1} = z_{1k} + k^{\frac{\delta}{2}}(x_{1k}) - k^{\frac{\delta}{2}}(x_{1k+1}) + \frac{\delta}{2}(z_2 + c^{\frac{\delta}{2}}(x_{1k}) + u_{1k}), \quad z_{2k+1} = u_{2k} - c^{\frac{\delta}{2}}(x_{1k+1}).$$

Accordingly, the feedback $u = \mathrm{col}(u_1, u_2)$ solution to

$$\frac{\delta}{2}u_1 = k^{\frac{\delta}{2}}(x_{1k+1}) - k^{\frac{\delta}{2}}(x_{1k}) - \frac{\delta}{2}(z_2 + c^{\frac{\delta}{2}}(x_1)), \quad u_2 = c^{\frac{\delta}{2}}(x_{1k+1})$$

guarantees I&I stabilization in closed-loop.

6 Conclusion

The paper revisits recent authors' works by emphasizing the role of sampling for stabilizing nonlinear time-delay systems. While assuming entire delays, we show that smooth stabilizability of the continuous-time delay-free system is enough for deducing the existence of a stabilizer for the retarded dynamics. The proposed solution employs a discrete-time predictor which is instrumental for designing the stabilizing feedback. Finally, a robust redesign is carried out by extending the Immersion and Invariance approach to time-delay systems. The proposed design methodologies are constructive. Perspectives are opened toward more general classes of time-delay systems (non-entire, distributed and multi-channel delays) together with a suitable comparison with reduction-based techniques.

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