



Convective Motions in Water: Linear and Nonlinear Models, Criteria of Convection Onset

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Abstract. The problem of convective motions adequate description on the base of linear or nonlinear mathematic models is considered. The criterion of convection onset is formulated in the form $|\mathbf{J}_{conv}|/|\mathbf{J}_{cond}| > 1$ where the $|\mathbf{J}_{conv}|$ and $|\mathbf{J}_{cond}|$ are the convective and conductive fluxes of heat correspondingly. As a result of governing equations solution for initial time moments of the problem the characteristic scales are chosen. It is shown that for Rayleigh-Bénard convection the Rayleigh number $Ra = \alpha g |\nabla T| d^4 / \chi \nu$ should be attributed not to the onset, but to the intensity of convection and to the rate of its development after the onset.

Keywords: Convection · Heat fluxes · Onset criterion

1 Introduction

During more than one century after the basic experimental work by Bénard [1] and the prime theoretical investigation by the Lord Rayleigh [2], the problem of the heat convection occupies one of the fundamental place in hydrodynamic studies. The results of linear approach to the announced problem were summarized in [3]. The further progress in convection phenomena analysis was developed by means of weak nonlinear models for which the different mathematical approaches were applied [4–6].

The results of experiments [7] and correct form of mathematical model equations *sufficiently* depend on the substance in which the convection is developed. The reason is the dependence of thermodynamic characteristics of liquid on the temperature particularly in the regimes with great overheating [8]. Hence, it is necessary at once to declare the substance for which the problem is considered.

The main goal of the presented work is to investigate the problem of convection in the water without admixtures.

2 The Governing Equations

The classic system of governing equations and boundary conditions of the problem under consideration has the form

$$\begin{aligned} \rho (\partial \mathbf{v}_i / \partial t + (\mathbf{v} \cdot \nabla) \mathbf{v}_i) &= -\partial p / \partial x_i + \partial \sigma'_{ik} / \partial x_k - \rho g \delta_{iz} \\ \rho T (\partial s / \partial t + \mathbf{v} \cdot \nabla s) &= \sigma'_{ik} \partial \mathbf{v}_i / \partial x_k + \nabla \cdot (\kappa \nabla T) \\ \partial \rho / \partial t + \nabla \cdot (\rho \mathbf{v}) &= 0, \quad \rho = \rho(p, s) \end{aligned} \quad (1)$$

Here $\sigma'_{ik} = \eta (\partial \mathbf{v}_i / \partial x_k + \partial \mathbf{v}_k / \partial x_i - 2\delta_{ik} \nabla \cdot \mathbf{v} / 3) + \zeta \delta_{ik} \nabla \cdot \mathbf{v}$ is the tensor of viscose tension, η and ζ are the first and second dynamic viscosity correspondingly, ρ is the liquid density, p is the pressure, T is the temperature, s is the entropy per unit volume, g is gravitational acceleration, κ is heat conductivity coefficient.

The system (1) consists of Navier-Stokes, heat transport, continuity and state equations. In the general case, it is regarded that the density, kinetic coefficients and coefficient of thermal expansion depend on the temperature.

From the state equation it follows

$$d\rho = \left(\frac{\partial \rho}{\partial p} \right)_s dp + \left(\frac{\partial \rho}{\partial s} \right)_p ds = \frac{1}{c^2} dp + \left(\frac{\partial \rho}{\partial T} \right)_p \left(\frac{\partial T}{\partial s} \right)_p ds = \frac{1}{c^2} dp - \frac{\rho T}{c_p} \alpha ds$$

where c is adiabatic speed of sound, c_p is heat capacity under constant pressure, $\alpha = -(\partial \rho / \partial T)_p / \rho$ is the coefficient of thermal expansion which also depend on the temperature.

Because in convective regimes the generation of sound may be neglected then the continuity equation reforms into equation

$$\begin{aligned} \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} &= \rho \nabla \cdot \mathbf{v} - \frac{\rho \alpha T}{c_p} \frac{ds}{dt} = 0 \Rightarrow \\ \Rightarrow \nabla \cdot \mathbf{v} &= \frac{\alpha T}{c_p} \frac{ds}{dt} = \frac{\alpha}{\rho c_p} \left(\sigma'_{ik} \partial \mathbf{v}_i / \partial x_k + \nabla \cdot (\kappa \nabla T) \right). \end{aligned} \quad (2)$$

On the base of thermodynamic relation $ds = c_p dT / T$ and Eq. (2) it is possible to rewrite the system (1) in the form

$$\begin{aligned} \rho (\partial \mathbf{v}_i / \partial t + (\mathbf{v} \cdot \nabla) \mathbf{v}_i) &= -\partial p / \partial x_i + \partial \sigma'_{ik} / \partial x_k - \rho g \delta_{iz} \\ \nabla \cdot \mathbf{v} &= \alpha (\partial T / \partial t + \mathbf{v} \cdot \nabla T) = \frac{\alpha}{\rho c_p} \left(\sigma'_{ik} \partial \mathbf{v}_i / \partial x_k + \nabla \cdot (\kappa \nabla T) \right). \end{aligned} \quad (3)$$

The boundary conditions on the rigid surfaces consist of the condition of zero velocity field and either fixing value of temperature

$$T|_{\Sigma} = T_{\Sigma} \quad (4)$$

or fixing thermal flux

$$\chi \partial T / \partial n + \gamma_T T |_{\Sigma} = q_T \quad (5)$$

where $\chi = \kappa / \rho c_p$ is the thermal diffusion coefficient, γ_T is the coefficient of heat exchange of boundary surface and water, q_T is heat flux and n is the unit normal to the boundary. In the presented investigation the boundary conditions

$$T|_{z=0} = T_b \quad T|_{z=d} = T_c \quad T_b > T_c \quad (6)$$

will be used, where T_b and T_c are the temperatures of bottom and cover respectively.

3 The Onset of Convection

As it was proclaimed in the **Introduction** the reason for convection onset is nonuniform temperature distribution along the bottom or/and cover. Let in real situation the temperature of bottom may be presented as the sum of mean temperature T_b and some disturbances $\tilde{T}(x, y, t)$ with zero mean value

$$T = T_b + \tilde{T}(x, y, t), \quad \langle \tilde{T}(x, y, t) \rangle = \lim_{S \rightarrow \infty} \frac{\int_S \tilde{T}(x, y, t) dx dy}{S} = 0 \quad (7)$$

where S is a square of heater.

The density of water without sub mixtures at atmospheric pressure is defined by the relation

$$\rho(T) = \rho_0 \exp \left(T \left(\alpha_0 - \frac{\alpha_1}{1 + \alpha_2} \left(\frac{T}{T_r} \right)^{\alpha_2} \right) \right) \quad (8)$$

where T is the water temperature in centigrade degree, $T_r = 273.15 K$ is reference temperature (ice melting temperature in absolute thermodynamic scale), and the constants are $\rho_0 = 999.87 \text{ kg/m}^3$, $\alpha_0 = 1.83 \cdot 10^{-4} K^{-1}$, $\alpha_1 = 1.6 \cdot 10^{-3} K^{-1}$, $\alpha_2 = 0.54$. The relative error of the above relation is not greater than 10^{-3} [9] for whole diapason of liquid water phase.

Substitution of (7) into (8) permits to describe the distribution of water density near the heater by the relation

$$\rho = \rho_0 \exp \left(\alpha (1 + \tilde{T}/T_b) \left(1 - \beta \left(1 + \frac{\tilde{T}}{T_b} \right)^{\alpha_2} \right) \right) \approx \quad (9)$$

$$\approx R_0 (1 + \gamma x + \delta x^2), \quad x = \frac{\tilde{T}}{T_b}$$

where $R_0 = \rho_0 \exp(\alpha(1 - \beta))$ is the density of water on the heater surface in the case of absense of temperature disturbances \tilde{T} , $\alpha = \alpha_0 T_b$, $\beta = \frac{\alpha_1}{\alpha_0(1 + \alpha_2)}$

$\left(\frac{T_b}{T_r} \right)^{\alpha_2}$, $\gamma = \alpha(1 - (1 + \alpha_2)\beta)$, $\delta = \frac{\alpha}{2} (\alpha - (1 + \alpha_2)(\alpha_2 + 2\alpha)\beta + \alpha(\alpha + \alpha_2)^2 \beta^2)$.

The relative error of the (9) is not greater than 10^{-2} .

The goal of further actions is to calculate the dispersion and root mean square deviation of density variations from ideal value R_0 near heater surface. Accordingly to (9) the difference of density is defined by relation

$$\Delta\rho = \rho - R_0 \approx R_0(\gamma x + \delta x^2) \quad (10)$$

then with regard to $\langle x \rangle = 0$ (because $\langle \tilde{T} \rangle = 0$) the relation is valid

$$\overline{\Delta\rho} = R_0\overline{\delta x^2} \quad \Rightarrow \quad \overline{\Delta\rho^2} = R_0^2\overline{\delta^2 x^2} \quad (11)$$

From the other hand

$$(\Delta\rho)^2 \approx R_0^2(\gamma x + \delta x^2)^2 = R_0^2(\gamma^2 x^2 + 2\gamma\delta x^3 + \delta^2 x^4) \approx R_0^2\gamma^2 x^2 \quad (12)$$

where the last approximate relation is provided by the smallness of x and δ/γ for real experimental regimes.

On the base of (11, 12) the dispersion of density variations is calculated

$$\sigma_\rho^2 = \overline{(\Delta\rho)^2} - (\overline{\Delta\rho})^2 = R_0^2\overline{\gamma^2 x^2} - R_0^2\overline{\delta^2 x^2} \approx R_0^2\overline{\gamma^2 x^2} \quad (13)$$

also as density rms deviation

$$\sigma_\rho = R_0|\gamma|\sigma_T \quad (14)$$

where $\sigma_T = \sqrt{\overline{x^2}}$ is rms deviation of heater surface temperature.

Because the buoyancy force is defined as product of gravitational acceleration and water density deviation from its mean value then rms deviation of Archimedes force equals to the value

$$\sigma_A = g\sigma_\rho = gR_0|\gamma|\sigma_T. \quad (15)$$

In Boussinesq approximation the inertial acceleration of liquid element at initial time moments is defined by the ratio of buoyancy force to mean water density, i.e. by the value $a_{in} = g|\gamma|\sigma_T$. When the rms deviation of temperature is fixed then the more the temperature of the heater, the more the rms of buoyancy force (and inertial acceleration). It means that the local perturbations of conductive mechanism of heat transfer due to convective flux are rising under heater temperature increasing even through immutable quality of the heater temperature stabilization.

Let's consider some small spherical liquid element of radius R near the bottom heater (analogous procedure can be done also for cover colder). Accordingly to probability theory its more probable overheating temperature is $T_b + \sigma_T$. The mean temperature on the same horizontal level is T_b then the buoyance force will be act on this element. Under action of this force the liquid element will float up and as the development of motion the viscose force will arise. Let $\bar{T}(t)$ is the mean temperature of element, so its density is defined by the relation

(8) $\bar{\rho}(t) = \rho(\bar{T}(t))$. The initial temperature distribution of ambient liquid is described by the formula

$$T_{st}(z) = T_b - |\nabla T|z, \quad |\nabla T| = \frac{T_b - T_c}{d} \tag{16}$$

where d is the distance between bottom and cover.

The density of stratified liquid is also defined by (8) so $\rho_{st}(z) = \rho(T_{st}(z))$. The vertical coordinate of floating up element will be signed by the symbol $z(t)$. During its vertical motion the considered element will be surrounded by ambient stratified liquid which temperature and density are equal to $T_{st}(z(t))$ and $\rho_{st}(t) = \rho(T_{st}(z(t)))$ correspondingly. As a result both the mean temperature and the mean density of the element will be changed due to thermal conductivity with surrounding liquid. Thus the system of equations describing the motion of considered element is the following

$$\begin{aligned} \bar{\rho}(t)V\ddot{z} &= (\bar{\rho}(t) - \rho_{st}(t))Vg - \pi K_f \bar{\rho}(t)R\nu\dot{z}, \quad \dot{z}|_{t=0} = z|_{t=0} = 0 \\ \dot{\bar{T}}(t) &= -\tilde{\chi}(\bar{T}(t) - T_{st}(t)), \quad \bar{T}(t)|_{t=0} = T_b + \sigma_T \end{aligned} \tag{17}$$

where $V = 4\pi R^3/3$ is the volume of the element, K_f is the coefficient of profile drag of liquid sphere in the same liquid (experimental data show that $K_f \in [1..3]$), L_χ is some characteristic scale of diffusion for spherical element and $\tilde{\chi} = \chi/L_\chi^2$.

The solution of the second initial problem of (17) has the form

$$\begin{aligned} \bar{T}(t) &= \left(T_b + \sigma_T + \tilde{\chi} \int_0^t T_{st}(\tau) e^{\tilde{\chi}\tau} d\tau \right) e^{-\tilde{\chi}t} = \\ &= T_{st} + \left(\sigma_T + |\nabla T| \int_0^t \dot{z}(\tau) e^{\tilde{\chi}\tau} d\tau \right) e^{-\tilde{\chi}t} \end{aligned} \tag{18}$$

which was produced with regard to (16).

The substitution of the result received into the first initial problem of (17) permits to get in the crude approximation the initial problem for coordinate $z(t)$ of liquid element

$$\ddot{z} = \alpha_0 g \left(\sigma_T + |\nabla T| \int_0^t \dot{z}(\tau) e^{\tilde{\chi}\tau} d\tau \right) e^{-\tilde{\chi}t} - \tilde{\nu}\dot{z}, \quad \dot{z}|_{t=0} = z|_{t=0} = 0 \tag{19}$$

where $\tilde{\nu} = 3K_f\nu/4R^2$.

The solution of (19) produces the required estimation for vertical coordinate and velocity of the floating liquid element on the initial stage of motion

$$\begin{aligned} z(t) &= \frac{\alpha_0 g \sigma_T}{k^2 - \Omega^2} \left(1 - e^{-kt} \left(\cosh(\Omega t) + \frac{k}{\Omega} \sinh(\Omega t) \right) \right) \\ \dot{z}(t) &= \frac{\alpha_0 g \sigma_T}{\Omega} e^{-kt} \sinh(\Omega t) \end{aligned} \tag{20}$$

where

$$k = \frac{\tilde{\nu} + \tilde{\chi}}{2}, \quad \Omega = \sqrt{\left(\frac{\tilde{\nu} - \tilde{\chi}}{2}\right)^2 + \alpha_0 g |\nabla T|}. \quad (21)$$

As it follows from (20, 21) the character of liquid element floating depends on relation between parameters k and Ω . If $\Omega > k$ the height of floating is restricted only by cover so $z_{max} = d$. The dependencies of $z(t)$ for some ratios $\Omega/k > 1$ are shown on Fig. 1.

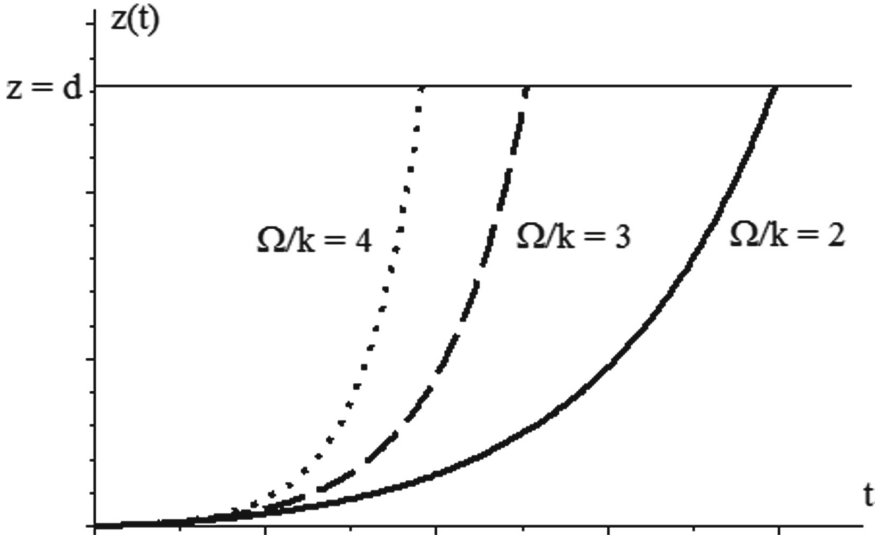


Fig. 1. The graphs of height $z(t)$ of floating element when $\Omega > k$ (solid, dash and dot lines). The thin horizontal line is the level of cover $z = d$.

The greater is the ratio Ω/k the greater is the value of standard Rayleigh number Ra and the quickly the floating liquid element will reach the cover.

In the opposite case when $k > \Omega$ the height of floating has the maximum defined by the condition

$$\dot{z}(t) = 0 \quad \Rightarrow \quad t \rightarrow \infty$$

and described by the relation

$$z_{max} = \lim_{t \rightarrow \infty} z(t) = \frac{\alpha_0 g \sigma T}{k^2 - \Omega^2}. \quad (22)$$

The greater is the ratio k/Ω the small is the height of the liquid element rising which always should be less than the liquid layer thickness d . The dependences of $z(t)$ for some ratios $k/\Omega > 1$ are shown on Fig. 2.

On the base of (21) the condition $\Omega > k$ may be rewrite in equivalent form, namely

$$\widetilde{Ra} = \frac{\alpha_0 g |\nabla T|}{\tilde{\chi} \tilde{\nu}} > 1 \quad (23)$$

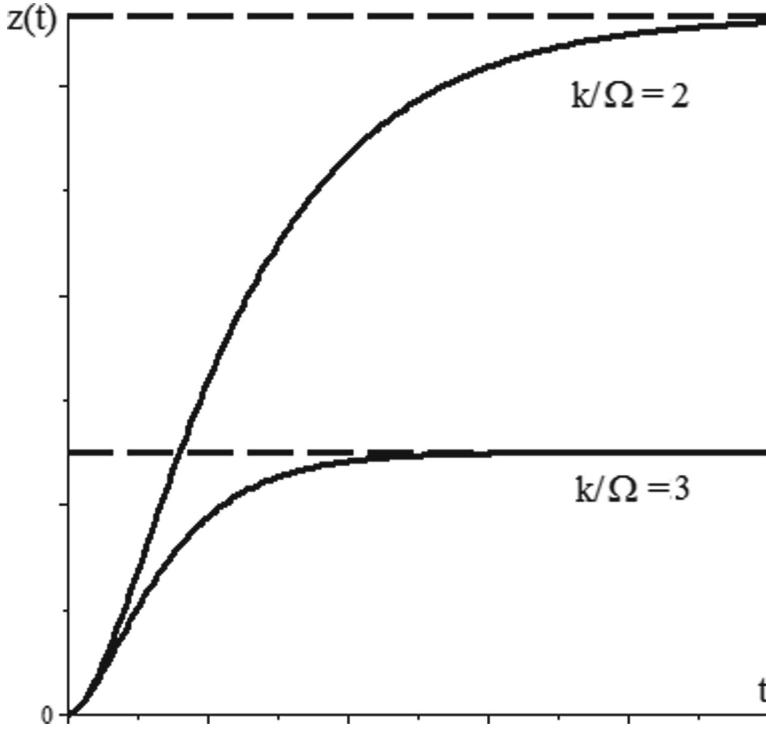


Fig. 2. The graphs of height $z(t)$ of floating element when $k > \Omega$ (solid lines). Dashed lines are asymptotic values for corresponding curves.

It is evident that received modified Rayleigh number \widetilde{Ra} is proportional to the standard defined $Ra = \alpha_0 g |\nabla T| d^4 / \chi \nu$ but don't coincide with it and has other physical sense which is usually attributed to the Rayleigh number.

By means of consequent differentiation the Eq. (19) may be transformed into equation

$$\ddot{z} + (\tilde{\nu} + \tilde{\chi})\dot{z} + (\tilde{\nu}\tilde{\chi} - \alpha_0 g |\nabla T|)z = \text{const.} \tag{24}$$

The characteristic equation correspondent to (24) has the form and solutions

$$\lambda^2 + (\tilde{\nu} + \tilde{\chi})\lambda + \tilde{\nu}\tilde{\chi} - \alpha_0 g |\nabla T| = 0$$

$$\lambda_{\pm} = -\frac{\tilde{\nu} + \tilde{\chi}}{2} \pm \sqrt{\left(\frac{\tilde{\nu} - \tilde{\chi}}{2}\right)^2 + \alpha_0 g |\nabla T|} \tag{25}$$

The instability of solutions of (19) has place when $\lambda_+ > 0$ that corresponds to $\alpha_0 g |\nabla T| / \tilde{\nu}\tilde{\chi} > 1$. Thus the critical value of the new Rayleigh number is defined by the relation

$$\widetilde{Ra}_c = 1. \tag{26}$$

It necessary to emphasize here that Rayleigh number \widetilde{Ra} defines the velocity of growth of convective transport and its intensity but not the onset of convec-

tion. The onset of convection is defined by the value of rms deviation of the heater temperature σ_T . If $\sigma_T = 0$ then accordingly to physical sense no convective motion may be observed in water. When $\sigma_T \neq 0$ then convection always has place and the criterion of convection onset is only *subjective* conception as it was proclaimed in **Introduction**.

Prior to formulate the criterion of convective onset it is necessary to give some subjective definition of convective motion presence. Let the definition is the following: *The convective motion in the observed space point is proclaimed as existing if the absolute value of convective heat flux is not less than conductive one.*

For the problem under consideration it means that the ratio should be valid

$$(T_b - T_c)\overline{\dot{z}(t)} \geq \chi|\nabla T| \quad \equiv \quad \overline{\dot{z}(t)} \geq \frac{\chi}{d}. \quad (27)$$

In (27) the difference $T_b - T_c$ has place because from the cover the negative heat flux due to the same reasons as a positive flux from the bottom is presented. The over line in (27) is the symbol of averaging through the time of floating.

In accordance to results presented on Fig. 2 there are two possible cases. The first one is the case $k > \Omega$ when the liquid element doesn't reach the cover and its maximum height is defined by relation (22). But the time of rising equals to infinity so the average velocity is equal to zero. But the such approach is incorrect for the case under consideration. The fact of the matter is that at the first time moments the liquid element is accelerated upward. Through some time period t_* its acceleration change its sign and the element is breaking. But during the acceleration period the possibility of (27) performance may have place. The time t_* of acceleration period follows from condition

$$\ddot{z}(t_*) = 0$$

from which it follows

$$t_* = \operatorname{arctanh}(k/\Omega)/\Omega, \quad z(t_*) = \frac{\alpha_0 g \sigma_T}{k^2 - \Omega^2} \left(1 - \frac{k \exp(-kt_*)}{\sqrt{k^2 - \Omega^2}} \right) \quad (28)$$

so the condition (27) obtains the form

$$\overline{\dot{z}(t)} = \frac{z(t_*)}{t_*} \geq \frac{\chi}{d}. \quad (29)$$

In the second case $k < \Omega$ the liquid element float up to the cover so $z_{max} = d$ and the time t^* of element rising is defined as solution of the equation

$$d = \frac{\alpha_0 g \sigma_T}{k^2 - \Omega^2} \left(1 - e^{-kt^*} (\cosh(\Omega t^*) + \frac{k}{\Omega} \sinh(\Omega t^*)) \right) \quad (30)$$

and the condition (27) obtains the form

$$t_\chi/t^* \geq 1 \quad (31)$$

where $t_\chi = d^2/\chi$ is the characteristic diffusive time of the liquid layer.

The inverted value to $\tilde{\chi}$, earlier introduced in (17), is the estimation of heat loss time by the liquid element. Accordingly to well-known solution [10] for dynamic sphere cooling in the ambient environment the time of heat losing is proportional to square of sphere radius so one can write

$$L_\chi \sim R \Rightarrow \tilde{\chi} = K_\chi \frac{\chi}{R^2} \tag{32}$$

where K_χ is some constant.

So the value of the \tilde{Ra} may be presented in the form

$$\tilde{Ra} = K_{Ra} \frac{\alpha_0 g |\nabla T| R^4}{\chi \nu}, \quad K_{Ra} = \frac{4}{3 K_f K_\chi} \tag{33}$$

It follows from (33) that the smaller is floating liquid element the smaller is the Rayleigh number \tilde{Ra} and the smaller is the intensity of convective motion in liquid. The existence of the fluid motion when standard Rayleigh number $Ra = 0$ (homogeneously temperature distribution) is shown on Fig. 3. For common imaging all curves are normalized on its own maximum.

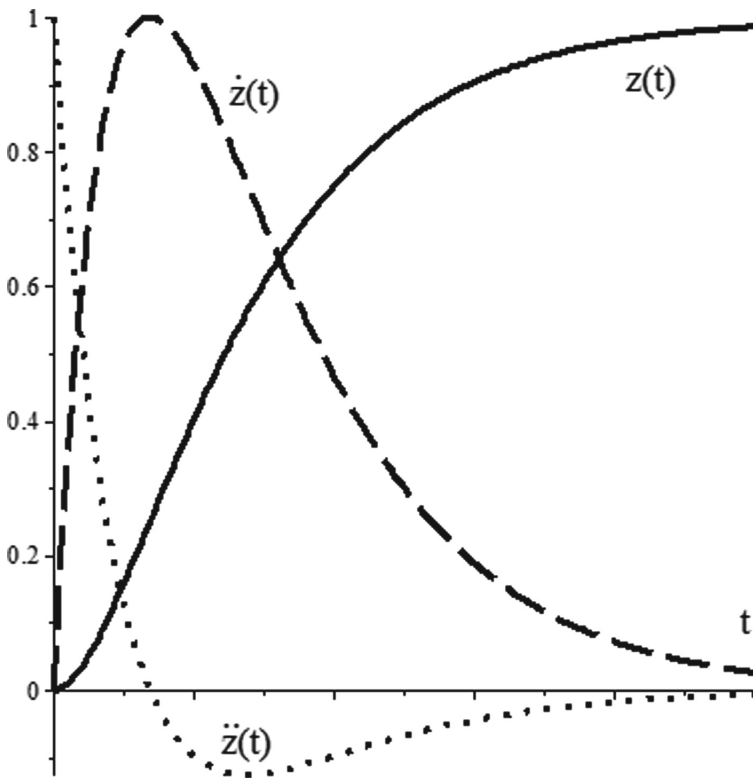


Fig. 3. The acceleration $\ddot{z}(t)$ (dot), velocity $\dot{z}(t)$ (dash) and height $z(t)$ (solid) of the floating liquid element when $Ra = 0$.

The explicit analytical forms of (29, 31) are some tedious to be presented here, but for concrete situations it is easy to check out numerically the announced conditions. If the condition (29) is valid then the relations (20–22, 28) describe the characteristics of vertically limited convection. For the case of (31) the characteristics of fully developed convection are described by (20–22, 30).

4 The Transformation of Governing System

For the further investigations it is useful to transform the system (3) to convenient form by means of some preliminary actions. Because the generation of heat due to viscose effects is sufficiently ineffective the term $\sigma'_{ik} \partial v_i / \partial x_k$ in the second equation of (3) may be neglected. The velocity field is divided on two parts

$$\mathbf{v} = \bar{\mathbf{v}} + \tilde{\mathbf{v}} \quad \nabla \cdot \bar{\mathbf{v}} = 0, \quad \nabla \cdot \tilde{\mathbf{v}} = \frac{\alpha}{\rho c_p} \nabla \cdot (\kappa \nabla T). \quad (34)$$

Because the convective transport of temperature and advective transport of momentum as well as viscose force are stipulated basically by solenoidal part $\bar{\mathbf{v}}$ of velocity then the approximation

$$\bar{\sigma}'_{ik} \approx \eta (\partial \bar{v}_i / \partial x_k + \partial \bar{v}_k / \partial x_i) \quad (35)$$

is used and the system (3) transforms into the new approximate system

$$\rho (\partial \bar{v}_i / \partial t + (\bar{\mathbf{v}} \cdot \nabla) \bar{v}_i) = -\partial p / \partial x_i + \partial \bar{\sigma}'_{ik} / \partial x_k - \rho g \delta_{iz} \quad (36)$$

$$\partial T / \partial t + \bar{\mathbf{v}} \cdot \nabla T = \nabla \cdot (\chi \nabla T), \quad \nabla \cdot \bar{\mathbf{v}} = 0, \quad \nabla \cdot \tilde{\mathbf{v}} = \alpha \nabla \cdot (\chi \nabla T).$$

For the reason of very slowly varying of the term ρc_p for water in the presence of temperature variations its value was introduced under action of ∇ operator.

The standard approach to the analysis of nonlinear hydrodynamic equations is to describe the phenomenon in dimensionless form. For this reason, some scales are usually introduced for differential (space and time) and field (velocity, pressure, temperature etc.) variables. The correct definition of these scales is very complex problem on the solution of which the final results are strongly dependent. For example, the frequently used scale for velocity field (namely χ/d in [3–5] etc.) is in incompatible contradiction with experimental results which show the value of velocity in dozen times greater. As a result of the announced scale application the convective and advective terms of equations are considered as negligible or very small in comparison with linear ones and the solutions received (for velocity and temperature fields) are far from the observed in experiments.

The complexity of the problem of the suitable scale choice is conditioned by the fact that the numerous experiments show the different partial and time scales for different convection regimes and furthermore on the different stages of the certain regime. The standard used scales ([3–5] etc.) at once transform the nonlinear governing system of equations into the form of linear or weakly

nonlinear mathematical model which is not adequate to the problem under consideration. This fact points to the necessity of *solving of spatial and time scales problem* for correct construction of dimensionless variant of equations. And this problem is arisen for certain regime of convection.

4.1 The Initial Stage of Convection

To investigate the characteristic features of the initial stage of convection it is necessary to transform the system (36) into dimensionless form. The time and space scales differ for the cases $\Omega > k$ and $k > \Omega$. For this reason the results will be presented separately.

In the case $\Omega > k$ the characteristic time scale follows from (20) in the form

$$t' = (\Omega - k)t \tag{37}$$

where t' is the dimensionless time.

When the time moments are so small that the floating height of liquid element is less than d then accordingly to (20) the action of differentials have the forms

$$\frac{\partial}{\partial z} = \frac{\Omega(\Omega - k)}{\alpha_0 g \sigma_T} e^{-t'} \frac{\partial}{\partial t'} \quad \frac{\partial}{\partial t} = (\Omega - k) \frac{\partial}{\partial t'}. \tag{38}$$

The scale of the vertical velocity component follows from (20)

$$v_z = \frac{\alpha_0 g \sigma_T}{\Omega} e^{t'} v' \tag{39}$$

where v' is the dimensionless velocity.

The temperature of water is presented in the form

$$T = T_{st}(z) + (\Delta T + \sigma_T)\theta \tag{40}$$

where θ is the dimensionless temperature perturbation.

The above introduced dimensionless field values have the order $v', \theta \sim O(1)$.

On the base of (37–40) the heat transport equation of the system (36) transforms into equation

$$\theta'_{t'} + \lambda v(1 + \lambda^{-1}\theta'_{t'}) = \lambda^{-2}(\chi\theta''_{t't'} + \chi'_T\theta'_{t'}{}^2) - \chi\lambda^{-1}\theta'_{t'} \tag{41}$$

where $\lambda = \alpha_0 g \sigma_T \exp(t') / \Omega(\Omega - k)$.

From (41) follows that the greater term of the ratio of convective and conductive heat fluxes is defined by relation

$$\frac{|\mathbf{J}_{conv}|}{|\mathbf{J}_{cond}|} \approx \chi\lambda^2 v. \tag{42}$$

For the water and $\sigma_T > 10^{-2} K$ this ratio is greater then 1 at the initial moment $t = 0$ and grows with a time.

When the time momets are sufficiently great so the floating height of liquid element equals d then the spatial vertical scale is defined by relation

$$z = dz'. \quad (43)$$

For this situation the time, velocity and temperature scales remain the same and the heat transport equation of the system (36) transforms into equation

$$\theta'_{t'} + \frac{\lambda v}{d}(1 + \theta'_{z'}) \approx \frac{1}{d^2(\Omega - k)} \left(\chi'_T \Delta T (1 + \theta'_{z'})^2 + \chi \theta''_{z'z'} \right) \quad (44)$$

from which follows

$$\frac{|\mathbf{J}_{conv}|}{|\mathbf{J}_{cond}|} \approx \frac{\lambda v d (\Omega - k)}{\chi} \quad (45)$$

which for water is greater than 1 and tends to growth with a time.

In the case $k > \Omega$ the characteristic vertical scale follows from (20) in the form

$$z = \frac{\alpha_0 g \sigma_T}{k^2 - \Omega^2} z'. \quad (46)$$

Let the time scale is described by the relation

$$t = m_t t' \quad (47)$$

where m_t is undefined positive time scale.

So velocity and temperature scales remain the same then on the base (46, 47) the heat transport equation of the system (36) transforms into equation

$$\begin{aligned} m_t \theta'_{t'} + \frac{\alpha_0 g \sigma_T}{d \Omega} v \left(1 + \frac{d(k^2 - \Omega^2)}{\alpha_0 g \sigma_T} \theta'_{z'} \right) = \\ = \left(\frac{k^2 - \Omega^2}{\alpha_0 g \sigma_T} \right)^2 \left(\chi'_T \Delta T (1 + \theta'_{z'})^2 + \chi \theta''_{z'z'} \right) \end{aligned} \quad (48)$$

From (48) it follows the result

$$\frac{|\mathbf{J}_{conv}|}{|\mathbf{J}_{cond}|} \approx \frac{(\alpha_0 g \sigma_T)^2}{d \chi \Omega (\Omega - k)} v \exp((\Omega - k) m_t t') \quad (49)$$

which value for water is less than 1 in the starting moments and decrease with a time.

Thus in the case $\Omega > k$ the heat transport equation is strongly nonlinear and no perturbation methods can be applied to analyze it. Such methods may be valid only for the opposite case $k > \Omega$.

4.2 The Quasi-stationary Regime of Convection

Because in the regime of stationary full-developed convection the velocity of convective transport sufficiently exceeds the diffusive velocity then the system (36) in regard of (35) may be rewrite in the form

$$\rho \bar{\mathbf{v}} \cdot \nabla \bar{v}_i = -\partial p / \partial x_i + \eta \Delta \bar{v}_i + \partial \eta / \partial x_k (\partial \bar{v}_i / \partial x_k + \partial \bar{v}_k / \partial x_i) - \rho g \delta_{iz} \quad (50)$$

$$\bar{\mathbf{v}} \cdot \nabla T \approx 0, \nabla \cdot \bar{\mathbf{v}} = 0.$$

The second equation of (50) means that in the regime of stationary convection the fields of velocity and temperature gradient are orthogonal to each other. Naturally this orthogonality has an approximate character for the reason of the previously approximations done. From this equation the representation of velocity field it follows

$$\bar{\mathbf{v}} = \mathbf{A} \times \nabla T \quad (51)$$

which transforms the announced equation into identity.

The substitution of (51) into the third equation of (50) leads to the consequence of relations

$$\nabla \cdot \bar{\mathbf{v}} = \nabla \cdot (\mathbf{A} \times \nabla T) = \nabla T \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times (\nabla T) = \nabla T \cdot \nabla \times \mathbf{A} = 0 \quad (52)$$

The first way to satisfy to the last equality in consequence (52) is to proclaim $\mathbf{A} = \nabla \Phi$ where Φ is some arbitrary scalar function. In this case the velocity field is described by relation

$$\bar{\mathbf{v}} = \nabla \Phi \times \nabla T \quad (53)$$

The second way is to use the representation for vector field \mathbf{A} in the form

$$\mathbf{A} = A\mathbf{e}_x + B\mathbf{e}_y + C\mathbf{e}_z \quad (54)$$

where A , B and C are the arbitrary functions.

As a result the velocity field has the form

$$\bar{\mathbf{v}} = (BT'_z - CT'_y)\mathbf{e}_x + (CT'_x - AT'_z)\mathbf{e}_y + (AT'_y - BT'_x)\mathbf{e}_z \quad (55)$$

and the condition (52) of solenoidal field type transforms into relation

$$(C'_y - B'_z)T'_x + (A'_z - C'_x)T'_y + (B'_x - A'_y)T'_z = 0. \quad (56)$$

In the case of 2D-motion (the roll convection) the conditions $T'_y = \bar{v}_y = 0$ are valid from (55) it follows

$$\bar{v}_x = BT'_z, \quad \bar{v}_z = -BT'_x, \quad CT'_x - AT'_z = 0. \quad (57)$$

In the first case when $\mathbf{A} = \nabla \Phi$ it follows from (54) that , $A = \Phi'_x$, $B = \Phi'_y$, $C = \Phi'_z$ and the last relation of (57) transforms into equation

$$\Phi'_z T'_x - \Phi'_x T'_z = 0 \quad (58)$$

and because $T'_y = 0$ then the first integral of (58) is the temperature T so $\Phi = \Phi(T)$.

In this case $B = \Phi'_y = \Phi'_T T'_y = 0$ (in accordance to 2D problem) and in the result $\bar{v}_x = \bar{v}_z = 0$. It means that the first way can't be realized.

In the second case for $T'_y = \bar{v}_y = 0$ from (56) it follows

$$(C'_y - B'_z)T'_x + (B'_x - A'_y)T'_z = 0, \quad CT'_x - AT'_z = 0 \quad (59)$$

and then the result is valid: the introduction of stream function $\Psi = \Psi(T)$ is valid so

$$\bar{v}_x = \Psi'_z = \Psi'_T T'_z, \quad \bar{v}_z = -\Psi'_x = -\Psi'_T T'_x. \quad (60)$$

Thus for $2D$ -motion the presentation (11) provides the realization of the second and third relations of the system (50). The exclusion of pressure from the first equation of (50) leads to the *unique* equation in which (in regard to (60)) the *two* unknown values are presented namely the temperature field T and some stream function $\Psi(T)$. For the definite substance (water in the problem under consideration) the temperature dependencies $\rho(T)$ and $\eta(T)$ are known. The necessary derivatives of these values are calculated by means of relations

$$\frac{\partial \rho}{\partial x_i} = \rho'_T \frac{\partial T}{\partial x_i}, \quad \frac{\partial \eta}{\partial x_i} = \eta'_T \frac{\partial T}{\partial x_i}.$$

The explicit form of the announced *unique* equation does not presented for the reason of its tedious size.

In $3D$ -motion for the regimes of the developed cell convection the description of velocity field into the cell it is convenient by means of toroidal-poloidal potential because the axis z is singled out so

$$\bar{v}_x = \Phi'_y + \Psi''_{xz}, \quad \bar{v}_y = -\Phi'_x + \Psi''_{yz}, \quad \bar{v}_z = -\Psi''_{xx} - \Psi''_{yy} = -\Delta_{\perp} \Psi. \quad (61)$$

So the experimental results point to absence of axial rotation in the individual cell then the toroidal potential may be neglected and (61) are transformed into

$$\bar{v}_x = \Psi''_{xz}, \quad \bar{v}_y = \Psi''_{yz}, \quad \bar{v}_z = -\Delta_{\perp} \Psi. \quad (62)$$

The representations (61, 62) transform the condition $\nabla \cdot \bar{\mathbf{v}}$ into identity and substitution of (62) into (51) reforms the equation of heat transport into equation

$$T'_y \Delta \Psi'_x - T'_x \Delta \Psi'_y = 0. \quad (63)$$

The experimental observations also show that in a good approximation the convective structures by Rayleigh-Bénard are characterized by zero helicity i.e. the condition

$$\bar{\mathbf{v}} \cdot \nabla \times \bar{\mathbf{v}} = 0 \quad (64)$$

is valid.

In the case of $2D$ roll convection this condition is valid automatically don't supply additional relations. But in the regime of cell convection

$$\nabla \times \bar{\mathbf{v}} = -\Delta \Psi'_y \mathbf{e}_x + \Delta \Psi'_x \mathbf{e}_y \quad (65)$$

so with regard the condition (64) the additional kinematic condition for velocity field is formed

$$\Delta \Psi'_x \Psi''_{yz} - \Delta \Psi'_y \Psi''_{xz} = 0. \quad (66)$$

From (63) and (65) also follows $\nabla \times \bar{\mathbf{v}} \cdot \nabla T = 0$. It means that in any space point the vectors $\bar{\mathbf{v}}$, $\nabla \times \bar{\mathbf{v}}$ and ∇T form the orthogonal triplet. Hence, the representation for temperature field is valid

$$\begin{aligned} \nabla T &= f(x, y, z) \bar{\mathbf{v}} \times \nabla \times \bar{\mathbf{v}} \Rightarrow T'_x = f \Delta_{\perp} \Psi \Delta_{\perp} \Psi'_x, \\ T'_y &= f \Delta_{\perp} \Psi \Delta_{\perp} \Psi'_y, \quad T'_z = f (\Psi''_{yz} \Delta_{\perp} \Psi'_y + (\Psi''_{xz} \Delta_{\perp} \Psi'_x)). \end{aligned} \quad (67)$$

The relations (66, 67) may be very helpful for the solving of nonlinear equations which arise after pressure elimination from Navier-Stokes equation.

5 Conclusions

Presented here the new approach to formation of convective flows in liquids (particularly in water) shows that the standard Rayleigh number doesn't characterize the condition for convection start but defines the intensity and velocity of growth of convective motion. The proposed new criterion based on new form of the Rayleigh number has the explicit physical sense.

The one more general conclusion follows from the investigations done that the linear or slightly nonlinear mathematical models may be applied when the modified Rayleigh number \bar{Ra} is less than unity. In the opposite case the convective motion is described by strongly nonlinear equations.

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