Far Fields of Surface Gravity Waves Under Unsteady Generation Regimes

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Abstract. The far fields of surface wave perturbations excited by an oscillating localized source moving in a heavy liquid of infinite depth are studied. It is shown that the excited fields are a sum of two wedge-like ship waves located insider the corresponding wave wedges. Each of the excited two waves is a complicated wave system of transverse and longitudinal perturbations. The properties of the dispersion curves are studied and the phase pictures describing the structure of wave surface perturbations are calculated. The characteristics of the excited wave fields are studied depending on the basic parameters of the wave generation such as the velocity of motion of the perturbation source and the frequency of its oscillations. Uniform asymptotic solutions are constructed in terms of the Airy function and its derivative, which permits describing the far fields of surface perturbations both outside and inside the corresponding wave wedges.

Keywords: Gravity surface waves · Far fields · Uniform asymptotics

1 Introduction

The surface wave motions in the marine environment can either originate due to natural causes (wind waves, flow past underwater obstacles, bottom relief variations, density and flow fields) or be generated by the flow past natural obstacles (platforms, underwater pipelines, complex hydraulic facilities). The general system of hydrodynamic equations describing the surface perturbations is a rather complicated mathematical problem from the standpoint of proving the existence and uniqueness theorems for solutions in the corresponding function classes and from the computational standpoint [1–6]. In the framework of the linear theory, the surface wave perturbations are analytically studied by integral representation methods and various asymptotic methods [7–11]. The main results of solving the problems of generation of surface wave perturbations are represented in most general integral form, and to obtain the integral solutions, it is thus necessary to develop asymptotic methods for their investigation which admit a qualitative analysis and rapid estimations of the obtained solutions. Moreover, to analyze the data of the sea surface remote sensing, it is required to know the causes of various surface phenomena. To obtain a detailed description of a wide class of physical phenomena related to the dynamics of surface perturbations in inhomogeneous and unsteady natural environments, it is necessary to have sufficiently developed mathematical models. The fact that the structure of the heavy sea surface is

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three-dimensional is also significant, and there are currently no possibilities for large-scale computational experimental modeling of three-dimensional ocean flows at large times with a sufficient accuracy. But in several cases, the initial qualitative concept of the considered class of wave phenomena can be obtained by using simpler asymptotic models and analytic methods for studying them. In this connection, it is necessary to mention the classical hydrodynamic problems of constructing asymptotic solutions which describe the evolution of surface perturbations excited by sources of various nature in heavy homogeneous liquids. The model solutions permit further obtaining representations of surface wave fields with regard to variability and unsteadiness of real natural environments. So several results of asymptotic analysis of linear problems describing different regions of generation and propagation of surface perturbations also underlie the currently actively developing nonlinear theory of generation of ocean waves of extremely large amplitude, the so-called rogue waves [3]. The contemporary state of the art in the study of linear and nonlinear surface perturbations can be found in [5]. In [12, 13], the problem of constructing uniform asymptotics of far fields of surface perturbations due to an oscillating source moving with a bounded velocity was considered. The goal in the present paper is to construct uniform asymptotics of far fields of surface perturbations excited by the fast motion of a localized oscillating source of perturbations in a heavy homogeneous liquid of infinite depth.

2 Problem Formulation and Integral Forms of Solutions

We consider the steady-state pattern of wave perturbations on the surface of an ideal heavy liquid of infinite depth when the perturbation source moves with velocity V in the positive direction of the axis x. The waves are generated by a moving oscillating point source of perturbations located at the depth H (the axis z is directed upwards from the unperturbed liquid) whose capacity varies by the law $q = \exp(i\omega t)\exp(\varepsilon t)$ $(-\infty < t < \infty)$. Further, we seek the limit as $\varepsilon \to 0$ in the obtained solution. The perturbation of the potential $\Phi(x, y, z, t)$ with respect to the homogeneous flow moving with velocity V ($\nabla\Phi = (u, v, w)$, where u, v, w are components of the vector of perturbations of the velocity $(V, 0, 0)$) is described by the following equation with an appropriate linearized boundary condition on the surface of the liquid [1, 5, 12, 13]

$$\begin{gathered}\delta\Phi(x,y,z,t) = \exp(i\omega t)\exp(\varepsilon t)\delta(x)\delta(y)\delta(z+H), \quad z<0 \\ \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2 \Phi + g\frac{\partial\Phi}{\partial z} = 0, \quad z = 0\end{gathered} \tag{1}$$

Here Δ is the three-dimensional Laplace operator, and $\delta(x)$ - is the Dirac delta function. We seek the solution of problem (1) in the for $\Phi(x, y, z, t) = \exp(i\omega t)\exp(\varepsilon t)\varphi(x, y, z)$, where the function $\varphi(x, y, z)$ is determined from the problem

$$\Delta\varphi(x,y,z) = \delta(x)\delta(y)\delta(z+H), \quad z<0,$$
$$\left(i\omega + \varepsilon + V\frac{\partial}{\partial x}\right)^2 \varphi + g\frac{\partial\varphi}{\partial z} = 0, \ z = 0.$$

The Fourier transform of the potential $\varphi(x,y,z)$,

$$\Omega(\mu,\nu,z) = \int_{-\infty}^{\infty} \exp(i\mu x)dx \int_{-\infty}^{\infty} \exp(i\nu y)\varphi(x,y,z)\, dy;$$

is determined from the boundary-value problem

$$\frac{\partial^2\Omega(\mu,\nu,z)}{\partial z^2} - k^2\Omega(\mu,\nu,z) = \delta(z+H), \qquad z<0,$$
$$(\mathrm{i}\omega + \varepsilon - \mathrm{i}\mu V)^2\Omega(\mu,\nu,z) + g\frac{\partial\Omega(\mu,\nu,z)}{\partial z} = 0, \quad z = 0,$$
$$\Omega(\mu,\nu,z) \to 0, \quad z \to -\infty, \, k^2 = \mu^2 + \nu^2,$$

whose solution in the domain $-H<z<0$ has the form

$$\Omega(\mu,\nu,z) = -\frac{(\omega - \mu V)^2 sh(kz) + gk\, ch(kz)}{k\exp(kH)((\varepsilon + i(\omega - \mu V))^2 + gk)}.$$

The free surface elevation $\eta(x,y,t)$ is related to the potential $\Phi(x,y,z,t)$ by the condition [1, 5, 12, 13]

$$\eta(x,y,t) = -\frac{1}{g}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\Phi(x,y,z,t) =$$
$$= \frac{-\exp(i\omega t + \varepsilon t)}{g}(i(\omega - i\varepsilon)\varphi(z,y,z) + V\frac{\partial\varphi(x,y,z)}{\partial x}), z = 0$$

Then the Fourier transform $\Lambda(\mu,\nu,t)$ of the function $\eta(x,y,t)$ becomes

$$\Lambda(\mu,\nu,t) = \frac{i(\omega - \mu V)\exp(i\omega t)\exp(-kH)}{(\varepsilon + i(\omega - \mu V))^2 + gk}$$

In the obtained expression, the parameter ε is preserved only in the denominator, which is necessary to determine the displacement of the pole of the integrand with respect to the real axis (into the upper or lower half-plane). Then we can obtain the inverse Fourier transform

$$\eta(x,y,t) = \frac{i\exp(i\omega t)}{4\pi^2}\int_{-\infty}^{\infty} \exp(-i\nu y)d\nu \int_{-\infty}^{\infty} \frac{(\omega - \mu V)\exp(-kH - i\mu x)d\mu}{(\varepsilon + i(\omega - \mu V))^2 + gk}. \qquad (2)$$

The zeros of the denominator in the integrand in (2) determine the dispersion relation: $B(\omega, \mu, \nu) \equiv (\omega - \mu V)^2 - g\sqrt{\mu^2 + \nu^2} = 0$. The set of values of the parameter $V\omega/g > 0$ is divided by two characteristic values $1/4$ and $\sqrt{6}/9$ into three intervals. For $V\omega/g < 1/4$, the dispersion curve consists of three branches, one closed and two non-closed; this case was considered in [12]. In this case, the wave picture is a sum of two wedge-like (longitudinal) waves with half-opening angle of the wave wedge less than $\pi/2$ and the annular-like (transverse) waves around the source. For $V\omega/g > \sqrt{6}/9$, the dispersion curve consist of two unclosed branches without extrema. In this case, the wave picture is a sum of two wedge-like ship waves with the half-opening angle of the wave wedge less than $\pi/2$. If $1/4 < V\omega/g < \sqrt{6}/9$, then the dispersion curve consists of two unclosed curves one of which has two local extrema. One branch of the dispersion curve corresponds to the usual wedge-like waves with the half-opening angle of the wave wedge less than $\pi/2$, and the second branch corresponds to the ship waves with the half-opening angle of the wave wedge greater than $\pi/2$ (the wave front is directed upstream away from the source). This system of hybrid waves simultaneously has the features of both the annular (transverse) and wedge-like (longitudinal) waves [13]. Further, we consider the case $V\omega/g > \sqrt{6}/9$. Then the integrand in the inner integral in (2) has two real poles $\mu_{1,2}$. Figure 1 illustrates the results of calculations of the corresponding dispersion curves $\mu_{1,2}(\nu)$. Here and below, the following parameters of calculations were used: $H = 5$ m, $\omega = 0.5$ s^{-1}, and $V = 30$ m/s.

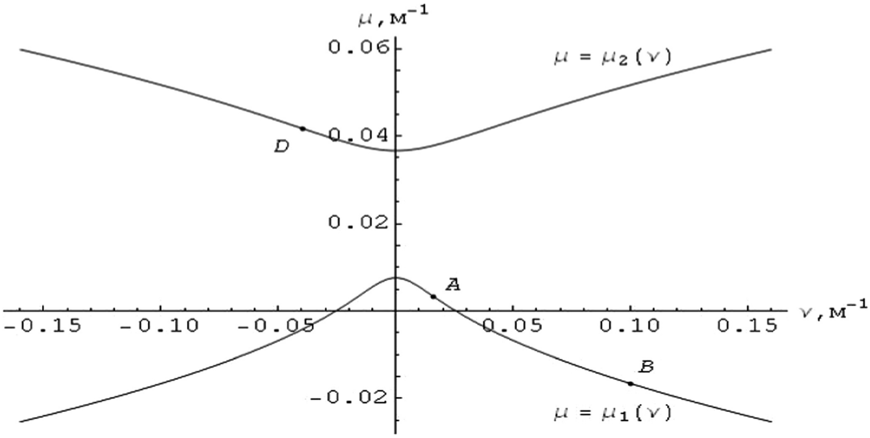

Fig. 1. Dispersion curves $\mu_1(\nu)$ and $\mu_2(\nu)$, *A* and *D* are deflection points, and *B* is a root of the equation $\mu_1'(\nu) = \mu/\nu$.

3 Construction of Solution Asymptotics

To calculate the inner integral in (2), it is necessary to determine the displacement of the poles $\Delta\mu$ for $\varepsilon > 0$. For this, we equate the determinant of the integrand in the inner integral in (2) with zero for $\varepsilon = 0$: $B(\omega, \mu, \nu) = 0$, and for $\varepsilon > 0$: $B(\omega - i\varepsilon, \mu + \Delta\mu, \nu) = 0$. Then we obtain: $\Delta\mu = i\varepsilon \frac{\partial B}{\partial \omega} / (\frac{\partial B}{\partial \mu})$. Under the assumption that $\omega = \omega(\mu, \nu)$, we have $\frac{\partial B}{\partial \omega} / (\frac{\partial B}{\partial \mu}) = -1/(\frac{\partial \omega}{\partial \mu})$ and $\delta\mu = -i\varepsilon/(\frac{\partial \omega}{\partial \mu})$. With regard to $\frac{\partial \omega}{\partial \mu} > 0$, we see that both poles move into the lower half-plane. Then, for $x < 0$, the contour of integration over the variable μ in (2) becomes closed in the upper half-plane, and the poles $\mu_{1,2}(\nu)$ do not make contributions to the wave field. For $x > 0$, the contour of integration over μ in (2) becomes closed in the lower half-plane and, taking into account the contributions of the poles, we obtain

$$
\begin{aligned}
&\eta(x, y, t) = I_1(x, y, t) + I_2(x, y, t), \\
&I_m(x, y, t) = \int_{-\infty}^{\infty} F(\mu_m(\nu), \nu) \exp(-ixS_m(\nu, \alpha)) d\nu, \\
&F(\mu_m(\nu), \nu) = \frac{\exp(i\omega t)}{2\pi} \frac{(\omega - \mu V) \exp(-kH)}{2V(\omega - \mu V) + \mu g/k}, \\
&S_m(\nu, \alpha) = \mu_m(\nu) + \nu tg\alpha, tg\alpha = y/x, m = 1, 2.
\end{aligned}
\tag{3}
$$

For large value of $x > 0$, the asymptotic behavior of the integrals in the sum (3) is completely determined by the stationary points of the phase function $S_m(\nu, \alpha)$ which are determined from the equation $\mu'_m(\nu) = -tg\alpha$. First, we consider the first term in (3). The function $\mu'_1(\nu)$ has one maximum on the interval of integration over the variable ν associated with the corresponding value of the argument α, which is further denoted by A_1. The value A_1 determines the boundaries of the wave wedge (Kelvin wedge) which are described by the equation $y = \pm x \; tg \; A_1$. For $0 < \alpha < A_1$, the phase function $S_1(\nu, \alpha)$ has two stationary points on the real axis ν: $0 < \nu_2(\alpha) < \nu_1(\alpha)$. For $A_1 < \alpha < \pi/2$, there are two complex conjugate stationary points $\nu_1(\alpha), \nu_2(\alpha)$, and for definiteness, we assume that $\text{Im}\, \nu_1(\alpha) > 0$.

We introduce the notation: $\Phi_1 = -\mu_1(\nu) - \nu y + \omega t$. Then from the phase stationarity condition $\mu'_1(\nu) = -\text{tg}\alpha$, we can obtain the parametric equations of the family of constant phase lines $\Phi_1 = C$ ($C = \text{const}$) for different values of C

$$
x(\nu) = \frac{\omega t - C}{\mu_1(\nu) - \nu\mu'_1(\nu)}, \quad y(\nu) = -\frac{\mu'_1(\nu)(\omega t - C)}{\mu_1(\nu) - \nu\mu'_1(\nu)}
$$

Figure 2 shows the lines of equal phase for $t = 0$, $C = 2\pi n$, $n = -5, -4, \ldots, 5$. The right-hand branch ($\nu > 0$) of the dispersion curve $\mu_1(\nu)$ corresponds to with the upper part of the picture ($y > 0$). Since the phase portrait of the waves is symmetric with respect to the axis x, we further consider only this domain. Point A in Fig. 1 is the deflection point of the curve $\mu_1(\nu)$, i.e., a root of the equation $\mu''_1(\nu) = 0$. Therefore, the value A is associated with the wave wedge boundary (dashed line in Fig. 2) inside which the traveling waves described by the integral I_1 are propagating. Point B in Fig. 1 is a root of the equation $\mu'_1(\nu) = \mu/\nu$ and is associated with dashed line 4 in Fig. 2. The part of the dispersion curve from zero to point A (Fig. 1) is associated with transverse crests of the waves (solid lines 2 in Fig. 2). The part of the dispersion curve from point A to point B in Fig. 1 is associated with longitudinal crests (solid lines 1a in Fig. 2) located between dashed lines 3 and 4 in domain *I*. The part of the dispersion curve to the right of point B (Fig. 1) is associated with longitudinal crests (solid lines 1*b* in Fig. 2) located between the dashed line 4 and the axis x (domain *II* in Fig. 2). On the crests of longitudinal waves in domain *I* and on the crests of the transverse waves, the phases Φ_1 take the values $2\pi l$ $(l = -1, -2, .., -5)$. The phases of the crests of longitudinal waves in domain *II* in Fig. 2 are equal to $2\pi k$ $(k = 1, 2, .., 5)$. On dashed line 4, the wave phase Φ_1 is zero for $t = 0$. The longitudinal waves in domain *I* and the transverse waves propagate from the origin to infinity. The longitudinal waves in domain *II* propagate in the direction of dashed line 4. We present the basic characteristics of the wave field for the following parameters of calculations: the wave length along the horizontal axis is $\lambda_1 = 2\pi/\mu_1(0) = 828.8$ m, the half-opening angle of the wave wedge is $A_1 = 20.3°$, and the wave front is given by the equation $y = x \text{ tg } A_1$.

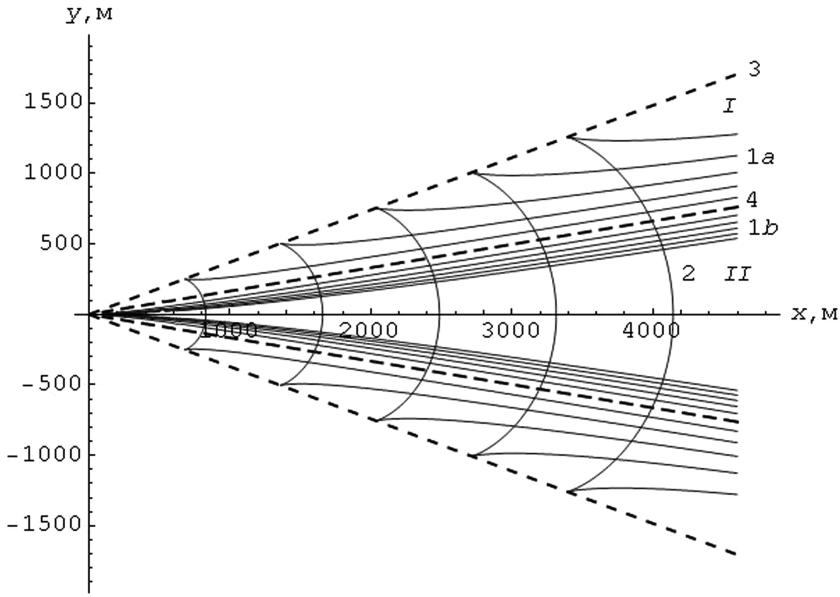

Fig. 2. Lines of equal phase for the integral I_1: lines 1a correspond to longitudinal waves in domain *I*, lines 1b correspond to longitudinal wave in domain *II*, lines 2 correspond to transverse waves, line 3 indicates the wave front, and line 4 separates domains *I* and II.

Inside the wave wedge, the field can be calculated by the method of stationary phase, then the contribution is made by both of the stationary $\nu_1(\alpha)$, $\nu_2(\alpha)$, and the field is exponentially small outside the wave wedge. The asymptotics of the integral $I_1(x, y, t)$ for large $x > 0$ calculated by the method of stationary phase has the form [14, 15]

$$I_1(x,y,t) \approx T_1 + T_2,$$
$$T_i = \left(\frac{2\pi}{x\left|(\mu_1'' \nu_i(\alpha))\right|}\right)^{1/2} F(\mu_1(\nu_i(\alpha)), \nu_i(\alpha))\, E_i,$$
$$E_i = \exp(-ix(\mu_1(\nu_i(\alpha)) + \nu_i(\alpha) tg\alpha)) - i\pi sign(\mu_1''(\nu_i(\alpha)))/4), i = 1, 2.$$

The asymptotics calculated by the method of stationary phase is not uniform, because the stationary points merge on the wave front: $\nu_1(A_1) = \nu_2(A_1)$ and $\mu_1''(\nu_1(A_1)) = \mu_1''(\nu_2(A_1)) = 0$. Therefore, the asymptotics calculated by the method of stationary phase cannot be applied near the boundary of the Kelvin wave wedge. The uniform asymptotics of $I_1(x,y,t)$ for $x > 0$ applicable at far a distance from the wave front and near it has the form [14, 15]

$$\begin{gathered} I_1(x,y,t) \approx \frac{2\pi \exp(i(r\lambda(\alpha) + \omega t))}{x^{1/3}} (0.5(G(\sqrt{\sigma(\alpha)}) + G(-\sqrt{\sigma(\alpha)}))Ai(x^{2/3}\sigma(\alpha)) - \\ -i\frac{(G(\sqrt{\sigma(\alpha)}) - G(-\sqrt{\sigma(\alpha)})}{2x^{1/3}\sqrt{\sigma(\alpha)}} Ai'(x^{2/3}\sigma(\alpha))), \\ \lambda(\alpha) = (S_1(\nu_1(\alpha),\alpha) + S_1(\nu_2(\alpha),\alpha))/2, \\ \sigma(\alpha) = (3(S_1(\nu_2(\alpha),\alpha) - S_1(\nu_1(\alpha),\alpha))/4)^{2/3}, \\ G(\sqrt{\sigma(\alpha)}) = F(\mu_1(\nu_1(\alpha)), \nu_1(\alpha))\sqrt{\frac{2\sqrt{\sigma(\alpha)}}{\mu''_1(\nu_1(\alpha))}}, \\ G(-\sqrt{\sigma(\alpha)}) = F(\mu_1(\nu_2(\alpha)), \nu_2(\alpha))\sqrt{\frac{2\sqrt{\sigma(\alpha)}}{\mu''_1(\nu_2(\alpha))}}, \end{gathered} \tag{4}$$

where $Ai(\tau) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \cos(\tau t - t^3/3)dt$ is the Airy function and $Ai'(\tau)$ is the derivative of the Airy function. The obtained asymptotics becomes non-uniform if the Airy function and its derivative are replaced by the corresponding expansions for large values of the argument.

Further, we consider the contribution of the dispersion curve $\mu_2(\nu)$ to the resultant wave field. Figure 3 presents the lines of equal phase of the integral $I_2(x,y,t)$ for $t = 0$, $C = 2\pi n$, $n = -1, -2, \ldots, -5$. The upper half of Fig. 3 ($y > 0$) corresponds to the left branch of the dispersion curve $\mu_2(\nu)$ ($\nu < 0$). The part of the dispersion curve from zero to point D in Fig. 1 is associated with transverse waves (solid lines 2 in Fig. 3). The part of the dispersion curve from point D to infinity is associated with longitudinal waves (solid lines 1 in Fig. 3). The deflection point is associated with the wave front $y = x$ tg A_2 (dashed line 3 in Fig. 3), where the half-opening angle of the wave wedge is $A_2 = 10.1°$. All lines of equal phase go from the origin to infinity. The length of the transverse wave along the horizontal axis x is $\lambda_2 = 2\pi/\mu_2(0) = 171.5$ m. The uniform asymptotics of the integral $I_2(x,y,t)$ for large $x > 0$ are estimated similarly to (4). We note that the waves described by the integral I_2 significantly (approximately by a factor of three) exceed in amplitudes the waves determined by the integral I_1.

The above numerical calculations show that an increase in the velocity V of motion of the source (for a fixed frequency ω of the source oscillations) leads to a decrease in the half-opening angles of both of the wave wedges. In this case, the distance between the neighboring wave crests increases; in particular, there is an increase in the lengths of transverse waves λ_1 and λ_2 along the axis x. Figures 4 and 5 presents the results of

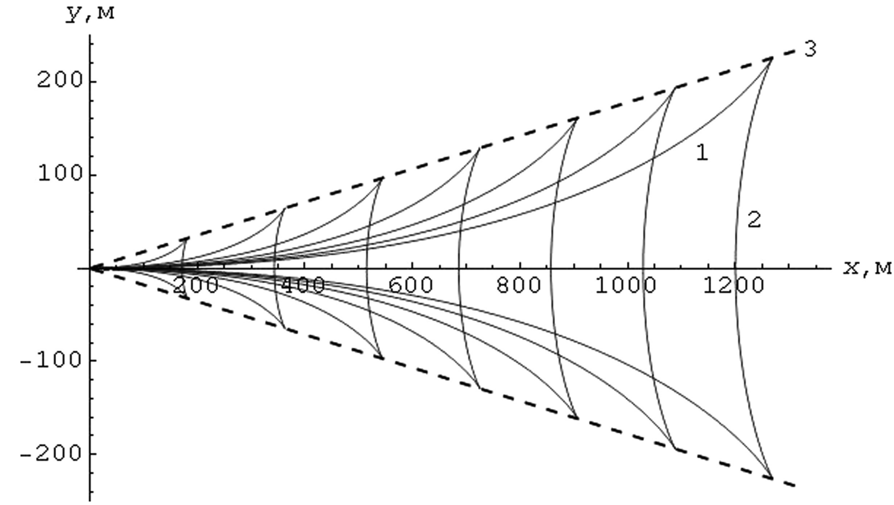

Fig. 3. Lines of equal phase for the integral I_2: lines 1 are longitudinal waves, lines 2 are transverse waves, and line 3 is the wave front.

calculations of uniform asymptotics of the integrals $I_1(x, y, t)$ and $I_2(x, y, t)$ for $t = 0$. The sum of these terms describes the total field of the free surface elevation at a far distance from the moving oscillating source of perturbations.

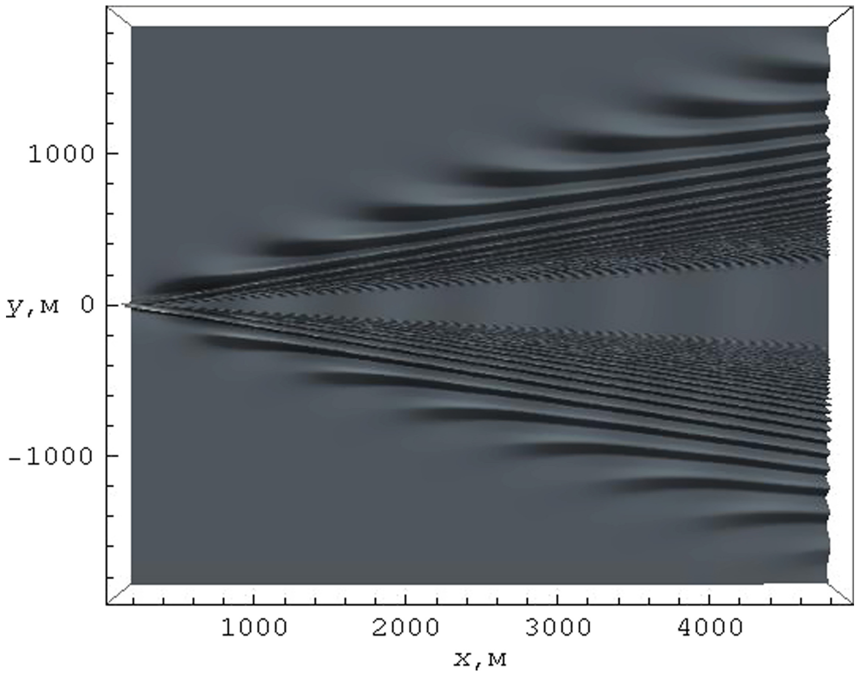

Fig. 4. Uniform asymptotics of the integral I_1 at a far distance from the moving source.

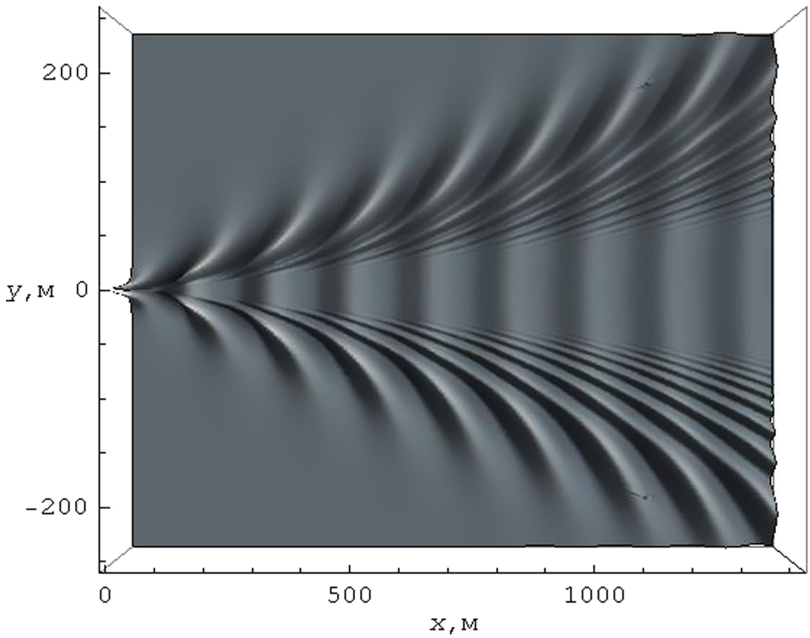

Fig. 5. Uniform asymptotics of the integral I_2 at a far distance from the moving source.

4 Conclusion

We have shown that the far fields of surface perturbations due to a rapidly moving localized oscillating source in a heavy liquid of infinite depth are sums of two wedge-like (ship) waves each of which is contained inside the corresponding Kelvin wave wedge. It is shown that the amplitude of one wave is several times greater than the amplitude of the other wave. Each of the excited two waves is a complex wave system of transverse and longitudinal wave perturbations. The characteristics of the excited surface perturbations were studied depending on the basic parameters of the wave generation such as the velocity of motion of the source of perturbations and the frequency of its oscillations. The constructed asymptotic solutions permit describing the far-range fields of surface perturbations excited by a moving localized unsteady source both outside and inside the corresponding wave wedges. The obtained asymptotics of far-range fields of surface wave perturbations allow one efficiently to calculate the basic characteristics of wave fields and, in addition, qualitatively to analyze the obtained solutions, which is important for obtaining the well-posed statements of mathematical models of wave dynamics of surface perturbations of real natural environment. The work was carried out within the framework of the state assignment (project AAAA-A17-117021310375-7).

References

1. Lighthill, J.: Waves in Fluids. Cambridge University Press, Cambridge (1978)
2. Pedlosky, J.: Waves in the Ocean and Atmosphere: Introduction to Wave Dynamics. Springer, Heidelberg (2010)
3. Kharif, C., Pelinovsky, E., Slynyaev, A.: Rogue Waves in the Ocean. Springer, Berlin (2009)
4. Bulatov, V., Vladimirov, Yu.: Waves Dynamics of Stratified Mediums. Nauka Publisher, Moscow (2012)
5. Mei, C., Stiassnie, M., Yue, D.: Theory and Applications of Ocean Surface Waves. Advanced Series of Ocean Engineering, vol. 42. World Scientific Publishing, London (2017)
6. Morozov, E.: Oceanic Internal Tides. Observations, Analysis and Modeling. Springer, Berlin (2018)
7. Brown, M.: The Maslov integral representation of slowly varying dispersive wavetrains in inhomogeneous media. Wave Motion **32**, 247–266 (2000)
8. Chen, X., Wu, G.: On singular and highly oscillatory properties of the Green function for ship motions. J. Fluid Mech. **445**, 77–91 (2001)
9. Zakharov, V., Dyachenko, A., Vasilyev, O.: New method for numerical simulations of a nonstationary potential flow of incompressible fluid with a free surface. Eur. J. Mech. B Fluids **21**, 283–291 (2002)
10. Lannes, D.: Well-posedness of the water waves equations. J. Am. Math. Soc. **18**, 605–654 (2005)
11. Dobrokhotov, S., Grushin, V., Sergeev, S., Tirozzi, B.: Asymptotic theory of linear water waves in a domain with non-uniform bottom with rapidly oscillating sections. Russ. J. Math. Phys. **23**, 455–475 (2016)
12. Bulatov, V., Vladimirov, Yu., Vladimirov, I.: Far fields of the surface disturbances produced by a pulsating source in an infinite-depth fluid. Fluid Dyn. **52**(5), 617–622 (2017)
13. Bulatov, V., Vladimirov, Yu.: Hybrid surface waves from a harmonic perturbation source. Atmos. Oceanic Phys. **54**(2), 196–200 (2018)
14. Borovikov, V.: Uniform stationary phase method. In: IEE Electromagnetic Waves Series, vol. 40, London (1994)
15. Haberman, R.: Applied Partial Differential Equations. Pearson Educations, Upper Saddle River (2013)