Multiple Criteria Decision Making

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New Perspectives in Multiple Criteria Decision Making

Innovative Applications and Case Studies

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Multiple Criteria Decision Making

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Preface

An Outline of the Status and Perspective of Multicriteria Decision Analysis

For more than four decades Multiple Criteria Decision Analysis (MCDA) has consistently been one of the most active areas in Operations Research and Management Science (OR/MS). Since the pioneering work by John von Neumann and Oskar Morgenstern on utility theory, the development of decision analysis by Howard Raiffa and Ron Howard, the contributions of Abraham Charnes and William Cooper on goal programming, and those of Tjalling Koopmans and Arthur Geoffrion on the foundations of efficiency measurement and multi-objective optimization, Kenneth Arrow's contributions to social choice theory, and Bernard Roy's foundations of outranking relations, the field of MCDA made significant progress in terms of methodological development and applications.

MCDA deals with decision-making/aiding problems involving the consideration of multiple (conflicting) criteria, attributes, points of view, goals, and objectives. Such problems naturally arise in all areas of business activity, the public sector, as well as in choices made by individuals. In contrast to the traditional framework of single-objective problems, where the best option can be described by a single measure, when dealing with multiple criteria the problem becomes ill-defined because a single best solution does not exist. Therefore, various behavioral, modeling, and algorithmic issues arise, which cannot be addressed unless a systematic methodology is adopted. This procedure is not only prescriptive providing answers to a given decision problem, but also constructive, in the sense that the actors involved in the decision process progressively gain a better understanding of the problem and their preferences, that ultimately leads to nontrivial solutions to complex instances.

The field of MCDA provides an arsenal of methodologies and tools to handle the above issues, including soft approaches for problem structuring and decision modeling, techniques and models for aggregating criteria, optimization approaches, and algorithms for problems involving multiple objectives, and decision support system (DSS) implementations. Throughout its history, MCDA has followed a dynamic path of development. New types of decision models have been introduced, allowing the aggregation of different types of information (qualitative, quantitative, fuzzy, etc.), new multi-objective optimization tools have been explored for interactive decision support and combinatorial problems (e.g., metaheuristics), and advanced DSSs have been developed using improved data management/visualization and web-based technologies. Moreover, the field has progressed in terms of behavioral issues, on aspects related to preference modeling and elicitation, the treatment of uncertainties, imprecision, and ill-determination, while also strengthening its connections with emerging data analytic technologies.

At the same time, the range of applications has been constantly widening and new areas of interest arise. Except for standard business applications (finance, logistics, marketing, human resources, etc.), many new areas now benefit from MCDA, including environmental management, energy planning, sustainable development, and various areas of the public sector and policy making.

For MCDA to maintain its success path there are several areas for future development. For instance, the extension of existing decision models to allow the modeling of more complex preference structures could provide additional flexibility to decision analysts and decision makers with more general and less restrictive tools for handling difficult decision aiding instances. More complex models require axiomatization, deep understanding of their analytical properties, and tools to make them comprehensible/accessible by decision makers. Procedures for preference modeling and elicitation using information derived from data in a robust framework could facilitate the construction of decision model and reduce the cognitive effort involved. Behavioral aspects of preference modeling are also worth the investigation, together with exploring algorithmic advances in areas such as metaheuristics, soft computing, data analytics/visualization, and computer science (e.g., web-based technologies, tools for knowledge representation and modeling, etc.).

Addressing some of these ideas and areas requires an interdisciplinary approach, combining elements from various areas in OR/MS, mathematical economics, and computer science, among others. Adopting such an interdisciplinary approach could not only lead to advances on the theory of MCDA but also promote the field in other areas.

Aims and Scope

The aim of this book is not to constitute a reference for providing an overview of standard and well-known MCDA approaches. Several other books and edited volumes have already covered this area rather comprehensively. Instead, this edited volume seeks to focus on emerging areas of research in MCDA and the perspectives in the theory and applications of the field, thus providing researchers working in this area with a collection of high-quality chapters indicating how the MCDA is currently forming and how it can be shaped in the future. It is worth noting that this covers both theoretical aspects and applied research. While the importance of the perspectives in the theory of MCDA is mostly obvious, we should emphasize that the trends and perspectives in terms of applications are also important to identify new areas that have the potential for applied MCDA research, understands the context of these domains and design new MCDA approaches that can be successfully applied in practice. With these remarks in mind, below we provide an outline of the organization and the contents of this edited volume.

Organization

The book includes 16 contributions organized in four parts covering a wide range of MCDA methodologies, recent advances, and applications.

The first part of the book includes four chapters devoted to some fundamental methodologies and MCDA concepts. In the first chapter (New Trends in Preference, Utility, and Choice: From a Mono-approach to a Multi-approach) A. Giarlotta provides a comprehensive overview of some new trends in preference modeling, utility representation, and choice rationalization. The chapter starts with the traditional "mono-approach" traditionally used in mathematical economics for describing an agent's preference structure. The recent trend towards using a "multi-approach" that relies on multiple tools is introduced and some characteristic approaches are presented. New advances in this alternative paradigm are also analyzed in relation to MCDA.

The second chapter (Analytic Hierarchy Process and Its Extensions) by A. Ishizaka covers the analytic hierarchy process (AHP) and its extensions. AHP has traditionally been one of the most widely used methods in MCDA. The chapter first introduces the main ideas and methodological steps of AHP and then presents new advances and extensions in areas such as the analytic network process, group decision-making, variants for sorting problems, and visualization tools.

In the third chapter (Beyond Multicriteria Ranking Problems: The Case of PROMETHEE), Y. de Smet summarizes the recent developments in PROMETHEE methods, which follow the principles of outranking relations theory. PROMETHEE method have been originally introduced for multicriteria choice and ranking problems. Recently other types of problems, such as sorting and clustering, have also been addressed through variants of the PROMETHEE methods. The chapter describes some of these variants and discusses the relations between ranking, sorting, and clustering problems.

The final chapter (Preference Disaggregation for Multicriteria Decision Aiding: An Overview and Perspectives) of the first part is devoted to preference disaggregation analysis. M. Doumpos and C. Zopounidis describe the principles of this methodological stream of MCDA and its uses for constructing different types of decision models. The perspectives in this area are also discussed, in the context of robustness analysis, the use of alternative types of decision models, the optimization tools used to infer preference information from decision instances, as well as the potential of extending this area to large data.

In the first chapter (Normed Utility Functions: Some Recent Advances) of the second part, R. Mesiar, A. Kolesárová, A. Stupňanová, and R. R. Yager summarize some new results and trends in aggregation theory and introduce some new ideas that can be useful for providing multicriteria decision aiding. More specifically, the authors present two recently developed aggregation approaches, namely the *k*-additive and *k*-maxitive aggregation functions. Moreover, construction techniques are also presented.

The next chapter (Interpretation of Multicriteria Decision Making Models with Interacting Criteria) by M. Grabisch and C. Labreuche focuses on MCDA models that allow the modeling of interactions between criteria, such as the generalized-additive independence (GAI) model. The chapter further describes ways to develop an interpretation of general utility-based models through the introduction of importance indices for the decision criteria. The issue of constructing a monotone decomposition of the GAI model is also discussed.

In the last chapter (New Directions in Ordinal Evaluation: Sugeno Integrals and Beyond) of the second part of the book, M. Couceiro, D. Dubois, H. Fargier, M. Grabisch, H. Prade, and A. Rico present new directions on the use of Sugeno integrals for multicriteria evaluation problems in an ordinal setting. The chapter surveys the axiomatic characterizations of Sugeno integrals and their expression in possibilistic logic. Moreover, new developments in this area are presented such as the use of local utility functions, the notion of bipolar qualitative evaluation, as well as the use of Sugeno integrals and if-then rules for qualitative data analysis.

The first chapter (Advances and New Orientations in Goal Programming) of the third part is devoted to goal programming (GP). D. Jones and C. Romero provide an overview of the literature on different variants of GP models and proposed a conceptual distance-metric framework that unifies/describes the existing GP models. The chapter also analyzes the connections to bounded rationality and social choice functions and discusses future developments to expand the use and flexibility of GP models.

The next chapter (Robust Goal Programming with Interactive Fuzzy Coefficients), by M. Inuiguchi, is also devoted to GP, but in a fuzzy context where the goals and coefficients in the objective are fuzzy. To treat the fuzziness in such elements of an GP model, the approach of oblique fuzzy vectors is introduced. This approach extends existing methodologies for fuzzy GP by allowing the modeling of the interactions between fuzzy coefficients. Solution procedures are also discussed.

In the third chapter (Multiobjective Bilevel Programming: Concepts and Perspectives of Development) of the second part, M. J. Alves, C. Henggeler Antunes, and J. P. Costa cover the area of multi-objective bilevel programming. Multi-objective problems that have a hierarchical structure have attracted significant research interest. The chapter provides a novel view of the main concepts in this area, including the optimistic/pessimistic leader's perspectives, as well as algorithmic issues. The chapter also discusses traditional and emerging application fields as well as pitfalls in existing approaches, which may lead to new advances and improvements.

The fourth part of the book includes six chapters devoted to applications of MCDA in various emerging areas. In the first chapter (Multi-criteria Evaluation in Public Economics and Policy) of this part, G. Munda presents the contributions of MCDA techniques in public economics and policy. The chapter starts with an outline of cost–benefit analysis (CBA), which is the standard tool used in welfare economics. CBA is then systematically compared against the MCDA paradigm using ten comparison criteria, thus leading to the identification of the benefits and possibilities that MCDA tools provide in this important area.

In the next chapter (Perspectives on Multi-criteria Decision Analysis and Life-Cycle Assessment), L. C. Dias, F. Freire, and J. Geldermann discusses the combination of MCDA and life-cycle assessment (LCA) for environmental management. First the LCA framework is discussed and then the main characteristics of the MCDA perspective to environmental decision-making are outlined. Finally, an overview of the trends and perspective on the combination of the two approaches is given.

The chapter (The Monitoring of Social Innovation Projects: An Integrated Approach) of M. F. Norese, F. Barbiero, L. Corazza, and L. Sacco, presents a case study regarding the application of a MCDA approach based on the ELECTRE outranking methods for monitoring of social innovation projects by the Municipality of Turin in Italy. Except for a MCDA approach, the proposed analysis further combines other tools, such as cognitive mapping and actor network analysis to analyze the behavior of funded innovated start-up companies and to evaluate their business projects as part of an inclusive and sustainable economy.

The next chapter (Multiobjective Optimization in the Energy Sector: Selected Problems and Challenges), by C. Henggeler Antunes, illustrates the applications of multi-objective optimization approaches in the energy sector, focusing on electricity smart grids. The chapter covers issues such as unit commitment and dispatch problems, resilient systems, the usage of demand-side resources, problems associated with electric vehicles, as well as issues related to energy markets.

The area of energy systems is also the subject of the next chapter (Optimization and Multicriteria Evaluation of District Heat Production and Storage), by R. Lahdelma, G. Kayo, E. Abdollahi, and P. Salminen. The authors present a case study about the use of MCDA techniques for the evaluation of renewable energy technologies for district heating in Finland. The proposed methodology combines stochastic multicriteria acceptability analysis (SMAA) with a production planning optimization model taking into consideration various technical and economic criteria.

The book closes with the chapter (Comparison of Routing Methods in Telecommunication Networks—An Overview and a New Proposal Using a Multi-criteria Approach Dealing with Imprecise Information) by J. Clímaco, J. Craveirinha, and L. Martins, on the evaluation and comparison of routing models in telecommunication networks. The author proposes a MCDA approach based on the VIP (Variable Interdependent Parameter) software, with an additive aggregation of criteria coping with imprecise information. The formulation of the MCDA model is illustrated through an application to a problem involving the choice of a point-to-point routing method in a transport telecom network.

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Part I
Basic Notions and Methods

New Trends in Preference, Utility, and Choice: From a Mono-approach to a Multi-approach

Alfio Giarlotta

Abstract We give an overview of some new trends in preference modeling, utility representation, and choice rationalization. Several recent contributions on these topics point in the same direction: the use of multiple tools—may they be binary relations, utility functions, or rationales explaining a choice behavior—in place of a single one, in order to more faithfully model economic phenomena. In this stream of research, the two traditional tenets of economic rationality, completeness and transitivity, are partially (and naturally) given up. Here we describe some recent approaches of this kind, namely: (1) utility representations having multiple orderings as a codomain, (2) multi-utility and modal utility representations, (3) a finer classifications of preference structures and forms of choice rationalizability by means of generalized Ferrers properties, (4) a descriptive characterization of all semiorders in terms of shifted types of lexicographic products, (5) bi-preference structures, and, in particular, necessary and possible preferences, (6) simultaneous and sequential multi-rationalizations of choices, and (7) multiple, iterated, and hierarchical resolutions of choice spaces. As multiple criteria decision analysis provides broader models to better fit reality, so does a multi-approach to preference, utility, and choice. The overall goal of this survey is to suggest the naturalness of this general setting, as well as its advantages over the classical mono-approach.

Keywords Preference modeling · Utility representation · Choice rationalization · Completeness · Transitivity · Lexicographic order · Semiorder · $\mathbb{Z}$-product · (m, n)-Ferrers property · Bi-preference · Necessary and Possible preference · Robust ordinal regression · Multi-utility representation · Modal utility representation · Multi-rationalization · Choice resolution

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1 Introduction

In the field of mathematical economics, the modelization of an agent's preference structure is traditionally done by means of a mono-approach, which uses a single binary relation satisfying the two basic tenets of economic rationality: (1) *completeness*, and (2) *transitivity*. (See, e.g., Chap. 1 of the classical microeconomics textbooks Mas-Colell et al. (1995) and Kreps (2013)). Under topological conditions of separability, these two properties guarantee the existence of a utility representation of preferences by a continuous real-valued function (Aleskerov et al. 2007; Bridges and Mehta 1995; Debreu 1954). Similarly, the traditional approach of revealed preference theory (Arrow 1959; Samuelson 1938) often employs complete and transitive binary relations to justify an agent's choice behavior. In some cases, the satisfaction of the two properties of completeness and transitivity has even guided the design of new economic theories: a striking instance of kind is given by the classical book "*Games and Economic Behavior*" of von Neumann and Morgenstern (1944).

By partially giving up these two properties, here we depart from traditional approaches, and examine: (a) alternative types of utility representations, (b) more refined kinds of preference structures, and (c) new forms of bounded rationality for choices. In fact, the general question that motivates this survey is the following:

(Q0) *Can we design sound theories of preference modeling, utility representation, and choice rationalization, which give up, partially or totally, the basic tenets of economic rationality?*

This paper illustrates some possible answers to question (Q0).

Specifically, first we deal with preference representations in a lexicographically ordered codomain (Chipman 1971; Fishburn 1974), thus extending the classical real-valued representation. This approach provides a description of preferences that fail to have a real-valued representation (Beardon et al. 2002a, b). Successively, we describe some novel types of preference structures, which are formed by nested and intertwined pairs of binary relations (Giarlotta and Watson 2018b). In this bi-preference approach, the two properties of transitivity and completeness are coherently spread over the two components. This feature makes these structures well suited to applications in operations research and economics. In particular, special types of bi-preferences, called necessary and possible (Giarlotta and Greco 2013), have already been successfully employed as a modeling tool in multiple criteria analysis (Greco et al. 2008). Under suitable conditions, bi-preferences can be represented by a doubly indexed family of utility functions: this is the so-called modal utility representation (Giarlotta and Greco 2013), which adapts to bi-preferences the recently introduced multi-utility representation of a preorder (Evren and Ok 2011; Ok 2002).

In parallel to a multi-approach to preference and utility, we also develop a theory of choice multi-rationalization. Samuelson's theory of revealed preferences (Arrow 1959; Houthakker 1950; Samuelson 1938) postulates that choices are observed, and preferences can be derived from them. The class of rationalizable choices is especially significative in this respect, since it codifies all types of choice behavior that can be explained by means of the maximization of a single binary relation. However, the theory of revealed preferences yields a sharp rational/irrational dichotomy, since any non-rationalizable choice behavior is bluntly classified as "irrational". With the goal of smoothening this dichotomy, several new theories of *bounded rationality* (Simon 1955, 1982) have naturally emerged over the last few years (Cherepanov et al. 2013; Kalai et al. 2002; Manzini and Mariotti 2007; Masatlioglu and Nakajima 2013; Rubinstein and Salant 2006). Here we describe a general setting for the multi-rationalizability of a choice (Cantone et al. 2018c), which may employ more than one binary preference to explain the behavior of an economic agent, thus broadening the classical notion of mono-rationalizability. We also sketch the main features of a recently introduced methodology in choice theory, called "resolution". This methodology, which is an adaptation of an analogous technique in general topology (Fedorcuk 1968; Watson 1992), studies the inner structure of a complex choice process (Cantone et al. 2018a) on the basis of a notion of delegations of tasks. This yields a decomposition (and explanation) of a complex selection process into independent and simpler decisional units, typically distributed in a hierarchical way.

Multiple criteria decision analysis (Greco 2005; Greco et al. 2010a) provides powerful analytical tools to handle complex real life problems, offering more flexible modelizations than mono-criterion techniques. Similarly, *mutatis mutandis*, a multi-approach to the theories of preference, utility representation, and choice rationalization yields a more realistic representation of economic phenomena rather than the classical mono-approach. The purpose of this work is to give an overview of a multi-approach to these theories, also suggesting its naturalness, feasibility, and potential.

Organization of the Paper

The remainder of this survey is organized into three main sections, a conclusive section, and an appendix.

Section 2 (The Mono-approach). We start in Sect. 2.1 with a historical discussion about the two properties of transitivity and completeness. Successively, we provide an overview of basic notions and classical results in preference modeling (Sect. 2.2), utility representations (Sect. 2.3), and choice rationalization (Sect. 2.4). These theories use a single tool for the description of an agent's behavior/attitude. In summarizing their main achievements, we shall also detect some shortcomings, and indicate possible ways of coping with the arising issues.

Section 3 (The Transition). Here we sketch a few recent approaches to the theories described in Sects. 2.2–2.4. These techniques, which suggest the use of multiple tools to represent economic behavior, address some shortcomings of classical theories and pave the way for more general approaches to these topics. Specifically, we describe: utility representations using lexicographic orderings as a codomain (Sect. 3.1), universal characterizations of semiorders based on shifted lexicographic products (Sect. 3.2), Ferrers properties describing a discrete evolution of transitivity (Sect. 3.3), choice correspondences rationalizable by well-structured revealed preferences (Sect. 3.4), and a process detecting the inner structure of a choice in terms of delegations of tasks (Sect. 3.5). The goal of this section is to provide the reader with a natural justification and a smooth transition toward a multi-approach.

Section 4 (The Multi-approach). Here we finally describe some very recent developments in the theories described in Sects. 2.2–2.4, which employ multiple tools rather than a single one. Specifically, in Sect. 4.1 we introduce bi-preference structures, and describe their advantages over mono-preferences. In Sect. 4.2, we deal with particular types of bi-preferences, called necessary and possible, which have been already used in multiple criteria decision analysis. In Sect. 4.3, we recall the notion of a multi-utility representation, and show how bi-preferences are representable by a suitably indexed type of multi-utility representation, called modal. Within the theory of choice rationalization, we provide in Sect. 4.4 an overview of the recent bounded rationality approaches, which use multiple binary rationales to explain a choice behavior. Finally, in Sect. 4.5 we describe a natural extension of the notion of choice resolution to a multiple and iterated setting.

Section 5 concludes this contribution.
The **Appendix** contains two figures, which graphically describe some results.
Neither original results nor proofs appear in this survey.

2 The Mono-approach

To keep the presentation as much self-contained as possible, this section recalls the classical setting of the theories of preference modeling, utility representation, and choice rationalization.

2.1 The Two Classical Tenets of Rationality

A preference structure on a set X of alternatives is usually modeled by a binary relation R on X. Traditionally, R is assumed to "behave well", in the sense that it satisfies suitable ordering properties. The two classical properties that are assumed to hold for R are:

(Completeness) for any distinct $x, y \in X$, either xRy or yRx (or both)[1];
(Transitivity) for any $x, y, z \in X$, if xRy and yRz, then xRz.

The reasons for which R is often supposed to be both complete and transitive are several, some being related to their economic significance, some others to their mathematical tractability. However, both properties have been questioned by eminent scholars over time.

[1]Notice that, since x and y are distinct, this formulation of completeness does not imply reflexivity.

In their monumental work *Theory of Games and Economic Behavior* (von Neumann and Morgenstern 1944), von Neumann and Morgenstern already acknowledged, albeit rather elusively, that preferences may naturally be incomplete (pp. 19–20):

> We have conceded that one may doubt whether a person can always decide which of two alternatives … he prefers. If the general comparability assumption is not made, a mathematical theory … is still possible. It leads to what may be described as a many dimensional vector concept of utility. This is a more complicated and less satisfactory set-up, but we do not propose to treat it systematically at this time.

In fact, von Neumann and Morgenstern limited their analysis to complete (and transitive) preferences, due to the mathematical amenability of this simplified setting, and never published details about the mentioned "many dimensional vector concept of utility".

In his seminal paper on incomplete preferences, Aumann (1962) suggested (p. 449) an interpretation of von Neumann and Morgenstern's statement:

> What they probably had in mind was some kind of mapping from the space of lotteries to a canonical partially ordered euclidian space, rather than the real-valued mappings we use here; but it is not clear to me how this approach can be worked out.

Aumann's criticism of the completeness property was quite direct (p. 446):

> Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint.

Since Aumann's work, many other authors started abandoning the axiom of completeness as a basic feature of rational behavior. On the topic, Bewley (1986) and Ok (2002) attentively elaborate on the links between the notion of rationality and the incompleteness of preferences.

In their systematic analysis of the multi-utility representation of preferences, Evren and Ok (2011) mention several behavioral phenomena which naturally yield incompleteness, e.g., status-quo bias (Apesteguía and Ballester 2009; Masatlioglu and Ok 2005), intransitive choice (Manzini and Mariotti 2007), choice deferral (Kopylov 2009), and indecisiveness in revealed preferences (Eliaz and Ok 2006). Similarly, incompleteness has been a main focus in various decision models used in operations research and management science (Danan 2010; Greco et al. 2008; Masin and Bukchin 2008), financial economics (Rigotti and Shannon 2005), political economics (Levy 2004; Roemer 1999), and game theory (Bade 2005). Further, several recent studies on (in)decisions under risk and uncertainty use incomplete preorders to model preferences (Dubra et al. 2004; Ghirardato et al. 2003, 2004; Gilboa et al. 2010; Maccheroni 2004; Nau 2006; Ok et al. 2012). Last but not least, following the seminal work of Bernard Roy (1985, 1990a, b), there is a large number of multiple criteria decision methodologies which explicitly take into account incompleteness of preferences as a natural feature of the decision maker's attitude (Greco et al. 2010a).

The axiom of transitivity was possibly harder to abandon, even if probably questioned before completeness. In his well-known paper, Tversky (1969) was still advocating the importance of transitivity in the modelization of preferences, since its violation could cause unpleasant phenomena of "money pump" (Davidson et al. 1955).[2] This attitude was however contrasted by other authors, who had already been designing economic models in which transitivity was partially or totally abandoned. The probabilistic choice model proposed by Luce (1959) can be regarded as a pioneering example of intransitive preferences in economic theory. The obstinate insistence of some economists to employ transitive models even brought Sen (1971) to declare that revealed preference theory is "obsessed with transitivity". In their recent paper, Bleichrodt and Wakker (2015) argue that the year 1982 was a sort of "breaking point" in the economic literature, since transitivity was given up in three seminal papers related to *regret theory*: the axiomatic approach of Fishburn (1982), a decision analysis oriented paper by Bell (1982), and the fundamental contribution of Loomes and Sudgen (1982). From an experimental point of view, there are many papers in mathematical psychology explaining intransitivity of preferences by random models, insofar as the subject's preferences vary over time from one type of ordering to another: see, e.g., Regenwetter et al. (2010, 2011) for some models of this kind, and Davis-Stober et al. (2018) for a recent method to test these models.

In the same stream of research that opposes the blunt assumption of fully transitive preferences, we ought to mention the extraordinary amount of literature on semiorders, interval orders, and similar preference structures, which describe forms of rational behavior characterized by weaker forms of transitivity. Anticipated by the intuitions of Fechner (1860), Poincaré (1908), Georgescu-Roegen (1936), Armstrong (1939), and Halphen (1955), research on intransitive preference structures had its definitive consecration by the seminal papers of Luce (1956) and Fishburn (1970), who formally introduced the notions of semiorder and interval order, respectively. Their approaches are based on the idea of weakening the axiom of transitivity, rather than abandoning it all together. Indeed, Luce's famous coffee/sugar example suggests that the transitivity of the associated indifference should be somehow weakened and regulated, whereas the transitivity of the strict preference may be retained as a natural assumption of rational behavior.

The recently introduced weak (m, n)-Ferrers properties go exactly in the direction of considering binary structures with a transitive strict preference but a possibly intransitive indifference (Giarlotta and Watson 2014a). Originally designed to provide a combinatorial extension of the Ferrers condition and semitransitivity—which coincide, respectively, with weak $(2, 2)$-Ferrers and weak $(3, 1)$-Ferrers—these properties display a finite taxonomy of enhanced forms of the transitivity of the strict preference. In fact, roughly speaking, weak (m, n)-Ferrers properties classify transitive strict preferences by means of the types of forbidden mixed cycles of preference/indifference (see Sect. 4.2 in Cantone et al. (2016)). It follows that such an approach may be relevant for economic applications insofar as weak (m, n)-Ferrers properties prompt a possible recognition of money-pump effects due to the presence of mixed cycles of a certain length and type.

[2]See Sect. 3.3 of this survey for a discussion on this point in relation to the so-called (m, n)-Ferrers properties.

Strict (m, n)-Ferrers properties (Giarlotta and Watson 2014a, 2018a; Öztürk 2008) go even further in weakening the assumption of transitivity, since they do not even postulate the transitivity of the strict preference. These properties yield an infinite taxonomy of intransitive preference structures, which are connected to other types of money-pump phenomena.

In this paper, we shall also mention some new approaches to preference modeling in which both basic tenets of economic rationality are only partially retained, being "spread over" two binary relations (see Sects. 4.1 and 4.2 on bi-preferences and NaP-preferences, respectively).

2.2 Preference Modeling

Here we summarize the basic terminology in preference theory. Two good sources of information on this topic—as well as on utility representations, which is the topic of the next section—are the textbooks by Bridges and Mehta (1995) and Aleskerov et al. (2007).

Henceforth, X is a nonempty (possibly infinite) set of alternatives (courses of action, etc.), and $\Delta(X) = \{(x, x) : x \in X\}$ is the *diagonal* of X.

Definition 2.1 A reflexive binary relation on X is referred to as a *weak preference* on X, and is henceforth denoted by $\succsim$; the pair $(X, \succsim)$ is generically called an *ordered set*. The following relations are derived from a weak preference $\succsim$ on X: its *strict preference* $\succ$ (the asymmetric part of $\succsim$), its *indifference* $\sim$ (the symmetric part of $\succsim$), and its *incomparability* $\perp$ (the symmetric part of the complement of $\succsim$). These relations are formally defined as follows for each $x, y \in X$:

$$
\begin{aligned}
x \succ y &\quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (x \succsim y) \ \wedge \ \neg(y \succsim x) \\
x \sim y &\quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (x \succsim y) \ \wedge \ (y \succsim x) \\
x \perp y &\quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \neg(x \succsim y) \ \wedge \ \neg(y \succsim x).
\end{aligned}
$$

Given an ordered set $(X, \succsim)$, the set of *maximal elements* of $A \subseteq X$ is defined by

$$\max(A, \succsim) \ := \ \{x \in A : \ (\nexists y \in A) \ y \succ x\}.$$

The *composition* of two weak preferences $\succsim_1$ and $\succsim_2$ on X is the binary relation $\succsim_1 \circ \succsim_2$ on X defined as follows for all $x, y \in X$:

$$x(\succsim_1 \circ \succsim_2)\, y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists z \in X)\ x \succsim_1 z \succsim_2 y.$$

Notice that a weak preference $\succsim$ is (i) complete if and only if its incomparability $\perp$ is empty, and (ii) transitive if and only the inclusion $\succsim \circ \succsim \subseteq \succsim$ holds. Whenever $\succsim$ is complete, the set of maximal elements of $A \subseteq X$ can be also written as $\max(A, \succsim)$

$:= \{x \in A : (\forall y \in A)\, x \succsim y\}$. Finally, observe that, even when X is finite, the set $\max(A, \succsim)$ may be empty, due to the possible presence of strict cycles (see Definition 2.2).

Definition 2.2 A weak preference $\succsim$ on X is called (x, y, z, w are arbitrary elements of X):

- *complete* (or *total* or *connected*) if $x \succsim y$ or $y \succsim x$ always holds ($x \neq y$);
- *antisymmetric* if $x \succsim y$ and $y \succsim x$ implies $x = y$ (equivalently, $\sim$ is the diagonal of X);
- *acyclic* if there are no $x_1, x_2, \ldots, x_n \in X$, with $n \geq 3$, such that $x_1 \succ x_2 \succ \cdots \succ x_n \succ x_1$;
- *quasi-transitive* if $\succ$ is transitive, i.e., ($x \succ y$ and $y \succ z$) implies $x \succ z$[3];
- *Ferrers* if ($x \succsim y$ and $z \succsim w$) implies ($x \succsim w$ or $z \succsim y$);
- *semitransitive* if ($x \succsim y$ and $y \succsim z$) implies ($x \succsim w$ or $w \succsim z$);
- an *interval order* if it is Ferrers;
- a *semiorder* if it is Ferrers and semitransitive;
- a (*partial*) *preorder* if it is transitive;
- a *partial order* if it is an antisymmetric preorder;
- a *total preorder* if it is a complete preorder;
- a *linear order* if it is an antisymmetric total preorder.

Accordingly, the pair $(X, \succsim)$ is called, e.g., a *semiordered set*, a *preordered set*, a *partially ordered set* (also called a *poset*), a *linearly ordered set* (also called a *linear ordering* or a *chain*), etc.

Notice that (i) any total preorder is trivially a semiorder, (ii) any semiorder is trivially an interval order, (iii) an interval order is both complete and quasi-transitive, and (iv) any quasi-transitive weak preference is acyclic. Moreover, the indifference derived from a preorder is an equivalence relation, but the same does not hold for the indifference associated to a semiorder (hence, a fortiori, for that of an interval order). Observe also that if X is finite, then an acyclic relation on X always has maximal elements for each nonempty subset of X.

Next, we recall some notions due to Fishburn (1970), which play an important role in the theory of preferences, especially for defining notions of (semi)continuity as well as for preferences that are interval orders and semiorders (but also for bi-preference structures, see Sects. 4.1 and 4.2): the "traces" of a weak preference.

Definition 2.3 Let $\succsim$ be a weak preference on X. For each $x \in X$, let

(weak lower section of x) $x^{\downarrow,\succsim} := \{w \in X : x \succsim w\}$,
(weak upper section of x) $x^{\uparrow,\succsim} := \{w \in X : w \succsim x\}$,
(strict lower section of x) $x^{\downarrow,\succ} := \{w \in X : x \succ w\}$,
(strict upper section of x) $x^{\uparrow,\succ} := \{w \in X : w \succ x\}$.

---

[3] In case $\succsim$ is complete, then the following statements are equivalent: (i) $\succsim$ is quasi-transitive; (ii) for each $x, y, z \in X$, $x \succ y \succsim z$ implies $x \succsim z$; (iii) for each $x, y, z \in X$, $x \succsim y \succ z$ implies $x \succsim z$.

Define three binary relations[4] on X as follows for each $x, y \in X$:

$$
\begin{array}{llll}
\textit{(left trace of } \precsim\textit{)} & x \succsim^{*} y & \stackrel{\text{def}}{\iff} & y^{\downarrow,\succsim} \subseteq x^{\downarrow,\succsim}, \\
\textit{(right trace of } \succsim\textit{)} & x \succsim^{**} y & \stackrel{\text{def}}{\iff} & x^{\uparrow,\succsim} \subseteq y^{\uparrow,\succsim}, \\
\textit{(global trace of } \succsim\textit{)} & x \succsim_{0} y & \stackrel{\text{def}}{\iff} & x \succsim^{*} y \ \wedge \ x \succsim^{**} y.
\end{array}
$$

The next lemma collects some enlightening results about traces: see, e.g., Fishburn (1985), Monjardet (1978), Pirlot and Vincke (1997).

Lemma 2.4 *Let $\succsim$ be a weak preference on X.*

- $\succsim^{*}$, $\succsim^{**}$, $\succsim_0$ *are preorders contained in* $\succsim$.
- $\succsim^{*} \circ \succsim \subseteq \succsim$ *and* $\succsim \circ \succsim^{**} \subseteq \succsim$.
- $\succsim_0 \circ \succsim \subseteq \succsim$ *and* $\succsim \circ \succsim_0 \subseteq \succsim$.
- $\succsim$ *is an interval order* $\iff$ $\succsim^{*}$ *is a total preorder* $\iff$ $\succsim^{**}$ *is a total preorder.*
- $\succsim$ *is a semiorder* $\iff$ $\succsim_0$ *is a total preorder.*
- $\succsim$ *is a preorder* $\iff$ $\succsim = \succsim_0$.
- $\succsim$ *is a total preorder* $\iff$ $\succsim = \succsim_0$ *is complete.*

Many classical results on preferences are related to the possibility of (continuously) representing them by a utility function, a topic that is analyzed in the next section. There are also other issues arising from the traditional mono-approach to preference modeling, mostly due to the limited expressive power of a single binary relation. In this respect, a general question is:

(Q1) *Can we use binary relations to represent preferences in a more flexible and realistic way?*

We shall address question (Q1) in Sects. 4.1 and 4.2, where we suggest how a bi-preference approach may enhance the modeling power of a binary representation of agents' preference structures by taking into account two different kinds of "attitudes".

2.3 Utility Representations

In this section we deal with the classical setting of real-valued utility representations of binary preferences. Two are the basic issues, the first purely order-theoretic and the second topological:

---

[4]We follow the approach described in Bouyssou and Pirlot (2004), defining all traces in terms of weak sections, instead of defining strict traces first and then deriving weak traces. The difference is immaterial whenever dealing with complete and quasi-transitive preferences, in particular for interval orders and semiorders. Notice also that the notion of global trace has been recently revised from a different perspective, and renamed *transitive core* (Nishimura 2018).

(Q2) *Can we can represent a total preference relation by a real-valued utility function?*
(Q3) *Can we make this utility function continuous?*

To start, we give the basic elements to properly formulate and then address question (Q2).[5]

Definition 2.5 A binary relation $\succsim$ on X is *representable in* $\mathbb{R}$ if there is a function $u\colon X \to \mathbb{R}$ such that, for all $x, y \in X$, we have

$$x \succsim y \quad \Longleftrightarrow \quad u(x) \geq u(y).$$

In this case, the function u is a *utility representation* of $(X, \succsim)$ in $\mathbb{R}$. (We also say that $(X, \succsim)$ is *order-embeddable* or *embeddable* in $\mathbb{R}$.) The chain $(\mathbb{R}, \geq)$ is the *base* of the representation.

An obvious necessary condition for the representability of a weak preference $\succsim$ in $\mathbb{R}$ is that $\succsim$ must be a total preorder, i.e., it satisfies the two classical properties of transitivity and completeness. This condition is also sufficient for the cases in which the ground set X is finite or countably infinite (see, e.g., Chap. 1 of Bridges and Mehta (1995)). In the general case, however, we need an additional property of "separability" to ensure representability.

The first characterization of representability in $\mathbb{R}$ is most likely the following (Cantor 1895; Milgram 1939):

Theorem 2.6 (Cantor 1895; Milgram 1939) *A linear ordering $(X, \succsim)$ is order-embeddable in $\mathbb{R}$ if and only if it includes a countable subset that is weakly order-dense in X.*[6]

Similar characterizations were given by Birkhoff (1948). Nevertheless, due to an imperfect communication in the scientific community, until the early 1950s economists considered all preference relations as representable in $\mathbb{R}$. In other words, the concepts of "preference" and"utility" were (wrongly) considered equivalent. For a salient instance of this kind, let us cite Hicks (1956, p. 19):

> If a set of items is strongly ordered, it is such that each item has a place of its own in the order; it could, in principle, be given a number.

If the above statement were to hold, then *every* total preorder would be representable in $\mathbb{R}$, and the concepts of preference and utility would coincide, which is false.

---

[5]The literature also examines weaker forms of representability of a single binary relation, e.g., the existence of (continuous, semicontinuous) *Richter-Peleg* utility functions (Alcantud et al. 2016; Peleg 1970; Richter 1966). We shall deal with this topic in Sect. 4.3, where we also discuss some shortcomings of this notion, and introduce multi-utility representations.

[6]A set $Y \subseteq X$ is *weakly order-dense* in X if, for each $x_1, x_2 \in X$ such that $x_1 \succ x_2$, there is $y \in Y$ with the property that $x_1 \succsim y \succsim x_2$. Such a set is often called *Debreu order-dense*, and the existence of a countable Debreu order-dense is referred to as *Debreu-separability* (Bridges and Mehta 1995).

In his celebrated paper on the *Open Gap Lemma*, Debreu (1954) finally exhibited an example of a natural preference that is non-representable in $\mathbb{R}$: the lexicographic plane $\mathbb{R}^2_{\text{lex}} = (\mathbb{R}^2, \precsim_{\text{lex}})$. Several characterizations of representability followed, for instance (Fleischer 1961):

Theorem 2.7 (Fleischer 1961) *A chain* $(X, \precsim)$ *is representable in* $\mathbb{R}$ *if and only if it has at most countably many jumps and the topological space* $(X, \tau_{\precsim})$ *is separable.*[7]

For an extensive overview of the topic, the reader is referred to Bridges and Mehta (1995), Mehta (1998).

In 2002, Beardon et al. (2002a, b) systematically analyzed the structure of total and transitive preferences that fail to be representable in $\mathbb{R}$, and obtain a striking subordering classification of them. Their characterization (Beardon et al. 2002a) can be suggestively rephrased as follows:

Theorem 2.8 (Beardon et al. 2002a) *A chain is non-representable in* $\mathbb{R}$ *if and only if it is* (i) *long or* (ii) *large or* (iii) *wild.*[8]

(Here by "long" we mean that it contains a copy of the first uncountable ordinal[9] ω_1 or its reverse ordering $\omega_1{}^*$; by "large" we mean that it contains a copy of a non-representable subordering of the lexicographic plane $\mathbb{R}^2_{\text{lex}}$; and by "wild" we mean that it contains a copy of an *Aronszajn line*, which is defined as an uncountable chain such that neither ω_1 nor $\omega_1{}^*$ nor an uncountable subordering of $\mathbb{R}$ embeds into it.) Some more recent results in this direction, which use lexicographic orders as modeling tools, are mentioned in Sect. 3.1.

Next, we deal with question (Q3), that is, the existence of a *continuous* real-valued representation. To describe the topological setting, we recall the notions of (i) the continuity of a preorder, and (ii) the order topology induced by a preorder. (For all undefined topological notions, the reader may consult the classical textbook by Munkres (2000).)

Definition 2.9 Given a topological space (X, τ), a preorder $\precsim$ on X is *continuous*[10] if $\precsim$ is a closed subset of the topological product $X \times X$.

---

[7] A *jump* in an ordered space $(X, \precsim)$ is a pair $(a, b) \in X^2$ such that $a \succ b$ and there is no point $c \in X$ such that $a \succ c \succ b$. The topology $\tau_{\precsim}$ is the *order topology* induced by $\precsim$. The topological space $(X, \tau_{\precsim})$ is *separable* if it contains a countable set D that intersects each nonempty open set. See Munkres (2000) for topological notions.

[8] This is not the terminology originally used by the authors.

[9] An *ordinal* is a well-ordered set $(X, <)$ such that each $x \in X$ is equal to its initial segment $\{y \in X : y < x\}$. The finite ordinals are the natural numbers. The first infinite ordinal is the set ω_0 of all natural numbers, endowed with the usual order. The first uncountable ordinal is the set ω_1 of all countable ordinals, endowed with the natural order. The famous *continuum hypothesis*, formulated by George Cantor in 1878, says that the cardinality of $\mathbb{R}$ is equal to ω_1 (as a cardinal). In 1963, Paul Cohen proved that the continuum hypothesis is independent from the axioms of ZFC (Zermelo-Fraenkel axiomatic set theory, plus the Axiom of Choice), in sense that there are models in which it is true, and models in which it is false (because $|\mathbb{R}| > \omega_1$ holds). See the classical textbook by Kunen (1980) for ZFC axiomatic set theory.

[10] Here we use the notion of continuity employed in some standard textbooks in microeconomic theory, such as Mas-Colell et al. (1995, p. 46). Other authors sometimes employ a weaker notion

It can be shown that a *complete* preorder $\precsim$ on (X, τ) is continuous if and only if (i) all weak upper sections $x^{\uparrow,\precsim}$ and lower sections $x^{\downarrow,\precsim}$ are closed subsets of (X, τ) if and only if (ii) all strict upper sections $x^{\uparrow,\succ}$ and lower sections $x^{\downarrow,\succ}$ are open subsets of (X, τ). Conditions (i) and (ii) are sometimes called, respectively, *closed semicontinuity* and *open semicontinuity*, whereas their joint satisfaction is called *bi-semicontinuity*: see Sect. 4.1. Notice that bi-semicontinuity does not imply continuity for incomplete preorders.[11]

Definition 2.10 Given a preordered set $(X, \precsim)$, the *order topology* $\tau_{\precsim}$ on X induced by $\precsim$ is the topology having as a subbasis the family of all strict upper and lower sections (equivalently, the topology having as a basis the family of all open intervals).

An immediate consequence of Definitions 2.9 and 2.10 is that for any totally preordered set $(X, \precsim)$, the order topology $\tau_{\precsim}$ is the coarsest topology on X such that $\precsim$ is continuous.

There are many results dealing with continuous real-valued utility representations of a total preorder. The most classical theorems in this field are due to Eilenberg (1941) and Debreu (1954, 1964):

Theorem 2.11 (Eilenberg 1941) *In a connected separable topological space, any continuous total preorder is continuously representable in* $\mathbb{R}$.

Theorem 2.12 (Debreu 1954, 1964) *In a second countable topological space, any continuous total preorder is continuously representable in* $\mathbb{R}$.

A miscellany of representation results followed (in the 1970s): let us recall, among others, the approaches due to Jaffray (1975a), Neuefeind (1972), Peleg (1970), Richter (1980), and Sondermann (1980). A common denominator of many approaches to the topic is the *Open Gap Lemma*, which was (incorrectly) proved by Debreu (1954), and then corrected by the same author ten years later (Debreu 1964). For our purpose, the most relevant consequence of this result is the following:

Corollary 2.13 *If a total preorder on a topological space is representable in* $\mathbb{R}$, *then it is continuously representable in* $\mathbb{R}$.

The above result brings back the problem of the continuous representability of a total preorder to that of its mere representability, on which Theorem 2.8 by Beardon et al. (2002a) certainly sheds some light. However, Theorem 2.8 mostly provides

---

of continuity: see, e.g., Sect. 1.6 of Bridges and Mehta (1995). However, from the point of view of applications, the distinction between the various notions of continuity is often immaterial. See also Evren and Ok (2011, p. 555), and Gerasímou (2013, pp. 2–3).

[11] Herden and Pallack (2002) provide a very simple counterexample to the equivalence between continuity and bi-semicontinuity for incomplete preferences: in fact, they show that the relation of equality is a bi-semicontinuous non-continuous preorder in any topological space that is T_1 but not Hausdorff. On the topic, see also Gerasímou (2013), who characterizes continuity in terms of closed semicontinuity and a property of "local expansion" of transitivity (Theorem 1 in Gerasímou (2013)).

negative information, since several total preorders typically fail to be representable. Thus, it appears natural to seek more refined classifications of non-representable preferences. More precisely, the (new) questions are:

(Q2′) *Can we detect weaker forms of representability for non-representable preferences?*
(Q3′) *Can we make these weaker forms of representability continuous?*

A possible approach to questions (Q2′) and (Q3′) is to establish a "degree of representability" of total preferences by using more descriptive codomains rather than the set of real numbers. In this respect, codomains (different from $\mathbb{R}$) ensuring that the content of Corollary 2.13 is preserved—in the sense that the representability of a total preorder implies its continuous representability—look quite appealing. This brought Herden and Mehta (2004) to formulate the notion of a *Debreu chain*, which is a linear ordering such that the representability in it also ensures the existence of a continuous representation. (Thus, by Corollary 2.13 the linear ordering of the reals is the prototype of a Debreu chain; however, it is not the only one.)

In the same direction of research, some other authors extended the notion of a Debreu chain to that of a *pointwise Debreu* and *locally Debreu* chain (Caserta et al. 2008), also considering lexicographic products satisfying these properties (Giarlotta and Watson 2009). We shall deal with these recent approaches that aim at enlarging the representability of preference relations in Sect. 3.1, where we consider representations with lexicographic codomains. Further, in Sect. 3.2 we will present a universal description of semiorders by means of embeddings into modified forms of lexicographic products.

Nevertheless, the issues mentioned in the last two paragraphs are not the only ones. In fact, further problems on representability arise for the lack of representations of preferences that fail to fully possess the classical tenets of economic rationality. More precisely, the issue—which is obviously related to the question (Q1) formulated in Sect. 2.2—is the following:

(Q4) *How can we represent more refined preference structures by means of utility functions?*

We shall present possible ways to address question (Q4) in Sect. 4.3, where we deal with multiple and modal utility representations of both a single preference and a pair of preferences.

2.4 Choice Rationalization

Here we recall some elementary definitions on choices. We also summarize the basics of the theory of revealed preferences, pioneered by Samuelson (1938) and successively developed by several eminent scholars: see, among many others, Arrow (1959, 1963), Chernoff (1954), Hansson (1968), Herzberger (1973), Houthakker (1950), Plott (1973), Richter (1966), Sen (1971, 1986, 1993). For further details, the reader is referred to some textbooks on the topic, such as Aleskerov et al. (2007) and Suzumura (1983), as well as the very recent monograph by Chambers and Echenique (2016).

Definition 2.14 Let Ω be a family of nonempty subsets of X, which contains all singletons and is closed under the operation of taking finite unions (hence Ω contains all nonempty finite subsets of X).[12] A *choice correspondence* on X is a map $c\colon \Omega \to \Omega$ such that the inclusion $c(A) \subseteq A$ holds for any $A \in \Omega$. In particular, a *choice function* is a single-valued choice correspondence, that is, $|c(A)| = 1$ for all $A \in \Omega$. The set Ω is the *domain* of c, elements of Ω are *menus*, and elements of a menu are *items*. A *choice space* is a pair (Ω, c), where c is a choice correspondence on X having Ω as domain. A choice space (Ω, c) is *complete* if Ω is the family 2^X of all nonempty subsets of X, and is *finite* if Ω is the family of all finite nonempty subsets of X.

The nonempty set $c(A)$ collects all items of A deemed "selectable" by the economic agent; in case the problem requires that a single item is to be chosen, this is usually done at a later time and with a different procedure. However, in the special case of a choice function, a single item is immediately selected from each menu: this is the original setting under which Samuelson was working in his seminal paper Samuelson (1938), later extended to the general case of choice correspondences.

Next, we recall the classical notion of the preference revealed by a choice, which is typically employed in order to identify all cases of rational behavior.

Definition 2.15 Let (Ω, c) be a choice space. The *preference revealed by c*, denoted by $\succsim_c$, is the binary relation on X defined as follows for each $x, y \in X$:

$$x \succsim_c y \quad \overset{\text{def}}{\Longleftrightarrow} \quad \text{there is a menu } A \in \Omega \text{ such that } x, y \in A \text{ and } x \in c(A).$$

Then c is called *rationalizable* if it can be retrieved from $\succsim_c$ by maximization, that is, for all menus $A \in \Omega$, the equality $c(A) = \max(A, \succsim_c)$ holds. Equivalently, c is rationalizable if there is a (not necessarily complete) binary relation $\succsim$ on X such that $c(A) = \max(A, \succsim)$ for all $A \in \Omega$.

The next example illustrates the notions introduced so far.

Example 2.16 Consider the following choice correspondences on $X = \{x, y, z\}$[13]:

[12] The literature on choice theory also consider other types of domains, e.g., for the case of choices arising from *consumer demand theory*. For the sake of simplicity, here we limit our analysis to the case in which Ω satisfies some rather mild closure properties (see Cantone et al. (2016), Eliaz and Ok (2006) for a justification of this assumption).

[13] Selected items are underlined: thus, $\underline{x}\,y\,z$ means $c(\{x, y, z\}) = \{x\}$, $\underline{y}\,\underline{z}$ means $c(\{y, z\}) = \{y, z\}$, etc. Notice that, by the very definition of a choice correspondence, we always have $c(\{a\}) = \{a\}$ for each $a \in X$: thus, it suffices to indicate how choices are defined for menus of size at least two.

$$\begin{array}{lllll} (c_1) & \underline{x}\, y\, z, & \underline{x}\, y, & \underline{x}\, z, & \underline{y}\, z, \\ (c_2) & \underline{x}\, \underline{y}\, z, & \underline{x}\, \underline{y}, & \underline{x}\, z, & \underline{y}\, \underline{z}, \\ (c_3) & \underline{x}\, y\, z, & \underline{x}\, \underline{y}, & \underline{x}\, \underline{z}, & \underline{y}\, \underline{z}. \end{array}$$

The three relations of revealed preferences $\succsim_{c_1}$, $\succsim_{c_2}$, and $\succsim_{c_3}$ are respectively defined by

$$\begin{array}{llll} (c_1) & x \succ_{c_1} y, & x \succ_{c_1} z, & y \succ_{c_1} z, \\ (c_2) & x \sim_{c_2} y, & x \succ_{c_2} z, & y \sim_{c_2} z, \\ (c_3) & x \sim_{c_3} y, & x \sim_{c_3} z, & y \sim_{c_3} z. \end{array}$$

Notice that $\succsim_{c_1}$ is a linear order, $\succsim_{c_2}$ is quasi-transitive but not transitive, and $\succsim_3$ is an equivalence relation. Further, c_1 and c_2 are rationalizable, whereas c_3 is not.

(Mono-)rationalizability coincides with the existence of an underlying preference relation that fully describes the observed choice behavior. It is clear that a tiny percentage of choices are rational according to this notion, since the size of the family of choices on a set X is much larger, in general, than the family of acyclic binary relations on X. In other words, Definition 2.15 implies that the large majority of choices are labeled as "irrational". This situation naturally calls for new, more refined notions of rationalizability, which should aim at smoothening the sharp dichotomy between rational and irrational choices, possibly identifying weaker notions of rationality. We shall deal with some recent approaches of this kind in Sect. 4.4.

Most of the existing results on the rationalizability of a choice are stated in terms of the satisfaction of *axioms of choice consistency*. These are properties codifying rules of coherent behavior, which ought to be respected in order to qualify a selection process as consistent. Here are a few of the plethora of axioms introduced in the literature during the last 80 years:

◊ **Property** (α) (*Standard Contraction Consistency*):
If $x \in A \subseteq B$ and $x \in c(B)$, then $x \in c(A)$.

◊ **Property** (β) (*Symmetric Expansion Consistency*):
If $A \subseteq B$, $x, y \in c(A)$, and $y \in c(B)$, then $x \in c(B)$.

◊ **Property** (γ) (*Standard Expansion Consistency*):
If $x \in c(A_i)$ for all $i \in I$, then $x \in c\left(\bigcup_{i \in I} A_i\right)$.

◊ **Property** (ρ) (*Standard Replacement Consistency*):
If $y \in c(A)$ and $y \notin c(A \cup \{x\})$, then $x \in c(A \cup \{x\})$.

◊ WARP (*Weak Axiom of Revealed Preference*):
If $x \in A$ and there are $y \in c(A)$ and $B \in \Omega$ such that $y \in B$ and $x \in c(B)$, then $x \in c(A)$.

◊ PI (*Path Independence*):
$c(A \cup B) = c(c(A) \cup c(B))$.

(A universal quantification over menus and items is implicit.)

The first three properties are classical, respectively introduced by Chernoff (1954) for (α), and by Sen (1971) for (β) and (γ); on the contrary, property (ρ) is very recent (Cantone et al. 2016). WARP, due to Samuelson (1938), is the most well known axiom in choice theory. PI is a very elegant axiom due to Plott (1973).

The semantics of these axioms of choice consistency is simple. Property (α) says that if an item x is selected from a menu B, then x is also selected from any submenu $A \subseteq B$ containing it. Property (β) states that any two items x, y selected from a menu A are simultaneously either selected or rejected in any larger menu B. Property (γ) says that if an item x is selected from all menus in a family $\mathcal{A}$, then x is also selected from the menu obtained as the union of the elements of $\mathcal{A}$. Property (ρ) states that if an item y is selected from a menu A but is rejected as soon as a new item x is adjoined to A, then the new item x is selected from the larger menu $A \cup \{x\}$. WARP says that an item x is always selected from a menu A whenever there is an item y selected from A such that x is revealed to be preferred to y. Finally, PI states that if the dynamic process of selection proceeds in a "divide and conquer" manner,[14] then the final outcome is independent of the way the menu is initially divided for consideration.

Example 2.17 For the choices defined in Example 2.16, the following holds:

(1) c_1 satisfies all listed axioms of choice consistency;
(2) c_2 satisfies (α), (γ), (ρ), and PI, but (β) and WARP fail;
(3) c_3 only satisfies (α), but none of the other properties hold for it.

We conclude this overview by listing some relationships between forms of rationalizability of a choice and the axioms of choice consistency introduced above, which hold under very mild conditions on the choice domain: see, among several references on the topic, the classical papers by Arrow (1959) and Sen (1971), as well as the recent results in Cantone et al. (2016).

Theorem 2.18 *The following equivalences hold for a choice space* (Ω, c)*:*

(i) *c is rationalizable* $\iff$ (α) & (γ) *hold.*
(ii) *c is rationalizable by a total preorder* $\iff$ *WARP holds* $\iff$ (α) & (β) *hold.*
(iii) *c is rationalizable by a preorder* $\iff$ (α) & (γ) & (ρ) *hold.*

The following questions naturally arise:

(Q5) *Can we refine the classification of rationalizable choices given by Theorem* 2.18?

(Q6) *Can we smoothen the classical rational/irrational dichotomy, providing a classification of non-rationalizable choices by means of "degrees of rationality"?*

Questions (Q5) and (Q6) will be addressed in Sects. 3.4 and 4.4, respectively.

[14] By a "divide and conquer" manner, we mean: the menu is split up into smaller sets, a choice is made over each of these sets, the selected items are collected, and finally a choice is made from them.

3 The Transition

In this section we start a process of transition toward a multi-approach. Specifically, we describe: some alternative tools in preference modeling, utility representations, and choice rationalization, all of which suggest the opportunity to pursue a multi-approach to a full extent. These techniques do solve a few of the issues arising from the classical mono-approach. However, they are not completely satisfactory, inasmuch as they fail to address some other important problems.

3.1 Utilities with Lexicographic Codomains

As already recalled in the previous sections, several well-behaved preferences that naturally appear in applied fields fail to be representable by a real-valued utility function. In fact, even in the desirable scenario in which an agent's preferences are transitive and complete, their representability by real-valued embeddings is not guaranteed in general. This consideration brought Herden and Mehta (2004) to formulate the following question:

(Q7) *Why do we only consider* $\mathbb{R}$*-valued utility functions as representations of preferences?*

As extensively discussed in Mehta (1998), the literature on utility representations mostly deals with utility functions with values in the linear ordering $(\mathbb{R}, \geq)$. Regrettably, the very same literature lacks a systematic and convincing discussion explaining why $\mathbb{R}$ is the only considered codomain. The rationale of such a choice is possibly connected to the fact that economists naturally identify the utility of a bundle of goods by a real number. In addition, the mathematical amenability of the linearly ordered topological space $(\mathbb{R}, \geq, \tau_{\geq})$—which is metrizable, complete, separable, etc.—provides further reasons of opportunity to universally implement this choice.

However, Herden and Mehta (2004) argue that these arguments do not suffice. In fact, the two authors identify several types of problems connected to the inveterate use of $\mathbb{R}$ as *the* codomain of utility functions. Following the presentation given in Caserta et al. (2008), we collect these issues in two groups: (a) mathematical, which in turn can be ordinal or cardinal[15]; and (b) theoretical.

(a) Historically, the most significant example of ordinal obstruction to the representability in $\mathbb{R}$ is the lexicographic plane $\mathbb{R}^2_{\text{lex}}$ (Debreu 1954): this linear ordering is not representable in $\mathbb{R}$ because it does not satisfy the *countable chain condition* (i.e., there are uncountably many pairwise disjoint nonempty open intervals). Another example of non-representability in $\mathbb{R}$ due to an ordinal obstruction is the *long line*, that is, the lexicographic product $\omega_1 \times_{\text{lex}} [0, 1)$ with its minimum

[15] We should also distinguish between purely ordinal codomains, and those which also have an algebraic structure. Among the latter, let us mention (without getting into details) representations that employ *non-Archimedean ordered fields*, introduced by Narens (1985).

(0, 0) removed. The importance of the latter linear ordering in economic theory is widely acknowledged (Estévez and Hervés 1995; Monteiro 1987). The structural reason for which the long line cannot be embedded into $\mathbb{R}$ is that it contains a copy of ω_1, the first uncountable ordinal. (For a throughout discussion of ordinal obstructions to representability, see Beardon et al. (2002a, b).)
Cardinal obstructions to the representability in $\mathbb{R}$ are quite frequent as well. Herden and Mehta (2004) give some examples of commodity spaces studied in economic theory, which fail to be representable in $\mathbb{R}$ because their cardinality is greater than the continuum. A first example of this kind is the infinite-dimensional commodity space $L^\infty(\mu)$ of essentially bounded measurable functions on a measure space; in most models used in general equilibrium theory (Bewley 1972), this linear ordering is too large to be embedded in $\mathbb{R}$. Another example of a linear preference that is not embeddable in $\mathbb{R}$ for cardinal reasons is the space $(\mathbb{R}^n)^\mathbb{R}$ of all functions from $\mathbb{R}$ to the commodity space $\mathbb{R}^n$, used in capital theory (Diamond 1965).

(b) From the theoretical point of view, the use of $\mathbb{R}$ to represent preferences may even clash with the very concept of utility. In his paper on the foundations of utility, Chipman (1960) argues that utility is not a real number, but a vector that is inherently lexicographic in nature. Accordingly, he proposes to employ the lexicographic power 2^α_{lex} as a base of utility representations. (Here $2 = \{0, 1\}$ is the linear ordering with two elements, and α is a suitable ordinal number.) Chipman points out the convenience to use of a transfinite sequence of length α in place of a real number to represent preferences: mathematically, every linear ordering becomes representable; economically, the concept of utility becomes easier to understand. Last but not least, representability of a preference space $(X, \precsim)$ in $\mathbb{R}$ requires the topological space $(X, \tau_{\precsim})$ to have a countable base, which has no intuitive meaning from the economic point of view (Chipman 1971). For an extensive analysis of a notion of lexicographic utility and alternative types of utility representations, the reader is referred to the (dated but always valuable) survey by Fishburn (1974).

In the light of the above discussion, it seems natural to consider alternative utility representations, which use a base chain different from $\mathbb{R}$. The most frequent base chains employed in the literature are lexicographic products, e.g., 2^α_{lex} (as in Chipman (1971)), $\mathbb{R} \times_{\text{lex}} 2$ (as in Wakker (1988)), $\mathbb{R}^n_{\text{lex}}$ (as in Knoblauch (2000)), and the long line (as in Campión et al. (2006)). Thus, it appears useful to develop a theory of utility representations in which the base chain is a lexicographic product of linear orderings. To start, we recall the basic definition of lexicographic product.

Definition 3.1 Let $\mathcal{X} = \{(X_j, \succsim_j) : j \in J\}$ be a nonempty family of chains indexed over a well-ordered set $(J, \leq)$. The *lexicographic product* of $\mathcal{X}$ is the chain $\prod^{\text{lex}}_{j\in J} X_j = \left(\prod_{j\in J} X_j, \succsim_{\text{lex}}\right)$, where the strict linear order $\succ_{\text{lex}}$ is defined as follows for all $x = (x_j)_{j\in J}$, $y = (y_j)_{j\in J} \in \prod_{j\in J} X_j$:

$$x \succ_{\text{lex}} y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \text{there is } \ \delta \in J \ \text{ such that } \ x_\delta \succ_\delta y_\delta \ \text{ and } \ x_j = y_j \text{ for all } j < \delta.$$

In particular, $X \times_{\text{lex}} Y$ denotes the lexicographic product of the two chains X and Y. In case the well-ordered index set J is a nonzero ordinal α, we denote the corresponding lexicographic product by $\prod_{\xi<\alpha}^{\text{lex}} X_\xi$. Further, X_{lex}^{α} is the *lexicographic power* of α-many copies of X.

The use of lexicographic products as a codomain of utility representations can be naturally motivated when modeling multidimensional preferences. In fact, in order to endow a Cartesian product of some given chains with a linear order, lexicographic utility structures come very handy, since they are linked to the existence of some factors which are "overwhelmingly more important" than others.

For instance, assume that there are n factors $X_1, \ldots, X_n$ of concern to the decision maker. An element $x_j \in X_j$ is a "level of the factor X_j" (e.g., in an allocation problem, x_j represents the resources allocated to the j-th activity). Then $X = X_1 \times \cdots \times X_n$ is the set on which a preference $\succsim$ has to be established by the decision maker. A lexicographic modeling of utilities requires finding whether there exist n individual utility functions $u_j \colon X \to \mathbb{R}$, $j = 1, \ldots, n$, such that, for each $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$, we have $x \succsim y$ if and only if $(u_1(x), \ldots, u_n(x)) \succsim_{\text{lex}} (u_1(y), \ldots, u_n(y))$, where $\succsim_{\text{lex}}$ is the lexicographic ordering on $\mathbb{R}^n$. In this way, preferences are classified according to a measure of their "lexicographic complexity". For instance, if a chain $(X_1, \succsim_1)$ can be order-embedded into the lexicographic power $\mathbb{R}_{\text{lex}}^2$ but not in $\mathbb{R}$, and another chain $(X_2, \succsim_2)$ can be order-embedded into $\mathbb{R}_{\text{lex}}^4$ but not in $\mathbb{R}_{\text{lex}}^3$, then the lexicographic complexity of the latter is greater than the lexicographic complexity of the former. Formally, we can define the notion of the representability number of a chain as follows (Giarlotta 2005):

Definition 3.2 A chain $(X, \succsim)$ is *α-representable in* $\mathbb{R}$ if it can be embedded into the lexicographic power $\mathbb{R}_{\text{lex}}^{\alpha}$, where α is an ordinal number. The least ordinal α such that X is α-representable in $\mathbb{R}$ is the *representability number of X in* $\mathbb{R}$, denoted by $\mathsf{repr}_{\mathbb{R}}(X)$. More generally, given a base chain B, the *representability number of X in B*, denoted by $\mathsf{repr}_B(X)$, is the least ordinal α such that X can be embedded into the lexicographic power B_{lex}^{α}.

The α-representability of a chain $(X, \succsim)$ in $\mathbb{R}$ corresponds to having a representation of the preference ordering $\succsim$ in X by a well-ordered family of utility functions $u_\xi \colon X \to \mathbb{R}$ indexed by the ordinal numbers $\xi < \alpha$. Then, for any $x, y \in X$, we have $x \succ y$ if and only if $u_\delta(x) > u_\delta(y)$ holds, where δ is the least ordinal number below α at which $u_\delta(x)$ and $u_\delta(y)$ differ. One can think of the ordinal indices as determining the relative importance of the utility functions u_ξ.

In connection with the findings of Theorem 2.8, it is well-known that long chains are not α-representable in $\mathbb{R}$ for any countable ordinal α (see Fleischer (1961)): thus, their representability number in $\mathbb{R}$ is ω_1. It follows that the family of all chains can be partitioned in three classes:

(i) long chains;
(ii) short (i.e., not long) chains with uncountable representability number in $\mathbb{R}$;
(iii) chains with countable representability number in $\mathbb{R}$.

The two classes (ii) and (iii) are very rich in variety. For instance, it is not surprising that Aronszajn lines belong to class (ii). On the other hand, rather unexpectedly, in (ii) we can also find several hierarchies of *small* chains, i.e., in the terminology of Theorem 2.8, chains that are neither long nor wild: see Giarlotta (2004a, Chap. 5). Even more surprisingly, class (iii) contains many types of linear orderings. For instance, Giarlotta and Watson (2013) exhibit a hierarchy of chains having representability number in $\mathbb{R}$ equal to ω (the first infinite ordinal). Finally, in Giarlotta (2004b) lexicographic products that are representable in $\mathbb{R}$ (i.e., such that $\mathsf{repr}_{\mathbb{R}}(X) = 1$) are characterized in terms of suitable features of their factors.

Concerning the case of base chains different from $\mathbb{R}$, in Giarlotta (2005) the author determines the value of $\mathsf{repr}_B(X)$ for several base chains B and represented chains X, again in relation to Theorem 2.8. Specifically, the following results hold[16]:

Theorem 3.3 (i) *If κ is a regular cardinal that is not embeddable into B, then $\mathsf{repr}_B(\kappa) = \kappa$.*
(ii) *If B is an uncountable chain such that $A \times_{\text{lex}} 2$ is not embeddable in B for any uncountable $A \subseteq B$, then $\mathsf{repr}_B(B^{\alpha}_{\text{lex}}) = \alpha$ for any ordinal α.*
(iii) *If X is an Aronszajn line or a Souslin line, then $\mathsf{repr}_{\mathbb{R}}(X) = \omega_1$.*

In particular, Theorem 3.3(ii) yields the following known fact (Kuhlmann 1995):

Corollary 3.4 *$\mathsf{repr}_{\mathbb{R}}(\mathbb{R}^{\alpha}_{\text{lex}}) = \alpha$ for any ordinal α.*

Some additional instances of theoretical results concerning the representations of lexicographic preferences are given in Candeal and Induráin (1999), Giarlotta and Watson (2014b), Kuhlmann (1995).

3.2 Universal Semiorders

Semiorders are among the most studied categories of binary relations in preference modeling. This is due to their capability to model many phenomena in economics and psychology, whenever the agent exhibits preferences/choices with a "threshold of perception or discrimination" (also called *just noticeable difference*, see Manders (1981)). The reader may consult Chap. 2 of the monograph by Pirlot and Vincke (1997) for an extensive list of possible applications.

The notion of a semiorder originally appeared (under a different name) in 1914, in the work of Wiener (1914) (see Fishburn and Monjardet (1992)). Nevertheless, the introduction of semiorders in economics is usually attributed to Luce (1956), who

[16]See Kunen (1980) for the undefined notions of *regular cardinal* and *Souslin line*.

was the first to use this model to study choices in settings where the agent's indifference is naturally intransitive. Luce's original definition is based on the reciprocal behavior of the associated relations of strict preference and indifference. Nowadays, a semiorder is defined as either a reflexive relation that is Ferrers and semitransitive, or, equivalently, an asymmetric relation that is Ferrers and semitransitive (sometimes called a *strict semiorder*).

Since Luce's seminal paper, research on semiorders has been abundant, due the universally acknowledged importance of this type of ordered structure. Several contributions on the topic are concerned with real-valued representations of semiorders (Beja and Gilboa 1992; Campión et al. 2008; Candeal and Induráin 2010; Gensemer 1987; Krantz 1967; Lehrer and Wagner 1985; Manders 1981; Monjardet 1978; Nakamura 2002), whereas many others deal with the more general notion of an interval order (see, e.g., Beja and Gilboa (1992), Bosi et al. (2001) and references therein), a preference structure introduced by Fishburn in the 1970s (Fishburn 1970, 1973b, 1985). Semiorders have been also studied in connection to the assessment of knowledge and learning: on the topic, the interested reader may consult the monographs by Doignon and Falmagne on *Knowledge Spaces* (Doignon and Falmagne 1999) and *Learning Spaces* (Falmagne and Doignon 2011), as well as some papers describing stochastic theories for the evolution of preference structures (Doignon and Falmagne 1997; Falmagne 1996, 1997; Falmagne and Doignon 1997).

Concerning the utility representation of semiorders, a main contribution on the topic is the classical paper by Scott and Suppes (1958), in which semiorders are described by the existence of a "shifted" type of utility function (see also Rabinovitch (1977)). Formally, a *Scott-Suppes representation* of a semiordered set $(X, \succsim)$ is a function $u\colon X \to \mathbb{R}$ such that the equivalence "$x \succsim y \Leftrightarrow u(x) + 1 \geq u(y)$" holds for all $x, y \in X$. (Here 1 is the threshold of perception or discrimination.) It is well known that not all semiorders admit a Scott-Suppes representation: in fact, its existence imposes strong structural restrictions, as pointed out by Swistak (1980). In this respect, a recent result by Candeal and Induráin (2010) characterizes Scott-Suppes representable semiorders in terms of the properties of *regularity* and *s-separability*. Despite these restrictions, Scott-Suppes representations have been given a lot of attention, due to their relevance in several fields of research, e.g., modelizations of choice with errors (Agaev and Aleskerov 1993), choice theory under risk (Fishburn 1968), extensive measurement in mathematical psychology (Krantz 1967; Lehrer and Wagner 1985), decision making under risk (Rubinstein 1988).

Very recently, the structure of an arbitrary semiorder has been fully described by Giarlotta and Watson (2016). This description has the flavour of a Scott-Suppes representation, insofar as it uses "shifted" forms of lexicographic products. In fact, any semiorder can be order-embedded into a modified form of lexicographic product of three total preorders: here the modification is given by a shift operator, which typically creates intransitive indifferences. Since the middle factor of this modified product is the usual ordering $(\mathbb{Z}, \geq)$ of the integers, and the shift operator is applied to it, these structures are called $\mathbb{Z}$-*products*. In particular, a $\mathbb{Z}$-*line* is a $\mathbb{Z}$-product in which the first and the third factors are linear orderings. The formal notions are as follows:

Definition 3.5 The $\mathbb{Z}$-*product* of two totally preordered sets $(A, \succsim_A)$ and $(B, \succsim_B)$ is the triple $\left(P, \oplus 1, \succsim_{\text{lex}}^{\oplus 1}\right)$, where:

- P is the Cartesian product $A \times \mathbb{Z} \times B$;
- $\oplus 1$ is the unary operation on P defined by $(a, n, b) \oplus 1 := (a, n+1, b)$ for each $(a, n, b) \in P$;
- $\succsim_{\text{lex}}^{\oplus 1}$ is the canonical completion[17] of the asymmetric relation $\succ_{\text{lex}}^{\oplus 1}$ on P defined by

$$(a, n, b) \succ_{\text{lex}}^{\oplus 1} (a', n', b') \quad \overset{\text{def}}{\Longleftrightarrow} \quad (a, n, b) \succ_{\text{lex}} (a', n', b') \oplus 1$$

  for each $(a, n, b), (a', n', b') \in P$, with $\succ_{\text{lex}}$ being the standard lexicographic order on P.

$A \odot_{\mathbb{Z}} B$ denotes the $\mathbb{Z}$-product of the total preorders $(A, \succsim_A)$ and $(B, \succsim_B)$. The $\mathbb{Z}$-product of two linear orderings is a $\mathbb{Z}$-*line*.

It turns out that $\mathbb{Z}$-products (and $\mathbb{Z}$-lines) are *universal* semiorders, in the sense that any semiorder order-embeds into a $\mathbb{Z}$-product. The process to construct such an embedding is rather technical, but it can be summarized in the following three main steps:

(1) first consider a "macro-ordering", given by the transitive closure[18] of the semiorder;
(2) then partition each equivalence class of the macro-ordering into "vertical slices" indexed by the integers, allowing only certain relationships between pairs of slices;
(3) finally establish a "micro-ordering" to further refine the distinction among elements of the semiorder, and obtain an order-embedding into a $\mathbb{Z}$-product.

The binary relations used at each stage are total preorders. This fact is clear for the macro-ordering at stage (1). At stage (2), the partition of each indifference class of the transitive closure uses a *locally monotonic integer slicer* (*LMIS*), which is an integer-valued map having some ordering properties. The micro-ordering employed at stage (3) is a modified form of trace, called *sliced trace*, which allows "backward paths" with respect to an LMIS. The reader is referred to Giarlotta and Watson (2016) for several examples of LMIS and the associated sliced traces. Then, we have (Giarlotta and Watson 2016):

Theorem 3.6 *The following statements are equivalent for a reflexive and complete* $(X, \succsim)$:

(i) $(X, \succsim)$ *is a semiordered space;*
(ii) $(X, \succsim)$ *order-embeds into a* $\mathbb{Z}$-*product;*
(iii) $(X, \succsim)$ *order-embeds into a* $\mathbb{Z}$-*line;*
(iv) $(X, \succsim)$ *order-embeds into* $(X, \succsim_{\text{tc}}) \odot_{\mathbb{Z}} (X, \succsim_{\zeta})$.

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[17] The *canonical completion* of an asymmetric relation transforms incomparability into indifference.
[18] The *transitive closure* of a binary relation $\succsim$ is the smallest transitive relation $\succsim_{\text{tc}}$ containing $\succsim$.

(In (iv), $\precsim_{\mathrm{tc}}$ is the transitive closure of the semiorder $\precsim$, and $\precsim_{\zeta}$ is the sliced trace associated to some $\mathsf{LMIS}\ \zeta \colon \mathbb{R} \to \mathbb{Z}$.) The following three consequences of Theorem 3.6 are noteworthy:

Corollary 3.7 *$\mathbb{Z}$-lines are universal semiorders.*

Corollary 3.8 *The $\mathbb{Z}$-line $\mathbb{Q} \odot_{\mathbb{Z}} \mathbb{Q}$ is a universal countable semiorder.*

Corollary 3.9 (Rabinovitch 1978) *The dimension*[19] *of a strict semiorder is at most* 3.

In addition to the above consequences, the descriptive characterization of all semiorders established in Theorem 3.6 may provide a unifying view of several results that are currently scattered throughout the literature. For instance, many notions of *separability*—Cantor, Debreu, Jaffray, strong, weak, topological, interval order, semiorder, etc.—that have been extensively studied in the past (see, e.g., Beja and Gilboa (1992), Candeal et al. (2012) and references therein) can be characterized by suitable properties of embedding into $\mathbb{Z}$-lines. Similarly, the geometric representations of semiorders given by Beja and Gilboa (see Theorems 3.7, 3.8, 4.4, and 4.5 in Beja and Gilboa (1992)) as well as the characterization of Scott-Suppes representability given by Candeal and Induráin (2010) can be described in terms of properties of embeddability into special $\mathbb{Z}$-lines.[20]

3.3 (m, n)-Ferrers Preferences

As recalled in Sect. 2.2, an interval order can be equivalently defined as (1) a reflexive relation satisfying the Ferrers property, or (2) an asymmetric relation satisfying the strict Ferrers property[21]: to distinguish the two cases, we shall speak of a *strict interval order* in case (2). The two settings are equivalent because the canonical completion of a strict interval order is an interval order, and, conversely, the asymmetric part of an interval order is a strict interval order.

Similarly, a semiorder can be equivalently defined as (1) a reflexive relation satisfying both the Ferrers and the semitransitive properties, or (2) an asymmetric relation satisfying both the strict Ferrers and the strict semitransitive properties: for clarity, we speak of a *strict semiorder* in case (2). Again, the difference between (1) and (2) is immaterial, since the canonical completion of a strict semiorder is a semiorder, and the asymmetric part of a semiorder is a strict semiorder.

Interval orders and semiorders have been employed in the literature on preference modeling as a sound alternative to total preorders, due to their ability to realistically

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[19]The *dimension* of a strict semiorder $\succ$ is the least number of strict linear orders whose intersection gives $\succ$.

[20]This is a work in progress (Giarlotta and Watson 2018c).

[21]The *strict Ferrers property* and the *strict semitransitive property* are respectively defined exactly as the Ferrers property and the semitransitive property in Definition 2.2, with $\succ$ in place of $\succsim$.

describe situations in which the agent displays intransitive preferences. In fact, interval orders (hence semiorders) always have a transitive strict part, but the associated indifference fails, in general, to be transitive. The main difference between modelizations based on interval orders and those based on semiorders is that in the former case the threshold of discrimination need not be constant.

Quite recently, in the process of defining broader types of preferences for which the associated indifference may be intransitive—and, specifically, to generalize some variations of semiorders proposed by Fishburn (1997)—, Öztürk introduced the notion of (m, n)-*Ferrers properties*. These properties require that the first and the last elements of two sequences of preferences having length m and n must be suitably related to each other. In particular, the classical Ferrers condition is the $(2, 2)$-Ferrers property, whereas semitransitivity is the $(3, 1)$-Ferrers property.

However, Öztürk's definition is limited to an asymmetric (and transitive) relation, and so it does not allow one to systematically deal with "degrees of transitivity" of preferences. This motivated a further extension of her approach by Giarlotta and Watson (2014a), who distinguish two types of (m, n)-Ferrers properties: *weak and strict*, respectively related to sequences of preferences that are either reflexive or asymmetric.

Definition 3.10 Let $\succsim$ be a weak preference on X, and $\succ$ its asymmetric part. For fixed integers $m \geq n \geq 1$, we say that $\succsim$ satisfies the *weak* (m, n)-*Ferrers property* (or it is *weakly* (m, n)-*Ferrers*) if the implication

$$(x_1 \succsim \cdots \succsim x_m) \ \wedge \ (y_1 \succsim \cdots \succsim y_n) \quad \Longrightarrow \quad x_1 \succsim y_n \ \vee \ y_1 \succsim x_m \tag{1}$$

holds for all $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$. The notion of *strict* (m, n)-*Ferrers property* is defined similarly, substituting $\succsim$ by $\succ$ in (1).

Notice that the (strict or weak) $(2, 2)$-Ferrers property is the classical Ferrers condition, whereas the (strict or weak) $(3, 1)$-Ferrers property is semitransitivity. Said differently, weak and strict (m, n)-Ferrers properties coincide for $m + n = 4$, i.e., for interval orders and semiorders. However, they behave quite oppositely as m and n grow:

Lemma 3.11 *Let $\succsim$ be a total weak preference on X. For all integers m, n, p, q such that $m \geq n \geq 1$, $p \geq q \geq 1$, $m \geq p$, $n \geq q$, and $m + n \geq 3$, we have:*

- *if $\succsim$ is weakly (m, n)-Ferrers, then $\succsim$ is weakly (p, q)-Ferrers;*
- *if $\succsim$ is strictly (p, q)-Ferrers and $\succ$ is transitive, then $\succsim$ is strictly (m, n)-Ferrers.*

In other words, weak (m, n)-Ferrers properties display an increasing strength as m and n grow, whereas strict (m, n)-Ferrers properties becomes weaker and weaker (under the hypothesis of quasi-transitivity) as m and n grow.

Weak (m, n)-Ferrers properties are simpler to study, since they display a finite taxonomy. In fact, the family of weak (m, n)-Ferrers properties forms a finite lattice under implication, having as maximum the $(3, 3)$-Ferrers property, which corresponds to transitivity. Figure 6 in the Appendix (taken from Cantone et al. (2016))

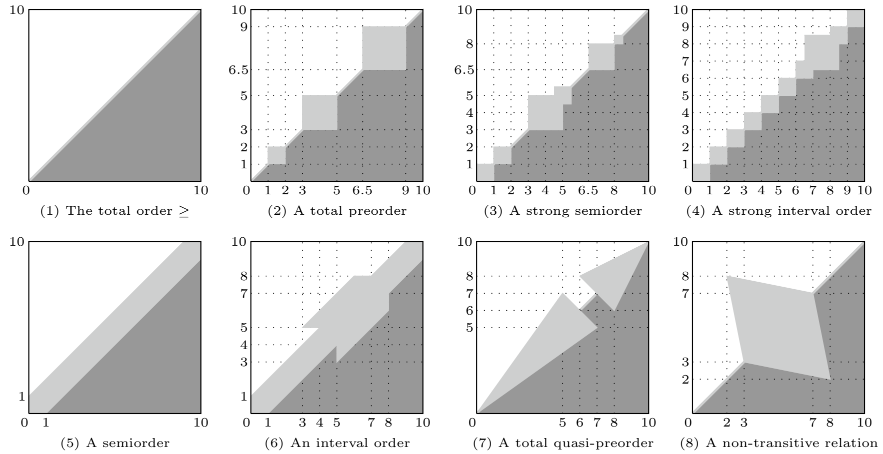

Fig. 1 A geometric representation of some extensions of the linear ordering ([0, 10], ≥)

describes all implications among combinations of weak (m, n)-Ferrers properties (in the gray boxes): see Theorem 3.1 in Giarlotta and Watson (2014a). All reverse implications do not hold: see Examples 3.3–3.10 in Giarlotta and Watson (2014a). Roughly speaking, weak (m, n)-Ferrers properties are linked to the transitivity of the associated relation of indifference. In this respect, Fig. 6 describes a sort of *discrete evolution* of the transitive property: from possibly no shade of transitivity (at (1, 1)-Ferrers), to quasi-transitivity (at (2, 1)-Ferrers), to the classical Ferrers condition (at (2, 2)-Ferrers) and semitransitivity (at (3, 1)-Ferrers), until its full satisfaction (at (3, 3)-Ferrers), after having described several forms of transitivity on the path to full transitivity.

To give an idea of the possible "shape" of some weak (m, n)-Ferrers preferences, Fig. 1 (taken from Giarlotta (2014)) describes the geometric form of a few of them, whenever these preferences happen to be extensions of a linear continuum.[22] For all eight pictures in Fig. 1, the dark gray area represents the strict preference, whereas the light gray area is the indifference: for instance, in picture (2) we have $7 \succ 5$ and $7 \sim 8$, in picture (8) we have $3 \succ 2.5$ and $7 \sim 2.5$, etc. Further, by *strong semiorder* we mean weakly (3, 2)- and (4, 1)-Ferrers, whereas by *strong interval order* we mean weakly (3, 2)-Ferrers.

Contrary to weak (m, n)-Ferrers properties, strict (m, n)-Ferrers properties are much more complicated to classify. Roughly speaking, these properties are linked to the transitivity of the associated relation of strict preference, hence they refine the graph given in Fig. 6 (in the Appendix) in its lowest part (especially for the so-called "extended preferences").

[22] A *linear continuum* is a linear ordering with the properties that (i) every nonempty subset with an upper bound has a least upper bound, and (ii) for every pair of distinct elements, we can always find another element strictly in between them.

It turns out that even the case of strict $(m, 1)$-Ferrers properties is difficult to analyze, since it gives rise to an infinite taxonomy of preferences. Furthermore, even if strict $(m, 1)$-Ferrers properties somehow become less and less strong as m increases, they do *not* display a monotonic behavior. Specifically, the strongest strict $(m, 1)$-Ferrers property is (2, 1), which implies all the other strict $(m, 1)$-Ferrers properties for $m \geq 3$: in fact, a strictly (2, 1)-Ferrers preference is a total preorder. The second strongest property is (3, 1), since it implies all strict $(m, 1)$-Ferrers properties for $m \geq 4$: in fact, a strictly (3, 1)-Ferrers preference is always quasi-transitive. However, starting from the strict (4, 1)-Ferrers property, this apparent regularity of behavior vanishes, since (4, 1) implies neither (5, 1) nor quasi-transitivity.

This erratic behavior of strict $(m, 1)$-Ferrers properties induced Giarlotta and Watson (2018a) to perform a combinatorial analysis of them, which yielded the following nontrivial characterization:

Theorem 3.12 *The following statements are equivalent for all distinct integers* $m, n \geq 2$*:*

(i) *the strict* $(n, 1)$*-Ferrers property implies the strict* $(m, 1)$*-Ferrers property;*
(ii) $n < m$ *and exactly one of the following conditions holds:*

- *(ii.1)* $m < 2n - 3$ *and* $(2n - 3 - m)$ *divides* $(n - 3)$*;*
- *(ii.2)* $m = 2n - 3$ *and* n *is odd;*
- *(ii.3)* $m > 2n - 3$.

An interesting consequence of Theorem 3.12 is that the implications among strict $(m, 1)$-Ferrers "eventually stabilize", in the sense that a strict $(m, 1)$-Ferrers property implies all strict $(p, 1)$-Ferrers properties for p large enough. To formally state this result we need a notion:

Definition 3.13 Given an integer $m \geq 2$, the *Ferrers stabilizer* of m, denoted by $\mathsf{st}(m)$, is the least integer $p \geq m$ with the property that the strict $(m, 1)$-Ferrers property implies the strict $(q, 1)$-Ferrers property for all $q \geq p$.

Roughly speaking, the Ferrers stabilizer of an integer is an index of its "limit strength" for what concerns the satisfaction of the transitive property: the higher this number, the less strong the property. For instance, $\mathsf{st}(2) = 2$, $\mathsf{st}(3) = 3$, $\mathsf{st}(4) = 6$, $\mathsf{st}(11) = 17$, $\mathsf{st}(23) = 41$, $\mathsf{st}(63) = 117$, etc. The formula to compute the Ferrers stabilizer of an integer is surprisingly simple (Giarlotta and Watson 2018a):

Corollary 3.14 $\mathsf{st}(m) = 2m - 3 - \mathsf{run}(m)$ *for each* $m \geq 2$.

The notation $\mathsf{run}(m)$ in Corollary 3.14 stands for the *running index* of m, defined by

$$\mathsf{run}(m) := \begin{cases} -1 & \text{for even } m \\ 0 & \text{for } m = 3 \\ \max\left\{p < m - 3 : \{1, \ldots, p\} \subseteq \mathsf{Div}(m-3)\right\} & \text{otherwise,} \end{cases}$$

with $\mathsf{Div}(m-3)$ being the set of divisors of $m-3$, including 1. Thus, in particular, the running index of an odd number $m \geq 5$ is the largest integer less than or equal to $\frac{m-3}{2}$, which leads a running sequence of divisors of $m-3$. For instance, $\mathsf{run}(5) = \mathsf{run}(7) = \mathsf{run}(11) = \mathsf{run}(13) = \mathsf{run}(17) = 2, \quad \mathsf{run}(9) = \mathsf{run}(21) = \mathsf{run}(33) = 3$, $\mathsf{run}(15) = \mathsf{run}(27) = 4$, $\mathsf{run}(63) = 6$, etc.

The preceding discussion might suggest that weak and strict (m, n)-Ferrers properties are a mere numerical/combinatorial curiosity, being totally unsuited for potential applications to real life problems. However, such an impression would be incorrect. In fact, (m, n)-Ferrers properties turn out to be linked to *money-pump phenomena*, which have been carefully analyzed in several fields of research, such as economics, psychology, and philosophy (Davidson et al. 1955; Gustafsson 2010; Hansson 1993; McClennen 1990; Piper 2014; Rabinowicz 2008; Restle 1961; Schick 1986; Schumm 1987; Tversky 1969). Originally observed by Davidson et al. (1955), these phenomena are described by Tversky (1969) in relation to the failure of the (strict) transitive property:

> Transitivity, however, is one of the basic and the most compelling principles of rational behaviour. For if one violates transitivity, it is a well known conclusion that he is acting, in effect, as a "money-pump". Suppose an individual prefers y to x, z to y, and x to z. It is reasonable to assume that he is willing to pay a sum of money to replace x by y. Similarly, he should be willing to pay some amount of money to replace y by z, and still a third amount to replace z by x. Thus, he ends up with the alternative he started with but with less money.

It is apparent that the presence of a strict cycle of preferences puts the economic agent at the risk of losing all her money, since she may get involved in another cycle of money-pump, and continue in this fashion until her financial resources are exhausted.

Admittedly, the above money-pump effect requires strict cycles of preferences, which are forbidden starting from the satisfaction of the weak (2, 1)-Ferrers property (which is equivalent to quasi-transitivity). However, many contributions to the economic literature show that a money-pump effect may also arise in the presence of *mixed cycles* of strict preferences and indifferences: see, e.g., Restle (1961), who argues that a strict cycle can be easily induced by a mixed cycle using a "small bonus" approach.[23] Moreover, several other ways to induce a money-pump from mixed cycles of strict preferences/indifferences have been proposed in the literature, e.g., by Schumm (1987) in a multiple-criteria set up, as well as by Gustafsson (2010) using the notion of dominance in cases of preferences under uncertainty.[24]

In Sect. 4.3 of their paper on choices that are rationalizable by (m, n)-Ferrers preferences, Cantone et al. (2016) introduce a simple model of transactions of goods, which is well suited to describe the semantics of weak (m, n)-Ferrers properties. Specifically, they show that, in this model, whenever the binary relation modeling

[23]For some recent examples of this approach, see Hansson (1993) and Rabinowicz (2008).

[24]On the other hand, Schick (1986) and McClennen (1990) argue against the possibility of a money-pump phenomenon, observing that, after transactions between indifferent alternatives, an economic agent may well refuse a transaction between strictly preferred alternatives. However, as Piper (2014) notes, the above solutions are based on the (unlikely) circumstance that the economic agent remembers the past and accordingly plans the future.

the agent's preference structure satisfies a fixed (m, n)-Ferrers property, there exists a strategy that prevents the agent from getting involved in mixed indifference/strict preference cycles of a certain type. In fact, the authors exhibit a numeric relationship between the level of transitivity of an economic agent's preference structure (i.e., the satisfaction of a certain weak (m, n)-Ferrers property) on one hand, and the caution that she has to exercise whenever indulging in certain types of transactions (i.e., the avoidance of money-pump phenomena) on the other hand.

A similar type of argument applies to strict $(m, 1)$-Ferrers properties. To that end, Giarlotta and Watson (2018a) introduce a simple notion of "cash-value" preference as follows:

Definition 3.15 Given goods X and Y, if there is a (perfectly divisible and fungible) good G such that X is weakly preferred to G, and G is weakly preferred to Y, then we say X is *cash-value preferred* to Y.

Typically, G will be money. Essentially cash-value preference is the strengthened weak preference an agent arrives at when required to assign cash-value to goods: X is cash-value preferred to Y if X is weakly preferred to an amount of cash which is weakly preferred to Y. Then, we have:

Proposition 3.16 *A preference $\succsim$ satisfies the strict $(m, 1)$-Ferrers property if and only if we never have a sequence of the type $x_1 \succ x_2 \succ \cdots \succ x_m$ where x_m is cash-value preferred to x_1.*

3.4 (m, n)-Rationalizable Choices

Here we answer question (Q5) in Sect. 2.4, refining the classification of rationalizable choices provided by Theorem 2.18. This topic is based on a recent paper by Cantone et al. (2016).

The basic idea of this approach to revealed preference theory is to systematically separate two issues: (1) the rationalizability of a choice, and (2) the internal structure of its revealed preference. This goal is achieved by designing a class of *axioms of replacement consistency*, all having the same flavor: in fact, these properties examine how the addition of an item to a menu causes a substitution in the subset of selected elements. We have already examined a property of this kind in Sect. 2.4: the standard axiom (δ) of replacement consistency, which characterizes rationalizable choices with a quasi-transitive revealed preference (see Theorem 2.18(iii)). The natural extension of this approach to additional properties of the same kind aims at characterizing rationalizable choices whose revealed preference satisfies different levels of transitivity.

Specifically, first we examine those cases in which the revealed preference is an interval order, a semiorder, or a total preorder: this yields an axiomatization that is alternative to those given by Jamison and Lau (1973, 1975), Fishburn (1975), Schwartz (1976), and Bandyopadhyay and Sengupta (1991, 1993). Successively, in order to complete a taxonomic classification of rationalizable choices, we also characterize choices with a weakly (m, n)-Ferrers revealed preference by means of additional axioms of replacement consistency. In this way, we provide a uniform treatment of the topic by introducing properties of choice consistency that belong to a single category.

To start, we state three new axioms of replacement consistency:

♢ **Property** (ρ_{F}) (*Ferrers Replacement Consistency*):
If $x \in c(A)$, $y \in A$, $z \in c(B)$, and $z \notin c(B \cup \{y\})$, then $x \in c(B \cup \{x\})$.

♢ **Property** (ρ_{st}) (*Semitransitive Replacement Consistency*):
If $y \in c(A)$, $z \in A$, $z \in c(B)$, and $y \notin c(A \cup \{x\})$, then $x \in c(B \cup \{x\})$.

♢ **Property** (ρ_{t}) (*Transitive Replacement Consistency*):
If $y \in c(A)$ and $y \notin c(A \cup \{x\})$, then $c(A \cup \{x\}) = \{x\}$.

(As usual, a universal quantification over menus and items is implicit.)

The rationale of the above properties is similar to that of the standard axiom (ρ) of replacement consistency, in the sense that, under suitable conditions, a new item "replaces" an old item in the selection taste of the economic agent. The statement of (ρ_{t}) only involves two items and a single menu, hence its semantics is quite simple to understand. In fact, the antecedent of (ρ_{t}) is exactly the same as that of (ρ), but its consequent is drastically stronger: if y is selected from A but is rejected from it as soon as x is adjoined to A, then x "fully replaces" y in the selection taste of the agent, being the unique item selected from the larger menu $A \cup \{x\}$. On the other hand, the rationale of axioms (ρ_{F}) and (ρ_{st}), despite being of the same nature, is more subtle, since their statements simultaneously involve three items and two menus. To give a better insight into their semantics, in what follows we reformulate all axioms of replacement consistency (ρ), (ρ_{F}), (ρ_{st}), and (ρ_{t}) using a model-theoretic notation.

First, we associate to any choice correspondence $c\colon \Omega \to \Omega$ two new preference relations $\succsim_c^+$ and $\trianglerighteq_c$, both inspired by the replacement paradigm:

$$
\begin{aligned}
x \succsim_c^+ y \quad &\overset{\text{def}}{\Longleftrightarrow} (\exists A \in \Omega) \;\; y \in A \;\wedge\; c(A \cup \{x\}) = \{x\} \\
x \trianglerighteq_c y \quad &\overset{\text{def}}{\Longleftrightarrow} (\exists A \in \Omega) \;\; y \in c(A) \;\wedge\; y \notin c(A \cup \{x\}).
\end{aligned}
$$

Second, we employ the following model-theoretic notation:

$$
\begin{aligned}
&A \models x \succsim_c y \;\text{ stands for } y \in A \;\wedge\; x \in c(A),\\
&A \models x \succsim_c^+ y \;\text{ stands for } y \in A \;\wedge\; c(A \cup \{x\}) = \{x\},\\
&A \models x \trianglerighteq_c y \;\text{ stands for } y \in c(A) \;\wedge\; y \notin c(A \cup \{x\}),
\end{aligned}
$$

where $A \in \Omega$ and $x, y \in X$. According to the standard model theory semantics of the employed notation, $A \models x \succsim_c y$ means that menu A "witnesses" a revealed preference of x over y; the meaning of $A \models x \succsim_c^+ y$ and $A \models x \trianglerighteq_c y$ is similar. Finally, we reformulate the four axioms of replacement consistency using the above notation:

$$
\begin{array}{llcl}
(\rho) & A \models x \trianglerighteq_c y & \Longrightarrow & A \cup \{x\} \models x \succsim_c y \\
(\rho_{\mathsf{F}}) & \big(A \models x \succsim_c y\big) \wedge \big(B \models y \trianglerighteq_c z\big) \wedge z \in c(B) & \Longrightarrow & B \cup \{x\} \models x \succsim_c z \\
(\rho_{\mathsf{st}}) & \big(A \models x \trianglerighteq_c y\big) \wedge \big(A \models y \succsim_c z\big) \wedge z \in c(B) & \Longrightarrow & B \cup \{x\} \models x \succsim_c z \\
(\rho_{\mathsf{t}}) & A \models x \trianglerighteq_c y & \Longrightarrow & A \cup \{x\} \models x \succsim_c^{+} y.
\end{array}
$$

Note that this alternative formulation of the four axioms of replacement consistency reveals a complementarity of (ρ_{F}) and (ρ_{st}), since they both state a type of "transitive coherence" of the two binary relations $\succsim_c$ and $\trianglerighteq_c$.[25]

One of the main results in Cantone et al. (2016) connects these properties of replacement consistency to levels of transitivity of the rationalizing preference, thus partially answering question (Q5):

Theorem 3.17 *Let $c \colon \Omega \to \Omega$ be a rationalizable choice correspondence, and $\succsim_c$ its revealed preference. The following equivalences hold:*

(i) $\succsim_c$ *is quasi-transitive* $\iff$ *c satisfies axiom* (ρ)*;*
(ii) $\succsim_c$ *is Ferrers* $\iff$ *c satisfies axiom* (ρ_{F})*;*
(iii) $\succsim_c$ *is semitransitive* $\iff$ *c satisfies axiom* (ρ_{st})*;*
(iv) $\succsim_c$ *is transitive* $\iff$ *c satisfies axiom* (ρ_{t}).

Theorem 3.17 readily yields

Corollary 3.18 *The following equivalences hold for an arbitrary choice correspondence c:*

(i) *c is rationalizable by a preorder* $\iff$ *properties* (α)*,* (γ)*, and* (ρ) *hold;*
(ii) *c is rationalizable by an interval order* $\iff$ *properties* (α)*,* (γ)*, and* (ρ_{F}) *hold;*
(iii) *c is rationalizable by a semiorder* $\iff$ *properties* (α)*,* (γ)*,* (ρ_{F})*, and* (ρ_{st}) *hold;*
(iv) *c is rationalizable by a total preorder* $\iff$ *properties* (α)*,* (γ)*, and* (ρ_{t}) *hold.*

The analysis conducted in Cantone et al. (2016) goes further in the direction of classifying rationalizable preferences in terms of the transitive structure of their revealed preferences. In fact, the authors design, for each relevant pair (m, n) of positive integers, a property $(\rho_{m,n})$ of (m, n)*-replacement consistency*, finally proving the following result:

Theorem 3.19 *A choice correspondence is rationalizable by an* (m, n)*-Ferrers preference if and only if properties* (α)*,* (γ)*, and* $(\rho_{m,n})$ *hold for it.*

We refer the reader to the paper (Cantone et al. 2016) for further details about the described approach, as well as for future directions of research on the topic.

[25]For the formal notion of the transitive coherence of two binary relations, see Sect. 4.1 on bi-preferences.

3.5 Resolutions of Choices

In this section, which is entirely based on a very recent research by Cantone et al. (2018a), we introduce a novel notion for choices, called "resolution". This notion is designed to better understand the inner structure of an observed choice behavior, because it provides a constructive way to possibly decompose the overall selection process in terms of smaller choices.

The general concept of resolution originated, however, from a different field of research. In fact, resolutions were first introduced by Fedorcuk (1968) for the class of topological spaces. The successive development of this notion by Watson (1992) has proven to be very useful in providing a common point of view of many seemingly different topological spaces (as well as for linearly ordered spaces (Caserta et al. 2006)): see the large amount of references in Watson (1992). The idea underlying the notion of topological resolution is natural: given a base topological space, a family of fibre topological spaces indexed by the base space, and a family of continuous maps also indexed by the base space, the output is a larger topological space, the resolution, in which every point is substituted by the associated fibre space. For instance, the *double arrow space*—i.e., the lexicographic product $\mathbb{R} \times_{\text{lex}} 2$ endowed with the order topology, examined by Wakker (1988) in his study on lexicographic preferences—can be seen as a resolution of $\mathbb{R}$ at all points x into the discrete space $2 = \{0, 1\}$ by the functions $f_x : \mathbb{R} \setminus \{x\} \to 2$, defined by $f_x(x') := 0$ if $x' < x$, and $f_x(x') := 1$ if $x' > x$.

Cantone et al. (2018a) adapt the notion of topological resolution to choice theory: in this new setting, a resolution describes how to build up a complete choice from independent choices on smaller ground sets. In a nutshell, the process of resolving a choice space into a larger choice space metaphorically consists of taking a magnifying glass, observing one special item of the primitive space as being a menu with its own choice structure, and obtaining a larger choice according to this new piece of information.

Before getting into technicalities, it may be useful to suggest possible interpretations of this notion in some familiar settings. For instance, a resolution of a choice can be seen as follows:

(1) in a corporation, as the delegation of tasks from the top management to a department;
(2) in a restaurant menu, as the opening of a submenu at a specific (type of) item;
(3) in a portfolio, as the choice of investments including stocks recommended by a broker;
(4) in a budgeted hiring, as the hiring of employees including a particular class of workers.

On a more formal basis, assume that we are given a complete[26] choice space (X, c_X), called "base" choice space. We identify a distinguished alternative x in X:

[26] According to the notation employed in Definition 2.14, we should denote these choice space by $(2^X, c_X)$. However, for the sake of simplicity, here we prefer to use the more direct notation (X, c_X).

the peculiarity of x is that it can be viewed as a menu itself, which may open up at a "fibre" choice space (Y, c_Y). To make a selection in the resolved space (Z, c_Z), the process goes as follows. First, we choose from X, where one of the selections is the (currently closed) menu x. If we do not pick x, then we do not bother opening the fibre menu Y. On the other hand, if we choose x, then we open Y, and use the inner structure of the fibre space (Y, c_Y) to make choices there as well. The following definition makes the above process formal:

Definition 3.20 Let (X, c_X) and (Y, c_Y) be two complete choice spaces defined on disjoint ground sets X and Y. Select $x \in X$, and let $Z := (X \setminus \{x\}) \cup Y$. Define a map $\pi \colon Z \to X$ by

$$\pi(z) := \begin{cases} z & \text{if } z \in X \setminus \{x\} \\ x & \text{if } z \in Y. \end{cases}$$

The *resolution of* (X, c_X) *at* x *into* (Y, c_Y), denoted by

$$(Z, c_Z) = (X, c_X) \otimes_x (Y, c_Y),$$

is the complete choice space on Z whose choice correspondence $c_Z \colon 2^Z \to 2^Z$ is given by

$$c_Z(A) := \begin{cases} \big(c_X(\pi(A)) \setminus \{x\}\big) \cup c_Y(A \cap Y) & \text{if } x \in c_X(\pi(A)) \\ c_X(\pi(A)) & \text{otherwise.} \end{cases} \tag{2}$$

The two *factors* (X, c_X) and (Y, c_Y) are, respectively, the *base choice space* and the *fibre choice space (at* x*)*, whereas the distinguished item $x \in X$ is the *base point* of the resolution. The surjective map π is the *projection* of the resolution. If both X and Y contain at least two items, then the resolution $(X, c_X) \otimes_x (Y, c_Y)$ is *nontrivial*; otherwise, it is *trivial*.

Figure 2 intuitively describes the semantics of a resolution $(Z, c_Z) = (X, c_X) \otimes_x (Y, c_Y)$. According to Definition 3.20, if $A \subseteq Z$ is any menu in (Z, c_Z), then the choice set $c_Z(A)$ is obtained as follows. First, look at the trace of A in X, computing $\pi(A)$: thus, we have $\pi(A) = A$ if A does not intersect Y, and $\pi(A) = (A \cap X) \cup \{x\}$ otherwise. Second, we distinguish two cases: (i) x is selected from $\pi(A)$; (ii) x is not

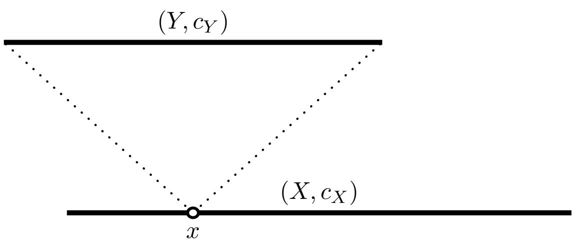

Fig. 2 A resolution $(Z, c_Z) = (X, c_X) \otimes_x (Y, c_Y)$

selected from $\pi(A)$. In case (i), $c_Z(A)$ is obtained as the union of what is selected from the trace of A in X (minus x) and what is selected from the trace of A in Y: this is the first line of (2). In case (ii), we do not open Y at all, and $c_Z(A)$ is exclusively computed on the basis of the trace of A in X: this is the second line of (2).

Definition 3.21 A choice space (Z, c) is *resolvable* if it is isomorphic[27] to a non-trivial resolution; otherwise, it is *irresolvable*. If (Z, c) is a resolvable space, then we also say that c is *resolvable*.

The next example clarifies the above notions.

Example 3.22 Let $X = \{x, x'\}$ and $Y = \{y, z\}$. Below we resolve some choices c_X on X at the base point $x \in X$ into some choices c_Y on Y. We list the corresponding results as resolutions (Z, c_Z), where $Z := (X \setminus \{x\}) \cup Y = \{x', y, z\}$ (the notation is suggestive of the meaning):

$$\begin{array}{rlcl}
\text{(i)} & x\,\underline{x'} \otimes_x \underline{y}\,z & = & \underline{x'}\,y\,z,\ \underline{x'}\,y,\ \underline{x'}\,z,\ \underline{y}\,z,\\
\text{(ii)} & x\,\underline{x'} \otimes_x \underline{y}\,\underline{z} & = & \underline{x'}\,y\,z,\ \underline{x'}\,y,\ \underline{x'}\,z,\ \underline{y}\,\underline{z},\\
\text{(iii)} & \underline{x}\,x' \otimes_x \underline{y}\,\underline{z} & = & x'\,\underline{y}\,\underline{z},\ x'\,\underline{y},\ x'\,\underline{z},\ \underline{y}\,\underline{z},\\
\text{(iv)} & \underline{x}\,\underline{x'} \otimes_x \underline{y}\,\underline{z} & = & \underline{x'}\,\underline{y}\,\underline{z},\ \underline{x'}\,\underline{y},\ \underline{x'}\,\underline{z},\ \underline{y}\,\underline{z},\\
\text{(v)} & \underline{x}\,\underline{x'} \otimes_x \underline{y}\,z & = & \underline{x'}\,\underline{y}\,z,\ \underline{x'}\,\underline{y},\ \underline{x'}\,\underline{z},\ \underline{y}\,z.
\end{array}$$

For instance, (i) means that the choice c_X is defined by $c_X(\{x, x'\}) = \{x'\}$, the choice c_Y is defined by $c_Y(\{y, z\}) = \{y\}$, and the choice c_Z is defined by $c_Z(\{x', y, z\}) = \{x'\}$, $c_Z(\{x', y\}) = \{x'\}$, $c_Z(\{x', z\}) = \{x'\}$, and $c_Z(\{y, z\}) = \{y\}$.

Notice that the five resolutions (i)–(v) produce rationalizable choices. In particular, (i)–(iv) are rationalizable by a total preorder, in fact they are the unique (up to isomorphisms) choices on a 3-element set satisfying WARP. The resolution (v) is slightly different from the others, because WARP does not hold for it: in fact, it is the unique (up to isomorphisms) rationalizable choice on a 3-element ground set having a quasi-transitive but intransitive revealed preference. Finally, observe that there is only one (up to isomorphisms) choice on a 3-element set $Z = \{x', y, z\}$ that is rationalizable but irresolvable, namely,

$$\text{(vi)}\quad \underline{x'}\,y\,z,\ \underline{x'}\,y,\ \underline{x'}\,\underline{z},\ \underline{y}\,z.$$

Among the six rationalizable choices on a 3-element set, the irresolvable choice (vi) is the unique having a revealed preference that fails to be quasi-transitive (cf. Theorem 2.18(iii)): indeed, properties (α) and (γ) hold, whereas (ρ) does not (since z is chosen in $A = \{x', z\}$ and not in $A \cup \{y\} = Z$, but y fails to be selected from Z).

The main problem that arises in this context is how to characterize the process that reverses a resolution, expressing a resolved choice in terms of its factors. Formally, the question is:

[27] Two choice spaces (X, c_X) and (W, c_W) are *isomorphic* if there exists a bijection $\sigma\colon X \to W$ that preserves the choice structure, i.e., the equality $\sigma(c_X(A)) = c_W(\sigma(A))$ holds for each menu $A \in 2^X$.

(Q8) *Can we determine whether a choice is resolvable (i.e., it is a resolution of simpler choices)?*

A constructive answer to question (Q8) is provided in Cantone et al. (2018a), where resolvable choices are characterized by three properties, each of which has an immediate economic interpretation. These properties yield the notion of "contractible" menu, defined as follows[28]:

Definition 3.23 Let (X, c) be a complete choice. A menu $E \in 2^X$ is *contractible* if the following three conditions hold for each $A \in 2^X$:

(R1) $A \cap E \neq \emptyset \implies c(A) \setminus E = c(A \cup E) \setminus E$,
(R2) $c(A) \cap E \neq \emptyset \implies c(A) \cap E = c(A \cap E)$, and
(R3) $A \cap E \neq \emptyset \implies \big(c(A \cup E) \cap E \neq \emptyset \iff c(A) \cap E \neq \emptyset\big)$.

In the four settings described at the beginning of this section, a contractible menu reveals (1) an autonomous department within a corporation, or (2) an implicit catch-of-the-day submenu within a menu, or (3) an implicit trusted stock broker on whom an investor relies, or (4) an implicit hiring budget for, e.g., engineers. The semantics of the three conditions (R1)–(R3) is natural. For instance, in a corporation with a CEO and a VP of marketing:

(R1) the non-marketing tasks selected by the CEO are independent of which marketing tasks are available, as long as there is at least one;
(R2) the tasks selected by the VP are independent of which non-marketing tasks are available;
(R3) whether or not a marketing task is selected is unaffected by which marketing tasks are available, as long as there is at least one.

The following characterization of the resolvability of a choice is a consequence of the constructive approach undertaken in Cantone et al. (2018a):

Corollary 3.24 *A choice is resolvable if and only if there is a contractible proper*[29] *menu.*

We conclude the first part of this section with an illustrative example, which provides an instance of how to constructively answer question (Q8).

Example 3.25 Assume that a diner D goes every weekend to a restaurant, which offers a wide variety of dishes. Over time, we observe the following selections of D over all menus on $Z = \{p, c, s, t\}$, where p is pizza, c is chips, s is salmon, and t is tuna:

---

[28]Contractibility is a form of "outer indiscernibility", in the sense that a contractible menu cannot be distinguished from outside, but it typically has an internally distinguishable structure. In fact, contractibility is a weaker version of *revealed indiscernibility*, introduced by Cantone et al. (2018b) in the process of dealing with *congruence relations* (i.e., structure-preserving equivalence relations) on a choice space.

[29]A menu is *proper* if it contains more than one item and is different from the ground set.

$$p\,\underline{c}\,\underline{s}\,t,\ \ p\,\underline{c}\,\underline{s},\ \ p\,\underline{c}\,\underline{t},\ \ \underline{p}\,s\,t,\ \ \underline{c}\,\underline{s}\,t,\ \ \underline{p}\,c,\ \ \underline{p}\,s,\ \ \underline{p}\,t,\ \ \underline{c}\,\underline{s},\ \ \underline{c}\,\underline{t},\ \ \underline{s}\,t.$$

Thus, $p\,\underline{c}\,\underline{s}\,t$ says that salmon and chips are selected by D when pizza and tuna are available in addition to them, $\underline{p}\,s\,t$ states that D chooses pizza over salmon and tuna, etc. At a first look, this choice cannot be readily justified. However, a less superficial analysis may reveal some underlying principles of selection. Thus, the natural question is: *Can we explain this choice better as a resolution of simpler choices?* The technique developed in Cantone et al. (2018a) allows us to constructively answer this question in a positive way.

Specifically, first we argue that the proper menu $\{s, t\}$ is contractible, since it satisfies properties (R1)–(R3); thus, by Corollary 3.24, the diner's choice is resolvable. Second, denoted by f the abstract item "fish", we compute the so-called *base choice* induced by the menu $\{s, t\}$ on the base set $X = \{p, c, f\}$, which is $p\,\underline{c}\,\underline{f},\ \underline{p}\,c,\ \underline{p}\,f,\ \underline{c}\,\underline{f}$. Finally, the main result in Cantone et al. (2018a) yields

$$p\,\underline{c}\,\underline{s}\,t,\ p\,\underline{c}\,\underline{s},\ p\,\underline{c}\,\underline{t},\ \underline{p}\,s\,t,\ \underline{c}\,\underline{s}\,t,\ \underline{p}\,c,\ \underline{p}\,s,\ \underline{p}\,t,\ \underline{c}\,\underline{s},\ \underline{c}\,\underline{t},\ \underline{s}\,t \quad = \quad p\,\underline{c}\,\underline{f},\ \underline{p}\,c,\ \underline{p}\,f,\ \underline{c}\,\underline{f} \otimes_f \underline{s}\,t.$$

In more descriptive terms, as soon as we establish the contractibility of the menu $\{s, t\} = \{\text{salmon, tuna}\}$, the selection made by D can be justified in terms of more elementary choices as follows: (1) the two fish items salmon and tuna become a fish submenu, in which D selects salmon over tuna; (2) the induced base choice on $\{p, c, f\} = \{\text{pizza, chips, fish}\}$ shows that the diner selects fish and chips, if both are available, and selects pizza alone over either fish and chips if only one of those is available.

The second part of this section is devoted to exploring the relationship between resolutions and revealed preference theory. In this context, the following natural question can be formulated:

(Q9) *If an axiom of consistency holds for both the base choice and the fibre choice, does it also hold for the resolved choice?*

The answer to question (Q9) is positive for the majority of the mentioned properties of choice consistency. In fact, we have (Cantone et al. 2018a):

Theorem 3.26 *A resolution satisfies an axiom in* $\{(\alpha), (\gamma), (\rho)\}$ *if and only if so do its factors. A resolution is path independent if and only so are its factors.*

Theorems 3.26 and 2.18 readily yield the following interesting consequence:

Corollary 3.27 *A resolution is rationalizable if and only if so are both factors. A resolution is rationalizable by a preorder if and only if so are both factors.*

Thus, for instance, in a corporate structure setting, if the CEO makes a rational selection and delegates all marketing choices to a rational VP, then the overall selection is still rational.

However, it turns out that WARP (and (β) as well) is not preserved by resolutions, unless in very special cases. This last fact raises further doubts—in addition to those already raised in the literature (see, e.g., Eliaz and Ok (2006))—on considering this property as an undisputed feature of rational choice behavior. To clarify the tight boundaries of the preservation of WARP, we need a notion.

Definition 3.28 An alternative x in a choice space (X, c) is a *repellent point* if either $c(A) = \{x\}$ or $x \notin c(A)$ holds for any menu $A \in 2^X$.

Thus, whether a repellent point is selected or not from a menu depends on the other available items; however, if it is selected, then it is unique. Then, we have (Cantone et al. 2018a):

Theorem 3.29 *A resolution satisfies WARP if and only if both factors satisfy WARP and either the base point is repellent or the fibre choice correspondence is the identity.*

Theorem 3.29 says that WARP (equivalently, rationalizability by a total preorder) is only preserved by resolutions in either extreme or trivial cases.

4 The Multi-approach

In this section we briefly describe some possible answers to the general question (Q0) formulated at the beginning of this survey. All these answers will rely on a multi-approach.

4.1 Bi-preferences

Here we specifically address question (Q1) posed in Sect. 2.2. To that end, we depart from the traditional mono-relation approach to preference theory, and sketch a theory of bi-preferences. This section is entirely based on two very recent paper by Giarlotta and Watson (2018b, 2019).

4.1.1 Definition, Examples, and Motivation

Informally, a bi-preference is a pair of nested binary relations on the same set of alternatives, which provides two different yet connected types of information about the preference structure of an agent (or a set of agents). In the general setting, we have: (1) a "rigid" preference $\succsim^R$, which codifies the very core of the agent's preference attitude, and is assumed to be fully rational; and (2) a "soft" preference $\succsim^S$, which summarizes the agent's tolerance, her willingness/capability to compromise, and is assumed to be partially rational. In other words, $\succsim^R$ and $\succsim^S$ describe, respectively, what "must" and "may" happen. As a consequence of their semantics, the rigid component $\succsim^R$ of a bi-preference is transitive, whereas the soft component $\succsim^S$ is a coherent extension of $\succsim^R$. The formal definition is as follows:

Definition 4.1 A *bi-preference* on X is a pair $\left(\succsim^R, \succsim^S\right)$ of binary relations on X such that[30]

(Core Transitivity) $\succsim^R$ is a preorder,
(Soft Extension) $\succsim^S$ contains $\succsim^R$, and
(Transitive Coherence) $\succsim^R \circ \succsim^S \subseteq \succsim^S$ and $\succsim^S \circ \succsim^R \subseteq \succsim^S$.

The relation $\succsim^R$ is the *rigid preference*, and $\succsim^S$ is the *soft preference*. A bi-preference $\left(\succsim^R, \succsim^S\right)$ is *complete* if so is $\succsim^S$. The irreflexive relation $\succsim^G := \succsim^S \setminus \succsim^R$ is the *gap* of $\left(\succsim^R, \succsim^S\right)$.

Transitivity is the fundamental property that shapes the structure of a bi-preference. In fact, Core Transitivity ensures that the rigid part $\succsim^R$ of an economic agent's preference structure is rational, whereas Soft Extension and Transitive Coherence require that the soft preference $\succsim^S$ expands $\succsim^R$ in a way that rationality is locally preserved with respect to $\succsim^R$. However, Transitive Coherence does not guarantee the global transitivity of $\succsim^S$, since the soft preference may even fail to be quasi-transitive. Notice that no assumption of completeness is made in the general setting; nevertheless, some types of bi-preferences turn out to be complete, e.g., NaP-preferences (see Sect. 4.2). The difference between what may and must happen, codified by the gap $\succsim^G$, provides information about the agent's indecisiveness, and prompts a partial ordering describing the stability of the given information: see Sect. 4.2 for a formal description of the related poset.

The next example exhibits several natural instances of bi-preferences (see also Giarlotta and Watson (2018b)).

Example 4.2 Let $\succsim$ be a weak preference on X, and $\succsim_0$ its trace. Further, let $\equiv$ be a generic equivalence relation on X, and $\Delta(X) = \{(x, x) : x \in X\}$ the diagonal of X. Finally, let $c \colon \Omega \to \Omega$ be a choice correspondence on X, and $\succsim_c$ the preference revealed by c.

(i) $(\succsim, \succsim)$ is a bi-preference if and only if $\succsim$ is a preorder.
(iii) $(\succsim, X^2)$ is a complete bi-preference if and only if $\succsim$ is a preorder.
(iii) $(\Delta(X), \succsim)$ is a bi-preference.
(iv) $(\Delta(X), X^2)$ is the bi-preference on X with the largest possible gap.
(v) $(\succsim_0, \succsim)$ is a bi-preference.
(vi) If c satisfies properties (α) and (ρ), then the pair $(\succsim'_c, \succsim_c)$ is a complete bi-preference, where $\succsim'_c$ is defined by $x \succsim'_c y$ if $y \in c(A)$ implies $x \in c(A)$ for all $A \in \Omega$ with $x, y \in A$.

[30] This notion was originally introduced by Giarlotta and Greco (2013) under the name of *partial* NaP-*preference*. Here we switch to a simpler and more agile terminology, which allows us to qualify special types of bi-preferences, such as "uniform", "monotonic", "comonotonic", etc. (see later in this section).

(vii) $(\equiv, \precsim)$ is a bi-preference, where $\precsim$ is any symmetric extension of $\equiv$ satisfying transitive coherence (i.e, the two compositions of $\equiv$ and $\precsim$ are contained into $\precsim$).

(viii) $(\equiv, X^2)$ is a complete bi-preference.

The interest in bi-preferences stems from economic motivations, since, in many applications to decision making, these structures provide a more accurate modelization rather than mono-preferences. A relevant advantage of a bi-preference approach is that the two layers of information—rigid and soft—allow one to explicitly identify the four binary relations of strict preference, indifference, incomparability, and *indecisiveness*. This yields an enrichment of the descriptive power of the model, since the mono-preference approach is able, at its best, to only distinguish three relations, namely, strict preference, indifference, and incomparability. As a matter of fact, in most classical modelizations, indecisiveness is assumed to coincide with incomparability, which in turn implies that an economic agent with a complete preference structure displays by definition no indecisiveness whatsoever. On the contrary, in a bi-preference approach, indecisiveness is naturally definable at an aggregate level in the form of a transition of states, going from an incomparability in the (usually incomplete) rigid component to an indifference in the (possibly complete) soft one. The latter point is better clarified in dealing with special types of bi-preferences, called "uniform", which are the subject of the next section.

4.1.2 Uniform Bi-preferences

How do bi-preference actually arise in a decision process? In Giarlotta and Watson (2018b), the authors provide two possible answers to this question, which are designed according to the formative process of the two components of a bi-preference. In fact, they distinguish between (I) *simultaneous* and (II) *sequential* bi-preferences, depending on the timing of their formation.[31] In case (I), the two preferences are constructed at the same time. Typical examples of this kind are necessary and possible preferences stemming from applications of the robust ordinal regression approach in multiple criteria decision analysis (see next section). Another instance of type (I) is the bi-preference revealed by a "replaceable" choice (see Example 4.2(vi)).[32]

On the other hand, there are many bi-preferences which happen to be sequential, especially in a collective decision making setting. Whenever the components of a bi-preference are formed in temporally distinct stages, we identify a *primitive* preference (which can be either the rigid or the soft component) and a *derived* preference. Typical examples of type (II) are "tracing" bi-preferences associated to primitive soft components (see Example 4.2(v)). Other examples of sequential bi-preferences are those in which the primitive component describes the (rational) inner attitude of an agent, whereas the soft component is a derived extension of the former, obtained

[31] The philosophy underlying this distinction will be used again in dealing with (1) simultaneous and sequential multi-rationalizations (Sect. 4.4), and (2) multiple and iterated resolutions (Sect. 4.5).

[32] A choice is *replaceable* if the consistency properties (α) and (ρ) hold for it.

by consistently enriching the core evaluation in view of a specific goal ("sharpening" or "smoothening" the primitive judgement). The next definition provides instances of this kind.

Definition 4.3 A bi-preference $(\succsim^R, \succsim^S)$ is *uniform* if the strict preferences associated to the two components are nested one inside the other. In particular, it is *monotonic* if the inclusion $\succ^R \subseteq \succ^S$ holds, and *comonotonic* if the reverse inclusion $\succ^S \subseteq \succ^R$ holds.

The logics underlying the two types of uniform bi-preferences are similar (in their objective) but dual (in their philosophy). Specifically, their similarity consists of the fact that their common goal is to enrich the primitive rigid judgement by providing additional soft information; furthermore, the criterion used in this enrichment is "uniformly" applied to all alternatives. On the other hand, their duality lies in the philosophy used to construct the bi-preference: in fact, in a monotonic approach the soft component "sharpens" the rigid judgement, whereas in a comonotonic one the soft component "smoothens" it. Let us sketch how the two philosophies of sharpening and smoothening the primitive judgement operate.

Assume that a preorder $\succsim^R$ is given on a set X of alternatives. There are three possible configurations between two generic alternatives $x, y \in X$, namely:

(1) a rigid indifference $x \sim^R y$;
(2) a rigid preference $x \succ^R y$ (or, dually, $y \succ^R x$);
(3) a rigid incomparability $x \perp^R y$.

Uniform bi-preferences create two additional cases, according to the specific logic used in each extension. In a monotonic bi-preference $\left(\succsim^R, \succsim^S\right)$, we obtain the following five configurations:

(M1) a *pure indifference* $x \sim^R y$;
(M2) a *pure strict preference* $x \succ^R y$ (or, dually, $y \succ^R x$);
(M3) a *pure gap preference* $x \succ^G y$ (or, dually, $y \succ^G x$), i.e., $x \succ^S y$ and $x \perp^R y$;
(M4) a *pure indecisiveness* $x \sim^G y$, i.e., $x \sim^S y$ and $x \perp^R y$;
(M5) a *pure incomparability* $x \perp^S y$.

Notice that the configurations (M3), (M4), and (M5) all stem from the rigid configuration (3), and are obtained by sharpening the judgment of incomparability given at the primitive level. Figure 3—taken from Giarlotta and Watson (2018b)—describes the three rigid configurations (1)–(3) and their successive enrichment by the five monotonic configurations (M1)–(M5).[33] Any thick black arrow in Fig. 3 represents a rigid preference (from the source over the tail), whereas a thin gray arrow stands for a gap preference.

As extensively explained in Giarlotta and Watson (2018b), monotonic bi-preferences may naturally arise, for instance, in the process of selecting the set of best alternatives—possibly a single one—in each feasible menu. In this case, the economic agent employs the rigid preference $\succsim^R$ to pre-select some items according

[33]Configurations (2)′, (M2)′, and (M3)′ are dual to, respectively, (2), (M2), and (M3).

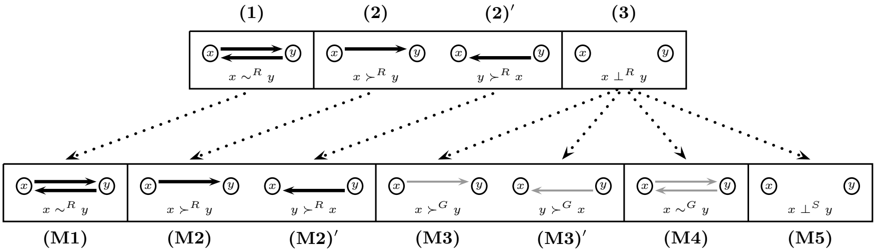

Fig. 3 The five types of admissible configurations in a *monotonic* bi-preference

to a logic of maximality: x is pre-selected in a menu A whenever there is no other alternative $y \in A$ such that $y \succ^R x$. Then, in order to make a more accurate selection, the economic agent may ask external sources to rank the remaining non-dominated options, introducing new strict preferences via $\succsim^S$ in a transitively coherent way: in Fig. 3 this is obtained by passing from a configuration of type (3) to one of type (M3) or (M3′).[34]

Also in a comonotonic bi-preference $(\succsim^R, \succsim^S)$ there are five possible configurations, even if they are created in a different way:

(C1) a *pure indifference* $x \sim^R y$;
(C2) a *pure strict preference* $x \succ^S y$ (or, dually, $y \succ^S x$);
(C3) a *balanced preference* $x \succ^R y$ and $x \sim^S y$ (or, dually, $y \succ^R x$ and $x \sim^S y$);
(C4) a *pure indecisiveness* $x \sim^G y$, i.e., $x \sim^S y$ and $x \perp^R y$;
(C5) a *pure incomparability* $x \perp^S y$.

Notice that the configurations (C2) and (C3) stem from the rigid configuration (2), whereas the configurations (C4) and (C5) stem from the rigid configuration (3). All comonotonic configurations are obtained by smoothening the judgment of either strict preference or incomparability given at the primitive level. This fact is especially clear for the configuration (C3), where a rigid preference of x over y is smoothened by a reverse soft preference of y over x (e.g., because there are scenarios in which y may be preferred to x). Figure 4 describes the three rigid configurations (1)–(3) and their successive enrichment by the five comonotonic configurations (C1)–(C5).[35]

Comonotonic bi-preferences arise in cases when there is a necessity to consider different types of arguments to evaluate a preference of an alternative over another one. Imagine, for instance, that in the process of establishing a rigid relationship between two alternatives x and y, the economic agent decides that the most likely judgement is that x is strictly preferred to y, i.e., $x \succ^R y$ holds. However, she is not perfectly convinced of this judgement, because there are (less likely yet possible)

[34]This procedure is reminiscent of the *rational shortlist method*, a bounded rationality approach to individual choice recently introduced by Manzini and Mariotti (2007). However, in the latter case, the two sequential rationales need not be nested one inside the other, and they fail in general to be transitively coherent. On the point, see Sect. 4.4 of this survey.

[35]Configurations (2)′, (C2)′, and (C3)′ are dual to, respectively, (2), (C2), and (C3).

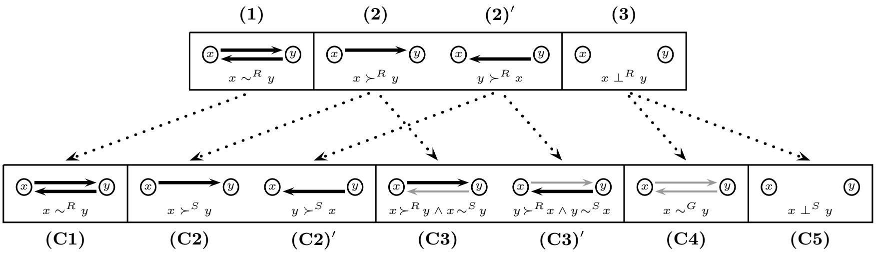

Fig. 4 The five types of admissible configurations in a *comonotonic* bi-preference

scenarios in which she feels that y is either indifferent to x or even preferred over x. In this circumstance, the role of $\succsim^S$ is to weaken the primitive judgement modelled by $\succsim^R$, thus transforming the strict rigid preference $x \succ^R y$ into a soft indifference $x \sim^S y$: this is exactly what happens in Fig. 4 in passing from a configuration (2) of strict rigid preference to a configuration (C3) of balanced preference.

4.1.3 An Extension of Schmeidler's Theorem

We conclude this section on bi-preferences by providing an interesting theoretical application (see Giarlotta and Watson, 2019). In 1971, Schmeidler (1971) proved the following elegant—and maybe surprising—result connecting the continuity of a preorder to its completeness:

Theorem 4.4 (Schmeidler 1971) *A nontrivial bi-semicontinuous preorder on a connected topological space is complete.*

Here "nontrivial" means that the asymmetric part of the preorder is nonempty, whereas "bi-semicontinuous" means that it is both *closed semicontinuous* (i.e., all lower and upper weak sections are closed) and *open semicontinuous* (i.e., all lower and upper strict sections are open).[36]

The proof of Theorem 4.4 given by Schmeidler is neat and compact. However, two different arguments—one order-theoretic and one topological—are quite intertwined, and this fact prevents one from fully understanding to what extent the two hypotheses of connectedness (of the space) and bi-continuity (of the preorder) are needed in the proof. In an attempt to refine Schmeidler's argument, Giarlotta and Watson (2019) have very recently generalized Theorem 4.4 to a bi-preference setting. Here we illustrate one of these extensions, which yields Schmeidler's theorem as a corollary (but Schmeidler's theorem does not allow one to derive it). To begin with, we need a few new notions.

Definition 4.5 A bi-preference $(\succsim^R, \succsim^S)$ is *quasi-monotonic* if $\succ^R \cap (\succsim^S \circ \succ^S) \subseteq \succ^S$ and $\succ^R \cap (\succ^S \circ \succsim^S) \subseteq \succ^S$.

[36]Cf. with the notion of continuity given in Sect. 2.3.

Notice that quasi-monotonicity is a weakening of the property of monotonicity introduced in Definition 4.3, which requires the inclusion $\succ^R \subseteq \succ^S$ to hold regardless of any condition pointing in that direction. Instead, quasi-monotonicity states that a strict rigid preference $x \succ^R y$ implies a strict soft preference $x \succ^S y$ only if there are elements in X already suggesting that possibility, in the sense that either $x \succsim^S z \succ^S y$ or $x \succ^S z \succsim^S y$ holds for some $z \in X$. It follows that quasi-monotonicity is a rather mild assumption in several economic scenarios.

Definition 4.6 A bi-preference $(\succsim^R, \succsim^S)$ is *strongly comonotonic* if it is comonotonic and quasi-monotonic.

If $\succsim^S$ is a preorder, then any comonotonic bi-preference $(\succsim^R, \succsim^S)$ is obviously strongly comonotonic. However, transitivity of the soft component is not needed to ensure strong comonotonicity of a bi-preference (Giarlotta and Watson 2019).[37] Finally, we can state what we were after:

Theorem 4.7 *Let $(\succsim^R, \succsim^S)$ be a strongly comonotonic bi-preference on a connected topological space. If $\succsim^R$ is closed semicontinuous, and $\succ^S$ is nontrivial and open semicontinuous, then $(\succsim^R, \succsim^S)$ is a* NaP*-preference and $\succsim^S$ is complete.*

Schmeidler's theorem follows from Theorem 4.7 by taking a bi-preference $(\succsim^R, \succsim^R)$ such that $\succsim^R$ is a bi-semicontinuous preorder on a connected topological space.

4.2 Necessary and Possible Preferences

In this section, we give a more refined answer to question (Q1) posed in Sect. 2.2, describing special types of (comonotonic) bi-preferences: necessary and possible preferences. The specialty of these bi-preference structures lies in the fact that they have already proven to be useful in several applications within multiple criteria decision analysis.

4.2.1 NaP-Preferences, Robust Ordinal Regression, and Decisions Theory

We start with the main notion:

Definition 4.8 A *necessary and possible preference* (NaP*-preference)* on X is a comonotonic bi-preference $(\succsim^N, \succsim^P)$ on X such that the following additional property holds:

[37] In fact, Giarlotta and Watson (2018b) constructively characterize strongly comonotonic bi-preferences as those that can be obtained from simpler types of bi-preferences by an operation of *resolution* (Resolutions of preference structures do have the same flavor as the operation of choice resolution described in Sect. 3.5.).

(Mixed Completeness) for each $x, y \in X$, either $x \succsim^N y$ or $y \succsim^P x$ holds.

In this case, $\succsim^N$ and $\succsim^P$ are, respectively, the *necessary preference* and the *possible preference*. The complement of $\succsim^P$ in X^2 is the *impossible preference* $\succ^I$. A NaP-preference is *normalized* if its necessary component is a partial order. We denote by $\mathsf{NaP}(X)$ the family of all NaP-preferences on X, and by $\mathsf{NaP}_{\mathrm{nor}}(X)$ the subfamily of all normalized NaP-preferences on X.

Notice that Mixed Completeness implies that the possible component $\succsim^P$ of a NaP-preference $(\succsim^N, \succsim^P)$ is complete; in particular, configuration (C5) in Fig. 4 is ruled out.

NaP-preferences arise quite naturally in applications that require the considerations of several points of view, for instance within the framework of Multiple Criteria Decision Analysis (MCDA). In fact, the first appearance of NaP-preferences dates to 2008, in the seminal paper on *Robust Ordinal Regression* (ROR) by Greco et al. (2008). (See also Angilella et al. (2010b) for a non-additive ROR model based on the Choquet integral, as well as Greco et al. (2010b) for an overview of the ROR methodology.)

The ROR approach was originally designed to provide a consistent extension of the UTA method of Jacquet-Lagreze and Siskos (1982), which only considered special types of utility functions fitting the information provided by the decision maker. Instead, in a ROR approach, *all* compatible utility functions are taken into account, which in turns yields the creation of a more refined preference structure: a necessary and possible preference.

Nowadays, the ROR is among the most used methodologies employed in MCDA, as witnessed by the very large amount of applications in several fields: see, among many others, Angilella et al. (2016) for an application to urban and territorial planning, Angilella et al. (2014) for a customer satisfaction analysis based on a multiplicity of interacting criteria, and Corrente et al. (2016) for applications of ROR to decisions under uncertainty and risk. We refer the reader to the recent paper by Corrente et al. (2013) for a survey on ROR in preference leaning and ranking.

In a ROR approach, the pieces of information provided by an economic agent on a set X of n-dimensional alternatives (i.e., in the presence of a set of $n \geq 2$ evaluation criteria $g_i\colon X \to \mathbb{R}$, $1 \leq i \leq n$) are used to build a set $\mathcal{U}$ of *global value functions* $u\colon \mathbb{R}^n \to \mathbb{R}$, which do not contradict data. In this multi-dimensional setting, two binary relations $\succsim^N$ and $\succsim^P$ on X naturally arise by using, respectively, universal and existential quantification over $\mathcal{U}$:

$$\begin{aligned} x \succsim^N y &\overset{\text{def}}{\iff} (\forall u \in \mathcal{U}) \;\; u(x) \geq u(y), \\ x \succsim^P y &\overset{\text{def}}{\iff} (\exists u \in \mathcal{U}) \;\; u(x) \geq u(y), \end{aligned} \tag{3}$$

where $x, y \in X$ are arbitrary. Then the pair $(\succsim^N, \succsim^P)$ is a NaP-preference on A.

MCDA is not the unique setting in which NaP-preferences (and bi-preferences in general) naturally arise. In fact, the realm of Decision Theory under uncertainty offers another environment that is well suited to be described by these types of bi-preference structures. For instance, in an Anscombe-Aumann setting (1963), prototypes of a necessary preference and a possible preference are well known, being respectively described by Bewley's *Knightian preferences* (1986) and Lehrer-Teper's *justifiable preferences* (2011). Specifically, given a set of priors, a Knightian approach states that an act f is preferred to another act g if this preference holds for *all* priors; on the other hand, a model of justifiable preferences considers f better than g if this preference holds for *at least one* prior. In a von Neumann-Morgenstern's setting (1944), a further example of a necessary preference extendable by means of a possible preference is given by the incomplete preorder modeled as in Dubra et al. (2004): here the authors consider a set of utility functions $\mathcal{U}$ such that a lottery p is preferred to another lottery q if the expected utility of p is not smaller than the expected utility of q for all functions in $\mathcal{U}$.

Indeed, earlier approaches to Decision Theory use bi-preferences. For instance, Gilboa et al. (2010) define two relations in an Anscombe-Aumann setting: an *objective preference* $\succsim^*$ and a *subjective preference* $\succsim^\wedge$. The objective relation $\succsim^*$ is a Knightian preference, which models cases such that the decision maker can convince everybody that he is right. The subjective relation $\succsim^\wedge$ codifies the maxmin expected utility of Gilboa and Schmeidler (1989), and represents preferences such that the decision maker cannot be convinced by anybody that he is wrong. The objective preference $\succsim^*$ is a preorder, whereas the subjective preference $\succsim^\wedge$ is a complete preorder that extends $\succsim^*$. However, although this model explicitly uses a bi-preference approach, its underlying philosophy is quite different from that of NaP-preferences: in particular, no transitive coherence is assumed to hold between the objective and the subjective preferences.

On the other hand, some very recent contributions in Decision Theory employ bi-preferences more in the spirit of a NaP-preferences approach. For instance, in the model proposed by Cerreia-Vioglio et al. (2018) within an Anscombe-Aumann setting, two types of consistent preferences are used: the first reflects the decision maker's judgments about well-being (her *mental preferences*), whereas the second represents the decision maker's choice behavior (her *behavioral preferences*). The authors propose axioms that describe the relationship between these preferences, that is, between mind and behavior. Under standard expected utility assumptions, two representations are obtained: the first uniquely infers choice behavior from mental preferences; the second uses mental preferences to direct choice behavior, however leaving room for biases and framing effects. Some of the results proved in Cerreia-Vioglio et al. (2018) concern NaP-preferences.

Finally, we mention a necessary and possible extension of a very recent approach to sequential decision making, introduced by Chambers and Miller (2018). The two authors develop a normative theory of incomplete preferences, called "benchmarking". Their theory aims at simplifying a decision making process by modeling its preliminary stage: for instance, in the hiring process for an academic job, a committee may employ some objective criteria to make a first screening among candidates. Chambers and Miller characterize *benchmarking rules*, which are binary relations satisfying four natural properties: transitivity, monotonicity with respect to set-containment, incomparability of marginal gains, and incomparability of marginal losses. In Giarlotta and Watson (2018d), it is shown that these rules are indifference-induced, that is, the combination of their symmetric part with set-containment fully describes them. In the same spirit, additional judgements of pure similarity provided by (groups of) decision makers allow one to enhance these structures, and give rise to NaP-*benchmarking rules* (Giarlotta and Watson 2018d).

4.2.2 Characterizations, Properties, and Semantics

In what follows, we give an overview of the main features and the semantics of NaP-preferences. The following order-theoretic characterization holds (Giarlotta and Greco 2013):

Theorem 4.9 (AC)[38] *A pair* $\left(\succsim^N, \succsim^P\right)$ *of binary relations on* X *is a* NaP*-preference on* X *if and only if there is a family* $\mathcal{T}$ *of total preorders on* X *such that* $\succsim^N = \bigcap \mathcal{T}$ *and* $\succsim^P = \bigcup \mathcal{T}$.

In 1998, Donaldson and Weymark (1998) proved a famous result, which says that any preorder can be written as an intersection of total preorders. This result, later proved again by Bossert (1999) using a simpler technique, strengthens Lemma 15.4 in Fishburn (1973a) as well as Theorem A(4) in Suzumura (1983). Donaldson and Weymark's result is an immediate consequence of Theorem 4.9:

Corollary 4.10 (Donaldson and Weymark 1998; Bossert 1999) *Every preorder is the intersection of a collection of total preorders.*

The proof of Corollary 4.10 given by Donaldson and Weymark (1998) is direct, and makes no use of related results for partial orders. Instead, the proof given by Bossert (1999) is elementary, since it makes use of a notorious result by Dushnik and Miller (1941), which says that every partial order is the intersection of a collection of linear orders.

Theorem 4.9 is the natural abstraction of the representation (3) stemming from a family $\mathcal{U}$ of real-valued utility functions (cf. Sect. 4.3). In fact, Theorem 4.9 can be equivalently formulated by saying that $\left(\succsim^N, \succsim^P\right)$ is a NaP-preference on X if and only if there exists a family $\mathcal{T}$ of total preorders on X such that, for all $x, y \in X$, the following two equivalences hold:

$$\begin{array}{lll} x \succsim^N y & \Longleftrightarrow & (\forall \succsim \in \mathcal{T}) \;\; x \succsim y, \\ x \succsim^P y & \Longleftrightarrow & (\exists \succsim \in \mathcal{T}) \;\; x \succsim y. \end{array} \tag{4}$$

Thus, if the total preorders in $\mathcal{T}$ are Debreu-separable,[39] then the two representations (3) and (4) essentially coincide (since every total preorder in $\mathcal{T}$ is representable in $\mathbb{R}$ by Theorem 2.6).

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[38] The Axiom of Choice (AC) is needed in the proof of Theorem 4.9 to apply Zorn's Lemma in the case of an uncountable ground set X.

[39] See Footnote 6.

Many other examples of NaP-preferences naturally arise in both theoretical settings and practical applications. We have already discussed real decision problems in MCDA which may benefit from a necessary and possible approach. For some instances of theoretical applications, notice that the four examples of preorders (written as intersection of total preorders) presented by Donaldson and Weymark (1998) immediately generalize to NaP-preferences:

(1) the *strong Pareto preorder*, which is the first line of representation (3);
(2) the *weak Pareto preorder*, which is a suitable manipulation of utility representation (3);
(3) the *dominance preorder* (Blackorby and Donaldson 1977), which extends the strong Pareto preorder by regarding permutations of utility vectors as being indifferent to each other;
(4) the *hull of dominance preorder* (Blackorby and Donaldson 1977), which extends the dominance quasi-ordering by using bi-stochastic matrices.[40]

An alternative characterization of NaP-preferences, which emphasizes different aspects of these structures, is given by Giarlotta and Watson (2017b, Lemma 2.4):

Theorem 4.11 *A pair $\left(\succsim^N, \succsim^P\right)$ of binary relations on X is a NaP-preference if and only if Core Transitivity, Soft Extension and the following three additional properties hold:*

(Rigid Strict Extension) $\succ^N$ *includes* $\succ^P$,
(Soft Completeness) $\succsim^P$ *is complete, and*
(Mixed Transitivity) $\succsim^N \circ \succ^P \subseteq \succ^P$ *and* $\succ^P \circ \succsim^N \subseteq \succ^P$.

In particular, a bi-preference is a NaP-preference if and only if it is comonotonic and complete.

Rigid Strict Extension and Mixed Transitivity describe the characteristic features of the strict possible preference $\succ^P$: this relation is stronger than the strict necessary preference $\succ^N$ insofar as it models a situation of preference in which no compensation in the opposite sense is allowed (see configurations (C2) and (C2)′ vs. configurations (C3) and (C3)′ in Fig. 4). In fact, not only $\succ^P$ is a strict partial order included in $\succ^N$, but also it enjoys a property of mixed transitivity whenever combined with the weak necessary preference.

The semantics of a NaP-preference is related to the type of information provided by the economic agent on the set X of alternatives. Indeed, Definition 4.8 yields a partition of X^2 into three (possibly empty) classes, namely, $\succsim^N$, $\succsim^G$, and $\succ^I$. The union $\succsim^N \cup \succ^I$ codifies the "total information" provided by the economic agent, and the gap $\succsim^G$ represents a gray area of "indecisiveness". The total information provided by the agent can be, in turn, split into two subtypes: the necessary preference $\succsim^N$ models its positive part (what *must* happen), and the impossible preference $\succ^I$

[40] A square matrix is *bi-stochastic* if (i) all of its entries are non-negative, and (ii) all the row and column sums are equal to one.

its negative part (what *cannot* happen). According to this interpretation of NaP-preferences, we can make the family $\mathsf{NaP}(X)$ into a poset, using a binary relation that ranks bi-preferences according to a measure of their "informative content/stability".

Definition 4.12 Let $\sqsubseteq$ be the binary relation on $\mathsf{NaP}(X)$ defined by

$$\left(\succsim_1^N, \succsim_1^P\right) \sqsubseteq \left(\succsim_2^N, \succsim_2^P\right) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \succsim_1^N \subseteq \succsim_2^N \quad \text{and} \quad \succsim_1^P \supseteq \succsim_2^P . \tag{5}$$

If (5) holds, then $\left(\succsim_2^N, \succsim_2^P\right)$ is an *informative refinement* of $\left(\succsim_1^N, \succsim_1^P\right)$.

Clearly, the pair $\left(\mathsf{NaP}(X), \sqsubseteq\right)$ is a poset. Upon observing that the condition $\succsim_1^P \supseteq \succsim_2^P$ can be equivalently rewritten as $\succ_1^I \subseteq \succ_2^I$, the semantics of the binary relation $\sqsubseteq$ becomes apparent. Indeed, an informative refinement of a NaP-preference is characterized by an enlargement of both types—positive and negative—of information, allowing no compensation whatsoever between the two types. The maximal elements of this poset represent situations in which the gap is empty: in these cases, the economic agent's preference structure has no gray area of indecisiveness, and so it is perfectly stable. These maximal NaP-preferences "are" the total preorders on X. On the other hand, the unique minimum element of this poset represents a situation of complete absence of either positive or negative information: in this case, everything may happen, and so the agent's preference structure is totally unstable. The next result describes the features of the poset $\left(\mathsf{NaP}(X), \sqsubseteq\right)$ (see Lemma 5.4 in Giarlotta and Greco (2013)).

Theorem 4.13 *$(\mathsf{NaP}(X), \sqsubseteq)$ is a meet-semilattice,*[41] *having $\left(\Delta(X), X^2\right)$ as its unique minimum element, and all pairs $(\succsim, \succsim)$ as its maximal elements, with $\succsim$ any total preorder on X.*

An extended discussion on the topic can be found in Giarlotta and Greco (2013). For a graphical representation of the meet-semilattice $\left(\mathsf{NaP}(X), \sqsubseteq\right)$, the reader may glimpse at either Fig. 3 in Giarlotta and Greco (2013) (for the simplest case $|X| = 2$), or Fig. 6 in Giarlotta (2014) (for the already complicated case $|X| = 3$). For the sake of completeness, this last figure is reported in the Appendix as Fig. 7.

In view of Theorem 4.13 and its interpretation in terms of informative content, it becomes very interesting to determine whether, in the case of a finite ground set X, this poset is "well-graded" in the sense of Doignon and Falmagne (1997). Let us recall their notion of well-gradedness:

Definition 4.14 Let X a finite set, and $d\colon 2^X \times 2^X \to \mathbb{R}$ a metric on the collection of nonempty subsets of X. A family $\mathcal{F} \subseteq 2^X$ is *well-graded* if, for each $R, S \in \mathcal{F}$ at distance n, there is a sequence of sets $R = F_0, F_1, \ldots, F_n = S$ in $\mathcal{F}$ such that $d(F_{i-1}, F_i) = 1$ for each $i = 1, \ldots, n$.

[41] *Meet-semilattice* means that $(\mathsf{NaP}(X), \sqsubseteq)$ is a poset, and for each $\left(\succsim_1^N, \succsim_1^P\right), \left(\succsim_2^N, \succsim_2^P\right) \in \mathsf{NaP}(X)$, there is a greatest element $\left(\succsim_3^N, \succsim_3^P\right)$ in $\mathsf{NaP}(X)$ such that $\left(\succsim_3^N, \succsim_3^P\right) \sqsubseteq \left(\succsim_1^N, \succsim_1^P\right)$ and $\left(\succsim_3^N, \succsim_3^P\right) \sqsubseteq \left(\succsim_2^N, \succsim_2^P\right)$.

Upon regarding families of preference relations on X as sets of pairs, and taking as metric on 2^X the classical distance between sets, that is, the size of the symmetric difference, we have:

Theorem 4.15 (Doignon and Falmagne 1997) *The classes of partial orders, semiorders and interval orders on a finite set are well-graded.*

In other words, a uniform family $\mathcal{F}$ of preferences (i.e., preferences of the same type, e.g., semiorders) on a set is well-graded if for any two distinct relations $R, S \in \mathcal{F}$ whose symmetric difference has size n, R and S can be connected by a "path" of length n, that is, a sequence of n elementary steps within $\mathcal{F}$ which smoothly transforms R into S by changing (i.e., eliminating or adding) one edge a time. The well-gradedness of a uniform family of binary relations is a fundamental property, which is needed in order to develop a stochastic theory describing the evolution of preferences through the random occurrence of quantum tokens of information (Falmagne 1996, 1997; Falmagne and Doignon 1997). In the very same direction, Giarlotta and Watson (2017a) prove the following fact, which paves the way toward the development of a stochastic theory of NaP-preferences:

Theorem 4.16 *The class of normalized NaP-preferences on a finite set is well-graded.*

Apart from those aspects already mentioned, NaP-preferences have also been studied from several other perspectives:

- properties of transitive coherence linking the two components, and their relationship with the genesis of interval orders and semiorders (Giarlotta 2014);
- asymmetric and normalized forms of NaP-preferences (Giarlotta 2015);
- symmetric counterparts of NaP-preferences, called NaP-indifferences (Giarlotta and Watson 2017b), which also codify forms of revealed similarity in individual choice theory (see Sect. 3.2 of Giarlotta and Watson (2017b), which is related to the notion of a *congruence relation* on a choice space Cantone et al. (2018b)).

4.2.3 Some Related Approaches in Fuzzy Set Theory

We conclude this section by summarizing the main features of two very recent approaches in *fuzzy set theory* (Zadeh 1965), which have employed NaP-preferences (and bi-preferences, in general) as a source of inspiration:

(1) fuzzy politics;
(2) NaP-hesitant fuzzy sets.

Concerning (1), Alcantud et al. (2018) design a model for the genesis of parties, which is based of a fuzzy elaboration of a necessary and possible approach. In this model, for each topic of interest for the political campaign, a candidate is described by means of a PaP-*profile* (*private and public profile*): the private one is known only among politicians, whereas the public one is available to every citizen and accordingly displayed on the media. Both components of a PaP-profile are trapezoidal or quasi-trapezoidal[42] fuzzy sets on the interval $[-1, 1]$, where -1 represents extreme left, 0 perfect centre, and 1 extreme right. The private profile is always contained—in the sense of fuzzy set theory—in the public one. A candidate's private profile describes the very core of his political ideas on a topic, which he naturally shares with other politicians in an attempt to form aggregations of powers to better pursue his goals. On the other hand, the public profile extends the private one by summarizing the politician's tolerance/willingness to compromise on the topics of the campaign. All candidates are then paired up according to the similarity of their PaP-profiles on all topics: this procedure creates the so-called *matching graph* of politicians. Finally, an algorithm, which is based on the size and the cohesion of the cliques of the matching graph, determines the family of newborn parties.

Concerning (2), Zadeh's fuzzy set theory (Zadeh 1965) deals with imprecise-ness/vagueness of data and evaluations by imputing degrees to which objects belong to a set. The appearance of fuzzy sets induced the rise of several related theories, which codify subjectivity, uncertainty, imprecision, or roughness of evaluations. The rationale of these theories is to create new and more flexible methodologies, which allow one to realistically model a variety of concrete decision problems. In this direction, Torra (2010) recently extended the notion of fuzzy sets by that of *hesitant fuzzy sets*: these are maps assigning to any element of X a subset of $[0, 1]$ (instead of a single element of $[0, 1]$ as for fuzzy sets). Hesitant fuzzy sets permit the modelization of phenomena that cannot be handled by classical fuzzy set theory: for instance, collective decision making is a natural outlet for hesitant fuzzy models (Alcantud et al. 2016). Alcantud and Giarlotta (2019) propose an extension of Torra's notion of hesitant fuzzy set, which fits quite well group decision making. In fact, indecisiveness in judgements is described by two nested hesitant fuzzy sets, which form a NaP-*hesitant fuzzy set*: the smaller (necessary) component collects membership values determined according to a rigid evaluation, whereas the larger (possible) component comprises socially acceptable membership values. This novel approach displays structural similarities with Atanassov's *intuitionistic fuzzy set theory* (Atanassov 1986, 1999), but has rather different features and goals.

4.3 Multiple and Modal Utility Representations

Classical utility representations of preferences fall short under many points of views, as pointed out in Sects. 2.3 and 3.1. Here we examine two ways of dealing with some shortcomings of traditional (and less traditional) approaches, which are similar yet they address different issues: (1) multi-utility representations, and (2) modal utility representations.

[42] For the notions of trapezoidal and quasi-trapezoidal profiles, as well as for their canonical representations by means of quadruples of real numbers in $[-1, 1]$, see Sect. 2.3 in Alcantud et al. (2018).

4.3.1 Multi-utility Representations

A traditional stream of research concentrates on the analysis of the continuous and semicontinuous representability of all preorders. Since an incomplete preorder obviously admits no representation by means of a single utility function, two alternative approaches to the topic have been proposed over time, using either (i) a single utility function in a weaker form, or (ii) a family of utility functions. Approach (i), due to Richter (1966) and Peleg (1970), is quite classical:

Definition 4.17 A preorder $\succsim$ on a set X is *Richter-Peleg representable* if there exists a map $u\colon X \to \mathbb{R}$ such that the following two implications hold for each $x, y \in X$:

$$x \succsim y \implies u(x) \geq u(y) \qquad \text{and} \qquad x \succ y \implies u(x) > u(y).$$

In this case, the function u is a *Richter-Peleg representation* of $\succsim$.

A lot is known about this notion, due to the work on analytic order theory by Herden (1989), Jaffray (1975b), Levin (1983), and Sonderman (1980). However, the use of a Richter-Peleg representation of a preorder is limited by the fact that it determines a loss of information, since one cannot recover the primitive preference from its representation (Majumdar and Sen 1976). This has recently brought several authors to consider hybrid approaches to the topic, as that of a *Richter-Peleg multi-utility representation*: see Minguzzi (2013), as well as Alcantud et al. (2016).

These hybrid solutions to the mentioned problem bring us to discuss the second approach, namely, (ii) multi-utility representations of a preorder. Originally introduced by Ok (2002), the topic of multi-utility representation has benefited from many important contributions by Kaminski (2007), Evren and Ok (2011), Bosi and Herden (2012), and Evren (2014). Here is the formal notion:

Definition 4.18 A *multi-utility representation* of a preorder $\succsim$ on X is a family $\mathcal{U}$ of functions $u\colon X \to \mathbb{R}$ such that the following equivalence holds for each $x, y \in X$:

$$x \succsim y \iff (\forall u \in \mathcal{U})\ u(x) \geq u(y). \tag{6}$$

(Cf. (6) with the first line of (3) in Sect. 4.2.)

Proposition 1 of Evren and Ok (2011) easily establishes that every preorder admits a (semicontinuous) multi-utility representation.[43] On the other hand, the problem of finding *continuous* multi-utility representations of a preorder poses some difficulties. Upon extending Herden's (1989) approach, Evren and Ok (2011) derive a theoretical characterization for the existence of a continuous multi-utility representation of a preorder, which is linked to the solution of a Urysohn-type separation problem. Regrettably, this characterization offers no insight in practical cases. Thus, the two authors establish two *sufficient* conditions, which are useful in applications. The first result imposes severe restrictions of the topological space and mild conditions on the preorder:

[43]See also Ok (2002, Theorem 3) and Mandler (2006, Theorem 1) for the existence of special multi-utility representations under suitable order-separability conditions.

Theorem 4.19 (Evren and Ok 2011) *Every continuous preorder on a σ-compact and locally compact Hausdorff space has a continuous multi-utility representation.*

The following consequence of Theorem 4.19 makes it a useful tool in applications when the ground topological space has nice features:

Corollary 4.20 (Evren and Ok 2011) *Every continuous preorder on a topological space that is either compact or a nonempty closed subset of a Euclidean space has a continuous multi-utility representation.*

The second result of Evren and Ok is complementary to the first, insofar as it requires less from the topological space and more from the preorder:

Theorem 4.21 (Evren and Ok 2011) *Every "nice" semicontinuous preorder satisfying strong local non-satiation has a continuous multi-utility representation.*

The two properties of "strong local non-satiation" and "niceness" are rather undemanding conditions, which are often satisfied by preferences encountered in dynamic consumer theory and decision making under uncertainty.

We conclude the discussion on the multi-utility representation of a preorder by emphasizing that this approach has been used in many recent preference models under uncertainty, which deal with either a single potentially incomplete preference, or a suitable pair of linked preferences: see, e.g., Cerreia-Vioglio et al. (2018), Dubra et al. (2004), Ghirardato et al. (2003), Gilboa et al. (2010), Ok et al. (2012).

4.3.2 Modal Utility Representations

The notion analyzed here is a simple variation of a multi-utility representation. Specifically, the motivating question is the following:

(Q10) *Given a preorder $\succsim$ and a suitable extension of $\succsim$, can we obtain a multi-utility representation of $\succsim$ that simultaneously represents (in a different way) its extension?*

Such a representation would describe the preorder "globally" and its extension "locally". The formal notion is the following (Giarlotta and Greco 2013):

Definition 4.22 Let $\left(\succsim^R, \succsim^S\right)$ be a pair of binary relations on X. A *modal utility representation* of $\left(\succsim^R, \succsim^S\right)$ is a nonempty family

$$\mathcal{U} \;=\; \left\{u_h^k : h \in H \;\wedge k \in K_h\right\}$$

of utility functions $u_h^k \colon X \to \mathbb{R}$ indexed over the set $\bigcup_{h\in H}\left\{(h,k) : k \in K_h\right\}$ such that the following properties hold for each $x, y \in X$:

(M1) $x \succsim^R y \iff (\forall h \in H)\ (\forall k \in K_h)\ u_h^k(x) \geq u_h^k(y)$;
(M2) $x \succsim^S y \iff (\exists h \in H)\ (\forall k \in K_h)\ u_h^k(x) \geq u_h^k(y)$.

A pair $(\succsim^R, \succsim^S)$ that admits a modal utility representation $\mathcal{U}$ is *modally representable*: in this case, H is the set of *modes* of $\mathcal{U}$, whereas K_h is the *extent* of mode $h \in H$. In particular, $\mathcal{U}$ is *unimodal* if $|H| = 1$, and *simple* if $|K_h| = 1$ for each $h \in H$.

Notice that unimodal and simple representations only need one parameter to be described: indeed, a unimodal representation $\mathcal{U}$ can be written as $\mathcal{U} = \{u_0^k : k \in K_0\}$, whereas a simple modal representation can be written as $\mathcal{U} = \{u_h^0 : h \in H\}$. The modal representability of a pair of weak preferences can be characterized as follows (Giarlotta and Greco 2013):

Theorem 4.23 *A pair $(\succsim^R, \succsim^S)$ of binary relations is modally representable if and only if it is a bi-preference.*

The next definition describes a special case of modal representability[44]:

Definition 4.24 A bi-preference $(\succsim^R, \succsim^S)$ on X is *quantifier-representable* if there is a family $\mathcal{U}$ of utility functions on X such that the following properties hold for each $x, y \in X$:

(U1) $x \succsim^R y \iff (\forall u \in U)\ \ u(x) \geq u(y)$;
(U2) $x \succsim^S y \iff (\exists u \in \mathcal{U})\ \ u(x) \geq u(y)$.

Quantifier-representability implies modal representability, but not vice versa (Giarlotta and Greco 2013):

Proposition 4.25 *Let $(\succsim^R, \succsim^S)$ be a bi-preference on X.*

- *$(\succsim^R, \succsim^S)$ has a simple modal representation if and only if it is quantifier-representable.*
- *$(\succsim^R, \succsim^S)$ has a unimodal representation if and only if it has a multi-utility representation if and only if $\succsim^R = \succsim^S$ is a (possibly incomplete) preorder.*
- *$(\succsim^R, \succsim^S)$ has a simple unimodal representation if and only if $\succsim^R = \succsim^S$ is a representable total preorder.*

4.4 Multi-rationalizable Choices

In this section we provide a possible answer to the question (Q6) posed in Sect. 2.4.

In 1955, Herbert Simon described a behavioral choice model of *bounded rationality* (Simon 1955, 1982). In this pioneering work, an economic agent makes her choices according to a *list* of elements in the ground set X, a binary preference over X, and a satisfactory threshold $x^* \in X$. Then, she selects a unique element, which is either the first element in the list that is not inferior to x^*, or, if there is none, the last element in the list.

Quite recently, Rubinstein and Salant (2006) create a very rich framework in which an agent makes choices from a (finite) list rather than from a set. This choice model

[44] This type of representation has already been mentioned: see (3) in Sect. 4.2.

encompasses both the classical setting of revealed preference theory and Simon's approach, as well as several other models of rational behavior (*place-dependent rationality*, *reference point dictatorship* (Tversky and Kahneman 1991), *successive choice* (Salant 2003), *contrast effect*, etc.).

Many additional models of bounded rationality have recently been proposed in the framework of choice theory. These models pursue either a "simultaneous approach" (i.e., all justifying preferences are applied at the same time to explain the selection from the feasible menus) or a "sequential approach" (i.e., the justifying preferences are applied in some order, possibly with different procedures, to explain the selection process): see, among the most recent contributions of both kinds, Apesteguía and Ballester (2013), Au and Kawai (2011), Cherepanov et al. (2013), García-Sanz and Alcantud (2015), Kalai et al. (2002), Manzini and Mariotti (2007, 2012), Masatlioglu and Nakajima (2013), and Tyson (2013).

Here, following the most recent trends in the literature, we lay down a comprehensive framework for a theory of choice multi-rationalization, which aims at refining the classical theory of revealed preferences. This refinement is pursued by associating a *degree of binary rationality* to each choice: this is defined as the least number (*degree*) of binary relations (*binary*) that are needed to explain (*rationality*) the observed choice behavior of an economic agent or a group of economic agents. This in turn yields a natural classification of choices that are non-rationalizable according to Definition 2.15. In this way, the amount of choices possessing features of rationality is enlarged, and the rational/irrational dichotomy arising from revealed preference theory is smoothened.

To start, we classify the approaches that use binary relations (henceforth called *rationales*) to justify a choice behavior, listing the main variables under consideration.

(1) **Ground set** X:
   (a) finite,
   (b) infinite.

(2) **Choice domain** Ω:
   (a) total (the powerset of the ground set minus the empty set),
   (b) partial (a nonempty subset of a total domain, usually subject to closure properties).

(3) **Selection mode** c:
   (a) choice function (single valued),
   (b) choice correspondence (multi valued),
   (c) quasi-choice function (zero/one valued),
   (d) quasi-choice correspondence (zero/multi valued).[45]

[45] A *quasi-choice function* on X is a map $c\colon \Omega \to \Omega \cup \{\emptyset\}$ such that $c(A) \subseteq A$ and $0 \leq |c(A)| \leq 1$ for all $A \in \Omega$. Thus, the agent selects either a single item or no item at all from each menu. Similarly, a *quasi-choice correspondence* on X is a map $c\colon \Omega \to \Omega \cup \{\emptyset\}$ such that $c(A) \subseteq A$ and $0 \leq |c(A)| \leq |A|$ for all $A \in \Omega$.

(4) **Internal structure of rationales**:
(a) none (no property),
(b) partially transitive (acyclic, quasi-transitive, (m, n)-Ferrers),
(c) fully transitive (total preorder, linear order).

(5) **Retrieval modality**:
(a) by binary maximization (classical approach),
(b) by psychological maximization (behavioral approach),
(c) by type (general approach, model theoretic).

(6) **Number of rationales**:
(a) mono-rationalization (one preference),
(b) multi-rationalization (a nonempty set of preferences).

(7) **Interactions among rationales**:
(a) free (no interaction),
(b) monotonic (order-preserving with respect to reverse set-containment),
(c) strongly coherent (monotonic and transitively coherent),
(d) listable (guided by an underlying linear order), etc.

(8) **Philosophy/Timing**:
(a) simultaneous (selection made in one step, considering all rationales at the same time),
(b) sequential (selection made in sequential steps, each step with its own modality).

(Of course, features (7)–(8) apply only in the case that several rationales can be used, that is, in (6b).) Many approaches to choice rationalizability can be identified by the above features. For instance, most traditional models using a single binary rationale typically fall in the category identified by the features (1a)–(1b), (2a)–(2b), (3a)–(3b), (4b)–(4c), and (5a). Several recent approaches of bounded rationality can be classified by the same parameters, often using a retrieval modality of (5b) "psychological maximization" (Cherepanov et al. 2013; Rubinstein and Salant 2006; Salant 2003; Simon 1955). In the remainder of this section, we provide the reader with an overview of some selected approaches of this kind, separately dealing with (8a) simultaneous multi-rationalization, and (8b) sequential multi-rationalization.

4.4.1 Simultaneous Multi-rationalization

A relatively recent multi-approach to the theory of choice rationalization is the *rationalization by multiple rationales* (RMR), due to Kalai et al. (2002). Their approach falls into the category identified by the features (1a), (2a), (3a), (4c), (5a), (6b), (7a), and (8a). The goal of RMR is to provide a discrete measure of the rationalizability of any total choice function on a finite set: this is accomplished by determining the minimum number of linear orders such that the unique item selected in each menu is maximal for some rationale. Kalai et al. (2002) show that any single valued choice on an n-element set can always be rationalized by $(n-1)$ linear orders, and the likelihood that this maximum value is attained tends to one as the size n of the ground set goes to infinity. The RMR approach is neat and direct, but has some shortcomings. In fact, it can only be employed if (1a) the ground set X is finite, (2a) the choice domain Ω is total, (3a) the selection mode is a single valued choice function, and (4c) the binary rationales are linear orders; moreover, (7a) no interaction/relation among the rationales needs to exist.

Some of the above issues can be addressed by designing a general theory of *simultaneous multi-rationalization*, which also takes into account the following cases: (2b) partial choice domains, (3b) multi-valued choice correspondences, (4b) acyclic, quasi-transitive or (m, n)-Ferrers rationales, and (7b)–(7c)–(7d) various types of interactions among rationales. The next few definitions, given by Cantone et al. (2018c), are motivated by this goal.[46]

Definition 4.26 For any finite nonempty set X, let $\mathsf{Pref}(X)$ be the family of all reflexive and complete binary relations on X. In what follows, we denote by $\mathsf{Pref}_{\mathrm{ac}}(X)$, $\mathsf{Pref}_{\mathrm{qt}}(X)$, $\mathsf{Pref}_{\mathrm{tra}}(X)$, and $\mathsf{Pref}_{\mathrm{lin}}(X)$, the subfamilies of $\mathsf{Pref}(X)$ composed of relations that are, respectively, acyclic, quasi-transitive, transitive, and linear.

The notion of (simultaneous) multi-rationalization described by the next definition assigns to each menu a binary rationale belonging to a pre-selected family of preferences: for instance, in RMR, all admissible rationalizing preferences are in $\mathsf{Pref}_{\mathrm{lin}}(X)$. For less demanding theories of choice multi-rationalizability, one may assume that either $\mathsf{Pref}_{\mathrm{tra}}(X)$ or $\mathsf{Pref}_{\mathrm{qt}}(X)$ are employed instead. Below we impose a minimal condition of internal consistency on the rationales: acyclicity. This assumption agrees with common practice (see, e.g., Rubinstein and Salant (2006)).

Definition 4.27 A choice correspondence $c\colon \Omega \to \Omega$ on a finite ground set X is *freely multi-rationalizable* (*FMR*) if there exists a map $f\colon \Omega \to \mathsf{Pref}_{\mathrm{ac}}(X)$ with the following property:

(Local Rationalization) $\quad c(A) = \max(A, f(A))$ for all $A \in \Omega$.

A function f with the above properties is a *free rationalizer* for c, and the cardinality of its image is its *rank*. An FMR choice correspondence c is *freely p-rationalizable* if there exists a free rationalizer for c having rank p. Further, the *free rationalizability number* $\mathsf{rat}^{\to}_{\mathrm{free}}(c)$ of c is the least positive integer p such that c is freely p-rationalizable, that is,

$$\mathsf{rat}^{\to}_{\mathrm{free}}(c) \;:=\; \min\left\{p \in \mathbb{N} : c \text{ has a free rationalizer with rank } p\right\}.$$

Next, we define an order-preserving map $\mathsf{rat}^{\to}_{\mathrm{free}}\colon \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\}$ as follows for each $n \geq 1$:

46 In order to avoid dealing with the theory of infinite cardinals, we limit our analysis to case (1a), that is, we assume that the ground set X is finite.

$$\mathsf{rat}^{\rightarrow}_{\text{free}}(n) \; := \; \min \left\{ p \in \mathbb{N} : (\forall c \in \mathsf{Choice}(n)) \; \mathsf{rat}^{\rightarrow}_{\text{free}}(c) \leq p \right\}$$

where $\mathsf{Choice}(n)$ denotes the set of all choice correspondences on a set of size n. Finally, denoting by $\mathsf{Choice}^{-}(n)$ the set of all choice functions (i.e., single valued) on a set of size n, we define a map $\mathsf{rat}^{-}_{\text{free}} \colon \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ as follows for each $n \geq 1$:

$$\mathsf{rat}^{-}_{\text{free}}(n) \; := \; \min \left\{ p \in \mathbb{N} : (\forall c \in \mathsf{Choice}^{-}(n)) \; \mathsf{rat}^{\rightarrow}_{\text{free}}(c) \leq p \right\}.$$

(Obviously, the inequality $\mathsf{rat}^{-}_{\text{free}}(n) \leq \mathsf{rat}^{\rightarrow}_{\text{free}}(n)$ holds for all integers $n \geq 1$.)

The definition of a freely multi-rationalizable choice correspondence calls for the existence of a family of *minimally consistent* binary rationales, which *globally* justifies the selection process by *locally* using the classical maximization paradigm. Notice that a free rationalizer provides each feasible menu with its own acyclic justification, which is in general independent of the rationales associated to the other menus: in this sense we use the adjective “free”.

Intuitively, $\mathsf{rat}^{\rightarrow}_{\text{free}}(c)$ says how many “rational states of mind” are needed to explain the observed choice behavior c of an economic agent. Consequently, $\mathsf{rat}^{\rightarrow}_{\text{free}}(n)$ can be thought as a discrete measure of how irrational an arbitrary choice on an n-element set may be, and the lower bounds exhibit the “least rational” choice behaviors. It is apparent that each choice correspondence $c \colon \varOmega \to \varOmega$ is always FMR, and the upper bound $\mathsf{rat}^{\rightarrow}_{\text{free}}(c) \leq |\varOmega|$ holds.

Example 4.28 Consider the following total choice correspondences on $X = \{x, y, z\}$:

$$\begin{array}{ll} (c_1) & \underline{x}\,y,\;\; \underline{x}\,z,\;\; \underline{y}\,\underline{z},\;\; \underline{x}\,y\,z \\ (c_2) & \underline{x}\,\underline{y},\;\; \underline{x}\,z,\;\; \underline{y}\,\underline{z},\;\; \underline{x}\,y\,z \\ (c_3) & \underline{x}\,y,\;\; x\,\underline{z},\;\; \underline{y}\,z,\;\; \underline{x}\,\underline{y}\,\underline{z}. \end{array}$$

Then, we have $\mathsf{rat}^{\rightarrow}_{\text{free}}(c_i) = i$ for $i = 1, 2, 3$.

The estimation of the free rationalizability number is not difficult for total choice functions, i.e., in the case examined by Kalai et al. (2002). A first interesting fact is that in this case it is immaterial whether we take $\mathsf{Pref}_{\text{lin}}(X)$ or $\mathsf{Pref}_{\text{ac}}(X)$ as family of rationales. Indeed, denoted by $\mathsf{rat}_{\mathsf{RMR}}(c)$ the free rationalizability number by means of linear orders on X, we have (Cantone et al. 2018c):

Lemma 4.29 *$\mathsf{rat}^{\rightarrow}_{\text{free}}(c) = \mathsf{rat}_{\mathsf{RMR}}(c)$ for each total choice function c.*

In view of Lemma 4.29, the main results of Kalai et al. (2002) can be restated as follows:

Theorem 4.30 (Kalai et al. 2002) *The equality $\mathsf{rat}^{-}_{\text{free}}(n) = n - 1$ holds for each $n \geq 1$. Further, the fraction of total choice functions on X having the maximum free rationalizability number tends to 1 as the size n of X goes to infinity.*

The findings of Theorem 4.30 are appealing, albeit somehow expected: the larger the size of the ground set, the higher the percentage of chaotic choice *functions* (where "chaotic" means that they require the maximum number of rationales). Regrettably, the situation becomes far more complicated for choice *correspondences*. A first, simple result on $\mathsf{rat}^{\rightarrow}_{\text{free}}(n)$ is the following (Cantone et al. 2018c):

Proposition 4.31 *$\mathsf{rat}^{\rightarrow}_{\text{free}}(n) \leq 2^{n-1}$ for each $n \geq 1$.*

The upper bound given by Proposition 4.31 already fails to be tight for $n = 3$, because $\mathsf{rat}^{\rightarrow}_{\text{free}}(3) = 3$. In fact, a (rather complicated) combinatorial analysis of $\mathsf{rat}^{\rightarrow}_{\text{free}}(n)$ suggests that finding better bounds is highly nontrivial (Cantone et al. 2018c). An even more difficult problem is the following:

Problem 4.32 Determine $\mathsf{rat}^{\rightarrow}_{\text{free}}(n)$ for each $n \geq 1$.

Going back to RMR, Kalai et al. (2002, p. 2487) conclude their contribution by explicitly recognizing that a serious issue of their approach is given by feature (7a), i.e., a total absence of interactions among rationales[47]:

> We fully acknowledge the crudeness of this approach. The appeal of the RMR proposed for "Luce and Raiffa's dinner" does not emanate only from its small number of orderings, but also from the simplicity of describing in which cases each of them is applied. … More research is needed to define and investigate "structured" forms of rationalization.

Along the path suggested by them, we now sketch a "structured" type of simultaneous multi-rationalization, called *monotonic*: the reader is referred to Cantone et al. (2018c) for other types of structured multi-rationalizations (*strongly coherent*, *listable*, etc.).

Definition 4.33 A choice correspondence $c \colon \Omega \to \Omega$ is *monotonically multi-rationalizable* (*MMR*) if there exists a free rationalizer $f \colon \Omega \to \mathsf{Pref}_{\text{ac}}(X)$ satisfying the following additional property:

(Monotonic Coherence) for all $A, B \in \Omega$, if $A \subseteq B$, then $f(A) \supseteq f(B)$.

Such a function f is a *monotonic rationalizer* for c. A choice correspondence is *monotonically p-rationalizable* if there exists a monotonic rationalizer for c having rank p. If a choice correspondence c is MMR, then its *monotonic rationalizability number* $\mathsf{rat}^{\rightarrow}_{\text{mon}}(c)$ is defined as the least positive integer p such that c is monotonically p-rationalizable, that is,

$$\mathsf{rat}^{\rightarrow}_{\text{mon}}(c) \;:=\; \min\{p \in \mathbb{N} : c \text{ has a monotonic rationalizer with rank } p\}.$$

On the other hand, if c has no monotonic rationalizer, then we set by definition $\mathsf{rat}^{\rightarrow}_{\text{mon}}(c) := \infty$.

[47] "Luce and Raiffa's dinner" mentioned in the quotation below will be described in Example 4.35.

Monotonic Coherence requires that an expansion of a menu induces a contraction of the associated preference; said differently, whenever going to larger menus, the local rationale might be less informative. This possible loss of information is due to the natural difficulty of a human mind to make comparisons among a large number of items.[48] For instance, in the process of selecting within larger and larger menus, the economic agent may start dropping some relationships not so well established in her mind, transforming an indifference between two items into an incomparability. This approach finds its theoretical justification in well established theories in psychology, which advocate the use of bounded rationality heuristics and simplified strategies in making judgements, especially in complicated settings: see the classical work by Tversky (1972), Tversky and Kahneman (1974), and Kahneman et al. (1982).

Notice that, in the simplest case, the notion of monotonic multi-rationalizability generalizes the classical notion of rationalizability. In fact, for any choice correspondence $c\colon \Omega \to \Omega$,

$$c \text{ rationalizable } \iff c \text{ monotonically 1-rationalizable } \iff \mathsf{rat}^{\rightarrow}_{\text{free}}(c) = \mathsf{rat}^{\rightarrow}_{\text{mon}}(c) = 1.$$

The next example provides some simple instances of choices on a 3-element ground set, which are monotonically rationalizable or fail to be so.

Example 4.34 Consider the following total choice correspondences on $X = \{x, y, z\}$:

$$\begin{array}{ll}
(c_1) & \underline{x}\,y,\ \ \underline{x}\,z,\ \ \underline{y}\,\underline{z},\ \ \underline{x}\,y\,z \\
(c_2) & \underline{x}\,y,\ \ \underline{x}\,z,\ \ \underline{y}\,\underline{z},\ \ \underline{x}\,\underline{y}\,z \\
(c_3) & \underline{x}\,y,\ \ x\,\underline{z},\ \ \underline{y}\,z,\ \ \underline{x}\,\underline{y}\,\underline{z} \\
(c_\infty) & \underline{x}\,y,\ \ \underline{x}\,z,\ \ \underline{y}\,\underline{z},\ \ x\,\underline{y}\,\underline{z} \\
(c'_\infty) & \underline{x}\,\underline{y},\ \ \underline{x}\,z,\ \ \underline{y}\,z,\ \ x\,y\,\underline{z}.
\end{array}$$

Then, we have $\mathsf{rat}^{\rightarrow}_{\text{mon}}(c_i) = i$ for $i = 1, 2, 3$, and $\mathsf{rat}^{\rightarrow}_{\text{mon}}(c_\infty) = \mathsf{rat}^{\rightarrow}_{\text{mon}}(c'_\infty) = \infty$. It is worth examining the (abstraction of the) pathologies displayed by the choices c_∞ and c'_∞, which prevent them from being MMR: see Cantone et al. (2018c).

To provide a motivation for a monotonic approach, next we examine a well known instance of choice reversal, described by Luce and Raiffa (1957). In this example, the consumer exhibits a switch in her (single valued) selection process whenever going from a menu to a larger one. This choice reversal phenomenon –which might look somehow unexpected, but is motivated by the so-called *epistemic value* of the menu described by Sen (1993)—causes the choice function to be non-rationalizable by a single binary relation. Kalai et al. (2002) provide a "non-structured" RMR justification of this choice behavior; below we also exhibit a "structured" justification, which may give further insight in the selection process.

[48] It is worth noticing that this is exactly the same underlying philosophy which has inspired those MCDA methodologies that limit comparisons to few points of view a time: the prototype of such an approach is the *Pairwise Criterion Comparison Approach* (PCCA), originally developed by Matarazzo (1986, 1988a, b, 1990a, b, 1991a, b) and later on by his followers Angilella and Giarlotta (2009); Angilella et al. (2010a), Giarlotta (1998, 2001), Greco (1997, 2005).

Example 4.35 (*Luce and Raiffa's dinner*) A diner chooses a main course from a restaurant's menu. If the menu consists of chicken (x) and steak (y) only, then she chooses chicken. On the other hand, if the menu consists of chicken, steak, and frog's legs (z), then she selects steak. Thus, the (partial) selection process is modeled by any choice c on $X = \{x, y, z\}$ such that $\underline{x}y$ and $x\underline{y}z$. The classical theory of revealed preferences cannot formally explain the diner's behavior, since the (partial) choice function c fails to be rationalizable. On the other hand, this choice behavior can be formally justified by employing a multi-rationalization model instead. For instance, after suitably extending the definition of c to a total choice function $\widehat{c}\colon 2^X \to 2^X$, Kalai et al. (2002) exhibit a free 2-rationalization of $\widehat{c}$ by linear orders. However, this multi-rationalization does not provide any links between the two rationales, and so the diner's choice reversal phenomenon remains somehow unexplained by it.

A monotonic multi-rationalization may provide a sound explanation of this behavior. Let $\succsim_1$ and $\succsim_2$ be the preferences on X defined by $\succsim_1 := X^2 \setminus \{(y, x)\}$ and $\succsim_2 := \{(y, z), (z, x)\} \cup \Delta(X)$. Then, the map $f\colon \Omega \to \mathsf{Pref}_{\mathrm{ac}}(X)$, defined by $f(\{x, y\}) := \succsim_1$ and $f(X) := \succsim_2$, is a monotonic rationalizer of c having rank two (hence $\mathsf{rat}^{\rightarrow}_{\mathrm{mon}}(c) = 2$). The two rationales $\succsim_1$ and $\succsim_2$ suggest an interesting interpretation of the choice reversal phenomenon. In fact, $\succsim_1$ can be seen as a *direct* rationale, in the sense that the strict preference $x \succ_1 y$ locally motivates the selection of chicken over steak whenever they are the only available main courses. On the other hand, $\succsim_2$ is an *indirect* rationale, in the sense that the added item z induces a sequence of strict preferences $y \succ_2 z \succ_2 x$, which in turn motivates the selection of steak from the expanded menu. Said differently, the selection of steak from the full menu "factors through" the new item (frog's legs), since the latter provides an indirect rationale for the observed switch: for example, the presence of frog's legs in the menu gives information about the (good) quality of the chef, or it simply grosses the diner out and induces her to avoid chicken as well.

We conclude this example by observing that there are completions of c that are MMR, and others that fail to be so. For instance, the choice correspondence $\widetilde{c}$, which extends c by defining $x\underline{z}$ and $\underline{y}z$, is MMR with $\mathsf{rat}^{\rightarrow}_{\mathrm{mon}}(\widetilde{c}) = 2$. On the contrary, any extension of c such that $y\underline{z}$ (i.e., frog legs are always selected versus steak) fails to be MMR.[49]

The next definition, which refines the notion of monotonic multi-rationalizability, provides a surprising link with the classical theory of revealed preferences.

Definition 4.36 A choice correspondence $c\colon \Omega \to \Omega$ is *elementarily MMR* (*eMMR*) if it has a monotonic rationalizer $f\colon \Omega \to \mathsf{Pref}_{\mathrm{ac}}(X)$ such that all restrictions $f(A) \upharpoonright A$ are antisymmetric.

The following characterization of eMMR choices is noteworthy (Cantone et al. 2018c):

[49]This issue is connected to the problem of the *lifting* of choices having certain properties: see Cantone et al. (2017) for the general formulation of the problem and its logic-theoretic analysis in some special cases.

Theorem 4.37 *A total choice correspondence is* eMMR *if and only if it satisfies* (γ).

The above result has some interesting consequences, which shed further light on the role of two standard axioms of choice consistency: Chernoff's contraction property (α), and Sen's expansion property (γ). To clarify these links, we need the following notion:

Definition 4.38 A choice correspondence is *elementary* if it is single valued on all doubletons.

Then, we have (Cantone et al. 2018c):

Corollary 4.39 *For an elementary total choice correspondence c, we have:*

(i) c is rationalizable by a total preorder if and only if (α) *holds;*
(ii) c is MMR *if and only if it is* eMMR *if and only if* (γ) *holds.*

In particular, we obtain a characterization of single valued MMR total choices:

Corollary 4.40 *A total choice function is* MMR *if and only if axiom* (γ) *holds.*

Corollary 4.40 is interesting in two different ways. First, it characterizes a class of total choice functions, which are rationalizable is a wider sense than that prompted by the theory of revealed preferences. Second, possibly more important, it sheds some light on the intrinsic semantics of the standard axiom (γ) of expansion consistency, which is almost invariantly considered in association to its dual counterpart, the standard axiom (α) of contraction consistency.

4.4.2 Sequential Multi-rationalization

We now switch to scenario (8b), that is, *sequential* types of multi-rationalizations.[50] In this stream of research, the literature is quite abundant: see, among many others, Apesteguía and Ballester (2013) for choices by sequential procedures, Manzini and Mariotti (2007), Au and Kawai (2011), and Garcia and Alcantud (2015) for sequentially rationalizable choices, Lleras et al. (2017) for consideration filters, Masatlioglu et al. (2012) for attention filters, Masatlioglu and Nakajiama (2013) for choices by iterative search, Manzini and Mariotti (2012) for choices by lexicographic semiorders, Rubinstein and Salant (2006) for choices from lists, and Tyson (2013) for general shortlisting procedures. To keep exposition compact, we shall limit our analysis to the two models proposed by Rubinstein and Salant (2006), and by Manzini and Mariotti (2007).

[50]Somehow in between approaches (8a) and (8b) lies the very interesting model constructed by Cherepanov et al. (2013), who provide a general framework for a formal and testable theory of rationalization, in which a decision maker selects her preferred alternative from among those that she can rationalize.

In their model, Rubinstein and Salant (2006) study choice functions from lists, where a *list* is a sequence of distinct elements of a finite set. A choice function from lists singles out one element from every list. They show that a certain class of choice functions from lists can be characterized by two equivalent properties: *Partition Independence* (PI') and List Independence of Irrelevant Alternatives (LIIA). Property PI' extends to lists the classical condition of *Path Independence* (PI) due to Plott (1973): it requires that arbitrarily dividing a list into several sublists, choosing from each sublist, and finally choosing from the list of chosen elements yields the same result as choosing from the original list. Property LIIA states that omitting unchosen elements from a list does not alter the choice. The class $\mathsf{Choice}^{\mathrm{RS}}_{\mathrm{seq}}(X)$ of choice functions from lists in X characterized by the satisfaction of PI' (equivalently, LIIA) is an enlargement of the class of rationalizable choice functions on X in the sense of revealed preference theory. In fact, each function in $\mathsf{Choice}^{\mathrm{RS}}_{\mathrm{seq}}(X)$ is parameterized by a preference relation $\succsim$ over X and a labelling of every $\sim$-indifference set by "First" or "Last". Then, given a list, each such function identifies the set of $\succsim$-maximal elements within the list, and chooses the first or the last element among them according to the label of the $\sim$-indifference set they belong to.

Rubinstein and Salant successively extend their approach to cases in which the order of the elements in the list is not directly observable, e.g., when the list is virtual. Under such circumstances, they analyze choice *correspondences* attaching to every set of alternatives all the elements that are chosen for some ordering of that set. They prove the following interesting two facts: (1) choice functions from lists that satisfy PI' induce choice correspondences that satisfy WARP, and, conversely, (2) a choice correspondence satisfying WARP can be "explained" by a choice function from lists that satisfies PI'. Thus, their results provide a new interpretation of the notion of choice correspondence.

Finally, Rubinstein and Salant consider situations in which the decision maker deterministically chooses from lists generated from sets by a random process. A random choice function assigns to every set of alternatives a probability measure over the set, where the probability of an element is the likelihood that it will be chosen from the set. Then they show that a choice function from lists satisfies PI' if and only if the induced random choice function is monotone (in the sense that the probability of choosing an element from a set weakly increases as the set of available items shrinks).

Manzini and Mariotti (2007) study choice *functions* that can be justified by maximizing more than one preference relation in a given order. In the simplest case of two rationales, they call this procedure "rational shortlist method". The employed terminology is suggestive of the procedure: intuitively, the first rationale identifies a shortlist of candidate alternatives from which the second rationale selects a unique element. Below we recall the notion of 2-sequential rationalizability in the general case of a total choice *correspondence* (2015).

Definition 4.41 A choice correspondence $c\colon 2^X \to 2^X$ is 2-*sequentially rationalizable* (also called a *rational shortlist method*, RSM) if there is an ordered pair $(\succ_1, \succ_2)$

of asymmetric relations on X such that $c(A) = \max(\max(A, \succ_1), \succ_2)$ for all menus $A \in 2^X$.

A 2-sequentially rationalizable choice correspondence describes a decision procedure of an agent, who goes through two sequential rounds of elimination to select the alternatives: in the first round, she only retains those items that are maximal according to $\succ_1$; in the second, she only keeps the items that are maximal according to $\succ_2$. RSMs naturally apply in several practical situations: for instance, in a portfolio selection, a cautious investor can first eliminate all risky alternatives, and then select the one(s) that give the maximum expected return. RSMs also fit the scenario of sequential "noncompensatory" heuristics widely studied in the psychology literature (see, e.g., Tversky's (1972) "elimination by aspects" procedure), as well as that of consumers' "two-stage consideration and choice" decision procedures in the management literature (Yee et al. 2007).

A crucial feature of the rational shortlist method is that the order of application of the two rationales is *fixed* for all menus. In their Theorem 1, Manzini and Mariotti (2007) characterize the 2-sequential rationalizability of choice *functions* by the satisfaction of two testable properties of choice consistency, namely, standard expansion (γ) and a weak form of WARP. More recently, García-Sanz and Alcantud (2015) obtain a partial characterization of the 2-sequential rationalizability of choice *correspondences*, which holds under a mild condition, called *Choice Without Dominated Elements* (CWDE). In this respect, it is worth noticing that the process of resolution described in Sect. 3.5 does preserve the 2-sequential rationalizability of choice correspondences under condition CWDE (2018a).

Manzini and Mariotti (2007) also consider the natural generalization of the rational shortlist method, which is a rationalization procedure for choice functions by means of more than two sequential criteria. (Again, we describe the notion in the general case of a choice correspondence.)

Definition 4.42 For each integer $n \geq 2$, a choice correspondence $c\colon 2^X \to 2^X$ is *n-sequentially rationalizable* whenever there is an ordered n-tuple $(\succ_1, \succ_2, \ldots, \succ_n)$ of asymmetric relations on X such that, for all menus $A \in 2^X$, if the sets M_i, $i = 0, 1, \ldots, n$, are recursively defined by

- $M_0(A) := A$, and
- $M_i(A) := \max(M_{i-1}(A, \succ_i))$ for $i = 1, \ldots, n$,

then $c(A) = M_n(A)$ (that is, $c(A) = \max(\max(\ldots(\max(A, \succ_1), \succ_2), \ldots), \succ_n))$. A choice correspondence is *sequentially rationalizable* if it is n-sequentially rationalizable for some $n \geq 2$.

Using the same terminology as for simultaneous multi-rationalizability, we introduce the following natural notion:

Definition 4.43 The *sequential multi-rationalizability number* of a choice correspondence c, denoted by $\mathsf{rat}^{\uparrow}(c)$, as the least integer p such that c is sequentially p-rationalizable if there is one, and ∞ otherwise.

Notice that there are choice functions c that fail to be sequentially rationalizable, that is, $\mathsf{rat}^{\uparrow}(c) = \infty$. Manzini and Mariotti (2007) obtain a partial characterizations of sequentially rationalizable choice functions, and, by means of a recursion lemma, they also express the 3-sequential rationalizability of a choice function in terms of the existence of a suitable choice correspondence. On the other hand, a full characterization of the 3-sequential rationalizability of a choice function—let aside that of a choice correspondence—is still unknown. More generally, the following problem appears to be highly nontrivial:

Problem 4.44 Characterize sequentially rationalizable choice functions and correspondences.[51]

Further, similarly to the case of simultaneous multi-rationalizability, it appears of some interest to obtain estimates of the values of $\mathsf{rat}^{\uparrow}(c)$ for choice functions/correspondences that are sequentially rationalizable.

4.5 Multiple, Iterated, and Hierarchical Resolutions of Choices

In this final section, we suggest how to generalize the notion of choice resolution introduced in Sect. 3.5 by using multiple and iterated resolutions first, and then combining them into hierarchical resolutions. Technical details are barely sketched, because a deeper analysis would be lengthy and complicated. Recall that the process of resolution of a base choice space (X, c_X) at a single point $x \in X$ into a single fibre choice space (Y_x, c_{Y_x}) operates as follows:

(1) view the item x as a menu, which (potentially) opens up at Y_x;
(2) to make a choice in the resolved space $(Z, c_Z) = (X, c_X) \otimes_x (Y_x, c_{Y_x})$, first select from X, where one of the choices is the (closed) menu x:

  (2.a) if x is not picked up, then leave the point-menu Y_x closed;
  (2.b) if x is picked up, then open the point-menu Y_x, and make there choices as well.

The natural directions in which the notion of resolution can be generalized are apparent:

- **multiple (horizontal) resolution**, obtained by simultaneously resolving the points of a nonempty set $X' \subseteq X$ into fibre choice spaces;

[51] A very surprising answer to a closely related question is given by Mandler, Manzini, and Mariotti (2012). In their paper, the authors show that “fast and frugal” sequential procedures are *not* incompatible with utility maximization. In fact, two rather unexpected facts hold: (1) any agent who uses the benchmark model of quickly-executing checklists always has a utility function, and (2) any utility maximizer can make decisions with a quickly-executing checklist (under suitable conditions on the domain). In Mandler et al.’s (2012) words: “*Checklists are a fast and frugal way to maximize utility.*”

- **iterated (vertical) resolution**, obtained by sequentially resolving an ordered list of points into fibre choice spaces, where each point in the list belongs to the fibre choice in which the preceding point has been resolved;
- **hierarchical (tree) resolution**, obtained as an arbitrary combination of horizontal and vertical resolutions, thus expanding a choice space into a tree-structured choice space.

To give an idea of the technicalities involved in a multiple and/or iterated resolution, below we examine the two simplest cases: (1) a horizontal resolution at two base points, and (2) a vertical resolution at two base points. For the sake of readability, we simplify the convoluted notation by dropping some subscripts. We start with case (1).

Definition 4.45 Let (X, c), (Y_1, c_1), (Y_2, c_2) be three complete choice spaces on disjoint ground sets X, Y_1, Y_2, and let x_1, x_2 be two distinct points of X. Set $Z := (X \cup Y_1 \cup Y_2) \setminus \{x_1, x_2\}$. Define a surjective map $\pi\colon Z \to X$ by

$$\pi(z) := \begin{cases} z & \text{if } z \in X \setminus \{x_1, x_2\} \\ x_1 & \text{if } z \in Y_1 \\ x_2 & \text{if } z \in Y_2. \end{cases}$$

The *multiple (horizontal) resolution of* (X, c) *at* (x_1, x_2) *into* $\big((Y_1, c_1), (Y_2, c_2)\big)$, denoted by

$$(Z, c_Z) = (X, c) \otimes^{\rightarrow}_{x_1, x_2} \big\langle (Y_1, c_1), (Y_2, c_2) \big\rangle,$$

is the complete choice space on Z whose choice correspondence $c_Z\colon 2^Z \to 2^Z$ is defined by

$$c_Z(A) := \begin{cases} c(\pi(A)) \cup c_1(A \cap Y_1) \cup c_2(A \cap Y_2) \setminus \{x_1, x_2\} & \text{if } x_1, x_2 \in c(\pi(A)) \\ c(\pi(A)) \cup c_1(A \cap Y_1) \setminus \{x_1\} & \text{if } x_1 \in c(\pi(A)) \text{ and } x_2 \notin c(\pi(A)) \\ c(\pi(A)) \cup c_2(A \cap Y_2) \setminus \{x_2\} & \text{if } x_1 \notin c(\pi(A)) \text{ and } x_2 \in c(\pi(A)) \\ c(\pi(A)) & \text{otherwise.} \end{cases}$$

(X, c) is the *base choice space*, x_1, x_2 are the two *(simultaneous) base points*, (Y_1, c_1), (Y_2, c_2) are the two *(simultaneous) fibre choices*, and π is the *(simultaneous) projection*.

The extension of the notion of multiple resolution to the general case, where several (even all) points of the base space are simultaneously resolved into fibre choices, is straightforward.

Next, we examine case (2), that is, iterated resolutions at two points in a sequence.

Definition 4.46 Let (X, c), (Y_1, c_1), (Y_{11}, c_{11}) be three complete choice spaces on disjoint ground sets X, Y_1, Y_{11}. Select $x \in X$ and $y_1 \in Y_1$. Set

$$Z := (X \cup Y_1 \cup Y_{11}) \setminus \{x, y_1\} \qquad \text{and} \qquad Z_1 := (Y_1 \cup Y_{11}) \setminus \{y_1\}.$$

Further, define two surjective maps $\pi \colon Z \to X$ and $\pi_1 \colon Z_1 \to Y_1$ by

$$\pi(z) := \begin{cases} z & \text{if } z \in X \setminus \{x\} \\ x & \text{otherwise} \end{cases} \qquad \text{and} \qquad \pi_1(z_1) := \begin{cases} z_1 & \text{if } z_1 \in Y_1 \setminus \{y_1\} \\ y_1 & \text{otherwise.} \end{cases}$$

The *iterated (vertical) resolution of* (X, c) *at* (x, y_1) *into* $\big((Y_1, c_1), (Y_{11}, c_{11})\big)$, *denoted by*

$$(Z, c_Z) = (X, c) \otimes^{\uparrow}_{x, y_1} \big\langle (Y_1, c_1), (Y_{11}, c_{11}) \big\rangle,$$

is the complete choice space on Z whose choice correspondence $c_Z \colon 2^Z \to 2^Z$ is defined by

$$c_Z(A) := \begin{cases} c(\pi(A)) \cup c_1(\pi_1(A \cap Y_1)) \cup c_{11}(A \cap Y_{11}) \setminus \{x, y_1\} & \text{if } x \in c(\pi(A)) \text{ and } y_1 \in c_1(\pi_1(A \cap Y_1)) \\ c(\pi(A)) \cup c_1(A \cap Y_1) \setminus \{x_1\} & \text{if } x \in c(\pi(A)) \text{ and } y_1 \notin c_1(\pi_1(A \cap Y_1)) \\ c(\pi(A)) & \text{otherwise.} \end{cases}$$

(X, c) is the *base choice space*, x, y_1 are the two *(sequential) base points*, (Y_1, c_1), (Y_{11}, c_{11}) are the two *(sequential) fibre choices*, and π, π_1 are the two *(sequential) projections*.

The extension of the notion of multiple resolution to the general case, where the vertical resolution keeps going up to a certain height, is not conceptually difficult but technically complicated.

The most general form of resolution is the hierarchical resolution, which is obtained by resolving a base choice space by means of a *rooted tree* (a connected acyclic graph with a distinguished node, called root). The formal description is technically complicated, so we avoid presenting it here. However, to give an idea of how they work, Fig. 5 provides a graphical representation of a simple case of hierarchical resolution, which has width 4 and height 3 (width and height are defined as for trees). The employed notation—which, however, can be simplified—is supposed to suggest how a hierarchical resolution is formally defined.

Possible applications of hierarchical resolutions of choices are apparent for, e.g., corporate structures, investment portfolios, etc. For instance, imagine the case of a large multi-national company, whose very articulated organization suggests the CEO to fully delegate decision authority to either national branches (with their own hierarchical structure) or transversal departments (with their own hierarchical structure). Then the possibility to detect the inner structure of the corporation by just observing its choice behavior on projects may have a high strategic impact in the decision making process of its competitors.

To conclude, we connect hierarchical processes to similar approaches in MCDA. Corrente et al. study multiple criteria hierarchy processes within the ROR approach in Corrente et al. (2012), and within ELECTRE and PROMETHEE methodologies in Corrente et al. (2013). More recently, Angilella et al. (2016) consider ROR and SMAA (Stochastic Multiobjective Acceptability Analysis) in a multiple criteria hierarchy process for the Choquet integral preference model, whereas Cor-

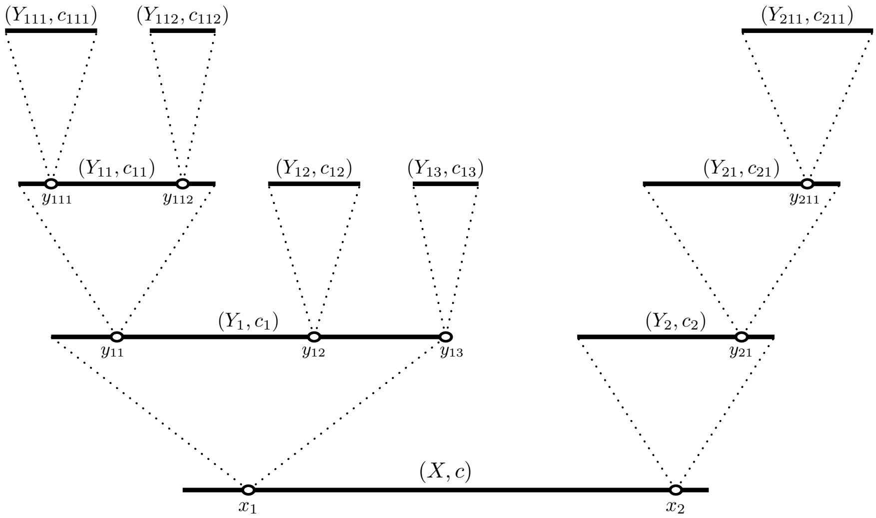

Fig. 5 A hierarchical resolution of width 4 and height 3

rente et al. (2016) examine a multiple criteria process for ELECTRE Tri methods. Further, Angilella et al. (2018) evaluate sustainable development by means of composite indices using the hierarchical-SMAA-Choquet integral approach. Additional contributions in this field take into account hierarchical structures: see Fujimoto et al. (1998) for a theoretical analysis of the hierarchical decomposition of the Choquet integral, Del Vasto-Terrientes et al. (2015) for an outranking-based methodology with a hierarchy of criteria, and Del Vasto-Terrientes et al. (2016) for a hierarchical multi-criteria sorting approach.

5 Conclusion

In this survey we have discussed some recent approaches to the theories of preference modeling, utility representation, and choice rationalization. These approaches are inspired by a multiple criteria philosophy, since they take into account several "points of view"—preference relations, utility functions, or binary rationales for choices—to explain an agent's behavior. We hope to have provided the reader with enough convincing arguments on the naturalness and feasibility of a multi-approach to these theories, and its advantages over the classical mono-approach.

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Appendix

This section contains two figures, which summarize some results of this survey. Figure 6 describes all implications between combinations of weak (m, n)-Ferrers properties. For instance, the arrow from the box (3, 2) (i.e., strong interval orders) to the the box (3, 1) and (2, 2) (i.e., semiorders) says that any strong interval order is a semiorder, but the vice versa is false in general. Notice that the very last segment of

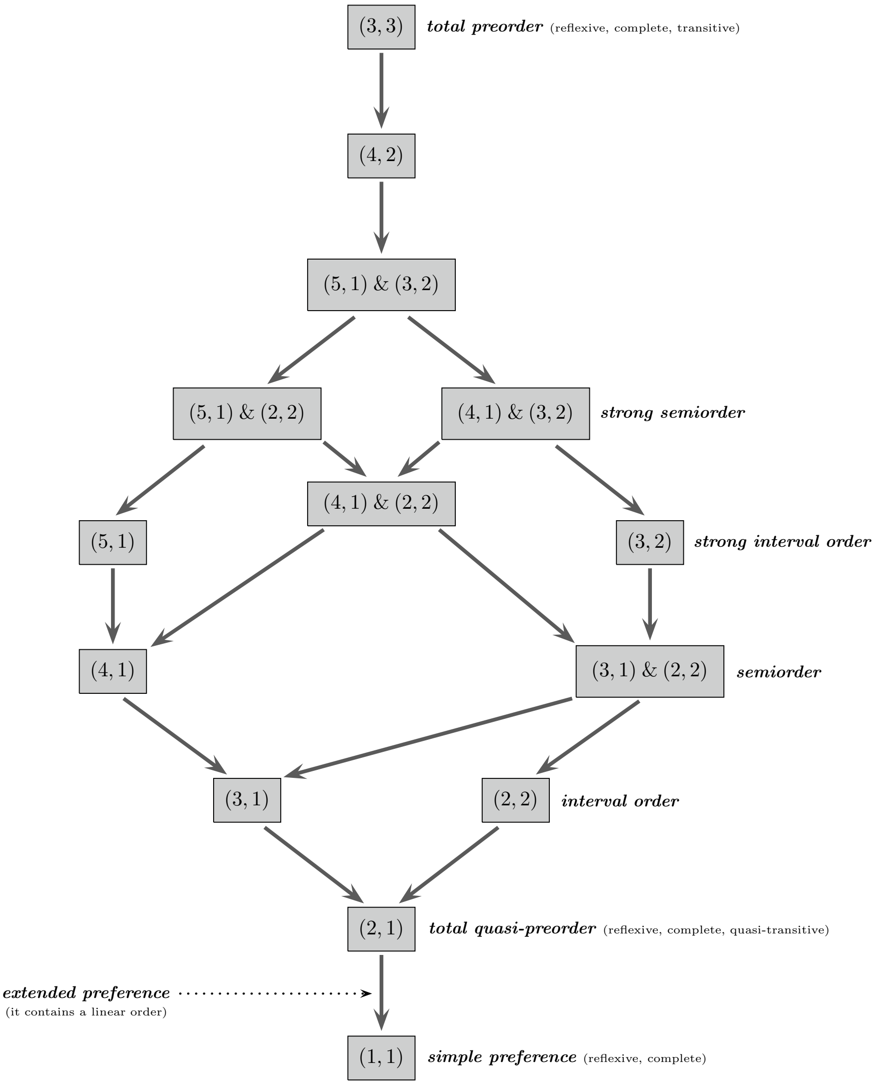


Fig. 6 Implications among combinations of weak (m, n)-Ferrers properties

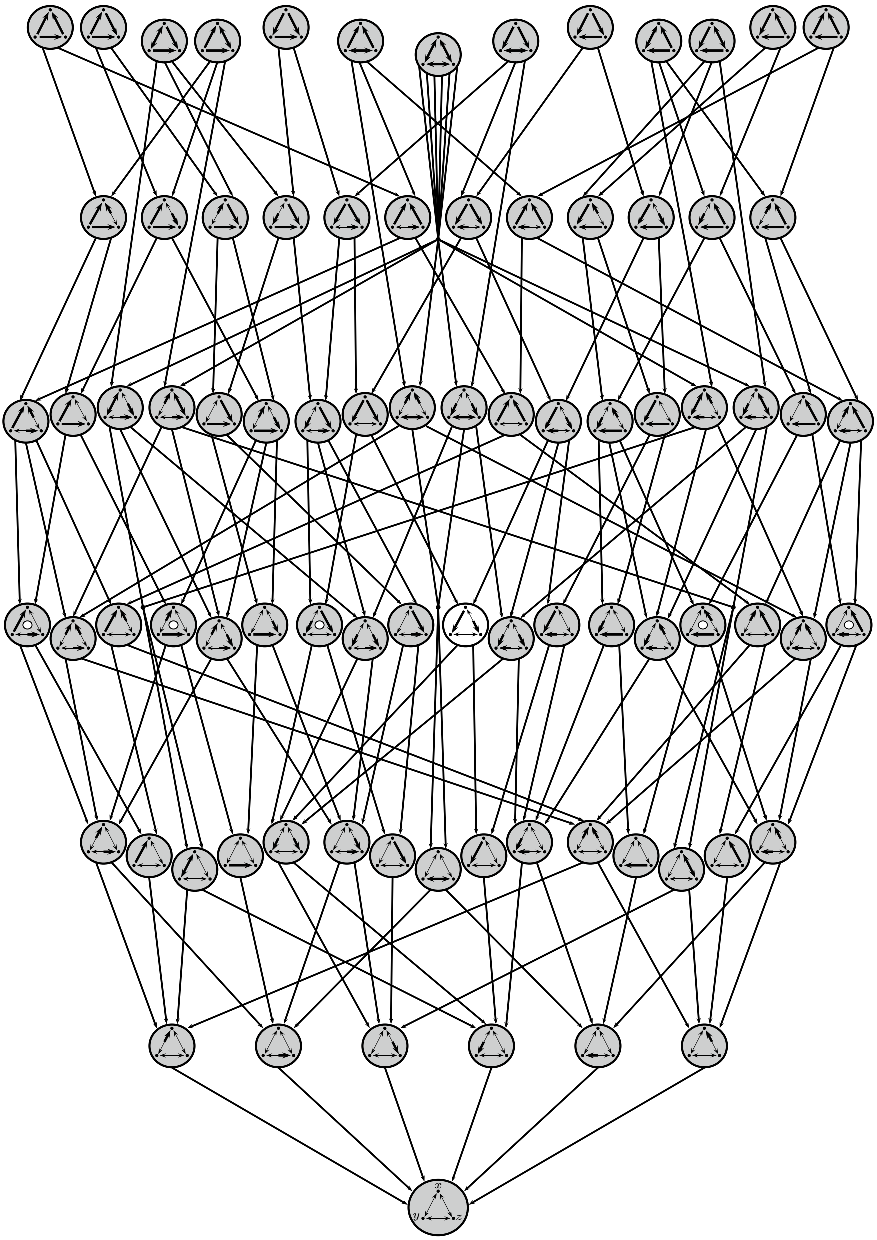
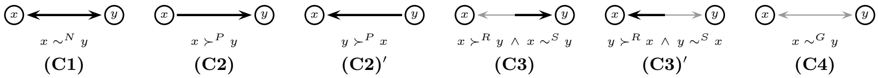


Fig. 7 The meet-semilattice of the NaP-preferences on the set $X = \{x, y, z\}$

the picture—that is, going from a total quasi-preorder to a simple preference—can be refined into an infinite hierarchy by using strict $(m, 1)$-Ferrers properties.

Figure 7 exhibits the meet-semilattice of all NaP-preferences on $X = \{x, y, z\}$. For compactness, we simplify the notation in Fig. 4 to identify the comonotonic configurations (C1), (C2), (C2)′, (C3), (C3)′, and (C4) as follows (configuration (C5) never appears for NaP-preferences):



Observe that many configurations in Fig. 7 are isomorphic to each other (where an isomorphism between bi-preferences is defined in the obvious way). For instance, at level 3 of the meet-semilattice in Fig. 7 (the root is a level 0), the isomorphism class of the NaP-preference emphasized by a white background comprises six elements (the other five elements being identified by a white dot). A simple computation shows that the number of non-isomorphic NaP-preferences on a 3-element set is 20. The following combinatorial problem appears nontrivial:

Problem 5.1 For any integer $n \geq 3$, determine the number of pairwise non-isomorphic (either all or normalized) NaP-preferences on an n-element set.

Notice that, in the special case of NaP-preferences having a semiorder as a possible component, the above problem is related to a possible generalization of the *Catalan* number (Stanley 1999).

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Analytic Hierarchy Process and Its Extensions

Alessio Ishizaka

Abstract Analytic Hierarchy Process (AHP) is a popular and long used multi-criteria decision analysis method. Despite this fact, there are still space for new research in all its methodological steps. These include problem structuring, pairwise comparisons, priorities derivation, consistency and reduction techniques of pairwise comparisons. Moreover, future research agenda can also be found in the extensions of AHP: Analytic Network Process (for dealing with interactions) and AHPSort (for sorting problems). Finally, we discuss visualisation techniques for the Analytic Hierarchy Process.

1 Introduction

It is well-known today that the Analytic Hierarchy Process (AHP) is an extremely useful method. Several reviews have compiled their success stories (Zahedi 1986; Golden et al. 1989; Shim 1989; Vargas 1990; Saaty and Forman 1992; Forman and Gass 2001; Kumar and Vaidya 2006; Omkarprasad and Sushil 2006; Ho 2008; Liberatore and Nydick 2008; Sipahi and Timor 2010; Dung et al. 2016). The first full description of Analytic Hierarchy Process has been published in 1977 (Saaty 1977) and the vast majority of the applications still use AHP as described in this first publication. A full review of AHP can be found on (Ishizaka and Labib 2011a, b; Ishizaka and Nemery, 2013). This chapter reminds the main developments and then sketches the major directions in methodological developments and further research in this important field.

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2 Problem Modelling

As with all decision-making processes, the facilitator will sit a long time with the decision-maker(s) to structure the problem. AHP has the advantage of permitting a hierarchical structure of the criteria (Fig. 1), which provides users with a better focus on specific criteria and sub-criteria when allocating the weights. However, different structure may lead to a different final ranking (Barzilai 1998). Moreover, several authors (Stillwell et al. 1987; Weber et al. 1988; Pöyhönen et al. 1997) have observed that criteria with a large number of sub-criteria tend to receive more weight than when they are less detailed. As problem modelling set the scene and all the rest depends on it, it is a very important step. It is certainly a topic of research that needs to be deepened. In particular, problem structuring methods and soft systems could be used (Marttunen et al. 2017).

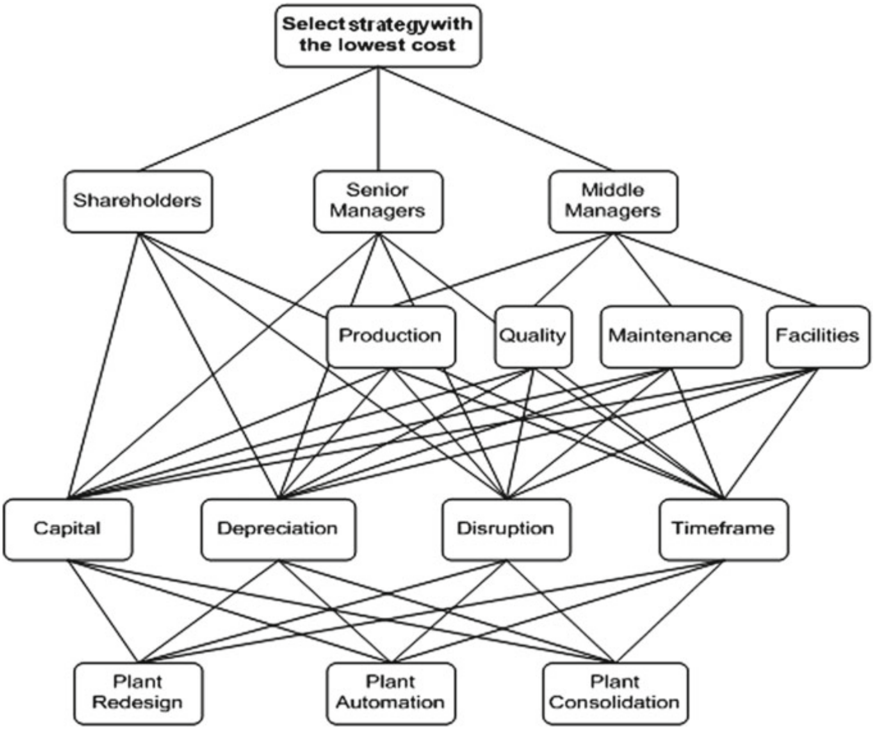

Fig. 1 Example of a hierarchy (Ishizaka and Labib 2011a, b)

3 Pair-Wise Comparisons

Psychologists argue that it is easier and more accurate to express one's opinion on only two alternatives than simultaneously on all the alternatives. It also allows consistency cross checking between the different pair-wise comparisons (see Sect. 5). AHP uses a ratio scale, which, contrary to methods using interval scales (Kainulainen et al. 2009), requires no units in the comparison as the judgement is a relative value or a quotient *a*/*b* of two quantities *a* and *b* having the same units (intensity, meters, utility, etc.). Comparisons are recorded in a positive reciprocal matrix (1).

$$\mathbf{A} = \begin{bmatrix} 1 & a_{12} & & a_{1n} \\ a_{21} & \ldots & a_{ij} & \ldots \\ \ldots\ a_{ji} & = 1/a_{ij} & \ldots & \ldots \\ a_{n1} & \ldots & \ldots & 1 \end{bmatrix} \tag{1}$$

where a_{ij} is the comparison between element *i* and *j*

One of AHP's strengths is the possibility to evaluate quantitative as well as qualitative criteria and alternatives on the same preference scale. These can be numerical, verbal (Table 1) or graphical (Fig. 3). The use of verbal responses is intuitively appealing, user-friendly and more common in our everyday lives than numbers. It may also allow some ambiguity in non-trivial comparisons.

To derive priorities, the verbal comparisons must be converted into numerical ones. In Saaty's AHP the verbal statements are converted into integers from one to nine. Theoretically there is no reason to be restricted to these numbers and verbal gradation. Although the verbal gradation has been little investigated, several other numerical scales have been proposed (Table 2).

Among all the proposed scales, the linear scale with the integers one to nine and their reciprocals has been used by far the most often in applications. The choice of the "best" scale is a very heated debate. Some scientists argue that the choice depends on

Table 1 The 1–9 fundamental scale

Intensity of importance	Definition
1	Equal importance
2	Weak
3	Moderate importance
4	Moderate plus
5	Strong importance
6	Strong plus
7	Very strong or demon-started importance
8	Very, very strong
9	Extreme importance

Table 2 Different scales for comparing two alternatives (for the comparison of A and B, c = 1 indicates A = B; c > 1 indicates A > B; when A < B, the reciprocal values 1/c are used)

Scale type	Definition	Parameters
Linear (Saaty 1977)	$c = a \cdot x$	$a > 0; x = \{1, 2, \ldots, 9\}$
Power (Harker and Vargas 1987)	$c = x^a$	$a > 1; x = \{1, 2, \ldots, 9\}$
Geometric (Lootsma 1989)	$c = a^{x-1}$	$a > 1; x = \{1, 2, \ldots, 9\}$ or $x = \{1, 1.5, \ldots, 4\}$ or other step
Logarithmic (Ishizaka et al. 2010)	$c = \log_a(x + (a - 1))$	$a > 1; x = \{1, 2, \ldots, 9\}$
Root square (Harker and Vargas 1987)	$c = \sqrt[a]{x}$	$a > 1; x = \{1, 2, \ldots, 9\}$
Asymptotical (Dodd and Donegan 1995)	$c = \tanh^{-1}\left(\frac{\sqrt{3}(x-1)}{14}\right)$	$x = \{1, 2, \ldots, 9\}$
Inverse linear (Ma and Zheng 1991)	$c = 9/(10 - x)$	$x = \{1, 2, \ldots, 9\}$
Balanced (Salo and Hamalainen 1997)	$c = w/(1 - w)$	$w = \{0.5, 0.55, 0.6, \ldots, 0.9\}$

the person and the decision problem (Harker and Vargas 1987; Pöyhönen et al. 1997). Recently Meesariganda and Ishizaka (2017) have selected the most appropriate scale by using first a problem where the ratio of the alternatives is known to see which scale fits the best; this may be the way forward.

4 Priorities Derivation

The goal is to find a set of priorities $p_1, \ldots, p_n$ such that p_i/p_j match the comparisons a_{ij} in a consistent matrix. Several methods have been developed and all satisfy this condition. The problem arises when the comparison matrix is slightly inconsistent. In the case of the introduction of small inconsistency, we can decently think that it induces only a small distortion. Based on this idea, Saaty (1977) uses the perturbation theory to justify the use of the principal eigenvector **p** as the desired priorities vector (2). He argues that slight variations in a consistent matrix imply slight variations of the eigenvector and the eigenvalue.

$$\mathbf{A} \cdot \mathbf{p} = \lambda \cdot \mathbf{p} \tag{2}$$

where

A is the comparison matrix
p is the priorities vector
λ is the maximal eigenvalue.

Crawford and Williams (1985) have adopted another approach in minimizing the multiplicative error (3):

$$a_{ij} = \frac{p_i}{p_j} e_{ij} \tag{3}$$

where

a_{ij} is the comparison between object i and j
p_i is the priority of object i
e_{ij} is the error.

The multiplicative error is commonly accepted to be log normal distributed (similarly the additive error would be assumed to be normal distributed). The geometric mean (4) will minimize the sum of these errors (5).

$$p_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}} \tag{4}$$

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln(a_{ij}) - \ln\left(\frac{p_i}{p_j}\right) \right)^2 \tag{5}$$

Simulations did not highlight major difference between the geometric mean and the eigenvalue method (Budescu et al. 1986; Golany and Kress 1993; Herman and Koczkodaj 1996; Mikhailov and Singh 1999; Cho and Wedley 2004; Jones and Mardle 2004; Ishizaka and Lusti 2006).

Other methods have been proposed, each one based either on the idea of the distance minimisation (like the geometric mean) or on the idea that small perturbation inducing small errors (like the eigenvalue method or the arithmetic mean of rows). Cho and Wedley (2004) have enumerated 18 different methods and probably many others are to come. In this area, the future research is: how to choose the method to calculate the priorities?

5 Consistency

As priorities make sense only if derived from consistent or near consistent matrices, a consistency check must be applied. Saaty (1977) has proposed a consistency index (CI), which is related to the eigenvalue method:

$$\mathrm{CI} = \frac{\lambda\max - n}{n - 1}, \tag{6}$$

where

Table 3 Random indices from (Saaty 1977)

n	3	4	5	6	7	8	9	10
RI	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

n = dimension of the matrix
λ_{max} = maximal eigenvalue.

The consistency ratio, the ratio of CI and RI, is given by:

$$CR = CI/RI \tag{7}$$

where RI is the random index (the average CI of 500 randomly filled matrices)

If CR is less than 10%, then the matrix can be considered as having an acceptable consistency.

Saaty (1977) calculated the random indices shown in Table 3.

However several other indices have been developed (Brunelli and Fedrizzi 2014). Which one to use? What are the difference between them? Are any more indices more appropriate to develop? These are all open questions.

6 Reducing the Number of Pairwise Comparisons

AHP has a well know drawback: the number of comparisons to provide can be very high. There are several way to reduce the number of pairwise comparisons:

(A) Clustering technique

Already in the original publication Saaty (1977) proposes to group similar alternatives into clusters of 7 elements. The decision-maker would compare the alternatives within the clusters and the clusters themselves. Ishizaka (2012) developed a variant where each cluster has a common alternative in neighbouring cluster, which permits to join cluster without compare them in a separate matrix. In both cases, the number of pairwise comparisons still increases exponentially.

(B) Partial completion of the matrix

The matrix is partially completed and missing values are then estimated from the given evaluations. Several techniques have been developed (Harker 1987; Fedrizzi and Giove 2007; Bozóki et al. 2010; Gomez-Ruiz et al. 2010; Benítez et al. 2014; Kun 2015; Csató and Rónyai 2016; Jandova et al. 2017). This technique has two main questions: Which evaluations do I need to complete? How many evaluations do I need to complete in order to have a satisfactory vector of priorities? Carmone et al. (1997) finds that 50% of the evaluations need to be completed, which is only a reduction of half of the pairwise comparison required.

(C) Benchmarking comparison

Table 4 Four ways to combine preferences (Ishizaka and Labib 2011a, b)

		Mathematical aggregation	
		Yes	No
Aggregation on:	Judgements	Geometric mean on judgements	Consensus vote on judgements
	Priorities	Weighted arithmetic mean on priorities	Consensus vote on priorities

Instead of comparing each alternative with each other's, they are compared with only with a set of fixed alternatives as in AHPSort (Ishizaka et al. 2012) or Best-Worst method (Rezaei 2015). The number of required evaluations is greatly reduced but is still proportional to the number of alternatives.

(D) Reference levels and interpolation

The pairwise comparison is not asked on all the alternatives but the decision-maker is asked to compare only some reference levels on the considered criteria. The other non reference evaluations are obtained by interpolating the values assigned by AHP to the reference levels (Corrente et al. 2016). This method can only be used with quantitative criteria and known scores of the alternatives (e.g. price).

As the number of pairwise comparisons is the most hindering feature of AHP, this area is in need of improvements.

7 AHP in Group Decision Making

As a decision affects often several persons, the standard AHP has been adapted in order to be applied in group decisions. Consulting several experts avoids also bias that may be present when the judgements are considered from a single expert. There are four ways to combine the preferences into a consensus rating (Table 4).

Grošelj et al. (2015) have compared seven aggregations technique but probably many others are to be developed. If the focus has been mainly on the mathematical aggregations, many more work has be accomplished on the negotiation side.

8 Analytic Network Process

The main difference between Analytic Network Process (ANP) and AHP is the structure. AHP has a hierarchical structure and ANP is based on a network structure (Saaty and Takizawa 1986; Saaty 1996). The adopted structure depends on the modelling of

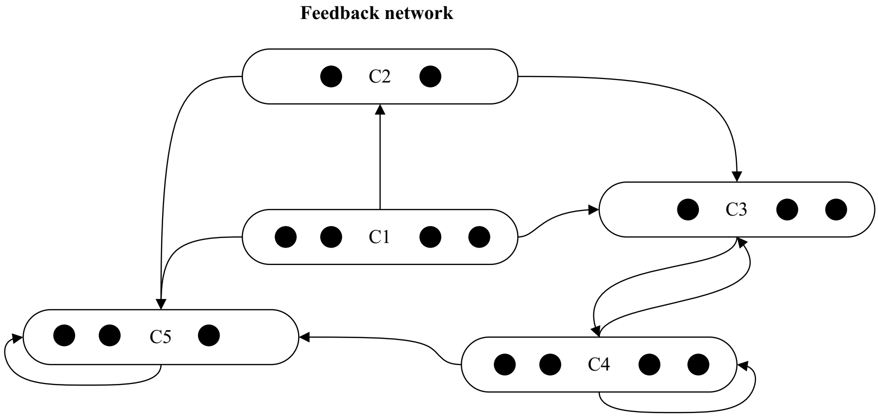

Fig. 2 Example of network with feedbacks of the ANP with 6 clusters

the problem (Ihrig et al. 2017): for example, a hierarchical structure is a linear top-down relationship with no feedback from a lower to a higher level, while the network structure is composed of different elements and clusters (groups of elements) that are connected to one another. The network structure can have connections between any factors in the decision problem. These connections represent the different relationships that exist between the clusters and the elements in the decision problem. Different relationships exist between the clusters and their elements (Figure 2):

- Inner dependence: this is a dependency in the same cluster—e.g. between two criteria or two alternatives.
- Outer dependence: this is a dependency between two clusters—e.g. between the cluster of alternatives and the cluster of criteria, or vice versa.

The influence of each node on other nodes in a network, can be gathered in a supermatrix (Saaty 2001). If dependencies do not exist between nodes, zero is entered. The supermatrix is then squared many times until it reaches stability.

ANP is the only method that can model all kind of dependency. However, it has a practical issue: the number of pairwise comparisons required is very high. Decreasing the number of pairwise comparisons, whilst keeping the full modelling of interdependencies is certainly the main future research area of ANP.

9 AHPSort

AHP is mainly used for ranking problems and occasionally for choice problems. AHPSort is a variant used for sorting problems (Ishizaka et al. 2012). A sorting problem aims to assign each alternative into one of the predefined ordered classes. In the case of problems with a large set of alternatives, AHPSort enables us to avoid the construction of a pairwise comparison matrix including all the alternatives. The alternatives are not compared with each other but only with the profiles representing the classes. Thus, the pairwise comparison matrix is much smaller. In the case of problems where the set of alternatives could change (by either adding or removing an alternative), using AHPSort can avoid modifying the pairwise comparison matrix of the alternatives and recalculating the priorities. When an alternative is removed, its attached pairwise comparison matrix is also removed but the other pairwise comparison matrices are untouched. When an alternative is added, a new pairwise comparison matrix is added and only the pairwise comparisons of the alternative with the profiles representing the classes need to be provided.

Recently, AHPSort has been extended for group decision (López and Ishizaka 2017), fuzzy problems (Krejčí and Ishizaka 2018) and problems with cost and benefit criteria (Ishizaka and López 2018). Certainly there are other extensions in this area.

10 Visualisation

Visual techniques have long been used in AHP for evaluating the pairwise comparisons (Fig. 3) and performing a sensibility analysis (Fig. 4). They have been integrated into the main software that supports AHP, and greatly facilitated the decision-making process (Ishizaka and Labib 2009). However, visual techniques cannot only facilitate the decision-making process but can also be used as a descriptive tool that explains the whole problem (Nemery et al. 2012). In (Ishizaka et al. 2016), GAIA was first coupled with AHP.

The idea of GAIA is to visualise on a plane as much information as possible related to a problem (Mareschal and Brans 1988). For this purpose, we can use the dimensionality reduction technique of the principal component analysis (PCA). The PCA is applied on the local priorities of AHP entered in a matrix. Data are displayed on a plane with the two axes having the maximal and next-to-maximal dispersions (Collins et al. 2017). These two axes correspond to the first two principal components.

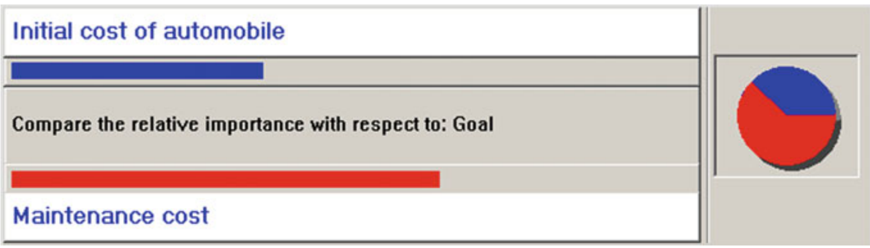

Fig. 3 Graphical scale

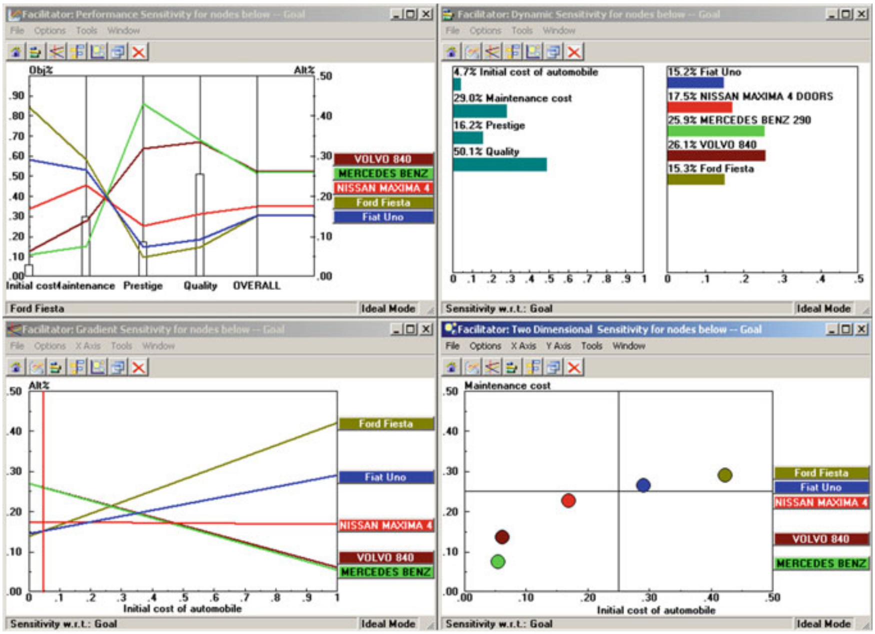

Fig. 4 An example of four possible graphical sensitivity analyses in expert choice

As decisions need to be explained to be implemented and visual management helps in this context, it is to expect that more visual applications to AHP will be developed in future.

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Beyond Multicriteria Ranking Problems: The Case of PROMETHEE

Yves De Smet

Abstract PROMETHEE is a well-known multicriteria outranking method. If it was primarily developed for (complete or partial) ranking purposes, recent extensions have been proposed in sorting and clustering contexts. Among them, the methods called PROMETHEE TRI and PROMETHEE CLUSTER were first presented in 2004. Unfortunately, these suffered from some drawbacks that we highlight in this contribution. To overcome these problems, authors have developed other extensions such as FlowSort, PCLUST, etc. The purpose of this paper is to provide a summary of some of these contributions, to highlight their existing links and list several remaining research questions. From a global perspective, we will show that the boundaries between *ranking*, *sorting* and *clustering* are blurred.

Keywords PROMETHEE · Sorting · Multicriteria clustering · FlowSort · PClust

1 Introduction

PROMETHEE[1] belongs to the family of so-called multicriteria *outranking* methods. They have been initiated and developed by Prof. Jean-Pierre Brans since the 80s. For the last thirty years, a number of researchers have contributed to its methodological developments and applications to real problems. In 2010, Behzadian et al. (2010) already reported more than 200 applications published in 100 journals. These cover finance, health care, environmental management, logistics and transportation, education, sports, etc. The successful application of PROMETHEE is certainly due to its simplicity and the existence of user-friendly software such as PROMCALC (Brans et al. 1994), Decision Lab 2000, Visual PROMETHEE or D-SIGHT (Hayez et al. 2012).

[1] Preference Ranking Organization Method for Enrichment Evaluations.

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PROMETHEE has initially been developed for (partial or complete) ranking problems. Later, additional tools have been proposed like, for instance, *GAIA*[2] for the descriptive problematic or *PROMETHEE V* (Brans and Mareschal 1992) for portfolio selection problems. Following this trend, J. Figueira, Y. De Smet and J.P. Brans proposed, in 2004, an extension of PROMETHEE for sorting and clustering problems. Due to some methodological drawbacks, this paper was never published in a scientific journal but remained accessible as a technical report of the SMG research unit (Figueira et al. 2004). Today, this work has been cited more than 75 times (according to Google Scholar) and has led to the development of different algorithms for sorting and clustering (based on the principles of the PROMETHEE methodology). The aim of this contribution is to summarize some of these approaches and to highlight directions for future research. Here, we focus on PROMETHEE based algorithms. Of course a number of observations, propositions and open questions can be extended to other approaches.

From a global perspective, we would like to highlight different links between ranking, sorting and clustering approaches. Indeed, we will illustrate how a ranking method can be used for sorting purposes, how a sorting method can be exploited in a clustering procedure, how an ordered clustering method can be seen as an alternative ranking method and, finally, how a sorting algorithm can help decision makers to rank alternatives. These approaches tend to show that the boundaries between these so-called *problematics* are blurred.

Finally, we decided to exclude illustrations of the proposed algorithms from this contribution (these can easily be found in the related articles) but rather to keep a global perspective on the links between the different approaches.

The paper is organized as follows. First, a brief reminder about the basics of PROMETHEE I and II will be presented in Sect. 2. Then we will discuss PROMETHEE TRI and CLUSTER in Sect. 3 and as well as their limits. This will lead us to present new approaches such as *FlowSort* (Nemery et al. 2008) in Sect. 4 for the sorting problematic and two extensions to multicriteria clustering problems in Sects. 5.1 and 5.2. At the end of these three sections, we will list a number of open questions that are specific to the presented methods. In the conclusion a more general perspective will be adopted; we will try to highlight common and distinctive features as well as general directions for future research.

2 A Brief Reminder About the PROMETHEE I and II Rankings

Let us consider a set of alternatives $A = \{a_1 \ldots a_n\}$ and a set of criteria $F = \{f_1 \ldots f_q\}$. Without loss of generality, we suppose that these q criteria have to be maximized. For each criterion f_k, the Decision Maker (DM) evaluates the preference of an alternative a_i over an alternative a_j by measuring the difference of their evaluations on f_k.

[2]Graphical Analysis for Interactive Assistance.

$$d_k(a_i, a_j) = f_k(a_i) - f_k(a_j) \tag{1}$$

This pairwise comparison allows the DM to quantify how alternative a_i performs on f_k compared to alternative a_j. Then, we use a preference function P_k to transform this value into a preference degree. Depending on the shape of the preference function (cf. Table 1), the DM could define the indifference threshold q_k and the preference threshold p_k for each criterion.

To quantify the global preference of a_i over a_j, we define the notion of preference index $\pi(a_i, a_j)$. It allows us to aggregate all the unicriterion preferences $P_k(a_i, a_j)$ by considering the weights ω_k associated to each criterion.

$$\pi(a_i, a_j) = \sum_{k=1}^{q} P_k[d_k(a_i, a_j)] \cdot \omega_k \tag{2}$$

$$\omega_k \geq 0 \, , \sum_{k=1}^{q} \omega_k = 1 \tag{3}$$

The last step of the PROMETHEE methods relies on the calculation of the outranking flow scores of each alternative. It allows the DM to quantify simultaneously how a_i is preferred (on average) to all the remaining alternatives x of the set A and how these alternatives x are preferred (on average) to a_i. These two notions are respectively represented by the positive flow score ϕ^+ and the negative flow score ϕ^- in PROMETHEE I.

$$\phi^+(a_i) = \frac{1}{n-1} \sum_{x \in A} \pi(a_i, x) \tag{4}$$

$$\phi^-(a_i) = \frac{1}{n-1} \sum_{x \in A} \pi(x, a_i) \tag{5}$$

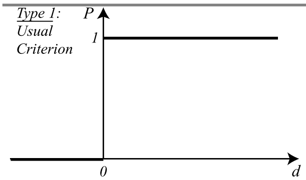
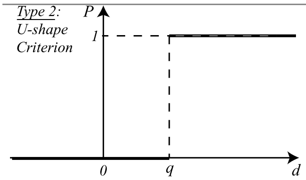
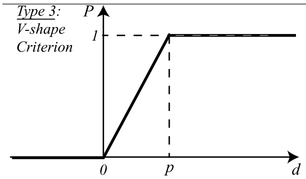
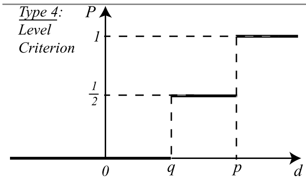
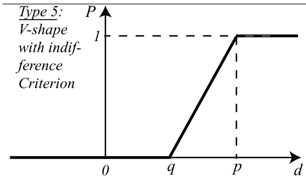
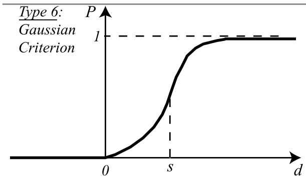
The positive and negative flow scores could be combined into the outranking net flow score ϕ which is used in PROMETHEE II.

$$\phi(a_i) = \phi^+(a_i) - \phi^-(a_i) \tag{6}$$

Based on the positive and negative flow scores, the PROMETHEE I method generates a partial ranking of the alternatives (which is the intersection between the two rankings induced by ϕ^+ and ϕ^-). In PROMETHEE II, a complete order is generated from the net flow scores of the alternatives. Finally let us note that the net flow score of a_i can also be expressed as follows:

$$\phi(a_i) = \sum_{k=1}^{q} \phi_k(a_i) \cdot \omega_k \tag{7}$$

Table 1 Types of generalized criteria ($P(d)$:Preference function—for the sake of simplicity $d = d_k(a_i, a_j)$)

Generalized criterion	Definition	Parameters to fix
	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 & d > 0 \end{cases}$	–
	$P(d) = \begin{cases} 0 & d \leq q \\ 1 & d > q \end{cases}$	q
	$P(d) = \begin{cases} 0 & d \leq 0 \\ \frac{d}{p} & 0 \leq d \leq p \\ 1 & d > p \end{cases}$	p
	$P(d) = \begin{cases} 0 & d \leq q \\ \frac{1}{2} & q < d \leq p \\ 1 & d > p \end{cases}$	p, q
	$P(d) = \begin{cases} 0 & d \leq q \\ \frac{d-q}{p-q} & q < d \leq p \\ 1 & d > p \end{cases}$	p, q
	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 - e^{-\frac{d^2}{2s^2}} & d > 0 \end{cases}$	s

where

$$\phi_k(a_i) = \frac{1}{n-1} \sum_{x \in A} P_k[d_k(a_i, x)] - P_k[d_k(x, a_i)] \tag{8}$$

with $\phi_k(a_i) \in [-1, 1]$ that is called the unicriterion net flow score of a_i on criterion f_k. Finally, let us note that:

$$\sum_{a_i \in A} \phi(a_i) = 0 \tag{9}$$

The PROMETHEE methods have been subject to a number methodological developments. Of course, an exhaustive summary of these approaches goes beyond the scope of this chapter. Nevertheless, we would like to mention two main issues that have been addressed in the literature and that are related to the forthcoming developments.

Firstly, the computation of a ranking based on positive, negative or net flow scores can be viewed as a *recipe*. Few arguments are given to motivate the justification of such an approach. Some authors started to address these issues (Bouyssou and Perny 1992; Mareschal et al. 2008). For instance, Mareschal et al. (2008) proved that the computation of the PROMETHEE net flow scores can be justified by the minimization of a given penalty function. In the same spirit, while evaluating the quality of a given multicriteria partition, a natural question occurs; what kind of indicator are we trying to optimize? How can we evaluate a *good* multicriteria partition? Given the fact that ordered multicriteria clustering is related to ranking (this will be discussed later) the same kind of questions arise; When applying a given ranking procedure, do we have the warranty that we minimize some kind of inconsistency indicator? Answering this question will give some credit to the considered ranking procedure.

Secondly, PROMETHEE methods, as well as other multicriteria approaches, have been criticized because these are subject to *rank reversal* issues (De Keyser et al. 1996) i.e. the relative position of two alternatives can be influenced by the presence or deletion of a third alternative. We will not enter the philosophical debate about the legitimacy of *rank reversal*. However, it is worth noting that a number of authors (Eppe and De Smet 2017; Roland et al. 2012; Mareschal et al. 2008; Verly et al. 2013) have investigated conditions under which rank reversal cannot happen or can be predicted/controlled. Among them, Doan and De Smet (2016) have proposed to use an extension of the FlowSort method to build rankings that are not subject to rank reversal (this will be briefly summarized in Sect. 4).

Finally, let us point out that authors have often claimed that the computation of net flow scores was $O(q \cdot n^2)$. This has an impact on the possible extensions of PROMETHEE to classification. Indeed, classification applications are often based on large data sets (typically including hundreds or thousands of alternatives). Fortunately, Calders and Van Assche (2018) recently proved that it was $O(q \cdot nlog(n))$ which alleviate this limit.

3 A First Approach

In what follows, we will assume that the reader is familiar with two basic classification methods: the k-nearest neighbor and the k-means algorithms. We refer the interested reader to Duda et al. (2000) for a detailed description of these approaches.

3.1 PROMETHEE TRI and PROMETHEE CLUSTER

First of all, let us consider the extension of PROMETHEE to sorting problems. Here are the main steps of PROMETHEE TRI. Let $C_1, C_2, \ldots, C_m$ denote the m ordered categories. Without loss of generality, we assume $C_h \succ C_l$ when $h < l$. Each category C_h is characterized by a representative element called the *central* profile and denoted by r_h. Let $R = \{r_1, r_2, \ldots, r_m\}$ denote the set of central profiles. In general, these central profiles can be actual alternatives or fictitious ones. In order to keep simple notations, we will consider here after that $R \subset A$ (let us note that this assumption can easily be relaxed). In sorting problems, we assume that these categories are known (as well as their representative profiles).

In what follows, we will evaluate the proximity between a given alternative a_i and the different central profiles. Therefore one computes the deviation between a_i and r_h as follows:

$$e(a_i, r_h) = \sum_{k=1}^{q} |\phi_k(a_i) - \phi_k(r_h)|\omega_k \tag{10}$$

The assignment rule works as follows:

$$a_i \in C_T \Leftrightarrow T = \arg \min_{h=1,\ldots,m} \{e(a_i, r_h)\} \tag{11}$$

In other words, a_i is assigned to the category that is characterized by the closest central profile (computed as being the L_1 weighted distance in the unicriterion net flow scores space). The informed reader will immediately see that this procedure is really close to the *nearest neighbor* technique (1NN) applied to the unicriterion net flow scores.

When categories are unknown, clustering techniques can be used to better understand the structure of the set of alternatives. PROMETHEE CLUSTER is based on the principles of the k-means algorithm. First, we start with a set of central profiles (for instance these can be built randomly). Then, one applies PROMETHEE TRI to all the alternatives. At the end of this step, each alternative is put in a given cluster. The next step consists to update the central profiles (in order to better represent the new clusters). These can be built as the mean or median evaluations of the alternatives belonging to the given set. The procedure is then repeated until the cluster membership does not change.

Both methods can be seen as being direct extensions of *traditional* supervised classification and clustering techniques. The only novelty comes from the fact that one works in the unicriterion net flow scores space (which is based on intra-criterion preference parameters) and the use of a L_1 weighted distance (which includes inter-criteria preference parameters).

3.2 Discussion

First of all, let us note that we have only presented the first version of PROMETHEE TRI and PROMETHEE CLUSTER. It is worth mentioning that other versions have been proposed in the literature. Unfortunately they do not overcome the limitations that we address in this section. Therefore, we will not provide further description of these extensions.

To begin, let us point out that a distinctive feature of muticriteria analysis is that the comparison of two alternatives leads most of the time to asymmetric relations (for instance if a_i is preferred to a_j then a_j is not preferred to a_i - this argument will be further discussed in Sect. 5). This leads to the fact that relations might exist between categories or clusters. In sorting, categories are assumed to be ordered. This should have consequences on the nature of reference profiles that are used in PROMETHEE TRI (typically the profile of a better category should at least *dominate* those of less preferred categories). Unfortunately this point is not properly addressed in the method. In addition, in PROMETHEE CLUSTER, the potential relations between clusters are simply not considered; all clusters are supposed to be incomparable (which is not reasonable since preference relations exist between the alternatives).

At this point, it is worth investigating the term *multicriteria clustering*. More precisely, what is the difference between multicriteria clustering and *traditional* clustering approaches. Cailloux et al. (2007) were among the first to investigate this question. Put in simple words, a multicriteria clustering procedure is such that inverting the optimization of a given criterion might lead to a different output. This is opposed to a *traditional* clustering procedure which always leads to the same output (whatever one chooses to maximize or to minimize a given criterion). This is referred to the *criteria dependency* property. Unfortunately, PROMETHEE CLUSTER does not respect this property (Cailloux et al. 2007) and so cannot be differentiated from a *traditional* clustering approach. Finally, PROMETHEE CLUSTER suffers from the same drawbacks as the k-means algorithm (i.e. the sensitivity to initial conditions, the possible disappearance of clusters during the procedure execution, etc.).

As already stressed, the assignment procedure of PROMETHEE TRI is based on the weighted deviation between unicriterion net flow scores (based on the whole data set). Therefore, the assignment of a given alternative can be influenced by the presence of other alternatives to be sorted. This could lead to *assignment reversals* (which are similar to rank reversal but in a sorting context).

4 PROMETHEE for Sorting: The FlowSort Method

In what follows, we describe a limited version of the FlowSort procedure developed by Ph. Nemery de Bellevaux in his Ph.D. thesis (Nemery et al. 2008). From our point of view, this method constitutes the most natural extension of PROMETHEE to the sorting problematic.

We assume that each category C_h is characterized by two limit profiles: the upper profile r_h and the lower profile r_{h+1} (let us note that the lower profile of C_h corresponds to the upper profile of C_{h+1} see Fig. 1). Let $R = \{r_1, \ldots, r_{m+1}\}$ be the set of limit profiles. These are assumed to respect the following conditions:

Condition 4.1:

$$\forall a_i \in A : g_k(r_{m+1}) \leq g_k(a_i) \leq g_k(r_1) \; \forall k \in \{1, \ldots, q\} \tag{12}$$

Condition 4.2:

$$\forall r_h, r_l \in R | h < l : g_k(r_h) \geq g_k(r_l) \; \forall k \in \{1, \ldots, q\} \tag{13}$$

Condition 4.3:

$$\forall r_h, r_l \in R | h < l : \pi(r_h, r_l) > 0 \tag{14}$$

The first condition imposes that all the evaluations of the alternatives to be assigned are lying between the evaluations of r_{m+1} and r_1. As a natural consequence, no evaluation can be better than the one of the upper profile of the best category or worse than the lower profile of the worst category. Let us note that this condition is not restrictive since r_1 (respectively r_{m+1}) can always be defined as the ideal point of the problem (respectively the nadir point). The two next conditions impose that some consistency should exist between the order of the categories and the preferences between the limit profiles:

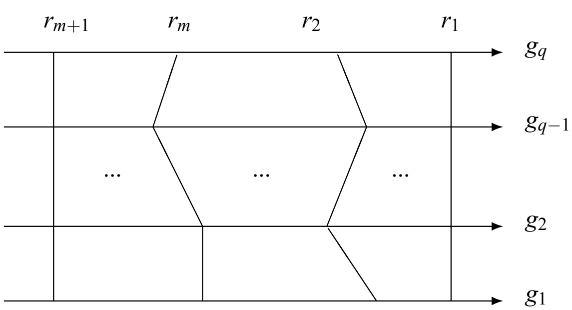


Fig. 1 Illustration of limit profiles r_h in FlowSort

- the evaluation of the upper limit profile of a better category should be at least as good as the evaluation of the upper profile of a worse category;
- the preference of the upper profile of a better category over the upper profile of a worse category should always be strictly positive.

Let us consider an alternative $a_i \in A$ to be sorted. The underlying idea of the FlowSort procedure is to compare a_i with respect to the elements of R by using the PROMETHEE I or the PROMETHEE II ranking. Let us define $R_i = R \cup \{a_i\}$ (therefore $|R_i| = m + 2$). For all $x \in R_i$, the flow scores are computed as follows:

$$\phi^+_{R_i}(x) = \frac{1}{m+1} \sum_{y \in R_i} \pi(x, y) \tag{15}$$

$$\phi^-_{R_i}(x) = \frac{1}{m+1} \sum_{y \in R_i} \pi(y, x) \tag{16}$$

$$\phi_{R_i}(x) = \phi^+_{R_i}(x) - \phi^-_{R_i}(x) \tag{17}$$

The ranking based on the positive and negative flow scores can lead to two different situations:

$$Z_{\phi^+}(a_i) = Z_T \quad if \quad \phi^+_{R_i}(r_T) \geq \phi^+_{R_i}(a_i) > \phi^+_{R_i}(r_{T+1}) \tag{18}$$

$$Z_{\phi^-}(a_i) = Z_U \quad if \quad \phi^-_{R_i}(r_U) < \phi^-_{R_i}(a_i) \leq \phi^-_{R_i}(r_{U+1}) \tag{19}$$

where $Z_{\phi^+}(a_i)$ (respectively $Z_{\phi^-}(a_i)$) represents the assignment based on the positive (respectively negative) flow score only. The assignment rule based on the PROMETHEE I ranking should integrate both of these aspects. As a consequence, let $B = \min\{T, U\}$ be the index of the category corresponding to the best assignment and let $W = \max\{T, U\}$ be the index of the category corresponding to the worst assignment. The first assignment rule will lead to conclude that a_i is assigned to the set of categories $[Z_B, \ldots, Z_W]$. Of course, if $W = B$ the assignment is unique.

Alternatively, the decision maker could force the assignment to a unique category by using a rule based on the net flow score:

$$Z_\phi(a_i) = Z_V \; if \; \phi_{R_i}(r_V) \geq \phi_{R_i}(a_i) > \phi_{R_i}(r_{V+1}) \tag{20}$$

As expected, the assignment procedures based on the PROMETHEE I and PROMETHEE II rankings are consistent. More formally (Nemery de Bellevaux 2008):

$$\forall a_i \in A : Z_B(a_i) \succeq Z_V(a_i) \succeq Z_W(a_i) \tag{21}$$

In other words, the assignment based on the net flow score will always lead to a category that is at least as good as ($\succeq$) the worst category and no better than the best category found by the first assignment rule.

These two assignment rules are the basics of FlowSort. Let us remind the reader that this section only constitutes a limited presentation of the method. We have to stress that a similar procedure exists when categories are represented by central profiles (instead of limit profiles). In addition, the principle on which relies FlowSort (the use of a ranking method to do sorting) can easily be extended to other approaches. Finally, it is worth noting that a number of theoretical properties have been analyzed to characterize the assignment rules. We refer the interested reader to (Nemery de Bellevaux 2008) for a detailed analysis.

Compared to PROMETHEE TRI, FlowSort exhibits a number of advantages. Among them, we can cite the fact that:

- the definition of central or limit profiles is consistent with the order of the categories;
- the assignment procedure is only based on reference alternatives (and so does not depend on other alternatives to be sorted) and, so, limit the number of comparisons to perform;
- the method respects a certain number of *natural* properties—among others (Nemery et al. 2008):
  - *Strong homogeneity property*: When two alternatives are compared similarly to the reference profiles they are affected to the same categories;
  - *Monotonicity property*: If a_i dominates a_j, then a_i will be affected to category which is at least as good as the category to which a_j will be assigned;
  - *Stability property*: The fusion or the separation of two neighboring categories does not affect the assignment of the alternatives to other categories;
  - *Conformity property*: If the performances of an alternative a_i are "in between" the performances of two consecutive limiting profiles, it will be assigned to the category delimited by these profiles.

The FlowSort method is the illustration that a given ranking method, in this case PROMETHEE, can easily be used to develop a sorting method. Recently, Doan and De Smet (2016) adopted the opposite point of view; they investigated how FlowSort could be used for ranking purposes. Let us consider a set of reference profiles that will serve as a comparison basis, let us compute $\phi_{R_i}(a_i)$ and $\phi_{R_j}(a_j)$, we have:

$$\phi_{R_i}(a_i) \geq \phi_{R_j}(a_j) \Rightarrow a_i S a_j$$

where $S = P \cup I$ is the binary outranking relation. It can be shown that the induced S relation respects the monotonicity principle and is invariant with respect to non-discriminating reference profiles (i.e. references r_h such that $\pi(a_i, r_h) = \pi(a_j, r_h)$ and $\pi(r_h, a_i) = \pi(r_h, a_j)$). By construction, the ranking induced is independent to rank reversal since the relative ranking of two alternatives only depends on the set of reference profiles (and not on other alternatives to be ranked). Of course one might say that the problem is somehow relocated since the ranking depend on the set of reference profiles and so might be affected if this set changes. We do think that in some situations decision makers would reasonably agree on a set of reference profiles that will serve as a comparison basis. Therefore the problem of rank reversal (related to this set) would be more limited. The determination of these reference profiles (as well as their number) is still an open question and probably increases the complexity of the elicitation process. For instance, one could investigate the minimum number of reference profiles needed to discriminate all the alternatives, the possible equivalence classes between reference profiles, etc. This approach illustrates how a sorting method can be used to compute rankings. To be complete, let us note that this is close to an approach that was previously proposed by Rolland (2013).

Let us mention that the initial FlowSort model has already been extended to different variants: F-FlowSort (Campos et al. 2015) based on Fuzzy Set theory, FlowSort-GDSS (Lolli et al. 2015) to address sorting problems with multiple decision makers or an extension supporting imprecision about the parameters of the method (modelled by intervals) (Janssen and Nemery 2013). Finally let us mention, that Van Assche and De Smet (2016) have recently proposed a procedure to elicit the FlowSort parameters based on categorisation examples.

5 PROMETHEE for Clustering

On the one hand, sorting has a long tradition in multicriteria decision aid and is recognized by most authors as being one of the three main multicriteria problematics (Roy 1996; Vincke 1992). Methods like the Trichotomic Segmentation (Moscarola and Roy 1977), N-TOMIC (Massaglia and Ostanello 1991), ELECTRE TRI (Yu 1992), ORCLASS (Larichev and Moshkovich 1994) or PROAFTN (Belacel 2000) are just a few examples of approaches developed between the late seventies and 2000. We refer the interested reader to Zopounidis and Doumpos (2002) for a rich introduction to this field. On the other hand, the development of clustering methods that are dedicated to multicriteria analysis started later. To the best of our knowledge, De Smet and Montano (2004) are among the first who, in 2004, have started to consider the integration of multicriteria preferences in a clustering procedure. Several researchers (Boujelben 2017; Eppe et al. 2014; Eppe and De Smet 2017; Fernandez et al. 2010; Meyer et al. 2013; Rocha et al. 2013; De Smet et al. 2012) continued to deepen this new research field.

Of course, multicriteria clustering techniques can be helpful. Obviously, they allow to better understand the structure of a given multicriteria problem. For instance, during the exploratory phase of a problem characterized by a lot of alternatives, one could submit to the decision maker the representative elements of the different clusters (instead of the whole dataset) in order to simplify the decision process. Multicriteria clustering methods can also be used to help decision makers building categories in a sorting context.

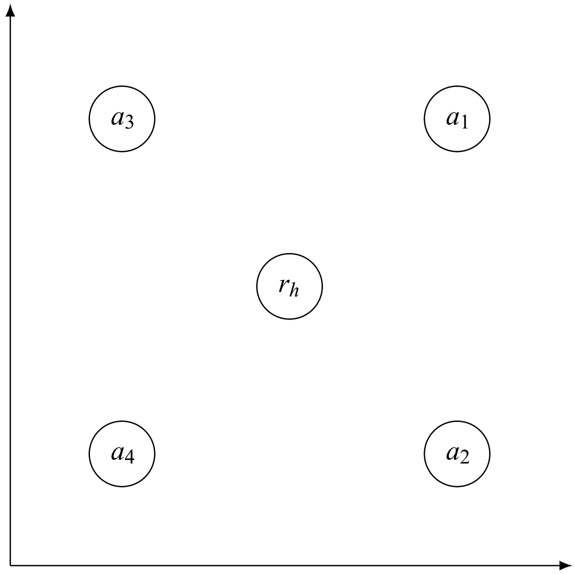


Fig. 2 Distance versus dominance in multicriteria analysis

The need to develop new procedures to address multicriteria clustering problems relies on the fact that most *traditional* clustering algorithms are based on the notion of a *distance* which is, by definition, symmetric. On the contrary, a multicriteria preference is by nature asymmetric. To better understand this distinctive feature, let us consider the example of 5 alternatives plotted on Fig. 2. First of all, its clear that all the 4 peripheral alternatives are located at the same distance to the central one r_h. From this point of view, one cannot differentiate them with respect to r_h. Let us now imagine that both criteria have to be maximized (as well as the use of the natural dominance relation). In this case, the situation is somehow different. Indeed, alternative a_1 dominates a_4 and r_h while a_2 and a_3 are incomparable together. This simple example shows that the asymmetric nature of multicriteria relations brings valuable additional information about the considered alternatives. In addition, this gives the opportunity to build preferential relations between the clusters (leading to state for instance that a given cluster is preferred to another one). Let us note that a broad range of multicriteria clustering outputs can be considered; from nominal clusters (where no relation exist between the clusters) to totally ordered clusters (where the relations between clusters are supposed to be complete and transitive). Between these two extreme situations one can consider cases where the relations are not complete (leaving place for incomparable clusters) or not transitive. Some authors refer this as relational multicriteria clustering (Meyer et al. 2013).

Of course the existence of potential relations between clusters has an impact on the size of the solutions space. Indeed, in *traditional* clustering approaches, when trying to determine a partition of m clusters, one faces a number of potential solutions that is equal to the Stirling number of the second kind $S(n, m)$. If potential relations might exist between the clusters, this number is multiplied by $3^{C_2^m}$ (here we suppose that between two clusters C_h and C_l only three situations could happen: C_h is preferred to C_l, C_l is preferred to C_h or C_h and C_l are incomparable). For instance, if we consider the simple case of 10 alternatives that have to be split in 3 clusters (with possible relations between them); we already get 251.910 possible partitions.

5.1 PCLUST

Based on the principles of FlowSort, Sarrazin et al. (2018) have developed the model PCLUST which is an extension of PROMETHEE I for interval clustering (a first model, called P2CLUST, was initially introduced (De Smet 2014) for ordered clustering). In this context, a given alternative can be assigned to a unique cluster or to a set of successive clusters. Here we consider a set of categories that could be divided in two groups: the principal categories C_h and the interval categories $C_{h,l}$, $\forall h, l \in \{1 \dots m\}$ and $h \neq l$. The principal categories are ordered and their reference profiles respect the dominance principle. While the interval categories $C_{h,l}$ are located "between" the principal categories C_h and C_l. Considering the preference relation of PROMETHEE, it means that the profile $r_{h,l}$ is incomparable to r_h and r_l. The clustering procedure of the PCLUST method is composed of the following steps: Initialization of the central profiles (based on different approaches; random, equidistributed, etc.), the assignment of the alternatives to the categories according to PROMETHEE I and the update of central profiles (based on three different approaches). The procedure is repeated until a stopping condition is met (such as a maximum number of iterations or unchanged results for the last n_{iter} cycles).

In the following, we describe each step of the clustering procedure. The reader who is familiar with the k-means procedure directly sees that this approach follows the same main steps. Nevertheless, as in P2CLUST (De Smet 2014), two distinctive features have to be highlighted. At first, the allocation is based on a multicriteria sorting method. Secondly, the update of the reference profiles has to respect the multicriteria nature of the problem (i.e. the dominance condition).

At first, we determine the central profiles either randomly or by equidistributing the evaluations on every criterion. When initializing the reference profiles randomly, we need to sort the evaluations on every criteria in order to respect the dominance principle between clusters.

Next, the assignment of the alternatives to the categories is done with respect to an assignment rule. Let consider an alternative $a_i \in A$ and the set of reference profiles $R = \{r_1 \dots r_m\}$. As in FlowSort, we define the set $R_i = R \cup \{a_i\}$. We compute the preference degrees between the actions of R_i and we calculate the positive and negative flow scores. Finally, we assign an alternative to a category by following these two conditions:

Condition 5.1: $C_{\phi^+}(a_i) = C_T$ if:
$|\phi^+_{R_i}(r_T) - \phi^+_{R_i}(a_i)| = \arg\min_{\forall l} |\phi^+_{R_i}(r_l) - \phi^+_{R_i}(a_i)|$

Condition 5.2: $C_{\phi^-}(a_i) = C_U$ if:
$|\phi^-_{R_i}(r_U) - \phi^-_{R_i}(a_i)| = \arg\min_{\forall l} |\phi^-_{R_i}(r_l) - \phi^-_{R_i}(a_i)|$

Based on these conditions, two different categories C_T and C_U could be obtained. We define the following assignment rule:

Assignment: $\forall a_i \in A, \forall h, l \in \{1 \ldots K\}$

$$\begin{cases} \text{if } C_{\phi^+}(a_i) = C_{\phi^-}(a_i) = C_T = C_U, & a_i \in C_T = C_U \\ \text{else}, & a_i \in C_{T,U} \end{cases}$$

As in the k-means algorithm, we now have to update the reference profiles in order to correctly represent the different clusters. This step is not obvious since alternatives might belong to principal categories but also to interval categories. Put in other words, once we try to define the reference element of C_h it is logic to take into account all the elements belonging to C_h but also those that appear in interval categories covering C_h. Different strategies have been tested in Sarrazin et al. (2018). We refer the interested reader to this article for a complete description. This algorithm has been tested on real benchmark data sets and has shown good results in terms of quality of the obtained partition, convergence and stability.

To the best of our knowledge, the notion of *interval* clustering is rather new (let us note that some others (Boujelben and De Smet 2016) use the term *disjunctive partitions*). It allows to avoid forcing the assignment in a specific category if clear reasons for such kind of assignment are not met. Of course, the clear interpretation of such kinds of outputs is still at its early stages and deserves more attention.

5.2 An Approach Based on a Hierarchical Procedure

In the context of deterministic clustering methods, Rosenfeld and De Smet (2017) developed a procedure based on a hierarchical procedure. At the core of this approach, one has to determine how to divide or merge subsets of alternatives. This was addressed in the following way; let us imagine that one tries to divide the set of alternatives A into two complementary subsets denoted B and $\overline{B}$ such that B is strongly preferred to $\overline{B}$. This can be interpreted to be equivalent to maximize the global preference of B over $\overline{B}$ and minimize the global preference of $\overline{B}$ over B. More formally, we want to identify a subset $B^* \subset A$ such that:

$$B^* = \arg\max_{B \subset A} \sum_{a_i \in B, a_j \in \overline{B}} (\pi_{ij} - \pi_{ji}) \tag{22}$$

Let us note that, for the sake of simplicity, we denote $\pi(a_i, a_j) = \pi_{ij}$. The solution of this problem could be found by testing all potential subsets $B \subset A$ but this would be time-consuming for a large number of alternatives. Fortunately, it can be done in a more efficient way since B^* is determined by the set of alternatives characterized by positive net flow scores. Indeed, we have:

$$\sum_{a_i \in B, a_j \in \overline{B}} (\pi_{ij} - \pi_{ji}) = \sum_{a_i \in B, a_j \in A} (\pi_{ij} - \pi_{ji}) = (n-1) \sum_{a_i \in B} \phi(a_i) \tag{23}$$

As a consequence, the optimal set is strictly composed of alternatives which are characterized by positive net flow scores. Therefore, computing the PROMETHEE ranking allows us solving the previous optimization problem.

As in *traditional* hierarchical clustering techniques, two approaches can be considered: top-down or bottom up. In the top-down approach, one starts with the whole set of alternatives A. This is split according to the previous property and leads to two subsets that are naturally ordered (alternatives with positive net flow scores are assigned to the best cluster while the others are put in the worst cluster). The splitting procedure continues until the desired number of clusters is obtained. Of course, at each step, one has to identify which cluster is likely to be split. The idea is to divide the least homogeneous cluster. In order to quantify the homogeneity of a cluster C_h, one computes the following indicator:

$$\Delta_h = \frac{1}{n_h} \sum_{a_i, a_j \in C_h} \pi(a_i, a_j) \tag{24}$$

where n_h represents the cardinal of C_h. In the case of a perfectly homogeneous cluster, we should have $\pi_{ij} = \pi_{ji} = 0$ for all alternatives a_i and a_j belonging to it. As soon as these preferences increase the cluster becomes less homogeneous. Therefore the Δ_h indicator has to be minimized.

Alternatively, one can emphasize a bottom up approach. Here, the procedure starts with an ordered partition of n clusters that is built based on the PROMETHEE II ranking (let us note that sometimes ties can appear leading to initial cluster with several alternatives). Then, one decides to merge a pair of consecutive clusters. This couple of clusters is determined in such a way that the resulting new cluster has the best possible homogeneity. The procedure continues until the desired number of clusters is obtained.

5.3 Multicriteria Clustering Quality indicators

In the previous subsections, we have described three possible multicriteria procedures based on the PROMETHEE methodology. Nevertheless, a crucial question remains: once a given multicriteria partition has been obtained how can we evaluate its quality?

To compare the quality of different multicriteria ordered clustering procedures, we might imagine using a reference evaluation scheme such as a metric. There exist different quality indexes in the literature, such as the Dunn Index (1974), the Davies-Bouldin Index (Liu et al. 2011), etc. Generally, these indexes measure both the intra-clusters homogeneity and the inter-clusters heterogeneity. They both should be maximized. Unfortunately, these indexes are not appropriate for multicriteria clustering (since they do not take into account its distinctive features i.e. its the existence of asymmetric relations between the alternatives). Here, we propose an alternative quality index dedicated for ordered clustering.

The evaluation of the quality of a multicriteria ordered clustering is done as follows. First, clusters have to be as homogeneous as possible. In broad terms, the alternatives belonging to a same cluster have to be as 'close' as possible to each other. For each cluster C_h, we propose to evaluate its homogeneity as in the previous section.

$$\Delta_h = \frac{1}{n_h} \sum_{x_i, x_j \in C_h} \pi(x_i, x_j) \tag{25}$$

The notion of distance that appears in standard quality indexes is replaced by a preference relationship between the alternatives belonging to the same cluster. This index has to be minimized. In an ideal case, $\pi(x_i, x_j) = \pi(x_j, x_i) = 0, \forall x_i, x_j \in C_h$.

Secondly, the set of clusters has to be as heterogeneous as possible. In other words, we might say that the clusters have to be as 'far' as possible from each other. Let us denote $\delta(h, l)$ the index of heterogeneity between the clusters C_h and C_l when $h < l$:

$$\delta(h, l) = \pi(r_h, r_l) - \pi(r_l, r_h) \tag{26}$$

The distance is also replaced by a preference relationship between the mean value of the alternatives of the cluster C_h (r_h) and the mean value of the alternatives of the cluster C_l (r_l) (where C_h is better than C_l). This index has to be maximized because the alternatives of the best cluster have to be preferred the alternatives of the other set. Here we explicitly take into account the order between the clusters. Finally, to have a global indicator, we have to pool these indexes into a global quality indicator. For the heterogeneity of the set of clusters, it is not always pertinent to compare the alternatives of clusters which are not next to each other. Then, we define the quality index D as follows:

$$D = \frac{\sum_{h=1}^{m} \Delta_h}{\sum_{h=1}^{m-1} \delta(h, h+1)} \tag{27}$$

Let us point out that in the case of the hierarchical approach, Rosenfeld and De Smet have proposed an hybrid procedure (mixing the top-down and bottom-up approach) in order to optimize the aformentionned indicator. Of course, this indicator is a first attempt to assess the quality of a given multicriteria ordered partition and needs further investigation.

6 Conclusion and Directions for Future Research

In traditional multicriteria textbooks (Roy 1996; Vincke 1992) one usually distinguishes three main problematics; choice, ranking or sorting. Sometimes authors also mention the descriptive or the portfolio problematics. More recently, researchers have started to investigate multicriteria clustering methods. At first sight, these main families seem to be well distinguished. A first conclusion of this contribution is that the situation is less clear. Indeed, one may point out:

- **The link between sorting and ranking**: In his Ph.D. thesis, Ph. Nemery developed the FlowSort method. From a global perspective, this is based on the use of a ranking method (PROMETHEE I or II) applied to a set of reference profiles in order to determine the relative position of a given alternative to be sorted. Of course, this can easily be extended to other ranking methods. As a consequence, it is always possible to build a sorting method based on a ranking procedure (the assignment being based on the relative position with respect to reference elements). Alternatively, Doan and De Smet (2016) adopted the opposite approach i.e. how to use a sorting method (in this case FlowSort) in order to obtain rankings. In the latter case, the objective was to build a procedure that was less subject to rank reversal issues.
- **The link between clustering and ranking**: As already stressed, in multicriteria analysis, the relative positions of alternatives are most of the time asymmetric (in opposition to a distance measure which is, by definition, symmetric). This leads to the opportunity of having ordered clusters. Here the difference between ordered clusters and a ranking can be addressed. On the one hand, in ranking problems indifferent alternatives can be put in the same equivalence class. On the other hand, in clustering problems, when the number of clusters is equal or close to the number of alternatives, one can get an output that is close to a ranking. Finally, in applications such as the academic ranking of world universities, one faces an output that is a mix between a ranking among the top 100 first alternatives and then a clustering of remaining universities (by groups of 50). Finally, let us point out that assessing the quality of a multicriteria clustering procedure can be done by evaluating the outputs based on given quality indicators. In the same spirit, one could investigate what kind of quality indicator does a given ranking procedure optimize?
- **The link between sorting and multicriteria clustering**: As described, clustering methods, such as PCLUST, partly rely on a sorting procedure (in this case FlowSort). In addition, multicriteria clustering procedures can be used to help decision makers defining categories in a sorting context.

These observations show that the frontiers between these different types of problems are blurred.

A number of research questions still needs to be addressed. We will not come back on the elements that have been listed in this contribution and that are specific to the PROMETHEE related methods. Rather, from a global perspective, one can mention:

- **Benchmark multicriteria data sets and validation indicators**: When speaking about multicriteria classification (i.e. sorting and clustering methods), one cannot neglect the comparison with traditional classification techniques. The latter are often tested on benchmark data sets with respect to well-defined indicators. This issue is still emerging (Sobrie 2016) in the multicriteria context and needs future research.
- **Clusters size restriction**: In sorting methods, authors have already addressed the question of categories size restrictions (Köksalan et al. 2017). Usually, this leads to solve (mix) integer linear programs that exhibit some limits when the number of alternatives increases. To the best of our knowledge, the question of size restrictions has not yet been addressed in multicriteria clustering.
- **Multicriteria validation properties**: As shown in Sect. 4, theoretical validation properties exist in a sorting context (for instance the monotonicity and conformity properties). These have still to be extended in the multicriteria clustering context.
- **Detection of multicriteria outliers**: The management of outliers is crucial in statistics since they are likely to affect estimators, models' parameters, etc. In a clustering context, the presence of outliers can have impact on the clusters detection. A similar effect could appear in a multicriteria clustering context. Nevertheless, to the best of our knowledge, the notion of *multicriteria* outliers has received low attention (De Smet et al. 2017).

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Preference Disaggregation for Multicriteria Decision Aiding: An Overview and Perspectives

Michalis Doumpos and Constantin Zopounidis

Abstract In multicriteria decision aiding, preference disaggregation analysis involves the inference of preferential information from holistic judgments that the decision maker provides. This area of research has attracted strong interest and various methodologies have been proposed over the past three decades for different types of decision problems and multicriteria models. This chapter overviews the developments and perspective in this field, covering established techniques as well as the state-of-the-art developments and future prospects.

Keywords Multicriteria decision aiding · Preference disaggregation · Linear programming · Robustness

1 Introduction

Multiple criteria decision aid (MCDA) is involved with supporting the decision process for problems that require the consideration of multiple conflicting criteria. Such problems arise is numerous areas in management, engineering, and social sciences, thus being of major research and practical interest. MCDA follows a constructive and prescriptive approach that facilitates the learning process of the decision maker (DM) and the formulation of non-trivial solutions to ill-structured problems.

A significant part of MCDA research and practice is devoted to the building of decision models. These may involve optimization problems with multiple objectives or problems where a finite set of alternative options should be evaluated. This chapter focuses on the latter type of problems. In this context, multicriteria decision models

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integrate data and expert judgment (preferences) into comprehensive analytical representations of a specific decision problem. Given that models are oriented towards the DMs who are involved in the decision process, it is crucial to ensure that the DMs preferences and judgment are properly incorporated in the models.

There are two main approaches in MCDA for eliciting preferential information from the DM or a group of DMs. The first follows a direct approach in which the DM interacts with a decision analyst and provides direct information describing his/her system of preferences. For instance, DMs may be asked to specify the trade-offs between the decision criteria or other types of judgments regarding the importance of the criteria and the way they should be valued.

An alternative approach is to indirectly infer preferential information from decision instances that the DM provides. This is referred to as preference disaggregation analysis (PDA) (Jacquet-Lagrèze and Siskos 2001). PDA methodologies use ordinal regression techniques to infer decision models that are compatible with the evaluations that a DM provides for some decision examples. This scheme is applicable to the analysis of preference judgments in various forms (ordinal and/or cardinal) and requires limited cognitive effort by the DM. The general framework of PDA is applicable to various types of decision models, including functional, relational, and symbolic models.

In this chapter we provide an overview of the PDA paradigm, its uses for constructing multicriteria models in the form of value functions and outranking relations, discuss recent research trends in the field, and outline the future development perspectives of this area.

The chapter is organized as follows. Section 2 introduces the general framework of PDA and describes the use of PDA techniques for constructing various types of decision models in MCDA. Section 3 focuses on the recent trends and perspectives in this field, whereas Sect. 4 concludes the chapter.

2 The Framework of Preference Disaggregation Analysis

PDA is a general methodological framework for building decision models using decision examples provided by a DM or a group of DMs. The provided examples form a reference set of decision alternatives, $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$, where each reference alternative is described by a data vector involving n decision criteria, i.e., $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{in})$. Moreover, the DM provides holistic judgments of the reference alternatives, which can be expressed in various forms, including:

- a rank order, i.e., a ranking of the alternatives from the best to the worst,
- a classification (sorting) into predefined performance categories (e.g., high, medium, low performance, etc.),
- pairwise comparisons between the alternatives, possibly combined with information about preference intensity.

Based on this information, the objective of PDA is to identify a decision model $F(\mathbf{x})$ that is as consistent as possible with the DM's judgments. Once the model is built, it can be used to evaluate any alternative outside the reference set X. This approach can be considered as a regression scheme, in the sense that a decision model is fitted on the given reference data and then extrapolated to other instances.

In the following subsections we describe the use of the PDA framework for inferring decision models in the form of value functions and outranking relations.

2.1 Value Function Models

Multiattribute value functions constitute a popular class of decision models in MCDA, founded on the principles of multiattribute value/utility theory (MAVT/MAUT). In the context of decision making under certainty, a value function $V(\mathbf{x})$ aggregates all decision criteria into a composite index that represents the performance value of the alternative actions according to the value system of the DM, such that:

$$V(\mathbf{x}) > V(\mathbf{y}) \Rightarrow \text{alternative } \mathbf{x} \text{ is preferred over alternative } \mathbf{y} \ (\mathbf{x} \succ \mathbf{y})$$
$$V(\mathbf{x}) = V(\mathbf{y}) \Rightarrow \text{alternatives } \mathbf{x} \text{ and } \mathbf{y} \text{ are indifferent } (\mathbf{x} \sim \mathbf{y})$$

Assuming the decision criteria are mutually preferentially independent (Keeney and Raiffa 1993), the value function has an additive representation:

$$V(\mathbf{x}) = w_1 v_1(x_1) + w_2 v_2(x_2) + \cdots + w_n v_n(x_n) \tag{1}$$

where $w_1, \ldots, w_n \geq 0$ are scaling constants, often referred to as "weights", representing the tradeoffs between the criteria and $v_1(x_1), \ldots, v_n(x_n)$ are marginal value functions that translate the original scales of the criteria to a common value (i.e., performance) scale, usually defined in the range [0, 1], with 0 corresponding to poor performance and 1 indicating the ideal. Under weaker criteria independence assumptions, more general value functions can be considered (e.g., the multilinear function), at the expense of the increased complexity of the resulting model.

Given a reference set consisting of m alternatives rank-ordered by the DM from the best ($\mathbf{x}_1$) to the worst one ($\mathbf{x}_m$), the inference of a compatible value function can be expressed into the following optimization problem:

$$\begin{array}{lll}
\min & \sigma_1 + \sigma_2 + \cdots + \sigma_m & \\
\text{s.t.} & V(\mathbf{x}_i) - V(\mathbf{x}_{i+1}) + \sigma_i - \sigma_{i+1} \geq \delta & \forall\, \mathbf{x}_i \succ \mathbf{x}_{i+1} \\
 & V(\mathbf{x}_i) - V(\mathbf{x}_{i+1}) + \sigma_i - \sigma_{i+1} = 0 & \forall\, \mathbf{x}_i \sim \mathbf{x}_{i+1} \\
 & v_k(x_{ik}) - v_k(x_{jk}) \geq 0 & \forall x_{ik} \geq x_{jk} \\
 & v_k(x_{k*}) = 0,\ v_k(x_k^*) = 1 & k = 1, \ldots, n \\
 & w_1 + w_2 + \cdots + w_n = 1 & \\
 & w_k, v_k(\cdot), \sigma_i \geq 0 & i = 1, \ldots, m,\ k = 1, \ldots, n
\end{array} \tag{2}$$

where $\sigma_1, \ldots, \sigma_m$ represent the error terms associated with the evaluation of the reference alternatives, and $\delta > 0$ is a user-defined small positive constant. The first two constrains define the error variables in accordance with the rank-order of the alternatives of the reference set. The third constraint ensures that the marginal value function are non-decreasing (assuming criteria in maximization form) and the fourth constraint defines the scale of the marginal value functions to be in [0, 1], with 0 corresponding to the worst level x_{k*} of each criterion and 1 to the ideal (best) level x_k^*. Finally, the fifth constraint normalizes the weights of the criteria so that they sum up to 1.

Formulation (2) is the basis of the well-known UTA method (Jacquet-Lagrèze and Siskos 1982), in which the above optimization problem is expressed in linear programming form for general additive models with non-linear marginal value functions. Thus, the UTA method is a very flexible approach that enables the development of powerful, yet comprehensible additive evaluation models, which are more general than a simple (linear) weighted average. Siskos et al. (2016) provide an overview of various extensions and variants of the UTA family of methods.

Except for a total pre-order of the reference actions, the disaggregation framework can also be employed to infer preferential information using other types of inputs. For instance, Figueira et al. (2009) proposed the GRIP method, which enables the modeling and analysis of partial pre-orders and intensities of preference. Pairwise comparisons are used in the MACBETH method (Bana e Costa et al. 2016) combining ordinal and cardinal judgments.

A similar approach is also applicable for classification (sorting) problems, where one is interested in building decision models for assigning the alternative options under consideration into ordered performances categories $C_1, C_2, \ldots, C_q$, with C_1 being the class of the best alternatives and C_q is the one consisting of the worst options (Zopounidis and Doumpos 2002). One way to define a decision rule based on a value function model for classifying the alternatives to the predefined categories is to compare the alternatives' value scores to thresholds that discriminate the classes:

$$t_\ell < V(\mathbf{x}_i) < t_{\ell-1} \Leftrightarrow \text{Alternative } i \text{ is assigned to category } \ell$$

where $0 = t_q < t_{q-1} < \cdots < t_1 < 1 = t_0$ are the discriminating thresholds defined on the same 0–1 scale as the value function model $V(\mathbf{x})$.

Under the above classification rule, a value function model can be inferred from a set of assignment (classification) examples provided by the DM, through the solution of the following optimization problem (Doumpos and Zopounidis 2002):

$$
\begin{aligned}
\min \ & \sum_{i=1}^{m} \omega_i (\sigma_i^+ + \sigma_i^-) \\
\text{s.t.} \ & V(\mathbf{x}_i) + \sigma_i^+ \geq t_\ell + \delta && \forall \mathbf{x}_i \in \{C_1 \cup C_2 \cup \cdots \cup C_{q-1}\} \\
& V(\mathbf{x}_i) - \sigma_i^- \leq t_{\ell-1} - \delta && \forall \mathbf{x}_i \in \{C_2 \cup C_3 \cup \cdots \cup C_q\} \\
& t_\ell - t_{\ell+1} \geq \varepsilon && \ell = 1, \ldots, q-2 \\
& v_k(x_{ik}) - v_k(x_{jk}) \geq 0 && \forall x_{ik} \geq x_{jk} \\
& v_k(x_{k*}) = 0, \ v_k(x_k^*) = 1 && k = 1, \ldots, n \\
& w_1 + w_2 + \cdots + w_n = 1 \\
& w_k, v_k(\cdot), \sigma_i^+, \sigma_i^-, t_\ell \geq 0 && \forall i, k, \ell
\end{aligned}
\tag{3}
$$

The first constraint defines the error variable σ_i^+ for an alternative i in relation to the lower category threshold, whereas the second constrain defines the violations (σ_i^-) of the upper category thresholds. The third constraint ensures that the discriminating thresholds define an increasing sequence, with ε being a user-defined non-negative constant. The objective function of the above formulation minimizes the sum of all error variables. The errors for specific alternatives or groups of alternatives can optionally be weighted using case-specific weights $\omega_1, \ldots, \omega_m > 0$.

Alternative decision rules can also be considered, such as the example-based rule proposed by Greco et al. (2010) and the hierarchical discrimination approach of Zopounidis and Doumpos (2000).

2.2 *Outranking Relations*

Roy (1968) first introduced outranking approaches in MCDA with the development of the ELECTRE methods (ELimination Et Choix Traduisant la REalité). Outranking approaches are based on relational preference models. Typically, outranking and preference relations are defined between pairs of alternatives (i, j), such that:

$$\mathbf{x}_i \, S \, \mathbf{x}_j \Rightarrow \text{alternative } i \text{ is at least as good as alternative } j$$

In the same manner one can define a binary preference relation:

$$\mathbf{x}_i \, P \, \mathbf{x}_j \Rightarrow \text{alternative } i \text{ is preferred over alternative } j$$

Overviews of such methodologies can be found in Brans and De Smet (2016), Figueira et al. (2016), Martel and Matarazzo (2016). Outranking models have some unique features compared to value function systems. For instance, they allow the modeling of non-compensatory and intransitive preferences in a flexible and natural manner. Moreover, they are suitable for handling fuzziness and uncertainty in data and preference judgments.

On the other hand, inferring outranking models from decision instance is more involved compared to the inference of value functions, because the structure of such models and their exploitation procedures are more complex and require the specification of many parameters.

For illustration purposes, we briefly describe a typical example of an outranking model from the ELECTRE TRI method (Roy 1996; Roy and Bouyssou 1993). ELECTRE TRI is a multicriteria method for assigning a finite set of alternatives into ordered categories. The categories are defined by boundary profiles $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{q-1}$, each corresponding to a separating boundary between successive categories. The separating profiles are defined as vector of boundary levels for the decision criteria, i.e., $\mathbf{r}_\ell = (r_{\ell 1}, r_{\ell 2}, \ldots, r_{\ell n})$.

Each alternative is compared against the profiles to assess the validity of the outranking relations $\mathbf{x}_i \, S \, \mathbf{r}_\ell$ and $\mathbf{r}_\ell \, S \, \mathbf{x}_i$. The construction of these relations in the ELECTRE TRI method is done through a two-step process. First the concordance index is derived, which represents the strength of the evidence that supports the outranking relation. For the relation $\mathbf{x}_i \, S \, \mathbf{r}_\ell$, the concordance index is a weighted average of partial concordance indices:

$$C(\mathbf{x}_i, \mathbf{r}_\ell) = \sum_{k=1}^{n} w_k c_k(x_{ik}, r_{\ell k}) \tag{4}$$

where

$$c_{ik}(x_{ik}, r_{\ell k}) = \begin{cases} 0 & \text{if } x_{ik} \le r_{\ell k} - p_k \\ \frac{x_{ik} - r_{\ell k} + p_k}{p_k - q_k} & \text{if } r_{\ell k} - p_k < x_{ik} < r_{\ell k} - q_k \\ 1 & \text{if } x_{ik} \ge r_{\ell k} - q_k \end{cases} \tag{5}$$

with $p_k \ge q_k \ge 0$ representing the preference and indifference thresholds, respectively.

In the second stage, discordance indices are calculated for criteria that have veto power. Such criteria may invalidate the outranking relation, irrespective of the indications provided by other criteria. The discordance index for criterion k involving the relation $\mathbf{x}_i \, S \, \mathbf{r}_\ell$, is defined as follows:

$$d_k(x_{ik}, r_{\ell k}) = \begin{cases} 0 & \text{if } x_{ik} \ge r_{\ell k} - p_k \\ \frac{x_{ik} - r_{\ell k} + p_k}{p_k - v_k} & \text{if } r_{\ell k} - v_k < x_{ik} < r_{\ell k} - p_k \\ 1 & \text{if } x_{ik} \le r_{\ell k} - v_k \end{cases} \tag{6}$$

where v_k is the veto thresholds for criterion k ($v_k \ge p_k \ge q_k \ge 0$).

Finally, the concordance and discordance indices are combined into a credibility index measuring the overall strength of the outranking relation:

$$\sigma(\mathbf{x}_i, \mathbf{r}_\ell) = C(\mathbf{x}_i, \mathbf{r}_\ell) \prod_{k \in F} \frac{1 - d_k(x_{ik}, r_{\ell k})}{1 - C(\mathbf{x}_i, \mathbf{r}_\ell)} \tag{7}$$

where F denotes the set of criteria with $d_k(x_{ik}, r_{\ell k}) > C(\mathbf{x}_i, \mathbf{r}_\ell)$. The outranking relation $\mathbf{x}_i \, S \, \mathbf{r}_{\ell-1}$ holds true if $\sigma(\mathbf{x}_i, \mathbf{r}_\ell) \ge \lambda$, where $0.5 \le \lambda \le 1$ is a user-defined cut-off point. Then, two assignment procedures can be employed (independently or jointly) to classify the alternatives:

- Optimistic assignment: alternative i is assigned to category C_ℓ, where ℓ corresponds to the largest index such that $\mathbf{r}_{\ell-1}\, S\, \mathbf{x}_i$ and $\mathbf{x}_i \,\neg S\, \mathbf{r}_{\ell-1}$, with $\neg$ denoting the negation operation.
- Pessimistic assignment: alternative i is assigned to category C_ℓ, where ℓ is the lowest index such that $\mathbf{x}_i\, S\, \mathbf{r}_\ell$.

The parameters of this MCDA approach include the weights of the criteria, the preference, indifference, and veto thresholds, as well as the cut-off point λ. The complex structure of the model makes it impossible to infer these parameters from a set of assignment examples using analytical optimization approaches (i.e., linear or non-linear programming).

Mousseau and Słowiński (1998) first proposed a disaggregation approach for inferring the parameters of the ELECTRE TRI method under the pessimistic assignment procedure, without considering veto effects. Their approach was based on a non-linear and non-convex optimization formulation. Other approaches for ELECTRE-based methods have used linear programming formulations, focusing on the inference of specific sets of parameters, while assuming the others fixed. For instance, linear and mixed-integer linear programming formulations have been proposed for inferring the weights of the criteria (Bisdorff et al. 2013; Mousseau et al. 2001; Zheng et al. 2014), the category profiles (Dias et al. 2002), and veto thresholds (Dias and Mousseau 2006). Similar techniques have also been used for other outranking methods, such as PROMETHEE (Eppe and De Smet 2012; Frikha et al. 2010, 2011). More general approaches have used heuristics and metaheuristics for inferring the parameters of outranking models in more complex settings (Belacel et al. 2007; Covantes et al. 2016; Doumpos et al. 2009; Doumpos and Zopounidis 2002; Goletsis et al. 2004; Sobrie et al. 2013; Van Assche and De Smet 2016). However, it should be noted that such methodologies require large-scale data to provide meaningful results.

3 Perspectives

In this section we discuss some state-of-the-art topics in PDA together with some perspectives in this area of MCDA. The discussion covers three main areas, namely robustness, modeling forms, and optimization models for large-scale problems.

3.1 Robustness Issues

One of the most important issues in disaggregation techniques involves the robustness of the inferred decision models. The robustness concern in the context of PDA arises for two main reasons. First, assuming a set of holistic evaluations provided by a DM for some reference actions, multiple decision models might exist that are compatible (or approximately compatible) with the DM's evaluations. Second, variations of the reference set used in the analysis (e.g., addition of new alternatives, removal of alternatives, etc.) may lead to very different results.

As an example consider the optimization formulation (3) and assume that a sorting of a set of reference alternatives is available that does not contain inconsistent judgments. This implies that the error variables can be omitted, thus leading to a system of feasible constraints:

$$
\begin{array}{ll}
V(\mathbf{x}_i) \geq t_\ell + \delta & \forall\, \mathbf{x}_i \in C_\ell,\ \ell = 1, \ldots, q-1 \\
V(\mathbf{x}_i) \leq t_\ell - \delta & \forall\, \mathbf{x}_i \in C_\ell,\ \ell = 2, \ldots, q \\
t_\ell - t_{\ell+1} \geq \varepsilon & \ell = 1, \ldots, q-2 \\
v_k(x_{ik}) - v_k(x_{jk}) \geq 0 & \forall x_{ik} \geq x_{jk} \\
v_k(x_{k*}) = 0,\ v_k(x_k^*) = 1 & k = 1, \ldots, n \\
w_1 + w_2 + \cdots + w_n = 1 & \\
w_k, v_k(\cdot), \sigma_i^+, \sigma_i^-, t_\ell \geq 0 & \forall\, i, k, \ell
\end{array}
\tag{8}
$$

The size of the feasible set defined by such constraints is associated with the robustness of the results. A large set of feasible solutions implies that there are many alternative decision models that are compatible with the DM's judgments on the reference alternatives. However, these models may provide different recommendations when extrapolated to other instances. Moreover, even if there exists a single compatible model, it may be sensitive to changes in the set of reference examples used to derive it.

Measuring robustness and providing robust recommendations has attracted much research in PDA and MCDA in general. To this end, analytical and simulation techniques have be considered. A typical analytical approach that is easy to implement for preference elicitation approaches that rely on linear programming formulations, is to use post-optimality analysis (Siskos and Grigoroudis 2010). Post-optimality analysis enables the identification of characteristic points from the solution polyhedron. Such points can be derived by analyzing the range of the criteria weights (i.e., minimum and maximum weights) within the set of feasible solutions. The variation of the resulting solutions can be used to derive measures of robustness, such as the stability index proposed by Grigoroudis and Siskos (2002). Other measures of robustness based on analytical procedures for assessing the size of the set of feasible solutions have been proposed by Doumpos and Zopounidis (2016). Moreover, the average of the solutions can be employed as an approximate barycenter solution corresponding to a decision model that best represents the DM's holistic judgments about the reference actions. The selection of representative decision models has also been considered in context of model regularization (Doumpos and Zopounidis 2007), whereas an analytic center formulation was proposed in Bouse et al. (2010).

Alternatively, instead of seeking to define a representative model, another approach that has been considered, focuses on providing a range of recommendations using all compatible decision models (Greco et al. 2008, 2010). This framework is general enough to be applicable to different types of decision models. It was first introduced as an extension of the UTA family of methods, which relies on additive value function, but it has been also been applied to outranking approaches and rule-based models (Greco et al. 2011; Kadziński and Ciomek 2016; Kadziński et al. 2016), as well as to group decision making (Greco et al. 2012), and models that consider preference intensities (Figueira et al. 2009).

Except for analytical procedures, simulation techniques have also been used, based on the framework of the SMAA methods (Lahdelma and Salminen 2001). Such approaches rely on the examination of a large set of random scenarios for the parameters of a decision model, sampled from the set of all compatible decision models. The simulation process yields recommendations in probabilistic form using the information provided through a reference set of decision examples (Tervonen et al. 2009). Analytical and simulation approaches have also be combined into hybrid schemes (Corrente et al. 2016; Kadziński et al. 2016; Kadziński and Tervonen 2013), whereas robustness measures based on simulation techniques have also been proposed (Kadzinski et al. 2017; Vetschera et al. 2010).

Even though a lot of research has already been made on the issue of robustness, there is still open room for further advances in this area. One particular aspect that deserves further consideration is data-driven robustness. By this we consider the robustness of the models with respect to the data used to infer them (Doumpos and Zopounidis 2016). Most of the literature on the subject of robustness analysis for PDA has focused on measures and procedures assuming a well-defined set of reference actions. However, changes in the reference set by adding and removing alternatives could have a significant impact on the results and their robustness. Therefore, additional research is needed on this topic to obtain a comprehensive view of robustness both in terms of the information provided by a given set of decision examples as well as the stability of the results to changes in the reference set as well as when extrapolating a decision model derived from PDA to new decision instances outside the reference set. These issues further relate to the complexity of the models, as more complex models are more sensitive to data perturbations, thus potentially being prone to providing poor generalizing results.

3.2 Alternative Modeling Forms

Most of the existing research in the area of PDA for multicriteria decision aiding has focused on well-known MCDA models such as additive value functions and outranking relations. Other modeling approaches, however, are also relevant for decision modeling and preference elicitation. The main advantage of using alternative types of decision models is the consideration of more general preference structures that cannot be adequately described through classical decision models.

One approach that has attracted considerably interest involves the use of the Choquet integral as a preference model that allows the consideration of interactions between the criteria (Marichal and Roubens 2000). Some work on this studies can be found in the works of Kojadinovic (2004) and Angilella et al. (2010), while Grabisch and Labreuche (2008) provide a review such models.

Symbolic model expressed in the form of decision rules have also attracted wide research interest. Decision rule models originate from the area of machine learning. The most complete and well-axiomatized methodology for constructing decision rule preference models from decision examples, is based on the rough sets theory (Greco et al. 1999, 2001; Pawlak and Słowiński 1994). The decision rule preference model has been initially considered in the context of multicriteria classification (sorting) problems. This theory, however, has also been considered in the context of ranking and choice decision problems (Fortemps et al. 2008; Greco et al. 2001), multi-objective optimization (Greco et al. 2008), conjoint measurement (Greco et al. 2004) and Bayesian decision theory (Greco et al. 2007). An axiomatic characterization of the decision rule preference model in relation to other types of models can be found in Greco et al. (2004).

Except for decision rules, other types of machine learning preference models include neural networks (NNs) and kernel models. NNs have been used to learn general types of value function (Malakooti and Zhou 1994) and binary relations (Hu 2009) as well as for multicriteria clustering problems (Malakooti and Raman 2000) and for preference elicitation in multi-objective optimization (Chen and Lin 2003). Kernel methods have been used to infer value functions for ranking problems (Herbrich et al. 2000) and for preference modeling through binary relations (Pahikkala et al. 2010). It has been shown that kernel preference models can represent a wide class of existing classical models, including value models, Choquet integrals, and outranking models (Waegeman et al. 2009). A review of this promising area regarding the connections of PDA with machine learning was presented in Doumpos and Zopounidis (2011).

3.3 Optimization Models and Large-Scale Data

Classical PDA methodologies rely on linear programming (LP) formulations for inferring preferential information from decision examples. LP is a very convenient, yet powerful tool. At least three main strengths can be noted regarding the use of LP for PDA:

- *Availability and computational power*: LP solvers are easily available and solution methods are powerful enough to allow the handling of large-scale instances without posing any computational difficulty.
- *Flexibility*: LP formulations provide a lot of flexibility to analysts and DMs, as they can be easily modified and adapted to allow the handling of different types of input information and the calibration of the final decision model.

- *Post-optimality analysis*: the theory of LP provides a lot of well-established tools for conducting post-optimality analysis in various forms, which as it was explained in the previous section, is an important issue for robustness analysis.

LP formulations have been used in PDA to infer the preferential parameters for various types of decision models. The most typical examples involve additive value functions in ordinal regression and classification problems (i.e., the UTA family of methods and its extensions (Doumpos and Zopounidis 2002; Greco et al. 2008; Jacquet-Lagrèze and Siskos 2001; Siskos et al. 2016)). Similar formulations have also been used for simplified outranking models (Dias et al. 2002; Doumpos and Zopounidis 2004; Mousseau et al. 2001).

On the negative side, LP formulations are not general enough to cover all types of decision models. This shortcoming can be partially addressed through extensions involving mixed-integer linear programming (MILP) formulations. Among other uses, such approaches enable the handing of alternative measures of the deviations between the DM's holistic evaluations and the model's outputs (e.g., the Kendall's τ rank correlation coefficient for ordinal regression (Jacquet-Lagrèze and Siskos 1982), the inference of more complex preference structures (Bisdorff et al. 2013; Dias and Mousseau 2006), the handling of special types of requirements during the model inference process (Köksalan et al. 2017), the treatment and resolution of inconsistencies in the judgments provided by the DM (Mousseau et al. 2003), as well as the implementation of specific types of robustness analysis (Kadziński et al. 2012).

The main shortcoming of MILP formulations, however, is their high computational burden, due to their combinatorial nature. Moreover, the inference of complex decision models cannot be fully represented in analytical mathematical programming formulations. The only available option to overcome such difficulties is to resort to special type algorithms (Belahcène et al. 2018) or metaheuristics (Belacel et al. 2007; Doumpos 2010; Doumpos et al. 2009; Fernandez et al. 2012; Sobrie et al. 2018). Such approaches extend the range of possibilities for inferring preferential information through PDA techniques, enabling both the handing of complex decision models and the use of large data.

The extensions of PDA approaches to large data is also an issue that could be further explored. In a typical MCDA context, a DM can only provide a few holistic evaluations as decision examples for implementing a PDA methodology. However, as decision problems in various fields become more data intensive, the handling of large data can open new areas for PDA in connection with other related areas, such as the field of preference learning (Fürünkranz and Hüllermeier 2011), which has been mostly developed within the machine learning community. It should be noted, however, that extensions to large and big data is not solely about algorithmic advances that will allow existing inference procedures to scale up well with the size of the data. The type of output obtained from PDA approaches and the way decision aiding is implemented and provided, is of equal importance. To this end, it is important to further consider implementations into new types of decision support systems and other publicly available software tools based on common standards (Cailloux et al.

2014), that except for traditional modeling and data management capabilities, will further provide advanced visualization and reporting modules targeting both end users (i.e., DMs) and data/decision analysts.

4 Conclusions

The elicitation of preferential information is fundamental for the application of any MCDA methodology. PDA has evolved as a major area of research in MCDA providing a general framework for inferring decision models from data. Initially developed in the context of value function models for ordinal regression and ranking, PDA approaches are now available for a wide class of different types of decision models and decision problematics. PDA facilitates the inference of preferential information enabling decision makers to provide easy to understand inputs in the form of holistic evaluations, rather than requiring the specification of complex parameters, which may be difficult to understand.

In this chapter we reviewed the main approaches in this field, covering different types of decision models. Moreover, state-of-the-art advances and future perspective were discussed on issues such as robustness analysis, the use of new modeling forms, and the different types of model inference procedures. These emerging areas of research can widen the range of applications of PDA to new areas and enable the handling of more complex information available in various forms and collected through different sources. To this end, the interconnections with other related areas such as data and decision analytics could also be of interest.

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Part II
New Aggregation Approaches

Normed Utility Functions: Some Recent Advances

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Abstract In this chapter, we summarize some new results and trends in aggregation theory, thus contributing to the domain of normed utility functions. In particular, we discuss k-additive and k-maxitive aggregation functions and also present some construction methods. Penalty- and deviation-based approaches can be seen as implicitly given construction methods. For non-symmetric (weighted) aggregation functions, four symmetrization methods based on the optimization are introduced. All discussed results and construction methods are exemplified.

1 Introduction

In this chapter, we will consider a fixed number n of criteria $C_1, C_2, \ldots, C_n$. For any alternative **a**, the degree x_i of satisfaction of C_i by the alternative **a** is assumed to be a real number from the unit interval [0, 1]. Note that $x_i = 1$ means the complete satisfaction of the criterion C_i by the alternative **a**, and on the other hand, $x_i = 0$ represents

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the complete failure of **a** in C_i. Thus, any alternative **a** can be identified by a score vector $\mathbf{x} = (x_1, \ldots, x_n) \in [0, 1]^n$, and the global utility $U(\mathbf{a})$ of the alternative **a** can be seen as a value $A(\mathbf{x})$ of an appropriate function $A\colon [0, 1]^n \to \mathbb{R}$. To keep the Pareto property of the utility U, the increasing monotonicity of A in each variable should be required. Supposing U to be a normed utility function, i.e., max $U = 1$, min $U = 0$, the function A has to satisfy two boundary conditions, $A(\mathbf{0}) = A(0, \ldots, 0) = 0$, and $A(\mathbf{1}) = A(1, \ldots, 1) = 1$. Hence, normed utility functions are in a one-to-one correspondence with aggregation functions.

Definition 1 A function $A\colon [0, 1]^n \to [0, 1]$ is an (n-ary) aggregation function whenever it is an order preserving homomorphisms between bounded lattices $([0, 1]^n, \leq)$ and $([0, 1], \leq)$, i.e., if A is increasing in each variable and $A(\mathbf{0}) = 0$, $A(\mathbf{1}) = 1$.

An overview of classes of aggregation functions, their properties, relations, construction methods and many other information on aggregation functions can be found, e.g., in the recent monographs (Beliakov et al. 2007, 2016; Calvo and Beliakov 2010; Grabisch et al. 2009). Several links between aggregation functions and multi-criteria decision methods were also highlighted in edited volumes (Greco et al. 2005, 2010).

The aim of this chapter is to recall some recent results achieved in aggregation theory and to introduce some new ideas with a high potential to be applied in multi-criteria decision support. In the next section, we discuss k-additive aggregation functions. Note that though k-additive capacities were introduced twenty years ago, the idea of k-additive aggregation functions is very fresh. Observe that the Owen extension of capacities (Owen 1988) preserves k-additivity, which is not the case of the Lovász extension (Lovász 1983), i.e., of the Choquet integral (Choquet 1953). Similarly, the concept of k-maxitive capacities has been known for about twenty years (Mesiar 1997, 2003). In Sect. 3, we discuss k-maxitive aggregation functions which have recently been introduced in Mesiar and Kolesárová (2016). Note that the Sugeno integral (Sugeno 1974) as well as the Shilkret integral (Shilkret 1971) preserve k-maxitivity. In Sect. 4, we recall two construction methods for idempotent (i.e., unanimous) aggregation functions. The penalty-based approach for construction of aggregation functions already has quite a long history, but the deviation-based approach has been introduced recently. We recall both these methods because of their link which can be briefly compared to the link between extremal points of real functions and roots of the related derivatives. In Sect. 5 we discuss some symmetrization methods for aggregation functions, including a few new proposals. Finally, some concluding remarks are provided.

2 k-Additive Aggregation Functions

For a fixed $n \in \mathbb{N}$, let N denote the set $\{1, 2, \ldots, n\}$. Recall that a capacity $\mu\colon 2^N \to [0, 1]$ is a monotone set function satisfying the conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$. In 1997, Grabisch introduced the notion of k-additive capacities, $k \in N$, see Grabisch

(1997). Note that 1-additive capacities are just the standard discrete probabilities on N. The Grabisch k-additivity of a capacity μ has been based on the vanishing of the related Möbius transform $M_\mu\colon 2^N \to \mathbb{R}$ on all subsets of N containing at least $(k+1)$ elements, i.e., if for each $E \subseteq N$, $M_\mu(E) = \sum_{F\subseteq E}(-1)^{|E\setminus F|}\mu(F) = 0$ whenever $|E| > k$. Note that this approach to k-additivity is strongly linked to the finiteness of the space $(N, 2^N)$. A generalization of k-additivity independent of the cardinality of a considered measurable space $(X, \mathscr{A})$ was discussed in Mesiar (2003), Valášková (2007), and on the space $(N, 2^N)$ it coincides with the Grabisch approach. Following Mesiar (2003), Valášková (2007), a capacity $\mu\colon 2^N \to [0,1]$ is *k-additive* if and only if for any pairwise disjoint subsets $E_1, \dots, E_{k+1}$ of N we have

$$\sum_{\emptyset\neq I\subseteq\{1,\dots,k+1\}} (-1)^{k+1-|I|}\,\mu\left(\bigcup_{i\in I} E_i\right) = 0. \tag{1}$$

In particular, a capacity μ is 2-additive if and only if for any pairwise disjoint subsets E_1, E_2, E_3 of N we have

$$\begin{aligned} m(E_1\cup E_2\cup E_3) - (m(E_1\cup E_2) + m(E_2\cup E_3) + m(E_1\cup E_3)) \\ + m(E_1) + m(E_2) + m(E_3) = 0. \end{aligned} \tag{2}$$

Inspired by Eq. (1), we have recently introduced the concept of k-additivity for aggregation functions, see Kolesárová et al. (2016). Observe that in the case of aggregation functions the order k is not constrained by the dimension n as it is in the case of capacities.

Definition 2 Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function and let $k \in \mathbb{N}$ be a fixed integer. Then A is said to be k-additive whenever for all $\mathbf{x}_1, \dots, \mathbf{x}_{k+1} \in [0,1]^n$ such that also $\sum_{i=1}^{k+1}\mathbf{x}_i \in [0,1]^n$ we have

$$\sum_{\emptyset\neq I\subseteq\{1,\dots,k+1\}} (-1)^{k+1-|I|}\,A\left(\sum_{i\in I}\mathbf{x}_i\right) = 0. \tag{3}$$

Note that the 1-additivity of A is just the standard additivity of aggregation functions and then, necessarily, $A = M_{\mathbf{w}}$ is a weighted arithmetic mean, $M_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^n w_i x_i$, where $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ is a normed weighting vector, i.e., satisfying the property $\sum_{i=1}^n w_i = 1$. For $k = 2$, A is 2-additive if and only if for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in [0,1]^n$ such that $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 \in [0,1]^n$ we have

$$\begin{aligned} A(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3) - (A(\mathbf{x}_1+\mathbf{x}_2) + A(\mathbf{x}_2+\mathbf{x}_3) + A(\mathbf{x}_1+\mathbf{x}_3)) \\ + A(\mathbf{x}_1) + A(\mathbf{x}_2) + A(\mathbf{x}_3) = 0. \end{aligned} \tag{4}$$

Observe that the set function $\mu\colon 2^N \to [0,1]$ defined by $\mu(E) = A(\mathbf{1}_E)$, where $\mathbf{1}_E\colon N \to \{0,1\}$ is the characteristic function of E, $\mathbf{1}_E(i) = \begin{cases} 1 & \text{if } i \in E, \\ 0 & \text{otherwise,}\end{cases}$ and A

is an aggregation function, is a capacity on N. Note that we identify the characteristic function $\mathbf{1}_E$ with the n-ary vector $(\mathbf{1}_E(1), \ldots, \mathbf{1}_E(n))$. It is not difficult to check that the k-additivity of A ensures the k-additivity of μ whenever $k \le n$, i.e., A is an appropriate extension of a k-additive capacity μ, but not vice-versa. Clearly, if A is a k-additive aggregation function then it is also m-additive for any integer $m > k$. A similar claim holds for capacities. On the other hand, each capacity μ on N is n-additive, which is not the case of aggregation functions.

Example 1

(i) Consider the aggregation function $A_k\colon [0, 1]^n \to [0, 1]$, $A_k(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i^k$, where $k \in \mathbb{N}$. Then A_k is k-additive, and if $k > 1$ it is not $(k-1)$-additive.
(ii) The aggregation function $Min\colon [0, 1]^n \to [0, 1]$, $Min(\mathbf{x}) = \min\{x_1, \ldots, x_n\}$, is not k-additive for any $k \in \mathbb{N}$.

Due to Example 1(ii), it is evident that the Lovász extension of capacities (Lovász 1983) does not preserve k additivity whenever $k > 1$. It is enough to consider $n = 2$ and the smallest capacity m_* on $N = \{1, 2\}$, given by $m_*(E) = 0$ whenever E differs from N. Then the related Lovász extension, i.e., the Choquet integral with respect to m_*, is just the aggregation function Min. As $n = 2$, μ_* is 2-additive, but Min is not of this property. On the other hand, we have the following positive result. More details can be found in Kolesárová et al. (2016).

Theorem 1 *Let $\mu\colon 2^N \to [0, 1]$ be a k-additive capacity. Then the corresponding Owen extension of μ, $O_\mu\colon [0, 1]^n \to [0, 1]$ given by*

$$O_\mu(\mathbf{x}) = \sum_{\emptyset \ne E \subseteq N} M_\mu(E) \left(\prod_{i \in E} x_i\right), \tag{5}$$

is a k-additive aggregation function.

As $M_\mu(E) = 0$ whenever μ is a k-additive capacity and $|E| > k$, it is clear that the corresponding Owen extension O_μ is a polynomial of variables $x_1, \ldots, x_n$ whose degree does not exceed k. This observation is also a characterization of general k-additive aggregation functions, see Kolesárová et al. (2016).

Theorem 2 *An aggregation function function $A\colon [0, 1]^n \to [0, 1]$ is k-additive for an integer $k \in \mathbb{N}$ if and only if it is a polynomial of a degree equal at most k.*

Note that n-ary k-additive aggregation functions are polynomials of n variables $x_1, \ldots, x_n$ of a degree at most k, which are increasing on $[0, 1]^n$ and satisfy the boundary conditions. The following theorem gives an explicit description of 2-additive n-ary aggregation functions.

Theorem 3 *A function $A\colon [0, 1]^n \to [0, 1]$ is a 2-additive n-ary aggregation function if and only if for each $\mathbf{x} \in [0, 1]^n$,*

$$A(\mathbf{x}) = \sum_{1\le i\le j\le n} \alpha_{ij} x_i x_j + \sum_{i=1}^{n} \beta_i x_i, \tag{6}$$

where the coefficients α_{ij} *and* β_i *are constrained by the following conditions:*

$$\sum_{1\le i\le j\le n} \alpha_{ij} + \sum_{i=1}^{n} \beta_i = 1,$$

and

$$\beta_i \ge 2\max\{0, -\alpha_{ii}\} + \sum_{p<j,\, i\in\{p,j\}} \max\{0, -\alpha_{pj}\} \;\; for\ all\ i \in \{1, \dots, n\}.$$

In particular, if $n = 1$ then $A\colon [0, 1] \to [0, 1]$ is a 2-additive aggregation function if and only if $A(x) = (\alpha + 1)x - \alpha x^2$ for some $\alpha \in [-1, 1]$.

As an easy consequence of Theorem 2, it may be concluded that for normed weighting vectors $\mathbf{w}_1, \dots, \mathbf{w}_k \in [0, 1]^n$ the function $A\colon [0, 1]^n \to [0, 1]$ given by $A(\mathbf{x}) = \prod_{i=1}^{k} M_{\mathbf{w}_i}(\mathbf{x})$, i.e., the product of k weighted arithmetic means, is a proper k-additive aggregation function. Also note that any convex combination of k-additive n-ary aggregation functions is k-additive.

The 2-additive n-ary aggregation functions can also be characterized as follows.

Theorem 4 *For a fixed* $n \in \mathbb{N}$*, the class of all* 2*-additive* n*-ary aggregation functions is a simplex with vertices* $F_{ij}, G_{ij}\colon [0, 1]^n \to [0, 1]$*,* $1 \le i \le j \le n$*, given by*

$$F_{ij}(\mathbf{x}) = x_i x_j, \quad G_{ij}(\mathbf{x}) = x_i + x_j - x_i x_j.$$

Note that 2-additive Owen extensions are characterized, for each $i \in N$, by the equality $\alpha_{ii} = \beta_{ii}$ in a convex decomposition

$$O_\mu = \sum_{1\le i\le j\le n} (\alpha_{ij} F_{ij} + \beta_{ij} G_{ij}).$$

Recall that the standard arithmetic mean can be characterized as a symmetric idempotent (i.e., anonymous and unanimous) 1-additive aggregation function. Then symmetric idempotent 2-additive aggregation functions can be viewed as 2-additive arithmetic means.

Theorem 5 *Let* $n \ge 2$ *be fixed. Then a function* $A\colon [0, 1]^n \to [0, 1]$ *is a* 2*-additive arithmetic mean if and only if, for each* $\mathbf{x} \in [0, 1]^n$*,*

$$A(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{\delta}{n}\sum_{i=1}^{n} x_i^2 - \frac{2\delta}{n(n-1)} \sum_{i<j} x_i x_j, \tag{7}$$

for some $\delta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Observe that (7) can be written as

$$A(\mathbf{x}) = M(\mathbf{x}) + \delta s_{\mathbf{x}}^2,$$

where $M(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the standard arithmetic mean and $s_{\mathbf{x}}^2$ is the variance of the sample $\mathbf{x} = (x_1, \ldots, x_n)$ given by $s_{\mathbf{x}}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - M(\mathbf{x}))^2$.

In particular, when $n = 2$, $A\colon [0, 1]^2 \to [0, 1]$ is a 2-additive arithmetic mean if and only if, for each $(x_1, x_2) \in [0, 1]^2$,

$$A(x_1, x_2) = \frac{x_1 + x_2}{2} + \gamma(x_1 - x_2)^2, \text{ where } \gamma \in \left[-\frac{1}{4}, \frac{1}{4}\right].$$

Any class of aggregation functions related to the standard additivity can be generalized by considering k-additivity instead. For example, we can introduce the property of comonotone k-additivity and subsequently, k-OWA operators or k-Choquet integrals. Although the first attempt in this direction was done in Kolesárová et al. (2016), the topic is still open to a deeper investigation.

3 *k*-Maxitive Aggregation Functions

The notion of k-maxitive capacities was introduced twenty years ago by Mesiar (1997), see also Mesiar (2003). Recall that a capacity $\mu\colon 2^N \to [0, 1]$ is said to be k-maxitive (for some $k \in N$) if and only if for any $E \subseteq N$ with cardinality $|E| > k$ there is a subset $F \subset E$ with cardinality not exceeding k, $|F| \leq k$, such that $\mu(E) = \mu(F)$. Similarly to the case of k-additive capacities, a k-maxitive capacity μ is determined by its values $\mu(F)$ on the sets $F \subseteq N$ with $|F| \leq k$. Note that 1-maxitive capacities are the standard maxitive capacities, i.e., possibility measures on N as introduced by Zadeh (1978). An equivalent definition of k-maxitive capacities, independent of the cardinality of a considered space, was given in Valášková (2007). By this source, a capacity $\mu\colon 2^N \to [0, 1]$ is k-maxitive if and only if for any subsets $E_1, \ldots, E_{k+1}$ of N there is an index set $I \subset \{1, \ldots, k+1\}$ with $|I| = k$, such that

$$\mu\left(\bigcup_{i=1}^{k+1} E_i\right) = \mu\left(\bigcup_{i\in I} E_i\right).$$

Inspired by the last equality, we have introduced the notion of k-maxitive aggregation functions, see Mesiar and Kolesárová (2016).

Definition 3 Let $A\colon [0, 1]^n \to [0, 1]$ be an aggregation function and $k \in N$ a fixed integer. A is said to be k-maxitive if for any $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \in [0, 1]^n$ there is an index set $I \subsetneqq \{1, \ldots, k+1\}$ such that

$$A\left(\bigvee_{i=1}^{k+1} \mathbf{x}_i\right) = A\left(\bigvee_{i\in I} \mathbf{x}_i\right). \tag{8}$$

Clearly, (8) can be written equivalently in the form

$$A\left(\bigvee_{i=1}^{k+1} \mathbf{x}_i\right) = \bigvee_{i=1}^{k+1} A\left(\bigvee_{j\neq i} \mathbf{x}_j\right).$$

The notion of k-maxitivity can be introduced straightforwardly for aggregation functions acting on bounded lattices, and, in particular, on ordinal (linguistic) scales. Formally, one can also introduce the notion of k-maxitive aggregation functions for $k > n$, but then it is not difficult to see that for any $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \in [0, 1]^n$ there is an index set $I \subset \{1, \ldots, k+1\}$, $|I| = n$, such that $\bigvee_{i=1}^{k+1} \mathbf{x}_i = \bigvee_{i\in I} \mathbf{x}_i$. Hence, any aggregation function $A\colon [0, 1]^n \to [0, 1]$ is n-maxitive.

Example 2

(i) The aggregation function $Min\colon [0, 1]^n \to [0, 1]$ is n-maxitive but not $(n-1)$-maxitive. The same is true for the arithmetic mean $M\colon [0, 1]^n \to [0, 1]$.
(ii) For any weighting vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0, 1]^n$, $\bigvee_{i=1}^{n} w_i = 1$, the weighted maximum $Max_{\mathbf{w}}\colon [0, 1]^n \to [0, 1]$ given by $Max_{\mathbf{w}} = \bigvee_{i=1}^{n}(w_i \wedge x_i)$ is a 1-maxitive aggregation function.
(iii) The median function $Med\colon [0, 1]^n \to [0, 1]$ is a k-maxitive aggregation function but not $(k-1)$-maxitive, where $k = \frac{n+1}{2}$ if n is odd, and $k = \frac{n}{2} + 1$, if n is even.

Recall that each 1-maxitive aggregation function $A\colon [0, 1]^n \to [0, 1]$ can be represented in the form

$$A(\mathbf{x}) = \bigvee_{i=1}^{n} f_i(x_i),$$

where $f_i\colon [0, 1] \to [0, 1]$, $i \in N$, are increasing functions constrained by $f_i(0) = 0$ for each $i \in N$, and $f_{i_0}(1) = 1$ for at least one index $i_0 \in N$. A similar representation is valid for k-maxitive aggregation functions, $k \geq 2$, when k-ary increasing functions are considered. In particular, one can consider a single k-ary aggregation function for constructing k-maxitive aggregation functions.

Theorem 6 *Let $n \in \mathbb{N}$, $k \in N$, and let B be a k-ary aggregation function. Then the function $A_B\colon [0, 1]^n \to [0, 1]$ defined by*

$$A_B(\mathbf{x}) = \bigvee_{\substack{I\subseteq N\\ |I|=k}} B(\mathbf{x}_I), \tag{9}$$

where for $I = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$, the notation $\mathbf{x}_I = (x_{i_1}, \ldots, x_{i_k})$ is used, is a k-maxitive n-ary aggregation function.

Example 3 Let $M_k\colon [0,1]^k \to [0,1]$ be the arithmetic mean, $M_k(x_1,\dots,x_k) = \frac{1}{k}\sum_{i=1}^{k} x_i$. Then for any $n \geq k$, the k-maxitive n-ary aggregation function A_{M_k} is given by

$$A_{M_k}(\mathbf{x}) = \bigvee_{1\leq i_1<\cdots<i_k\leq n} \left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) = \frac{1}{k}\sum_{i=1}^{k} x_{(i)},$$

where $(\cdot)\colon N \to N$ is a permutation satisfying $x_{(1)} \geq \cdots \geq x_{(n)}$.

Observe that A_{M_k} is a special OWA operator (Yager 1988) related to the normed weighting vector $\mathbf{w} = (\underbrace{1/k,\dots,1/k}_{k-times}, \underbrace{0,\dots,0}_{(n-k)-times})$.

For symmetric k-maxitive aggregation functions, formula (9) is not only a construction method but a representation as well.

Theorem 7 *A symmetric aggregation function* $A\colon [0,1]^n \to [0,1]$ *is* k*-maxitive for some* $k \in N$ *if and only if there is a symmetric* k*-ary aggregation function* $B\colon [0,1]^k \to [0,1]$ *such that* $A = A_B$*, and then*

$$A(\mathbf{x}) = B(x_{(1)},\dots,x_{(k)}),$$

where $(\cdot)\colon N \to N$ *is a permutation described in Example 3.*

To illustrate Theorem 7, consider the aggregation function $A = Med\colon [0,1]^3 \to [0,1]$. Recall that the ternary median is a symmetric 2-maxitive aggregation function. Then for the binary minimum $Min\colon [0,1]^2 \to [0,1]$ we have $A = A_{Min}$, i.e., $Med(x_1,x_2,x_3) = Min(x_{(1)},x_{(2)})$.

Let us still illustrate a link between k-maxitive capacities and k-maxitive aggregation functions. First, observe that a capacity μ on N induced by an n-ary k-maxitive aggregation function A, $\mu(E) = A(\mathbf{1}_E)$ for each $E \subseteq N$, is k-maxitive. On the other hand, there are several integrals resulting in a k-maxitive aggregation function once a k-maxitive capacity is considered.

Theorem 8 *Let* $\otimes\colon [0,1]^2 \to [0,1]$ *be a fixed semicopula and* $\mu\colon 2^N \to [0,1]$ *a* k*-maxitive capacity. Then the function* $\mathfrak{I}_{\otimes,\mu}\colon [0,1]^n \to [0,1]$ *given by*

$$\mathfrak{I}_{\otimes,\mu}(\mathbf{x}) = \bigvee_{i=1}^{n} x_i \otimes \mu(\{j \in N \mid x_j \geq x_i\}), \tag{10}$$

is a k*-maxitive aggregation function.*

For the convenience of the reader we recall that a semicopula $\otimes\colon [0,1]^2 \to [0,1]$ is a binary aggregation function with neutral element $e = 1$, i.e., satisfying $x \otimes 1 = 1 \otimes x = x$ for each $x \in [0,1]$, see Durante and Sempi (2005).

Note that the aggregation function $\mathfrak{I}_{\otimes,\mu}$ introduced by (10) is the smallest universal integral on $[0,1]$ based on a semicopula $\otimes$ and a capacity μ, see Klement et al. (2010).

In particular, $\mathfrak{I}_{\wedge,\mu}$, where $\wedge$ denotes the minimum, is the Sugeno integral (Sugeno 1974), and for $\otimes = \cdot$ (product), $\mathfrak{I}_{\cdot,\mu}$ is the Shilkret integral (Shilkret 1971). Theorem 8 cannot be extended for the Choquet integral Ch_μ (Choquet 1953). Consider, for example, $n = 2$ and a 1-maxitive capacity (i.e., a possibility measure) μ determined by $\mu(\{1\}) = 0.5$ and $\mu(\{2\}) = 1$. Then

$$Ch_\mu(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 \geq x_2, \\ x_2 & \text{otherwise.} \end{cases}$$

It is not difficult to check that Ch_μ is not 1-maxitive. More details can be found in Mesiar and Kolesárová (2016).

Remark 1 A simultaneous requirement of k-maxitivity and p-additivity is rather restrictive for aggregation functions. For example, the only 1-maxitive and 1-additive aggregation functions are the projections $P_i : [0, 1]^n \to [0, 1]$, $P_i(\mathbf{x}) = x_i$. 1-maxitivity and proper 2-additivity yield the squares of projections, while the proper 2-maxitivity and proper 2-additivity are simultaneously satisfied only by the products $P_i \cdot P_j, i \neq j, i, j \in N$.

4 Penalty- and Deviation-Based Constructions of Aggregation Functions

The idea of minimizing the "distances" $\sum_{i=1}^n h(x_i, y)$ of a given sample $(x_1, \ldots, x_n)$ from a point y (centroids, Fréchet means) inspired Yager's initiation of the penalty-based constructions of aggregation functions (Yager 1993).

Formally, for any function $h : [0, 1]^2 \to [0, \infty[$ such that $h(x, y) = 0$ if and only if $x = y$ and $h(x, y) \leq \max\{h(x_1, y), h(x_2, y)\}$ if $x \in [x_1, x_2]$, the value $h(x, y)$ can be seen as a penalty "punishing" the replacement of the input x by y. The total penalty $\sum_{i=1}^n h(x_i, y)$ then describes the penalty for representing the sample $(x_1, \ldots, x_n)$ by a single value y. Then the optimal choice of y is that one minimizing the total penalty. For example, if $h(x, y) = (x - y)^2$, the minimizer of $\sum_{i=1}^n (x_i - y)^2$ is just the arithmetic mean $M(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n x_i$. In general, the minimizer of a total penalty need not exist, or there can be more such minimizers. Consider, for example, $h(x, y) = |x - y|$. When assuming a permutation $\langle\cdot\rangle : N \to N$ such that $x_{\langle 1\rangle} \leq \cdots \leq x_{\langle n\rangle}$, then if n is odd, $n = 2k - 1$, the minimizer of $\sum_{i=1}^n |x_i - y|$ is just $x_{\langle k\rangle}$, i.e., the median of the sample $(x_1, \ldots, x_n)$, and if n is even, $n = 2k$, any $y \in [x_{\langle k\rangle}, x_{\langle k+1\rangle}]$ is a minimizer of the considered total penalty. If we take the mid point of the interval $[x_{\langle k\rangle}, x_{\langle k+1\rangle}]$ as a single representative of all possible minimizers, we again obtain the median.

Now, consider $h(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise,} \end{cases}$ and put $\mathbf{x} = (0, 0, 0.5, 1, 1)$. Then

$$\sum_{i=1}^{n} h(x_i, y) = \begin{cases} 3 & \text{if } y \in \{0, 1\}, \\ 4 & \text{if } y = 0.5, \\ 5 & \text{otherwise}, \end{cases}$$

i.e., we have two unconnected minimizers $y_1 = 0$ and $y_2 = 1$. Observe that in the case where we have a unique minimizer of $\sum_{i=1}^{n} h(x_i, y)$, the mode of the sample $(x_1, \dots, x_n)$ is obtained. However, the mode is not an aggregation function, as, e.g., its monotonicity is violated. To ensure that the result of the above sketched construction yields an aggregation function, in Calvo et al. (2002) the following approach to penalty functions was proposed.

Definition 4 Let $K: \mathbb{R} \to \mathbb{R}^+$ be a convex function with unique minimum $K(0) = 0$, and let $s: [0, 1] \to \mathbb{R}$ be a continuous strictly monotone function. Then the function $h: [0, 1]^2 \to \mathbb{R}^+$ given by $h(x, y) = K(s(x) - s(y))$ is called a dissimilarity function and the function $P_h: [0, 1]^{n+1} \to \mathbb{R}^+$ given by $P_h(\mathbf{x}, y) = \sum_{i=1}^{n} h(x_i, y)$ is called an h-penalty function.

Theorem 9 *Let $h: [0, 1]^2 \to \mathbb{R}^+$ be a dissimilarity function. Then, for any $\mathbf{x} \in [0, 1]^n$, the h-penalty function P_h set of minimizers is a closed interval $[a_\mathbf{x}, b_\mathbf{x}] \subseteq [0, 1]$, i.e., for any $y_1 \in [a_\mathbf{x}, b_\mathbf{x}]$ and $y_2 \in [0, 1]$ we have $P_h(\mathbf{x}, y_1) \leq P_h(\mathbf{x}, y_2)$. Moreover, the function $A_h: [0, 1]^n \to [0, 1]$ given by*

$$A_h(\mathbf{x}) = \frac{a_\mathbf{x} + b_\mathbf{x}}{2}$$

is a continuous symmetric idempotent aggregation function.

The approach to the construction of idempotent aggregation functions described in Theorem 9 can be further generalized. It allows to introduce weights into the symmetric aggregation functions A_h by considering a weighted h-penalty function $P_{h,\mathbf{w}}$ given by $P_{h,\mathbf{w}}(\mathbf{x}, y) = \sum_{i=1}^{n} w_i\, h(x_i, y)$. Next, one can consider a dissimilarity vector function $H = (h_1, \dots, h_n)$ and the related penalty function P_H given by $P_H(\mathbf{x}, y) = \sum_{i=1}^{n} h_i(x_i, y)$. More details can be found in Calvo et al. (2002).

Example 4 Let $h_1(x, y) = w_1|x - y|$ and $h_2(x, y) = w_2(x - y)^2$, the weights w_1, w_2 being positive, and let $H = (h_1, h_2)$. Then the idempotent aggregation function $A_H: [0, 1]^2 \to [0, 1]$ is given by

$$A_H(x_1, x_2) = \text{med}\left\{x_1, x_2 - \frac{w_1}{2w_2}, x_2 + \frac{w_1}{2w_2}\right\}.$$

Note that if $w_1 \geq 2w_2$, $A_H(x_1, x_2) = x_1$, i.e., A_H is the projection to the first coordinate.

For some other generalizations of a penalty-based approach to constructing aggregation functions we recommend (Bustince et al. 2017; Calvo and Beliakov 2010).

Another approach to constructing aggregation functions is based on deviation functions. The original idea of Daróczy (1972) was based on a continuous function $d\colon [0,1]^2 \to \mathbb{R}$ such that $d(x, y) = 0$ if and only if $x = y$, and $d(x, \cdot)\colon [0,1] \to \mathbb{R}$ is strictly increasing. Then the Daróczy mean $D_d\colon [0,1]^n \to [0,1]$ is just the root of the equation $\sum_{i=1}^n d(x_i, y) = 0$, i.e., for any $\mathbf{x} \in [0,1]^n$, $\sum_{i=1}^n d(x_i, D_d(\mathbf{x})) = 0$. However, neither the Daróczy mean nor its generalizations due to Losonczi (1973) and others, are aggregation functions, in general. To avoid this failure, we have proposed the concept of moderate deviation functions (Decký et al. 2018).

Definition 5 A function $d\colon [0,1]^2 \to \mathbb{R}$ is called a moderate deviation function if it satisfies the properties:

(i) the function $d(x, \cdot)\colon [0,1] \to \mathbb{R}$ is increasing for each $x \in [0,1]$,
(ii) the function $d(\cdot, y)\colon [0,1] \to \mathbb{R}$ is decreasing for each $y \in [0,1]$,
(iii) $d(x, y) = 0$ if and only if $x = y$.

Let us emphasize that the functions $d(x, \cdot)$ and $d(\cdot, y)$ are not necessarily strictly monotone neither continuous.

Theorem 10 *Let $d\colon [0,1]^2 \to \mathbb{R}$ be a moderate deviation function. Then the function $M_d\colon [0,1]^n \to [0,1]$ given by*

$$M_d(\mathbf{x}) = \frac{1}{2}\left(\sup\left\{y \in [0,1] \mid \sum_{i=1}^n d(x_i, y) < 0\right\} + \inf\left\{y \in [0,1] \mid \sum_{i=1}^n d(x_i, y) > 0\right\}\right) \tag{11}$$

(the standard convention $\inf \emptyset = 1$ *and* $\sup \emptyset = 0$ *being considered) is a symmetric idempotent aggregation function.*

Note that Theorem 10 can also be modified—either by introducing the weights and considering the function $\sum_{i=1}^n w_i d(x_i, y)$, or by considering a vector moderate deviation function $D = (d_1, \ldots, d_n)$ and consequently, using the function $\sum_{i=1}^n d_i(x_i, y)$. Also observe that both penalty- and deviation-based construction methods can be seen as implicit construction methods. For a fixed input vector $\mathbf{x} \in [0,1]^n$, instead of looking for a formula describing the corresponding aggregation function, we simply solve a univariate problem with parameters $x_1, \ldots, x_n$.

Another possible modification of (11) can be obtained by replacing the binary arithmetic mean by any idempotent aggregation function $B\colon [0,1]^2 \to [0,1]$ and defining the function $M_{d,B}\colon [0,1]^n \to [0,1]$ by

$$M_{d,B}(\mathbf{x}) = B\left(\sup\left\{y \in [0,1] \mid \sum_{i=1}^n d(x_i, y) < 0\right\}, \inf\left\{y \in [0,1] \mid \sum_{i=1}^n d(x_i, y) > 0\right\}\right). \tag{12}$$

Example 5

(i) Define the functions $d_1, d_2 \colon [0,1]^2 \to \mathbb{R}$ by $d_1(x, y) = y - x$ and $d_2(x, y) = (y - x)|y - x|$. Then both d_1 and d_2 are moderate deviation functions. Put $D = (d_1, d_2)$. To find the value $M_D(0, 1)$, we apply formula (11) to the function $d_1(0, y) + d_2(1, y) = y - (y - 1)^2$, which finally gives $M_D(0, 1) = \frac{3-\sqrt{5}}{2}$. Similarly, the input vector $(1, 0)$ is related to the function $d_1(1, y) + d_2(0, y) = y - 1 + y^2$ and thus $M_D(1, 0) = \frac{\sqrt{5}-1}{2}$.

(ii) Let $d \colon [0,1]^2 \to \mathbb{R}$ be the signum function applied to $y - x$, i.e., $d(x, y) = \operatorname{sgn}(y - x)$. Then d is a moderate deviation function and the related aggregation function M_d is just the median function, $M_d = Med$. Observe that neither $d(x, \cdot)$ nor $d(\cdot, y)$ are continuous (strictly monotone) functions.

(iii) For $c \in]0, \infty[$, define the function $d_c \colon [0,1]^2 \to \mathbb{R}$ by $d_c(x, y) = (x + c)(y - x)$. Then d_c is the Daróczy deviation function and $D_{d_c} = M_{d_c}$ is the Daróczy mean, independently of the considered parameter c. If $c \in]0, 1[$ then the function $d_c(\cdot, \frac{c+1}{2})$ is given by $d_c(x, \frac{c+1}{2}) = \frac{(c+1)c}{2} + \frac{1-c}{2}x - x^2$ and hence it is not decreasing. Thus, d_c is not a moderate deviation function and moreover, $D_d = M_d$ is not increasing for some sufficiently big $n \in \mathbb{N}$, hence it is not an aggregation function. For example, if $n = 2$ then M_d is an aggregation function whenever $c \geq \sqrt{0.5}$, but not if $c < \sqrt{0.5}$. Similarly, if $n = 3$ then M_d is an aggregation function if and only if $c \geq \sqrt{\frac{2}{3}}$. On the other hand, if $c \geq 1$ then d_c is a moderate deviation function and the related deviation-based mean $D_{d_c} = M_{d_c}$ is an aggregation function independently of $n \in \mathbb{N}$.

Example 6 For some real constants $b < 0 < a$, define the function $d_{a,b} \colon [0,1]^2 \to \mathbb{R}$ by

$$d_{a,b}(x, y) = \begin{cases} a & \text{if } y > x, \\ 0 & \text{if } y = x, \\ b & \text{if } y < x. \end{cases}$$

Then $d_{a,b}$ is a moderate deviation function. Let $B \colon [0,1]^2 \to [0,1]$ be an OWA operator given by

$$B(x, y) = \alpha \max\{x, y\} + (1 - \alpha) \min\{x, y\}.$$

Then the function $M_{d_{a,b},B} \colon [0,1]^n \to [0,1]$ given by formula (12) is an OWA operator with weights generated for any $n \in \mathbb{N}$ by a function $Q \colon [0,1] \to [0,1]$,

$$Q(x) = \begin{cases} 0 & \text{if } x < \frac{a}{a-b}, \\ \alpha & \text{if } x = \frac{a}{a-b}, \\ 1 & \text{otherwise.} \end{cases}$$

Hence, $M_{d_{a,b},B}(\mathbf{x}) = \sum_{i=1}^{n} \left(Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right) x_{(i)}$, where the permutation $(\cdot) \colon N \to N$ satisfies $x_{(1)} \geq \cdots \geq x_{(n)}$. Note that if $\alpha = \frac{1}{2}$ and $b = -a$ then $M_{d_{a,b},B} = Med$ is the median function.

5 Some Symmetrization Methods

Standard symmetrization methods are related to particular permutations of input vectors $\mathbf{x} \in [0,1]^n$. Namely, considering the permutations $(\cdot), \langle\cdot\rangle : N \to N$ introduced above, i.e., permutations satisfying $x_{(1)} \geq \cdots \geq x_{(n)}$ and $x_{\langle 1\rangle} \leq \cdots \leq x_{\langle n\rangle}$, respectively, for any aggregation function $A : [0,1]^n \to [0,1]$ we obtain symmetric n-ary aggregation functions $A_{()}, A_{\langle\rangle}$ given by

$$A_{()}(\mathbf{x}) = A(x_{(1)}, \ldots, x_{(n)}) \text{ and } A_{\langle\rangle}(\mathbf{x}) = A(x_{\langle 1\rangle}, \ldots, x_{\langle n\rangle}),$$

respectively. For more details see Grabisch et al. (2009). Note that symmetrization of a weighted arithmetic mean $M_\mathbf{w}$ yields OWA operators $OWA_\mathbf{w}$ and $OWA_{\mathbf{w}^{rev}}$ given by

$$OWA_\mathbf{w}(\mathbf{x}) = \sum_{i=1}^n w_i x_{(i)} \text{ and } OWA_{\mathbf{w}^{rev}}(\mathbf{x}) = \sum_{i=1}^n w_i x_{\langle i\rangle} = \sum_{i=1}^n w_{n-i+1} x_{(i)},$$

respectively.

Recently, we have introduced in Mesiar et al. (2018) two different symmetrization methods resulting, for any aggregation function $A : [0,1]^n \to [0,1]$, in functions $A^*, A_* : [0,1]^n \to [0,1]$ given by

$$A^*(\mathbf{x}) = \bigvee_{\sigma \in \mathscr{P}_n} A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \tag{13}$$

and

$$A_*(\mathbf{x}) = \bigwedge_{\sigma \in \mathscr{P}_n} A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \tag{14}$$

where $\mathscr{P}_n$ is the set of all $N \to N$ permutations.

Note that instead of the operators $\vee$ and $\wedge$ in (13) and (14), respectively, one can use any other $(n!)$-ary symmetric aggregation function B, e.g., the arithmetic mean. Then the function $A_{[B]} : [0,1]^n \to [0,1]$ given by

$$A_{[B]}(\mathbf{x}) = B(A(\mathbf{x}_{\sigma_1}), \ldots, A(\mathbf{x}_{\sigma_{n!}})),$$

where $\{\sigma_1, \ldots, \sigma_{n!}\} = \mathscr{P}_n$, is a symmetric aggregation function satisfying $A_* = A_{[Min]} \leq A_{[B]} \leq A_{[Max]} = A^*$.

Example 7

(i) Let $M_\mathbf{w}$ be a weighted arithmetic mean. Then the extremal symmetrized aggregation functions $(M_\mathbf{w})^*, (M_\mathbf{w})_* : [0,1]^n \to [0,1]$ are given as follows:

$$(M_\mathbf{w})^*(\mathbf{x}) = M_\mathbf{w}(\mathbf{x}^\mathbf{w}) = OWA_{\mathbf{w}^*}(\mathbf{x})$$

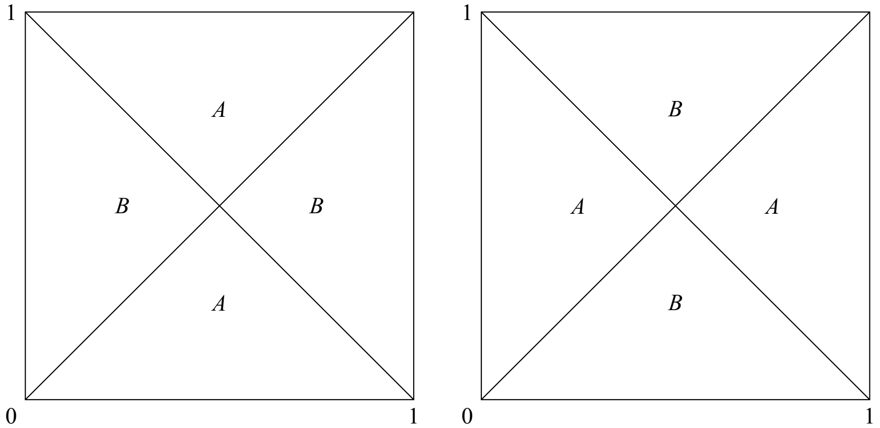

Fig. 1 Illustration of Example 7(ii): $A^* = B^*$ (left), $A_* = B_*$ (right)

and

$$(M_{\mathbf{w}})_*(\mathbf{x}) = M_{\mathbf{w}}(\mathbf{x_w}) = OWA_{(\mathbf{w}^*)^{rev}}(\mathbf{x}),$$

where $\mathbf{x^w}$ is a permutation of $\mathbf{x} = (x_1, \ldots, x_n)$ which is comonotone with $\mathbf{w}$, i.e., $(x_i^{\mathbf{w}} - x_j^{\mathbf{w}})(w_i - w_j) \geq 0$ for any $i, j \in N$, and $\mathbf{w}^*$ is a decreasing permutation of $\mathbf{w}$. This means that the value $(M_{\mathbf{w}})^*(\mathbf{x})$ is obtained as a sum of the products of the corresponding order statistics of samples $\mathbf{w}$ and $\mathbf{x}$, i.e., the greatest weight multiplies the greatest input, etc., and finally, the smallest weight multiplies the smallest input. The case of $(M_{\mathbf{w}})_*$ is reversed, n-tuples $\mathbf{x_w}$ and $\mathbf{w}$ are countermonotone, i.e., the greatest weight is multiplied by the smallest input, etc.

(ii) Let $A(x_1, x_2) = \frac{x_1 + x_2^2}{2}$. Let $B, K \colon [0, 1]^2 \to \mathbb{R}$ be the functions given by $B(x_1, x_2) = A(x_2, x_1)$ and $K = B - A$. Then the symmetric aggregation functions $A^* = B^*$ and $A_* = B_*$ are depicted in Fig. 1, and $A^* = A_* + |K|$.

We recall a few of the properties of the above mentioned extremal symmetrization methods (Mesiar et al. 2018):

- $A = A^*$ or $A = A_*$ or $A^* = A_*$ if and only if A is a symmetric aggregation function;
- A^* (A_*) is idempotent if and only if A is idempotent;
- If $A^{dual} \colon [0, 1]^n \to [0, 1]$, $A^{dual}(\mathbf{x}) = 1 - A(\mathbf{1} - \mathbf{x})$, is the dual aggregation function to A, then

$$(A^{dual})_* = (A^*)^{dual} \quad \text{and} \quad (A^{dual})^* = (A_*)^{dual};$$

- For any permutation $\sigma \in \mathscr{P}_n$, putting $A_\sigma(\mathbf{x}) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, we have $A^* = (A_\sigma)^*$ and $A_* = (A_\sigma)_*$.

For weighted aggregation functions also other symmetrization methods can be introduced, when considering all possible permutations of weights and permutations of inputs. Then the notation $A^\diamond$ and $A_\diamond$ is used for the respective maximal and minimal outputs of aggregation. This approach can also be applied, for example, in the case of penalty or deviation functions based approaches. If all considered penalty (deviation) functions are the same then both approaches to symmetrization coincide (e.g., in the case of weighted arithmetic means), i.e., $A^\diamond = A^*$ and $A_\diamond = A_*$. In the opposite case, new symmetric aggregation functions are obtained and then $A_\diamond < A_* < A < A^* < A^\diamond$.

Example 8 Consider the aggregation function $A_H\colon [0,1]^2 \to [0,1]$, discussed in Example 4, where $H = (h_1, h_2)$ and $h_1(x,y) = w_1|x-y|$, $h_2(x,y) = w_2(x-y)^2$. Then

$$(A_H)^*(x_1,x_2) = \max\{A_H(x_1,x_2), A_H(x_2,x_1)\}$$
$$= \max\left\{\operatorname{med}\left\{x_1, x_2 - \frac{w_1}{2w_2}, x_2 + \frac{w_1}{2w_2}\right\}, \operatorname{med}\left\{x_2, x_1 - \frac{w_1}{2w_2}, x_1 + \frac{w_1}{2w_2}\right\}\right\}$$

and

$$(A_H)^\diamond(x_1,x_2) = \max\{(A_H)^*(x_1,x_2), (A_{(h_2,h_1)})^*(x_1,x_2)\}$$
$$= \min\left\{\max\{x_1,x_2\}, \min\{x_1,x_2\} + \frac{\max\{w_1,w_2\}}{2\min\{w_1,w_2\}}\right\}.$$

Clearly, $(A_H)^* < (A_H)^\diamond$. Considering $w_1 = 2$ and $w_2 = 3$ gives

$$(A_H)^*\left(\frac{1}{6}, \frac{2}{3}\right) = \frac{1}{2} < \frac{11}{12} = (A_H)^\diamond\left(\frac{1}{6}, \frac{2}{3}\right).$$

Similarly, $(A_H)_\diamond = \min\{(A_H)_*, (A_{(h_2,h_1)})_*\} < (A_H)_*$.

Remark 2 The proposed symmetrization methods yielding A^* and A_* have a link to the well-known Hungarian algorithm (Burkard et al. 2009). Indeed, for some particular aggregation functions (including weighted arithmetic means, among others) one can use this algorithm to compute the values of A^* and A_*. In particular, if a weighted arithmetic mean $M_{\mathbf{w}}$ is considered then $(M_{\mathbf{w}})_*(\mathbf{x})$ is just the output of the Hungarian algorithm related to the matrix

$$\begin{bmatrix} w_1x_1 & w_2x_2 & \dots & w_nx_n \\ \vdots & \vdots & \ddots & \vdots \\ w_1x_n & w_2x_n & \dots & w_nx_n \end{bmatrix} = [x_i w_j]_{i,j=1}^n.$$

Similarly, the value $-(M_{\mathbf{w}})^*(\mathbf{x})$ is the output of the mentioned algorithm for the matrix $-[x_i w_j]_{i,j=1}^n$.

6 Concluding Remarks

In this chapter, we have recalled some recent results and sketched some new trends in the area of aggregation functions which can be seen as special utility functions. Note that we have considered real inputs from the unit interval [0, 1] only, generalization to an arbitrary real scale $[a, b]$ being obvious. To the recent trends in aggregation theory surely also belongs the investigation of aggregation functions acting on particular lattices (such as lattices of intervals, fuzzy sets, etc.) or on general (bounded distributive) lattices. Note that in the case of lattices several general results have been obtained which, when considering the real scale [0, 1] equipped with the standard ordering of reals, have also brought a new knowledge in the standard aggregation theory on [0, 1]—for example, the characterization of the Sugeno integrals as the only class of aggregation functions that preserves the congruences (Halaš et al. 2016).

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Interpretation of Multicriteria Decision Making Models with Interacting Criteria

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Abstract We consider general MCDA models with discrete attributes. These models are shown to be equivalent to a multichoice game and we put some emphasis on discrete Generalized Independence Models (GAI), especially those which are 2-additive, that is, limited to terms of at most two attributes. The chapter studies the interpretation of these models. For general MCDA models, we study how to define a meaningful importance index, and propose mainly two kinds on importance indices: the signed and the absolute importance indices. For 2-additive GAI models, we study the issue of the decomposition, which is not unique in general. We show that for a monotone 2-additive GAI model, it is always possible to obtain a decomposition where each term is monotone. This has important consequences on the tractability and interpretability of the model.

1 Introduction

Traditionally, MCDA is primarily interested in studying (through characterizations) various preference models, and learning them thanks to dedicated elicitation approaches. However, once the model has been obtained, the work is far from being finished. The end-user is highly interested in having insights on the behaviour of the model, and cannot be satisfied with only a black-box model. It is necessary to be able to provide explanations to the user about the reasons behind the decision taken by the model. We are mainly interested in the interpretability of MCDA model in this chapter, where by *interpretability* we mean the ability to provide general information

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on the model (e.g., what are the most important criteria), and not the explanation of a specific decision.

The interpretability of a MCDA model is all the more difficult when the MCDA model is rich and captures subtle and complex decision strategies. This is particularly the case when the interaction among criteria is taken into account. Some well-known MCDA models, such as the Choquet integral w.r.t. a (k-ary) capacity or the Generalized-Additive Independence (GAI) model can represent interaction among criteria, and we will focus on these models in this chapter.

We define in Sect. 3 a general interpretation of a utility-based MCDA model with discrete attributes, without any restriction on the type of model. More precisely, this interpretation takes the form of importance indices on the decision attributes, and we basically define two kinds of importance indices: the signed importance index and the absolute importance index. The signed importance index computes the average variation induced by an attribute over the model. It is linear in the MCDA model and is similar to the concept of *value* in cooperative game theory. Hence for a GAI model that takes the form of the sum of utilities over small subsets of the attributes, the importance index can be computed on each term separately. On the other hand, the absolute importance index computes the cumulated variation induced by an attribute over the model. It is in general not linear, but becomes linear when the overall utility is a linear combination with nonnegative coefficients of monotone utilities. The question is then whether we have such property for the GAI model.

We address the question of the monotone decomposition of a GAI model in Sect. 4, focusing on the 2-additive case. More precisely, if we have a nonnegative and monotone 2-additive GAI model, is it possible without loss of generality to assume that each term in the GAI decomposition is itself nonnegative and monotone? If such property is true, this would allow to interpret directly each term in the GAI decomposition, and as we will explain, the complexity of the learning procedure is greatly reduced. The main result of this section shows that indeed, such a decomposition exists (Theorem 5). However, we do not know in general how to obtain a monotone decomposition, and classical decompositions like the one proposed by Braziunas or the ANOVA decomposition do not yield in general a monotone decomposition. The last part of Sect. 4 is devoted to this question and gives some hints to solve it.

2 Background

2.1 *Multicriteria Decision Making and Conjoint Measurement*

(see, e.g., Bouyssou and Pirlot 2016 for more details) We consider a multicriteria decision problem described by attributes $X_1, \ldots, X_n$. Potential alternatives are elements $x = (x_1, \ldots, x_n)$ of the Cartesian product $X_1 \times \cdots \times X_n =: X$. We denote by $N = \{1, \ldots, n\}$ the index set of the attributes, and suppose throughout the paper that $n \geq 2$. We employ the usual notation for compound alternatives, that is, for any $x, y \in X$ and $A \subseteq N$, by (x_A, y_{-A}) we mean the alternative taking value x_i for $i \in A$ and y_i otherwise. We write x_{-i} instead of $x_{-\{i\}}$ and extend this notation to the attributes as well: X_A, X_{-A}, etc.

The preference of the decision maker (DM) is represented by a binary relation $\succcurlyeq$ on X, supposed to be complete and transitive. Ordinal measurement amounts to finding a numerical representation $U : X \to \mathbb{R}$ of the preference in the sense that $x \succcurlyeq y$ is equivalent to $U(x) \geq U(y)$. U is called a *value function*. A classical and simple example of value function is the *additive value function*

$$U(x) = \sum_{i=1}^{n} u_i(x_i) \tag{1}$$

where $u_i : X_i \to \mathbb{R}$ are the *marginal value functions* on each attribute. It is well known that this model forces $\succcurlyeq$ to satisfy *mutual preferential independence*: for any $\emptyset \neq A \subset N$, any $x, y, z, t \in X$,

$$(x_A, z_{-A}) \succcurlyeq (y_A, z_{-A}) \Leftrightarrow (x_A, t_{-A}) \succcurlyeq (y_A, t_{-A}).$$

This strong condition is rarely met in practice, and usually one assume a much weaker version, where preferential independence is required only for singletons. Specifically, $\succcurlyeq$ is said to satisfy *weak preferential independence* if for every $i \in N$, every $x, y, z, t \in X$,

$$(x_i, z_{-i}) \succcurlyeq (y_i, z_{-i}) \quad \Longleftrightarrow \quad (x_i, t_{-i}) \succcurlyeq (y_i, t_{-i}).$$

Under this condition, it is meaningful to define a preference relation $\succcurlyeq_i$ over each attribute X_i as follows: for any $x_i, y_i \in X_i$, $x_i \succcurlyeq_i y_i$ if $(x_i, z_{-i}) \succcurlyeq (y_i, z_{-i})$ for some $z \in X$. Then $\succcurlyeq_i$ is a complete, transitive binary relation on X_i. It is easy to check that *monotonicity* holds:

$$x_i \succcurlyeq_i y_i \ \forall i \in N \Rightarrow x \succcurlyeq y. \tag{2}$$

2.2 Generalized Additive Independence (GAI) Models

The additive value function model being too restrictive, one must look towards more general models. The *Generalized Additive Independence* model, proposed first by Fishburn (1967) (see also the pioneering work of Bacchus and Grove 1995), is a natural generalization where the monodimensional marginal value functions are replaced by multidimensional marginals:

$$U(x) = \sum_{S \in \mathcal{S}} u_S(x_S) \qquad (x \in X), \tag{3}$$

where $\mathscr{S} \subseteq 2^N \setminus \{\emptyset\}$. The additive value function model is recovered with $\mathscr{S} = \{\{1\}, \ldots, \{n\}\}$. There is no specific requirement on the collection $\mathscr{S}$ (hence $S, T \in \mathscr{S}$ may overlap), nor on the marginal terms u_S.

The GAI model is very versatile, and may even violate the weak preference independence property. However, if we suppose that this property is true, (2) implies that a GAI model satisfying weak preferential independence is monotone:

$$x_i \succcurlyeq_i y_i \;\forall i \in N \Rightarrow U(x) \geq U(y) \tag{4}$$

2.3 *Discrete GAI Models*

We suppose from now on that the attributes take a finite number of values (discrete attributes):

$$X_i = \{a_i^0, \ldots, a_i^{k_i}\} \qquad (i \in N),$$

supposing that $a_i^0 \preccurlyeq_i \cdots \preccurlyeq_i a_i^{k_i}$. Recall that under weak independence, the binary relations $\preccurlyeq_i$, $i = 1, \ldots, n$ are complete preorders, and monotonicity (4) holds. For the sake of convenience, we normalize U by letting

$$U(a_1^0, \ldots, a_n^0) = 0, \quad U(a_1^{k_1}, \ldots, a_n^{k_n}) = 1. \tag{5}$$

Let us now simplify the notation. We replace each value a_i^ℓ of attribute X_i by simply its index ℓ. Doing so, an alternative $(a_1^{\ell_1}, \ldots, a_n^{\ell_n})$ is represented by $(\ell_1, \ldots, \ell_n)$ without ambiguity. Letting $L := L_1 \times \cdots \times L_n$, with $L_i = \{0, 1, \ldots, k_i\}$, $i \in N$, this amounts to defining a bijection $\varphi : X \to L$ with $\varphi(a_1^{\ell_1}, \ldots, a_n^{\ell_n}) = (\ell_1, \ldots, \ell_n)$. Thanks to the ordering $a_i^0 \preccurlyeq_i \cdots \preccurlyeq_i a_i^{k_i}$, $v := U \circ \varphi^{-1} : L \to \mathbb{R}$ is a monotone function, which by (5) satisfies

$$v(0_N) = 0, v(k_N) = 1,$$

letting $0_N = (0, \ldots, 0)$, $k_N = (k_1, \ldots, k_n)$.

From now on, we assume for simplicity that $k_1 = k_2 = \cdots = k_n =: k$ (this is without loss of generality, as the results presented hereafter remain valid for the general case). Such functions v are nothing other than k-ary capacities (Grabisch and Labreuche 2003), which are particular *multichoice games* (Hsiao and Raghavan 1990): a multichoice game is a function $v : L \to$ satisfying $v(0_N) = 0$, and hence does not necessarily fulfill monotonicity. We denote by $\mathscr{G}(L)$ the set of multichoice games defined on L, and by $\mathscr{G}_M(L)$ the set of monotone multichoice games.

To summarize, we have considered a particular class of GAI models, namely those satisfying weak preferential independence and having discrete attributes with k values. Under these assumptions, the GAI model is equivalent to a k-ary capacity.

In Labreuche and Grabisch (to appear), the authors have considered continuous GAI models with the same assumption of weak preferential independence, by means of interpolation methods (Choquet integral, multilinear model) applied on a discrete model.

2.4 Models Based on Multichoice Games

Thanks to multichoice games, we can be more general and drop the assumption of weak preferential independence, while keeping discrete attributes. Indeed, let us consider as above $L = L_1 \times \cdots \times L_n$ to be the set of alternatives (up to the mapping φ), with $\ell \in L_i$ corresponding to some value a_i^ℓ of attribute X_i. As weak preferential independence does not hold any more, we cannot define an order on each attribute X_i. As a consequence, the function $v = U \circ \varphi^{-1}$ is no more monotone and therefore is not a k-ary capacity but merely a multichoice game. Let us give a simple example borrowed from Ridaoui et al. (2017a) to show that the situation is not so uncommon.

Example 1 The level of comfort of humans depends on three main attributes: temperature of the air (X_1), humidity of the air (X_2) and velocity of the air (X_3). Then $v(x_1, x_2, x_3)$ measures the comfort level. One can readily see that v is not monotone in its three arguments. For x_2 and x_3 fixed, v is maximal for intermediate values of the temperature (typically around 23 °C). Similarly, the value of humidity maximizing v is neither too low nor too high. Finally, for x_1 relatively large, some wind is well appreciated, but not too much. Hence for any i, and supposing the other two attributes being fixed, there exists an optimal value $\widehat{\ell_i} \in L_i$ such that v is increasing in x_i below $\widehat{\ell_i}$, and then decreasing in x_i above $\widehat{\ell_i}$.

Lack of monotonicity makes the analysis difficult. We will see in Sect. 3 how to define an importance index of attributes, which is valid for nonmonotonic models.

2.5 p-Additive Models

k-ary capacities are generalization of capacities introduced by Choquet (1953), while multichoice games generalize transferable utility (TU) games, introduced by Von Neumann and Morgenstern (1947). In these classical notions, we have $k = 1$, which amounts to considering set functions $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$ (TU-games), and being monotone and satisfying $v(N) = 1$ (capacities). For these functions, an important notion is the Möbius transform (Rota 1964), which permits to define p-additive games and capacities (Grabisch 1997). We introduce below these notions for the general case.

The *Möbius transform* of a multichoice game v is a function $m^v : L \to \mathbb{R}$ which is the unique solution of the linear system

$$v(z) = \sum_{y \leq z} m^v(y) \qquad (z \in L). \tag{6}$$

Its solution is shown to be (Grabisch and Labreuche to appear)

$$m^v(z) = \sum_{y \leq z \,:\, z_i - y_i \leq 1 \forall i \in N} (-1)^{\sum_{i\in N}(z_i - y_i)} v(y) \qquad (z \in \{0, 1, \ldots, k\}^N). \tag{7}$$

It follows that any multichoice game v can be written as:

$$v = \sum_{x \in L^N \setminus \{0_N\}} m^v(x) u_x, \tag{8}$$

with u_x a k-ary capacity defined by

$$u_x(z) = \begin{cases} 1, & \text{if } z \geq x \\ 0, & \text{otherwise.} \end{cases}$$

By analogy with classical games, u_x is called the *unanimity game* centred on x. Note that this decomposition is unique as the unanimity games are linearly independent, and form a basis of the vector space $\mathcal{G}(L)$. Another basis is given by the Dirac games δ_x, with $x \in L$, $x \neq 0_N$:

$$\delta_x(y) = \begin{cases} 1 & \text{, if } y = x \\ 0, & \text{otherwise.} \end{cases}$$

For further reference, let us introduce $\mathcal{G}_+(L)$ the set of games with a nonnegative Möbius transform. Then any game $v \in \mathcal{G}(L)$ can be expressed in a unique way as

$$v = v^+ - v^- \tag{9}$$

with $v^+, v^- \in \mathcal{G}_+(L)$.

We say that a multichoice game v is *(at most) p-additive* for some $p \in \{1, \ldots, n\}$ if its Möbius transform satisfies $m^v(z) = 0$ whenever $|\text{supp}(z)| > p$, where

$$\text{supp}(z) = \{i \in N \mid z_i > 0\}.$$

The following result is shown in Grabisch and Labreuche (to appear).

Lemma 1 *Let $p \in \{1, \ldots, n\}$. A multichoice game v is p-additive if and only if it has the form*

$$v(z) = \sum_{x \in L, 0 < |\text{supp}(x)| \leq p} v_x(x \wedge z) \qquad (z \in L) \tag{10}$$

where $v_x : L \to \mathbb{R}$ with $v_x(0_N) = 0$.

Supposing that v is monotone, thanks to the bijection φ, the above result says that a discrete GAI model U is p-additive if and only if each term u_S, $S \in \mathscr{S}$, has at most p variables, i.e., $|S| \leq p$. In Sect. 4, we will study 2-additive GAI models. These models are of particular interest, because although they are much more general than the additive value function model, they remain tractable since each term depends of at most two variables.

3 Importance Indices for Discrete Multicriteria Decision Models

We suppose in this section to have a MCDA model which is a multichoice game v on L (see Sect. 2.4).

The first level of interpretation of a MCDA model is to indicate to the user which attributes are the most important or influential in the decision model. This amounts to computing *importance indices* of each criterion in the model. The knowledge of these values is very important. First, criteria of small importance index can be neglected. Second, the decision maker can rank the criteria by increasing importance according to his expertise. The comparison of this order with the order obtained from the importance indices is very informative. When there is some discrepancy, this means that the model have been underspecified, or there are some misunderstanding. The elicitation of the model has then to be updated.

The MCDA models we are interested in can represent very rich and diverse decision strategies. The deciphering of complex MCDA models cannot be done from the sole knowledge of importance indices. One also needs information about how criteria interact together. We do not describe interaction indices in this chapter. We recommend reference (Ridaoui et al. 2018) to the interested reader.

This section is based on Ridaoui et al. (2017a, b). We start by introducing the general idea of how to define an importance index (Sect. 3.1). Importance indices are closely related to the concept of a *value* in cooperative game theory. The existing literature in this field is summarized in Sect. 3.2. We consider two classes of importances indices. Section 3.3 defines the importance of criterion i as the average added-value (over all possible situations) of making a unitary improvement on criterion i. The sign of this index represents the general monotonicity of the model: U is globally nondecreasing (resp. nonincreasing) if the index is nonnegative (resp. nonpositive). This index is thus called *signed* importance index. It might happen that a function U that is nonincreasing in some area and nondecreasing in another area has an overall signed importance close to 0, even though this criterion is very important. We have thus also defined an *absolute importance index* (Sect. 3.4) to measure the net contribution of a criterion regardless of the monotonicity. Finally, we construct a very general class of importance indices written as a norm over all possible unitary improvement on a criterion (Sect. 3.5).

3.1 How to Define Importance Indices

The main ingredient behind importance indices, as for example it is defined for continuous functions of several variables (see, e.g., Grabisch et al. 2009, Sect. 10.3), is the average variation induced by a given variable over its domain, or equivalently, the average of the partial derivative of the function over its domain. We propose as a starting point to take this approach and to adapt it to v, which is a function defined over the discrete domain L. To this end, we introduce its *derivative w.r.t.* i, $i \in N$:

$$\Delta_i v(x) = v(x + 1_i) - v(x) \qquad (x \in L, x_i < k),$$

where 1_i is a shorthand for $(1_i, 0_{-i})$. Following the foregoing discussion, the general form of the importance index of attribute X_i w.r.t. v should read:

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} p^i_x \Delta_i v(x), \tag{11}$$

where p^i_x is a real positive constant, for every such x and i. If the weights p^i_x depend only on $p^i_{x_{-i}}$, then

$$\sum_{x_i=0}^{k_i-1} p^i_{x_{-i}} \Delta_i v(x) = p^i_{x_{-i}} (v(x_{-i}, k_i) - v(x_{-i}, 0_i)), \tag{12}$$

i.e., only the variation between k_i and 0_i matters.

Clearly, if v is a monotone function, then $\phi_i(v)$ is a nonnegative quantity for every $i \in N$, while it is a nonpositive quantity if v is antimonotonic. For this reason, one may call $\phi(v)$ a *signed importance index*. What about nonmonotonic models, as the one given in Example 1? Taking 23 °C as the optimal temperature, the derivative of v w.r.t. the temperature is positive for temperatures below 23 °C, and becomes negative above 23 °C. As a consequence, positive and negative variations may cancel each other, resulting in an importance index for temperature which is close to 0. This is quite counterintuitive, as surely temperature matters in the evaluation of comfort.

The foregoing discussion shows that a (signed) importance index is not adequate in any situation. Therefore, another definition seems to be necessary. The simplest idea to avoid this drawback is to cumulate the magnitude of the variations, regardless of their sign, instead of summing them algebraically. This leads to the following formula:

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} p^i_x |\Delta_i v(x)|. \tag{13}$$

We call such an index an *absolute importance index*. It coincides with the former one for monotonic games.

At this point of the discussion, one may say that only absolute importance indices are relevant, the signed ones leading to counterintuitive results. We think, however, that both are useful and should be used, provided we are aware of its precise meaning: The signed index indicates the overall trend of the model w.r.t. an attribute (increasing or decreasing), while the absolute index measures the amount of variation of the model induced by an attribute.

Finally, still other definitions can be proposed if one remarks that the absolute value is the L_1 norm, and other norms can be used as well. We define *norm-based importance indices* as those of the following form:

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} p^i_x \|\Delta_i v(x)\|. \tag{14}$$

3.2 Values in Game Theory

We indicate in this section connections with cooperative game theory. In this field, a central notion is the one of value. Let us take for simplicity the case of classical TU-games. Interpreting $v(N)$ as the total benefit achieved by the cooperation of all players, a *value* is a way of sharing $v(N)$ among all the players, taking into account their contribution to the game. Formally, it is a mapping $\phi : 2^N \to \mathbb{R}^N$, and the usual requirement is that the value is *efficient*, which means that the benefit $v(N)$ is shared without waste and nothing more can be given: $\sum_{i \in N} \phi_i(v) = v(N)$. The best known value is the Shapley value (1953), defined by:

$$\phi^{Sh}_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!}\big(v(S \cup i) - v(S)\big), \forall i \in N. \tag{15}$$

Letting $k = 1$, the derivative becomes $\Delta_i v(S) = v(S \cup i) - v(S)$, hence the Shapley value has the form of a signed importance index. Indeed, in voting games, it is used as a power index, which is the counterpart of importance index for voting problems.

The Shapley value has been generalized to multichoice games, in different ways by several authors. We mention here (Hsiao and Raghavan 1993) (historically the first one), van den Nouweland et al. (1995), Klijn et al. (1999), Peters and Zank (2005), Grabisch and Labreuche (2008), etc. The value of Peters and Zank reads:

$$\phi^{\mathrm{PZ}}_i(v) = \sum_{x_{-i} \in \Gamma(L_{-i})} \frac{(n-\kappa(x_{-i})-1)!\kappa(x_{-i})!}{n!}(v(x_{-i}, k) - v(x_{-i}, 0)), \tag{16}$$

where $\kappa(x_S)$ is the size of the kernel of x_S, i.e., $\kappa(x_S) = |\{i \in S \mid x_i = k\}|$, and $\Gamma(L_S) = \{0, k\}^S$. Note that only vertices of L_{-i} are used in the computation.

All these values satisfy efficiency, as this is a basic requirement in a cooperation context. However, in a MCDA context, efficiency is not a relevant notion. Especially when v is not monotone, satisfying efficiency would lead to strange results. Indeed, as in Example 1, the value $v(k_N)$ is low and even close to 0 since if all three parameters (temperature, humidity, wind velocity) take their maximal values, there is no comfort at all. As a consequence, the sum of the importance indices would be close to zero, and if absolute importance indices are taken (which are nonnegative by definition), the conclusion is that all criteria have a negligible importance, which is again quite counterintuitive.

As a conclusion, efficiency as defined in game theory must be abandoned, and none of the values defined in the literature can be taken as an importance index. In what follows, we propose and axiomatize different importance indices which are suitable for our MCDA context.

3.3 Signed Importance Indices

The aim of this section is to axiomatize the family of signed importance indices given by (11) and to propose a particular one based on suitable axioms.

The three first axioms we propose are the classical axioms used in the original axiomatization of Shapley. The first one says that ϕ is a linear operator on $\mathscr{G}(L)$.

Linearity axiom (L): ϕ is linear on $\mathscr{G}(L)$, i.e., for any $v, w \in \mathscr{G}(L)$, $\forall \alpha \in \mathbb{R}$,

$$\phi_i(v + \alpha w) = \phi(v) + \alpha\phi(w).$$

An attribute $i \in N$ is said to be null for $v \in \mathscr{G}(L)$ if

$$v(x + 1_i) = v(x), \forall x \in L, x_i < k.$$

Null axiom (N): If an attribute i is null for $v \in \mathscr{G}(L)$, then $\phi_i(v) = 0$.

This axiom says that an attribute for which an increment of 1 does not improve the evaluation is not important. It turns out that these two axioms are characteristic of the family of signed importance indices.

Proposition 1 *Under axioms* **(L)** *and* **(N)***, for all $i \in N$, there exist $p_x^i \in \mathbb{R}$, for all $x \in L$ with $x_i < k_i$, such that for all $v \in \mathscr{G}(L)$,*

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k_i}} p_x^i\big(v(x + 1_i) - v(x)\big). \tag{17}$$

We try now to refine the family by adding suitable properties. The first one is related to symmetry or anonymity: the numbering of the attributes should have no influence on the computation of the importance index.

Let σ be a permutation on N. For all $x \in L$, we denote $\sigma(x)_{\sigma(i)} = x_i$. For all $v \in \mathscr{G}(L)$, the game $\sigma \circ v$ is defined by $\sigma \circ v(\sigma(x)) = v(x)$.

Symmetry axiom (S): For any permutation σ on N, $\phi_{\sigma(i)}(\sigma \circ v) = \phi_i(v)$, $\forall i \in N$.

The next axiom is an invariance property. It says that the calculus of the importance index does not depend on the position on the "grid" L. It is another kind of symmetry axiom, relative to the levels $0, 1, \ldots, k$, not to the attributes.

Invariance axiom (I): Let us consider two games $v, w \in \mathcal{G}(L)$ such that, for some $i \in N$,

$$v(x + 1_i) - v(x) = w(x) - w(x - 1_i), \forall x \in L, x_i \notin \{0, k\}$$

$$v(x_{-i}, 1_i) - v(x_{-i}, 0_i) = w(x_{-i}, k_i) - w(x_{-i}, k_i - 1), \forall x_{-i} \in L_{-i}.$$

Then $\phi_i(v) = \phi_i(w)$.

With these two additional axioms, we obtain the following result.

Proposition 2 *Under axioms **(L)**, **(N)**, **(I)** and **(S)**, for all $v \in \mathcal{G}(L)$, for all $i \in N$,*

$$\phi_i(v) = \sum_{x_{-i} \in L_{-i}} p_{n(x_{-i})}\big(v(x_{-i}, k_i) - v(x_{-i}, 0_i)\big),$$

where $n(x_{-i}) = (n_0, n_1, \ldots, n_k)$ with n_j the number of components of x_{-i} being equal to j.

The effect of the two axioms is the following: **(I)** forces p^i_x to depends only on x_{-i} and i, which by (12) implies that only the difference between k_i and 0_i on attribute i matters. Then the symmetry axiom makes the constant $p^i_{x_{-i}}$ to depend only on the "cardinality" of x_{-i}.

It remains to find a last axiom for determining the constants uniquely. As the usual efficiency axiom of game theory is not suitable in this context, we propose instead a substitute which is in the spirit of variation calculus:

Efficiency axiom (E): For all $v \in \mathcal{G}(L)$,

$$\sum_{i \in N} \phi_i(v) = \sum_{\substack{x \in L \\ x_j < k}} \big(v(x + 1_N) - v(x)\big).$$

It can be explained as follows: taking an alternative $x \in L$ and increasing the value of each attribute by one unit, i.e., going to $x + 1_N$, the amount of variation is due to the contribution of all attributes, and the sum of all importance indices should be equal to the sum of this variation for all alternatives x. Interestingly, the axiom is nevertheless not so far from the original efficiency axiom because when taking $k = 1$, it reduces to the classical efficiency axiom $\sum_i \phi_i(v) = v(N)$.

Finally, we can show:

Theorem 1 *Under axioms* **(L)**, **(N)**, **(I)**, **(S)** *and* **(E)**, *for all* $v \in \mathscr{G}(L)$

$$\phi_i(v) = \phi_i^{\mathrm{S}}(v) := \sum_{x_{-i} \in L_{-i}} \frac{(n - \sigma(x_{-i}) - 1)!\kappa(x_{-i})!}{(n + \kappa(x_{-i}) - \sigma(x_{-i}))!} \big(v(x_{-i}, k_i) - v(x_{-i}, 0_i)\big), \forall i \in N. \tag{18}$$

3.4 Absolute Importance Indices

We turn to the axiomatization of the family of absolute importance indices and as before try to find a particular index of interest.

The major difficulty in axiomatizing (13) is that ϕ does not satisfy linearity. Therefore, it is not possible to start from the decomposition of a game on some basis. We remark that if v is monotone, then $|v(x + 1_i) - v(x)| = v(x + 1_i) - v(x)$ for every $x \in L, x_i < k$. However, $\mathscr{G}_M(L)$ is not a linear subspace of $\mathscr{G}(L)$ but a convex cone, and we cannot apply directly the linearity axiom on it. The idea is the following: using the expression of v in the basis of unanimity games (8), this expression turns to be a conic combination iff v is in $\mathscr{G}_+(L)$. As any game can be written as the difference of two games in $\mathscr{G}_+(L)$ (see (9)), it is then possible to extend this expression to monotone games. Hence, ϕ should commute with conic combination and differences of games in $\mathscr{G}_+(L)$.

Conic Combination axiom (CC): For every $v, w \in \mathscr{G}_+(L)$, for every $\alpha \in \mathbb{R}_+$,

$$\phi(v + \alpha w) = \phi(v) + \alpha\phi(w).$$

Decomposition axiom (D): If $v, v' \in \mathscr{G}_+(L)$ and $v - v'$ is monotone, then $\phi(v - v') = \phi(v) - \phi(v')$.

These two axioms permit to obtain the following result.

Proposition 3 *Under axioms* **(CC)** *and* **(D)**, *for all* $i \in N$, *there exists constants* $a_x^i \in \mathbb{R}$, *for all* $x \in L$, *such that* $\forall v \in \mathscr{G}_M(L)$,

$$\phi_i(v) = \sum_{x \in L} a_x^i v(x). \tag{19}$$

Taking two multichoice games v and w for which the marginal contribution of an attribute i to a game v is the same or the opposite of that to a game w, the average importance of attribute i shall be the same for v and w.

Marginal contribution axiom (MC): Let $i \in N$ and $v, w \in \mathscr{G}(L)$ such that

$$|\Delta_i(v)(x)| = |\Delta_i(w)(x)|, \forall x \in L, x_i < k.$$

Then

$$\phi_i(v) = \phi_i(w).$$

The following result shows that the family of absolute importance indices is characterized by the above three axioms.

Proposition 4 *Under axioms* **(CC)***,* **(D)** *and* **(MC)***, there exist real constants* p_x^i*,* $i \in N, x \in L, x_i < k_i$*, such that for every* $v \in \mathcal{G}(L)$*,*

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k_i}} p_x^i |\Delta_i v(x)|. \tag{20}$$

Surprisingly, there is no need of the null axiom. This is because it is implied by **(MC)** and **(CC)** (or by **(MC)** and **(D)**) as it is easy to check.

As for the signed importance index, the introduction of the two symmetry axioms **(S)** and **(I)** permits to reduce the number of constants, as p_x^i is turned into $p_{n(x_{-i})}$, where $n(x_{-i}) = (n_0, n_1, \ldots, n_k)$ with n_j the number of components of x_{-i} being equal to j.

It remains to determine uniquely the constants by imposing some normalization condition. The first one is based on the Dirac games δ_x. Observe that if $x_i \neq 0, k$, the sum of absolute variations along the i axis is 2, otherwise it is 1. Normalizing by the total number of points in the grid L_{-i}, which is $(k+1)^{n-1}$ so that the result is not dependent of the size of the grid, we obtain the following:

Calibration axiom 1st version (C1): For every $x \in L \setminus \{0_N\}$

$$\phi_i(\delta_x) = \begin{cases} 2/(k+1)^{n-1} & \text{if } 0 < x_i < k \\ 1/(k+1)^{n-1} & \text{otherwise.} \end{cases}$$

Theorem 2 *Under axioms* ***(CC)****,* ***(D)****,* ***(S)****,* ***(I)****,* ***(MC)*** *and* ***(C1)****, for all* $v \in \mathcal{G}(L)$

$$\phi_i(v) = \frac{1}{(k+1)^{n-1}} \sum_{\substack{x \in L \\ x_i < k}} |v(x+1_i) - v(x)|, \forall i \in N,$$

Another possibility is based on unanimity games.

Calibration axiom 2nd version (C2): For all $x \in L \setminus \{0_N\}$, for all $i \in S(x)$,

$$\phi_i(u_x) = \frac{1}{s(x)}.$$

Theorem 3 *Under axioms* ***(CC)****,* ***(D)****,* ***(S)****,* ***(I)****,* ***(MC)*** *and* ***(C2)****, for all* $v \in \mathcal{G}(L)$

$$\phi_i(v) = \sum_{\substack{x_{-i} \in \{0,k\}^{N\setminus\{i\}} \\ x_i \in L_i, x_i < k}} \frac{(n - s(x_{-i}) - 1)!s(x_{-i})!}{n!} |v(x + 1_i) - v(x)|, \forall i \in N.$$

We observe that **(C1)** yields an importance index similar to the Banzhaf value, while the use of **(C2)** gives a result close to the Shapley value. It is possible, however, to obtain exactly the coefficients of the signed importance index (see Theorem 1) by using suitable axioms (Grabisch et al. submitted).

3.5 Norm-Based Importance Indices

We turn now to the family of norm-based importance indices, which have the general form (5). A fundamental difficulty is that the use of a norm different from L_1 forbids to take an axiomatic approach similar to the one we used for absolute importance indices, because there would exist no class of games where a property similar to linearity would hold. Nevertheless, it is possible to obtain a general form through a number of axioms which are presented below. In the rest of this section $i \in N$ is fixed.

Nonnegativity (NN): The importance index takes nonnegative values, i.e., $\phi_i : \mathscr{G}(L) \to \mathbb{R}_+$.

Absolute Homogeneity (AH): For every $\alpha \in \mathbb{R}$ and every game $v \in \mathscr{G}(L)$,

$$\phi_i(\alpha v) = |\alpha|\phi_i(v)$$

Subadditivity (SA): For any games $v, w \in \mathscr{G}(L)$,

$$\phi(v + w) \leq \phi(v) + \phi(w)$$

Strong Null axiom (SN): $\phi_i(v) = 0$ if and only if i is null for v.

The nonnegativity axiom says that importance indices are nonnegative quantities. Absolute homogeneity says that multiplying a game be a constant just multiplies the importance index by the magnitude of this constant. The subadditivity axiom expresses the fact that summing two games v, w may hinder the importance of an attribute by some hedging effect: the positive variation of i at some point x for v can be cancelled by a negative variation at the same point for w. Lastly, the strong null axiom is a strong version of the usual null axiom, in the sense that *only* games whose attribute i is null can lead to a null importance index for i.

We obtain the following.

Theorem 4 *Under axioms* ***(NN), (AH), (SA)*** *and* ***(SN)****, there exists a norm* $\|\cdot\|$ *on* $\mathbb{R}^{k(k+1)^{n-1}}$ *and a linear one-to-one mapping* h *on* $\mathbb{R}^{k(k+1)^{n-1}}$ *such that*

$$\phi_i(v) = \|h \circ \Delta_i(v)\|.$$

4 Monotone Decomposition of a 2-Additive GAI Model and its Interpretation

Following Sect. 3, a first level of interpretation of a MCDA model consists in using the generic indices, such as the importance indices. To go further, one needs to take profit of the particular form (in particular the mathematical expression) of utility U. We will focus in this section on the GAI model. As it takes an additive form, it would be convenient to interpret U by interpreting each term u_S separately. We will see that the decomposition of a GAI model is far from being unique, which makes its interpretation delicate. We focus on the special class of the 2-additive GAI models that are monotone, as monotonicity is a very natural property in MCDA. We will see whether monotonicity improves the interpretability of a GAI model.

4.1 Difficulty of the Interpretation of a GAI Model

We formalize in this section the intuition given earlier on the difficulty of interpreting a GAI model.

4.1.1 Illustration of the Difficulty on an Example

A model very similar to the 2-additive GAI model is defined in Greco et al. (2014). In this reference, the sign of interacting terms $u_{i,j}$ is interpreted as the sign of interaction. This is borrowed from the expression of the Choquet integral with respect to a 2-additive capacity v, which takes the following form written in terms of the Möbius transform (Choquet 1953):

$$\mathrm{Ch}_v(a_1, \ldots, a_n) = \sum_{i \in N} m^v(\{i\})\, a_i + \sum_{\{i,j\} \subseteq N} m^v(\{i, j\})\, a_i \wedge a_j,$$

where m^v is the Möbius transform of v (see Sect. 2.5). The interaction coefficient between factors i and j is equal $m^v(\{i, j\})$, so that the sign of the interaction is given by the sign of $m^v(\{i, j\})$. However, this interpretation holds only for the expression of the Choquet integral with respect to the Möbius transform, that is, the expression of Ch on the basis $\{a_i\ ,\ i \in N\} \cup \{a_i \wedge a_j\ ,\ \{i, j\} \subseteq N\}$. This is no more true for another basis. In order to illustrate this, let us take the following example.

Example 2 Let us take the following function of two variables:

$$U(x_1, x_2) = 2x_1 + x_2 - \min(x_1, x_2). \tag{21}$$

Following the intuition of Greco et al. (2014), one would say that there is a negative interaction between the two attributes. However, using the relation $\max(x_1, x_2) + \min(x_1, x_2) = x_1 + x_2$, we obtain an equivalent expression

$$U(x_1, x_2) = x_1 + \max(x_1, x_2) \tag{22}$$

in which the bivariate term is now nonnegative.

Apart from the problem of the sign of the interaction, relations (21) and (22) are two different equivalent expressions of the same model U. If one wants to present the interacting term $u_{i,j}$ to the user, which one among terms "$-\min(x_1, x_2)$" and "$\max(x_1, x_2)$" shall be shown?

4.1.2 Nonuniqueness of the GAI Decomposition

We have already seen that the GAI decomposition (3) is not unique. In the additive utility model, each utility term is given up to a constant. Comparing (21) and (22), we see that the terms in the GAI decomposition can take very different expressions.

This rises the question of the *decomposition* of a GAI model $U(x)$. Is it possible to relate all decompositions? Fishburn (1967) has shown that any two equivalent decompositions $U(x) = \sum_{S\in\mathscr{S}} u_S(x_S) = \sum_{S\in\mathscr{S}} u'_S(x_S)$ are related as follows:

$$u'_S(x_S) = u_S(x_S) + \sum_{S'\in\mathscr{S}\setminus\{S\},\, S\cap S'\neq\emptyset} f_{S,S'}(x_{S\cap S'}) + c_S \tag{23}$$

where $f_{S,S'} : X_{S\cap S'} \to \mathbb{R}$, and $\sum_{S\in\mathscr{S}} \left[\sum_{S'\in\mathscr{S}\setminus\{S\},\, S\cap S'\neq\emptyset} f_{S,S'}(x_{S\cap S'}) + c_S\right] = 0$. There is an intrinsic difficulty coming from the nonuniqueness of the GAI decomposition. In the interpretation of model U, which specific decomposition shall be used? This raises the question of whether there exists a canonical decomposition, which would allow for an intuitive interpretation of U. This question will be addressed in Sect. 4.4.

4.2 Monotonicity Conditions

Monotonicity is an essential property in MCDA. We assume that U satisfies monotonicity condition (2). If U is monotone and takes the form (3), it would be counterintuitive and misleading for the end-user if we present him some terms u_S that are not monotone in some of their coordinates.

In Example 2, it is apparent from (22) that U is monotone in the two attributes. Expression (21) is formed of three terms, the first two being increasing while the third one is decreasing. Presenting this to the user would be confusing because of the decreasing term. On the other hand, (22) has only two terms, both of them being nondecreasing. Clearly, the latter expression is more transparent because it has fewer terms and each term in the expression is both nonnegative and nondecreasing.

We have seen that, in two equivalent expressions (such as (21) and (22)), a similar term $u_{1,2}$ ("$-\min(x_1, x_2)$" and "$\max(x_1, x_2)$" respectively) does not have the same monotonicity. This cannot happen with the additive utility model and shows in particular that the GAI model does not necessarily satisfy weak independence. In the Artificial Intelligence community, researchers are interested in the representation of preferences that may violate weak independence. A well-known example of such a preference is the following: consider the choice of a menu described by two attributes X_1, X_2 where X_1 pertains on the type of wine and X_2 to the type of main course in a restaurant. Then usually, one prefers 'red wine' to 'white wine' if the main course is 'meat', but 'white wine' is preferred to 'red wine' if the main course is 'fish' (the preference over attribute 'wine' is conditional on the value on attribute 'main course') (Boutilier et al. 2001). This can be represented by a GAI model: $U(x_1, x_2) = u_{1,2}(x_1, x_2) + u_2(x_2)$, where

- preferences over X_2: $u_2(\text{meat}) = 1$, $u_2(\text{fish}) = 0$,
- preferences over X_1 conditionally on X_2: $u_{1,2}(\text{red, meat}) = 4$, $u_{1,2}(\text{white, meat}) = 2$, $u_{1,2}(\text{white, fish}) = 3$ and $u_{1,2}(\text{red, fish}) = 0$.

The examples which violate weak independence are far from being the general case in MCDA. Rather, most of MCDA problems satisfy weak independence, as there are in general natural preferences on each attribute. For this reason, following the traditional view of decision theory, we assume in this chapter that weak independence holds.

An important consequence of weak independence is that monotonicity holds for $\succcurlyeq$ (see (2)), and consequently for U too—see (4). We note that in (23), due to the presence of functions $f_{S,S'}$, we do not have $u_S(x_S) \geq u_S(y_S)$ iff $u'_S(x_S) \geq u'_S(y_S)$, for any two $x_S, y_S \in X_S$ (Braziunas 2012, p. 87). Moreover, even if U satisfies weak independence, it might be the case that u_S does not fulfil this condition, or satisfies it but does not have the same monotonicity as U.

4.3 Representation of Monotone 2-Additive GAI Models

We have seen in Example 2 a situation where, starting from a monotone 2-additive GAI model (namely expression (21)), we can find an equivalent expression (namely (22)) such that each term is nonnegative and monotone. The main question we wish to address in this section is the following one: is the previous repair process working in all situations?

4.3.1 Main Result

The following theorem states that a decomposition of a 2-additive monotone GAI model into monotone nondecreasing terms is always possible.

Theorem 5 (Grabisch and Labreuche to appear) *Let us consider a 2-additive discrete GAI model U satisfying monotonicity (4) and (5). Then there exist nonnegative and nondecreasing functions $u_i : X_i \to [0,1]$, $i \in N$, $u_{ij} : X_i \times X_j \to [0,1]$, $\{i,j\} \subseteq N$, such that*

$$U(x) = \sum_{i \in N} u_i(x_i) + \sum_{\{i,j\} \subseteq N} u_{ij}(x_i, x_j) \qquad (x \in X) \tag{24}$$

The rest of Sect. 4.3 is devoted to describing important consequences of this important result.

4.3.2 Consequence of the Main Result in the Computation of the Importance Indices

Let us consider a 2-additive monotone GAI model U (3). The first level of interpretation of U aims at computing the importance indices and showing to the user the most important attributes. Let us start with the signed importance index $\phi_i^{\mathrm{s}}(U \circ \varphi^{-1})$ (see Sect. 3.3). The main drawback of formula (18) is that it has an exponential number of terms in the number n of criteria. Fortunately, we can drastically reduce this complexity for GAI models. Indeed, as ϕ_i^{s} fulfills **(L)**, we can write

$$\phi_i^{\mathrm{s}}(U \circ \varphi^{-1}) = \sum_{S \in \mathscr{S}} \phi_i^{\mathrm{s}}(u_S \circ \varphi^{-1}).$$

For a 2-additive GAI model, $S \in \mathscr{S}$ contains at most two elements, which makes the computation of each term $\phi_i^{\mathrm{s}}(u_S \circ \varphi^{-1})$ extremely fast. Hence the computation of $\phi_i^{\mathrm{s}}(U \circ \varphi^{-1})$ becomes easily tractable (in $O(n^2)$) even for large values of n.

Two absolute importance indices have been proposed in Sect. 3.4. These two expressions have different coefficients compared to ϕ_i^{s}. As we mentioned in Sect. 3.4, it is possible define an absolute importance index having the same coefficient as in ϕ_i^{s} (Grabisch et al. submitted):

$$\phi_i^{\mathrm{a}}(v) := \sum_{x_{-i} \in L_{-i}} \frac{(n - \sigma(x_{-i}) - 1)!\kappa(x_{-i})!}{(n + \kappa(x_{-i}) - \sigma(x_{-i}))!} \left| v(x_{-i}, k_i) - v(x_{-i}, 0_i) \right|. \tag{25}$$

Then if U is monotone, we have

$$\phi_i^{\mathrm{s}}(U \circ \varphi^{-1}) = \phi_i^{\mathrm{a}}(U \circ \varphi^{-1}).$$

On the other hand, nothing a priori forbids each term u_S in the GAI decomposition to be nonpositive or nonmonotonic. In this case one would have

$$\phi_i^{\mathrm{s}}(u_S \circ \varphi^{-1}) \neq \phi_i^{\mathrm{a}}(u_S \circ \varphi^{-1}).$$

This would be counter-intuitive and misleading for the user. Fortunately, according to Theorem 5, there exists a GAI decomposition in which each term u_S is non negative and monotone. For such decomposition, we obtain $\phi_i^{\mathrm{s}}(u_S \circ \varphi^{-1}) = \phi_i^{\mathrm{a}}(u_S \circ \varphi^{-1})$.

4.3.3 A Complexity Problem in the Learning Procedure

(see Grabisch and Labreuche to appear for more details) Another very important consequence of Theorem 5 concerns the learning of the GAI model. Generally speaking, a learning procedure consists in collecting information on the preference relation $\succcurlyeq$, which is then used in an optimization problem, whose aim is to find a value function U of a given type representing at best the given preference. The variables of the optimization problem are then the parameters of the model U.

We begin by computing the number of unknowns in a 2-additive GAI model equivalent to a k-ary capacity. Such a model has the form (3) with $\mathscr{S}$ being the set of singletons and pairs. Since $|L_i| = k + 1$, this yields

$$\eta(k,n) = (k+1)\binom{n}{1} + (k+1)^2\binom{n}{2} = \frac{n(k+1)}{2}\Big(2 + (k+1)(n-1)\Big)$$

unknowns. U being monotone nondecreasing, this induces a number of monotonicity constraints on the unknowns, of the type

$$U(a_1^{j_1}, \dots, a_{i-1}^{j_{i-1}}, a_i^{j_i+1}, a_{i+1}^{j_{i+1}}, \dots, a_n^{j_n}) \geq U(a_1^{j_1}, \dots, a_{i-1}^{j_{i-1}}, a_i^{j_i}, a_{i+1}^{j_{i+1}}, \dots, a_n^{j_n}) \tag{26}$$

for every $i \in N$, $j_1 \in \{0, \dots, k_1\}, \dots, j_{i-1} \in \{0, \dots, k_{i-1}\}$, $j_i \in \{0, \dots, k_i - 1\}$, $j_{i+1} \in \{0, \dots, k_{i+1}\}, \dots, j_n \in \{0, \dots, k_n\}$. The number of elementary conditions contained in (26) is equal to

$$\sum_{i \in N}\Big(k_i \times \prod_{j \in N\setminus\{i\}} (k_j + 1)\Big).$$

In the case where $k_i = k$ for every i, this number becomes

$$\kappa(k,n) = n \times k \times (k+1)^{n-1}.$$

Although the number of variables $\eta(k,n)$ is still quadratic in n and k, the number of constraints $\kappa(k,n)$ is exponential in n. It follows that any practical identification of a GAI model based on some optimization procedure,[1] where the variables are the unknowns of the GAI model and the constraints are the monotonicity constraints (26) plus possibly some learning data, has to cope with an exponential number of constraints. The following tables, obtained with $k = 4$, shows that the underlying optimization problem becomes rapidly intractable.

n	4	6	8	10
$\eta(4, n)$	170	405	740	1175
$\kappa(k, n)$	2000	75 000	2 500 000	78 125 000

n	12	14	20
$\eta(k, n)$	1710	2345	4850
$\kappa(k, n)$	2 343 750 000	68 359 375 000	$1.526E + 15$

However, if a decomposition into nonnegative nondecreasing terms is possible, one has only to check monotonicity of each term:

$$\forall i \in N\ \forall l \in \{0, \dots, k_i - 1\} \qquad u_i(a_i^{l+1}) \geq u_i(a_i^l), \tag{27}$$

$$\forall \{i, j\} \subseteq N\ \forall l_i \in \{0, \dots, k_i - 1\}\ \forall l_j \in \{0, \dots, k_j\}$$
$$u_{i,j}(a_i^{l_i+1}, a_j^{l_j}) \geq u_{i,j}(a_i^{l_i}, a_j^{l_j}), \tag{28}$$

$$\forall \{i, j\} \subseteq N\ \forall l_i \in \{0, \dots, k_i\}\ \forall l_j \in \{0, \dots, k_j - 1\}$$
$$u_{i,j}(a_i^{l_i}, a_j^{l_j+1}) \geq u_{i,j}(a_i^{l_i}, a_j^{l_j}). \tag{29}$$

Then the number of monotonicity conditions drops to

$$\sum_{i \in N} k_i + \sum_{\{i,j\} \subseteq N} \big(k_i(k_j + 1) + k_j(k_i + 1)\big).$$

In the case where $k_i = k$ for every i, this number becomes

$$\kappa'(k, n) = n \times k \times \Big[(n - 1)(k + 1) + 1\Big],$$

which is quadratic in n. The following table ($k = 4$) shows that the optimization problem becomes tractable even for a large number of attributes.

n	4	6	8	10	12	14	20
$\kappa'(k, n)$	256	624	1152	1840	2688	3696	7680

[1] The learning problem can be classically transformed into a linear program, where the training set is seen as linear constraints on the GAI variables (Bigot et al. 2012; Greco et al. 2014). It could also be possible to perform statistical learning, like in Tehrani et al. (2012), where the underlying optimization problem is a convex problem under linear constraints.

4.4 Interpretation Through a Canonical Decomposition

The most convenient way to interpret a GAI model is to use a "canonical" decomposition in some sense. Following Theorem 5, is it possible to always find decomposition into nonnegative and nondecreasing terms? We review in this section two existing decompositions of a multivariable function into a GAI decomposition. We conclude by providing some research directions to construct a canonical decomposition of a GAI model fulfilling the monotonicity of each of its terms.

4.4.1 Braziunas' Decomposition

Braziunas has proposed a decomposition based on the Fishburn representation (Braziunas 2012). Fixing an order on $\mathscr{S}$, say, $\mathscr{S} = \{S_1, \ldots, S_p\}$, the overall value function reads $U(x) = \sum_{S \in \mathscr{S}} u_S^C(x_S)$ with, for every $j \in \{1, \ldots, p\}$

$$u_{S_j}^C(x_{S_j}) = U(x[S_j]) + \sum_{K \subseteq \{1,\ldots,j-1\},\ K \neq \emptyset} (-1)^{|K|} U\left(x\left[\cap_{k \in K} S_k \cap S_j\right]\right) \tag{30}$$

where $\cdot^C$ stands for "canonical", $\mathbb{O} \in X$ is any element in X seen as an anchor, and $x[S] \in X$ defined by $(x[S])_i = x_i$ if $i \in S$ and $(x[S])_i = \mathbb{O}_i$ otherwise (Braziunas 2012, p. 94). Note that the expression depends on the chosen ordering of the elements of $\mathscr{S}$. The two equivalent decompositions (21) and (22) were obtained with a particularly simple example. The previous remark provides a more systematic way to derive several equivalent decompositions of GAI models, as illustrated in the next example.

Example 3 Consider the following function $U(x_1, x_2, x_3) = x_2 + x_1\, x_3 + \max(x_1, x_2)$. We have $\mathscr{S} = \{S_1, S_2, S_3\}$ with $S_1 = \{2\}$, $S_2 = \{1, 3\}$ and $S_3 = \{1, 2\}$. Then the canonical decomposition gives, with $\mathbb{O} = (0, 0, 0)$:

$$\begin{aligned}
&u_{S_1}^C(x_2) = U(x[S_1]) = U(\mathbb{O}_1, x_2, \mathbb{O}_3) = 2\, x_2 \\
&u_{S_2}^C(x_1, x_3) = U(x[S_2]) - U(x[S_1 \cap S_2]) = U(x_1, \mathbb{O}_2, x_3) - U(\mathbb{O}) = x_1\, (x_3 + 1) \\
&u_{S_3}^C(x_1, x_2) = U(x[S_3]) - U(x[S_1 \cap S_3]) - U(x[S_2 \cap S_3]) + U(x[S_1 \cap S_2 \cap S_3]) \\
&\quad = U(x_1, x_2, \mathbb{O}_3) - U(\mathbb{O}_1, x_2, \mathbb{O}_3) - U(x_1, \mathbb{O}_2, \mathbb{O}_3) + U(\mathbb{O}) \\
&\quad = \max(x_1, x_2) - x_1 - x_2 = -\min(x_1, x_2)
\end{aligned}$$

We note that U is nondecreasing in all variables, even though, for the canonical decomposition, $u_{S_3}^C$ is nonincreasing in its two coordinates.

Let us take now the order $S'_1 = \{1, 2\}$, $S'_2 = \{1, 3\}$ and $S'_3 = \{2\}$. We obtain $u_{S'_1}^C(x_1, x_2) = x_2 + \max(x_1, x_2)$, $u_{S'_2}^C(x_1, x_3) = x_1\, x_3$ and $u_{S'_3}^C(x_2) = 0$. All terms are now nonnegative and monotone.

The previous example shows that the canonical decomposition does not guarantee to have only nondecreasing terms in the decomposition, and therefore does not provide an easily interpretable decomposition. Hence there is no well-defined semantics of the value functions u_S, contrarily to what is claimed in Braziunas (2012), Sect. 3.2.1.4.

4.4.2 ANOVA

We present in this section ANOVA (Fisher and Mackenzie 1923), which is a convenient way to construct a canonical decomposition of a GAI model.

In statistics, the analysis of variance (ANOVA) is a powerful tool to represent interaction between variables in a multivariate model (Fisher and Mackenzie 1923). Consider n independent random variables $Z_1, \dots, Z_n$ uniformly distributed in [0, 1], and a multivariate model $Y = F(Z)$, where $Z = (Z_1, \dots, Z_n)$. Let us denote by Z_S and Z_{-S} the groups of variables $(Z_i)_{i \in S}$ and $(Z_i)_{i \notin S}$ respectively. Hence, we may write $Z = (Z_S, Z_{-S})$. Moreover, we denote by $\mathbb{E}[Y]$ the expected value of Y taken over all variables $Z_1, \dots, Z_n$. The expected value of Y can be taken on a subset Z_S of variables, with the corresponding notation $\mathbb{E}_{Z_S}[Y]$.

Any multivariate function can be decomposed in the following way (ANOVA decomposition) (Fisher and Mackenzie 1923):

$$Y = F(Z) = F_\varnothing + \sum_{i=1}^{n} F_i(Z_i) + \sum_{i<j} F_{ij}(Z_i, Z_j) + \cdots + F_N(Z) = \sum_{S \subseteq N} F_S(Z_S),$$

with

$$\begin{aligned}
F_\varnothing &= \mathbb{E}[Y] \\
F_i(Z_i) &= \mathbb{E}[Y|Z_i] - F_\varnothing \\
F_{ij}(Z_i, Z_j) &= \mathbb{E}[Y|Z_i, Z_j] - F_i(Z_i) - F_j(Z_j) - F_\varnothing \\
&= \mathbb{E}[Y|Z_i, Z_j] - \mathbb{E}[Y|Z_i] - \mathbb{E}[Y|Z_j] + E[Y] \\
\vdots &= \vdots \\
F_S(Z_S) &= \mathbb{E}_{Z_{-S}}[Y|Z_S] - \sum_{T \subset S} F_T(Z_T) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mathbb{E}_{Z_{-T}}[Y|Z_T] \\
\vdots &= \vdots \\
F_N(Z) &= \sum_{T \subseteq N} (-1)^{|N \setminus T|} \mathbb{E}_{Z_{-T}}[Y|Z_T].
\end{aligned}$$

We note that the ANOVA decomposition corresponds exactly to a GAI decomposition. If we start with a function U taking a GAI decomposition with a collection $\mathscr{S}$ of represented subsets, then one can easily show that the ANOVA decomposition applied to U cannot return nonzero terms U_S where S is a superset of some terms in $\mathscr{S}$. In other word, the ANOVA decomposition will use subsets in $\widehat{\mathscr{S}} = \{S \subseteq S',\ S' \in \mathscr{S}\}$.

Let us apply the ANOVA decomposition to the example (22): $U(x_1, x_2) = x_1 + \max(x_1, x_2)$. We have

$$\begin{aligned} U_\varnothing = \mathbb{E}[Y] &= \int_0^1 \int_0^1 (x_1 + \max(x_1, x_2))\ dx_1\, dx_2 \\ &= \frac{1}{2} + \int_0^1 \int_0^{x_1} x_1\, dx_1\, dx_2 + \int_0^1 \int_{x_1}^1 x_2\, dx_1\, dx_2 \\ &= \frac{1}{2} + \int_0^1 {x_1}^2\, dx_1 + \int_0^1 \frac{1 - {x_1}^2}{2}\, dx_1 = \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{2} - \frac{1}{6}\right) = \frac{7}{6} \end{aligned}$$

$$\begin{aligned} &U_1(x_1) = \mathbb{E}[U|x_1] - U_\varnothing = \int_0^1 (x_1 + \max(x_1, x_2))\ dx_2 - \frac{7}{6} \\ &= x_1 + \int_0^{x_1} x_1\, dx_2 + \int_{x_1}^1 x_2\, dx_2 - \frac{7}{6} = x_1 + {x_1}^2 + \frac{1 - {x_1}^2}{2} - \frac{7}{6} = x_1 + \frac{{x_1}^2}{2} - \frac{2}{3} \end{aligned}$$

$$\begin{aligned} &U_2(x_2) = \mathbb{E}[U|x_2] - U_\varnothing = \int_0^1 (x_1 + \max(x_1, x_2))\ dx_1 - \frac{7}{6} \\ &= \frac{1}{2} + \int_0^{x_2} x_2\, dx_1 + \int_{x_2}^1 x_1\, dx_1 - \frac{7}{6} = -\frac{2}{3} + {x_2}^2 + \frac{1 - {x_2}^2}{2} = \frac{{x_2}^2}{2} - \frac{1}{6} \end{aligned}$$

$$\begin{aligned} U_{12}(x_1, x_2) &= U(x_1, x_2) - U_1(x_1) - U_2(x_2) + U_\varnothing \\ &= (x_1 + \max(x_1, x_2)) - \left(x_1 + \frac{{x_1}^2}{2} - \frac{2}{3}\right) - \left(\frac{{x_2}^2}{2} - \frac{1}{6}\right) + \frac{7}{6} \\ &= \max(x_1, x_2) - \frac{{x_1}^2 + {x_2}^2}{2} + 2 \end{aligned}$$

In particular, if $x_2 > x_1$,

$$\frac{\partial U_{12}(x_1, x_2)}{\partial x_1} = -x_1$$

so that U_{12} is not always increasing w.r.t. its two variables.

Hence the question of finding a decomposition into nondecreasing terms (which we call hereafter a *monotone* decomposition) is yet unsolved and as far as we know, its existence has not been studied. The next section tries to answer (at least partly) this question.

4.4.3 Toward a Monotone Decomposition of a GAI Model

We aim in this section at proposing some hints to define a decomposition of a GAI model in which all of its terms are nonnegative and nondecreasing.

The idea of ANOVA is valuable: to define decomposition in which F_i is the effect of moving x_i alone in F, and $F_{i,j}$ is the effect of moving x_i and x_j simultaneously, getting rid of the effect of varying only one of these variables. Higher order terms are defined likewise. This condition of F_S representing the sole contribution of varying all variables in S simultaneously can be put as an orthogonality condition:

$$\mathbb{E}_{Z_i}[F_S|Z_{S\setminus i}] = 0 \quad \forall i \in S.$$

We wish to keep the previous idea of orthogonality but represented in another way. Orthogonality will be written in terms of the difference with a reference situation – depicted as alternative $\mathbb{O}$ – as in Braziunas' approach. Following the ANOVA decomposition, a two-additive GAI model is written as

$$U(x) = u_\emptyset + \sum_{i\in N} u_i(x_i) + \sum_{\{i,j\}\subseteq N} u_{i,j}(x_i, x_j). \tag{31}$$

Term u_i should be the sole contribution of attribute x_i, removing the constant part. With a reference to option $\mathbb{O}$, we obtain the condition

$$u_i(\mathbb{O}_i) = 0. \tag{32}$$

Term $u_{i,j}$ shall not depict the effect on U of the variation of only x_i or x_j. Hence

$$u_{i,j}(x_i, \mathbb{O}_j) = 0 \quad \forall x_i \in X_i \tag{33}$$

$$u_{i,j}(\mathbb{O}_i, x_j) = 0 \quad \forall x_j \in X_j \tag{34}$$

By (32), (33) and (34), the term $u_\emptyset$ is the constant term:

$$u_\emptyset = U(\mathbb{O}). \tag{35}$$

In order to fulfill (32)–(35), nonnegativity and monotonicity of each u_S (i.e. (27)–(29)), the $\mathbb{O}$ alternative shall be the least element on each attribute. The next example shows that it is not always possible to find a decomposition fulfilling all previous conditions.

Example 4 (*Example* 2 *continued*) Let us consider (22): $U(x_1, x_2) = x_1 + \max(x_1, x_2)$. Let us try to find a decomposition of U in terms of $u_\emptyset$, $u_1(x_1) = x_1 + v_1(x_1)$, $u_2(x_2) = v_2(x_2)$ and $u_{1,2}(x_1, x_2) = \max(x_1, x_2) - v_1(x_1) - v_2(x_2)$ (where functions v_1 and v_2 are unknown) satisfying all previous conditions.

As u_2 is monotone, $v_2'(x_2) \geq 0$. For $x_1 > x_2$, $\frac{\partial u_{1,2}}{\partial x_2}(x_1, x_2) = -v_2'(x_2) \geq 0$. Hence v_2 is constant. We obtain $v_2 \equiv 0$ by (32).

Hence by (33), $u_{1,2}(x_1, \mathbb{O}_2) = x_1 - v_1(x_1) = 0$, so that $v_1(x_1) = x_1$ and $u_{1,2}(x_1, x_2) = \max(x_1, x_2) - x_1$. For $x_1 < x_2$, $\frac{\partial u_{1,2}}{\partial x_2}(x_1, x_2) = -1$, which violates the monotonicity of $u_{1,2}$.

We need to relax hard constraints (32)–(35), and transform them into an optimization objective. Finally, one aims at finding a decomposition of the form (31), where variables $u_\emptyset$, u_i, $u_{i,j}$ are found by minimizing

$$|u_\emptyset|^p + \sum_{i \in N} |u_i(\mathbb{O}_i)|^p + \sum_{\{i,j\} \subseteq N} \left[\sum_{x_i \in X_i} \left| u_{i,j}(x_i, \mathbb{O}_j) \right|^p + \sum_{x_j \in X_j} \left| u_{i,j}(\mathbb{O}_i, x_j) \right|^p \right] \tag{36}$$

where $p > 0$ is a fixed power factor, under the constraints that (31) shall hold for every $x \in X$, the u_i's and $u_{i,j}$'s are nonnegative and monotone (i.e. (27)–(29)). The minimization of (36) is indeed a relaxation of a weak version of (32)–(35).

One can enrich (36) with other terms – for instance, the entropy of the unknowns not represented in (36). If the obtained functional is convex, we get a unique solution to the optimization problem.

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New Directions in Ordinal Evaluation: Sugeno Integrals and Beyond

Miguel Couceiro, Didier Dubois, Hélène Fargier, Michel Grabisch, Henri Prade and Agnès Rico

Abstract This chapter provides a state-of-the-art account of the use of Sugeno integrals in decision evaluation, when it is difficult to use meaningful figures of merit when assessing the worth of a decision and when only a finite scale of, e.g., linguistic categories, can be used. Here, Sugeno integrals are thought of as idempotent lattice polynomial functions on a finite bounded chain, which makes it possible to assign importance weights to groups of criteria or states. Algebraic and behavioral characterizations of the Sugeno integral are presented and discussed, including the special cases of weighted minima and maxima. Extensions of this framework are also surveyed, namely: lexicographic refinements that increase the discrimination power of this approach; the use of local utility functions in order to cope with criteria having distinct rating scales; and the generalization of the criteria weighting scheme at work in Sugeno integrals. Another kind of extension considered is when ratings belong to a bipolar scale where good and bad figures are explicitly present, thus giving rise to the symmetric Sugeno integral or to the separate evaluation of pros and cons. Moreover, it is pointed out that Sugeno integrals encode decision rules and that this

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bridge leads to methods for extracting knowledge from qualitative data. The results of empirical studies of the latter are also presented and discussed, accordingly.

Keywords Sugeno integral · Lattice polynomial · Bipolarity · Qualitative decision theory · Decision rule

1 Introduction: Motivation for Qualitative Evaluation Methods

In the setting of Artificial Intelligence (for instance, recommender systems, cognitive robotics, but other fields as well), the use of decision rules based on numerical aggregation functions is not always natural. In particular, probabilities, utilities, importance weights cannot always be easily elicited from the user, by lack of time or lack of precision. Information systems advising persons cannot ask too many questions to users for modeling their preferences, nor collect from them meaningful numbers representing probabilities or criteria importance levels, or yet utility values. Even if they get them, making numerical operations on them needs justification, for instance, does the scale used authorize such calculations? To illustrate, suppose that a referee fills a form to assess the merits of a paper for a journal, and numerical ratings are required by the system. What is the precise meaning of these ratings? Does it make sense to compute averages from them?

In such situations it is more natural to resort to a qualitative approach to multicriteria evaluation. The rationale is to refrain from using numbers that look arbitrary or hard to collect, namely address decision problems in the ordinal setting. Gigerenzer and Todd (1999) have argued that human decisions are often made on the basis of an ordinal ranking of the strength of criteria rather than on numerical evaluations, and hence the qualitative nature of the decision process. In daily life, people seldom resort to explicit numerical computations of figures of merit. This idea has also been exploited in Artificial Intelligence for a long time in qualitative decision theory (Doyle and Thomason 1999). For instance, so-called conditional preference networks (CP-nets) (Boutilier et al. 2004) allow for an easier representation of ordinal preference relations on multidimensional sets of alternatives, using local conditional preference statements interpreted ceteris paribus. See Dubois et al. (2009) for a survey of qualitative decision rules under uncertainty.

There are two advantages to using a qualitative approach: (i) a gain in robustness and the need for less data; (ii) qualitative methods lend themselves to a logical representation (which makes proposed choices more easily explainable). There are two possible choices of qualitative settings for representing notions such as utility ratings stemming from several agents, importance levels and likelihood degrees:

- use distinct non-commensurate scales, which makes the framework very restrictive as impossibility theorems regarding rational aggregation processes are often obtained (e.g., in voting theory).

- use finite commensurate scales (taking advantage of notions facilitating commensurateness such as certainty equivalents), which leads to a finite ordered set of value classes.

In multi-criteria decision making, Sugeno integrals (Sugeno 1974, 1977) are commonly used as qualitative aggregation functions (Grabisch and Labreuche 2010) using finite scales under the commensurability assumption between them. The definition of these integrals is based on a monotonic set-function named capacity or fuzzy measure that aims to qualitatively represent the likelihood of sets of possible states of nature, the importance of sets of criteria, etc. These set functions are currently used in many areas such as uncertainty modeling (Dubois et al. 2000, 2001), multiple criteria aggregation (Bouyssou et al. 2009; Grabisch 1996; Greco et al. 2004) or in game theory (Schmeidler 1972). See also a recent book devoted to capacities in such areas (Grabisch 2016). Moreover, Sugeno integrals naturally lend themselves to a representation in terms of if-then rules involving thresholds (Dubois et al. 2014; Greco et al. 2004), which makes them easy to interpret.

This chapter surveys several results around Sugeno integral as well as some of its extensions in the problem of evaluating decisions under uncertainty or objects according to several criteria. We shall speak of *alternatives* to stand for acts, decisions or objects to be evaluated. In Sect. 2 we recall basic definitions and various mathematical properties of Sugeno integrals, including the links with conjunctive and disjunctive normal forms of lattice polynomials, and the close connection between Sugeno integrals and medians. Axiomatic characterisations of Sugeno integrals are also surveyed, as well as the expression of Sugeno integrals in possibilistic logic. Section 3 reviews generalisations of Sugeno integrals, namely their lexicographic refinements, the use of local utility functions that cope with the presence of several local scales, and the use of more general conjunction and implication connectives for combining weights of criteria and local ratings. Section 4 discusses the notion of bipolar qualitative evaluation by means of special cases or variants of Sugeno integrals, which use bipolar scales with explicit positive and negative values, or yet a positive and a negative scale, in order to balance the pros and the cons. The last section deals with qualitative data analysis, namely how to represent qualitative data by means of Sugeno integrals or by a set of if-then rules.

2 Sugeno Integrals

We consider a finite set of criteria $\mathcal{C} = \{1, \ldots, n\}$, also denoted by $[n]$. The alternatives considered are evaluated using these criteria. Here, the evaluation scale L is supposed to be common to all criteria and is assumed to be a finite totally ordered set, for instance a subset of the interval $[0, 1]$. In any case, the bottom of L is denoted by 0 and the top is denoted by 1. Using a single evaluation scale means that the ratings according to different criteria are commensurate, which is a strong assumption that will be lifted later on in the paper.

The maximum (resp. minimum) will be denoted by $\vee$ (resp. $\wedge$). An alternative is represented by a function $f : \mathcal{C} \to L$, or equivalently, by a tuple of ratings on the different criteria, i.e., by $f = (f_1, \dots, f_n) \in L^n$ where f_i is the rating of f according to criterion i. We assume that the rating scale L is equipped with an involutive order-reversing operation, denoted by $1 - \cdot$, such that whenever $\lambda \in L$, $1 - \lambda \in L$ as well.

2.1 Basic Definitions and Preliminaries

In usual multicriteria evaluation based on weighted average, importance weights are assigned to criteria. In this paper, importance levels can be assigned to sets of criteria (instead of single ones) by means of a capacity which is a mapping $\mu : 2^{\mathcal{C}} \to L$ such that $\mu(\emptyset) = 0$, $\mu(\mathcal{C}) = 1$, and if $A \subseteq B$ then $\mu(A) \leq \mu(B)$. This generalized importance assignment enables dependencies between criteria to be accounted for; namely, redundant criteria in a set A are such that $\mu(A) = \max_{i \in A} \mu(\{i\})$ (since the weight of group A is the one of the most important criterion in it), while a synergy between them is expressed when $\mu(A) > \max_{i \in A} \mu(\{i\})$. Capacities qualify to represent uncertainty in decision theory as well as importance weights in multicriteria decision-making.

The discrete Sugeno integral, widely used to aggregate qualitative local evaluations in multiple attribute evaluation, is defined by:

$$S_\mu(f) = \bigvee_{A \subseteq \mathcal{C}} (\mu(A) \wedge \bigwedge_{i \in A} f_i) \tag{1}$$

The name "integral" for such an expression may sound surprising. However, it was proposed first by Sugeno (1974) under the name "fuzzy integral" in analogy with Lebesgue integral under the following equivalent form:

$$S_\mu(f) = \bigvee_{\lambda \in L} \lambda \wedge \mu(\{i : f_i \geq \lambda\})$$

The idea was to replace integral (sum) and product in Lebesgue integral by fuzzy set union (max) and intersection (min). For further background see, e.g., Grabisch et al. (2009), Sugeno (1974, 1977).

There are alternative expressions of Sugeno integral as follows (Marichal 2000; Sugeno 1974, 1977):

$$S_\mu(f) = \bigvee_{A \subseteq \mathcal{C}} (\mu(A) \wedge \bigwedge_{i \in A} f_i) = \bigwedge_{A \subseteq \mathcal{C}} (\mu(\overline{A}) \vee \bigvee_{i \in A} f_i)) \tag{2}$$

$$= \bigvee_{i=1}^{n} f_{(i)} \wedge \mu(\{(i), \dots, (n)\}) = \bigwedge_{i=1}^{n} f_{(i)} \vee \mu(\{(i+1), \dots, (n)\}). \tag{3}$$

$$= \bigvee_{\lambda \in L} \lambda \wedge \mu(\{i : f_i \geq \lambda\}) = \bigwedge_{\lambda \in L} \lambda \vee \mu(\{i : f_i > \lambda\}). \tag{4}$$

where we have supposed $f_{(1)} \leq \cdots \leq f_{(n)}$ where (i) indices the ith least component of the vector f, and $\overline{A}$ is the complement of A in $\mathcal{C}$. It turns out (Kandel and Byatt 1978) that $S_\mu(f)$ is the median of the set

$$\{f_1, \ldots, f_n\} \cup \{\mu(\{(n)\}), \ldots, \mu(\{(2), \ldots, (n)\})\}.$$

For instance, if $f_i = \lambda$ for $i \in A$ and $\theta < \lambda$ otherwise, it is easily seen that $S_\mu(f)$ is the median of $\lambda, \theta, \mu(A)$.

Note that Sugeno integral has exponential complexity in terms of the number of criteria, but its expression can be reduced to one of linear size by ranking the values f_i. Besides not all values $\mu(A)$ are always useful for defining S_μ. The qualitative Mœbius transform $\mu_\#$ of the capacity μ (Grabisch 2004; Mesiar 1997) is a mapping from 2^C to L defined by

$$\mu_\#(E) = \begin{cases} \mu(E) \text{ if } \mu(E) > \bigvee_{B \subsetneq E} \mu(B) \\ 0 \text{ otherwise.} \end{cases}$$

It contains the minimal information needed to reconstruct the capacity μ. Due to the monotonicity of μ we can replace $\bigvee_{B \subsetneq E} \mu(B)$ in the above equation by $\bigvee_{i \in E} \mu(E \setminus \{i\})$. The function $\mu_\#$ is also the qualitative counterpart of a basic probability assignment in evidence theory (Dubois and Prade 1985), since it holds that $\mu(A) = \bigvee_{E \subseteq A} \mu_\#(E)$. The set $\mathcal{F}(\mu) = \{E : \mu_\#(E) > 0\}$ is called the set of focal subsets of μ. Then, the Sugeno integral S_μ can be expressed in a simplified form as

$$S_\mu(f) = \bigvee_{A \in \mathcal{F}(\mu)} (\mu(A) \wedge \bigwedge_{i \in A} f_i) \tag{5}$$

which contains no mathematically redundant min-terms.

A special case of capacity is a possibility measure (Dubois and Prade 1988, 2015; Zadeh 1978) which is a maxitive capacity, i.e., a capacity Π such that $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$. Since the set of criteria is finite, the possibility distribution $\pi : \pi_i = \Pi(\{i\})$, here representing individual criteria weights, is enough to recover the set-function: $\forall A \subseteq \mathcal{C}, \Pi(A) = \bigvee_{i \in A} \pi_i$: focal sets of Π are singletons. In this case, criteria are considered redundant with one another.

The conjugate $\mu^c(A)$ of capacity μ is a capacity defined by $\mu^c(A) = 1 - \mu(\overline{A}), \forall A \subseteq \mathcal{C}$. The conjugate of a possibility measure Π is a necessity measure $N(A) = 1 - \Pi(\overline{A})$, and then N is a minitive capacity, i.e., $N(A \cap B) = N(A) \wedge N(B)$. Moreover, $N(A) = \bigwedge_{i \notin A} \iota(i)$ where $\iota(i) = N(\mathcal{C} \setminus \{i\})$ (this is the degree of impossibility of i when dealing with uncertainty), and $\iota(i) = 1 - \pi_i$, where π defines the conjugate possibility measure $\Pi = N^c$. In a group A of criteria, we may have $N(\{i\}) = 0, \forall i \in A$ but $N(A) > 0$ which suggests that necessity measures account for criteria in positive synergy. Focal sets of necessity measures are nested, they are the cuts of the possibility distribution π, i.e., we have that $\mathcal{F}(N) = \{\{i : \pi_i \geq \lambda\} : \lambda \in L \setminus \{0\}\}$.

It is easy to see and well-known (Dubois and Prade 1980; Grabisch et al. 1992) that if the capacity is a possibility measure, the Sugeno integral simplifies in the form of a prioritized maximum (Dubois and Prade 1986):

$$S_{\Pi}(f) = SLMAX_{\pi}(f) = \bigvee_{i \in \mathcal{C}} \pi_i \wedge f_i.$$

Likewise it can be shown that if the capacity is a necessity measure, Sugeno integral simplifies in the form of a prioritized minimum (Dubois and Prade 1986):

$$S_N(f) = SLMIN_{\pi}(f) = \bigwedge_{i \in \mathcal{C}} (1 - \pi_i) \vee f_i.$$

The use of the optimistic criterion $SLMAX_{\pi}$ captures the so-called focus effect (Gigerenzer and Todd 1999): the order of magnitude of the importance of a group of criteria is the one of the most important argument, in the group. This assumption perfectly suits the intuition of a qualitative scale as it means that weaker arguments are always negligible compared with a single stronger one.

2.2 Sugeno Integrals as Lattice Polynomials Under Normal Form

A convenient way to introduce the discrete Sugeno integral is via the concept of lattice polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. More precisely, given a bounded chain L, by an n-ary polynomial function, we simply mean a function $\phi : L^n \to L$ defined recursively as follows:

(i) For each $i \in [n] = \{1, \ldots, n\}$ and each $\lambda \in L$, the projection $\phi(\lambda_1, \ldots \lambda_n) \mapsto \lambda_i$ and the constant function $\phi \mapsto \lambda$ are polynomial functions from L^n to L.
(ii) If ϕ and ψ are polynomial functions from L^n to L, then $\phi \vee \psi$ and $\phi \wedge \psi$ are polynomial functions from L^n to L.
(iii) Any polynomial function from L^n to L is obtained by finitely many applications of the rules (i) and (ii).

We refer to those polynomial functions constructed from projections by finitely many applications of (ii) as *lattice term functions* (or simply, *term functions*). A well-known example of a term function is the ternary median function, which is given by

$$\text{median}(\lambda, \lambda', \lambda'') = (\lambda \wedge \lambda') \vee (\lambda' \wedge \lambda'') \vee (\lambda'' \wedge \lambda)$$

$$= (\lambda \vee \lambda') \wedge (\lambda' \vee \lambda'') \wedge (\lambda'' \vee \lambda).$$

As shown by Marichal (2009), the discrete Sugeno integrals are exactly those polynomial functions $\phi : L^n \to L$ that are *idempotent*, i.e., that satisfy $\phi(\lambda, \dots, \lambda) = \lambda$.

In this subsection we revisit classical normal form representations of lattice polynomials and recall the median normal form representation that follows from *median decomposability*.

Disjunctive and conjunctive normal forms. Goodstein (1967) has shown that in the case of bounded distributive lattices, polynomial functions are exactly those which allow their representations in disjunctive and conjunctive normal forms. In this subsection we recall some related useful results.

Proposition 1 *Let $\phi\colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) ϕ is a polynomial function.
(ii) There exists a set function $\mu\colon 2^{[n]} \to L$ such that $\phi(f) = \bigvee_{I\subseteq[n]}(\mu(I) \wedge \bigwedge_{i\in I} f_i)$.
(iii) There exists a set function $\kappa\colon 2^{[n]} \to L$ such that $\phi(f) = \bigwedge_{I\subseteq[n]}(\kappa(I) \vee \bigvee_{i\in I} f_i)$.

The expressions given in (ii) and (iii) of Proposition 1 are usually referred to as the *disjunctive normal form* (DNF) representation and the *conjunctive normal form* (CNF) representation, respectively, of the polynomial function ϕ. Notice that the DNF and CNF representations of a polynomial function $\phi\colon L^n \to L$ are not necessarily unique.

For each $I \subseteq [n]$, let $\mathbf{1}_I$ be the element of L^n whose ith component is 1, if $i \in I$, and 0, otherwise. Let $\mu_\phi\colon 2^{[n]} \to L$ be the set function given by $\mu_\phi(I) = \phi(\mathbf{1}_I)$. It is monotone with inclusion. It is easy to see that if moreover $\mu_\phi([n]) = 1$ and $\mu_\phi(\emptyset) = 0$, then letting $\mu = \mu_\phi$ in Proposition 1, ϕ is a Sugeno integral in DNF. This representation is not unique: we still get the same Sugeno integral ϕ if we change μ_ϕ into any set-function μ in $\mathrm{DNF}(\phi) = \{\mu \in L^{2^{[n]}} : \phi(f) = \bigvee_{I\subseteq[n]} \mu(I) \wedge \bigwedge_{i\in I} f_i\} = \{\mu : \mu^\#_\phi \le \mu \le \mu_\phi\}$, using the qualitative Moebius transform $\mu^\#_\phi$.

Dually, let $\kappa_\phi\colon 2^{[n]} \to L$ be the function given by $\kappa_\phi(I) = \phi(\mathbf{1}_{[n]\setminus I}) = \mu_\phi(\overline{I})$. Function κ_ϕ is clearly antitone, and if μ_ϕ is a capacity, we recognize again the Sugeno integral in CNF in Proposition 1 (iii).

Remark 1 Note that μ_ϕ is the only monotone set function in $\mathrm{DNF}(\phi)$ and, similarly, κ is the only anti-monotone set function in $\mathrm{CNF}(\phi) = \{\kappa \in L^{2^{[n]}} : \phi(f) = \bigwedge_{I\subseteq[n]}(\kappa(I) \vee \bigvee_{i\in I} f_i)\}$.

Median normal form. It is not difficult to see that every lattice polynomial function $\phi\colon L^n \to L$ is *median decomposable*, i.e., for every $f \in L^n$,

$$\phi(f) = \mathrm{median}\left(\phi(f^{k\to 0}), f_k, \phi(f^{k\to 1})\right), \qquad (6)$$

where $f^{k\to\lambda} = (f_1, \ldots, f_{k-1}, \lambda, f_{k+1}, \ldots, f_n)$ for $k \in [n]$ is obtained by changing f_k into $\lambda \in L$ in the vector $(f_1, \ldots, f_n)$. In fact, the converse is also true, and thus we have the following characterization of lattice polynomial functions.

Theorem 1 (Marichal 2009, Theorem 17) *The solutions of the median decomposition equation (6) are exactly the lattice polynomial functions from L^n to L.*

For further characterizations, we refer the reader to the survey paper (Couceiro and Marichal 2010a). This median decomposition scheme naturally leads to a recursive procedure for obtaining median representations of functions independent from the way functions are given (Couceiro et al. 2011). Indeed, by setting a ranking of variables, we can repeatedly apply Theorem 1 to the variables of any given function in order to derive a nested formula made of medians applied to variables and constants. To illustrate, consider the 5-ary median function median_5. This naïve approach leads to a median normal form of median_5:

$$\begin{aligned}\text{median}_5(f_1, f_2, f_3, f_4, f_5) &= \text{median}\left(\text{median}_5(0, f_2, f_3, f_4, f_5), f_1, \text{median}_5(1, f_2, f_3, f_4, f_5)\right)\\ &= \text{median}\left(\text{median}\left(\text{median}_5(0, 0, f_3, f_4, f_5), f_2, \text{median}_5(0, 1, f_3, f_4, f_5)\right),\right.\\ &\quad \left. f_1, \text{median}\left(\text{median}_5(1, 0, f_3, f_4, f_5), f_2, \text{median}_5(1, 1, f_3, f_4, f_5)\right)\right)\\ &= \ldots\end{aligned}$$

In this way, we obtain a median normal form representation of median_5 with $1 + 2 + 4 + 8 + 16 = 31$ occurences of median, which is not optimal. Indeed, there exists a much smaller representation with only 4 occurences of median:

$$\text{median}_5(f_1, f_2, f_3, f_4, f_5) = \text{median}(\text{median}(\text{median}(f_2, f_3, f_4), f_4, f_5), \text{median}(f_2, f_3, f_5), f_1).$$

Now, it is not difficult to extend the results in Couceiro et al. (2006) and show that the median normal form produces representations that make use, up to polynomial equivalence, of the least number of monotonic connectives. However, an efficient procedure for computing the smallest median normal form is still unknown and the problem of deciding whether a median representation is minimal seems to be (mildly) untractable (Couceiro et al. 2017b, 2019).

2.3 Algebraic and Behavioral Characterizations

Sugeno integral has been characterized by a few properties, especially decomposability for comonotonic functions f and $g : L^n \to L$, i.e., such that $f_i > f_j \Rightarrow g_i \geq g_j, \forall i, j \in [n]$. Namely,

Theorem 2 *Let $I : L^{\mathcal{C}} \to L$. There is a capacity μ such that $I(f) = S_\mu(f)$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotonic $f, g \in L^{\mathcal{C}}$.*

2. $I(\lambda \wedge f) = \lambda \wedge I(f)$, *for every* $\lambda \in L$ *and* $f \in L^{\mathcal{C}}$.
3. $I(\mathbf{1}_{\mathcal{C}}) = 1$.

Equivalently, conditions (1–3) can be replaced by conditions (1'–3') below (Dubois et al. 2017):

1'. $I(f \wedge g) = I(f) \wedge I(g)$, for any comonotonic $f, g \in L^{\mathcal{C}}$.
2'. $I(\lambda \vee f) = \lambda \vee I(f)$, for every $\lambda \in L$ and $f \in L^{\mathcal{C}}$.
3'. $I(\mathbf{0}_{\mathcal{C}}) = 0$.

Most older formulations of this theorem (de Campos et al. 1991; de Campos and Bolaños 1992; Ralescu and Sugeno 1996) redundantly add an assumption of increasing monotonicity of the functional I (if $f \geq g$ then $I(f) \geq I(g)$) to the three conditions (1–3). But these papers do not point out the equivalent conditions (1'–3'). The existence of these two equivalent characterisations is due to the possibility of equivalently writing Sugeno integral in conjunctive and disjunctive normal form (see Eq. (2)). As a consequence, the De Morgan dual $1 - S_\mu(1 - f)$ of a Sugeno integral $S_\mu(f)$ is also a Sugeno integral in the sense that (2) can be expressed as

$$S_\mu(f) = 1 - S_{\mu^c}(1 - f) \quad (7)$$

for the conjugate capacity μ^c. Note that a functional $I(f)$ satisfies conditions (1–3) if and only if $1 - I(1 - f)$ satisfies conditions (1'–3'). Marichal (2000) provides several similar characterizations, especially one assuming maxitive and minitive comonotonicity (conditions 1 and 1') along with idempotence, one assuming homogeneity conditions 2 and 2' plus increasing monotonicity. Chateauneuf et al. (2008) propose an alternative axiomatization, more in the spirit of Schmeidler's work for the Choquet integral, mainly based on hedging effects. However, the proof that conditions (1–3) are necessary and sufficient seems to first appear in a thesis dissertation (Rico 2002 and then used in Grabisch et al. (2009); Grabisch (2016).

Another characterization has been provided in the context of decision under uncertainty, in a setting similar to Savage's approach to expected utility functionals. The set of criteria $\mathcal{C}$ is replaced by a *finite* set $[n]$ of states, and alternative decisions are just functions f from $[n]$ to a set of consequences X. We consider again a finite totally ordered scale $(L, \leq)$ with bottom 0 and top 1. A mapping $u : X \to L$ is named a utility function. We assume that X contains an ideal consequence x^* with $u(x^*) = 1$ and a worst consequence x_* with $u(x_*) = 0$. Note that the use of a single scale for rating the consequences of acts is more natural than for rating alternatives in multiple-criteria evaluation problems.

The decision-maker is supposed to supply a preference relation $\succeq$ on the set $X^{\mathcal{C}}$ of alternatives (called acts), that is, a non trivial preorder: $\succeq$ is transitive and complete.

We introduce new notations that will be useful in the following:

- A constant act x is such that $\exists x \in X, \forall i \in \mathcal{C}, \mathsf{x}(i) = x$. In particular, the acts x^* and x_* are such that $\mathsf{x}^*(i) = x^*, \forall i \in \mathcal{C}$ and $\mathsf{x}_*(i) = x_*, \forall i \in \mathcal{C}$.
- For acts f, g, fAg is the act defined by $fAg(i) = f(i)$ for all i in A and $fAg(i) = g(i)$ for all i in $\overline{A}$.

- When using the mapping that assigns to each state i the utility value $u(x)$ of its consequence $f(i) = x$ under act f, namely $u \circ f$, the values $u(f(i))$ will be simplified as f_i.

Note that the preference relation $\succeq$ induces a complete preordering $\geq_P$ on consequences: $x \geq_P y$ if and only if $\mathsf{x} \succeq \mathsf{y}$; this ordering can be extended to acts as follows: $f \geq_P g$ if and only if $f(i) \geq_P g(i), \forall i \in \mathcal{C}$. This is the Pareto-ordering. Then, one can define an act $f \vee g$ making the best of f and g, such that $\forall i \in \mathcal{C}$, $(f \vee g)_i = f_i$ if $f_i \geq g_i$ and g_i otherwise; and an act $f \wedge g$ making the worst of f and g, such that $\forall i \in \mathcal{C}$, $(f \wedge g)_i = f_i$ if $g_i \geq_P f_i$ and g_i otherwise. Acts are thus combined like fuzzy sets.

The axioms proposed by Dubois et al. (1998) are as follows:

A1 *Totality*: $\succeq$ is a non-trivial total preorder, i.e., it is transitive and complete, and $f \succ g$ for some acts.

WP3 *Weak compatibility with constant acts*: $\forall A \subseteq \mathcal{C}, \forall x, y \in X, \forall f$, $\mathsf{x} \succeq \mathsf{y}$ implies $\mathsf{x}Af \succeq \mathsf{y}Af$.

RCD *Restricted conjunctive dominance*: For any acts g, h and any constant act x, $\mathsf{x} \succ h$ and $g \succ h$ imply $\mathsf{x} \wedge g \succ h$.

RDD *Restricted max-dominance*: For any acts g, h and any constant act x, $h \succ \mathsf{x}$ and $h \succ g$ imply $h \succ \mathsf{x} \vee g$.

Axioms **A1** and **WP3** entail Pareto-dominance: if $f \geq_P g$ then $f \succeq g$ (see Lemma 4 in Dubois et al. 2000). Moreover, **RCD** and **RDD** make sense for one-shot decisions, i.e., without repetition, making the compensation of bad results by good ones impossible.

We recall here the main result about this axiomatization for decision under uncertainty (Dubois et al. 1998).

Theorem 3 *Let $(X^{\mathcal{C}}, \succeq)$ be a preference structure. The following propositions are equivalent:*

- *$(X^{\mathcal{C}}, \succeq)$ satisfies* **A1**, *plus* **WP3, RCD, RDD**.
- *there exists a finite chain L of preference levels, an L-valued monotonic set-function μ, and an L-valued utility function u on X, such that $f \succeq g$ if and only if $S_\mu(f) \geq S_\mu(g)$.*

The proof of this theorem as it appears in Dubois et al. (1998), Dubois et al. (2000) is incomplete. See Dubois and Rico (2018) for a complete proof.

The area of significance of this qualitative decision theory and more precisely the one of axioms **RCD** and **RDD**, is restricted to the case where X and $\mathcal{C}$ are finite and where the value scale is coarse. For instance, **RCD** means that limiting from above the potential utility values of an act g, that is better than another one h, to a constant value that is better than the utility of act h, still yields an act better than h. This is in contradiction with expected utility theory and debatable in the latter setting. Indeed, suppose g is a lottery where you win 1000 € against nothing with equal chances. Suppose the certainty equivalent of this lottery is 400 €, received for sure, and h is the fact of receiving 390 € for sure. Now, it is likely that, if f represents the certainty-equivalent of g, $f \wedge g$ will be felt strictly less attractive than h, as the former means you win 400 € against nothing with equal chances. Axiom **RCD** implies that such a lottery should ever be preferred to receiving $400 - \epsilon$ euros for sure, for arbitrary small positive values of ϵ. This axiom is thus strongly counterintuitive in the context of economic theory, with a continuous consequence set X. However the area of significance of qualitative decision theory is precisely when both X and S are finite.

Two presuppositions actually underlie axiom **RCD** (and similar ones for **RDD**)

(i) There is no compensation effect in the decision process: in case of equal chances, winning 1000 € cannot compensate the possibility of not earning anything. It fits with the case of one-shot decisions where the notion of certainty equivalent can never materialize: you can only get 1000 € or get nothing if you just play once. You cannot get 400 €. The latter can only be obtained in the average, by playing several times.

(ii) There is a big step between each level $\lambda_i \in V$ in the qualitative value scale and the next one λ_{i+1} with $V = \{1 = \lambda_1 > \cdots > \lambda_m = 0\}$. The preference pattern $f \succ h$ always means that f is significantly preferred to h so that the preference level of $f \wedge g$ can never get very close to that of h when $g \succ h$. The counterexample above is obtained by precisely bringing these two preference levels very close to each other so that $f \wedge g$ can become less attractive than the sure gain h. Level λ_{i+1} is in some sense considered negligible in front of λ_i.

Axioms **RDD** and **RCD** can be replaced in Theorem 3 by non-compensation assumptions (Dubois et al. 2000):

$$\textbf{Axiom NC}: \begin{cases} 1_L A y \sim y \text{ or } 1_L A y \sim 1_L A 0_L \\ \text{and} \\ x A 0_L \sim x \text{ or } x A 0_L \sim 1_L A 0_L \end{cases}$$

Non-compensation formalizes the following intuition: in order to evaluate act $1_L Ay$, there is no middle term between values $u(y)$ and $\mu(1_L A 0_L)$. Theorem 3 also holds if in the expression of **RCD** and **RDD** one considers any two comonotonic acts. Indeed Sugeno integrals are "linear" for operations maximum and minimum with respect to disjunctions and conjunctions of comonotonic acts as seen in condition 1 of Theorem 2 and the associated condition 1'. In this sense, Sugeno integral is a qualitative counterpart to Choquet integral. It is easy to check that these equalities hold with any two acts f and g, for the pessimistic and the optimistic possibilistic preference functionals respectively:

$$SLMIN_\pi(f \wedge g) = \min(SLMIN_\pi(f), SLMIN_\pi(g))$$

$$SLMAX_\pi(f \vee g) = \max(SLMAX_\pi(f), SLMAX_\pi(g)).$$

The criterion $SLMIN_\pi(f)$ can be axiomatized by strengthening axiom **RCD** as follows:

Axiom CD : $\forall f, g, h,\ f \succ h$ and $g \succ h$ jointly imply $f \wedge g \succ h$ (Conjunctive Dominance).

This axiom means that if two acts f, g are individually better than a third one, the act $f \wedge g$ which yields the worse result of both acts still remains better than the third one. It makes sense in the scope of a one-shot-decision. Together with Sugeno integral axioms, it implies that the set-function μ is a necessity measure and so, $S_\mu(f) = SLMIN_\pi(f)$, for some possibility distribution π. In order to figure out why axiom CD leads to a pessimistic criterion, Dubois et al. (2001) have noticed that it can be equivalently replaced by the following property:

Axiom PESS $\forall A \subseteq S, \forall f, g,\ fAg \succ g$ implies $g \succeq gAf$ (Pessimism).

This property can be explained as follows: if changing g into f when A occurs results in a better act, the decision maker has enough confidence in event A to consider that improving the results on A is worth trying. But, in this case, there is less confidence on the complement $\overline{A}$ than in A, and any possible improvement of g when $\overline{A}$ occurs is neglected. So, $g \succeq gAf$. For instance, g means losing (=A) or winning (=$\overline{A}$) 10,000 € with equal chances according to whether A occurs or not, and f means winning either nothing (=A) or 20,000 € (=$\overline{A}$) conditioned on the same event. Then fAg is clearly safer than g as there is no risk of losing money. However, if axiom **PESS** holds, then the chance of winning much more money (20,000 €) by choosing act gAf is neglected because there is still a good chance to lose 10,000 € with this lottery. Such a behaviour is clearly cautious.

Similarly, the optimistic criterion $SLMAX_\pi(f)$ can be axiomatized by strengthening the axioms **RDD** as follows:

AxiomDD : $\forall f, g, h,\ h \succ f$ and $h \succ g$ jointly imply $h \succ f \vee g$(Disjunctive Dominance.)

Together with properties appearing in Theorem 3 it implies that the set-function μ is a possibility measure and so, $S_\mu(f) = SLMAX_\pi(f)$ for some possibility distribution π. The optimistic counterpart to property axiom **PESS** that can serve as a substitute to axiom DD for the representation of criterion $SLMAX_\pi$ is:

Axiom OPT $\forall A \subseteq S, \forall f, g,\ g \succ fAg$ implies $gAf \succeq g$. (Optimism).

See also Chateauneuf et al. (2008) for more discussions and results on uncertainty averse and uncertainty seeking decision-makers in the sense of Sugeno integral.

2.4 Sugeno Integral and Decision Rules

So far, only a few works try to provide a logical reading of decision processes. One of such few attempts is given in Dubois et al. (1999) in the framework of decision under uncertainty, where uncertain knowledge and prioritized preference are respectively represented by means of two distinct possibilistic logic bases, and where the pessimistic or optimistic decision criteria that are maximized are particular cases of Sugeno integrals. Another attempt is given in Gérard et al. (2007) in the framework in a multiple criteria decision making, where a qualitative approach (in the spirit of possibilistic logic) is compared to the numerical analogue based on the Choquet integral. However, the most successful approach was initiated by Greco et al. (2004) where they provided a preliminary study (later completed by Bouyssou et al. (2009)) pointing out that the set of the elements for which a Sugeno integral is greater than a given score μ can be described by if–then rules.

Selection rules Consider Sugeno integral in the form given by (5), using the set $\mathcal{F}(\mu)$ of focal sets of μ. It is straightforward to see that the inequality $S_\mu(f) \geq \theta$ is equivalent to $\exists T \in \mathcal{F}(\mu)$ such that $\mu(T) \geq \theta$ and $\forall i \in T,\ f_i \geq \theta$. On this basis, it can be claimed that Sugeno integral based on capacity μ is equivalent to the set of if-then rules of the form:

$$R^s_T: \quad \textbf{If } \forall i \in T,\ f_i \geq \mu(T) \textbf{ then } S_\mu(f) \geq \mu(T).$$

for $T \in \mathcal{F}(\mu)$. Note that these rules are not redundant because either focal sets are not nested or, if they are, they correspond to distinct weights, and the greater the set, the larger the weight. Moreover, they are *single-thresholded* rules, which indicates the limited expressive power of Sugeno integrals. As such rules are bounding the global evaluation from below, they are meant to select "good" alternatives, so we can call them *selection rules*.

Conversely, a set of single-thresholded selection rules of the form "**If** $\forall i \in T_j,\ f_i \geq \theta_j$ **then** $\phi(f) \geq \theta_j$ for $j = 1, \ldots, k$" can be represented by the Sugeno integral with focal sets among $\{T_j : j = 1, \ldots, k\}$ and such that $\mu(A) = \max_{j:T_j \subseteq A} \theta_j$ (the integral representation does away with redundant rules).

This set of rules can be encoded in possibilistic logic (Dubois and Prade 2004) as a set of weighted cubes (Dubois et al. 2014). Define for each criterion i a family of Boolean predicates $P_i(\theta), \theta > 0 \in L$ such that $P_i(\theta)$ is true if $f_i \geq \theta$ and 0 otherwise (we write $f \models P_i(\theta)$). Then we consider weighted Boolean formulas of the form $[\bigwedge_{j \in T} P_j(\theta), \theta]$ and interpreted as

$$\pi_{[T,\theta]}(f) = \begin{cases} \theta \text{ if } f_i \geq \theta, \forall i \in T; \\ 0 \text{ otherwise} \end{cases}.$$

Each weighted cube $[\bigwedge_{j \in T} P_j(\mu(T)), \mu_\#(T)]$ for a focal set T encodes a selection rule R^s_T as stated above. The lower possibility distributions associated to a set of such weighted formulas is interpreted as the maximum of the lower possibility distributions associated to each weighted formula. The possibilistic base

$$G^-_\mu = \{[\bigwedge_{j \in T} P_j(\theta), \theta] : \mu(T) \geq \theta > 0,\ T \in \mathcal{F}(\mu)\}$$

with lower possibility distribution $\pi^-_\mu(f) = \max\{\pi^-_{[\phi,\theta]}(f) \ : \ \mu(T) \geq \theta > 0$ and $T \in \mathcal{F}(\mu)\}$ encodes a Sugeno integral since $S_\mu(f) = \pi^-_\mu(f)$ (see Proposition 4 in Dubois et al. 2014).

Elimination rules Symmetrically, we can obtain rules for the rejection of bad alternatives associated to the Sugeno integral, namely rules expressing the inequality $S_\mu(f) \leq \gamma$ (Dubois et al. 2015). The idea is to use the conjunctive min-max form of Sugeno integral in Eq. (2), which corresponds to possibility distributions over interpretations in standard possibilistic logic (Dubois and Prade 2004).

The focal sets of the conjugate of μ are sufficient to calculate the Sugeno integral, namely (Dubois et al. 2015):

$$S_\mu(f) = \min_{F \in \mathcal{F}(\mu^c)} \max(1 - \mu^c_\#(F), \max_{i \in F} f_i).$$

It is then clear that $S_\mu(f) \leq \theta$ if and only if $\exists F \in \mathcal{F}(\mu^c)$ with $\mu^c(F) \geq 1 - \theta$ s.t. $\forall x_i \in F \ f_i \leq \theta$. This result shows that for each focal set F of the conjugate μ^c we have the following single-thresholded *elimination rule*:

$$R^e_F : \quad \textbf{If } f_i \leq 1 - \mu^c_\#(F) \text{ for all } i \in F \textbf{ then } S_\mu(f) \leq 1 - \mu^c_\#(F).$$

Conversely, a set of single-thresholded elimination rules can be represented by a Sugeno integral.

The possibilistic logic encoding of elimination rules associated to Sugeno integral, is now obtained as set of weighted clauses. Define for each criterion i a family of Boolean predicates $P_i(\theta), \theta > 0 \in L$ such that $P_i(\theta)$ is true for f if $f_i > \theta$ and 0 otherwise.

The set of weighted clauses $\{(\bigvee_{j \in F} P_j(\theta), 1 - \theta)) \ : \ \theta < 1\}$ induces an upper possibility distribution:

$$\pi^+_F(f) = \min_{\theta < 1} \max(\theta, \max_{j \in F} P_j(\theta)) = \max_{j \in F} f_j.$$

Each weighted clause $(\bigvee_{j \in F} P_j(1 - \mu^c(F)), \mu^c(F)))$ for a focal set F of μ^c corresponds to the elimination rule R^e_F stated above.

A logical rendering of the Sugeno integral in the min-max form is obtained as follows. First consider the following base of clauses $B^F_\mu = \{(\bigvee_{j \in F} P_j(\theta), 1 - \theta)) : 1 - \mu^c_\#(F)) \leq \theta < 1\}$. It can be proved (Dubois et al. 2015) that the induced upper possibility is now of the form

$$\pi^+_{B^F_\mu}(f) = \max(1 - \mu^c_\#(F), \max_{i \in F} f_i).$$

The possibilistic base

$$G^+_\mu = \{(\bigvee_{j \in F} P_j(\theta), 1 - \theta)) \ : \ 1 - \mu^c(F)) \leq \theta < 1, F \in \mathcal{F}(\mu^c)\},$$

induced by all focal sets yields the possibility distribution with upper possibility distribution $\pi^+_\mu(f) = \min_{F \in \mathcal{F}(\mu^c)} \pi^+_{B^F_\mu}(f)$, which is precisely equal to $S_\mu(f)$.

This rule-based approach to the description of Sugeno integral delivers several lessons:

- A Sugeno integral can be equivalently expressed by a set of single-thresholded selection and elimination rules, respectively squeezing it from above and from below,
- The expressive power of Sugeno integrals is limited to a class of very specific decision rules.

3 Extensions

The finite scale approach to qualitative decision criteria is simple (especially in the case of weighted max and min). Strictly speaking, it belongs to the class of decision problems that were coined "sorting" by Roy (1996). Elements of the scale L correspond to a totally ordered set of classes of situations that are more or less attractive, and computing the global evaluation of an alternative comes down to assigning it to a class. In the context of DMU, the restriction of the pessimistic approach to the most plausible states, at work in possibilistic criteria, makes them more realistic than the maximin criterion, and more flexible than purely ordinal approaches with no commensurateness assumption.

However, approaches based on an absolute qualitative value scale have their own shortcomings.

- Naturally, a complete preorder on alternatives is obtained from the Sugeno integral. But this preorder is very coarse, especially if the number of elements in L is small. It cannot be large as the human mind cannot make sense of more than seven levels of absolute worth (Miller 1956). Many alternatives, some of which being intuitively better than other ones, will be put in the same class. Hence, one issue is to make Sugeno integral more discriminant via suitable refinement tools. This is the topic of the first subsection.
- Moreover, one has to accept the commensurability assumption between the dimensions of the decision problem. In decision under uncertainty this is not so problematic as there is a single value scale, and the uncertainty scale can be related to the value scale via the notion of certainty equivalent of an uncertain event. However this assumption is much more problematic in the MCDM problem, as each criterion has its own scale that may not be directly commensurate with other ones. In such a situation, Sugeno integral cannot be directly applied. Utility functions that relate the criteria scales to a single one must be introduced, which requires an extension of Sugeno integral described in the second subsection.
- Finally, the role of weights in Sugeno integral is confined to being bounds that limit the value scales from above and from below. In a weighted max, a little

important criterion can only deliver poor grades, while in a weighted min it can only deliver good grades. Other ways of letting importance degrees affect the local evaluations can be envisaged, changing the minimum or the maximum operations into more general conjunction and disjunction operations. This is the topic of the last subsection.

3.1 Lexicographic Refinements

The main reason for the lack of discrimination power of absolute qualitative criteria is the fact that they do not use all the available information to rank alternatives, since an alternative f can be considered indifferent to another alternative g, even if f is at least as good as g in all criteria and strictly so for some of them (violation of the strict Pareto ordering). This is typically the case when an alternative is rated by its worst performance or its best performance across criteria. This defect is absent from the expected utility model.

Refining min and max The lack of discrimination of the maximin rule itself (using $\min_{i \in \mathcal{C}} f_i$ to evaluate alternative f) was actually addressed a long time ago by Cohen and Jaffray (1980) who improved it by comparing acts on the basis of their worst consequences of *distinct* merits, i.e. one considers only the set $D(f, g) = \{i, f_i \neq g_i)\}$ to compare alternatives f and g. Define the refined strict preference relation between acts by

$$f \succ_{dmin} g \iff \min_{i \in D(f,g)} f_i > \min_{i \in D(f,g)} g_i$$

and the weak preference by $f \succeq_{dmin} g \iff \neg(g \succ_{dmin} f)$. This refined rule always rates an act f better than another act g whenever f strictly Pareto-dominates g. However, only a partial ordering of acts is then obtained. This last decision rule is actually no longer based on a preference functional (i.e. it cannot be encoded by a function, like expected utility). This decision rule has been independently proposed by Fargier et al. (1993) and used in fuzzy constraint satisfaction problems (Dubois and Fortemps 1999) under the name *discrimin ordering*.

This ordering can be further refined by the so-called *Leximin* ordering well-known in economics (Deschamps and Gevers 1978): The idea is to reorder vectors $f = (f_1, \ldots f_n)$ by non-decreasing values as $(f_{(1)}, \ldots, f_{(n)})$, where $f_{(k)}$ is the kth smallest component of the vector (i.e., $f_{(1)} \leq \ldots \leq f_{(n)}$). Define the *Leximin* ($\succeq_{lmin}$) and *Leximax* ($\succeq_{lmax}$) rules as:

- $f \succeq_{lmin} g \Leftrightarrow$ either $\forall j, f_{(j)} = g_{(j)}$ or $\exists i, \forall j < i, f_{(j)} = g_{(j)}$ and $f_{(i)} > g_{(i)}$
- $f \succeq_{lmax} g \Leftrightarrow$ either $\forall j, f_{(j)} = g_{(j)}$ or $\exists i, \forall j > i, f_{(j)} = g_{(j)}$ and $f_{(i)} > g_{(i)}$.

Similarly, a *Leximax* preorder can be envisaged as a refinement of the one induced by the maximum. Let $f, g \in L^n$. The two possible alternatives f and g are indifferent if and only if the corresponding reordered vectors are the same. The *Leximin*-ordering is a refinement of the discrimin ordering, hence of both the Pareto-ordering and the maximin-ordering (Dubois et al. 1996): $f \succ_{dmin} g$ implies $f \succ_{lmin} g$. Leximin optimal alternatives are always discrimin maximal alternatives, and thus indeed min-optimal and Pareto-maximal: $\succ_{lmin}$ is the most selective among these preference relations. The *Leximin* ordering can discriminate more than any symmetric aggregation function, since when, e.g., in the numerical setting, the sum of the f_i's equals the sum of the g_i's, it does not mean that the reordered vectors are the same. Similar comments apply for the Leximax ordering.

Weighted *Leximax*/*Leximin* criteria. Suppose that *Leximin* and *Leximax* orderings are defined on sets of tuples whose components belong to a totally ordered set $(V, \trianglerighteq)$, say $Leximin(\trianglerighteq)$ and $Leximax(\trianglerighteq)$. Now, suppose $(V, \trianglerighteq) = (L^p, \succeq_{lmin})$ or $(V, \trianglerighteq) = (L^p, \succeq_{lmax})$, with any positive integer p. Then, nested lexicographic ordering relations that enable L-valued matrices to be compared can be recursively obtained as $Leximin(\succeq_{lmin})$, $Leximax(\succeq_{lmin})$, $Leximin(\succeq_{lmax})$, or yet $Leximax(\succeq_{lmax})$.

Consider for instance the procedure $Leximax(\succeq_{lmin})$ defining the relation $\succeq_{lmax(\succeq lmin)}$. It applies to matrices A of dimension $p \times q$ with coefficients a_{ij} in $(L, \geq)$. These matrices can be totally ordered in a very refined way by this relation. Denote row i of A by $a_{i\cdot}$, and let $A^\star$ and $B^\star$ be rearranged matrices A and B such that terms in each row are reordered increasingly and rows are arranged lexicographically top-down in decreasing order. The relation $A \succ_{lmax(\succeq lmin)} B$ is defined as follows:

$$\exists k \leq p \text{ s.t. } \forall i < k, a^\star_{i\cdot} =_{lmin} b^\star_{i\cdot} \text{ and } a^\star_{k\cdot} >_{lmin} b^\star_{k\cdot}$$

Relation $\succeq_{lmax(\succeq lmin)}$ is a complete preorder. $A \simeq_{lmax(\succeq lmin)} B$ if and only if both matrices have the same coefficients up to the above described rearrangement. Moreover, $\succeq_{lmax(\succeq lmin)}$ refines the ranking obtained by the optimistic criterion:

$$\max_i \min_j a_{ij} > \max_i \min_j b_{ij} \text{ implies } A \succ_{lmax(\succeq lmin)} B.$$

and especially, if A Pareto-dominates B in the strict sense ($\forall i, j, a_{ij} \geq b_{ij}$ and $\exists i^*, j^*$ such that $a_{i^*j^*} > b_{i^*j^*}$), then $A \succ_{lmax(\succeq lmin)} B$.

The comparison of alternatives f and g using the weighted maximum $SLMAX_\pi$ can be refined using relation $\succeq_{lmax(\succeq lmin)}$ applied to $n \times 2$ matrices on $(L, \leq)$, n being the number of criteria, namely comparing matrices F_π and G_π with coefficients $f_{i1} = \pi_i$ and $f_{i2} = f_i$, $g_{i1} = \pi_i$ and $g_{i2} = g_i$.

Likewise the comparison of alternatives f and g using the weighted maximum $SLMIN_\pi$ can be refined using relation $\succeq_{lmin(\succeq lmax)}$ comparing matrices $[f]_{1-\pi}$ and $[g]_{1-\pi}$ with coefficients $f_{i1} = 1 - \pi_i$ and $f_{i2} = f_i$, $g_{i1} = 1 - \pi_i$ and $g_{i2} = g_i$.

Leximaxmin criteria and weighted average. It has been proved (Fargier and Sabbadin 2005) that the above refinements of $SLMIN_\pi$ and $SLMAX_\pi$ can be represented by weighted averages (e.g., expected utility functionals) using special kinds of probability distributions and real-valued utility functions. First note that, in a finite setting, the qualitative *Leximin* and *Leximax* rules can be simulated by means of a sum of utilities provided that the levels in the qualitative (finite) utility scale L are mapped to values sufficiently far away from one another on a numerical scale. Consider an increasing mapping ϕ from L to the reals. It is possible to define this mapping in such a way as to refine the max ordering:

$$\max_{i=1,\dots,n} f_i > \max_{i=1,\dots,n} g_i \text{ implies } \sum_{i=1,\dots,n} \phi(f_i) > \sum_{i=1,\dots,n} \phi(g_i) \tag{8}$$

For instance, the transformation $\phi(\lambda_i) = N^i$ with $N > n$ achieves this goal. It is a super-increasing mapping in the sense that $\phi(\lambda_i) > \sum_{j<i} \phi(\lambda_j), \forall i = 1, \dots, m$. In order to map L to [0, 1] so that $\phi(\lambda_0) = 0$ and $\phi(\lambda_n) = 1$ just let $\phi(\lambda_i) = \frac{N^i-1}{N^m-1}$. Note that it is a convex function $[n] \to \mathbb{R}^+$. It can actually be checked that the *Leximax* ordering is equivalent to applying the Bernoulli criterion with respect to such a convex utility function $\phi(.)$:

$$f >_{Leximax} g \text{ if and only if } \sum_{i=1,\dots,n} \phi(f_i) > \sum_{i=1,\dots,n} \phi(g_i). \tag{9}$$

A similar encoding of the *Leximin* procedure by a sum can be achieved using another super-increasing mapping (for instance, the transformation $\psi(\lambda_i) = \frac{1-N^{-i}}{1-N^{-m}}$ a concave function $L \to \mathbb{R}^+$):

$$f >_{Leximin} g \text{ if and only if } \sum_{i=1,\dots,n} \psi(f_i) > \sum_{i=1,\dots,n} \psi(g_i) \tag{10}$$

The *Leximin* ordering comes down to applying the Bernoulli criterion with respect to such a concave utility function $\psi(.)$. The qualitative pessimistic and optimistic criteria under total ignorance are thus refined by means of a classical criterion with respect to a risk-averse and risk-prone utility function respectively, as can be seen by plotting L against numerical values in the ranges $\phi(L)$ and $\psi(L)$.

The same results apply to possibilistic criteria $SLMIN_\pi$ and $SLMAX_\pi$ (Fargier and Sabbadin 2005) that can be simulated by weighted averages. Consider first the optimistic possibilistic criterion $SLMAX_\pi$ under a given possibility distribution π. We can again define an increasing mapping χ from L to the reals such that $\chi(\lambda_0) = 0$ and especially:

$$\begin{aligned}&\max_i \min(\pi_i, f_i) > \max_i \min(\pi_i, g_i)\\ &\text{implies}\\ &\textstyle\sum_{i=1,\dots,n} \chi(\pi_i) \cdot \chi(f_i) > \sum_{i=1,\dots,n} \chi(\pi_i) \cdot \chi(g_i)\end{aligned} \tag{11}$$

A sufficient condition is that: $\forall i \in \{1, \dots, m\}, \chi(\lambda_i)^2 \geq N \cdot \chi(\lambda_{i-1}) \cdot \chi(1)$ for some $N > n$. The increasing mapping $\chi(\lambda_i) = \frac{N}{N^{2m-i}}, i = 1, \dots, m$, and $\chi(\lambda_0) = 0$, with $N = n + 1$ can be chosen, with $n = |\mathcal{C}|$; $m = |L|$. It is such that $\chi(\lambda_m) = 1$.

Moreover, let $\{E_0, \ldots, E_k\}$ be the well-ordered partition of $\mathcal{C}$ induced by π, E_k containing the most important criteria, and E_0 the least important. Let $K = \frac{1}{\sum_{i=1,\ldots,k} |E_i| \cdot \chi(\pi_i)}$. Define $\chi^*(\lambda_i) = K \cdot \chi(\lambda_i)$, it holds that:

- The weights $p_i = \chi^*(\pi_i)$ define a probability assignment respectful of the possibilistic ordering of criteria. In particular, distribution p is uniform on equally important criteria (the sets E_j). Moreover, if $i \in E_j$ then p_i is greater than the sum of the probabilities of all less probable elements, that is, $p_i > P(E_{i-1} \cup \cdots \cup E_0)$. Such probabilities introduced by Snow (1999), are said to be *big-stepped* in Benferhat et al. (1999).
- the values $\chi(f_i)$ form a big-stepped numerical utility function (a super-increasing sequence of reals $u_l > \cdots > u_1$ such that $\forall l \geq i > 1, u_i > n \cdot u_{i-1}$) that can be encoded by a convex real mapping $[n] \to \mathbb{R}^+$.
- The preference functional

$$EU_+(f) = \sum_{i=1,\ldots,n} \chi^*(\pi_i) \cdot \chi(f_i) \tag{12}$$

  is an expected (big-stepped) utility criterion for a risk-seeking decision-maker, that refines the weighted maximum.

The pessimistic criterion $SLMIN_\pi$ can be similarly refined since $SLMIN_\pi(f) = 1 - SLMAX_\pi(1 - f)$ using the order-reversing map of L. Then, choosing the same mapping χ^* as above, one may have that

$$\begin{array}{l} \min_i \max(\pi_i, f_i) > \min_i \max(\pi_i, g_i) \\ \text{implies} \\ \sum_{i=1,\ldots,n} \chi^*(\pi_i) \cdot \phi(f_i) > \sum_{i=1,\ldots,n} \chi^*(\pi_i) \cdot \phi(g_i) \end{array} \tag{13}$$

where $\phi(\lambda_i) = 1 - \chi(1 - \lambda_i))$ (it is equal to $1 - \frac{n+1}{(n+1)^{2^i}}$ here). Function $\phi(\cdot)$ is a super-increasing numerical utility function that can be encoded by a concave real mapping $[n] \to \mathbb{R}^+$, and the weighted average criterion

$$EU_-(f) = \sum_{i=1,\ldots,n} \chi^*(\pi_i) \cdot \phi(f_i) \tag{14}$$

is a risk-averse one, that refines $SLMIN_\pi$.

The big-stepped functionals $EU_+(f)$ and $EU_-(f)$ turn out to represent the relations $\succeq_{lmax(\succeq lmin)}$ and $\succeq_{lmin(\succeq lmax)}$: it is proved in Fargier and Sabbadin (2005) that

$$EU_+(f) \geq EU_+(f) \text{ if and only if } [f]_\pi \succeq_{lmax(\succeq lmin)} [g]_\pi \tag{15}$$

$$EU_-(f) \geq EU_-(g) \text{ if and only if } [f]_{1-\pi} \succeq_{lmin(\succeq lmax)} [g]_{1-\pi}. \tag{16}$$

These results point out the deep agreement between qualitative possibilistic criteria and weighted averages. The former is just coarser than the latter, and as such cannot account for compensative effects. As a consequence, the additive preference functionals $EU_+(f)$ and $EU_-(f)$ refining the possibilistic criteria are qualitative despite their numerical encoding (numerical utility values are meaningless, but for ensuring a refined ranking over alternatives).

Refining Sugeno integral. Applying the increasing transformation χ that changes a maxmin expression into a sum of products to the minimal disjunctive form $\bigvee_{A\in\mathcal{F}(\mu)}(\mu_\#(A) \wedge \bigwedge_{i\in A} f_i)$ of Sugeno integral $S_\mu(f)$ yields:

$$E_\#^{lsug}(f) = \sum_{A\in 2^{\mathcal{C}}} \chi(\min_{i\in A} f_i) \cdot \chi^*(\mu_\#(A)) = \sum_{A\in 2^{\mathcal{C}}} \min_{i\in A} \chi(f_i) \cdot m_\#(A), \tag{17}$$

where $\chi(\lambda_m) = 1$, $\chi(\lambda_0) = 0$, $\chi(\lambda_j) = \frac{K}{K^{2^{m-j}}}$, $j = 1, m-1$, and we set $K = 2^{|\mathcal{C}|}$. Function χ^* normalizes χ in such a way that $\sum_{A\in 2^{\mathcal{C}}} m_\#(A) = 1$, where the positive weights $m_\#(\cdot) = \chi^*(\mu_\#(\cdot))$ define a random set. Ranking tuples by $E_\#^{lsug}(f)$ comes down to a *Leximax*($\geq_{lmin}$) comparison of $(2^n \times 2)$ matrices with rows of the form $(\mu_\#(A), \min_{i\in A} \chi(f_i))$. It is clear that $E_\#^{lsug}(f)$ is a Choquet integral w.r.t. a belief function with basic mass assignment $m_\#$. It refines the original Sugeno integral. More details can be found in Dubois and Fargier (2009a), Dubois and Fargier (2009b).

3.2 *Sugeno Utility Functionals*

In practice, each criterion in an MCDM problem may have its own scale, which requires an extension of Sugeno integral. We will assume that there are possibly distinct scales $L_1, \ldots, L_n$, one per criterion, that are are finite chains, and, with no danger of ambiguity, we will denote the top and bottom elements of L_i by 1 and 0, respectively, for all $i \in \mathcal{C}$. We say that a mapping $\varphi_i : L_i \to L$, $i \in [n]$, is a *local utility function* if it is order-preserving. It is a qualitative utility function since it is a mapping between finite chains. A function $\Phi : \mathbf{L} = \prod_{i=1}^n L_i \to L$ is said to be a *Sugeno utility functional* (SUF) if there is a Sugeno integral $S_\mu : L^n \to L$ and local utility functions $\varphi_i : L_i \to L$, $i \in [n]$ with $\varphi_i(0) = 0$, $\varphi_i(1) = 1$, such that

$$\Phi(f) = S_\mu(\varphi_1(f_1), \ldots, \varphi_n(f_n)). \tag{18}$$

an expression first proposed by Greco et al. (2004). We shall denote SUFs by $S_{\mu,\varphi}$. Note that Sugeno utility functionals are order-preserving. Moreover, it was shown in Couceiro and Waldhauser (2011) that the set of functions obtained by composing lattice polynomials with local utility functions is the same as the set of Sugeno utility functionals.

Sugeno utility functionals can be characterized in complete analogy with polynomial functions by extending the notion of median decomposability. We say that $\Phi : \mathbf{L} \to L$ is *pseudo-median decomposable* if for each $k \in [n]$ there is a local utility function $\varphi_k : L_k \to L$ such that

$$\Phi(f) = \text{med}\left(\Phi(f^{k\to 0}), \varphi_k(f_k), \Phi(f^{k\to 1})\right) \quad (19)$$

for every $f \in \mathbf{L}$.

Theorem 4 (Couceiro and Waldhauser 2014) *A function $\Phi : \mathbf{L} \to L$ a Sugeno utility functional if and only if Φ is pseudo-median decomposable.*

Remark 2 In Couceiro and Waldhauser (2011, 2014) a more general notion of pseudo-median decomposability was considered where the inner functions $\varphi_i : L_i \to L$, $i \in [n]$, are only required to satisfy boundary conditions.

Note that once the local utility functions $\varphi_i : L_i \to L$ ($i \in [n]$) are given, the pseudo-median decomposability formula (19) provides a disjunctive normal form of a polynomial function p_0 which can be used to factorize Φ. To this extent, let $\widehat{\mathbf{1}}_I$ denote the characteristic vector of $I \subseteq [n]$ in $\mathbf{L}$, i.e., $\widehat{\mathbf{1}}_I \in \mathbf{L}$ is the n-tuple whose i-th component is 1_{L_i} if $i \in I$, and 0_{L_i} otherwise.

Theorem 5 (Couceiro and Waldhauser 2014) *If $\Phi : \mathbf{L} \to L$ is pseudo-median decomposable w.r.t. local utility functions $\varphi_k : L_k \to L$ ($k \in [n]$), then $\Phi = p_0(\varphi_1, \ldots, \varphi_n)$, where the polynomial function p_0 is given by*

$$p_0\,(f_1, \ldots, f_n) = \bigvee_{I \subseteq [n]} \left(\Phi\big(\widehat{\mathbf{1}}_I\big) \wedge \bigwedge_{i \in I} f_i\right). \quad (20)$$

In other words, this theorem characterizes SUFs, namely, $p_0\,(f_1, \ldots, f_n) = S_{\mu,\varphi}(f)$ if p_0 is idempotent (i.e., we must have $\Phi\big(\widehat{\mathbf{1}}_{[n]}\big) = 1$ and $\Phi\big(\widehat{\mathbf{1}}_{\emptyset}\big) = 0$, and Φ is order preserving).

Remark 3 Procedures to obtain local utility functions $\varphi_i : L_i \to L$ ($i \in [n]$), which can be used to factorize a given Sugeno utility functional $f : \mathbf{L} \to L$ into a composition (18), were presented in Couceiro and Waldhauser (2011) when L is an arbitrary chain, and in Couceiro and Waldhauser (2014) when L is a finite distributive lattice.

Another kind of axiomatization was proposed quite early by Greco et al. (2004), namely conditions under which a preference relation $\succeq$ on $\mathbf{L} = \prod_{i=1}^{n} L_i \to L$ can be represented by a Sugeno utility functional. Namely, an equivalence between the two following statements is obtained (a part of Theorem 1 in Greco et al. 2004, here recalled in the finite setting):

- the preference relation $\succeq$ on $\mathbf{L}$ is a complete preordering such that for all $f, g, h, h' \in \mathbf{L}$, $\theta, \beta \in L_i$, $i \in [n]$, if $f^{i\to\theta} \succeq h$, and $g^{i\to\beta} \succeq h'$, then $g^{i\to\theta} \succeq h$ or $f^{i\to\beta} \succeq h'$;
- There is a Sugeno integral and local utility functions $\varphi_k : L_k \to L$ ($k \in [n]$) such that

$$S_\mu(\varphi_1(f_1), \ldots, \varphi_n(f_n)) \geq S_\mu(\varphi_1(g_1), \ldots, \varphi_n(g_n)) \quad \text{if and only if} \quad f \succeq g$$

Bouyssou et al. (2009) reconsider this result in the scope of conjoint measurement for their non-compensatory decomposable representation model, and show that Greco et al. axiom is a strong form of their non-compensation axiom stating that for all $f, g, h, h' \in \mathbf{L}$, $i \in [n]$, if $f \succeq h$ and $g \succeq h'$ then $f^{i \to g_i} \succeq h$ or $g^{i \to f_i} \succeq h'$.

Finally, another, simpler axiom has been used by Couceiro et al. (2016) (Theorem 3.6) to characterize Sugeno utility functionals:

$$\text{for all } f, g \in \mathbf{L}, \theta \in L_k, k \in [n), \text{ if } f^{k\to\theta} \succ f^{k\to 0} \text{ and } g^{k\to 1} \succ g^{k\to\theta} \text{ then } g^{k\to\theta} \succeq f^{k\to\theta}.$$

This axiom is closely connected to the median-decomposability of Sugeno integrals. Note that when $L_i = L$, $\forall i \in [n]$, Sugeno utility functionals are more expressive than Sugeno integrals even in the special cases of $SLMIN_\pi$ and $SLMAX_\pi$, as shown by the counterexample given in Couceiro et al. (2016):

Example 1 Let $L = \{0, \lambda, 1\}$ endowed with the ordering $0 < \lambda < 1$, and consider the preference relation $\succeq$ on $\mathbf{L} = L^2$ whose linearly ordered equivalence classes are

$$\begin{aligned}
[(1, 1)] &= \{(1, 1), (1, \lambda), (1, 0), (0, 1), (\lambda, 1)\},\\
[(\lambda, \lambda)] &= \{(\lambda, \lambda), (\lambda, 0)\},\\
[(0, \lambda)] &= \{(0, \lambda), (0, 0)\}.
\end{aligned}$$

This relation does not satisfy axiom **RCD**, e.g., take $f = (0, \lambda)$, $g = (0, 1)$ and $h = (\lambda, \lambda)$, where $g \succ f$ and $h \succ f$ but $f = h \wedge g$. Thus it cannot be represented by a Sugeno integral. However, letting $S_\mu(f) = f_1 \vee f_2$ (i.e. μ is the uniform possibility distribution), and utility functions φ_1 equal to the identity and $\varphi_2(1) = 1$, $\varphi_2(\lambda) = \varphi_2(0) = 0$, it can be checked that $\succeq$ is represented by the Sugeno utility functional $f_1 \vee \varphi_2(f_2)$.

3.3 Generalized Sugeno Integrals

In Sugeno integrals, the role of weights is devoted to shrinking the evaluation scales of each group of criteria. For instance, in the weighted maximum, the range of evaluation for a criterion i of weight $\pi_i < 1$ is restricted to $[0, \pi_i] \subset L$; in a weighted minimum, it is restricted to $[1 - \pi_i, 1] \subset L$. Any evaluation f_i greater than π_i in $SLMAX_\pi$ (resp. less than $1 - \pi_i$ in $SLMIN_\pi$) is brought back to π_i (resp. $1 - \pi_i$). This role of weights is very peculiar. In this part we explore various alternative weighting schemes. More precisely we are going to generalize rating modification schemes by weights in Sugeno integrals. We first consider generalized forms of aggregation functions $SLMIN_\pi$ and $SLMAX_\pi$, $\bigwedge_{i=1}^n (\pi_i \rhd f_i)$ and $\bigvee_{i=1}^n (\pi_i \otimes f_i)$ respectively, where $\rhd$ and $\otimes$ are suitable rating modification operations.

We first discuss the nature of such operations $\rhd$ and $\otimes$ following intuitive requirements.

In a conjunctive aggregation $\bigwedge_{i=1}^n (\pi_i \rhd f_i)$, the following conditions are natural:

(i) The global evaluation should not be affected by a useless criterion ($\pi_i = 0$) even if the rating is maximal ($f_i = 1$), hence we assume $0 \rhd f_i = 1$.
(ii) A very poor rating on a criterion with top importance $\pi_i = 1$ should be enough to bring the global evaluation down to 0, hence we assume $1 \rhd 0 = 0$.
(iii) The better is the local rating f_i, the greater is the modified rating $\pi_i \rhd f_i$ ($\pi_i \rhd f_i$ should increase with f_i).
(iv) The less important the criterion, the more lenient should be the modified local rating ($\pi_i \rhd f_i$ should decrease with π_i).

So the operation $\rhd$ should be a generalized implication.

Definition 1 A fuzzy implication is a two-place operation $\rightarrow$ on L such that:

(i) $0 \rightarrow 1 = 1$; $1 \rightarrow 0 = 0$; $1 \rightarrow 1 = 1$; $0 \rightarrow 0 = 1$.
(ii) $\alpha \rightarrow \beta$ is increasing in the wide sense with β for all $\alpha \in L$.
(iii) $\alpha \rightarrow \beta$ is decreasing in the wide sense with α for all $\beta \in L$.

In a disjunctive aggregation $\bigvee_{i=1}^{n}(\pi_i \otimes f_i)$ similar conditions should hold:

(i) For the same reason as in the conjunctive case, we should have $0 \otimes f_i = 0$.
(ii) A top rating on a criterion with top importance should be enough to bring the global evaluation to 1, hence we assume $1 \otimes 1 = 1$.
(iii) $\pi_i \otimes f_i$ should increase with f_i as in the conjunctive case.
(iv) A good rating on an important criterion should have more positive influence than one on a little important criterion ($\pi_i \otimes f_i$ should increase with π_i).

So the operation $\otimes$ should be a generalized conjunction.

Definition 2 A fuzzy conjunction is a two-place operation $\otimes$ on L such that:

(i) $0 \otimes 1 = 0$; $1 \otimes 0 = 0$; $1 \otimes 1 = 1$; $0 \otimes 0 = 0$.
(ii) $\alpha \otimes \beta$ is increasing in the wide sense with α, for all $\beta \in L$.
(iii) $\alpha \otimes \beta$ is increasing in the wide sense with β for all $\alpha \in L$.

Note that to each fuzzy implication $\rhd$ in the sense of Definition 1 corresponds a fuzzy conjunction in the sense of Definition 2 $\otimes = \mathcal{S}(\rhd)$ such that $\alpha\mathcal{S}(\rhd)\beta = 1 - (\alpha \rhd (1 - \beta))$. We shall say that such a pair of operations $(\rhd, \otimes)$ are *semidual*. Moreover these connectives can be exchanged, i.e., $\alpha \rhd \beta = \alpha\mathcal{S}(\otimes)\beta = 1 - (\alpha \otimes (1 - \beta))$, i.e., semiduality is an involutive transformation. Note that $\alpha \rhd \beta = 1$ if and only if $\alpha\mathcal{S}(\rhd)(1 - \beta) = 0$ that is, α and $1 - \beta$, when positive, are divisors of 0 for the fuzzy conjunction $\otimes = \mathcal{S}(\rhd)$.

It is interesting to notice that the equality (2) can be expressed using the conjugate capacity $\mu^c(A) = 1 - \mu(\overline{A})$ for every $A \subseteq \mathcal{C}$ and the semi-dual of $\wedge$, namely $\alpha\mathcal{S}(\wedge)\beta = \max(1 - \alpha, \beta)$, which is Kleene-Dienes implication $\rightarrow_{KD}$:

$$S_\mu(f) = \bigvee_{A \subseteq \mathcal{C}} \left(\mu(A) \wedge \bigwedge_{i \in A} f_i\right) = \bigwedge_{A \subseteq \mathcal{C}} (\mu^c(A) \rightarrow_{KD} \bigvee_{i \in A} f_i). \qquad (21)$$

This equality corresponds to extending the duality relation between a capacity μ and its conjugate to Sugeno integrals (Grabisch et al. 1992), i.e. (7).

Sugeno integral can be generalised with other inner fuzzy conjunction or implication functions linked by semiduality, using expressions we call *q-integrals* and *q-cointegrals*, denoted respectively by $\int_\mu^{\otimes} f$ and $\int_\mu^{\rightarrow} f$.

Definition 3 (Dubois et al. 2017) Let $\otimes$ be a fuzzy conjunction. A q-integral is the mapping $\int_\mu^{\otimes}: L^{\mathcal{C}} \to L$ defined by

$$\int_\mu^{\otimes} f = \bigvee_{A \subseteq C} \big(\mu(A) \otimes \bigwedge_{i \in A} f_i\big), \text{ for all } f \in L^{\mathcal{C}}.$$

Definition 4 (Dubois et al. 2017) Let $\rightarrow$ be a fuzzy implication, a q-cointegral is a mapping $\int_\mu^{\rightarrow}: L^{\mathcal{C}} \to L$ defined by

$$\int_\mu^{\rightarrow} f = \bigwedge_{A \subseteq C} \big(\mu^c(A) \rightarrow \bigvee_{i \in A} f_i\big), \text{ for all } f \in L^{\mathcal{C}}.$$

Note that, when $\otimes$ is the product and $L = [0, 1]$, the q-integral is Shilkret integral (Shilkret 1971) (and later reintroduced by Kaufmann (1978), under the name of “admissibility”). Grabisch et al. (1992) introduced the so-called Sugeno-like integrals, which are similar to q-integrals, where $\otimes$ is a triangular norm. Borzová-Molnárová et al. (2015) studied this type of integrals in the continuous case as well when $\otimes$ is a semicopula and $L = [0, 1]$. This kind of definition is also proposed by Dvořák and Holčapek (2012) assuming $(L, \otimes, 1)$ is a commutative monoid and considering the complete residuated lattice generated by this monoidal operation. In fact, what they study is an extension of Definition 3: Namely, they study fuzzy integrals of this type *extended to algebras of fuzzy sets,* that is, where μ is now a *fuzzy set* function that assigns an importance value $\mu(\tilde{A})$ to any *fuzzy* subset $\tilde{A}$ of $\mathcal{C}$.

Now if we consider the dual quantity $1 - \int_\mu^{\otimes}(1 - f)$, it is of the form of a cointegral $\int_\mu^{\rightarrow} f$ with respect to the semi-dual implication $\rightarrow = \mathcal{S}(\otimes)$. But it can be checked that we no longer have the equality (2), that is now

$$\int_\mu^{\otimes} f \neq \int_\mu^{\mathcal{S}(\otimes)} f$$

while we do have that $S_\mu(f) = \int_\mu^{\wedge} f = \int_\mu^{\rightarrow_{KD}} f$, for every capacity μ and every $f \in L^{\mathcal{C}}$, as expressed by identity (2). There are not many works considering q-cointegrals in the above sense. Grabisch et al. (1992, p. 302) notice the failure of the duality relation (7) for Sugeno-like integrals that use t-norms other than min, which hints at co-integrals using implications that are semi-duals of t-norms. The study of solutions to the equality $\int_\mu^{\otimes} f = \int_\mu^{\mathcal{S}(\otimes)} f$ is carried out in Boczek and Kaluszka (2017). It turns out that there exist very few operations other than Kleene-Dienes implication and Kleene conjunction for which these integrals coincide (the conjunctions must be of the form $\varphi(\alpha) \wedge \psi(\beta)$ for local utility functions $\varphi, \psi : L \to L$).

Dubois et al. (2016) study q-cointegrals induced by Gödel implication: $\alpha \to_G \beta = \begin{cases} 1 \text{ if } \alpha \le \beta, \\ \beta \text{ otherwise} \end{cases}$ and the q-integral induced by its semi-dual non-commutative conjunction (Dubois and Prade 1984):

$$\alpha \otimes_G \beta = 1 - (\alpha \to_G (1-\beta)) = \begin{cases} 0 \text{ if } a \le 1-\beta, \\ \beta \text{ otherwise.} \end{cases}$$

They also study q-cointegrals induced by the contrapositive symmetric of Gödel implication defined by $\alpha \to_{GC} \beta = \begin{cases} 1 \text{ if } \alpha \le \beta, \\ 1-\alpha \text{ otherwise} \end{cases}$ and its semidual conjunction, which is equal to $\beta \otimes_G \alpha$. Using a necessity measure for μ and implication $\to_G$, the aggregation $\int_\mu^{\to_G} f$ selects all alternatives that pass the prescribed threshold for each criterion and ranks the remaining ones according to their worst local ratings (forming a waiting list). Using $\to_{GC}$, the aggregation selects all alternatives that pass the prescribed threshold for each criterion and ranks the remaining ones according to the importance of violated criteria, putting the objects that violate the least important criteria upfront. Moreover it is shown in Dubois et al. (2016) that $\int_\mu^{\otimes_G f} > \int_\mu^{\to_G} f$ and $\int_\mu^{\otimes_{GC}} f > \int_\mu^{\to_{GC}} f$ in general. This inequality cannot even be generalised to other conjunctions. More precisely, if $\otimes$ is a triangular norm, or a copula, then $\int_\mu^{\to}(\theta A \lambda) > \int_\mu^{\otimes}(\theta A \lambda)$ in general ($\theta A \lambda$ is the alternative that takes value θ for $i \in A$ and λ otherwise). However, if $\otimes$ is greater than the minimum (which is the case with the semi-dual of the contrapositive symmetric of Gödel implication), then $\int_\mu^{\to}(\alpha A \beta) < \int_\mu^{\otimes}(\alpha A \beta)$ in general. Other q-integrals and q-co-integrals are studied in Dubois et al. (2017).

Elementary properties of q-integrals and co-integrals. The counterpart of the Sugeno integral expression in terms of the nested family of subsets $\{(i), \ldots, (n)\}$, induced by f, where $f_{(1)} \le \cdots \le f_{(n)}$, and the simplified form of Sugeno q-integral are still valid for q-integrals:

$$\int_\mu^{\otimes} f = \bigvee_{i=1}^{n} \mu(\{(i), \ldots, (n)\}) \otimes f_{(i)}, \qquad \int_\mu^{\otimes}(f) = \bigvee_{\lambda \in L} \mu(\{f \ge \lambda\}) \otimes \lambda.$$

If μ is a possibility measure Π based on possibility distribution π we retrieve the $\otimes$-weighted maximum: $\int_\Pi^{\otimes} f = MAX_\pi^{\otimes}(f)$. However, if μ is a necessity measure, the expression of the q-integral will not simplify when $\otimes \ne \min$. In other words we do not have that $\int_N^{\otimes} f = MIN_\pi^{\to}(f)$ for $\to = \mathcal{S}(\otimes)$ except when $\otimes = \wedge$.

Using semi-duality, q-cointegrals can be expressed in terms of q-integrals since $\int_\mu^{\to} f = 1 - \int_{\mu^c}^{\otimes}(1-f)$. In Dubois et al. (2017) we derive the following results for q-cointegrals from the ones on (conjunction-based) q-integrals:

$$\int_{\mu}^{\rightarrow} f = \bigwedge_{i=1}^{n} \mu^c(\{(1), \ldots, (i)\}) \rightarrow f_{(i)}, \qquad \int_{\mu}^{\rightarrow} f = \bigwedge_{\lambda \in L} \mu^c(\{f \leq \lambda\}) \rightarrow \lambda.$$

We also get that when μ is a necessity measure N based on possibility measure π, the q-cointegral reduces to the $\rightarrow$-weighted minimum: $\int_N^{\rightarrow} f = MIN_{\pi}^{\rightarrow}(f) = \wedge_{i=1}^{n} \pi_i \rightarrow f_i$. However the q-cointegral with respect to a possibility measure $\int_{\Pi}^{\rightarrow} f$ does not reduce to a weighted maximum.

The characterization result in Theorem 2 for Sugeno integrals can be extended to q-integrals and co-integrals.

Theorem 6 (Dubois et al. 2017) *Let $I\colon L^{\mathcal{C}} \rightarrow L$ be a mapping. There are a capacity μ and a fuzzy conjunction $\otimes$ such that $I(f) = \int_{\mu}^{\otimes} f$ for every $f \in L^{\mathcal{C}}$ if and only if*

(i) $I(f \vee g) = I(f) \vee I(g)$*, for any comonotone* $f, g \in L^{\mathcal{C}}$.
(ii) There are a capacity $\kappa\colon 2^{\mathcal{C}} \rightarrow L$ and a binary operation $\star$ on L such that $I(\lambda \wedge \mathbf{1}_A) = \kappa(A) \star \lambda$ for every $\lambda \in L$ and every $A \subseteq \mathcal{C}$.
(iii) $I(\mathbf{1}_{\mathcal{C}}) = 1$ *and* $I(\mathbf{0}_{\mathcal{C}}) = 0$.

In that case, we have $\mu = \kappa$ and $\otimes = \star$.

If the fuzzy conjunction $\otimes$ satisfies $\lambda \otimes 1 = \lambda$, then under the assumptions of Theorem 6, the functional I is of the form $I(f) = \int_{\mu}^{\otimes} f$ where $\mu(A) = I(\mathbf{1}_A)$. There is a specific result in Dubois et al. (2017) when the functional I is fully maxitive, to characterize possibilistic q-integrals of the form $I(f) = \int_{\Pi}^{\otimes} f$.

Since the fuzzy conjunction $\otimes$ is not supposed to be commutative, there is a companion q-integral $\int_{\mu}^{\star} f$ with $\lambda \star \lambda' = \lambda' \otimes \lambda$ and a similar characterization result.

For q-cointegrals defined from fuzzy implications, similar characterization results have been derived. However they use counterparts of properties (i'), (ii') and (iii') of Sugeno integral recalled under Theorem 2. We consider a fuzzy implication $\rightarrow$, which can always be assumed to be of the form $\alpha \rightarrow \beta = 1 - \alpha \otimes (1 - \beta)$ for a fuzzy conjunction $\otimes$. This semi-duality property leads to the following characterisation result.

Theorem 7 *Let $I\colon L^{\mathcal{C}} \rightarrow L$ be a mapping. There are a capacity μ and a fuzzy implication $\rightarrow$ such that $I(f) = \int_{\mu}^{\rightarrow} f$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied.*

(i) $I(f \wedge g) = I(f) \wedge I(g)$*, for any comonotone* $f, g \in L^{\mathcal{C}}$.
(ii) There are a capacity $\rho\colon 2^{\mathcal{C}} \rightarrow L$ and a binary operation $\rhd$ such that $I(\lambda \vee \mathbf{1}_A) = \rho^c(\overline{A}) \rhd \lambda, \forall \lambda \in L$.
(iii) $I(\mathbf{1}_{\mathcal{C}}) = 1$ *and* $I(\mathbf{0}_{\mathcal{C}}) = 0$.

In that case $\rho = \mu$, and $\rhd = \rightarrow$.

Note that the homogeneity condition $I(\lambda \vee \mathbf{1}_A) = \rho^c(\overline{A}) \rightarrow \lambda$ for q-cointegrals is better understood if we express the latter expression $(1 - \rho(\overline{A})) \rightarrow \lambda$ as $\rho(\overline{A}) \oplus \lambda$, where $\oplus$ is a disjunction built as the De Morgan dual $1 - (1 - \cdot) \otimes (1 - \cdot)$ of $\otimes =$

$\mathcal{S}(\rightarrow)$, which means it is indeed the disjunctive counterpart of the homogeneity condition $I(a \wedge \mathbf{1}_A) = \lambda(A) \otimes a$ for q-integrals. Clearly, if the fuzzy implication function is such that $\lambda \rightarrow 0 = 1 - \lambda$, then $I(\mathbf{1}_A) = 1 - \rho^c(\overline{A}) = \rho(A)$ (for instance $\rightarrow$ is the symmetric contrapositive of a residual fuzzy implication induced by a conjunction having two-sided identity 1, such as $\wedge$, for which we have $\lambda \rightarrow_{CG} 0 = 1 - \lambda$). There is a specific result in Dubois et al. (2017) when the functional I is fully minitive, to characterize possibilistic q-integrals of the form $I(f) = \int_N^{\rightarrow} f$. In the same paper, representation results for the companion q-cointegral defined from a q-cointegral by contrapositive symmetry are proposed.

To summarize, the above results indicate that properties of Sugeno integrals remain valid for more general functionals where the weights of groups of criteria may variously act to modify local evaluations. However these set of properties is split into some for q-integrals and others for q-co-integrals:

- Comonotonic maxitivity is specific to q-integrals and comonotonic minitivity is specific to q-cointegrals.
- Q-integrals and q-cointegrals do not satisfy the same homogeneity conditions, the former being homogeneous with respect to the meet $\wedge$ and the latter with respect to the join $\vee$.
- Q-integrals simplify if computed wrt a possibility measure and q-cointegrals simplify wrt a necessity measure but not conversely.

4 Bipolar Evaluation Methods

When rating alternatives on a scale L, it is useful to have a clear understanding of the meaning of its end-points. The explicit handling of positive and negative judgments when selecting an alternative must be distinguished from the simple need for ranking alternatives in terms, e.g., of preference. People also need to express that some alternative is good or bad for them, a notion that simple preference relations cannot express. Using a simple preference relation, the best available choice may fail to be really suitable for the decision-maker. In other circumstances, even the worst ranked option remains somewhat acceptable. To discriminate between these two situations, one absolute landmark or reference point expressing neutrality or indifference, and explicitly separating the positive and the negative judgments, must appear in the model.[1]

Modeling this situation requires a *bipolar scale* $(L, >)$, i.e., a totally ordered set with a prescribed interior element e called *neutral*, separating the positive range of evaluations $\lambda > e$ from the negative one $\lambda < e$. Mathematically, if the scale is

[1]Even ordinal decision methods need to inject some form of bipolarity. Note that multicriteria decision methods based on the merging of outranking relations use concordance and discordance tests between criteria (Roy 1996), where the notion of veto prevents the choice of alternatives that rate too low with respect to some criteria. It can be viewed as an attempt to capture the idea of bipolar preference (Öztürk and Tsoukiás 2008).

equipped with a binary operation $\star$ (an aggregation operator), e is an idempotent element for $\star$, possibly acting as an identity. Interestingly, classical utility theory does not exploit bipolarity. Utility functions are defined up to an increasing affine transformation (i.e., they rely on an interval scale), and the separation between positive and negative evaluations has no special meaning. In contrast, Cumulative Prospect Theory (CPT, for short) proposed by Tversky and Kahneman (1992) is an attempt to explicitly account for positive and negative evaluations using the real line as a genuine bipolar scale. The simplest qualitative bipolar scale contains three elements: $\{-, 0, +\}$. In bipolar scales, the negative side of the scale (below e) is the mirror image of the positive one (above e). An object is evaluated on such a bipolar scale as being either positive or negative or neutral. It cannot be positive and negative at the same time. This is called a *univariate bipolar* framework.

However, it is known from many experiments in cognitive psychology (Osgood et al. 1957; Cacioppo and Berntson 1994; Slovic et al. 2002) that humans often evaluate alternatives for the purpose of decision-making by considering positive and negative aspects *separately*. Under this bipolar view, comparing two alternatives comes down to comparing pairs of sets of arguments or features, namely, the set of pros and cons pertaining to one alternative versus the set of pros and cons pertaining to the other. This view of bipolarity requires the use of two unipolar qualitative scales L^+ and L^- (a positive one and a negative one) instead of a unique bipolar scale. This is the *bivariate unipolar* framework. Here each scale is unipolar in the sense that the neutral level is at one end of the scale. In a *positive* scale the bottom element is neutral. In a *negative* scale the top element is neutral. A bipolar scale can be viewed as the union of a positive and a negative scale $L^+ \cup L^-$ extending the ordering relations on each scale so $\forall \lambda^+ \in L^+, \lambda^- \in L^-, \lambda^+ > \lambda^-$.

There are in fact three forms of bipolarity can be found at work in the literature, called types I, II, III in Dubois and Prade (2008):

- **Type I: Symmetric univariate bipolarity**. It relies on the use of bipolar scales.
- **Type II: Symmetric bivariate bipolarity**. It relies on the use of two unipolar scales related via duality. Positive and negative strengths are computed similarly on the basis of the same data and can be conflicting. This is the case of argumentation systems where reasons for an alternative and reasons against it are collected from the same knowledge base prior to making a decision. Psychologists have shown that the simultaneous presence of positive and negative arguments prevents decisions from being simple to make, except when their strengths have different orders of magnitude.
- **Type III: Asymmetric bipolarity**. In this form of bipolarity, the negative part of the information is not of the same nature as the positive part, while in type II bipolarity only the polarities are opposite. In decision-making, this kind of bipolarity corresponds to the opposition between soft constraints (that eliminate unwanted alternatives) and criteria (that evaluate preferred ones), as discussed in Benferhat et al. (2006). Constraints have a prominent role and first select the most tolerated alternatives; positive preferences (such as goals and desires) then act to discriminate among this set of non-rejected alternatives. Hence a positive

evaluation, even if high, can never outperform a negative evaluation even if very weak. In this approach, negative features prevail over positive ones. The latter matter only when no constraint is violated. In the previous type of bipolarity, positive and negative arguments play symmetric roles.

In this section, we review two bipolar approaches to decision related to Sugeno integrals, namely the symmetric Sugeno integral, and a qualitative counterpart of CPT, where positive and negative sides of alternatives are evaluated separately in the spirit of argumentation theory.

4.1 The Symmetric Sugeno Integral

Let L be a totally ordered set with bottom element 0, and let $-L := \{-\lambda : \lambda \in L\}$ be its "symmetric" copy endowed with the reversed order. Consider the symmetric ordered structure $\tilde{L} := L \cup (-L) \setminus \{-0\}$, a bipolar scale analogous the real line where the zero that acts as a neutral element and such that $\lambda + (-\lambda) = 0$. In particular, $-(-\lambda) = \lambda$. The question is thus how to define lattice polynomial functions on such bipolar ordered structures while keeping the symmetry with respect to 0. In particular, we seek a symmetric extension of the Sugeno integral.

As we saw, lattice polynomial functions on distributive lattices (in particular, on linearly ordered sets) are can be represented in disjunctive normal form (see, e.g., Couceiro and Marichal 2010b, 2012; Goodstein 1967). Thus any lattice polynomial function could be in principle defined on a symmetric structure $\tilde{L}$ if one could define the symmetric extensions of maximum and minimum on L over the bipolar $\tilde{L}$.

The symmetric minimum $\owedge$ (playing the role of the product in the real line) is rather simple. Define the absolute value of $\lambda \in \tilde{L}$ as:$|\lambda| = \lambda$ if $\lambda \in L$ and $-\lambda$ otherwise. Grabisch (2003; 2004) proposed the following notion of *symmetric minimum*:

$$\lambda \owedge \lambda' := \begin{cases} -(|\lambda| \wedge |\lambda'|) & \text{if } \mathrm{sign}(\lambda) \neq \mathrm{sign}(\lambda') \\ |\lambda| \wedge |\lambda'| & \text{otherwise.} \end{cases} \tag{22}$$

The absolute value of $\lambda \owedge \lambda'$ equals $|\lambda| \wedge |\lambda'|$ and $\lambda \owedge \lambda' < 0$ if and only if the two elements λ and λ' have opposite signs. Like the usual minimum operator $\wedge$, the symmetric minimum $\owedge$ is associative.

Now, the symmetric notion of maximum playing the role of a sum is more challenging. Intuitively, the *symmetric maximum* $\ovee$ should extend the maximum on L with 0 as neutral element, and should fulfill the symmetry requirement: for every $\lambda \in \tilde{L}$, $\lambda \ovee (-\lambda) = 0$. However, unlike with the minimum, this symmetry requirement immediately implies that any extension $\ovee$ of the maximum operator $\vee$ cannot be associative. To illustrate this point, let $L = \mathbb{N}$ and observe that $(2 \ovee 3) \ovee (-3) = 3 \ovee (-3) = 0$ where as $2 \ovee (3 \ovee (-3)) = 2 \ovee 0 = 2$.

Nonetheless, Grabisch (2003) showed that the "best" definition of $\ovee$ (see Theorem 8 below) is:

$$\lambda \circledvee \lambda' = \begin{cases} -(|\lambda| \vee |\lambda'|) & \text{if } \lambda' \neq -\lambda \text{ and } |\lambda| \vee |\lambda'| = -\lambda \text{ or } = -\lambda' \\ 0 & \text{if } \lambda' = -\lambda \\ |\lambda| \vee |\lambda'| & \text{otherwise.} \end{cases} \quad (23)$$

In other words, if $\lambda' \neq -\lambda$, then $\lambda \circledvee \lambda'$ returns the element that is the larger in absolute value among the two elements λ and λ'. Moreover, it is not difficult to see that $\circledvee$ satisfies the following properties:

(C1) $\circledvee$ coincides with the maximum on L^2;
(C2) $\lambda \circledvee (-\lambda) = 0$ for every $\lambda \in \tilde{C}$;
(C3) $-(\lambda \circledvee \lambda') = (-\lambda) \circledvee (-\lambda')$ for every $\lambda, \lambda' \in \tilde{C}$.

Hence, $\circledvee$ almost behaves like $+$ on the real line, but for associativity. As shown in Grabisch (2003), if one requires that (C1), (C2) and (C3) hold, then (23) is the best possible definition for $\circledvee$.

Theorem 8 (Grabisch 2003, Prop. 5) *No binary operation satisfying (C1), (C2), (C3) is* associative *on a larger domain than* $\circledvee$*.*

The following result presents some further properties of $\circledvee$ and describes the sequences that reveal the nonassociativity of $\circledvee$.

Proposition 2 (Grabisch 2003, Couceiro and Grabisch 2013, Prop. 5) *The symmetric maximum has the following properties:*

(i) $\circledvee$ *is commutative on* $\tilde{L}$*.*
(ii) 0 *is the neutral element of* $\circledvee$*.*
(iii) $\circledvee$ *is associative on an expression involving* $\alpha_1, \dots, \alpha_n \in \tilde{L}$*, with* $|\{i : \alpha_i \neq 0\}| > 2$*, if and only if* $\bigvee_{i=1}^n \alpha_i \neq -\bigwedge_{i=1}^n \alpha_i$*.*
(iv) $\circledvee$ *is nondecreasing in each argument on* $\tilde{L}$*.*

Sequences fulfilling condition (iii) were referred to as *associative* in Couceiro and Grabisch (2013).

Now the ambiguity in evaluating $\circledvee$ on nonassociative sequences makes it hard to use it for defining lattice polynomial functions on $\tilde{L}$, since the result of $\circledvee_{i=1}^k a_i$ depends on the particular way the binary $\circledvee$ is applied to the terms of $(\alpha_i)_{1 \leq i \leq k}$. Grabisch (2003) suggested ways of making $\circledvee$ associative that were fully developed in Couceiro and Grabisch (2013), namely, by fixing beforehand a systematic way of putting parentheses on any sequence of $\tilde{L}^*$, procedure that was called a *rule of computation*.

Making $\circledvee$ associative: rules of computation. We now recall the formalism of Couceiro and Grabisch (2013). As, we will only consider countable sequences of elements of $\tilde{L}$, without loss of generality, we may assume that $\tilde{L} = \mathbb{Z}$. In this way, elements of $\tilde{L}^*$ are (finite) sequences of integers, denoted by $\sigma = (\lambda_i)_{i \in I}$ for some finite index set I, including the empty sequence ε, i.e.,

$$\tilde{L}^* = \Big(\bigcup_{n \in \mathbb{N}} (\tilde{L})^n\Big) \cup \{\varepsilon\}.$$

This convention will simplify our exposition and establish connections to the theory of integer means.

Also, as $\circledcirc$ is commutative, the order of symbols in the word does not matter, and we can consider the decreasing order of the absolute values of the elements in the sequence (e.g., $5, 5, -5, -3, 2, -2, 1, 0$). Since sequences are ordered, we can consider the following convenient formalism for representing sequences. For an arbitrary sequence

$$\sigma = (\underbrace{n_1, \ldots, n_1}_{p_1 \text{ times}}, \underbrace{-n_1, \ldots, -n_1}_{m_1 \text{ times}}, \ldots, \underbrace{n_q, \ldots, n_q}_{p_q \text{ times}}, \underbrace{-n_q, \ldots, -n_q}_{m_q \text{ times}})$$

let $\theta(\sigma) = (n_1, \ldots, n_q)$ be the sequence of absolute values (magnitudes) of integers in σ, and let $\psi(\sigma) = ((p_1, m_1), \ldots, (p_q, m_q))$ be the sequence of pairs of numbers of occurrence of these integers. For instance, if $\sigma = (3, 3, -3, 2, -2, -2, 1, 1, 1, 1)$, then

$$\theta(\sigma) = (3, 2, 1); \quad \psi(\sigma) = ((2, 1), (1, 2)(4, 0)).$$

Let $\mathfrak{S}$ denote the set of all integer sequences in this formalism, including the empty sequence, and let $\mathfrak{S}_0$ be the subset of all nonassociative sequences.

This formalism facilitates the precise definition of rules of computation in terms of 5 elementary rules $\rho_i : \mathfrak{S} \to \mathfrak{S}$ that act on σ in the following way:

(i) Elementary rule ρ_1: if $p_1 > 1$ and $m_1 > 0$, then p_1 is changed to $p_1 = 1$;
(ii) Elementary rule ρ_2: idem with p_1, m_1 exchanged;
(iii) Elementary rule ρ_3: if $p_1 > 0, m_1 > 0$, the pair (p_1, m_1) is changed into $(p_1 - c, m_1 - c)$, where $c = p_1 \wedge m_1$;
(iv) Elementary rule ρ_4: if $p_1 > 0, m_1 > 0$, and if $p_2 > 0$, then p_2 is changed into $p_2 = 0$;
(v) Elementary rule ρ_5: idem with m_2 replacing p_2.

Hence, elementary rules delete terms only in nonassociative sequences, and leave the associative ones invariant.

A *(well-formed) computation rule* R is a word built with the alphabet $\{\rho_1, \ldots, \rho_5\}$, i.e., $R \in \mathcal{L}(\rho_1, \ldots, \rho_5)$, such that $R(\sigma) \in \mathfrak{S} \setminus \mathfrak{S}_0$ for all $\sigma \in \mathfrak{S}$. The set of (well-formed) computation rules is denoted by $\mathfrak{R}$. Examples of rules are (words are read from left to right)

(i) $\langle\cdot\rangle_0 = \rho_3^*$, that corresponds to putting parentheses around each pair of maximal symmetric terms.
(ii) $\langle\cdot\rangle_= = (\rho_1\rho_2\rho_3)^*$, that corresponds to putting parentheses around terms with the same absolute value and sign, and then to putting parentheses around each each pair of maximal symmetric resulting terms.
(iii) $\langle\cdot\rangle_-^+ = (\rho_4\rho_5)^*\rho_1\rho_2\rho_3$, that corresponds to first putting parentheses around all positive terms and all negative terms, and then computing the symmetric maximum of the two results.

It is shown in Couceiro and Grabisch (2013) that each computation rule $R \in \mathfrak{R}$ corresponds to an arrangement of parentheses together with a permutation on the terms of sequences. Thus each $R \in \mathfrak{R}$ turns the symmetric maximum into an associative operation $\varovee_R : \mathfrak{S}^* \to \mathfrak{S}$ defined by $\varovee_R = \varovee \circ R$, since $R(\sigma) \in \mathfrak{S} \setminus \mathfrak{S}_0$ for all $\sigma \in \mathfrak{S}$.[2] Moreover, each computation rule has the form $R = T_1 T_2 \ldots$, where each T_i has the form $\omega \rho_1^{\alpha} \rho_2^{\beta} \rho_3$, with $\omega \in \mathcal{L}(\rho_4, \rho_5)$ and $\alpha, \beta \in \{0, 1\}$ (factorization scheme).[3]

Now to compute $\varovee_R(\sigma)$, one needs to delete symbols in the sequence $\theta(\sigma)$ exactly as they are deleted in $\psi(\sigma)$. This entails an ordering of $\mathfrak{R}$ that is discussed below.

Let $R, R' \in \mathfrak{R}$ and, for each sequence $\sigma = (a_i)_{i \in I}$, let $J_\sigma \subseteq I$ and $J'_\sigma \subseteq I$, be the sets of indices of the terms in σ deleted by R and R', respectively. Then, we write $R \leqslant R'$ if for all sequences $\sigma \in \mathfrak{S}$ we have $J_\sigma \supseteq J'_\sigma$. Clearly, it is reflexive and transitive, and thus it is a preorder. This induces an equivalence relation $\sim$ defined as follows: $R \sim R'$ if $R \leqslant R'$ and $R' \leqslant R$. The following proposition provides equivalent definitions of $\sim$.

Proposition 3 *Let $R, R' \in \mathfrak{R}$. Then the following assertions are equivalent.*

(i) $R \sim R'$.
(ii) $\varovee_R = \varovee_{R'}$.
(iii) $\mathrm{Ker}(\varovee_R) = \mathrm{Ker}(\varovee_{R'})$.[4]

Furthermore, they have exactly the same factorized irredundant form (see Couceiro and Grabisch 2013).

The structure of the poset $\mathfrak{R}/_\sim$ of equivalence classes endowed with the partial order induced by $\leqslant$ was investigated in Couceiro and Grabisch (2013) and shown to be highly complex. To give an idea, the subposet $\mathfrak{R}_{123}/_\sim$ of equivalence classes of rules $R \in \mathcal{L}(\rho_1, \rho_2, \rho_3)$ has infinitely many maximal elements, and $(\mathfrak{R}_{123}/_\sim, \leqslant)$ (and thus $(\mathfrak{R}/_\sim, \leqslant)$) embeds the powerset $(2^{\mathbb{N}}, \subseteq)$ of natural numbers, and hence it is of continuum cardinality. For further results on $\mathfrak{R}/_\sim$, see Couceiro and Grabisch (2013).

Characterizations of symmetric maxima $\varovee_R$ We now briefly describe the class of those integer functions $\phi : \tilde{L}^* \to \tilde{L}$ that coincide with symmetric maxima $\varovee_R$, for $R \in \mathfrak{R}$, by making use of the tight connections to the theory of integer means.

A function $\phi : \tilde{C}^* \to \tilde{C}$ that verifies anonymity,[5] internality,[6] monotonicity,[7] and decomposability[8] is called an *integer mean* in Bennett et al. (2014), where it was

[2]For convenience, we assume that $\varovee_R(\varepsilon) = 0$ and $\varovee_R(a) = a$, for every $a \in \tilde{C}$.

[3]Here, $\rho^0 = \varepsilon$ and $\rho^1 = \rho$.

[4]The *kernel* of $\varovee_R$ is defined by $\mathrm{Ker}(\varovee_R) = \{\sigma \in \mathfrak{S} \mid \varovee_R(\sigma) = 0\}$.

[5]ϕ is *anonymous* if for every $\sigma = (\alpha_i)_{i \in I} \in \mathfrak{S}$ and every permutation π on I, $\varovee_R(\sigma) = \varovee_R(\sigma \circ \pi)$, where $\sigma \circ \pi = (\alpha_{\pi_i})_{i \in I}$.

[6]ϕ is *internal* if for every $\sigma = (\alpha_i)_{i \in I} \in \mathfrak{S}$, $\min_{i \in I} \alpha_i \leqslant \phi(\sigma) \leqslant \max_{i \in I} \alpha_i$.

[7]ϕ is *monotone* if $\phi(\sigma) \leqslant \phi(\sigma')$ whenever $\sigma = (\alpha_i)_{i \in I} \in \mathfrak{S}$ and $\sigma' = (a'_i)_{i \in I} \in \mathfrak{S}$ are such that $\alpha_i \leqslant \alpha'_i$ for every $i \in I$.

[8]ϕ is *decomposable* if for every $\sigma = (\alpha_i)_{i \in I}$ and $K \subseteq I$, $\phi(\alpha_K) = b$ implies $\phi(\alpha_K, \alpha_{I \setminus K}) = \phi(|K| \cdot b, \alpha_{I \setminus K})$, where $|K| \cdot b$ means $b, b, \ldots, b$ (repeated $|K|$ times).

shown that all such functions are extremal.[9] However, the three last properties are too stringent for functions of the form $\phi = \circledvee_R$. This led us to considering some relaxations and variants of these properties, which culminated in the following characterizations of the class of symmetric maxima $\circledvee_R$, $R \in \mathfrak{R}$.

Theorem 9 (Couceiro and Grabisch 2017) *Let $\phi : \tilde{L}^* \to \tilde{L}$ be an anonymous and weakly associative function that satisfies (C1), (C2), (C3) on $\tilde{L}^2$, and let $R \in \mathfrak{R}$. Then $\phi = \circledvee_R$ if and only if one (or, equivalently, all) of the following assertions hold:*

(i) $\mathrm{Ker}(\circledvee_R) \subseteq \mathrm{Ker}(\phi)$ *and ϕ is decomposable on every $K \subseteq J^R_\sigma$,*
(ii) *ϕ is extremal w.r.t. $R \in \mathfrak{R}$, i.e., $\phi(\sigma) = \Big(\min_{i \in I \setminus J^R_\sigma} \alpha_i\Big) \circledvee \Big(\max_{i \in I \setminus J^R_\sigma} \alpha_i\Big)$,*
(iii) *ϕ is retractive w.r.t. R, i.e., $\phi(\sigma) = \phi(\alpha_{I \setminus J^R_\sigma})$,*

for every $\sigma = (\alpha_i)_{i \in I}$ in $\mathfrak{S}$.

Formulating the symmetric Sugeno integral. The question is how to define the Sugeno integral for functions which may take negative values, i.e., functions $f : [n] \to \tilde{L}$. We proceed by analogy with the Choquet integral. We recall its expression for a function $f : [n] \to [0, 1]$ w.r.t. a capacity μ:

$$C_\mu(f) = \sum_{i=1}^{n} (f_{(i)} - f_{(i-1)})\mu(A_{(i)}), \tag{24}$$

with $f_{(0)} := 0$. The usual way of defining the Choquet integral for functions taking negative values is the following one:

$$C_\mu(f) = C_\mu(f^+) - C_{\mu^c}(-f^-),$$

with f^+, f^- the positive and negative parts of f, i.e., $f^+ = f \vee 0$ and $f^- = (-f)^+$, and μ^c is the conjugate or dual of μ, defined when $\tilde{L} = \mathbb{R}$ by $\mu^c(S) = \mu([n]) - \mu(S^c)$. This is sometimes called the *asymmetric Choquet integral*, as it does *not* satisfy the property $C_\mu(-f) = -C_\mu(f)$. The motivation for its definition is that this is the only expression which is invariant by translation, i.e., $C_\mu(f + h) = C_\mu(f) + h\mu([n])$, h being a constant function of value h. In our context of ordinal scales, translation has no meaning, and consequently mimicking the definition of the (asymmetric) Choquet integral for the Sugeno integral of $\tilde{L}$-valued functions is meaningless.

On the other hand, the *symmetric Choquet integral* is defined by

$$\check{C}_\mu(f) = C_\mu(f^+) - C_\mu(f^-). \tag{25}$$

Its name comes from the fact that indeed this integral satisfies symmetry: $C_\mu(-f) = -C_\mu(f)$. Its explicit expression is found to be:

[9] ϕ is *extremal* if for every $\sigma = (\alpha_i)_{i \in I} \in \mathfrak{S}$, $\phi(\sigma) = \phi\Big(\min_{i \in I} \alpha_i, \max_{i \in I} \alpha_i\Big)$.

$$\check{C}_\mu(f) = \sum_{i=p+2}^{n} (f_{(i)} - f_{(i-1)})\mu(\{(i), \dots, (n)\}) + f_{(p+1)}\mu(\{(p+1), \dots, (n)\})$$
$$+ f_{(p)}\mu(\{(1), \dots, (p)\}) + \sum_{i=1}^{p-1}(f_{(i)} - f_{(i+1)})\mu(\{(1), \dots, (i)\}), \quad (26)$$

where $f_{(1)} \leq \dots \leq f_{(p)} < 0 \leq f_{(p+1)} \leq \dots \leq f_{(n)}$.

Based on this, we define the *symmetric Sugeno integral* of $f : [n] \to \tilde{L}$ w.r.t. a capacity μ as follows:

$$\check{S}_\mu(f) = S_\mu(f^+) \otimes (-S_\mu(f^-)), \quad (27)$$

where again f^+, f^- denote the positive and negative parts of f, i.e., $f^+ = f \vee 0$ and $f^- = (-f)^+$. Observe that thanks to property (C1), the symmetric Sugeno integral extends the usual Sugeno integral on $\tilde{L}$. Moreover, thanks to (C2) and (C3), the symmetric Sugeno integral behaves like the symmetric Choquet integral on the real line. In particular, by (C3) we have that $-S_\mu(f) = S_\mu(-f)$, and thus the explicit expression of $\check{S}_\mu(f)$ is also close to the symmetric Choquet integral:

$$\check{S}_\mu(f) = \Big[\bigotimes_{i=p+1}^{n} (f_{(i)} \oslash \mu(\{(i), \dots, (n)\})) \Big] \otimes \Big[\bigotimes_{i=1}^{p} (f_{(i)} \oslash \mu(\{(1), \dots, (i)\})) \Big], \quad (28)$$

with $f_{(1)} \leq \dots \leq f_{(p)} < 0 \leq f_{(p+1)} \leq \dots \leq f_{(n)}$ and $\oslash$ is the symmetric minimum. Notice that there is no ambiguity due to lack of associativity in this formula.

Now, by using the computation rule $\langle\cdot\rangle_+^-$, (28) can be rewritten as:

$$\check{S}_\mu(f) = \otimes_{\langle\cdot\rangle_+^-}(f_{(1)} \oslash \mu(\{(1)\}), \dots, f_{(p)} \oslash \mu(\{(1), \dots, (p)\}), f_{(p+1)} \oslash \mu(\{(p+1), \dots, (n)\}), \dots, f_{(n)} \oslash \mu(\{(n)\}))). \quad (29)$$

This fact together with our general framework for computation rules motivates several other definitions of the symmetric Sugeno integral, each of which reflecting different tendencies (e.g., pessimistic vs optimistic views). Indeed, it is tempting to consider replacing $\langle\cdot\rangle_+^-$ by any computation rule R:

$$\check{S}_\mu(f) = \otimes_R(f_{(1)} \oslash \mu(\{(1)\}), \dots, f_{(p)} \oslash \mu(\{(1), \dots, (p)\}), f_{(p+1)} \oslash \mu(\{(p+1), \dots, (n)\}), \dots, f_{(n)} \oslash \mu(\{(n)\}))). \quad (30)$$

However, we must be careful due to the fact that not all computation rules are monotonic, i.e., increasing one of the elements of a sequence σ cannot decrease the result. For example, the rule $\langle\cdot\rangle_=$ is not monotonic as the following example shows:

$$\otimes_{\langle\cdot\rangle_=}(5, -5, -5, 4, 3) = 4 \quad \text{whereas} \quad \otimes_{\langle\cdot\rangle_=}(5, -5, -4, 4, 3) = 3.$$

This naturally raises the question of determining those computation rules that are monotonic, which constitutes a topic of current research.

4.2 Deciding by Evaluating Pros and Cons

A formal elementary framework for a bivariate bipolar multicriteria decision analysis requires a positive scale L^+ and a negative scale L^- which are unipolar. We consider the symmetric case, namely there is an order-reversing bijection between L^+ and L^-. Basically they are copies of a finite totally ordered scale. In other words, the pair (L^+, L^-) can be viewed as the two parts of a bipolar scale. In the simplest qualitative setting, criteria are valued on a bipolar scale $L_B = \{-, 0, +\}$, whose elements reflect negativity, neutrality and positivity respectively. Then $L^+ = \{0, +\}$, $L^- = \{-, 0\}$, which are copies of the Boolean scale $\{0, 1\}$.

The importance of criteria is evaluated on a scale L that is commensurate with the two scales L^+ and L^-, in the sense that there is an order-preserving (for L^+) and an order-reversing (for L^-) injection preserving top and bottom between L^+, L^- and L: for instance $L^+ \subseteq L$ and $L^- \subseteq \{-\lambda : \lambda \in L\}$.

In the following we assume that alternatives map $\mathcal{C}$ to $L \times L$; namely f_i is of the form (f_i^-, f_i^+) with $-f_i^- \in L^-$ and $f_i^+ \in L^+$, with the assumption that $\min(f_i^-, f_i^+) = 0$ (the rating of f wrt a criterion is either positive or negative). Each value f_i expresses that criterion i brings an argument in favor of f ($f_i^+ > 0$) or in disfavor of f ($f_i^- > 0$) or yet is neutral to f (when $f_i^+ = f_i^- = 0$). In other words we can split the alternative f into positive f^+ and negative parts f^-, that can be independently evaluated on the scale L. In a nutshell, with respect to an alternative f, each criterion i has

- *a polarity*: the criterion i judges f positively, or negatively in the wide sense.
- *a degree of importance* $\pi_i \in L$ that does not depend on the alternative. More generally, we can use a capacity for weighting dependent criteria.

The proposed framework is clearly of type II in the bipolarity typology.

Example 2 Suppose that Luc has to choose a holiday destination and considers two options for which he has listed the pros and cons. Option f is in a very attractive region (a strong pro), and hotel has a swimming pool ; but it is very expensive, and the plane company has a terrible reputation (two strong cons). Option g is cheaper but it is in a non-democratic country, and Luc considers it a strong con. On the other hand, Option g includes a tennis court and a swimming pool. These are three pros, but not very decisive: they do matter, but not as much as the other arguments.

Formally, let:

- $\mathcal{C} = \{$Attractiveness (1), price (2), democracy (3), sport facilities (4), airline (5)$\}$.
- $L^+ = \{0, +, ++\}$, $L^- = \{--, -, 0\}$, $L = \{0, \lambda, 1\}$.

- Available alternatives:
  f gets ++ on attractiveness, −− on price and airline, + on sport facilities
  so $f^+ = (1, 0, 0, \lambda, 0)$ and $f^- = (0, 1, 0, 0, 1)$;
  g gets + on price −− on democracy and ++ for sports
  so $g^+ = (0, \lambda, 0, 1, 0)$ and $g^- = (0, 0, 1, 0, 0)$.

A special case of this framework is studied at length in Bonnefon et al. (2008a, b), where scales L^+, L^- are Boolean, while L is any bounded scale for assessing importance of criteria. Then an alternative f is modelled by two disjoint subsets of $\mathcal{C}$: $F^+ = \{i : f_i^+ = 1\}$ and $F^- = \{i : f_i^- = 1\}$ collecting the pros and cons for and against f. The set $F = F^+ \cup F^-$ is the set of relevant criteria (i.e. those that matter) for f.

There are two approaches to deciding preference among alternatives.

- Either we consider for each alternatives positive and negative summaries of arguments via an aggregation operator, here Sugeno integral, and define a partial ordering between alternatives.
- Or we can build a preference relation via pairwise comparisons assuming that, when comparing f and g, that criteria in disfavor of g give reasons to prefer f and conversely.

Comparing pairs of positive and negative ratings by Pareto-dominance. Suppose the set of criteria is weighted by means of a capacity μ valued on L, and that $L^+ = L^- = L$. In the above bipolar setting, an alternative f will then be evaluated by a pair $(S_\mu(f^-), S_\mu(f^+))$. Note that $S_\mu(f^-)$ is rated on a scale where 1 is interpreted as bad, 0 is neutral. If we wish to express it on a positive unipolar scale where 0 means bad, we have to replace $S_\mu(f^-)$ by $D_\nu(f^-) = S_{1-\mu^c}(1 - f^-)$ (using a monotone decreasing set-function $\nu = 1 - \mu^c$). $D_\nu(f^-)$ is named a desintegral in Dubois et al. (2016). A first way of comparing alternatives f and g is to use a kind of Pareto-dominance between (f^-, f^+) and (g^-, g^+).

$$f \succeq^{PB} g \iff S_\mu(f^+) \geq S_\mu(g^+) \text{ and } S_\mu(f^-) \leq S_\mu(g^-) \tag{31}$$

This ordering is exactly Pareto dominance if we use the desintegral for f^-. It satisfies obvious monotonicity conditions, namely if f and g are such that $f_i^+ \geq g_i^+, \forall i \in [n]$ and $f_i^- \leq g_i^-, \forall i \in [n]$, then $f \succeq^{PB} g$.

In the special case of Boolean positive and negative ratings, Bonnefon et al. (2008a, b) use a possibility assignment to weight criteria and a pair of possibility measures $(\Pi(F^-), \Pi(F^+))$ as a bipolar rating of this alternative. These evaluations[10] are justified by the focus effect according to which humans compare alternatives with respect to the most important criteria first, neglecting other ones. Note that the ordering $\succeq^{PB}$ collapses to Wald's pessimistic ordering if $F^+ = \emptyset$ (choosing based on the worst feature), and to its optimistic max-based counterpart if $F^- = \emptyset$.

[10]Interpreted in terms of *order of magnitude* of importance, hence the notation OM in Bonnefon et al. (2008a).

This decision rule has some deficiencies:

- The bipolar outranking relation $\succeq^{PB}$ concludes to incomparability in some cases when a preference would sound more natural. When f has both pros and cons, it is incomparable with the neutral alternative $\hat{f}$ such that $\hat{f}_i^+ = \hat{f}_i^- = 0, \forall i \in [n]$ even if the importance of the cons in f is negligible in front of the importance of its pros.
- Whenever two criteria i and j are such that $f_i^- = 1$ and $f_j^+ = 1$ (a very bad rating on criterion i and a very good one on j) and these criteria have maximal importance $\mu_\#(\{i\}) = \mu_\#(\{j\}) = 1$, then $(S_\mu(f^-), S_\mu(f^+)) = (1, 1)$, so that such alternatives are considered equally preferred. There can be many of them, hence a lack of discrimination.

A bipolar model in decision under uncertainty. The approach of Giang and Shenoy (2000, 2005) to decision under uncertainty is of the same vein. They have tried to obviate the need for making assumptions on the pessimistic or optimistic attitude of the decision-maker and thus, improve the discrimination power in the qualitative setting, by using, as a utility scale, a totally ordered set of possibility measures on a two element set $\{0, 1\}$ containing the values of the best and the worst consequences of acts. Each such possibility distribution represents a qualitative lottery in a set $L_\Pi = \{(\alpha, \beta), \max(\alpha, \beta) = 1, \alpha, \beta \in L\}$. Coefficient α represents the degree of possibility of obtaining the worst consequence, and coefficient β the degree of possibility of obtaining the best. This set can be viewed as a bipolar value scale ordered by the following complete preordering relation expressing preference:

$$(\alpha, \beta) \geq (\gamma, \delta) \text{ if and only if } (\alpha \leq \gamma \text{ and } \beta \geq \delta).$$

The bottom of this utility scale is $(1, 0)$, its top is $(0, 1)$ and its neutral point $(1, 1)$ means "indifferent". The fact this relation is complete is due to the fact that pairs (α, β) and (γ, δ) such that $(\alpha, \beta) > (\gamma, \delta)$ and $(\gamma, \delta) > (\alpha, \beta)$ cannot both lie in L_Π since then either $\max(\alpha, \beta) < 1$ or $\max(\gamma, \delta) < 1$. The canonical example of such a scale is the set of pairs $(\Pi(\overline{A}), \Pi(A))$ of degrees of possibility for event $A =$ "getting the best consequence", and its complement. The inequality $(\Pi(\overline{A}), \Pi(A)) > (\Pi(\overline{B}), \Pi(B))$ means that A is more likely (certain or plausible) than B (because it is equivalent to $\Pi(A) > \Pi(B)$ or $N(A) > N(B)$). In fact the induced likelihood ordering between events

$$A \succeq_{L\Pi} B \text{ if and only if } (\Pi(\overline{A}), \Pi(A)) \geq (\Pi(\overline{B}), \Pi(B))$$

is self-adjoint, that is, $A \succeq_{L\Pi} B$ is equivalent to $\overline{B} \succeq_{L\Pi} \overline{A}$.

Each consequence is supposed to have a utility value (α, β) in L_Π. The proposed preference functional maps acts, viewed as $n-$tuples $f = ((f_1^-, f_1^+), \dots, (f_n^-, f_n^+))$ of values in L_Π, to L_Π itself. The uncertainty is described by possibility weights $(\pi_1, \dots, \pi_n)$ with $\max_{i=1,\dots,n} \pi_i = 1$. The utility of an act f, called *binary possibilistic utility* is computed as the pair

$$W_{GS}(f) = (\max_{i=1,\ldots,n} \min(\pi_i, f_i^-), \max_{i=1,\ldots,n} \min(\pi_i, f_i^+)) \in L_\Pi.$$

Clearly, $W_{GS}(f)$ is of the form $(S_\mu(f^-), S_\mu(f^+))$ for $\mu = \Pi$. This form results from simple and very natural axioms on possibilistic lotteries, which are counterparts to the Von Neumann and Morgenstern axioms in decision under risk. Weng (2006) proposed a Savage-style axiomatization of this binary possibilistic utility functional. It puts together the axiomatizations of the optimistic and the pessimistic possibilistic criteria by Dubois et al. (2001), adding, to the axioms justifying Sugeno integral, two conditions: (i) the self-adjointness of the preference relation on binary acts, and (ii) an axiom enforcing axiom **OPT** on the subset of acts weakly preferred to a special act that plays the role of a neutral point separating favorable from unfavorable acts. Pessimistic and optimistic possibilistic criteria $SLMIN_\pi$ and $SLMAX_\pi$ are of course special cases of this bipolar criterion. They respectively correspond to either using the negative part of L_Π only (not telling (1, 1) from (0, 1) in case of pessimism) or using the positive part of L_Π only (not telling (1, 0) from (1, 1) in case of optimism).

The bipolar possibility relation. The problem with the bipolar Pareto-dominance is that it does not account for the fact that the two evaluations *share a common importance scale* L. Another idea for comparing alternatives f and g is to focus on criteria in $F \cup G$, i.e., those that matter for both alternatives. The principle at work is simple: any argument against f (resp. against g) is an argument pro g (resp., pro f). The most supported decision is then preferred, by comparing global evaluations on $F^+ \cup G^-$ and $F^- \cup G^+$: Instead of comparing f and g we compare $f^+ \vee g^-$ with $g^+ \vee f^-$ with respective components $f_i^+ \vee g_i^-$ and $f_i^- \vee g_i^+$:

Definition 5 (*Bipolar Sugeno Dominance*) $f \succeq^{BS} g \iff S_\mu(f^+ \vee g^-) \geq S_\mu(f^- \vee g^+)$.

It would be worth studying this preference relation and compare it to the preference relation induced by the symmetric Sugeno integral (27). In particular, Bipolar Sugeno Dominance does not require new operations on a bipolar scale since all computations are brought back to the positive part of the scale in Definition 5.

This kind of preference relation was first proposed in Bonnefon et al. (2008a) when scales L^+ and L^- are Boolean, μ is a possibility measure, and $F^- \cap G^+ = F^+ \cap G^- = \emptyset$. It yields the bipolar possibility relation

$$f \succeq^{B\Pi} g \iff \Pi(F^+ \cup G^-) \geq \Pi(G^+ \cup F^-).$$

This rule decides that f is at least as good as g as soon as there are important arguments either in favour of f or attacking g that are at least as strong as the best arguments in favour of g or attacking f. Obviously, $\succeq^{B\Pi}$ collapses to Wald's pessimistic ordering if $F = F^-$, $G = G^-$ and to its optimistic counterpart when $G = G^+$, $F = F^+$. In some sense, this definition is the most straightforward generalisation of possibility relations (Lewis 1973; Dubois 1986) to the bipolar case.

The bipolar possibility relation satisfies the following properties

(i) It is complete and its strict part is transitive.
(ii) The restriction of $\succeq^{B\Pi}$ to f such that $F = \{i\}, i = 1, \ldots n$ is a weak order.
(iii) *Ground Monotony*: $\forall f, g, h, h'$ such that $H = \{i\}, H' = \{i'\}, F \cap \{i, i'\} = \emptyset$ and $h' \succeq^{B\Pi} h$:
$f \vee h \succ g \Rightarrow f \vee h' \succ g$; $f \vee h \sim g \Rightarrow f \vee h' \succeq g$;
$g \succ f \vee h' \Rightarrow g \succ f \vee h$; $g \sim f \vee h' \Rightarrow g \succeq f \vee h$.
(iv) *Positive Cancellation*: $\forall f, g, h$ such that $f_i^+ = 1$ and 0 otherwise, $g_j^+ = 1$ and 0 otherwise, $h_k^- = 1$ and 0 otherwise and denoting by 0 the alternative that receives 0 for each criterion: $f \vee h \sim 0$ and $g \vee h \sim 0 \Rightarrow f \sim h$.
(v) *Negative Cancellation*: $\forall f, g, h$ such that $f_i^- = 1$ and 0 otherwise, $g_j^- = 1$ and 0 otherwise, $h_k^+ = 1$ and 0 otherwise: $f \vee h \sim 0$ and $g \vee h \sim 0 \Rightarrow f \sim h$.
(vi) *Strict negligibility*: $\forall f, g, f', g' : f \succ g$ and $f' \succ g' \Rightarrow f \vee f' \succ g \vee g'$.
(vii) *Idempotent Negligibility* $\forall f, gf', g' : f \succeq g$ and $f' \succeq g' \Rightarrow f \vee f' \succeq g \vee g'$.

Remark 4

- Note that the weak relation $\succeq^{B\Pi}$ is generally not transitive.
- Properties (iv) and (v) express a form of anonymity. It is required when a positive argument blocks a negative argument of the same strength: this blocking effect should not depend on the arguments themselves, but on their position in the importance scale only.
- The two last properties are direct consequences of working with importance levels that are orders of magnitude. $f \succ g$ means that f is much better than g, so much so as there is no way of overthrowing f by sets of weaker arguments (property (vi)).
- The last property presupposes that several arguments of the same strength are worth just one.

The above properties turn out to be characteristic of the bipolar possibility rule (Bonnefon et al. 2008a). They imply the existence of the importance scale, and the importance assignment to criteria as a possibility distribution.

Comparison with Cumulative Prospect Theory (CPT). There is a similarity between the bipolar possibility relation and the preference ordering of CPT. The latter assumes that the strength of reasons supporting an alternative f and the strength of reasons against it can be measured by means of two numerical capacities σ^+ and σ^- respectively mapping subsets F^+ and F^- to the unipolar scale $[0, +\infty)$. The capacity σ^+ reflects the importance of the group of positive arguments for f, and σ^- the importance of the group of negative arguments against it.

This approach moreover admits that it is possible to combine these evaluations by subtracting them and building a so-called "net predisposition" score expressed on a bipolar numerical scale (the real line):

$$\forall f, NP(f) = \sigma^+(F^+) - \sigma^-(F^-).$$

It is a special case of symmetric Choquet integral described in the previous section. Alternatives are then ranked according to this net predisposition: $f \succeq^{CPT} g \iff \sigma^+(F^+) - \sigma^-(F^-) \geq \sigma^+(G^+) - \sigma^-(G^-)$. The relation $\succeq^{B\Pi}$ can be viewed as the natural qualitative counterpart of $\succeq^{CPT}$; indeed, the bipolar possibility decision rule comes down to changing $+$ into max in the equivalent inequality $\sigma^+(F^+) + \sigma^-(G^-) \geq \sigma^+(G^+) + \sigma^-(F^-)$, that is, if $\sigma^+ = \sigma^-$ is additive, $\sigma^+(F^+ \cup G^-) \geq \sigma^+(G^+ \cup F^-)$.

So, there is a joint framework encompassing the CPT framework and the qualitative bipolar possibility relation, turning possibility measures into standard capacities $\kappa : 2^X \to L$ (Dubois and Fargier 2010):

$$f \succeq^{\kappa} g \iff \kappa(F^+ \cup G^-) \geq \kappa(G^+ \cup F^-)$$

adopting the view that an argument against alternative f is an argument in favour of g in the pairwise comparison of alternatives.

The following properties clearly hold for $\succeq^{\kappa}$: it is complete, and the restriction to single arguments is a weak order. However it is not clearly transitive, not even quasi-transitive in the general case. And while the non triviality, and both positive and negative monotony properties hold, the weak unanimity property, that would make $\succeq^{\kappa}$ a bipolar monotonic set relation, requires that κ satisfy an additional property on top of inclusion-monotonicity of capacities (Chateauneuf 1996):

Weak additivity: Let $A, B, C, D \subseteq X$ such that $A \cap C = \emptyset, B \cap D = \emptyset$; if $\kappa(A) \geq \kappa(B)$ and $\kappa(C) \geq \kappa(D)$ then $\kappa(A \cup C) \geq \kappa(B \cup D)$.

This property is, for capacities, equivalent to the following property involving only three subsets $A, B, C (= D)$ (Dubois 1986):

If $\kappa(A) \geq \kappa(B)$ then $\kappa(A \cup C) \geq \kappa(B \cup C)$, provided that $(A \cup B) \cap C = \emptyset$.

It implies that κ is a decomposable measure (Chateauneuf 1996), that is, there exists an operation $\star$ such that if $A \cap B = \emptyset$, $\kappa(A \cup B) = \kappa(A) \star \kappa(B)$. Due to compatibility with the underlying Boolean algebra of events, it is natural to consider that $\star$ is a co-norm. Choosing an Archimedean continuous co-norm on $L = [0, 1]$, it is clear that $\succeq^{\kappa}$ can verify additional properties:

- *Transitivity*: $\kappa(F^+ \cup G^-) \geq \kappa(G^+ \cup F^-)$ and $\kappa(G^+ \cup C^-) \geq \kappa(C^+ \cup G^-)$ imply $\kappa(F^+ \cup C^-) \geq \kappa(C^+ \cup F^-)$. Indeed the preconditions imply

$$\kappa(F^+) \star \kappa(G^-) \star \kappa(G^+) \star \kappa(C^-) \geq \kappa(G^+) \star \kappa(F^-) \star \kappa(G^-) \star \kappa(C^+)$$

  which yields the expected result by simplification (if $\star$ is a strict t-norm or κ is properly normalized). This simplification cannot be made if $\star = \max$.
- *Ground monotony* holds under the same assumptions about $\star$.

- *Positive and negative cancellation* properties reduce to the trivial statement that $\kappa(\{x\}) = \kappa(\{y\})$ and $\kappa(\{z\}) = \kappa(\{y\})$ imply $\kappa(\{x\}) = \kappa(\{z\})$.

In fact, relation $f \succeq^{\kappa} g$ is a conjoint generalisation of $\succeq^{CPT}$ and $\succeq^{B\Pi}$ that either comes down to one of them ($\succeq^{CPT}$ is obtained, if $L = [0, 1]$, $\star$ is a nilpotent Archimedean t-norm and κ is properly normalized, or a strict co-norm, taking the logarithm of κ) or a combination of them (if $\star$ is an ordinal sum of the basic conorms $\alpha + \beta - \alpha\beta$, $\min(1, \alpha + \beta)$, max) up to a rescaling.

Bipolar lexicographic outranking relations. The last property (Idempotent Negligibility) of the bipolar possibility rule is by far the most debatable feature of $\succeq^{B\Pi}$. It causes a drowning effect, usual in standard possibility theory. For instance, if the most important criteria satisfied by f are of the same importance as most important criteria satisfied by g, but there are more of the latter, the two alternatives are judged equally.

A tempting way of refining $\succeq^{B\Pi}$, is to use a leximax relation instead. Then the number of arguments of equal strength on each side is then taken into account. Among the two basic axioms of qualitative modeling, it comes down to giving up Idempotent Negligibility, while retaining Strict Negligibility. Preference can then be based on counting arguments of the same strength, but we still do not allow an important argument to be superseded by several less important ones, however large their number be (focus effect). The criteria satisfied or violated in f and g are scanned top down, until a level is reached such that the numbers of positive and negative arguments pertaining to the two alternatives are different; then, the option with the least number of violated criteria and the greatest number of satisfied ones is preferred.

There are two such decision rules respectively called "Bivariate Levelwise Tallying" and (univariate) "Levelwise Tallying" (Bonnefon et al. 2008b), according to whether positive and negative arguments are treated separately or not.

For any importance level $\lambda \in L$, let $F_\lambda = \{i \in F, \pi_i = \lambda\}$ be the λ-section of f, the set of relevant criteria of strength λ in f. Let $F^+_\lambda = \{i \in F_\lambda : f^+_i = 1\}$ (resp., $F^-_\lambda = \{i \in F_\lambda : f^-_i = 1\}$) be its positive (resp., negative) λ-section. Let $\delta(f, g)$ be the maximal level of importance where either the positive or the negative λ-sections of f and g differ, namely:

$$\delta(f, g) = \max\{\lambda : |F^+_\lambda| \neq |G^+_\lambda| \text{ or } |F^-_\lambda| \neq |G^-_\lambda|\}.$$

$\delta(f, g)$ is called the *decisive level* pertaining to (f, g). The Bivariate Levelwise Tallying preference rule reads:

$$f \succeq^{BL} g \iff |F^+_{\delta(f,g)}| \geq |G^+_{\delta(f,g)}| \text{ and } |F^-_{\delta(f,g)}| \leq |G^-_{\delta(f,g)}|.$$

It is easy to show that $\succeq^{BL}$ is reflexive, transitive, refines the bipolar Pareto ordering but is not complete. Indeed, $\succeq^{BL}$ concludes to an incomparability if and only if there is a conflict between the positive view and the negative view at the decisive level. From a descriptive point of view, this range of incomparability is a good point in favour of $\succeq^{BL}$.

Now, if one can assume a compensation between positive and negative arguments *at each importance level*, one argument canceling another one on the other side, the following refinement of relation $\succeq^{BL}$, called Univariate Levelwise Tallying, can be obtained:

$$f \succeq^{ULT} g \iff \exists \lambda \in L \setminus 0_L \text{s.t.} \begin{cases} \forall \theta > \lambda, & |A_\theta^+| - |A_\theta^-| = |B_\theta^+| - |B_\theta^-| \\ \text{and} & |F_\lambda^+| - |F_\lambda^-| > |G_\lambda^+| - |G_\lambda^-| \end{cases}$$

or$|A_\theta^+| - |A_\theta^-| = |B_\theta^+| - |B_\theta^-|, \forall \lambda \in L \setminus 0_L$ (the latter case is when $f \sim^{ULT} g$).

Interestingly, relation $\succeq^{ULT}$ is closely related to the decision rule originally proposed more than two centuries ago by Benjamin Franklin (1887).

The two decision rules proposed in this section obviously generate monotonic bipolar outranking relations. Each of them refines $\succeq^{B\Pi}$. The most decisive one is $\succeq^{ULT}$, which is moreover complete and transitive. This relation is the refinement of $\succeq^{B\Pi}$ that is a weak order and that satisfies the principle of preferential independence without introducing any bias on the importance order elementary criteria (that is, preserving the restriction of $\succeq^{B\Pi}$ to single criteria). See Bonnefon et al. (2008a) for an axiomatisation of these decision rules. It turns out that Levelwise Tallying is the most likely decision rule to be used by people as an empirical study suggests (Bonnefon et al. 2008b).

5 Qualitative Data Analysis

As a general family of aggregation functions, it is of interest to identify the family of Sugeno integrals that are compatible with a dataset made of vectors of criteria values together with the corresponding global evaluations. When the family is non empty, it can be described by bracketing the data by means of a lower capacity and an upper capacity. Such a dataset can be also described by means of sets of selection and deletion rules, which then correspond to a combination of Sugeno utility functionals. In this section, we briefly discuss nonparametric methods based on Sugeno integral for learning rule-based models that are widely used in multicriteria decision aid and ordinal classification (Gutiérrez et al. 2016) tasks.

5.1 Approach by Bracketing Datasets with Standard Sugeno Integrals

The problem considered here is the elicitation of a family of Sugeno integrals that are compatible with a dataset. Here, a dataset is a collection of L-valued tuples $f = (f_1, \ldots, f_n)$ associated with a global rating $\delta \in L$.

Definition 6 A pair (f, δ) is compatible with a Sugeno integral S_μ if and only if $S_\mu(f) = \delta$.

In the following, we study the constraints induced by a pair (f, δ) on the Sugeno integrals compatible with it and we fully characterize this family following ideas first given in Rico et al. (2005). For convenience, we assume that the f_i's are already increasingly ordered, i.e., $f_1 \leq \cdots \leq f_n$. Since

$$\bigwedge_{i=1}^{n} f_i \leq S_\mu(f_1, \ldots, f_n) \leq \bigvee_{i=1}^{n} f_i,$$

there exists a Sugeno integral that satisfies $S_\mu(f) = \delta$ if and only if $f_1 \leq \delta \leq f_n$. *We assume here that this consistency condition holds for the pairs* (f, δ) *considered.* For discussing the equation $S_\mu(f) = \delta$, it is useful to distinguish two cases.

DIF Case : $\forall i \in \mathcal{C},\ f_i \neq \delta$.

Let i be the index such that $f_1 \leq \ldots \leq f_{i-1} < \delta < f_i \leq \ldots \leq f_n$. We can then define two particular capacities $\check{\mu}_{f,\delta,DIF}$ and $\hat{\mu}_{f,\delta,DIF}$:

Definition 7

$$\forall X \in 2^{\mathcal{C}}, X \neq \emptyset, \mathcal{C} \quad \check{\mu}_{f,\delta,DIF}(X) = \begin{cases} \delta & \text{if } \{i, \ldots, n\} \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

and

$$\forall X \in 2^{\mathcal{C}}, X \neq \emptyset, \mathcal{C} \quad \hat{\mu}_{f,\delta,DIF}(X) = \begin{cases} \delta & \text{if } X \subseteq \{i, \ldots, n\} \\ 1 & \text{otherwise} \end{cases}.$$

It can be shown that :

$$\forall \mu \text{ s.t. } S_\mu(f) = \delta \text{ we have } \check{\mu}_{f,\delta,DIF} \leq \mu \leq \hat{\mu}_{f,\delta,DIF}.$$

Thus $\check{\mu}_{f,\delta,DIF}$ and $\hat{\mu}_{f,\delta,DIF}$ are the lower and upper bounds of the lattice of capacities which define the family of Sugeno integrals compatible with the pair (f, δ) in the DIF case.

EQU case : $\exists i \in \mathcal{C},\ f_i = \delta$.

Let i and j be the indices such that $f_1 \leq \cdots \leq f_{j-1} < f_j = \cdots = f_{i-1} = \delta < f_i \leq \cdots \leq f_n$. We can then define two particular capacities $\check{\mu}_{f,\delta,EQU}$ and $\hat{\mu}_{f,\delta,EQU}$:

Definition 8

$$\forall X \in 2^{\mathcal{C}},\ X \neq \emptyset, \mathcal{C} \quad \check{\mu}_{f,\delta,EQU}(X) = \begin{cases} \delta & \text{if } \{j, \ldots, i-1, \ldots n\} \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

and

$$\forall X \in 2^{\mathcal{C}},\ X \neq \emptyset, \mathcal{C} \quad \hat{\mu}_{f,\delta,EQU}(X) = \begin{cases} \delta & \text{if } X \subseteq \{i, \ldots, n\} \\ 1 & \text{otherwise} \end{cases}.$$

It can be shown that :

$$\forall v \text{ s.t. } S_\mu(f) = \delta \text{ we have } \check{\mu}_{f,\delta,EQU} \leq \mu \leq \hat{\mu}_{f,\delta,EQU}.$$

Thus $\check{\mu}_{f,\delta,EQU}$ and $\hat{\mu}_{f,\delta,EQU}$ are the lower and the upper bounds of the lattice of capacities which define the family of Sugeno integrals compatible with the pair (f, δ) in the EQU case.

Consistent family of a Sugeno integral with respect to a dataset. A dataset is consistent if there exists a non empty family of Sugeno integrals that are compatible with each pair (f, δ) in the dataset. Otherwise, it means that there is no representation of the dataset by a unique family of Sugeno integrals and that several families thereof are necessary, each covering a distinct subpart of the dataset. Let us consider a dataset $(f^i, \delta_i)_{i\in\{1,\dots,p\}}$ that contains p pairs. In order to simplify notations, we denote by $\check{\mu}_i$ the lower bound for (f^i, δ_i) and $\hat{\mu}_i$ the upper bound associated with (f^i, δ_i). Thus the lower and upper bounds of the family of compatible Sugeno integrals, if this family exists, are respectively

$$\check{\mu} = \bigvee_{i=1}^{p} \check{\mu}_i \quad \text{and} \quad \hat{\mu} = \bigwedge_{i=1}^{p} \hat{\mu}_i.$$

Thus, when a new piece of information (f, δ) is considered, $\check{\mu}$ and $\hat{\mu}$ are then revised by

$$\check{\mu}_{revised} = \check{\mu} \vee \check{\mu}_{f,\delta} \quad \text{and} \quad \hat{\mu}_{revised} = \hat{\mu} \wedge \hat{\mu}_{f,\delta}.$$

These results have been applied to a case study (Prade et al. 2009b) on mental workload data where the global evaluation relies on six criteria, where several families of Sugeno integrals were necessary for recovering the whole data set, thus revealing different aggregation attitudes with respect to mental workload. The families of Sugeno integral were identified thanks to a simulated annealing method (Prade et al. 2009b). Besides, it has been shown (Prade et al. 2009a) that the bracketing procedure can be viewed as a graded extension of the version space approach in machine learning (Mitchell 1982).

5.2 *Approach by Best Approximation Using Sugeno Utility Functionals*

We consider datasets that can be accurately modeled by a nondecreasing function. Since a SUF uses utility functions as arguments of the Sugeno integral, it can model rules with different thresholds. We consider sets of (selection) rules of the form

$$\textbf{if } f_1 \geq \alpha_1 \textbf{ and} \ldots \textbf{and } f_n \geq \alpha_n \textbf{ then } y \geq \delta \tag{32}$$

where $(\alpha_1, \ldots, \alpha_n) \in L^n$. Using results in Sect. 2.4, a SUF induces single-thresholded rules of the form

$$\textbf{if } \varphi_1(f_1) \geq \delta \textbf{ and} \ldots \textbf{and } \varphi_n(f_n) \geq \delta \textbf{ then } y \geq \delta$$

which can be turned into the form (32), if we choose monotone utility functions such that $\varphi_i(\alpha_i) \geq \delta$.

Let R be a set of rules of the form (32). There may be several functions that are compatible with R. We denote by Φ_R the smallest function compatible with R, defined by $\Phi_R = \max_{r\in R} \Phi_r$ such that for each rule r:

$$\Phi_r(f) = \delta^r, \textbf{if } \forall i \in \mathcal{C},\ f_i \geq \alpha_i^r, \textbf{ and } 0 \textbf{ otherwise}.$$

We will say that a function Φ is *equivalent* to R if $\Phi = \Phi_R$. It was shown in Couceiro et al. (2017a), Brabant et al. (2018) that:

(i) Any SUF $S_{\mu,\varphi}$ is equivalent to a rule set.
(ii) Any single rule is equivalent to a SUF.
(iii) Some rule sets are not equivalent to a single SUF.

Any SUF $S_{\varphi,\mu}$ is equivalent to the rule set

$$\bigcup_{I\subseteq[n]} \bigcup_{\delta\leq\mu(I)} \{\forall i \in I,\ f_i \geq \alpha_i \Rightarrow y \geq \delta\}, \tag{33}$$

where $\alpha_i = \min\{\lambda \in L_i \mid \varphi_i(\lambda) \geq \delta\}$. Note that this set is likely to contain redundant rules.

So each multiple-thresholded rule induces constraints on the utility functions. But the constraints induced by two or more rules can be inconsistent (Couceiro et al. 2017a). In other words, some combinations of rules cannot be expressed by a single SUF. Nonetheless, the second assertion shows that any rule set is equivalent to some function $M_{\boldsymbol{S}} : L^n \rightarrow L$ defined by

$$M_{\boldsymbol{S}}(f) = \max\{S_{\mu,\varphi}(f) \mid S_{\mu,\varphi} \in \boldsymbol{S}\},$$

where $\boldsymbol{S}$ is a set of SUFs. We call such a function a *max-SUF*.

In Brabant et al. (2018) a method of translation of a rule set R into a SUF is provided:

(i) Initialize μ and $\varphi = (\varphi_1, \ldots, \varphi_n)$ with minimal values.
(ii) For each rule $f_1 \geq \alpha_1, \ldots, f_n \geq \alpha_n \Rightarrow y \geq \delta$ in R:
    (a) let $A = \{i \in [n] \mid \alpha_i > 0\}$,
    (b) increase $\mu(A)$ up to δ,
    (c) for each $i \in A$, increase $\varphi_i(\alpha_i)$ up to δ.

After these steps we always have $S_{\mu,\varphi} \geq \Phi_R$. When $S_{\mu,\varphi} > \Phi_R$, no SUF is equivalent to R.

In some cases, it is not problematic that $S_{\mu,\varphi} > \Phi_R$. For example, if Φ_R is a model of a dataset $\mathcal{D}$, we may want to find an SUF that best fits $\mathcal{D}$. Obtaining $S_{\mu,\varphi} = \Phi_R$ is not always possible since SUFs are not expressive enough. However, equality can be always achieved using a max-SUF (Brabant et al. 2018). The method presented in what follows next section relies on this fact.

Learning rules from empirical data. Now there is no reason to think that a max-SUF is more interpretable than its equivalent rule set. Thus, an interesting question is whether SUFs can serve as an intermediary model that helps guiding the learning process of a rule based model. Indeed, in Brabant et al. (2018), such a learning algorithm is proposed. Let $\mathcal{D}$ be a dataset. The following three steps provide a method for modeling $\mathcal{D}$ by a max-SUF.

1. Selection of an order-preserving subset of data. Two data items (f, δ) and (g, γ) can be *anti-monotonic* together, i.e, $f \leq g$ and $\gamma \leq \delta$. We iteratively remove instances from $\mathcal{D}$, starting from those that are anti-monotonic with the highest number of other instances, until no anti-monotonic pair remains. We denote by $\mathcal{D}^-$ the dataset obtained in this way.

2. Modeling $\mathcal{D}^-$ by a rule set R. Initialize R to $\emptyset$. For each instance $((\alpha_1, \ldots, \alpha_n), \delta)$ in $\mathcal{D}^-$, search for $A \subseteq [n]$ with minimal cardinality, such that the rule

$$\forall i \in A,\ \ f_i \geq \alpha_i \Rightarrow y \geq \delta, \tag{34}$$

is not contradicted by any instance in $\mathcal{D}^-$. Add the rule (34) to R. At the end of this step, the class of each instance in $\mathcal{D}^-$ is exactly predicted by Φ_R.

3. Translation of R into a max-SUF. See Algorithm 1. The obtained max-SUF is not necessarily equivalent to R, but it fits $\mathcal{D}^-$ precisely.

Algorithm 1: Makes a partition **P** of R such that the max-SUF M_S verifies $M_S(f) = y$ for each instance (f, y).

```
P ← {}
for each r ∈ R do
    affected ← false
    for each P ∈ P do
        translate P into a SUF S_{μ,φ}
        if S_{μ,φ}(f) ≤ y for all instance (f, y) in D⁻ then
            add r to P
            affected ← true
            break loop
    if affected = false then
        add {r} to P
```

Table 1 Accuracy obtained with each method on each dataset. Datasets are numbered as in Blaszczyński et al. (2011)

	1	2	3	4	5	6	7	8	9	10	11	12	Avg.
Steps 1, 2	74.4	95.8	97.7	93.6	91.7	65.6	83.2	27	67	63.6	58.2	51.4	72.4
Steps 1, 2, 3	76	95.3	97.2	89.3	92.4	65.2	84.5	26.4	69.4	63	56.7	53.2	72.4
VC-DomLEM	76.7	96.3	97.1	91.7	95.4	67.5	87.7	26.9	66.7	55.6	56.4	54.6	72.7

Note that the max-SUF given by this method can be translated back into a rule set, which constitutes an equivalent model and is easier to interpret.

Empirical study The VC-DomLEM algorithm (Blaszczyński et al. 2011) is another method that can learn such a set of rules, which yields a good accuracy compared to other interpretable models. This method requires the tuning of hyperparameters, contrary to ours. B method is competitive with VC-DomLEM in terms of accuracy. Moreover, this method raised new questions about the relevance of capacities (i.e., monotonically increasing set functions) in data-modeling.

The method in Brabant et al. (2018) was compared to VC-DomLEM on the 12 datasets. In order to get an idea of the importance of Step 3 in our method, we separately evaluated the rule set given by Steps 1 and 2 alone, and the max-SUF given by Steps 1,2, and 3 (Table 1).

We see that Step 3 does not increase the accuracy on average. Therefore, the good results of this method are not due to the use of SUFs, but to the 2 first steps.

Now, the *length* of a rule is the number of attributes i where $\alpha_i > 0$ (since the condition $\alpha_i \geq 0$ is trivial). Shorter rules are easier to interpret and constitute more concise models. In Brabant et al. (2018) it was presented an empirical study of rule length distributions obtained after Steps 1, 2, and 3.

Moreover, it was compared to an analogous method using the dual max-SUFs. The dual of max-SUFs are the min-SUFs that correspond to sets of (rejection) rules of the form

$$\textbf{if } f_1 \leq \alpha_1 \textbf{ and } \dots \textbf{ and } f_n \leq \alpha_n \textbf{ then } y \leq \delta.$$

When learning min-SUFs by a dual method, the rule-length distribution differs from that obtained by learning max-SUFs. Long rules of one type sometimes go along with short rules of the other type. This empirical result hints at a combined method for rule length improvement.

6 Conclusion

This chapter has tried to advocate the merits of Sugeno integral as a tool for the qualitative evaluation of alternatives when utility values are not supposed to be numerical. After recalling the algebraic nature of Sugeno integral and its close links to the notion of median, recent developments of the approach, overcoming some of its limitations, have been surveyed: namely the enhancement of its expressiveness for the pairwise comparison of alternatives via lexicographic refinements, the use of utility functions when several attribute scales need to be reconciled, and the extension of the role of attribute weights in the aggregation process. The question of borrowing concepts from cumulative prospect theory for the distinct treatment of pros and cons in Sugeno integral has been also discussed.

Finally, methods for representing a qualitative dataset by means of one or several Sugeno integrals have been outlined, questioning the possibility of learning them from data. These results provide an extended range of tools for processing qualitative preference data in a non-trivial way, thus obviating the need to use numerical scales for attributes that are not easily and meaningfully measured. Besides, the close connections between Sugeno integrals and fuzzy decision rules suggest a way to extract meaning from data. However, the bracketting method may yield imprecise results, the SUFs are not expressive enough, and the max-SUFs may be complex. So, the quest for concise and faithful models devoted to qualitative data looks challenging.

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Part III
Goal Programming and Multi-objective Optimization

Advances and New Orientations in Goal Programming

Dylan Jones and Carlos Romero

Abstract This chapter starts by providing a categorization of current goal programming literature by type of variant used. Subsequently, goal programming is presented as a *secondary model* of a general p-metric distance function *primary model*. This orientation allows us to link goal programming with several fields like the determination of social choice functions or the interpretation and implementation of the Simonian concepts of bounded rationality and "satisficing". To undertake the latter task, this epistemic framework is understood as a Laudian "Research Tradition" instead of the usual understanding as a scientific theory. Finally, potential future developments to expand the use and flexibility of goal programming as well as to explore possible logical connections of goal programming with other decision-making areas are highlighted.

Keywords Goal programming · p-metrics · Bounded rationality · Research traditions

1 Introduction

Goal programming, as originally postulated by Charnes et al. (1955), Charnes and Cooper (1961) is the oldest technique with the field of multiple criteria decision making (MCDM). It has remained popular in the fifty years since its conception, possibly due to its relative simplicity of modelling and solution, and its flexibility to encompass different utility concepts and to integrate with other models from

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the fields of operational research and artificial intelligence (Jones and Tamiz 2010). The goal programming paradigm has expanded significantly in recent decades. The original models aligned with the Simonian satisficing concept and were at least in part used in a statistical regression type approach. Goal programming still retains a strong satisficing component but has also been expanded to include the philosophies of optimizing, ordering, and balancing (social equity). Equally, the range of applications of goal programming has expanded to cover a range of diverse and modern fields of application. For instance, a review of the 200+ goal programming papers published in ISI journals in the year 2016 includes applications in the fields of energy management, renewable energy planning, forest management, mining, healthcare, sustainable development, humanitarian logistics, production planning, finance, transportation, supply chain management, marketing, agricultural planning, water resource planning, tourism, research and development management, urban planning, educational planning, cloud service planning and project management.

This Chapter examples goal programming from two perspectives. Firstly, a literature based view of goal programming is developed, and a categorization of the current goal programming literature is made, with the principal discriminating factor being the goal programming variant(s). A classification of variants via their underlying distance-metrics and attributes of their goal targets and decision variables is given. Secondly, a conceptual distance-metric view of goal programming is given. The development of goal programming from its underlying distance-metric is explored and linkages to the concept of bounded rationality and social choice functions developed. Potential future developments in this direction to expand the use and flexibility of goal programming are discussed.

2 A Literature-Based View of Goal Programming Variants

The field of goal programming now contains many more variants than the original propositions of Charnes et al. (1955), Charnes and Cooper (1961). The concept of a variant has indeed itself evolved and is therefore worthy of discussion. This is supported by the diagrammatic map of Fig. 1, which divides the goal programming variants into various categories.

The first categorization stems from the usage of the word "variant" in the literature. The key question in attempting to reach a set of conflicting targets that cannot be simultaneously satisfied is how to distribute the unwanted deviations from the goals. In goal programming, this is represented by the form of the achievement function. Specifically, what distance measure is used to weight, prioritize or balance the unwanted deviations. The prime feature of the variants listed on the left-hand-side of Fig. 1 is the nature of this distance measure, and hence underlying philosophy utilized. These variants will henceforth be termed "distance-based variants". The measurement of distance in most goal programming variants uses the L_p family of distance functions based on metrics as defined by Eq. (1) and further explored in Sect. 3.

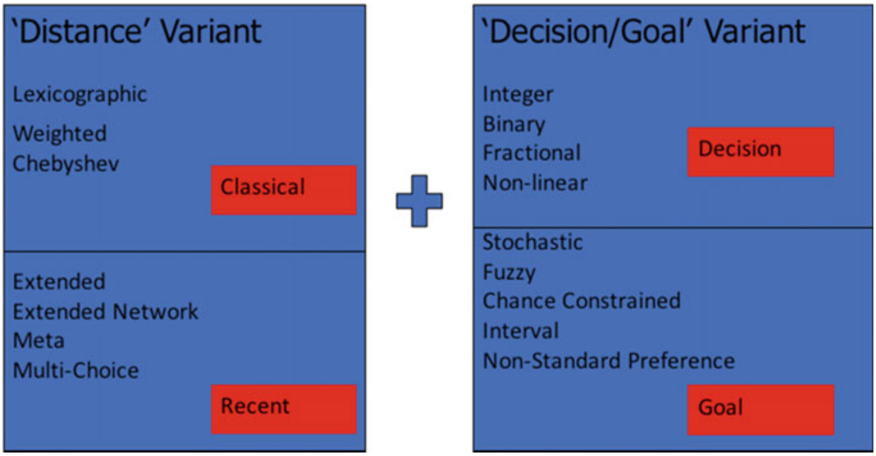

Fig. 1 Goal programming map

The upper left quadrant contains classical variants that are introduced before the year 2000. Their distinguishing attribute is that they focus on a single, main underlying philosophy. The lexicographic goal programming variant (Charnes and Cooper 1977) focuses on the prioritization or ordering of unwanted deviations from goals. The weighted goal programming variant (Charnes and Cooper 1961) focuses on the directed comparison of weighted, unwanted deviations from goals. The Chebyshev variant (Flavell 1976) focuses on achieving a balance between goals. The utility and distance function consequences of the above variants are detailed in Romero et al. (1998), Jones and Tamiz (2010).

The lower left quadrant focuses on distance-based variants introduced since the year 2000. These typically combine more than one underlying philosophy in order to provide the decision maker with greater flexibility when modelling their preferences. The extended goal programming variant (Romero 2001, 2004) combines the ordering, optimization and balancing philosophies by allowing a parametric mix of the average L_1 and worst L_∞ deviations in each of a number of priority levels. It can be reduced to consideration to a pair of, or a single, underlying philosophies by setting of its parameter levels including the number of priority levels (single or multiple). It can thus be reduced to all three classical variants. Extended goal programming is augmented by Jones et al. (2017) to form the extended network goal programming variant, which allows for a parametric mix of balance versus optimization for both objectives and stakeholders and the level of centralization to be employed across a network of multiple stakeholders with multiple objectives.

The meta-goal programming variant (Rodriguez-Uría et al. 2002) introduces the concept of a meta-goal, which allows the decision maker to directly set targets with respect to relevant underlying philosophies. The relative importance of meeting these targets can be controlled via the setting of meta-weights. The three meta-goals introduced by Rodriguez-Uría et al. (2002) relate to the weighted sum of unwanted deviations, the maximal (L_∞) unwanted deviation and the number of unmet goals (L_0). In this way, there is the potential to include further meta-goals in the framework relating to different underlying philosophies.

The multi-choice goal programming variant comprises the original Chang (2007) and revised (Chang 2008) formulations. It allows decision makers the flexibility of setting multiple, or a range of, target values for each goal. In the revised form this means that the goal programming achievement function contains a parametric mix of two terms. The first minimizes the weighted (L_1) sum of deviations whilst the second minimizes deviations from the most favourable goal value in the decision maker specified range. Thus, the multi-choice programming variant combines the concepts of meeting goal targets and improvement of the goal targets.

The right-hand side of Fig. 1 encompasses variants whose distinguishing feature is a property of the decision variables, deviational variables or individual goal targets, as opposed to the left-hand side variants which are concerned with the means of combination of unwanted deviations from the set of goals. The upper right quadrant focuses on properties of the decision variables. These broadly follow the rest of the mathematical programming paradigm, and many of the solution techniques developed for say integer, binary or non-linear programming can be utilized for integer goal programming, binary goal programming or non-linear goal programming respectively. However, there are sometimes specific attributes of the goal programming formulation that either facilitate solution or require special modelling or solution provision. This is the case with fractional goal programming, with a specific algorithm developed by Audet et al. (2004). A further example is given by Tamiz et al. (1999), who provide an adjustment of the Pareto detection and restoration techniques specifically tailored for integer goal programs.

The lower-right quadrant of Fig. 1 focuses on properties of the individual goals or target values. The stochastic, fuzzy and interval goal programming variants all allow for a measure of imprecision in the setting of goal targets due to uncertainty. Stochastic goal programming concentrates on the case where the goal target values or other parameters are random variables that can be specified according to some quantifiable probability distribution. These can then be solved by methods common to single objective stochastic programming, such as recourse or chance-constrained approaches (Masri 2017). Chance constrained goal programming (Charnes et al. 1976) allows for the introduction of probabilities associated with the achievement of goals and or constraints to be introduced into the goal programming model. Interval goal programming (Charnes and Collomb 1972) simply allows for the goal target value to be an interval rather than a single point on the underlying criterion scale. The fuzzy goal programming variant incorporates imprecision by allowing the goal target value to be expressed as a fuzzy rather than a crisp number, frequently but not mandatorily as a triangular fuzzy number. The criterion value is also normalized onto a zero-one range. Fuzzy goal programming (Narasimhan 1980; Hannan 1981) has proved to be the most common formal means of introducing uncertainty into the goal programming model, possibly because it is relatively straightforward from a modelling perspective and does not introduce much extra computational burden to the solution process. Non-standard preference goal programming is where the achievement function contribution of one or more unwanted deviational variables is not linear with respect to distance between the achieved and target levels of its corresponding goal (i.e. the value of that deviational variable). The earliest non-standard preference functions, summarized in Romero (1991) assumed that the per unit achievement function contribution increased with distance from the goal; i.e., increasing marginal penalty. This was hence termed a penalty function approach to goal programming. Later (Jones and Tamiz 1995) generalized this concept to include decreasing, discontinuous and non-linear penalty functions.

It should be noted that the four quadrants of Fig. 1 are not mutually exclusive, as a particular goal programming model could have attributes belonging to multiple quadrants. For example, a fuzzy, multi-choice goal program (Bankian-Tabrizi et al. 2012) or an integer, non-linear extended goal program (Choobineh and Mohagheghi 2016).

3 A Conceptual Distance-Metric Based View of Goal Programming Variants

The purpose of this section is to introduce a primary model from which several MCDM approaches can be deduced as secondary models, among them all the goal programming (GP) models introduced in the literature to date.[1]

The following notation will be used in what follows:

$i = 1 \ldots, q$ represents the set of criteria involved in the decision-making process.
$\boldsymbol{X}$ = vector of decision variables.
$f_i(\boldsymbol{X})$=mathematical expression of the ith criterion as a function of the vector of decision variables.
f_i^* = ideal value for the ith criterion, that is, the value achieved when the ith criterion is optimized, without considering the other criteria.
f_{i*} = anti-ideal value for the ith criterion, that is, the value achieved by the ith criterion when the other q − 1 criteria are optimized without considering the ith criterion.
$\hat{f}_i$ = point of reference for the ith criterion for the DM. This point will be interpreted in many ways, like a satisficing target, an ideal value, etc.
K_i = normaliser factor attached to the ith criterion, for instance, the range for each criterion (i.e. $K_i = f_i^* - f_{i*}$).
p = topological metric characterizing the distance function; that is, a real number belonging to the interval $[1, \infty]$.
$\boldsymbol{F}$ = feasible set, with a flexible mathematical structure at this stage.
W_i = weighting parameter that represents the relative importance attached by the DM to the ith criterion with respect to the other criteria.

[1]For the epistemological issues underlying the concepts of primary and secondary models and their respective links and problems of logic reductions see Nagel (1961), chapter 11 and especially pages 336–354.

The following general family of distance functions is now introduced (Romero 1991, pp. 86–88; Tamiz et al. 1998):

$$Min\, L_p = \left[\sum_{i=1}^{q} W_i^p \left| \frac{\hat{f}_i - f_i(X)}{K_i} \right|^p \right]^{1/p}$$
$$s.t.$$
$$\boldsymbol{X} \in \boldsymbol{F} \tag{1}$$

It should be remarked that without loss of generality, it has been assumed that all the criteria derive from attributes of the type "more is better", so all the objectives in model (1) are maximized.

This type of distance function model (1) was introduced in the operational research and management science (OR/MS) literature by Yu (1973) within the context of group decision-making problems. In his works, Yu demonstrated that the solutions obtained by minimizing (1) under very general conditions (basically the compactness of the feasible set $\boldsymbol{F}$ and the identification of the point of reference as the ideal one) enjoy useful economic and mathematical properties, such as: feasibility, uniqueness, symmetry, Pareto optimality, etc. (for technical details see Yu 1985, pp. 66–80).

On the other hand, Yu (1973) demonstrated that metric p in the context of distance function model (1) acts as a "balancing factor" between the maximum average achievement of all the criteria (that is, for $p = 1$) and the maximum discrepancy or maximum individual regret (that is, for $p = \infty$). Thus, for $p = \infty$, only the maximum deviation counts, and then model (1) turns into the following structure:

$$\text{Min}\, D$$
$$\text{s.t}$$
$$W_i \left| \frac{\hat{f}_i - f_i(X)}{K_i} \right| - D \leq 0 \quad \forall_i$$
$$\boldsymbol{X} \in \boldsymbol{F}, \tag{2}$$

where D represents the maximum discrepancy or individual regret.

Since the p-metric is a balancing factor is tempting to implement model (1) for different values of p. In fact in that way, intermediate solutions between the optimum average and the minimization of the maximum discrepancy can be obtained, assuming that such type of compromise solution exists for a particular problem. Thus, trade-offs between the maximum average achievement and the maximum discrepancy can be computed. However, given the non-smooth character of distance function of model (1) this strategy is computationally very complicated. A similar (albeit, in general, not equivalent) and computationally efficient model can be obtained through the following convex combination of the solutions corresponding to metrics p = 1, and p = ∞ (e.g., André et al. 2010, chapter 3):

$$\begin{aligned}
&\text{Min}(1-\lambda)\text{D}+\lambda\sum_{i=1}^{q}W_i\left|\frac{\hat{f}_i-f_i(X)}{K_i}\right| \\
&s.t. \\
&W_i\left|\frac{\hat{f}_i-f_i(X)}{K_i}\right|-D\leq 0\quad\forall i \\
&\boldsymbol{X}\in\boldsymbol{F}\lambda\in[0,1],
\end{aligned}\tag{3}$$

where λ plays the role of a control parameter. Thus, for $\lambda = 1$, we get the solution of maximum aggregated achievement corresponding to metric $p = 1$, for $\lambda = 0$, we get the solution of minimum discrepancy or minimum regret corresponding to metric $p = \infty$. For intermediate values of control parameter λ, we get compromises or trade-offs between these two opposite poles (maximum aggregated achievement and minimum disagreement), if they exist. Note that model (3) is still non-smooth due to the existence of absolute values. However, when in the following sections model (3) is particularized to different GP specifications then formulations that are relatively easier to compute will be obtained.

By particularizing the values of the parameters of the above models, practically all the MCDM approaches applicable for continuous problems can be obtained (Romero et al. 1998; Romero 2001). In other words, most of the MCDM approaches are supported by the topological minimization of a distance function. Being more specific in this paper and in logic terms models (1) and (3) will be considered as a *primary model* from which will be deduced as *secondary models* a significant but not exhaustive collection of GP formulations. But before that some epistemic reflections about the bounded rationality theory will be established, since the GP models to be deduced from (1) and (3) will be interpreted as a potential operational side for this type of theory.

4 Satisficing Logic and Bounded Rationality: Some Epistemic Reflections

Simon (1955, 1956, 1979) postulated that in today's complex organizations (big companies, state agencies, trade unions, etc.) the decisional context is defined by incomplete and asymmetrical information, limited resources, conflicts of interest among criteria, etc. Within this kind of context the DM is not able to maximize anything, much less a well defined criterion function as classical optimization assumes. On the contrary, Simon states that in decisional contexts so complex the perfect rational choice is not possible for practical reasons. Hence, the DM must make his/her decisions within a bounded rationality substratum. With that purpose Simon introduced the Northumbrian term "satisficing" (a merge of the words "satisfying" and "sufficing") to underpin his bounded rationality theory.

The terms bounded rationality and satisficing logic have been interpreted and applied in different ways, perhaps in too many ways. As Gigerenzer states: “Today, bounded rationality has become a diluted, fashionable term, used by proponents of quite disparate visions of reasonableness…” (Gigerenzer 2001, p. 37).

Just as a sample of different visions of bounded rationality, without being exhaustive and only citing works of leading social scientists we can compare the view of bounded rationality by Sargent (1993) for macroeconomic problems, by Rubinstein (1998) modelling bounded rationality within a substratum of games or by the authors of the book edited by Gigerenzer and Selten (2001) who try to address how humans make decisions in real life within the commented Simonian philosophy.

We dare to conjecture that the broad plurality of views about bounded rationality might be due to the epistemological status normally attached to the Simon’s proposal. Thus, under our view bounded rationality does not fit well within the category of an hypothesis or even of a single scientific theory but within a broader epistemic structure. In this sense, it might be more reasonable to frame Simon’s proposal within which Laudan defined as a “Research Tradition”.

Thus, for Laudan (1977, chapter 3 and more specifically pages 78–79) a Research Tradition (RT) has a number of traits, like: (a) A plurality of theories underlying each RT, (b) A common set of metaphysical and methodological commitments for every RT, and (c) Each RT provides different and sometimes contradictory formulations. Even though a rigorous analysis about the characterisation of bounded rationality as a Laudanian RT is obviously beyond the scope of this contribution, however it seems worthwhile to take into account this idea in order to understand the plurality of interpretations and models deriving from the Simon’s seminal proposal.

In fact, this type of theoretical construct encompasses a set of guidelines for the posterior development of specific theories. By accepting this type of epistemological commitment it is not complicated to understand the plurality of theoretical approaches deriving from the research tradition coined as bounded rationality.

Assuming the above plural perspective, a satisficing orientation with the status of a Laudian RT should imply in all its possible formulations scenarios where the DM attempts to achieve a set of relevant goals as closely as possible to the set of previously established targets. Following this orientation, an “operational satisficing” model can be accommodated with the help goal programming (GP). This task will be undertaken in the next section deriving several secondary models from *primary models* (1) and (3).

5 Goal Programming and Bounded Rationality Research Tradition: An Operational Linkage

Assuming the Simonian bounded rationality orientation as a Research Tradition, a satisficing approach might imply scenarios where the DM attempts to achieve a set of relevant goals as closely as possible to the set of established targets. Following this orientation, an “operational satisficing” model can be accommodated with the help of the GP theory (see pioneer works by Charnes et al. 1955 and Charnes and Cooper 1961)[2]. In order to link GP with the Simonian research tradition, the point of reference $\hat{f}_i$ of model (1) will be considered a satisficing target. As a first step in our analysis the following change of variables in *primary model* (1) is introduced (see Charnes and Cooper 1977):

$$n_i = \frac{1}{2}\left[\left|\hat{f}_i - f_i(X)\right| + (\hat{f}_i - f_i(X)\right] \tag{4}$$

$$p_i = \frac{1}{2}\left[\left|\hat{f}_i - f_i(X)\right| - (\hat{f}_i - f_i(X)\right] \tag{5}$$

By adding (4) and (5), and by subtracting (5) from (4), we have the following two identities:

$$n_i + p_i = \left|\hat{f}_i - f_i(X)\right| \tag{6}$$

$$n_i - p_i = \hat{f}_i - f_i(X) \tag{7}$$

According to (6) and (7) *primary model* (1) turns into the following structure:

Achievement function:

$$\begin{aligned}
&\text{Min}\left[\sum_{i=1}^{q} W_i^p \left(\frac{n_i + p_i}{K_i}\right)^p\right]^{1/p} \\
&s.t. \\
&f_i(x) + n_i - p_i = \hat{f}_i \forall i \\
&\boldsymbol{n} \geq \mathbf{0} \boldsymbol{p} \geq \mathbf{0} \\
&\boldsymbol{X} \in \boldsymbol{F}
\end{aligned} \tag{8}$$

Model (8) is known as an Archimedean GP formulation. The objective function of the GP is called the achievement function. Note that, when the ith goal derives from a "more is better" attribute, deviation variable n_i is unwanted and deviation variable p_i is wanted, hence only variable n_i must appear in the achievement function. On the contrary, when the ith goal derives from a "less is better" attribute, n_i is wanted and p_i unwanted, then only deviation variable p_i must appear in the achievement function. Finally, if the DM wants neither under-achievement nor over-achievement, both deviation variables must appear in the achievement function.

Model (8) encompasses several satisficing options, Thus, for p = 1, model (8) turns into:

Achievement function:

[2]The pioneers of linking the Simonian satisficing philosophy with goal programming are Lee (1972) and Ignizio (1976). An attempt of axiomatization of this interpretation of bounded rationality and satisficing, can be seen in González-Pachón and Romero (2004)

$$\begin{aligned}
&\text{Min}\left[\sum_{i=1}^{q} W_i\left(\frac{n_i+p_i}{K_i}\right)\right]\\
&s.t.\\
&f_i(x)+n_i-p_i=\hat{f}_i\forall i\\
&\boldsymbol{n}\geq \mathbf{0},\ \boldsymbol{p}\geq \mathbf{0}\\
&\boldsymbol{X}\in \boldsymbol{F}
\end{aligned}\tag{9}$$

Model (9) is known as a weighted GP, since this type of achievement function minimizes the weighted sum of the unwanted deviation variables. This model provides the best solution from the point of view of optimizing the aggregate achievement. Hence, when the satisficing process is orientated in that direction, then this solution is the "best" one. However, the performance of this type of solution can provide very poor results for the achievement of any of the goals involved in the decision-making process. Thus, if the DM is interested in obtaining satisficing solutions in a more balanced way, then it might be advisable to minimize the maximum discrepancy. This purpose can be obtained by setting p $= \infty$ in model (1). In that way, the following model is obtained:

Achievement function:

$$\begin{aligned}
&\text{Min D}\\
&s.t.\\
&W_i\left(\frac{n_i+p_i}{K_i}\right)-D\leq 0\ \forall i\\
&f_i(x)+n_i-p_i=\hat{f}_i\forall i\\
&\boldsymbol{n}\geq \mathbf{0},\ \boldsymbol{p}\geq \mathbf{0}\\
&\boldsymbol{X}\in \boldsymbol{F}
\end{aligned}\tag{10}$$

Model (10) is known as a MINMAX (or Chebyshev) GP model. The GP models (9) and (10) imply two different strategies for obtaining satisficing solutions. Thus, the former is advisable when the DM is interested in maximising the aggregate achievement (*maximum efficiency*) and the latter in minimising the maximum deviation between the achievement of the different goals (*maximum equity*). The first solution (WGP pole) can be extremely biased towards the achievement of some of the goals, whereas the second (Chebyshev pole) can provide poor aggregate performance across the different goals. By following the direction provided by model (4), we can obtain a linear convex combination of models (9) and (10) in order to obtain compromises, if they exist, between the two satisficing solutions obtained. This task can be undertaken through the commented linear convex combination or, by implementing the change of variables (6) and (7) into *primary model* (3). By applying any of the two equivalent orientations, the following Extended GP model is obtained (Romero 2001; Jones and Tamiz 2010):

Achievement function:

$$\begin{aligned}
&\text{Min}\,(1-\lambda)D+\lambda\left[\sum_{i=1}^{q}W_i\left(\frac{n_i+p_i}{K_i}\right)\right]\\
&s.t.\\
&W_i\left(\frac{n_i+p_i}{K_i}\right)-D\leq 0\forall i\\
&f_i(x)+n_i-p_i=\hat{f}_i\\
&\boldsymbol{n}\geq\boldsymbol{0}\boldsymbol{p}\geq\boldsymbol{0}\\
&\boldsymbol{X}\in\boldsymbol{F}
\end{aligned}\tag{11}$$

Control parameter λ plays a similar role to its use in model (3); that is, to trade-off "average achievement" against "balanced achievement". In fact, for $\lambda = 1$, we obtain a WGP model, whereas, the Chebyshev GP model is reproduced for $\lambda = 0$. For values of control parameter λ belonging to the open interval (0, 1), we get intermediate solutions, if they exist, between the two satisficing solutions considered.

Finally, the GP variant called lexicographic GP (LGP) is introduced. The achievement function of a LGP model is made up of an ordered vector whose dimension is equal to the Q number of pre-emptive priority levels for which the q goals have been grouped. Each component of this vector comprises the unwanted deviation variables of the goals placed at the corresponding priority level. Thus, we have (see Lee 1972; Ignizio and Perlis 1979):

Achievement function:

$$\begin{aligned}
&\textbf{Lex min a}=\left[\sum_{i\in h_1}W_i(\frac{n_i+p_i}{K_i}),\ldots,\sum_{i\in h_r}W_i(\frac{n_i+p_i}{K_i}),\ldots,\sum_{i\in hQ}W_i(\frac{n_i+p_i}{K_i})\right]\\
&s.t.\\
&\boldsymbol{n}\geq\boldsymbol{0},\ \boldsymbol{p}\geq\boldsymbol{0}\\
&\boldsymbol{X}\in\boldsymbol{F}
\end{aligned}\tag{12}$$

where h_r means the index set of goals placed at the rth priority level.

The satisficing orientation will seek now to find the lexicographic minimum of vector **a**, that is, the ordered minimization of its components. So, the first component of **a** is minimized, then the second component of **a** is minimized subject to the non-degradation of the minimum value of the first component obtained previously and so on. Note that a satisficing LGP model implies a non-compensatory preference structure, in the sense that there are no finite trade-offs among goals placed at different priority levels.In other words, the structure of preferences characterized by the achievement function of model (12) is not compatible with the existence of a utility function (Debreu 1959, pp. 72–73). This type of assumption is actually very strong, but also useful in situations where the DM has a pre-defined ordering of the goals in mind and does not wish to make direct "trade-off" comparisons between goals. This could be due to ethical considerations, such as the impossibility of quantifying the trade-offs between safety goals (e.g., measured in lives lost due to a natural hazard) and monetary goals (such as cost or profit) (see e.g., Jones and Tamiz 2010, pp. 13–14)[3].

6 Goal Programming as an Engine or Generator of Social Choice Functions

In this section it will be shown how the *primary model* (1) and the *secondary* GP models embedded in (1) can be a powerful engine or generator of social choice functions. Let us start by considering the usual group decision or social choice scenario where we have i = 1, 2, … q objects (alternatives, candidates, etc.) and j = 1, 2, … m entities (DM, social groups, electoral committees, etc.), which have to give judgement values over the q objects. At this stage the judgement values are expressed in a very general way, that is ordinal with complete or partial information, cardinal information provided according to different formats, etc. Now, in *primary model* (1), besides the change of criteria by objects, the following changes in the values and meaning of the other parameters and variables are implemented.

W_j now represents the social influence of the *j*th DM or social group (e.g., the size of the group). The normaliser factors K_i are eliminated, since we will not need to implement any normalization process. The point of reference $\hat{f}_i$ changes now to f_i^S, representing the consensus attached by the whole group to the *i*th object (i.e., the unknowns of the problem) and finally $f_i(\boldsymbol{X})$ changes now to f_i^j, representing the ordinal or cardinal valuation provided by the *j*th DM to the *i*th alternative. Taking into account these changes and multiplying the achievement function of (1) by minus 1, in order to indicate that the collective function is an increasing function (i.e., a utility function holding the "more is better" postulate), then the following "generator" of collective choice rules is obtained:

$$\text{Max U} = -\left[\sum_{i=1}^{q}\sum_{j=1}^{m} W_j^p \left|f_i^S - f_i^j\right|^p\right]^{1/p}$$

s.t

$$f_i^j \in \boldsymbol{F}(\text{set of conditions}) \qquad (13)$$

[3]The reason for the incompatibility between lexicographic orderings and utility functions is the non-continuity of preferences inherent to a lexicographic structure of preferences. Albeit, the continuity of preferences is neither a fact nor a hypothesis corroborated by empirical evidence but just a non-falsifiable assumption necessary to axiomatize the neoclassical consumption theory (see e.g., Deaton and Muellbauer 1986, p. 27).

By making in (13) p = 1, the following social choice function is obtained:

$$\mathrm{U}_1 = -\left[\sum_{i=1}^{q}\sum_{j=1}^{m} W_j \left| f_i^S - f_i^j \right| \right] \tag{14}$$

For (14) the average agreement is maximized. In other words, this social compromise consensus represents the "best solution" from the point of view of the majority (i.e., the classic Benthamite or utilitarian solution, see Bentham (1948)).

Setting p = ∞ in (13), we have:

$$\mathrm{U}_\infty = -\left[Max_{i,j} W_i \left| f_i^S - f_i^j \right| \right] \tag{15}$$

For (15) the disagreement of the DM more displaced with respect to the average consensus (i.e., the minority group) is minimized. Hence, this social compromise consensus represents the "best solution" from the point of view of the minority (i.e., the Rawlsian solution[4]).

The two above social solutions represent two opposite poles of the interest of the society as a whole. Hence, might be advisable to combine solutions (14) and (15), searching for possible compromise consensus solutions. Again we can undertake this task, as was done in the preceding sections by implementing the following convex combination:

$$U\lambda = -(1-\lambda)\left[Max_{i,j} W_i \left| f_i^S - f_i^j \right| \right] - \lambda \left[\sum_{i=1}^{q}\sum_{j=1}^{m} W_i \left| f_j^S - f_j^i \right| \right] \tag{16}$$

Control parameter λ plays a similar technical role as in the preceding sections. However, it is important to note that in this scenario λ trade-offs "average agreement" (Majority Benthamite solution) and the agreement of the "most unfavourable DM" (Minority Rawlsian solution). Hence, compromise consensuses between these opposite poles, if they exist, can be obtained for values of control parameter λ, belonging to the open interval (0, 1). Note that function (16) is not smooth, and consequently does not necessarily lead to a computable or solvable problem. However, it is rather straightforward to demonstrate that by introducing in a relative similar way the deviation variables used in the preceding section, model (16) turns into the following computable Extended GP formulation (Goanzález-Pachón and Romero 2009):

$$U_\lambda = -(1-\lambda)D - \lambda \left[\sum_{i=1}^{q}\sum_{j=1}^{m} w_j (n_i^j + p_i^j) \right]$$

s.t.

[4] Expression (16) represents in mathematical terms the "Second Principle of Justice" proposed by Rawls (1971, pp. 65–75).

$$w_j \sum_{j=1}^{m} (n_i^j + p_i^j) - D \leq 0$$
$$f_i^S + n_i^j - p_i^j = f_i^j \quad \forall i, j$$
$$f_i^j \in \boldsymbol{F}(\text{set of conditions}) \tag{17}$$

where D represents now the disagreement of the DM or social group with views more displaced from the consensus obtained. Model (17) from a computational point of view is just a parametric linear programming formulation, what turns its computation in a rather easy task. On the other hand, by resorting to different characterizations of the information provided by the DMs and the corresponding implication in the set of conditions $\boldsymbol{F}$, from (17) we can obtain social compromise consensus for the following cases: (a) ordinal and complete preferences, (b) ordinal and partial preferences and (c) cardinal and complete preferences through utility functions and through pairwise comparison matrices(see González-Pachón and Romero 1999, 2006, 2007 and 2011). In short, model (17) can be used as a generator of social compromise consensus in a large variety of preferential contexts.

The ideas presented in this section seems rather powerful since the use of GP for inducing social choice models might help in the direction of building sound, new bridges between social sciences and the (OR/MS) discipline. In this direction it should be noted recent research that by resorting to GP social principles of equity, freedom, absolute poverty, etc. are mathematically modelled, quantifying compromises and trade-offs among them (González-Pachón and Romero 2016; Jones et al. 2017).

7 Concluding Remarks

This Chapter has presented goal programming from both literature and underlying distance- metric perspectives. The goal programming variant map given by Fig. 1 is designed to help researchers and practitioners understand the nature and interrelations of the many goal programming variants that currently exist. It is intended that future, novel goal programming variants can be placed onto the map as they arise. Goal programming has been shown to be a flexible technique that still enjoys a healthy place amongst the field of MCDM methods. Its flexibility has allowed a range of social choice functions to be successfully modelled in the context of a mathematical method that allow decision maker(s) to come as close as possible to reaching their goals in a rigorous, justifiable way. Further social principles have been recently incorporated into the goal programming paradigm and our postulation is that this will continue in the near future. The linkage between goal programming and bounded rationality has been explored in this Chapter, with the goal that this will allow further research in this direction.

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Robust Goal Programming with Interactive Fuzzy Coefficients

Masahiro Inuiguchi

Abstract In this paper, goal programming problems with interactive fuzzy coefficients are treated. Two types of targets can be expressed by fuzzy sets in goal programming problems with fuzzy coefficients. One is the ambiguous target whose true value is not known precisely and the other is the target distribution to which the fuzzy set of objective function values is brought close. Corresponding to the difference of targets, we use two kinds of deviations naturally obtained from the extension principle. On the other hand, to treat the interaction among fuzzy coefficients, we introduce oblique fuzzy vectors (OFVs). An OFV can be obtained from the expert knowledge about the behavior of coefficients as well as from the principal component analysis of the stored coefficient data. It is shown that linear functions with OFVs can be obtained easily. The goal programming problems are formulated based on the necessity measure maximization model. It is shown that the reduced programming problems can be solved by a bisection method together with a simplex method. Moreover, it is shown that the constraints of the reduced programming problems have special structures such as a dual block angular structure and a bordered angular structure so that some decomposition methods are applicable.

1 Introduction

When we formulate real world programming problems, we may come across cases that coefficients of objective and constraint functions are not known precisely. In such cases, we may express those imprecise coefficients as fuzzy numbers based on the vague knowledge about the coefficients. As the result, we obtain programming problems with fuzzy coefficients.

Linear programming problems with fuzzy coefficients are known as possibilistic linear programming problems (Inuiguchi and Ramík 2000; Lodwick and Kacprzyk

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2010). Because the possibilistic linear programming problems have been investigated for more than 35 years, quite many approaches have been proposed and studied well (Inuiguchi and Ramík 2000; Lodwick and Kacprzyk 2010). Therefore, in the recent literature, the applications (Mousazadeh et al. 2015; Yaghin et al. 2013) and approaches to problems with generalized fuzzy sets (Dubey et al. 2012; Ramík and Vlach 2016) are more popular than new approaches with the conventional fuzzy sets.

However, even in the linear programming problems with the conventional fuzzy coefficients, the interaction among fuzzy coefficients has not yet studied, although some models have already proposed (Inuiguchi 2000). Fuzzy coefficients have been assumed to be non-interactive one another. This implies that the possible range of a coefficient is not influenced by the realization of another coefficient. For example, consider the weight and height of a medium-sized Japanese man. We estimate the weight is around 66.0 kg and the height is around 171 cm. If we know the height of the man is 172 cm, our estimation about his weight would become bigger than that before we know his height. However, the assumption of non-interaction implies that the estimated range of the weight of that man is not changed. In this example, the assumption of non-interaction is not very adequate. The assumption of non-interaction is not always reasonable. On the other hand, owing to this assumption, the reduced problems are tractable.

As described above, there is a need to treat interactive fuzzy coefficients. However, the introduction of interaction among fuzzy coefficients often diminishes the tractability of the reduced problems of possibilistic linear programming problems. Several models treating the interaction among fuzzy coefficients were proposed and shown that the tractability of the reduced problems is kept even if we introduce those models into usual possibilistic linear programming problems (Inuiguchi 2000). However, it is not yet known whether the tractability is kept for other types of possibilistic linear programming problems.

In this paper, we introduce oblique fuzzy vectors (OFVs) (Inuiguchi et al. 2003) to treat the interaction among fuzzy coefficients into modality goal programming (MGP) problems proposed by the authors (Inuiguchi and Kume 1989). An OFV can be obtained from the expert knowledge about the behavior of coefficients as well as from the principal component analysis of the stored coefficient data. MGP problems are extensions of interval goal programming (IGP) problems (Inuiguchi and Kume 1991). In MGP and IGP problems, we demonstrated that there are two possible definitions of deviation between a fuzzy/interval goal function value and a given fuzzy/interval target. One is the possible deviation based on the extension principle and the other is the necessary deviation derived from an equation with fuzzy numbers. To treat those deviations, the reduced problems become more complex than those of the usual possibilistic linear programming problems, although the authors (Inuiguchi and Kume 1989, 1991) succeeded to show the tractability of the reduced problems. We show that the introduction of OFVs into MGP problems does not diminish the tractability of the reduced problems.

This paper is organized as follows. In next section, we give the definition of oblique fuzzy vectors and show the simplicity in calculation of a linear function value with an oblique fuzzy vector. We describe the formulations based on the necessity measure maximization and reduced problems of MGP problems with ambiguous targets in Sect. 3. In these models, targets are not deterministic but uncertain or imprecise. There are two cases whether a regret function is defined or not. A fuzzy goal is given to the regret in the former case while a fuzzy goal is given to each target in the latter case. In Sect. 4, MGP problems with target distributions is formulated based on the necessity measure maximization. Reduced problems are shown in similar ways to MGP problems with ambiguous targets. In Sect. 5, concluding remarks are given with future topics.

2 Oblique Fuzzy Vectors and Their Use in Linear Functions

In this paper, we treat oblique fuzzy vectors (OFVs) (see Inuiguchi et al. 2003). An OFV C is defined by the following membership function:

$$\mu_C(\boldsymbol{c}) = \min_{i=1,2,\ldots,n} \mu_{S_i}(\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{c}), \tag{1}$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$, $\boldsymbol{d}_i = (d_{i1}, d_{i2}, \ldots, d_{in})^{\mathrm{T}}$, $i = 1, 2, \ldots, n$ and each S_i is a symmetric L-fuzzy number $(s_i, \alpha_i)_L$ whose membership function is defined by

$$\mu_{S_i}(r) = L\left(\frac{r - s_i}{\alpha_i}\right), \tag{2}$$

$L : \mathbf{R} \to [0, 1]$ is a reference function such that $L(0) = 1$, $L(r) = L(-r)$, $\lim_{r\to+\infty} L(r) = 0$ and L is upper semi-continuous and quasi-concave. Parameters s_i and $\alpha_i > 0$ are constants. S_i's are called non-interactive fuzzy numbers associated with C. The $n \times n$ matrix $D = (\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_n)$ is nonsingular and called an obliquity matrix associated with C. When $D = I$ (I is an identity matrix), the OFV C degenerates to a fuzzy vector of non-interactive fuzzy numbers S_i, $i = 1, 2, \ldots, n$. An oblique fuzzy vector is depicted in Fig. 1.

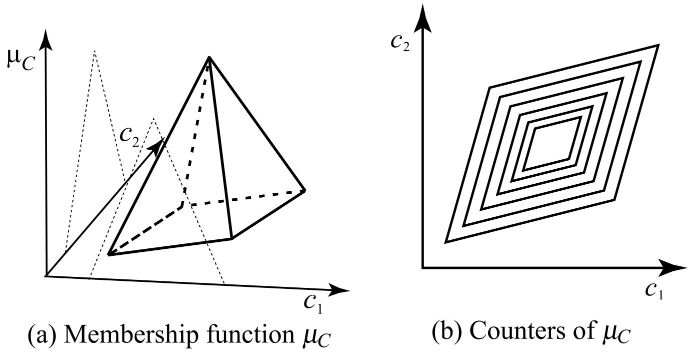

Fig. 1 An oblique fuzzy vector (2-dimensional case)

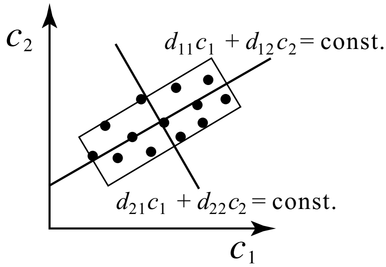

Fig. 2 Application of PCA to identification of an oblique fuzzy vector

The obliquity matrix D can be obtained from the expert knowledge about independent n linear function values $\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{c}, i = 1, 2, \ldots, n$ of $\boldsymbol{c}$ if they are available. Moreover, if stored data (more than n observations) about $\boldsymbol{c}$ are available, we can obtain the obliquity matrix D by the principal component analysis (PCA). Namely, by PCA, we obtain from the first principal component to the nth principal component and their coefficient vectors $\boldsymbol{d}_i, i = 1, 2, \ldots, n$ (see Fig. 2). Then we define $D = (\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_n)$. Under an obliquity matrix D, we obtain $S_i, i = 1, 2, \ldots, n$ from the knowledge about the possible range of $c_i, i = 1, 2, \ldots$. Namely, from the possible range of c_i expressed by L-fuzzy number $(\bar{c}_i, \beta_i)_L$, we obtain $S_i = (s_i, \alpha_i), i = 1, 2, \ldots, n$ by $s_i = \boldsymbol{d}_i^{\mathrm{T}}\bar{\boldsymbol{c}}$ and $\alpha_i = |\boldsymbol{d}_i|^{\mathrm{T}}\boldsymbol{\beta}$, where $\bar{\boldsymbol{c}} = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)^{\mathrm{T}}$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)^{\mathrm{T}}$ and $|\boldsymbol{d}_i| = (|d_{i1}|, |d_{i2}|, \ldots, |d_{in}|)^{\mathrm{T}}$.

Given a function $f : \mathbf{R}^n \to \mathbf{R}$, the function value of an OFV C is a fuzzy quantity $f(C)$ whose membership function $\mu_{f(C)}$ is defined by the extension principle,

$$\mu_{f(C)}(y) = \begin{cases} \sup\limits_{r \in f^{-1}(y)} \mu_C(\boldsymbol{r}), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$
$$= \begin{cases} \sup\limits_{\substack{u=(u_1,u_2,\ldots,u_n)^{\mathrm{T}}: \\ D^{-\mathrm{T}}u \in f^{-1}(y)}} \min\limits_{i=1,2,\ldots,n} \mu_{B_i}(u_i), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases} \tag{3}$$

where $f^{-1}(y)$ is the inverse image of f and $D^{-\mathrm{T}}$ is the inverse matrix of D^{T}.

When a function f is $f(\boldsymbol{r}) = \boldsymbol{k}^{\mathrm{T}}\boldsymbol{r} = \sum_{i=1}^{n} k_i r_i$, it is shown by Inuiguchi et al. (2003) that $f(C)$ is a symmetric L-fuzzy number $(f, \alpha^f)_L$ with

$$f = \boldsymbol{l}^{\mathrm{T}}\boldsymbol{s}, \ \alpha^f = |\boldsymbol{l}|^{\mathrm{T}}\boldsymbol{\alpha}, \ \boldsymbol{l}^{\mathrm{T}} = \boldsymbol{k}^{\mathrm{T}}D^{-\mathrm{T}}, \tag{4}$$

where $\boldsymbol{l} = (l_1, l_2, \ldots, l_n)^{\mathrm{T}}$, $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)^{\mathrm{T}}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$, $|\boldsymbol{l}| = (|l_1|, |l_2|, \ldots, |l_n|)^{\mathrm{T}}$.

Let us consider an n-dimensional OFV C whose obliquity matrix D is represented by

$$D = \begin{pmatrix} D_1 & O \\ O & D_2 \end{pmatrix}, \tag{5}$$

where O is a zero matrix and D_1 and D_2 are $n_1 \times n_1$ and $n_2 \times n_2$ matrices ($n = n_1 + n_2$). Let S_i, $i = 1, 2, \ldots, n$ be non-interactive fuzzy numbers associated with the OFV C. In connection with the OFV C, we also consider an n_1-dimensional OFV C^1 which has D_1 as the obliquity matrix and non-interactive fuzzy numbers S_i, $i = 1, 2, \ldots, n_1$ and an n_2-dimensional OFV C^2 which has D_2 and S_i, $i = n_1 + 1, n_1 + 2, \ldots, n$. For C, C^1 and C^2, we have

$$\mu_C(\boldsymbol{r}) = \min(\mu_{C^1}(\boldsymbol{r}_1), \mu_{C^2}(\boldsymbol{r}_2)), \tag{6}$$

where $\boldsymbol{r} = (\boldsymbol{r}_1^{\mathrm{T}}, \boldsymbol{r}_2^{\mathrm{T}})^{\mathrm{T}}$. When this equation holds, we say that C^1 and C^2 are non-interactive.

Moreover, let us consider a separable function $\varphi : \mathbf{R}^n \to \mathbf{R}$, such that $\varphi(\boldsymbol{r}) = h\,(f(\boldsymbol{r}_1), g(\boldsymbol{r}_2))$, where $f : \mathbf{R}^{n_1} \to \mathbf{R}$, $g : \mathbf{R}^{n_2} \to \mathbf{R}$ and $h : \mathbf{R}^2 \to \mathbf{R}$. Based on the extension principle (3), we obtain fuzzy quantities $f(C^1)$ and $g(C^2)$. We can regard a fuzzy vector $(f(C^1), g(C^2))$ as a 2-dimensional OFV with an obliquity matrix I and non-interactive fuzzy numbers $f(C^1)$ and $g(C^2)$. Under those assumptions, we can prove

$$\varphi(C) = h(f(C^1), g(C^2)). \tag{7}$$

This means that if a function φ is separable and an OFV C can be divided into non-interactive OFVs C^i, $i = 1, 2$, then $\varphi(C)$ can be calculated through a decomposition.

Now, let us consider an equation of non-interactive L-fuzzy numbers X, S_1 and S_2,

$$X + S_1 = S_2, \tag{8}$$

where $S_1 = (s_1, \alpha_1)_L$ and $S_2 = (s_2, \alpha_2)_L$ are given but $X = (x, \alpha)_L$ is unknown. As is known in literature, $X \neq S_2 - S_1 = (s_2 - s_1, \alpha_1 + \alpha_2)_L$. Moreover, (8) has a solution if and only if $\alpha_2 \geq \alpha_1$. Since the left-hand side yields an L-fuzzy number $(x + s_1, \alpha + \alpha_1)_L$, if $\alpha_2 \geq \alpha_1$, the solution is $X = (a_2 - a_1, \alpha_2 - \alpha_1)_L$. From this point of view, when $\alpha_2 \geq \alpha_1$, we define a difference other than $S_2 - S_1$ as

$$S_2 \check{-} S_1 = (s_2 - s_1, \alpha_2 - \alpha_1)_L. \tag{9}$$

Considering an equation, $S_2 - X = S_1$, we can define another difference $S_2 \hat{-} S_1$ by

$$S_2 \hat{-} S_1 = (s_2 - s_1, \alpha_1 - \alpha_2)_L, \tag{10}$$

when $\alpha_1 \geq \alpha_2$. Whereas $S_2 \check{-} S_1$ is defined when $\alpha_1 \leq \alpha_2$, $S_2 \hat{-} S_1$ is defined when $\alpha_1 \geq \alpha_2$. Combining $S_2 \check{-} S_1$ and $S_2 \hat{-} S_1$, we can define a difference $S_2 \tilde{-} S_1$ in any case as

$$S_2 \tilde{-} S_1 = (s_2 - s_1, |\alpha_1 - \alpha_2|)_L. \tag{11}$$

A fuzzy set S of $\mathbf{R}^n$ often represents a possible range of a variable vector $\boldsymbol{s}$. In such a case, the variable vector $\boldsymbol{s}$ is called a possibilistic variable vector and a membership function μ_S of S is considered as a possibility distribution.

A possible range of function values $f(\boldsymbol{s})$ of a possibilistic variable vector $\boldsymbol{s}$ is given by $f(S)$ when S is a possible range of $\boldsymbol{a}$. In order words, $\mu_{f(S)}$ defined by the extension principle is a possibility distribution of $f(\boldsymbol{s})$.

Given a possibility distribution μ_S, a possibility measure $\Pi_S(G)$ and a necessity measure $N_S(G)$ of a fuzzy event (fuzzy set) G are defined by

$$\Pi_S(G) = \sup_{\boldsymbol{r}} \min(\mu_S(\boldsymbol{r}), \mu_G(\boldsymbol{r})), \tag{12}$$

$$N_S(G) = \inf_{\boldsymbol{r}} \max(1 - \mu_S(\boldsymbol{r}), \mu_G(\boldsymbol{r})), \tag{13}$$

where μ_G is a membership function of G. $\Pi_S(G)$ and $N_S(G)$ evaluate the possibility and necessity degrees of the event $\boldsymbol{s}$ is in G when we know that $\boldsymbol{s}$ is in S, respectively.

In what follows, we use $N_S(G)$ only. Let us see a useful property of $N_S(G)$. To this end, we define an h-level set $[S]_h$ and a strong h-level set $(S)_h$ of a fuzzy set S as $[S]_h = \{\boldsymbol{r} \mid \mu_S(\boldsymbol{r}) \geq h\}$ and $(S)_h = \{\boldsymbol{r} \mid \mu_S(\boldsymbol{r}) > h\}$, respectively. We have the following equivalencies:

$$N_S(G) \geq h \Leftrightarrow (S)_{1-h} \subseteq [G]_h. \tag{14}$$

Moreover, if $[S]_h$ is bounded and closed for any $h \in (0, 1]$, we have the following equivalencies for any $h \in (0, 1]$:

$$N_S(G) \geq h \Leftrightarrow \mathrm{cl}\,(S)_{1-h} \subseteq [G]_h, \tag{15}$$

where $\mathrm{cl}\,(S)_{1-h}$ is a closure of a set $(S)_{1-h}$.

3 MGP with Ambiguous Targets

3.1 Problem Statement

We formulate the following goal programming problem with ambiguous coefficients and targets:

$$\begin{array}{l} \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} \overset{\pm}{\to} \boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i,\ i = 1, 2, \ldots, p_1, \\ \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} \overset{-}{\to} \boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i,\ i = p_1 + 1, \ldots, p_2, \\ \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} \to \boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i,\ i = p_2 + 1, \ldots, p, \\ \text{sub. to } \ A\boldsymbol{x} \leq \boldsymbol{b}, \end{array} \tag{16}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the decision variable vector and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^{\mathrm{T}}$ is a constant vector. A is an $m \times n$ matrix. $(\boldsymbol{c}_i^{\mathrm{T}}, \boldsymbol{g}_i^{\mathrm{T}}, q_i)^{\mathrm{T}}$ is a possibilistic variable vector whose possible range is represented by a $(2n + 1)$-dimensional OFV

C_i with an obliquity matrix D_i and non-interactive fuzzy numbers $S_{ij} = (s_{ij}, \alpha_{ij})_L$, $j = 1, 2, \ldots, 2n+1$. $(\boldsymbol{c}_i^{\mathrm{T}}, \boldsymbol{g}_i^{\mathrm{T}}, q_i)^{\mathrm{T}}, i = 1, 2, \ldots, p$ are non-interactive one another. Notation $f \overset{+}{\to} g$ shows a goal that the decision maker prefers the left-hand side f exceed the right-hand side g if it is possible, and otherwise he/she prefers f close to g. Similarly, $f \overset{-}{\to} g$ shows a goal that the decision maker prefers f less than g if it is possible, and otherwise he/she prefers f close to g. Moreover, $f \to g$ shows a goal that the decision maker prefers f close to g. In Problem (16), not only the left-hand sides but also the right-hand sides may be functions with ambiguous coefficients. Such a situation may be encountered in real world problems, e.g., making the amount of production close to the uncertain demand, the investment of the money less than or equal to the amount of uncertain gross sales, and so on. The left-hand side functions are called goal functions and the right-hand side functions are called targets.

Problem (16) can be interpreted as the following goal programming with ambiguous coefficients and zero targets:

$$\begin{array}{l} \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} - (\boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i) \overset{+}{\to} 0, i = 1, 2, \ldots, p_1, \\ \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} - (\boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i) \overset{-}{\to} 0, i = p_1 + 1, \ldots, p_2, \\ \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} - (\boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i) \to 0, i = p_2 + 1, \ldots, p, \\ \text{sub. to} \quad A\boldsymbol{x} \le \boldsymbol{b}. \end{array} \tag{17}$$

From (4), the possible range of the difference between a goal function and a target $\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} - (\boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i)$ is obtained as an L-fuzzy number $\Delta_i(\boldsymbol{x}) = (\delta_i(\boldsymbol{x}), \beta_i(\boldsymbol{x}))_L$ with definitions,

$$\delta_i(\boldsymbol{x}) = \boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{y}_i(\boldsymbol{x}), \tag{18}$$

$$\beta_i(\boldsymbol{x}) = \boldsymbol{\alpha}_i^{\mathrm{T}}|\boldsymbol{y}_i(\boldsymbol{x})|, \tag{19}$$

$$\boldsymbol{y}_i(\boldsymbol{x}) = D_i^{-1}(\boldsymbol{x}^{\mathrm{T}}, -\boldsymbol{x}^{\mathrm{T}}, -1)^{\mathrm{T}}, \tag{20}$$

where $\boldsymbol{s}_i = (s_{i1}, s_{i2}, \ldots, s_{i\ 2n+1})^{\mathrm{T}}$ and $\boldsymbol{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i\ 2n+1})^{\mathrm{T}}$.

3.2 Regret Minimization Model

As is done in goal programming literature, the decision-maker may be able to give a suitable weight vector so that his/her preference of the solution is represented by an aggregated regret function:

$$R(\boldsymbol{u}^+, \boldsymbol{u}^-) = \sum_{i=1}^{p_1} w_i u_i^- + \sum_{i=p_1+1}^{p_2} w_i u_i^+ \sum_{i=p_2+1}^{p} w_i(u_i^+ + u_i^-), \tag{21}$$

where $w_i \geq 0$, $i = 1, 2, \ldots, p$ are weights. $\boldsymbol{u}^+ = (u_1^+, u_2^+, \ldots, u_p^+)^{\mathrm{T}}$ and $\boldsymbol{u}^- = (u_1^-, u_2^-, \ldots, u_p^-)^{\mathrm{T}}$ are deviational variables such that

$$u_i^+ - u_i^- = \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} - (\boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i),\ u_i^+ \cdot u_i^- = 0. \tag{22}$$

Since $\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} - (\boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x} + q_i)$ has a possibility distribution $\mu_{\Delta_i(\boldsymbol{x})}$, so do u_i^+, u_i^- and $u_i^\pm = u_i^+ + u_i^-$. The possible ranges $U_i^+(\boldsymbol{x})$, $U_i^-(\boldsymbol{x})$ and $U_i^\pm$ of u_i^+, u_i^- and $u_i^\pm$ are defined by

$$U_i^+(\boldsymbol{x}) = \begin{cases} (\Delta_i(\boldsymbol{x}) \cap \mathbf{R}_+) \cup \{0\} & \text{if } \delta_i(\boldsymbol{x}) < 0, \\ \Delta_i(\boldsymbol{x}) \cap \mathbf{R}_+ & \text{if } \delta_i(\boldsymbol{x}) \geq 0, \end{cases} \tag{23}$$

$$U_i^-(\boldsymbol{x}) = \begin{cases} -\Delta_i(\boldsymbol{x}) \cap \mathbf{R}_+ & \text{if } \delta_i(\boldsymbol{x}) \leq 0, \\ (-\Delta_i(\boldsymbol{x}) \cap \mathbf{R}_+) \cup \{0\} & \text{if } \delta_i(\boldsymbol{x}) > 0, \end{cases} \tag{24}$$

$$U_i^\pm(\boldsymbol{x}) = (\Delta_i(\boldsymbol{x}) \cup -\Delta_i(\boldsymbol{x})) \cap \mathbf{R}_+, \tag{25}$$

where $\mathbf{R}_+ = \{r \mid r \geq 0\}$. It should be noted that u_i^- and u_i^+ for the same i are interactive each other, i.e., at least one of them should be zero. Thus, we have $U_i^\pm(\boldsymbol{x}) \neq U_i^+(\boldsymbol{x}) + U_i^-(\boldsymbol{x})$, where $+$ means the sum of non-interactive fuzzy numbers. However, by the assumption, $u_{i_1}^+$, $u_{i_2}^-$ and $u_{i_3}^\pm$ for $i_1 \neq i_2$, $i_2 \neq i_3$ and $i_3 \neq i_1$ are non-interactive.

The problem is formulated by a modality optimization model, more precisely a necessity measure maximization model. The necessity measure maximization model (Inuiguchi and Ramík 2000) can be seen as the generalization of a model based on max-min principle, a pessimistic criterion. The obtained solution has robustness against the fluctuation of ambiguous coefficients. We introduce a fuzzy goal G to $\bar{R}(\boldsymbol{u}^+, \boldsymbol{u}^-, \boldsymbol{u}^\pm)$. By the problem setting, the membership function μ_G of G should be non-increasing. Moreover, we assume that μ_G is upper semi-continuous. The objective function is formulated as

$$\text{maximize } N_{\bar{R}(\boldsymbol{x})}(G), \tag{26}$$

where

$$\bar{R}(\boldsymbol{x}) = \sum_{i=1}^{p_1} w_i U_i^-(\boldsymbol{x}) + \sum_{i=p_1+1}^{p_2} w_i U_i^+(\boldsymbol{x}) + \sum_{i=p_2+1}^{p} w_i U_i^\pm(\boldsymbol{x}). \tag{27}$$

We treat (26) as

$$\begin{aligned} &\text{maximize } h, \\ &\text{sub. to } N_{\bar{R}(\boldsymbol{x})}(G) \geq h,\ h \in [0, 1]. \end{aligned} \tag{28}$$

Let us define a pseudo-inverse $\mu_G^* : [0, 1] \to \mathbf{R}_+ \cup \{+\infty\}$ of μ_G by $\mu_G^*(h) = \sup\{r \mid \mu_G(r) \geq h\}$. Since μ_G is non-increasing and upper semi-continuous, we have $[G]_h = (-\infty, \mu_G^*(h)]$, $\forall h \in (0, 1]$. From (15), we should discuss the upper bound of $\mathrm{cl}(\bar{R}(U^+, U^-, U^\pm))_{1-h}$.

Define a pseudo-inverse $L^{\#} : [0, 1] \to \mathbf{R}_+ \cup \{-\infty\}$ of L by

$$L^{\#}(h) = \begin{cases} \sup\{r \mid L(r) > h\} & \text{if } h \in [0, 1), \\ -\infty & \text{if } h = 1. \end{cases} \tag{29}$$

Then $\mathrm{cl}(\Delta_i(\boldsymbol{x}))_h$ can be obtained as a closed interval $[\delta_i^{\mathrm{L}}(\boldsymbol{x} : h), \delta_i^{\mathrm{R}}(\boldsymbol{x} : h)]$ with

$$\delta_i^{\mathrm{L}}(\boldsymbol{x} : h) = \delta_i(\boldsymbol{x}) - L^{\#}(h)\beta_i(\boldsymbol{x}), \tag{30}$$

$$\delta_i^{\mathrm{R}}(\boldsymbol{x} : h) = \delta_i(\boldsymbol{x}) + L^{\#}(h)\beta_i(\boldsymbol{x}), \tag{31}$$

where $[+\infty, -\infty]$ is regarded as an empty set. Let $u_i^{\mathrm{R}+}(\boldsymbol{x} : h)$, $u_i^{\mathrm{R}-}(\boldsymbol{x} : h)$ and $u_i^{\mathrm{R}\pm}(\boldsymbol{x} : h)$ be the upper bounds of $\mathrm{cl}(U_i^+(\boldsymbol{x}))_h$, $\mathrm{cl}(U_i^-(\boldsymbol{x}))_h$ and $\mathrm{cl}(U_i^{\pm}(\boldsymbol{x}))_h$, respectively. By the definitions, we have $u_i^{\mathrm{R}+}(\boldsymbol{x} : h) = \max(\delta_i^{\mathrm{R}}(\boldsymbol{x} : h), 0)$, $u_i^{\mathrm{R}-}(\boldsymbol{x} : h) = \max(-\delta_i^{\mathrm{L}}(\boldsymbol{x} : h), 0)$, $u_i^{\mathrm{R}\pm}(\boldsymbol{x} : h) = |\delta_i(\boldsymbol{x})| + L^{\#}(h)\beta_i(\boldsymbol{x})$. Together with the fact that $\bar{R}$ is non-decreasing, the upper bound $r^{\mathrm{R}}(\boldsymbol{x} : h)$ of $\mathrm{cl}(\bar{R}(\boldsymbol{x}))_h$ is obtained as

$$r^{\mathrm{R}}(\boldsymbol{x} : h) = \sum_{i=1}^{p_1} w_i u_i^{\mathrm{R}-}(\boldsymbol{x} : h) + \sum_{i=p_1+1}^{p_2} w_i u_i^{\mathrm{R}+}(\boldsymbol{x} : h) + \sum_{i=p_2+1}^{p} w_i u_i^{\mathrm{R}\pm}(\boldsymbol{x} : h). \tag{32}$$

Hence, (28) is reduced to

$$\begin{aligned} &\text{maximize } h, \\ &\text{sub. to } r^{\mathrm{R}}(\boldsymbol{x} : 1 - h) \le \mu_G^*(h), \\ &\qquad 0 \le h \le 1. \end{aligned} \tag{33}$$

Let us divide $(2n + 1) \times (2n + 1)$ matrix D_i into two $n \times (2n + 1)$ matrices D_i^1 and D_i^2 and a $(2n + 1)$-dimensional vector $\boldsymbol{d}_i^3$ such that $D_i = (D_i^{1\mathrm{T}} \ \ D_i^{2\mathrm{T}} \ \ \boldsymbol{d}_i^3)^{\mathrm{T}}$. Introducing artificial variables, finally, Problem (17) is reduced to

$$\begin{aligned}
&\text{maximize } \ h, \\
&\text{sub. to} \\
&\quad \boldsymbol{s}_i^{\mathrm{T}}(\boldsymbol{y}_i^- - \boldsymbol{y}_i^+) + L^{\#}(1-h)\boldsymbol{\alpha}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-) \le v_i, \ i = 1, 2, \dots, p_1, \\
&\quad \boldsymbol{s}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) + L^{\#}(1-h)\boldsymbol{\alpha}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-) \le v_i, \ i = p_1 + 1, p_1 + 2, \dots, p_2, \\
&\quad \boldsymbol{s}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = v_i^+ - v_i^-, \ i = p_2 + 1, p_2 + 2, \dots, p, \\
&\quad \sum_{i=1}^{p_2} w_i v_i + \sum_{i=p_2+1}^{p} w_i \big(v_i^+ + v_i^- + L^{\#}(1-h)\boldsymbol{\alpha}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-)\big) \le \mu_G^*(h), \\
&\quad A D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-) \le \mathbf{b}, \\
&\quad D_i^1(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-), \ i = 2, 3, \dots, p, \\
&\quad D_i^2(\boldsymbol{y}_i^- - \boldsymbol{y}_i^+) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-), \ i = 1, 2, \dots, p, \\
&\quad \boldsymbol{d}_i^{3\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = 1, \ i = 1, 2, \dots, p, \\
&\quad 0 \le h \le 1, \ v_i \ge 0, \ i = 1, 2, \dots, p_2, \\
&\quad v_i^+, v_i^- \ge 0, \ i = p_2 + 1, \dots, p, \\
&\quad \boldsymbol{y}_i^+, \boldsymbol{y}_i^- \ge \mathbf{0}, \ i = 1, 2, \dots, p.
\end{aligned} \tag{34}$$

The complementary conditions $v_i^+ \cdot v_i^- = 0$, $i = p_2+1, \ldots, p$, $\boldsymbol{y}_i^{+\mathrm{T}} \boldsymbol{y}_i^- = 0$, $i = 1, 2, \ldots, p$ can be omitted by the same discussion as Theorem 10 of Inuiguchi et al. (2003). Namely, if a complementary condition is not satisfied, i.e., $z_i^+ \cdot z_i^- \neq 0$ ($\langle z_i^+, z_i^- \rangle$ is $\langle v_i^+, v_i^- \rangle$ or $\langle y_i^+, y_i^- \rangle$), we modify the solution by replacing z_i^+ and z_i^- with $\bar{z}_i^+ = \max(z_i^+ - z_i^-, 0)$ and $\bar{z}_i^- = \max(z_i^- - z_i^+, 0)$, respectively. By this modification, we obtain an optimal solution. The decision variable vector $\boldsymbol{x}$ is obtained as $\boldsymbol{x} = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-)$. Problem (34) can be solved by a bisection method with respect to h and a decomposed method for linear programs with a bordered angular structure.

Remark 1 Because we have $\boldsymbol{x} = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-)$, we may replace the set of constraints composed of $AD_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-) \leq \mathbf{b}$, $D_i^1(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-)$, $i = 2, 3, \ldots, p$ and $D_i^2(\boldsymbol{y}_i^- - \boldsymbol{y}_i^+) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-)$, $i = 1, 2, \ldots, p$, with a set of constraints composed of $A\boldsymbol{x} \leq \mathbf{b}$, $D_i^1(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = \boldsymbol{x}$, $i = 1, 2, \ldots, p$ and $D_i^2(\boldsymbol{y}_i^- - \boldsymbol{y}_i^+) = \boldsymbol{x}$, $i = 1, 2, \ldots, p$ in Problem (34) as well as Problems (35), (42), (50) and (51) described later.

Now, let us consider a special case where $p_1 = p_2 = 0$ and $G = (-\infty, \hat{g}]$, where $\hat{g} > 0$ is a constant. In this case, Problem (34) can be reduce to a linear fractional programming problem,

$$\begin{array}{ll}
\text{maximize} & \dfrac{\hat{g} - \sum_{i=1}^{p} w_i (v_i^+ + v_i^-)}{\sum_{i=1}^{p} w_i \boldsymbol{\alpha}_i^{\mathrm{T}} (\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-)}, \\
\text{sub. to} & \boldsymbol{s}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = v_i^+ - v_i^-, \ i = 1, 2, \ldots, p, \\
& AD_i^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-) \leq \mathbf{b}, \\
& D_i^1(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-), \ i = 2, 3, \ldots, p, \\
& D_i^2(\boldsymbol{y}_i^- - \boldsymbol{y}_i^+) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-), \ i = 1, 2, \ldots, p, \\
& \boldsymbol{d}_i^{3\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = 1, \ i = 1, 2, \ldots, p, \\
& v_i^+, v_i^- \geq 0, \ i = 1, 2, \ldots, p, \\
& \boldsymbol{y}_i^+, \boldsymbol{y}_i^- \geq \mathbf{0}, \ i = 1, 2, \ldots, p.
\end{array} \tag{35}$$

As is known in literature, this problem can be reduced to a linear programming problem with a bordered angular structure.

3.3 Formulation Without a Regret Function

We assume that, to each goal, the decision maker can specify a fuzzy goal G_i whose membership function value $\mu_{G_i}(r)$ shows his/her satisfaction degree, even if the decision maker cannot specify an aggregated regret function. By the nature of goals, μ_{G_i}, $i = 1, 2, \ldots, p_1$ are non-decreasing, μ_{G_i}, $i = p_1+1, \ldots, p_2$ non-increasing, and μ_{G_i}, $i = p_2 + 1, \ldots, p$ quasi-concave (unimodal). We also assume that μ_{G_i}'s are upper semi-continuous.

Applying a modality optimization model, we formulate Problem (17) as

$$\begin{aligned} \text{maximize} \ & \min_{i=1,2,\ldots,p} \psi_i(h_i), \\ \text{sub. to} \ & N_{\Delta_i}(G_i) \geq h_i, \ i = 1, 2, \ldots, p, \\ & A\boldsymbol{x} \leq \boldsymbol{b}, \\ & 0 \leq h_i \leq 1, \ i = 1, 2, \ldots, p, \end{aligned} \tag{36}$$

where $\psi_i : [0, 1] \to [0, 1]$ is upper semi-continuous and non-decreasing. Whereas μ_{G_i} is specified considering the ith deviation only, ψ_i is specified considering the relative importance among goals.

In order to reduce Problem (36) to the conventional mathematical programming problem, let us discuss equivalent conditions of $N_{\Delta_i}(G_i) \geq h_i, i = 1, 2, \ldots, p$. Since μ_{G_i} is upper semi-continuous, $[G_i]_h$ is closed. In view of (15), we should discuss $\mathrm{cl}(\Delta_i(\boldsymbol{x}))_h$. $\Delta_i(\boldsymbol{x})$ is an L-fuzzy number $(\delta_i(\boldsymbol{x}), \beta_i(\boldsymbol{x}))_L$. Thus, we have

$$\mathrm{cl}(\Delta_i(\boldsymbol{x}))_h = \left[\delta_i(\boldsymbol{x}) - L^{\#}(h)\beta_i(\boldsymbol{x}),\ \delta_i(\boldsymbol{x}) + L^{\#}(h)\beta_i(\boldsymbol{x})\right]. \tag{37}$$

Let $\mu^*_{G_i} : [0, 1] \to \mathbf{R} \cup \{+\infty\}$ and $\mu_{G_i*} : [0, 1] \to \mathbf{R} \cup \{-\infty\}$ are pseudo-inverses of μ_{G_i} defined by

$$\mu^*_{G_i}(h) = \sup\{r \mid \mu_{G_i}(r) \geq h\}, \tag{38}$$

$$\mu_{G_i*}(h) = \inf\{r \mid \mu_{G_i}(r) \geq h\}. \tag{39}$$

We have

$$[G_i]_h = \begin{cases} [\mu_{G_i*}(h), +\infty) & \text{if } i \leq p_1, \\ (-\infty, \mu^*_{G_i}(h)] & \text{if } p_1 < i \leq p_2, \\ [\mu_{G_i*}(h), \mu^*_{G_i}(h)] & \text{if } p_2 < i \leq p. \end{cases} \tag{40}$$

Hence, we have

$$N_{\Delta_i(\boldsymbol{x})}(G_i) \geq h \ \Leftrightarrow \begin{cases} \delta_i(\boldsymbol{x}) - L^{\#}(1-h)\beta_i(\boldsymbol{x}) \geq \mu_{G_i*}(h), \ \text{if } i \leq p_1, \\ \delta_i(\boldsymbol{x}) + L^{\#}(1-h)\beta_i(\boldsymbol{x}) \leq \mu^*_{G_i}(h), \ \text{if } p_1 < i \leq p_2, \\ \begin{pmatrix} \delta_i(\boldsymbol{x}) - L^{\#}(1-h)\beta_i(\boldsymbol{x}) \geq \mu_{G_i*}(h), \\ \delta_i(\boldsymbol{x}) + L^{\#}(1-h)\beta_i(\boldsymbol{x}) \leq \mu^*_{G_i}(h) \end{pmatrix}, \ \text{if } p_2 < i \leq p, \end{cases} \tag{41}$$

As the result, (36) is reduced to

$$
\begin{aligned}
&\text{maximize } h,\\
&\quad \text{sub. to } \boldsymbol{s}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) - L^{\#}(1-\psi_i^*(h))\boldsymbol{\alpha}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-) \geq \mu_{G_i*}(\psi_i^*(h)),\\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, p_1, p_2+1, p_2+2, \ldots, p,\\
&\qquad\quad \boldsymbol{s}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) + L^{\#}(1-\psi_i^*(h))\boldsymbol{\alpha}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-) \leq \mu_{G_i}^*(\psi_i^*(h)),\\
&\qquad\qquad\qquad\qquad\qquad\qquad i = p_1+1, p_1+2, \ldots, p,\\
&\qquad\quad AD_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-) \leq \mathbf{b},\\
&\qquad\quad D_i^1(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-),\ i = 2, 3, \ldots, p,\\
&\qquad\quad D_i^2(\boldsymbol{y}_i^- - \boldsymbol{y}_i^+) = D_1^1(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-),\ i = 1, 2, \ldots, p,\\
&\qquad\quad \boldsymbol{d}_i^{3\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = 1,\ i = 1, 2, \ldots, p,\\
&\qquad\quad 0 \leq h \leq 1,\ \boldsymbol{y}_i^+, \boldsymbol{y}_i^- \geq \mathbf{0},\ i = 1, 2, \ldots, p,
\end{aligned}
\tag{42}
$$

where ψ_i^* is a pseudo-inverse of ψ defined by

$$
\psi_i^*(h) = \inf\{r \in [0, 1] \mid \psi(r) \geq h\}. \tag{43}
$$

Problem (42) can be solved by a bisection method on h and a decomposition method for linear programs with a dual block angular structure (see Inuiguchi et al. 2003).

Example 1 Let us consider a small factory with three workers W1, W2 and W3. They produce three products Q1, Q2 and Q3. To produce a unit of Q1, it requires effort c_1 evaluated about 2 units. To produce a unit of Q2, it requires effort c_2 evaluated about 3 units. Finally, to produce a unit of Q3, it requires effort c_3 evaluated 2.5 units. Those efforts are individually expressed by symmetric triangular fuzzy numbers $(2, 1)_L$, $(3, 1.5)_L$ and $(2.5, 0.7)_L$, respectively, with $L(r) = 1 - |r|$. However, those efforts are interactive and the effort vector $\boldsymbol{c} = (c_1, c_2, c_3)^{\mathrm{T}}$ can be expressed by an oblique fuzzy vector. The following obliquity matrix D is obtained from the PCA application to stored data about $\boldsymbol{c} = (c_1, c_2, c_3)^{\mathrm{T}}$:

$$
D = \begin{pmatrix} 0.53 & 0.869 & 0.001 \\ 0.97 & -0.24 & -0.132 \\ 0.968 & -0.236 & 0.133 \end{pmatrix}. \tag{44}
$$

Then, the non-interactive fuzzy numbers associated with C are obtained as $S_1 = (6.39, 2.6626)_L$, $S_2 = (0.428, 1.3942)_L$ and $S_3 = (-0.0615, 0.2921)_L$.

The acquired work abilities of the workers are different. W1 can produce Q1 and Q2 but cannot produce Q3. W2 can produce Q2 and Q3 but cannot produce Q1. Finally, W3 can produce Q1 and Q3 but cannot produce Q2. They should produce 60 units of Q1, 40 units of Q2 and 50 units of 50. Under this circumstance, they would like to assign the jobs producing Q1, Q2 and Q3 so as to minimize the differences of efforts among three works.

Let x_{ij} be the amount of Qj assigned to worker Wi. Because of the work ability, we fix $x_{13} = x_{21} = x_{32} = 0$. We formulate this problem as the following MGP problem:

Table 1 The reduced problem of Example 1

$$
\begin{array}{l}
\text{maximize} \quad h,\\
\text{sub. to}\\
\quad (6.39-2.6626h)y_{11}^{+}-(6.39+2.6626h)y_{11}^{-}+(0.428-1.3942h)y_{12}^{+}\\
\quad\ -(0.428+1.3942h)y_{12}^{-}-(0.0615+0.2921h)y_{13}^{+}+(0.0615-0.2921h)y_{13}^{-}\geq -30+30h,\\
\quad (6.39-2.6626h)y_{21}^{+}-(6.39+2.6626h)y_{21}^{-}+(0.428-1.3942h)y_{22}^{+}\\
\quad\ -(0.428+1.3942h)y_{22}^{-}-(0.0615+0.2921hy_{23}^{+}+(0.0615-0.2921h)y_{23}^{-}\geq -30+30h,\\
\quad (6.39-2.6626h)y_{31}^{+}-(6.39+2.6626h)y_{31}^{-}+(0.428-1.3942h)y_{32}^{+}\\
\quad\ -(0.428+1.3942h)y_{32}^{-}-(0.0615+0.2921hy_{33}^{+}+(0.0615-0.2921h)y_{33}^{-}\geq -30+30h,\\
\quad (6.39+2.6626h)y_{11}^{+}-(6.39-2.6626h)y_{11}^{-}+(0.428+1.3942h)y_{12}^{+}\\
\quad\ -(0.428-1.3942h)y_{12}^{-}-(0.0615-0.2921h)y_{13}^{+}+(0.0615+0.2921h)y_{13}^{-}\leq 30-30h,\\
\quad (6.39+2.6626h)y_{21}^{+}-(6.39-2.6626h)y_{21}^{-}+(0.428+1.3942h)y_{22}^{+}\\
\quad\ -(0.428-1.3942h)y_{22}^{-}-(0.0615-0.2921h)y_{23}^{+}+(0.0615+0.2921h)y_{23}^{-}\leq 30-30h,\\
\quad (6.39+2.6626h)y_{31}^{+}-(6.39-2.6626h)y_{31}^{-}+(0.428+1.3942h)y_{32}^{+}\\
\quad\ -(0.428-1.3942h)y_{32}^{-}-(0.0615-0.2921h)y_{33}^{+}+(0.0615+0.2921h)y_{33}^{-}\leq 30-30h,\\
\quad x_{11}+x_{31}=60,\ x_{12}+x_{22}=40,\ x_{23}+x_{33}=50,\ x_{11},x_{12},x_{22},x_{23},x_{31},x_{33}\geq 0,\\
\quad x_{11}-0.53y_{11}^{+}+0.53y_{11}^{-}-0.869y_{12}^{+}+0.869y_{12}^{-}-0.001y_{13}^{+}+0.001y_{13}^{-}=0,\\
\quad x_{12}-x_{22}-0.97y_{11}^{+}+0.97y_{11}^{-}+0.24y_{12}^{+}-0.24y_{12}^{-}+0.132y_{13}^{+}-0.132y_{13}^{-}=0,\\
\quad -x_{23}-0.968y_{11}^{+}+0.968y_{11}^{-}+0.236y_{12}^{+}-0.236y_{12}^{-}-0.133y_{13}^{+}+0.133y_{13}^{-}=0,\\
\quad x_{11}-x_{31}-0.53y_{21}^{+}+0.53y_{21}^{-}-0.869y_{22}^{+}+0.869y_{22}^{-}-0.001y_{23}^{+}+0.001y_{23}^{-}=0,\\
\quad x_{12}-0.97y_{21}^{+}+0.97y_{21}^{-}+0.24y_{22}^{+}-0.24y_{22}^{-}+0.132y_{23}^{+}-0.132y_{23}^{-}=0,\\
\quad -x_{33}-0.968y_{21}^{+}+0.968y_{21}^{-}+0.236y_{22}^{+}-0.236y_{22}^{-}-0.133y_{23}^{+}+0.133y_{23}^{-}=0,\\
\quad -x_{31}-0.53y_{31}^{+}+0.53y_{31}^{-}-0.869y_{32}^{+}+0.869y_{32}^{-}-0.001y_{33}^{+}+0.001y_{33}^{-}=0,\\
\quad x_{22}-0.97y_{31}^{+}+0.97y_{31}^{-}+0.24y_{32}^{+}-0.24y_{32}^{-}+0.132y_{33}^{+}-0.132y_{33}^{-}=0,\\
\quad x_{23}-x_{33}-0.968y_{31}^{+}+0.968y_{31}^{-}+0.236y_{32}^{+}-0.236y_{32}^{-}-0.133y_{33}^{+}+0.133y_{33}^{-}=0,\\
\quad y_{11}^{+},y_{11}^{-},y_{12}^{+},y_{12}^{-},y_{13}^{+},y_{13}^{-},y_{21}^{+},y_{21}^{-},y_{22}^{+},y_{22}^{-},y_{23}^{+},y_{23}^{-},y_{31}^{+},y_{31}^{-},y_{32}^{+},y_{32}^{-},y_{33}^{+},y_{33}^{-},\geq 0.
\end{array}
$$

$$
\begin{array}{l}
c_1x_{11}+c_2x_{12}-(c_2x_{22}+c_3x_{23}) \overset{=}{\rightarrow} 0, \text{(Difference between W1 and W2)}\\
c_1x_{11}+c_2x_{12}-(c_1x_{31}+c_3x_{33}) \overset{=}{\rightarrow} 0, \text{(Difference between W1 and W3)}\\
c_2x_{22}+c_3x_{23}-(c_1x_{31}+c_3x_{33}) \overset{=}{\rightarrow} 0, \text{(Difference between W2 and W3)}\\
\text{sub. to} \quad x_{11}+x_{31}=60,\ x_{12}+x_{22}=40,\ x_{23}+x_{33}=50.
\end{array}
\tag{45}
$$

Giving a fuzzy goal G showing the satisfactory difference of efforts between any two workers, which is common in three goals of (45), we apply the formulation without regret function to Problem (45). The fuzzy goal G is defined by the following membership function:

$$
\mu_G(r)=\max\left(\frac{|r|}{30},0\right). \tag{46}
$$

Because each goal should be treated equally, we define $\psi_i(r)=r$. Then, applying Problem (42), we obtain the reduced problem shown in Table 1. In the reduced problem in Table 1, we use decision variables x_{11}, x_{12}, x_{22}, x_{23}, x_{31} and x_{33} corresponding to $\boldsymbol{x}$ based on Remark 1. Solving the problem shown in Table 1 by a bisection method with respect to h and a simplex method, we obtain the solution shown in Table 2. The solution is illustrated in Fig. 3. As shown in Fig. 3, we confirm the obtained solution satisfies the constraints. In this solution, the center values of effort differences

Table 2 Solution to the problem of Table 1

Objective function value: $h = 0.262557$

$x_{11} = 37.102938$	$y_{11}^{+} = 0$	$y_{21}^{+} = 0$	$y_{31}^{+} = 0.512846$
$x_{12} = 15.820264$	$y_{11}^{-} = 3.419658$	$y_{21}^{-} = 2.906812$	$y_{31}^{-} = 0$
$x_{22} = 24.179736$	$y_{12}^{+} = 44.831610$	$y_{22}^{+} = 18.321069$	$y_{32}^{+} = 0$
$x_{23} = 19.650983$	$y_{12}^{-} = 0$	$y_{22}^{-} = 0$	$y_{32}^{-} = 26.510541$
$x_{31} = 22.897062$	$y_{13}^{+} = 0$	$y_{23}^{+} = 0$	$y_{33}^{+} = 0$
$x_{33} = 30.349017$	$y_{13}^{-} = 43.311984$	$y_{23}^{-} = 174.522183$	$y_{33}^{-} = 131.210198$

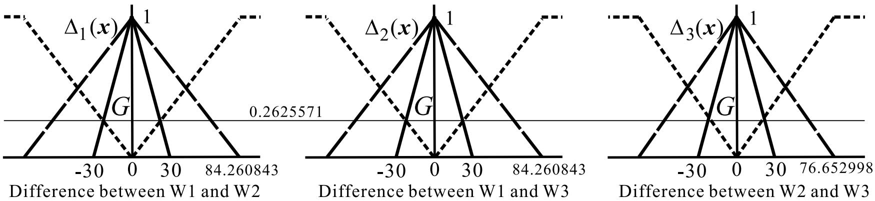

Fig. 3 Illustration of the solution of Example 1

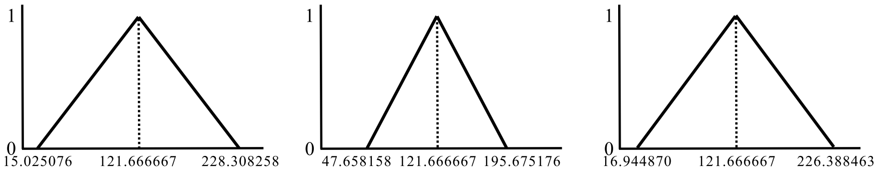

Fig. 4 Effort distributions of W1, W2 and W3 (Example 1)

between two workers are zeroes. The distribution of the effort difference between W2 and W3 is narrower than the other effort differences while the other two are same. The effort distribution of the workers with respect to the obtained solution is depicted in Fig. 4.

4 MGP with Target Distributions

Considering that the distribution of goal function values $f(\boldsymbol{x} : \boldsymbol{c}_i) = \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x}$ is obtained when coefficients $\boldsymbol{c}_i$ are ambiguous, we may have a target distribution of goal function values. Then, in this section, we treat the following goal programming problem with ambiguous coefficients and target distributions:

$$\begin{aligned} & f(\boldsymbol{x}:\boldsymbol{c}_i)=\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} \stackrel{+}{\Rightarrow} Q_i,\ i=1,2,\ldots,p_1,\\ & f(\boldsymbol{x}:\boldsymbol{c}_i)=\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} \stackrel{-}{\Rightarrow} Q_i,\ i=p_1+1,\ldots,p_2,\\ & f(\boldsymbol{x}:\boldsymbol{c}_i)=\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{x} \Rightarrow Q_i,\ i=p_2+1,\ldots,p,\\ & \text{sub. to } A\boldsymbol{x}\le\boldsymbol{b}, \end{aligned} \tag{47}$$

where $\boldsymbol{x}, \boldsymbol{b}, A$ are the same as those in Problem (16). $\boldsymbol{c}_i$ is a possibilistic variable vector whose possible range is represented by an n-dimensional OFV $\hat{C}_i$ with an obliquity matrix $\hat{D}_i$ and non-interactive fuzzy numbers $\hat{S}_{ij}=(\hat{s}_{ij},\hat{\alpha}_{ij})_L$, $j=1,2,\ldots,n$. $\boldsymbol{c}_i$, $i=1,2,\ldots,p$ are non-interactive one another. Q_i is an L-fuzzy number $(\hat{q}_i,\gamma_i)_L$. Notation $f \stackrel{+}{\Rightarrow} Q$ shows a goal that the decision maker prefers the possible range F of the left-hand side value f greater than the right-hand side fuzzy set Q if it is possible, and otherwise he/she prefers F close to Q. Similarly, $f \stackrel{-}{\Rightarrow} Q$ shows a goal that the decision maker prefers the possible range F of f less than Q if it is possible, and otherwise he/she prefers F close to Q. Moreover, $f \Rightarrow Q$ shows a goal that the decision maker prefers F close to Q. We define that F is not smaller than G if and only if $\max(F,G)=F$ and that F is not larger than G if and only if $\min(F,G)=F$, where $\max(F,G)$ and $\min(F,G)$ are those obtained by the extension principle. It is different from Problem (16) that the targets are fuzzy sets in Problem (47) and then we consider the possible range $f(\boldsymbol{x}:\hat{C}_i)$ instead of each goal function value $f(\boldsymbol{x}:\boldsymbol{c}_i)$. We note that we may have a case target distribution Q_i is variable. For example, when we want to make the distributions of $f(\boldsymbol{x}:\boldsymbol{c}_i)$ and $f(\boldsymbol{x}:\boldsymbol{c}_j)$ close, we set $Q_i=Q_j$ with a variable target distribution Q_i.

Problem (47) can be interpreted as the following goal programming with set-valued goal functions and zero targets:

$$\begin{aligned} & f(\boldsymbol{x}:\hat{C}_i) \mathbin{\tilde{-}} Q_i \stackrel{+}{\Rightarrow} \{0\},\ i=1,2,\ldots,p_1,\\ & f(\boldsymbol{x}:\hat{C}_i) \mathbin{\tilde{-}} Q_i \stackrel{-}{\Rightarrow} \{0\},\ i=p_1+1,\ldots,p_2,\\ & f(\boldsymbol{x}:\hat{C}_i) \mathbin{\tilde{-}} Q_i \Rightarrow \{0\},\ i=p_2+1,\ldots,p,\\ & \text{sub. to } A\boldsymbol{x}\le\boldsymbol{b}. \end{aligned} \tag{48}$$

From (4) and (11), $f(\boldsymbol{x}:C_i) \mathbin{\tilde{-}} Q_i$ becomes an L-fuzzy number $\hat{\Delta}_i(\boldsymbol{x})=(\hat{\delta}_i(\boldsymbol{x}),\hat{\beta}_i(\boldsymbol{x}))_L$, where $\hat{\delta}_i(\boldsymbol{x})$ and $\hat{\beta}_i(\boldsymbol{x})$ are defined as $\hat{\delta}_i(\boldsymbol{x})=\hat{\boldsymbol{s}}_i^{\mathrm{T}}\hat{\boldsymbol{y}}_i(\boldsymbol{x})-\hat{q}_i$, $\hat{\beta}_i(\boldsymbol{x})=\left|\hat{\boldsymbol{\alpha}}_i^{\mathrm{T}}|\boldsymbol{y}_i(\boldsymbol{x})|-\gamma_i\right|$ and $\hat{\boldsymbol{y}}(\boldsymbol{x})=\hat{D}_i^{-1}\boldsymbol{x}$, where $\hat{\boldsymbol{s}}_i=(\hat{s}_{i1},\hat{s}_{i2},\ldots,\hat{s}_{in})^{\mathrm{T}}$ and $\hat{\boldsymbol{\alpha}}_i=(\hat{\alpha}_{i1},\hat{\alpha}_{i2},\ldots,\hat{\alpha}_{i\,n})^{\mathrm{T}}$.

We regard Problem (48) as

$$\begin{aligned} & \zeta_i(\boldsymbol{x}) \stackrel{+}{\to} 0,\ i=1,2,\ldots,p_1,\\ & \zeta_i(\boldsymbol{x}) \stackrel{-}{\to} 0,\ i=p_1+1,\ldots,p_2,\\ & \zeta_i(\boldsymbol{x}) \to 0,\ i=p_2+1,\ldots,p,\\ & \text{sub. to } A\boldsymbol{x}\le\boldsymbol{b}, \end{aligned} \tag{49}$$

where $\zeta_i(\boldsymbol{x})$ is a possibilistic variable whose possible range is $\hat{\Delta}_i(\boldsymbol{x})$.

4.1 Regret Minimization Model

By the same discussion as Problem (17), the modality optimization model corresponding to (26) is reduced to

$$
\begin{aligned}
&\text{maximize } h,\\
&\text{sub. to}\\
&\hat{\mathbf{s}}_i^{\mathrm{T}}(\boldsymbol{y}_i^- - \boldsymbol{y}_i^+) + \hat{q}_i + L^{\#}(1-h)(\xi_i^+ + \xi_i^-) \le v_i,\ i = 1, 2, \ldots, p_1,\\
&\hat{\mathbf{s}}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) - \hat{q}_i + L^{\#}(1-h)(\xi_i^+ + \xi_i^-) \le v_i,\ i = p_1+1, p_1+2, \ldots, p_2,\\
&\hat{\mathbf{s}}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) - \hat{q}_i = v_i^+ - v_i^-,\ i = p_2+1, p_2+2, \ldots, p,\\
&\hat{\boldsymbol{\alpha}}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-) - \gamma_i = \xi_i^+ - \xi_i^-,\ i = 1, 2, \ldots, p,\\
&\sum_{i=1}^{p_2} w_i v_i + \sum_{i=p_2+1}^{p} w_i \left(v_i^+ + v_i^- + L^{\#}(1-h)(\xi_i^+ + \xi^-)\right) \le \mu_G^*(h),\\
&A\hat{D}_i(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-) \le \mathbf{b},\\
&\hat{D}_i(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = \hat{D}_i(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-),\ i = 2, 3, \ldots, p,\\
&0 \le h \le 1,\ v_i \ge 0,\ i = 1, 2, \ldots, p_2,\\
&v_i^+, v_i^- \ge 0,\ i = p_2+1, \ldots, p,\\
&\xi_i^+, \xi_i^- \ge 0,\ i = 1, 2, \ldots, p,\\
&\boldsymbol{y}_i^+, \boldsymbol{y}_i^- \ge \mathbf{0},\ {\boldsymbol{y}_i^+}^{\mathrm{T}}\boldsymbol{y}_i^- = 0,\ i = 1, 2, \ldots, p.
\end{aligned}
\tag{50}
$$

We note that Problem (50) includes complementary conditions ${\boldsymbol{y}_i^+}^{\mathrm{T}}\boldsymbol{y}_i^- = 0$, $i = 1, 2, \ldots, p$. This is because Theorem 10 of Inuiguchi et al. (2003) cannot be applied and the satisfaction of complementary conditions ${\boldsymbol{y}_i^+}^{\mathrm{T}}\boldsymbol{y}_i^- = 0$, $i = 1, 2, \ldots, p$ is not guaranteed. However, fortunately, with a modification, this problem can also be solved by a bisection method on h and a decomposition method for linear programs with a dual block angular structure. The modification is applied when the existence of a feasible solution is examined with a fixed h by a simplex method. To the problem checking the existence of a feasible solution with a fixed h, we erase the complementary conditions ${\boldsymbol{y}_i^+}^{\mathrm{T}}\boldsymbol{y}_i^- = 0$, $i = 1, 2, \ldots, p$ but add an objective function $\sum_{i=1}^{p} \bar{w}_i \mathbf{e}^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-)$, where $\mathbf{e} = (1, 1, \ldots, 1)^{\mathrm{T}}$ and $\bar{w}_i$, $i = 1, 2, \ldots, p$ are weights controlled by the bisection algorithm. At the beginning, we set $\bar{w}_i = 1$, $i = 1, 2, \ldots, p$. If the solution minimizing $\sum_{i=1}^{p} \bar{w}_i \mathbf{e}^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-)$ does not satisfy ${\boldsymbol{y}_{\bar{i}}^+}^{\mathrm{T}}\boldsymbol{y}_{\bar{i}}^- = 0$ for some $\bar{i}$, we increase $\bar{w}_{\bar{i}} = \bar{w}_{\bar{i}} + 100$ and solve again the linear programming problem with minimizing $\sum_{i=1}^{p} \bar{w}_i \mathbf{e}^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-)$ with same fixed level h. This update of $\bar{w}_i$'s continues until all complementary conditions ${\boldsymbol{y}_i^+}^{\mathrm{T}}\boldsymbol{y}_i^- = 0$, $i = 1, 2, \ldots, p$ are satisfied or some $\bar{w}_i$ exceeds a given threshold (we set 201 in Example 2 described later). In the former case, we found a feasible solution at the fixed level h and we increase h, but in the latter case, we consider that there is no feasible solution at the fixed level h and we decrease h.

4.2 Formulation Without a Regret Function

Applying the modality optimization model (36), Problem (48) is reduced to

$$
\begin{array}{ll}
\text{maximize} & \min_{i=1,2,\ldots,p} \psi_i(h_i), \\
\text{sub. to} & \hat{\mathbf{s}}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) - \hat{q}_i \ - L^{\#}(1-h_i)(\xi_i^+ + \xi_i^-) \geq \mu_{G_i*}(h_i), \\
& \qquad i = 1,2,\ldots,p_1, p_2+1, p_2+2, \ldots, p, \\
& \hat{\mathbf{s}}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) - \hat{q}_i \ + L^{\#}(1-h_i)(\xi_i^+ + \xi_i^-) \leq \mu_{G_i}^*(h_i), \\
& \qquad i = p_1+1, p_1+2, \ldots, p, \\
& \hat{\boldsymbol{\alpha}}_i^{\mathrm{T}}(\boldsymbol{y}_i^+ + \boldsymbol{y}_i^-) - \gamma_i = \xi_i^+ - \xi_i^-, \ i = 1,2,\ldots,p, \\
& A\hat{D}_i(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-) \leq \mathbf{b}, \\
& \hat{D}_i(\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = \hat{D}_i(\boldsymbol{y}_1^+ - \boldsymbol{y}_1^-), \ i = 2,3,\ldots,p, \\
& 0 \leq h_i \leq 1, \ i = 1,2,\ldots,p, \\
& \xi_i^+, \xi_i^- \geq 0, \ i = 1,2,\ldots,p, \\
& \boldsymbol{y}_i^+, \boldsymbol{y}_i^- \geq \mathbf{0}, \ {\boldsymbol{y}_i^+}^{\mathrm{T}} \boldsymbol{y}_i^- = 0, \ i = 1,2,\ldots,p.
\end{array}
\tag{51}
$$

This problem includes complementary conditions ${\boldsymbol{y}_i^+}^{\mathrm{T}}\boldsymbol{y}_i^- = 0$, $i = 1,2,\ldots,p$ but can be solved by a bisection method on the objective function value with the modification against the non-satisfaction of complementary conditions (described in the previous subsection) and a decomposition method for linear programs with a dual block angular structure.

Example 2 Let us consider a small factory under the same situation as Example 1. In Example 1, we minimized the differences among efforts of workers W1, W2 and W3. In this example, we minimize the differences among effort distributions of workers W1, W2 and W3. Namely, the problem is formulated as

$$
\begin{array}{l}
C_1x_{11} + C_2x_{12} \tilde{-} Q \Rrightarrow \{0\}, \text{(Distribution difference between W1 and W2)} \\
C_2x_{22} + C_2x_{23} \tilde{-} Q \Rrightarrow \{0\}, \text{(Distribution difference between W1 and W3)} \\
C_1x_{31} + C_3x_{33} \tilde{-} Q \Rrightarrow \{0\}, \text{(Distribution difference between W2 and W3)} \\
\text{sub. to} \ \ x_{11} + x_{31} = 60, \ x_{12} + x_{22} = 40, \ x_{23} + x_{33} = 50,
\end{array}
\tag{52}
$$

where $Q = (q,\gamma)_L$ and $L(r) = 1 - |r|$. We apply the formulation without regret function. We use the same fuzzy goal G defined by a membership function (46). Applying Problem (51), we obtain the reduced problem shown in Table 3. In the reduced problem in Table 3, we use decision variables x_{11}, x_{12}, x_{22}, x_{23}, x_{31} and x_{33} corresponding to $\boldsymbol{x}$ based on Remark 1. Solving the problem shown in Table 3 by a bisection method with respect to h and a simplex method with minimizing $\sum_{i=1}^{3} \bar{w}_i \sum_{j=1}^{3}(y_{ij}^+ + y_{ij}^-)$ for satisfaction of complementary conditions, we obtain the solution shown in Table 4. From Table 4, we know that the target distribution is obtained as $Q = (121.411651, 87.797881)_L$. The solution is illustrated in Fig. 5 by showing $C_1x_{11} + C_2x_{12} \tilde{-} Q$, $C_2x_{22} + C_3x_{22} \tilde{-} Q$ and $C_1x_{31} + C_3x_{33} \tilde{-} Q$. As shown in Fig. 5, we confirm the obtained solution satisfies the constraints. In this solution, the center values of differences of effort distributions of workers from the

Table 3 The reduced problem of Example 2

$$
\begin{aligned}
&\text{maximize} \quad h,\\
&\text{sub. to}\\
&6.39y_{11}^{+}-6.39y_{11}^{-}+0.428y_{12}^{+}-0.428y_{12}^{-}-0.0615y_{13}^{+}+0.0615y_{13}^{-}\\
&\qquad -h\xi_1^{+}-h\xi_1^{-}-q^{+}+q^{-}\geq -30+30h,\\
&6.39y_{21}^{+}-6.39y_{21}^{-}+0.428y_{22}^{+}-0.428y_{22}^{-}-0.0615y_{23}^{+}+0.0615y_{23}^{-}\\
&\qquad -h\xi_2^{+}-h\xi_2^{-}-q^{+}+q^{-}\geq -30+30h,\\
&6.39y_{31}^{+}-6.39y_{31}^{-}+0.428y_{32}^{+}-0.428y_{32}^{-}-0.0615y_{33}^{+}+0.0615y_{33}^{-}\\
&\qquad -h\xi_3^{+}-h\xi_3^{-}-q^{+}+q^{-}\geq -30+30h,\\
&6.39y_{11}^{+}-6.39y_{11}^{-}+0.428y_{12}^{+}-0.428y_{12}^{-}-0.0615y_{13}^{+}+0.0615y_{13}^{-}\\
&\qquad +h\xi_1^{+}+h\xi_1^{-}-q^{+}+q^{-}\leq 30-30h,\\
&6.39y_{21}^{+}-6.39y_{21}^{-}+0.428y_{22}^{+}-0.428y_{22}^{-}-0.0615y_{23}^{+}+0.0615y_{23}^{-}\\
&\qquad +h\xi_2^{+}+h\xi_2^{-}-q^{+}+q^{-}\leq 30-30h,\\
&6.39y_{31}^{+}-6.39y_{31}^{-}+0.428y_{32}^{+}-0.428y_{32}^{-}+0.0615y_{33}^{+}+0.0615y_{33}^{-}\\
&\qquad +h\xi_3^{+}+h\xi_3^{-}-q^{+}+q^{-}\leq 30-30h,\\
&2.6626y_{11}^{+}+2.6626y_{11}^{-}+1.3942y_{12}^{+}+1.3942y_{12}^{-}+0.2921y_{13}^{+}+0.2921y_{13}^{-}-\xi_1^{+}+\xi_1^{-}-\gamma=0,\\
&2.6626y_{21}^{+}+2.6626y_{21}^{-}+1.3942y_{22}^{+}+1.3942y_{22}^{-}+0.2921y_{23}^{+}+0.2921y_{23}^{-}-\xi_2^{+}+\xi_2^{-}-\gamma=0,\\
&2.6626y_{31}^{+}+2.6626y_{31}^{-}+1.3942y_{32}^{+}+1.3942y_{32}^{-}+0.2921y_{33}^{+}+0.2921y_{33}^{-}-\xi_3^{+}+\xi_3^{-}-\gamma=0,\\
&x_{11}+x_{31}=60,\ x_{12}+x_{22}=40,\ x_{23}+x_{33}=50,\ x_{11},x_{12},x_{22},x_{23},x_{31},x_{33}\geq 0,\\
&x_{11}-0.53y_{11}^{+}+0.53y_{11}^{-}-0.869y_{12}^{+}+0.869y_{12}^{-}-0.001y_{13}^{+}+0.001y_{13}^{-}=0,\\
&x_{12}-0.97y_{11}^{+}+0.97y_{11}^{-}+0.24y_{12}^{+}-0.24y_{12}^{-}+0.132y_{13}^{+}-0.132y_{13}^{-}=0,\\
&-0.968y_{11}^{+}+0.968y_{11}^{-}+0.236y_{12}^{+}-0.236y_{12}^{-}-0.133y_{13}^{+}+0.133y_{13}^{-}=0,\\
&-0.53y_{21}^{+}+0.53y_{21}^{-}-0.869y_{22}^{+}+0.869y_{22}^{-}-0.001y_{23}^{+}+0.001y_{23}^{-}=0,\\
&x_{22}-0.97y_{21}^{+}+0.97y_{21}^{-}+0.24y_{22}^{+}-0.24y_{22}^{-}+0.132y_{23}^{+}-0.132y_{23}^{-}=0,\\
&x_{23}-0.968y_{21}^{+}+0.968y_{21}^{-}+0.236y_{22}^{+}-0.236y_{22}^{-}-0.133y_{23}^{+}+0.133y_{23}^{-}=0,\\
&x_{31}-0.53y_{31}^{+}+0.53y_{31}^{-}-0.869y_{32}^{+}+0.869y_{32}^{-}-0.001y_{33}^{+}+0.001y_{33}^{-}=0,\\
&-0.97y_{31}^{+}+0.97y_{31}^{-}+0.24y_{32}^{+}-0.24y_{32}^{-}+0.132y_{33}^{+}-0.132y_{33}^{-}=0,\\
&x_{33}-0.968y_{31}^{+}+0.968y_{31}^{-}+0.236y_{32}^{+}-0.236y_{32}^{-}-0.133y_{33}^{+}+0.133y_{33}^{-}=0,\\
&y_{11}^{+},y_{11}^{-},y_{12}^{+},y_{12}^{-},y_{13}^{+},y_{13}^{-},y_{21}^{+},y_{21}^{-},y_{22}^{+},y_{22}^{-},y_{23}^{+},y_{23}^{-},y_{31}^{+},y_{31}^{-},y_{32}^{+},y_{32}^{-},y_{33}^{+},y_{33}^{-},\geq 0,\\
&\xi_1^{+},\xi_1^{-},\xi_2^{+},\xi_2^{-},\xi_3^{+},\xi_3^{-},q^{+},q^{-},\gamma\geq 0,\ y_{11}^{+}\cdot y_{11}^{-}=0,\ y_{12}^{+}\cdot y_{12}^{-}=0,\ y_{13}^{+}\cdot y_{13}^{-}=0,\\
&y_{21}^{+}\cdot y_{21}^{-}=0,\ y_{22}^{+}\cdot y_{22}^{-}=0,\ y_{23}^{+}\cdot y_{23}^{-}=0,\ y_{31}^{+}\cdot y_{31}^{-}=0,\ y_{32}^{+}\cdot y_{32}^{-}=0,\ y_{33}^{+}\cdot y_{33}^{-}=0.
\end{aligned}
$$

Table 4 Solution to the problem of Table 2

Objective function value: $h = 0.622293$

$x_{11} = 34.260581$	$y_{11}^{+} = 16.495264$	$y_{21}^{+} = 19.809489$	$y_{31}^{+} = 18.807462$	$\xi_1^{+} = 16.979378$
$x_{12} = 17.885179$	$y_{11}^{-} = 0$	$y_{21}^{-} = 0$	$y_{31}^{-} = 0$	$\xi_1^{-} = 0$
$x_{22} = 22.114821$	$y_{12}^{+} = 29.442925$	$y_{22}^{+} = 0$	$y_{32}^{+} = 18.027659$	$\xi_2^{+} = 0$
$x_{23} = 22.026875$	$y_{12}^{-} = 0$	$y_{22}^{-} = 12.081737$	$y_{32}^{-} = 0$	$\xi_2^{-} = 18.208779$
$x_{31} = 25.739419$	$y_{13}^{+} = 0$	$y_{23}^{+} = 0$	$y_{33}^{+} = 105.428789$	$\xi_3^{+} = 18.208779$
$x_{33} = 27.973125$	$y_{13}^{-} = 67.811170$	$y_{23}^{-} = 0$	$y_{33}^{-} = 0$	$\xi_3^{-} = 0$
$q^{+} = 121.411651$	$q^{-} = 0$	$\gamma = 87.797881$		

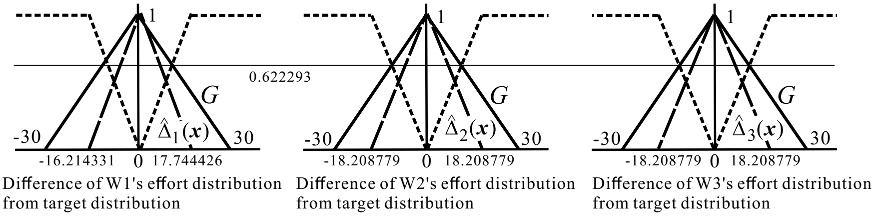

Fig. 5 Illustration of the solution of Example 2

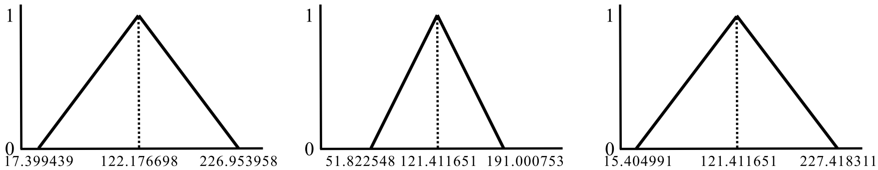

Fig. 6 Effort distributions of W1, W2 and W3 (Example 2)

target distribution are not always zeroes. The difference of the effort distribution of W1 from the target distribution is smallest in the obtained solution. The differences of the effort distribution of other workers from the target distribution are same. The effort distribution of the workers with respect to the obtained solution is depicted in Fig. 6.

Comparing the solutions obtained in Examples 1 and 2, they are similar. However, they are different to a certain extent. The effort distributions of W1 and W2 of the solution in Example 1 are wider than those of the solution in Example 2. On the other hand, the effort distribution of W3 of the solution in Example 1 is narrower than that of the solution in Example 2. In Example 1, we minimized the possible differences of efforts of workers under a fuzzy goal about the acceptable difference between efforts of two workers. Therefore, two of the differences will not be zeroes whenever one of efforts is not estimated precisely. On the other hand, in Example 2, we tried to minimize the differences between effort distributions of workers. The fuzzy goal is considered as a target distribution and we minimize the difference between effort and target distributions for each worker. The differences can be zeroes even if efforts are estimated only imprecisely because it attains when effort and target distributions coincide.

5 Concluding Remarks

In this paper, we treated modality goal programming problems with oblique fuzzy vectors. Oblique fuzzy vectors are useful to treat interactive fuzzy numbers in possibilistic linear programming problems without great loss of tractability. Moreover, an oblique fuzzy vector can be obtained by the principal component analysis of the stored data. On the other hand, there are several formulations depending on whether multiple goals are aggregated by a regret function or not and what kind of targets are treated in modality goal programming problems. In this paper, we considered only cases all goals are the same type but we can treat the modality goal programming problems with different type of goals. We applied only the necessity measure maximization models. We showed that all reduced problems can be solved by a bisection method together with simplex method.

The other models such as necessity fractile optimization models and expectation optimization models can be treated in the same way. The generalizations and applications of the proposed approaches are the future research topics.

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Multiobjective Bilevel Programming: Concepts and Perspectives of Development

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Abstract Bilevel programs model hierarchical non-cooperative decision processes with two decision makers, the leader and the follower, who control different sets of variables and have their own objective functions with interdependent constraints. Bilevel programs are very difficult to solve and even the linear case is NP-hard. In this chapter, a novel view on the main concepts in multiobjective and semivectorial bilevel problems is offered, including new types of solutions that are relevant for decision support. Optimistic and pessimistic leader's perspectives are explored; the extreme optimistic/deceiving and pessimistic/rewarding solutions in semivectorial problems and the optimistic Pareto fronts in multiobjective problems are defined and illustrated. Traditional and emerging application fields are reviewed. Potential difficulties and pitfalls associated with computing solutions to bilevel models with multiple objectives are outlined, shaping possible research avenues.

Keywords Multiobjective optimization · Bilevel programming · Semivectorial bilevel · Optimistic versus pessimistic approaches · Optimistic · Deceiving · Pessimistic · Rewarding solutions

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1 Introduction

Bilevel programs model hierarchical non-cooperative decision processes with two decision makers, the leader and the follower, who control different sets of variables and have their own objective functions with separate and/or interdependent constraints. The decision process is sequential: the leader first sets the values of his variables x to optimize his objective function; the follower then reacts by choosing the values of his variables y that optimize his objective function within the feasible solutions resulting from the leader's decision x. That is, the follower's problem is embedded in the leader's feasible region. Therefore, the leader needs to incorporate the follower's response into his optimization process because it affects solution feasibility and the leader's objective value. This type of sequential decision-making arises in many aspects of resource planning, management and policy-making, namely concerning the definition of pricing policies. For instance, in competitive electricity retail markets, retailers should establish commercial offers incorporating dynamic (time-of-use) tariffs aiming to maximize profits. However, they should take into account consumer's demand response to profit from time-differentiated tariffs by means of rescheduling the operation of appliances to minimize electricity bills (Alves and Antunes 2018b).

The multiobjective bilevel problem (MOBP) may have multiple objective functions at one or both levels. A special case of MOBP is the semivectorial bilevel problem (SVBP), in which there is a single objective function at the upper level and multiple objectives at the lower level. A set of lower level efficient solutions exists for each leader's decision x whenever multiple objective functions are considered at the lower level. In these circumstances, the leader's decision process should anticipate the follower's reaction within his efficient solution set, i.e. foreseeing the follower's trade-off between the lower level multiple objectives for any instantiation of the x variables. In the example mentioned above, to make the most of time-of-use tariffs, the consumer may want to minimize the electricity bill and also the discomfort associated with re-scheduling the operation of appliances outside the habitual time slots. Some approaches assume that the leader knows the follower's utility function, i.e. the leader knows the follower's choice, thus enabling to reduce the problem to a single objective bilevel problem or to a bilevel problem with multiple objectives at the upper level only. These cases lead to identifying an optimal solution or an efficient solution set for the leader, respectively. However, this assumption is seldom realistic in practice, since the follower's reaction within his efficient solution set depends on a preference structure that is not known by the leader and may be difficult to elicit.

With no information about the efficient solution the follower will choose, the leader may adopt a more optimistic or a more pessimistic attitude in face of his expectation of the more or less favorable follower's choice. Therefore, different types of solutions resulting from distinct leader's attitudes and follower's reactions should be computed to offer information about possible outcomes and ranges of objective values.

Bilevel programs are very difficult to solve and even the linear case is NP-hard (Dempe 2002). MOBP and SVBP become even more complicated due to the difficulties in computing solutions and issues associated with the decision process concerning the interaction leader—follower.

This chapter aims to offer a novel view on the main concepts in SVBP and MOBP, paying special attention to new types of solutions that are useful to support decisions based on this type of models. Optimistic and pessimistic leader's perspectives are explored, framing the *extreme* optimistic/deceiving and pessimistic/rewarding solutions in SVBP and the Optimistic Pareto front in MOBP, deriving from different leader's stances and follower's reactions, which require specific algorithmic techniques. Illustrative examples are used to shed light on these concepts, its relevance for decision support and the potential difficulties and pitfalls associated with computing solutions to bilevel models in a multiobjective setting.

In Sect. 2, the single objective bilevel problem is formulated and the inducible region is defined. Then the MOBP is presented, paying attention to the particular case of SVBP. A brief literature review on algorithmic approaches is presented. Section 3 is devoted to bilevel problems with multiple objective functions at the upper level and a single objective at the lower level. Section 4 addresses bilevel problems with multiple objective functions at the lower level, defining the optimistic, deceiving, pessimistic and rewarding solutions for the SVBP and the Optimistic Pareto front in MOBP, offering illustrative examples of these concepts. Section 5 presents an overview of selected traditional and emerging application fields. Section 6 unveils pitfalls associated with the computation of solutions to the SVBP/MOBP and offers a view on possible research avenues, namely regarding the decision support framework. Concluding remarks are presented in Sect. 7.

2 Multiobjective Bilevel Programming

2.1 Formulations and Fundamental Concepts

A general *bilevel programming* problem with a single objective function at each decision level can be defined as follows:

$$\begin{aligned} &\text{“}\max_{x\in X}\text{”} \quad F(x,y)\\ &s.t. \quad G(x,y)\le 0\\ &\qquad y\in \arg\max_{y\in Y}\{f(x,y): g(x,y)\le 0\} \end{aligned} \tag{BP}$$

$X \subseteq \mathbb{R}^{n_1}$ defines constraints only on the upper level decision variables x, which are controlled by the leader, and $Y \subseteq \mathbb{R}^{n_2}$ defines constraints only on the lower level decision variables y. $G(x, y) \le 0$ and $g(x, y) \le 0$ are general constraints, involving both x and y, in the upper and the lower level problems. $F(x, y)$ and $f(x, y)$ represent the leader's and the follower's objective functions, respectively.

The follower's rational reaction set to a given $x' \in X$ is:

$$\Psi(x') = \left\{ y' \in \mathbb{R}^{n_2} : y' \in \arg\max_{y \in Y} \{ f(x', y) : g(x', y) \leq 0 \} \right\}.$$

The set of feasible solutions over which the leader may optimize is called the *inducible region* (*IR*):

$$IR = \{ (x, y) \in \mathbb{R}^{n_1+n_2} : x \in X,\ G(x, y) \leq 0,\ y \in \Psi(x) \}.$$

The bilevel problem is the problem seen by the leader. Quotation marks in "max" $F(x, y)$ express the unclear definition of the objective function value $F(x, y)$ from the leader's perspective (since he has control only over x) if the set of optimal solutions to the lower level problem is not singleton (Dempe 2009). Even in the case where the lower level problem is a scalar optimization problem, more than one possible response of the follower may exist resulting from alternative optimal solutions to the follower's objective function. This poses a problem to the leader because the follower's choice may affect significantly the leader's decision. Most of the work on bilevel programming circumvents this difficulty by supposing that there is a single optimal solution to the lower level problem or adopting an optimistic approach. The optimistic approach presumes that the follower's response is always the most convenient for the leader. Under this assumption, the upper level optimization is executed with respect to x and y, which means that the leader can influence the decision of the follower (Dempe 2009). However, if the leader is risk-averse and wishes to limit the harm resulting from an undesirable option of the follower, a pessimistic approach should be considered. In this case the leader hedges against the worst case. Therefore, he chooses values for his variables that perform 'best' in view of the 'worst' follower's response (Tsoukalas et al. 2009). The pessimistic bilevel problem is even more difficult to solve than the optimistic one.

A general *multiobjective bilevel programming* problem with k objective functions at the upper level and m objective functions at the lower level can be formulated as follows:

$$\begin{aligned} &\text{“}\max_{x \in X}\text{”} \quad (F_1(x, y), \ldots, F_k(x, y)) \\ &s.t. \quad G(x, y) \leq 0 \\ &\qquad y \in \arg\max_{y \in Y} \{ (f_1(x, y), \ldots, f_m(x, y)) : g(x, y) \leq 0 \} \end{aligned} \tag{MOBP}$$

The bilevel problems comprising only one objective function at the upper level ($k = 1$) and multiple objective functions at the lower level ($m \geq 2$) has been called *semivectorial bilevel problem*. A general SVBP can be formulated as follows:

$$\begin{aligned} &\text{“}\max_{x \in X}\text{”} \quad F(x, y) \\ &s.t. \quad G(x, y) \le 0 \\ &\qquad y \in \arg\max_{y \in Y}\{(f_1(x, y), \dots, f_m(x, y)) : g(x, y) \le 0\} \end{aligned} \tag{SVBP}$$

Only *efficient* (*Pareto optimal*) solutions to the lower level problem for each x-vector are feasible to the MOBP/SVBP. Let $Y(x) = \{y \in Y : g(x, y) \le 0\}$. For a given $x' \in X$, a solution $y' \in Y(x')$ is *efficient to the lower level problem* (*f-efficient*) if and only if there is no other $y \in Y(x')$ that *dominates* y', i.e. such that $f_j(x', y) \ge f_j(x', y')$ for all $j = 1, \ldots, m$, and $f_j(x', y) > f_j(x', y')$ for at least one j.

Therefore, the set of f-efficient solutions for a given $x' \in X$ can be defined as: $\Psi_{\mathrm{Ef}}(x') = \{ y' \in Y(x')$: there is no $y \in Y(x')$ such that $f(x', y) \succ f(x', y')\}$, where $\succ$ denotes the dominance relation.

The *inducible region* of the MOBP or the SVBP is: $IR = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : x \in X, G(x, y) \le 0,\ y \in \Psi_{\mathrm{Ef}}(x)\}$.

When multiple objectives are considered at the lower level, the follower has, in general, a set of efficient solutions for each leader's decision x. If the leader does not have information about the follower's choice (e.g., by knowing his utility function), the leader should prepare for any compromise solution selected by the follower within the lower level efficient solution set. This uncertainty about the follower's behavior requires that the leader should adopt an attitude in face of his expectation of the follower's choice, which may be more or less favorable to the leader's single (SVBP) or multiple (MOBP) objectives. According to the more optimistic or more pessimistic attitude adopted by the leader anticipating the follower's choice, different types of solutions should be computed to offer information about possible outcomes and ranges of objective function values. In fact, it is not realistic to assume that the follower is indifferent to all efficient solutions obtained for a given decision of the leader or chooses according to the leader's objectives.

In a SVBP, the optimistic formulation considers that the solution to the lower level problem for each x is the f-efficient solution leading to the best value of the upper level objective function; the pessimistic formulation considers that the solution to the lower level problem for each x is the f-efficient solution that gives the worst value of the upper level objective function. In MOBP, the optimistic formulation has been almost exclusively used, which assumes that the solutions to the lower level problem for each x are the f-efficient solutions leading to efficient solutions to the upper level problem. Only a few studies have addressed the uncertainty associated with the lower level decision (in the sense that, in principle, the leader does not have information about the follower's decision). The concept of pessimism is not clearly defined and further research is necessary on this topic.

Four *extreme* solutions may be identified in the SVBP according to the more optimistic or more pessimistic leader's attitude: in the *optimistic solution*, the leader obtains the best objective function value when the follower decides in the best interest to the leader; however, the leader may engage in an optimistic attitude when setting his variables but the follower's choice is the worst for the leader, thus leading to the *deceiving solution*; in the *pessimistic solution*, the leader obtains the best objective function value when the follower's choice for each setting of upper level variables is the worst for the leader; however, the leader may engage in a pessimistic attitude when setting his variables but the follower's reaction is the most favorable to the leader, thus leading to the *rewarding solution* (Alves and Antunes 2016).

2.2 Classical and Metaheuristic Methods for Bilevel with Multiple Objective Functions

The MOBP with multiple objective functions at the upper level and a single objective function at the lower level just requires the consideration of a set of efficient solutions to the upper level problem; therefore, the issues associated with the uncertainty of the follower's choice are not at stake (unless alternative optimal solutions to the lower level problem exist).

Calvete and Galé (2010b) addressed the bilevel linear problem with multiple objective functions at the upper level and a linear objective function at the lower level. When all objective functions are linear and constraints at both levels define polyhedra, the authors prove that the efficient solution set is non-empty and propose different scalarization techniques to obtain efficient points. These techniques involve solving linear bilevel problems with a single objective function at each level.

Alves et al. (2012) proposed an exact procedure for bilevel linear programs with multiple objective functions at the upper level and a single objective function at the lower level, which is based on the reformulation of the problem as a multiobjective mixed 0–1 linear program that is dealt with a reference point algorithm. This procedure enables to characterize the whole Pareto front in bi-objective problems; in problems with any number of objective functions, a subset of nondominated solutions can be computed interactively according to the decision maker's preferences.

Alves and Costa (2014) developed a multiple objective particle swarm optimization (PSO) algorithm to solve bilevel linear problems with multiple objective functions at the upper level, which aims to generate a good approximation of the whole Pareto front by using a hybrid strategy for the global best selection and an adaptive mutation operator.

Other approaches have been developed to deal with specific applications using bilevel models with multiple objectives at the upper level and a single objective at the lower level, e.g. the works of (Roghanian et al. 2007) for supply chain planning and (Zhou et al. 2016) for a low-carbon power dispatch problem.

In SVBP, classical approaches and population-based metaheuristics have been developed mainly devoted to compute the optimistic solution. Classical approaches include using the Karush-Kuhn-Tucker (KKT) conditions, penalty methods and techniques exploring vertices in linear problems. The first approach involves replacing, whenever possible, a scalarization of the lower level problem by its KKT conditions. The second approach consists of formulating a nonlinear programming problem approximating the original one, which is solved iteratively by means of a penalty function method applied to the lower level problem leading, under certain conditions, to a sequence of approximated solutions converging to the optimal solution. The third approach is based on the property that only vertices of the constraint region need to be considered for the computation of the optimistic optimal solution to a linear SVBP. Population-based metaheuristics typically include evolutionary, PSO and differential evolution (DE) algorithms to cope with the difficulties to compute exact solutions.

The SVBP was first addressed by Bonnel (2006), providing necessary optimality conditions for an optimistic formulation. Considering weakly efficient solutions to the lower level problem, Bonnel and Morgan (2006) proposed a solution approach based on a penalty function. Ankhili and Mansouri (2009), Zheng and Wan (2011), Zheng et al. (2014), Ren and Wang (2016) developed penalty function methods to compute the optimistic solution to the SVBP with a multiobjective linear programming (MOLP) problem at the lower level. Calvete and Galé (2011) also focused on bilevel problems with a MOLP lower level problem (with all constraints linear and the upper level objective function quasiconcave). The problem is reformulated as an optimization problem over a nonconvex region given by the union of faces of the polyhedron defined by all constraints, so that an extreme point of the polyhedron is the optimistic solution to the problem. Both an enumerative exact algorithm and a genetic-based algorithm are proposed. Lv and Wan (2014) proposed another algorithm for the linear SVBP using the weighted-sum scalarization to reformulate the linear SVBP into a special BP where the lower level is a parametric linear scalar problem. Then the BP is transformed into a single level nonlinear, nonconvex programming problem with a linear objective function.

Liu et al. (2014) developed necessary optimality conditions for the pessimistic formulation of the SVBP, transforming the pessimistic SVBP into a generalized minimax optimization problem with constraints using a scalarization technique. Lv and Chen (2016) proposed a discretization iterative algorithm to compute the pessimistic solution to a SVBP with a convex lower level problem without upper level variables in the constraints. Alves et al. (2015) firstly introduced the concept of deceiving solution and proposed an algorithm based on PSO to approximate the optimistic, pessimistic and deceiving solutions to the SVBP. Algorithms based on DE and PSO were proposed by Alves and Antunes (2018a) to compute those solutions, as well as the rewarding solution.

Concerning bilevel problems with multiple objectives at both levels, most studies have also considered the optimistic formulation.

Shi and Xia (1997) proposed an interactive algorithm for the nonlinear MOBP, which simplifies the problem by transforming it into separate multiobjective decision-making problems at each level, using a satisfactoriness concept to model the leader's preferences. This work has been the basis for other models and algorithms with two or more levels that consider interactivity between the algorithm and the upper level decision maker to compute a satisfactory solution (Shi and Xia 2001; Abo-Sinna and Baky 2007). Eichfelder (2010) presented new theoretical results for the nonlinear non-convex MOBP and proposed an algorithm to problems with two objective functions at both levels with one upper level variable. Pieume et al. (2011) introduced a surrogate single level MOLP problem aimed to generate the whole set of feasible solutions to the upper level problem for the optimistic formulation of the linear MOBP. Two approaches for obtaining efficient solutions are developed, depending whether the leader is able or not to express his preferences regarding his objective functions.

Deb and Sinha (2010) proposed a hybrid evolutionary algorithm with a local search phase, with self-adaptive population size based on Euclidean distance and termination criteria based on the hypervolume measure. Zhang et al. (2013) presented a hybrid PSO algorithm with crossover operator for high dimensional MOBP using an elitist strategy. The multiobjective PSO algorithm developed by Carrasqueira et al. (2015) pays special attention to the lower level optimization process for each upper level decision variable vector, so that solutions obtained are actually efficient; otherwise they are not feasible to the upper level problem. All these algorithms aim to approximate the whole Pareto front of the problem considering an optimistic formulation of the MOBP. Gupta and Ong (2015) proposed the transformation of the lower level problem into a single-objective problem using scalarization techniques with adaptive parameters, e.g. weights in a weighted-sum of the objective functions. The leader's problem is modified by incorporating the weights of the lower level objective functions into the upper level problem, which are handled in a similar manner as the original leader's decision variables. Therefore, the weights evolve through variation operators (e.g. mutation and crossover) in the upper level and enter into the lower level optimization as fixed parameters. This means that the leader is able to choose the weights of the follower's objectives that most benefits his interests, thus assuming an optimistic approach. Ruuska and Miettinen (2012) proposed a procedure to construct new evolutionary algorithms for the optimistic MOBP by integrating an evolutionary multiobjective algorithm with a partial order that is compatible with bilevel optimization (Ruuska et al. 2012).

A different approach was adopted by Nishizaki and Sakawa (1999) in their interactive procedure, assuming that the leader has some subjective anticipation or belief of the follower's response. This anticipation may be optimistic, pessimistic or the leader knows the follower's preferences (e.g. arising from the past behavior of the follower).

More recently, Sinha et al. (2016) considered the decision uncertainty involved in modelling the follower's behavior and recognized that it is unrealistic to assume that the leader can decide the tradeoff the follower may choose. To handle this uncertainty, the authors assumed that the follower's preferences are characterized by a value function parameterized by an uncertain preference vector (e.g., a linear function with a stochastic weight vector for the different lower level objectives) and proposed a two-step approach: firstly, the leader uses his expectation about the follower's preferences to obtain the Pareto front for fixed parameters; then, the leader examines the extent of uncertainty by estimating a confidence region around the Pareto front previously obtained. Sinha et al. (2017) classify the MOBP into three different formulations. The first one is the optimistic formulation, which the authors call *standard* formulation. The second one considers deterministic decisions at lower level, assuming the leader has perfect information about the follower's preference structure. Using this preference information, which can be modelled as a value function, the lower level problem can be reduced to a single objective optimization problem. Whenever this information does not exist or cannot be elicited, the authors consider a third formulation in which the leader experiences lower level decision uncertainty, assuming that the follower's preferences are represented by a random variable (with a distribution that is parameterized by each upper level decision). Since this lower level uncertainty may have a significant impact on the upper level objective function(s), the leader should be aware of the consequences of distinct follower's decisions.

3 Bilevel Problems with Multiple Objective Functions at the Upper Level and a Single Objective Function at the Lower Level

The consideration of multiple objective functions only at the upper level does not impose challenges as hard as the ones associated with the need to consider a set of efficient solutions to the lower level problem with respect to the uncertainty of the follower's choice. However, computational difficulties are at stake for the identification of Pareto optimal solutions. It should be noticed that even in the case of a linear bilevel problem (LBP) with multiple objective functions at the upper level and a single objective function at the lower level (MO-SO-LBP), the characterization of the upper level Pareto optimal solutions should take into account that (Alves et al. 2012):

- supported but also unsupported Pareto optimal solutions may exist;
- the set of Pareto optimal solutions may be not connected even if *IR* is connected;
- the set of Pareto optimal solutions (or even weakly Pareto optimal solutions) may not be equal to the union of faces of the constraint region of the MO-SO-LBP, which includes all the constraints of the leader and the follower; that is, a face may be just partially Pareto optimal.

Example 1, in Fig. 1, illustrates a MO-SO-LBP with two objective functions at the upper level (Alves et al. 2012). S denotes the constraint region.

Example 1

$$\begin{aligned}
&\max_{x,y}\ F_1(x,y) = -2x \\
&\max_{x,y}\ F_2(x,y) = -x + 5y \\
&\text{s.t.}\quad \max_{y}\ f(y) = -y
\end{aligned}$$

$$\left.\begin{array}{ll} \text{s.t. } x - 2y \le 4 & (1) \\ 2x - y \le 24 & (2) \\ 3x + 4y \le 96 & (3) \\ x + 7y \le 126 & (4) \\ -4x + 5y \le 65 & (5) \\ x + 4y \ge 8 & (6) \\ x, y \ge 0 & \end{array}\right\} S$$

The inducible region, *IR*, is [DE] ∪ [EB] ∪ [BA]. The whole Pareto optimal solution set, which is a subset of *IR*, is {D} ∪]CB] ∪ [BA]. The values of the decision variables and the upper level objective functions in the points A, B, C and D are shown in Table 1.

The analysis of Fig. 1 enables to conclude that:

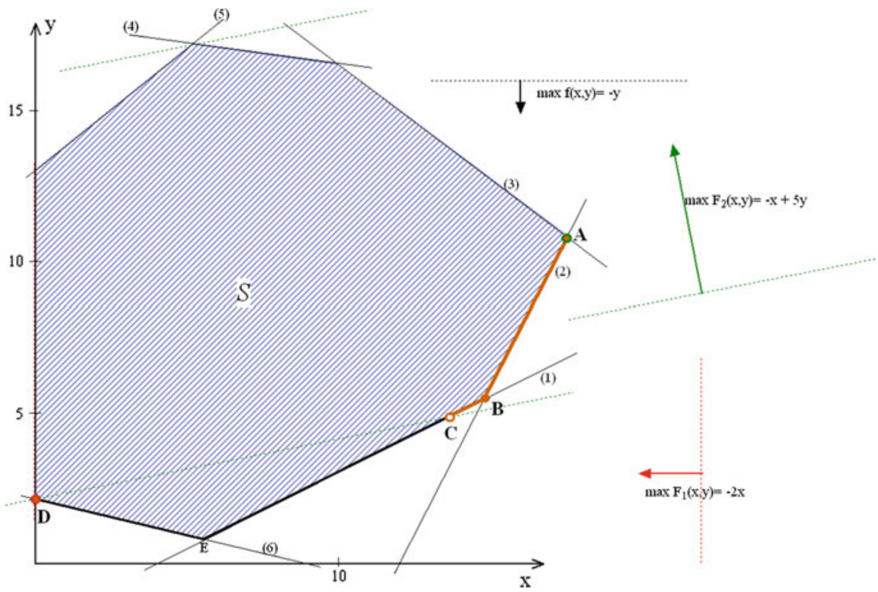

Fig. 1 Constraint region of Example 1

Table 1 Values of the (weakly) Pareto optimal extreme points of Example 1

	x	y	F_1	F_2
A	17.45455	10.90909	– 34.9091	37.09091
B	14.66667	5.333333	– 29.3333	12
C	*13.33333*	*4.666667*	– *26.6667*	*10*
D	0	2	0	10

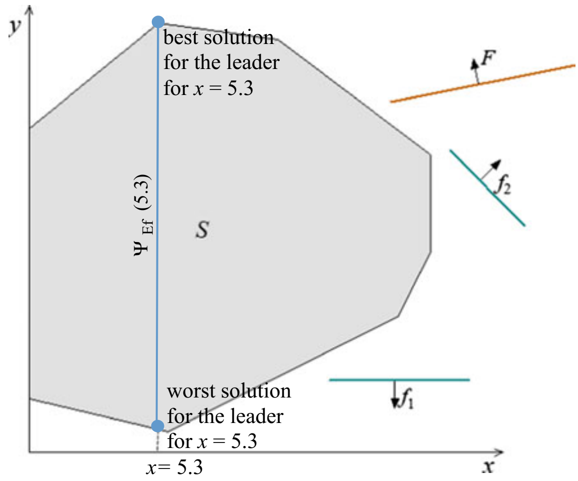

Fig. 2 Constraint region of Example 2

- D and A are the Pareto optimal solutions that maximize F_1 and F_2, respectively.
- Solutions from C (inclusive) to D (exclusive) of *IR* are not Pareto optimal as they are dominated by D. In comparison with C, D is superior only in F_1 being equal in F_2. Hence, C is a *weakly Pareto optimal* solution. Solutions in]DE] ∪ [EC[are strongly dominated by D.
- Only A and D are *supported* Pareto optimal solutions. All the others are *unsupported*, because there are convex combinations of A and D that would dominate them if they were feasible, i.e., if they belonged to *IR*.

This example shows that the Pareto optimal set of a MO-SO-LBP may be *not connected* and may have *unsupported* solutions. Furthermore, unsupported solutions may constitute the major part of the Pareto optimal set. Hence, they should not be disregarded.

4 Bilevel Problems with Multiple Objective Functions at the Lower Level

Let us first consider the bilevel problem with multiple objective functions at the lower level and one objective function at the upper level, i.e., the semivectorial bilevel problem SVBP.

Example 2 Consider the following problem with one objective function at the upper level and two objective functions at the lower level, whose constraint region S is displayed in Fig. 2.

$$\begin{aligned}
&\max F(x, y) = -x + 5y \\
&\text{s.t.} \\
&\quad \max f_1(y) = -y \\
&\quad \max f_2(y) = x + y \\
&\quad \text{s.t.} \\
&\quad\quad \left.\begin{array}{l}(1)-(6) \text{ of Example 1} \\ x \leq 16\end{array}\right\} S
\end{aligned}$$

Suppose the leader takes a particular decision, e.g. $x = 5.3$. The follower's objectives become:

$\max f_1 = -y$; $\max f_2 = 5.3 + y$. So, all the solutions in $\Psi_{\mathrm{Ef}}(5.3)$ represented in Fig. 2 are efficient to the follower for this leader's decision.

For each leader's decision x' there is a set of efficient solutions $\Psi_{\mathrm{Ef}}(x')$ to the follower, which in this example are represented by vertical line segments delimited by the boundary of S. Therefore, for a given x' the following question arises: what will be the choice of the follower within $\Psi_{\mathrm{Ef}}(x')$?

The two main ways suggested in the literature to frame the problem are the optimistic and the pessimistic approaches underlying different leader's perspectives. In the optimistic approach, the leader assumes that the follower is willing to support him and selects the solution among $\Psi_{\mathrm{Ef}}(x)$ that is the best for the leader.

- The *optimistic* solution, $(x^{\mathrm{o}}, y^{\mathrm{o}})$, is given by

$$\max_{x \in X, y \in Y} \{F(x, y) : y \in \Psi_{\mathrm{Ef}}(x), G(x, y) \leq 0\}.$$

In the pessimistic approach, the leader prepares for the worst case. The leader chooses the x that leads to a solution with best F in view of the follower's decisions y worst for the leader.

- The *pessimistic* solution, $(x^{\mathrm{p}}, y^{\mathrm{p}})$, if it exists, is given by

$$\max_{x \in X} \left\{ \min_{y \in Y} \{F(x, y) : y \in \Psi_{\mathrm{Ef}}(x)\} : G(x, y) \leq 0 \right\}.$$

In addition to the optimistic and pessimistic solutions, other types of solutions can be defined that provide useful insights to the leader of possible outcomes and ranges of values resulting from different decisions. In particular, the following solutions may be relevant for decision support: the result of a failed optimistic approach—*deceiving* solution—and the result of a successful pessimistic approach—*rewarding* solution.

A *deceiving* solution results whenever the leader makes an optimistic decision and the follower's reaction is against the interests of the leader. Thus, given the optimistic upper level decision x^{o},

- the *deceiving* solution is $(x^{\mathrm{d}}, y^{\mathrm{d}}) = (x^{\mathrm{o}}, y^{\mathrm{d}})$ where y^{d} is given by $\min_{y \in Y} \{F(x^{\mathrm{o}}, y) : y \in \Psi_{\mathrm{Ef}}(x^{\mathrm{o}})\}$.

Note that the deceiving solution may not satisfy upper level constraints $G(x, y) \leq 0$, i.e. it may be infeasible to the leader, because the aim is to show the follower's decision that is the worst outcome for the leader even if the follower does not take into account the upper level constraints. Nevertheless, this information can be useful to the leader's decision.

The *rewarding* solution is obtained whenever the leader takes a pessimistic approach and the follower's reaction is the most favorable to the leader. Thus, given the pessimistic upper level decision x^{p},

- the *rewarding* solution is $(x^{\mathrm{r}}, y^{\mathrm{r}}) = (x^{\mathrm{p}}, y^{\mathrm{r}})$ where y^{r} is given by $\max_{y \in Y}\{F(x^{\mathrm{p}}, y) : y \in \Psi_{\mathrm{Ef}}(x^{\mathrm{p}}), G(x^{\mathrm{p}}, y) \leq 0\}$.

The rewarding solution is selected among the lower level efficient solutions for $x = x^{\mathrm{p}}$ that satisfy the upper level constraints. The aim is to show the follower's decision that is the best outcome for the leader, which should be feasible.

Let the follower's efficient solutions that are the best for the leader be called *optimistic frontier* and the follower's efficient solutions that are the worst for the leader be called *pessimistic frontier*. These frontiers are defined as follows.

Optimistic frontier: The optimistic frontier (*Of*) is the set comprising the solutions (x, y') such that $x \in X$ and $y' \in O(x) = \left\{\arg\max_{y \in Y}\{F(x, y) : y \in \Psi_{\mathrm{Ef}}(x), G(x, y) \leq 0\}\right\}$.

Pessimistic frontier: The pessimistic frontier (*Pf*) is the set comprising the solutions (x, y'') such that $x \in X$ and $y'' \in P(x) = \left\{\arg\min_{y \in Y}\{F(x, y) : y \in \Psi_{\mathrm{Ef}}(x)\}\right\}$.

The optimistic and the pessimistic solutions are the feasible solutions (i.e., which satisfy all constraints including $G(x, y) \leq 0$) with maximum value of F in the *optimistic frontier* and the *pessimistic frontier*, respectively. The optimistic and rewarding solutions belong to *Of*, while the pessimistic and deceiving solutions belong to *Pf*.

The value of F in a feasible deceiving solution can be worse (and is never better) than the one in the pessimistic solution, because: the pessimistic solution $(x^{\mathrm{p}}, y^{\mathrm{p}} \in P(x^{\mathrm{p}}))$ is the best feasible solution according to F on the *pessimistic frontier Pf*; the deceiving solution $(x^{\mathrm{d}}, y^{\mathrm{d}} \in P(x^{\mathrm{d}}))$ also belongs to *Pf*; thus, if the deceiving solution satisfies $G(x^{\mathrm{d}}, y^{\mathrm{d}}) \leq 0$, it cannot be better than the pessimistic solution, otherwise it would be the pessimistic solution itself. At the best, the deceiving solution is as good as the pessimistic solution.

Figure 3 shows the optimistic—rewarding and the pessimistic—deceiving solutions of Example 2 on the *optimistic frontier* (*Of*) and the *pessimistic frontier* (*Pf*), respectively.

Let us consider another SVBL example with non-linear objective functions to be minimized. This problem is adapted from Problem 3 in (Deb and Sinha 2009)

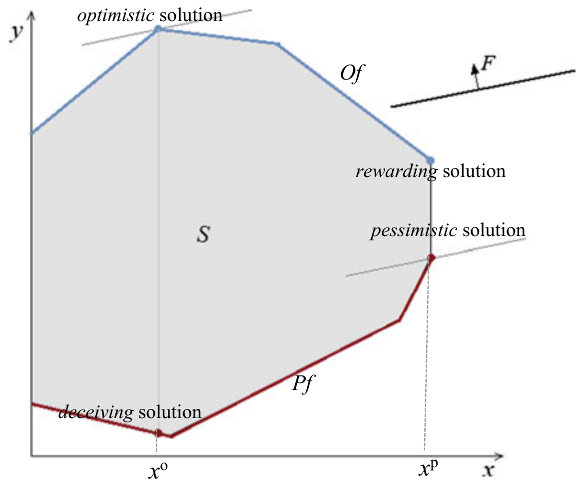

Fig. 3 Example 2: optimistic frontier (*Of*) and pessimistic frontier (*Pf*); optimistic, pessimistic, deceiving and rewarding solutions

considering just one upper level objective function, and it was presented in Alves et al. (2015).

Example 3

$$\begin{aligned} \min_{x} \quad & F(x, y) = (y_1 - 1)^2 + y_2^2 + x^2 \\ s.t. \quad & \min_{y} \ f_1(x, y) = y_1^2 + y_2^2 \\ & \min_{y} \ f_2(x, y) = (y_1 - x)^2 + y_2^2 \\ & s.t. - 1 \le y_1, y_2, x \le 2 \end{aligned}$$

For a given value of x, the efficient solutions to the lower level problem are:

$$(y_1, y_2) \in \mathbb{R}^2 : \begin{cases} y_1 \in [0, x], y_2 = 0 & \text{for } \ 0 \le x \le 2 \\ y_1 \in [x, 0], y_2 = 0 & \text{for } -1 \le x \le 0 \end{cases}$$

In Fig. 4, Ψ_{Ef} denotes the union of $\Psi_{\text{Ef}}(x)$ for all x, which is represented by the shaded area. The level curves of $F(x, y)$ are circles centered at the point $(y_1, x) = (1, 0)$. Since $y_2 = 0$ for all efficient solutions to the lower level problem, only y_1 and x are displayed in Fig. 4.

The *optimistic* solution is the point O: $(x, y_1, y_2) = (0.5, 0.5, 0)$, $(f_1, f_2) = (0.25, 0)$ and $F = 0.5$.

The solution that optimizes the leader's objective function within the subset of the follower's efficient solutions that are "worst for the leader", i.e. *Pf*, is the *pessimistic* solution (point P): $(x, y_1, y_2) = (0,\ 0,\ 0)$, $(f_1, f_2) = (0,\ 0)$ and $F = 1$.

The *deceiving* solution indicates the maximum risk the leader incurs if he adopts an optimistic approach, i.e. the leader chooses $x = 0.5$ (point D): $(x,\ y_1,\ y_2) = (0.5,\ 0,\ 0)$, $(f_1,\ f_2) = (0,\ 0.25)$ and $F = 1.25$.

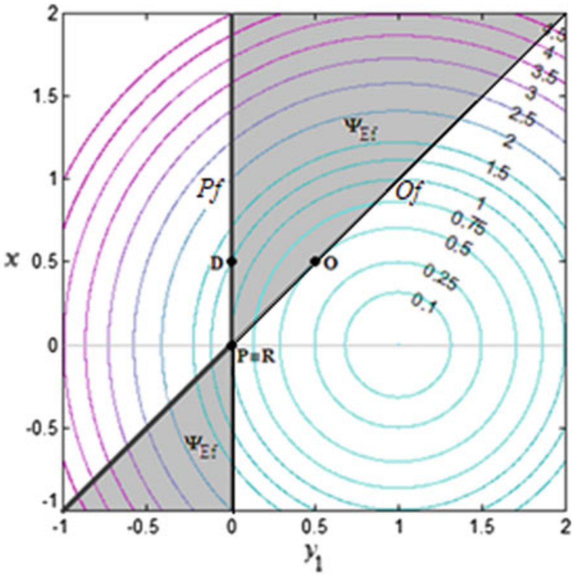

Fig. 4 Example 3: optimistic (O), pessimistic (P), deceiving (D) and rewarding (R) solutions

If the leader takes a pessimistic approach (i.e. he chooses $x = 0$), the only efficient solution to the follower is solution P. Therefore, in this example, the leader cannot obtain a better value of F, which means that the *rewarding* solution (R) coincides with the *pessimistic* solution (P).

In this problem, if the leader is willing to take some risk then the optimistic approach may be an interesting option because the worst outcome of the optimistic approach, given by the *deceiving* solution ($F = 1.25$), is not much worse than the *pessimistic* solution ($F = 1.0$); on the other hand, the *optimistic* solution provides a larger improvement to the leader's objective with respect to the *pessimistic* one ($F = 0.5$ vs. 1.0). Moreover, there is no opportunity to obtain a solution better than the *pessimistic* one if the leader adopts a pessimistic approach, because there is only one efficient solution to the follower (P ≡ R).

Concerning the bilevel problem with multiple objective functions at both levels (MOBP), the aim of most procedures has been to approximate the whole upper level Optimistic Pareto front or they consider that the follower's preferences are known, thus reducing the MOBP to a BP with multiple objective functions at the upper level and a single objective function at the lower level.

A feasible solution (x', y') to the MOBP is F-*efficient* if there is no other $(x, y) \in IR$ that dominates (x', y'), i.e. such that $F_j(x, y) \geq F_j(x', y')$ for all $j = 1, \ldots, k$, and $F_j(x, y) > F_j(x', y')$ for at least one j. This definition assumes an optimistic approach in which the follower selects the f-efficient solutions that are efficient for the leader, thus leading to a subset of *IR* containing all efficient solutions to the leader.

The *Optimistic Pareto Front* (*OPF*) to the MOBP is composed of the F-efficient solutions (x,y) whose y are contained in the union of all $\Psi_{\mathrm{Ef}}(x)$ for every x. In bilevel

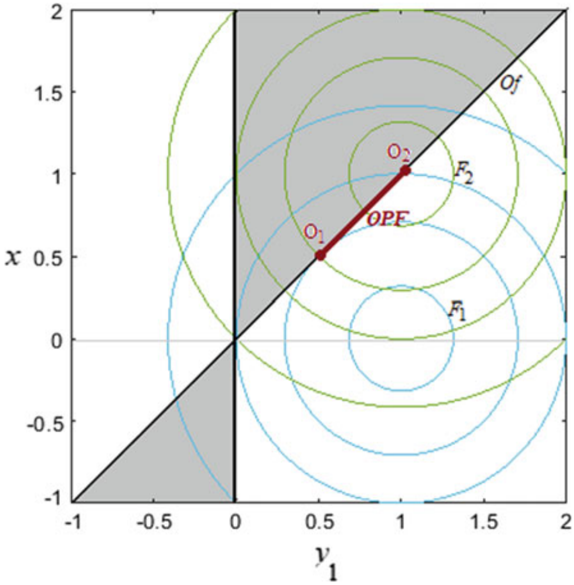

Fig. 5 Example 4: *Optimistic Pareto Front* (*OPF*) in the variable space

problems with multiple objective functions at both levels, the concepts associated with the pessimistic approach are more difficult to define and may not be consensual.

Let us consider a MOBP example (Problem 3 in Deb and Sinha (2009)), which is the previous Example 3 with a second upper level objective function.

Example 4

$$\begin{aligned} &\min_x \quad F_1(x, y) = (y_1 - 1)^2 + y_2^2 + x^2 \\ &\min_x \quad F_2(x, y) = (y_1 - 1)^2 + y_2^2 + (x - 1)^2 \\ &s.t. \quad \min_y \; f_1(x, y) = y_1^2 + y_2^2 \\ &\qquad\;\; \min_y \; f_2(x, y) = (y_1 - x)^2 + y_2^2 \\ &\qquad\;\; s.t. -1 \le y_1, y_2, x \le 2 \end{aligned}$$

The lower level efficient solutions are the same as in Example 3. The *Optimistic Pareto Front* (*OPF*) is obtained for $y_1 = x$ for $x \in [0.5, 1]$ (Deb and Sinha 2009).

In Fig. 5, the level curves of $F_1(x, y)$ and $F_2(x, y)$ are circles centered at the points $(y_1, x) = (1, 0)$ and $(y_1, x) = (1, 1)$, respectively. If separately considered, the objective functions $F_1(x, y)$ and $F_2(x, y)$ have the same optimistic frontier *Of*, which is represented in Fig. 4. Therefore, the *OPF* for the MOBP is located on the common *Of* between their individual optima (O_1 and O_2).

Figure 6 displays the upper level objective values for all solutions in *Of*, identifying the *Optimistic Pareto Front* (*OPF*).

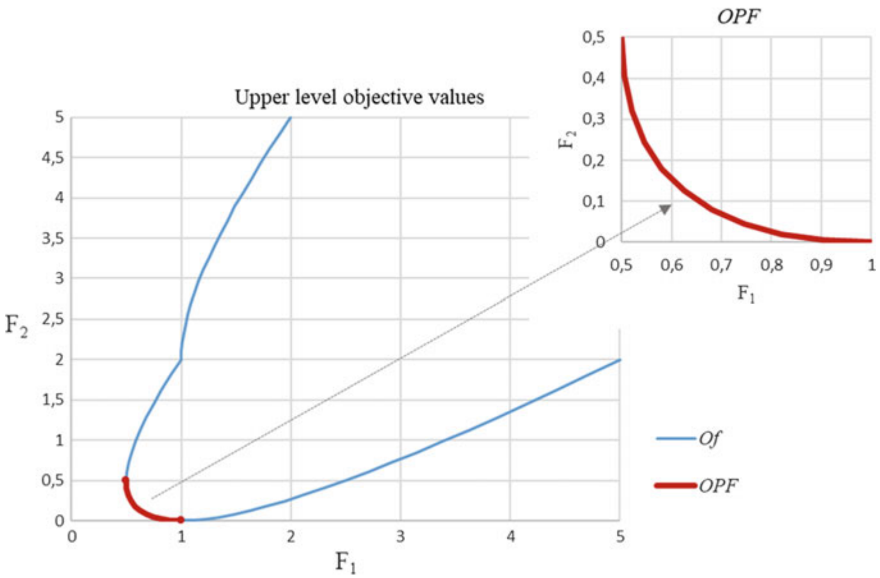

Fig. 6 Example 4: *Optimistic Pareto Front* (*OPF*) in the upper level objective space

5 Applications of MOBP

Several configurations of multiobjective bilevel optimization models arise in the general framework of problems in which upper level decisions concern design and policy issues, while operational issues are at stake at the lower level. Areas of application include network design, transportation, facility location and logistics, security planning, energy systems, environmental management, process engineering, as well as structural, shape and/or layout design optimization. A brief review of representative traditional and emerging areas of applications is made below, with focus on the nature of the multiple objective functions at upper and/or lower levels.

5.1 Network Design, Location and Transportation Policy

Multiobjective bilevel models have been used in network design and transportation policy problems, namely concerning the optimization of tolls revenue, also considering congestion pricing in toll design, health and environmental issues. Sinha et al. (2015) developed a model for an authority managing a network of roads, whose objectives are maximizing toll revenues and minimizing pollution levels, while the network users aim to minimize travel cost and travel time. Chen et al. (2010) presented a model for the network design problem in which the leader determines the optimal capacity enhancements in a transportation network by optimizing a set of system-wide objectives (traffic congestion, level of service and revenue) under demand uncertainty, while the lower level problem concerns the route choice behavior of users for a given capacity enhancement. Xu et al. (2012) dealt with a nonlinear minimum cost network flow problem in a large-scale construction project, in which in the upper level the construction contractor determines the material flow of each transportation network path aiming to minimize direct and transportation time costs and the lower level objective of the transportation manager is the minimization of transportation costs.

Gang et al. (2015) proposed a model for the location of a stone industrial park with a hierarchical structure consisting of a local authority and stone companies. The local government is the leader aiming to minimize pollution emissions and development and operating costs. The stone companies are the followers aiming to minimize costs.

Hammad et al. (2018) proposed a model to solve the multi-facility location problem with traffic equilibrium constraints. The aim is to locate a set of buildings with varying sensitivity thresholds due to the negative impacts of an existing semi-obnoxious facility. The traffic routing problem is modelled as a user equilibrium model at the lower level.

5.2 Environmental Management

Environmental economics and management decisions generally involve the interaction of a regulator authority/planning agency and individual firms maximizing profit but generating environmental externalities (e.g., air, land or water pollutants). Sinha et al. (2013) presented a study on environmental economics, in which the leader is the regulatory authority aiming to maximize total tax revenue and minimize the environmental damages caused by a mining company, which is the follower maximizing its total profit. Bostian et al. (2015a) considered the problem of designing a tax to reduce fertilizer use in agricultural production, in which the policy maker aims to minimize total fertilizer use in an agricultural watershed area and maximize total profits from agricultural production. For each tax rate, individual producers choose the quantity of fertilizer that maximizes their profit. Bostian et al. (2015b) evaluated the tradeoff between agricultural production and water quality considering the maximization of total profit and minimization of basin-level nitrate runoff resulting from fertilizer usage as the upper level objectives, while producer profit maximization is the lower level objective. The same objective functions at both levels are used in the model proposed by Whittaker et al. (2017), also in a context of a single leader (the government authority establishing a policy) and multiple followers (the farmers who should comply with the policy in a way that maximizes their own objectives).

5.3 Process Engineering

Process engineering deals with the design, operation, control and optimization of industrial processes, generally continuous ones, namely within chemical, petrochemical, agriculture, mineral processing, pharmaceutical, food and biotechnological industries. The use of multiobjective bilevel models depends on the type of processes and physical variables involved. In general, upper levels objectives are related with design and the lower level objectives with operation. Linnala et al. (2012) considered the design of a paper mill and its operation. At the design (upper) level, the objective functions to minimize are the fill percentages of wet broke and dry broke towers (in a broke system the rejected paper is collected and re-circulated back into the process as a raw material). At the operational (lower) level, the objective functions are the minimization of production loss, variation of fill percentage of the dry broke tower and variation of fill percentage of the wet broke tower, and the maximization of broke dosage. Halter and Mostaghim (2006) presented a thermodynamic model to determine parameters in the quantification of the physical and chemical properties of silicate melts. The upper level consists of minimizing the free energy of solid and liquid and minimizing the difference between the temperature obtained in the chemical reaction and the absolute temperature. In the lower level, the aim is the speciation data for each parameter set in the upper level.

5.4 Security Planning

Li et al. (2015b) proposed a model for the construction site security problem in which the upper level problem involves maximizing the efficiency of the construction facilities system and minimizing the countermeasure cost and economic loss due to the facilities to be secured. In the lower level problem, the attacker aims to inflict maximum loss of efficiency in the construction facilities system. Li et al. (2015a) developed a model for the dynamic construction site layout and security planning problem. The objective functions in upper level problem are the minimization of the layout costs and consequences of a potential attack. In the lower level problem, the attacker aims to maximize the economic consequence of the attack to the construction facilities system.

5.5 Energy Systems

A diversified set of applications in the energy sector has been reported in the literature, including tariff setting, electricity markets, and distributed generation. Zhou et al. (2016) proposed a model for a power dispatch problem in which the upper level decision maker is the regional power grid corporation, which allocates power quotas, and the lower level decision makers are power generation groups. The leader aims to maximize profit, minimize surplus power by balancing supply and demand and minimize total carbon emissions. Each follower establishes his power generation plans and prices to maximize profit. Lv et al. (2016) developed a model for the operation of a distribution network with grid-connected micro-grids. The upper (distribution network) level determines the optimal dispatch to optimize power losses and voltage profile. The lower (micro-grid) level uses the dispatch requirements to minimize the operation cost of distributed generators. Gao et al. (2017) presented a coordinated planning model for the interaction of distributed generation and distribution network frames. The objective functions at the upper level are the minimization of the cost of integrated investment and operation maintenance, active power losses, power purchasing and power failure. The objective functions at the lower level are the minimization of the cost of distributed generation investment and operation maintenance, the expected deviation rate of voltage and the maximization of loss reduction and power generation of distributed generation.

Wang et al. (2014) dealt with power consumption of data centers, proposing energy and locality aware multi-job scheduling in cloud computing to improve the energy efficiency of servers. The upper level objective functions are the minimization of the differences between each server's resource utilization and its optimal value and the maximization of data locality ratio. The lower level objective function is the minimization of the sum of the differences between each server's CPU utilization after scheduling and its optimal utilization. Stojiljković (2017) presented a case study of the design of a trigeneration system for an urban system. The upper level design objectives are the minimization of total costs, primary energy and greenhouse gases emission. The lower level operational objective deals with cost minimization.

Hawthorne and Panchal (2014) dealt with feed-in-tariff policy design for decentralized energy infrastructure. The upper level policy designer aims to maximize the quantity of energy generated and minimize policy cost. The lower level stakeholders decide on quantities to maximize net present value and minimize capital investment seeking a Nash equilibrium resulting from market interactions. Alves and Antunes (2018b) developed a model for the interaction between electricity retailers and consumers to optimize electricity time-of-use retail pricing. The retailer (upper level decision maker) establishes dynamic time-of-use electricity prices to maximize profits. The consumer (lower level decision maker) responds by determining an appliance operation schedule to minimize the electricity bill and the dissatisfaction in face of his preferences and requirements.

5.6 Logistics -Production-Distribution Planning

Jia et al. (2014) addressed production-distribution planning problems. The leader is the distribution company aiming to minimize its overall cost. The follower is the manufacturer aiming to minimize the cost and the storage cost.

Calvete and Galé (2010a) considered the problem of a distribution company owning distribution centers, which acquires products from manufacturing plants and delivers them to retailers. The distribution company, at the upper level, aims to minimize the overall transportation costs. The manufacturing plants, at the lower level, decide the allocation of production to plants aiming to minimize production costs and the cost of underutilization of plant capacity.

5.7 Structural Design Optimization

In structural design optimization, the optimal design problem is solved at the upper level, in general minimizing the cost or weight of a structure, while the structural analysis problem is solved at the lower level, which may involve optimizing forces, energy, displacements, etc. Ghotbi (2016) dealt with the design of a high-speed mechanism considering kinematic and dynamic objectives, which are treated as the leader in different problem variants. The kinematic objectives include the minimization of the structural error and the deviation of the transmission angle from its ideal value. The dynamic objective consists of the minimization of the peak torque required to drive the input link over a cycle. Dandurand et al. (2014) addressed the design of a hybrid vehicle layout at vehicle and battery levels. At the upper level, the layout of components in the under-hood of the vehicle is optimized, while the design of the battery pack is optimized at the lower level considering thermal and location criteria.

6 Perspectives in Multiobjective Bilevel Programming

Multiobjective bilevel programming is currently a subject of important research efforts from different communities, ranging from mathematical programming to (population-based) metaheuristic approaches. It has also received increasing attention from different application areas as there are many practical situations involving hierarchical optimization problems with multiple objective functions at one or both levels. However, several conceptual and computational challenges remain, particularly with respect to the general properties of MOBP/SVBP and their impact on the development of effective algorithms.

Improving classical algorithms (using branch and bound, enumeration or penalty techniques or KKT conditions), for certain classes of bilevel programming problems (e.g. linear or with lower-level convex problems), has the potential to trigger advances in multiobjective bilevel problems. Concerning approximation algorithms (heuristics or metaheuristics) when dealing with practical applications, the consideration of the intrinsic characteristics of the application can lead to more efficient algorithms and improve the accuracy of the final solutions. In addition, the combination of optimization techniques with well-tried techniques from other fields, such as machine learning, is also a promising research avenue.

Still, one of the main challenges in dealing with MOBP/SVBP concerns the feasibility of solutions. When approximation algorithms are used on multiobjective optimization problems (with a single level), there is no guarantee that true efficient solutions are obtained, but the feasibility of the solutions can be easily assessed by checking the constraints. In MOBP/SVBP, solutions that are not efficient to the lower level problem do not belong to the inducible region, i.e. they are not feasible to the bilevel problem. The possibility of getting solutions whose feasibility status is not easily recognized makes the problem more difficult to handle.

Regarding applications, several multiobjective formulations that have been proposed in the literature may be reformulated with advantage as bilevel models due to their intrinsic hierarchical relationship.

This section details some pitfalls associated with the evaluation of solutions and algorithm performance in SVBP/MOBP, a view on research avenues and novel applications with focus in the area of smart grids.

6.1 Pitfalls

Due to the theoretical and computational difficulties to solve the SVBP and MOBP, metaheuristic approaches have gained particular relevance, namely the ones based on the evolution of a population of solutions (genetic/evolutionary algorithms, particle swarm optimization, differential evolution). However, the inherently approximate nature of these techniques may lead to misleading results. In fact, since only efficient solutions to the lower level problem are feasible to the bilevel problem, approaches based on metaheuristics may yield apparently better solutions to the SVBP/MOBP but these solutions are invalid because they are not truly efficient to the lower level problem, although they may constitute good approximations. This may lead the algorithm to choose these solutions over truly efficient solutions, because they are better to the leader, presenting at the end of the computation process solutions that seem of good quality but are unfeasible. Indeed, this is an important pitfall that researchers must be aware of when comparing algorithms for SVBP/MOBP, both for the computation of the optimistic solution/optimistic Pareto front and for other types of solutions: a better front will not mean better performance of the algorithm if the efficiency of lower level solutions is not ensured. Although this drawback is intrinsic to these problems, whenever possible hybrid approaches may avoid it by coordinating metaheuristics with good computational performance for the upper level search with exact mathematical programming algorithms to solve the lower level problem for each instantiation of the upper level variables.

Moreover, even if only efficient solutions to the lower level problem are obtained, the assessment of pessimistic and deceiving solutions in SVBP is not straightforward since those solutions may not belong to the pessimistic frontier (i.e., they are not the worst for the leader for the corresponding setting of upper level decision variables) and therefore are not valid.

Similar difficulties arise in the computation of a pessimistic upper level Pareto front in MOBP. In this setting, one possibility is considering that the follower's decisions that are included in the pessimistic upper level Pareto front are the *most dominated* for the leader. For instance, consider two follower's efficient solutions for the same x: (x, y^1) and (x, y^2) with $F(x, y^1) \succ F(x, y^2)$; then, a pessimistic approach would assume that the follower chooses y^2, the worst option for the leader. However, the identification of these *most dominated* solutions may be computationally difficult.

6.2 Dealing with the Leader's Risk Versus Opportunity

As mentioned in Example 1, in SVBP the leader may take a high risk engaging in an optimistic attitude if the deceiving solution is significantly worse than the pessimistic one. In addition, if he engages in a pessimistic attitude, he still has the opportunity to obtain the rewarding solution, which may be close to the optimistic one, as in Example 1. Therefore, it is important for decision support purposes to offer the leader other types of solutions in addition to the four extreme ones proposed, that is, *moderate* solutions providing the highest expected value considering an optimism/pessimism index (e.g., probabilities of the follower's decision being in favor or against the interests of the leader). Another approach would be considering different plausible settings of the follower's preferences. The computation and comparison of solutions to SVBP/MOBP for these preferences would provide relevant information regarding balancing the risk and the opportunity associated with each leader's decision in face of the follower's possible choices.

6.3 Difficulties of the Pessimistic Approach

Almost all algorithms developed thus far for SVBP/MOBP have focused on the computation of the optimistic solution/Pareto front. As far as we know, there is no algorithm to compute the pessimistic Pareto front in MOBP. However, as mentioned above, considering only optimistic solutions may be unrealistic in many practical situations because this approach presumes that the follower does not have preferences and is indifferent to all efficient solutions obtained for a given decision of the leader.

Therefore, an emerging research field is the development of effective algorithms capable of computing not only the optimistic solution/optimistic Pareto front, but also the pessimistic solution/pessimistic Pareto front and other solutions that can offer the leader information about the risk versus opportunity provided by adopting a given strategy.

6.4 Novel Applications in Smart Grids

SVBP and MOBP are very relevant for decision support in actual decision problems, in which design and policy decision are at stake in the upper level and operational decisions should then be made at the lower level, possibly involving different stakeholders with potential conflicting interests.

An emerging application area in the realm of smart grids, in which consumer empowerment becomes a crucial issue, is the definition of dynamic tariffs as a component of the portfolio offers of a retailer (the leader) considering variable wholesale prices and network status. Consumers (the followers) confronted with time-differentiated prices are expected to engage in demand response actions by rescheduling appliance operation to less expensive periods and/or resetting thermostats thus trading-off cost and comfort dimensions. New applications of SVBP/MOBP optimization entail, for instance, managing congestion at distribution transformers by harnessing the flexibility associated with the charge/discharge of electric vehicle batteries as well as enabling load aggregator entities to participate in ancillary services or capacity markets. In these problems, the consideration of multiple followers, which may share decision variables, objective functions and constraints, is of utmost importance, since in the leader's outcome may be affected by the relationships among followers.

7 Concluding Remarks

In this chapter we presented the main concepts of multiobjective bilevel problems, in which multiple objective functions may arise in each level problem or in both. When multiple objective functions are at stake in the lower level problem, uncertainty regarding the follower's choice within his efficient solution set should be considered. Optimistic and pessimistic leader's perspectives were explored by characterizing the extreme optimistic/deceiving and pessimistic/rewarding solutions in semivectorial bilevel problems and the Optimistic Pareto front in multiobjective bilevel problems, which result from different leader stances and follower reactions. Their relevance for decision support as well as the potential difficulties and pitfalls associated with computing solutions to bilevel models in a multiobjective setting were discussed. Traditional and emerging application areas were reviewed and perspectives of development in multiobjective bilevel optimization were outlined.

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Part IV
Applications

Multi-criteria Evaluation in Public Economics and Policy

Giuseppe Munda

Abstract Public administrations need to assess policy options before their implementation; often there is some uncertainty if cost-benefit analysis (CBA) or multi-criteria evaluation (MCE) should be used. This Chapter aims at showing that MCE may help economics at overcoming some of its current difficulties in the empirical assessment of public policy options; thus MCE has to be placed in the future of welfare economics with no doubt. To corroborate this conclusion, a structured comparison of the main distinguishing features of CBA and MCE is carried out according to the following ten comparison criteria: efficiency, fairness, democratic basis, effectiveness, problem structuring, alternatives taken into account, policy consequences, comprehensiveness, transparency and mathematical aggregation rule.

Keywords Multiple criteria analysis · Public policy · Cost-benefit analysis · *ex-ante* impact assessment · Welfare economics

JEL Classification A12 · C44 · D04 D61 · R58

> *… there is such a long tradition in parts of economics and political philosophy of treating one allegedly homogeneous feature (such as income or utility) as the sole 'good thing' that could be effortlessly maximized (the more the merrier), that there is some nervousness in facing a problem of valuation involving heterogeneous objects, …*
>
> *And yet any serious problem of social judgement can hardly escape accommodating pluralities of values, …*
>
> *We cannot reduce all the things we have reason to value into one homogeneous magnitude.*
>
> (A. Sen, The Idea of justice, The Belknap Press of Harvard University Press, Cambridge, Massachusetts, 2009, p. 239).

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1 Introduction

When a public administration wishes to implement policies, there is a previous need of comparing different options to assess their social attractiveness. One of the key tasks of mainstream welfare economics is exactly this valuation and evaluation exercise. To define what exactly valuation and evaluation connote is not an easy task, "*… we value when comparing objects and evaluate when comparing the relative merits of actions. … There is a sense in which valuation is passive, while evaluation signifies more of an active engagement. We frequently value in order to evaluate. But not always. We sometimes value simply because we wish to understand a state of affairs, such as the quality of life in a country. Welfare economics studies life's quality, valuing objects and evaluating policies being only a means to measuring the quality of life and to discovering ways to improve it*" (Dasgupta 2001, p. C1).

In my opinion, a fair policy assessment process should consider the ethical obligation of taking a plurality of social values, perspectives and interests into account. A question then arises: which is the current practice in public economics?

In a recent[1] Financial Times debate on the future of economics, the main question was "*has economics failed?*" Some conclusions can be summarised as follows:

- Everybody agrees that all societies need to make decisions on important public issues; in this framework, the concepts of scarcity and opportunity cost cannot be ignored. Economics has developed many important tools to help real-world policy-making although is far from being an exact science.[2]
- "*It is not that the answers economists will give will necessarily be right. But they will be done within a systematic and rigorous framework. That is far better than merely waving our hands in the air… yet economics also suffers from fundamental difficulties … One difficulty is that economics studies just one dimension of human social interaction. Separating economic behaviour from the other aspects of society can be seriously misleading*[3]."
- As theorised by Amartya Sen, democracy to be effective in practice, needs a shared language of the public sphere. "*The justification for rules that govern society must be intelligible to everyone so that all can participate in the debate. On these grounds economics has certainly failed. While the discipline has influence over politics and policy it struggles to explain why its prescriptions are needed …. The problem is made worse by those who use economics as a weapon to win arguments for their own values*[4]." This creates a situation where the people who master the economic language may rationalise ideas they already believe a priori.

[1]May, 2018, more than 300 readers commented the initial debate between Tom Clark, editor of Prospect Magazine and Chris Giles, the FT's economics editor.

[2]See e.g. comments by Maurice Obstfeld, director of the International Monetary Fund research department and Tony Yates, former monetary economist at the Bank of England.

[3]Statement due to Martin Wolf, FT's chief economics commentator.

[4]Statement due to Gavin Jackson FT's economics reporter.

- Many economists believe that the only economic value is the concept of "shareholder value" which has been diffused by business schools. This mainstream view has mainly be constructed since "*most economics students are not trained in the history of economic thought so they think there is only one theory of value—which is not presented as a possible theory among many*[5]". There is a total lack of debate on this topic.
- The lack of diversity and value pluralism in the economic science "*unusually narrow criteria for advancement, … but economics is a living science, looking at a constantly changing society, and there is a lot to learn*[6]".

This Chapter aims at showing that multi-criteria evaluation (MCE) may help economics at overcoming some of these difficulties in the empirical assessment of public policy options. I strongly believe that MCE is the future of welfare economics. To corroborate this statement here a structured comparison of the main distinguishing features of the traditional welfare economics evaluation tool, i.e. cost-benefit analysis (CBA) and multi-criteria evaluation are discussed according to ten comparison criteria: efficiency, fairness, democratic basis, effectiveness, problem structuring, alternatives taken into account, policy consequences, comprehensiveness, transparency and mathematical aggregation rule. Section 2 deals with efficiency, fairness and democratic basis in the framework of cost-benefit analysis. Section 3 deals with the same three comparison criteria in the framework of multi-criteria evaluation. Section 4 presents a systematic comparison between CBA and MCE according to the other seven criteria. Finally some conclusions are illustrated.

2 Cost-Benefit Analysis: Efficiency, Fairness and Democratic Basis

The rationality behind cost-benefit analysis assumes that any individual makes rational decisions only if she/he weighs up the advantages and disadvantages of a particular action, so that some kind of best decision can always be made. Of course, the essence of cost-benefit analysis is that it is not confined to decisions that affect one individual; it relates to social decisions. Then, does the characteristic of rationality remain if we extend it to the social context? The basic argument underlying CBA is that this rationality does remain. That is, if individuals are left free to carry out their own personal cost-benefit analyses in respect to a given policy, then we can simply aggregate the results to secure a social assessment (Munda et al. 1995).

The notion of individual preference used in the Kaldor-Hicks compensation principle, which is the methodological foundation of cost-benefit analysis, is the preference expressed on the market place (or which would be expressed if there were a

[5]Mariana Mazzucato, professor at University College London.

[6]Diana Coyle, professor at University of Cambridge.

market) (see e.g. Hicks 1939; Kaldor 1939; Mishan 1971; Pearce and Nash 1989). This kind of "economic democracy" is preferred to classical political voting on the following grounds (see also Munda 2014):

1. The Kaldor-Hicks principle declares a social state $\boldsymbol{S_1}$ "socially preferable" to an existing social state $\boldsymbol{S_0}$ if those who gain from the transition to $\boldsymbol{S_1}$ can compensate those who lose and still have some gains left over. In political democracy, minorities must accept decisions taken by majority, on the contrary, in the framework of the Kaldor-Hicks compensation principle, losers receive compensation; this appears an improvement of the fairness of the policy process.
2. Economic democracy always reflects voters' preferences. If a voter *can be considered as a consumer*, then if she/he does not like a good, she/he does not buy it on the market.
3. To observe consumers' behaviour on the market is much cheaper, quicker and easier than political referenda on any specific policy option.
4. "*The use of money values permits some expression of the intensity of preference in the vote: it enables the individual to say how deeply he wants or does not want the project or good in question*" Pearce and Nash (1989, p. 7).

Although Kaldor and Hicks were interested in implementing objective Pareto efficiency, *explicitly not grounded on egalitarian considerations*, economic democracy appears to perform better than political democracy. But is this really true?

One should consider that by only taking preferences expressed on the market into account, the comparison of people is carried out according to *one objective* and *one institution* only, i.e. economic efficiency and markets. Different objectives and values, e.g. sustainability or fairness cannot be considered (Munda 2016); indeed individual's preferences as a consumer may differ from the ones as a citizen a lot (Sagoff 1988).

In establishing policy objectives, a key issue is: *does society wish to assign any resource to these objectives?* Clearly it is impossible to avoid the economic problem of "opposition between tastes and obstacles", as Pareto made clear. Cost-benefit analysis deals with this issue correctly. However, if the losers are poor (or even not yet been born), the compensation is always low. In fact, in a CBA framework, costs and benefits are often aggregated linearly in a net present value (NPV) formula:

$$SW = \sum_h U_h \tag{1}$$

where the subscript h denotes the individual to whom the utility function applies. Under the assumption that the marginal utility of money income (λ) is *equal for all individuals*, the variation of this social welfare function indicating the social worth of a project is:

$$\Delta SW = \sum_h \sum_i \frac{\partial U_{ih}}{\partial Y_{ih}} \cdot \Delta Y_{ih} = \lambda \sum_h \sum_i P_i \Delta Y_{ih} = \lambda \sum_i P_i \Delta Y_i \tag{2}$$

where h subscript denotes the individual to whom the utility function and quantity of the good Y_i apply. The translation into monetary terms is accomplished by the Eq. (3)

$$\lambda \frac{\partial U_i}{\partial Y_i} = P_i \tag{3}$$

where P_i is the (relative) price of good i.

Obviously, the assumption of the constancy of the marginal utility of income is mainly an ideological one. In contemporary welfare economics distributional consequences of policy options are taken into account. The most widespread approach is to attach different weights to different income groups (Bojo et al. 1990). However, it is not clear how to derive such weights and who should attach them.[7] On the other hand, not using any weighting system implies making the *implicit assumption* that the existing distribution of income is ideal. If, and only if, one is happy with such a value judgement, it is reasonable to use un-weighted market valuations to measure costs and benefits. *Therefore, there is no escape from value judgements,* the compensation principle is not the positivistic objective evaluation criterion Hicks hoped to be. On the other side it does not consider individuals as equal exactly the goal Kaldor aimed at, *it can be considered a direct application of the ancient principle that property owners should count more*.

3 Multi-criteria Evaluation: Efficiency, Fairness and Democratic Basis

The most widespread non-monetary approach to *ex-ante* Impact Assessment (IA) is multi-criteria evaluation (MCE). The basic methodological foundation of MCE is *incommensurability*, i.e. the notion that in comparing options, a plurality of dimensions and perspectives is needed (Frame and O'Connor 2011; Martinez-Alier et al. 1998; Munda 2016; O'Neill 1993). The fact that "*one's welfare economics will inevitably be different according as one is a liberal or a socialist, a nationalist or an internationalist, a Christian or a pagan*" (Hicks 1939, p. 696) is the normal state of affairs in public policy. There is no obvious reason why this issue of existence of a plurality of values should be considered a problem that can be solved by considering

[7] From the technical point of view, one should note that the fact that intensity of preference is taken into account inside a linear aggregation rule, has the consequence that weights must be considered as trade-offs. A question then arises: in their standard use, are distributional weights used as importance coefficients or as trade-offs? The basic idea underlying all the different weighting methods can be summarized by quoting the following sentence: "*if the decision-maker considers individual 2 more "deserving" than individual 1 he will weight 2's losses more heavily than 1's gains i.e. 2 > 1*" (Dasgupta and Pearce 1972, p. 65), thus weights should be considered as importance coefficients. Unfortunately, since CBA is based on a completely compensatory mathematical model, weights can only have the meaning of a trade-off ratio, as a consequence a theoretical inconsistency exists (see Munda 1996 for more details on this issue).

Table 1 Example of an impact matrix

		Alternatives			
Criteria	Units	a_1	a_2	a_3	a_4
g_1		$g_1(a_1)$	$g_1(a_2)$		$g_1(a_4)$
g_2					
g_3					
g_4					
g_5					
g_6		$g_6(a_1)$	$g_6(a_2)$		$g_6(a_4)$

consumers' preferences as the only relevant social values. A question arises here: *is it more scientific (and fair) an approach dealing with such a plurality of values explicitly or one which solve all conflicts by imposing a perspective considered superior on some ethical or technical grounds?*

The basic idea of multi-criteria evaluation (MCE) is to achieve the comparability of incommensurable metrics. From an operational point of view, the major strength of MCE is its ability to deal with policy issues characterised by various contradictory evaluations, thus allowing for an integrated assessment of the problem at hand. Being a decision tool, MCE focuses on the issue of the *opportunity cost* connected to the choice of any policy option, thus efficiency is surely an important objective to be considered. Differently from economic efficiency assessment tools such as CBA or frontier methods such as Data Envelopment Analysis (DEA), traditionally used in operational research (see Emrouznejad and Yang 2018 for a recent overview), MCE is based on a multidimensional definition of efficiency, where inputs and outputs are not transformed into a single measurement rod. Of course, CBA or DEA could also be one of the criteria used in a MCE exercise, but never the only ones. A clear advantage of MCE is that different objectives, such as efficiency, equity or sustainability can be treated separately in a transparent way.

A "*discrete multi-criterion problem*" can be formally described as follows (see e.g. Figueira et al. 2016; Ishizaka and Nemery 2013): A is a finite set of N feasible actions (or alternatives). M is the number of different points of view, or evaluation criteria, g_m, that are considered relevant to a specific policy problem. Where action $\boldsymbol{a}$ is evaluated to be better than action $\boldsymbol{b}$ (both belonging to the set A), by the m-th point of view, then $g_m(a) > g_m(b)$. In this way a decision problem may be represented in an N by M matrix P called an *evaluation or impact matrix*. In such a matrix, the typical element p_{mn} ($m = 1, 2, \ldots, M$; $n = 1, 2, \ldots, N$) represents the evaluation of the n-th alternative by means of the m-th criterion, in other words, each criterion score represents the performance of each alternative according to each criterion (see Table 1). The impact matrix may include quantitative, qualitative or both types of information.

In a discrete multi-criteria problem, there is a range of multi-criteria problem formulations, which may take one of the following forms (Roy 1996):

(α) the aim is to identify one and only one final alternative;
(β) the aim is the assignment of each alternative to an appropriate predefined category according to what one wants it to become afterwards (for instance, acceptance, rejection or delay for additional information);
(γ) the aim is to rank all feasible alternatives according to a total or partial pre-order;
(δ) the aim is to describe relevant alternatives and their consequences.

In synthesis, the information contained in the impact matrix useful for solving the so-called multi-criterion problem is:

- Intensity of preference (when quantitative criterion scores are present).
- Number of criteria in favour of a given alternative.
- Weight attached to each single criterion.
- Relationship of each single alternative with all the other alternatives.

Combinations of this information generate different mathematical aggregation rules. In general in a multi-criterion problem, there is no solution optimising all the criteria at the same time (ideal or utopia solution), and therefore "compromise solutions" have to be found. As noted by Arrow and Raynaud (1986), unlike other mathematical fields, only "reasonable" mathematical procedures can be developed in this framework. Reasonable here means that algorithms can be evaluated not only according to the *formal properties* they present, but, overall, according to the *empirical consequences* implied by their use.

Historically the first stage of the development of MCE is characterised by the so-called methodological principle of *multi-criteria decision making* (*MCDM*). The main objective of this approach is first to elicit preferences from a decision-maker, and then solve a well-structured mathematical decision problem (see e.g. Keeney and Raiffa 1976).

The limitations of the classical concept of an optimum solution and the consequential importance of the *decision process* were emphasised by authors such as Herbert Simon (e.g. 1976) and Bernard Roy. According to Roy (1996) saying that a decision is a good or bad one is in general impossible on the basis of referring only to a mathematical model. All aspects of a decision process which leads to a given decision also contribute to its quality and success. Thus, establishing the validity of a procedure is impossible, either based on a notion of *approximation* (i.e., discovering pre-existing truths) or on a mathematical property of *convergence* (i.e., does the decision automatically lead, in a finite number of steps, to the optimum a^*?). The final solution is more like a "creation" than a discovery. Under the concept of a *Multiple-Criteria Decision Aid* (*MCDA*) the principal aim is not to discover a solution, but to construct or create something which is viewed as liable to help an actor taking part in a decision process either to shape, argue, and/or transform her/his preferences, or to make a decision in conformity with his/her goals (Roy 1996).

A point to be considered is that in CBA economic votes of all individuals might be used (of course under the assumption that the current distribution of income is considered acceptable). On the other hand, MCE may be based on the preferences, perspectives and interests of a restricted number of policy-makers only. The need for public participation has been increasingly recognised in public policy analysis

(Guimarães-Pereira et al. 2006; Funtowicz and Ravetz 1991; O'Neill 2001). *Social Multi-Criteria Evaluation* (*SMCE*) tries to extend MCDA by incorporating the notion of social actor. Thus, a SMCE process must be as participative and as transparent as possible; although, participation is a necessary but not a sufficient condition for successful evaluation (Munda 2004, 2008). This is the main reason why the concept of SMCE is proposed in place of Participatory Multi-Criteria Evaluation or Stakeholder Multi-Criteria Decision Aid (Banville et al. 1998).

In a SMCE framework, fairness can be seen as an ethical obligation to take a plurality of social values, perspectives and interests into account in a coherent and transparent manner[8] (some real-world examples can be found in Cerreta and De Toro 2010; Gamboa 2006; Gamboa and Munda 2007; Garmendia and Stagl 2010; Lerche et al. 2017; Monterroso et al. 2011; Özkaynak 2008; Scolobig et al. 2008; Soma and Vatn 2009; Straton et al. 2010; Zendehdel et al. 2010). For example, the European Commission current practice on Impact Assessment considers three main objectives i.e. efficiency, effectiveness and coherence and it is based on the assessment of various broad impacts such as economic, environmental and social (including distribution of costs and benefits among social actors) ones.

The main accomplishment of SMCE is that a wide range of evaluation criteria (incommensurable from a technical point of view) has a direct translation in terms of plurality of values and perspectives (incommensurable from a social point of view) used in the evaluation exercise.

In this framework, mathematical models still play a very important role, i.e. the one of guaranteeing consistency between assumptions used and results obtained. This is a key success factor since multi-criteria mathematics does answer to the standard objection that the aggregation of apples and oranges is impossible in a definitive way.

4 A Systematic Comparison Between Cost-Benefit Analysis and Multi-criteria Evaluation According to Other Seven Criteria

In this section, a comparison of the key characteristics of cost-benefit analysis and multi-criteria evaluation will be carried out on the base of other seven comparison criteria, that is: effectiveness, problem structuring, alternatives taken into account, policy consequences, comprehensiveness, transparency and mathematical aggregation rule.

[8]It has to be clarified that the concept of fairness is different from the one of an equal distribution of income. A society could have a *fair inequality* if the economic system promotes and rewards individual efforts. Clearly ethical connotations are there; this implies that people, social scientists and governments differ significantly on what they consider to be fair. Overall there is agreement on the fact that evaluation of fairness should be linked to the *social process* leading to a certain outcome and not to the outcome itself (i.e. when differences in the final income distribution of a society exist, this does not mean that the society has unfair rules).

(a) **Effectiveness**

It is of key importance understanding that efficiency alone cannot be a relevant policy objective. *Effectiveness* (i.e. the degree to which goals and levels of output are achieved or problems are solved) is at least equally important; otherwise there is the risk to drive the policy evaluation framework towards a situation where efficiency means just "cheap" (Agasisti et al. 2017). For this reason, it is important to have a clear understanding of the difference between efficiency and effectiveness. To clarify this point, let's consider the following four situations obtained by combining efficiency with effectiveness in a public policy framework:

Effective	(A) Goals are achieved, (e.g. high education levels, good environmental quality standards, low percentage of population at risk of social exclusion…) but too many resources are used. The system is effective but there is a waste of resources	(B) Goals are achieved by using a reasonable amount of resources. Of course, this is the best situation. An obvious issue is the definition of what a "reasonable amount" means
Non-effective	(C) Goals are not achieved and a lot of resources are used. This is the worst situation	(D) Goals are not achieved but the amount of resources used is low. The system looks efficient (because it uses wisely poor resources) but it is non-effective. In this case efficiency is confused with parsimony
	Non-efficient	Efficient

It is immediately evident that efficiency is a relevant policy objective only and only if it is considered in combination with effectiveness; otherwise two different systems (e.g. countries, regions, cities, …) might present the same level of efficiency, with very different levels in goal achievements.

Moreover, one should note that market based policy tools such as CBA may be successful in efficient allocation of resources, but do not help e.g. in the protection of cultural or natural heritage. Of course, monetary compensation is the only possible tool when an irreparable and irreversible damage has already occurred, but is compensation effective for avoiding future social costs? Society has a much longer life expectancy than individuals, thus the value society attaches to e.g. natural resources is likely to deviate from individual values. Walras himself already noted that the market cannot be used as a basis for rational collective decision-making and that "*human destinies are not absolutely independent, but to some extent dependent on one another. There is a social morality which is distinct from individual morality*" (cited in Burgenmeier 1994, p. 347).

The presence of irreversibility and uncertainty can justify public policies based on the *precautionary principle*, which implies higher financial costs, but how much would the non-application cost? The burden could be enormous, as stated by the European Environment Agency.[9] In this framework an adequate policy tool is cost-effectiveness (CEA); that is given a certain *physical* target (e.g. the amount of cultural heritage to be protected or the amount of contamination considered acceptable), this should be achieved by the lowest possible use of resources. Two rankings may appear easily:

i. According cost (the lower, the better).
ii. According to the physical target (e.g., the more cultural heritage protected, the better).

As a consequence, CEA can easily be transformed into multi-criteria evaluation since at least two criteria and two different rankings have to be tackled inevitably. In summary dealing with effectiveness necessarily implies the use of MCE, in no way CBA alone can deal with it properly.

(b) **Problem Structuring**

The application of CBA requires the following main steps:

i. Choice of costs and benefits to be taken into account.
ii. Transformation of costs and benefits into money figures.
iii. Selection of the social discount rate.
iv. Selection of the time horizon considered relevant for the policy problem at hand.
v. Choice of a mathematical aggregation rule.
vi. Sensitivity analysis of results.

The implementation of a SMCE framework involves the following main steps:

i. Selection of the social actors relevant for the problem at hand.
ii. Definition of social actors' values, desires and preferences.
iii. Generation of policy options and selection of evaluation criteria as a process of co-creation resulting from a dialogue between analysts and social actors. In this way, evaluation criteria become a technical translation of social actors' needs, preferences and desires.
iv. Construction of the multi-criteria impact matrix synthesising the scores of all criteria for all policy alternatives, i.e. the performance of each option according to each criterion.
v. Construction of an equity impact matrix, illuminating all the distributional consequences of each single option on the various social actors.
vi. Application of a mathematical procedure.
vii. Sensitivity and robustness analysis with respect to the exclusion/inclusion of different criteria, criterion weights and dimensions.

[9] Late lessons from early warnings: the precautionary principle 1896–2000, European Environment Agency, Environmental issue report, No. 22, 2001.

(c) **Alternatives Taken Into Account**

CBA can evaluate any finite set of alternatives (even only one policy option). Arrow's axiom of independence of irrelevant alternatives[10] always applies since the desirability of each policy option is independent from all other ones (Arrow 1963). In MCE, any finite or even infinite (in the case of continuous approaches such as multi-objective linear programming) number of alternatives can be taken into account. With respect to the axiom of independence of irrelevant alternatives, some mathematical aggregation rules (e.g. multi-attribute utility theory) respect it; while other approaches (e.g. outranking methods) provide rankings which are also a function of the options taken into account.

(d) **Policy Consequences**

CBA results depend on the selection of costs and benefits and their proper conversion into monetary values. This is not easy at all. Valuation approaches present various methodological issues (see e.g. Copp 1987; Fusco 1986; Hammitt 2013; Hansson 2007; Lo and Spash 2013; O'Neill 1993; Sagoff 1988; Spash 2008) and technical uncertainties (see e.g. Aldred 2009; Frey 1986; Grüne-Yanoff 2009; Hansen 2011; Martinez-Alier et al. 1998; Munda 1996; Vatn and Bromley 1994). In MCE, a plurality of evaluation criteria, based on a multidimensional set of metrics, can be used. The criterion scores can be quantitative (measured on interval or ratio scales) or qualitative (measured on nominal or ordinal scales), uncertainty both stochastic and fuzzy can be dealt with.

In the selection of evaluation criteria, two main problems can be found in the real-world practice: (1) the evaluation model is designed with the objective of being as close as possible to the real-world problem; this may increase the number of evaluation criteria so much that its transparency is close to zero. (2) Only a limited number of criteria is used so that the model is simpler and faster to use; this may lead to an oversimplification of the real-world situation.

In the decision theory literature it is claimed that criteria should present two important properties (Roy 1996):

(1) "*Legibility*", i.e. the set of criteria should allow the possibility of discussion for e.g. assessing the inter-criteria information leading to the choice of a mathematical aggregation rule. Only non-redundant criteria should be used, this normally reduces the number of criteria to be taken into account.
(2) "*Operationality*", i.e. all actors participating in the evaluation process should consider the set of evaluation criteria chosen as a sound basis for the evolution of the study.

[10]Arrow's axiom of "the independence of irrelevant alternatives" states that the choice made in a given set of alternatives A depends only on the ordering made with respect to the alternatives in that set. Alternatives outside A (irrelevant since the choice must be made within A) should not affect the choice inside A.

(e) **Comprehensiveness**

A variety of policy objectives such as efficiency, equity, effectiveness, sustainability and so on can hardly be incorporated in a cost-benefit analysis simultaneously. MCE being multidimensional in nature, can easily tackle different and conflicting policy objectives measured on different metrics.

(f) **Transparency**

In CBA all policy consequences need to be converted into money terms and then aggregated. This reduces the potentialities that all consequences are clearly understood by the general public (although the illustration of background data may increase transparency). In MCE, all policy consequences are shown in the original metrics; this high degree of transparency is one of its most important success factors.

(g) **Mathematical Aggregation Rule**

CBA is based on traditional aggregation rules such as NPV or the Internal Rate of Return (IRR). However, it is not always clear which aggregation rule is the most adequate in a given policy framework. Many authors try to show that NPV is a superior approach; while other authors try to prove that various aggregation rules, under specific conditions, arrive at the same ranking of policy options. An exceptional case is the field of education economics where the internal rate of return is widely used. If NPV is used, since it can be considered an additive utility function, the condition of preference independence should always hold[11] (Munda 1996).

Since MCE is not based on the traditional quality criteria of approximation and convergence, its mathematical foundations are not unique and thus various approaches exist; this is a weak point of MCE surely (see e.g. Bell et al. 2003). An issue, that makes MCE aggregation rules very complex, is the fact they are *formal, descriptive and normative* models simultaneously. Musgrave (1981) proposes to divide assumptions into three categories: *negligibility assumptions, domain assumptions and heuristic assumptions.* The first type is required to simplify and focus on the essence of the phenomena studied. The second type of assumptions is needed when applying a theory to specify the domain of applicability. The third type is needed either when a theory cannot be directly tested or when the essential assumptions give rise to such a complex model that successive approximation is required.

Here, I will indicate some properties that can be considered desirable for a discrete multi-criteria aggregation rule (often called multi-criteria *method*) in the framework of public policy (see Munda 2008 for more details on this point). They are the following ones:

[11] This property is a necessary condition for the existence of a linear aggregation rule. From an operational point of view this means that an additive aggregation function permits the assessment of the marginal contribution of each cost and benefit separately. Each marginal contribution can then be added together to yield a total value. This implies that among the different aspects of a policy option there are no phenomena of synergy or conflict, this is rather unrealistic from a scientific point of view.

Descriptive domain assumptions:

- Mixed information on criterion scores should be dealt with.

Normative domain assumptions:

- Simplicity is desirable to guarantee coherence between the problem structuring and the results obtained, and means the use of as less ad hoc parameters as possible.
- The most useful result for policy-making is a complete ranking of alternatives.
- Weights are meaningful only as importance coefficients and not as trade-offs.
- Complete compensability[12] is not desirable.

Heuristic descriptive assumptions:

- When not all intensities of preference are meaningful, indifference and preference thresholds are useful exogenous parameters.[13]
- Dominated alternatives have to be considered.[14]

Finally one should note that these selection properties can be applied only to methods who achieve a set of minimum formal requirements, the most important being the following[15].
Formal domain assumptions:

- Unanimity.
- Monotonicity.
- Neutrality.

Negligibility formal assumptions:

- Anonymity.

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[12]Complete compensability is not desirable for the problem we are dealing with, since it implies that e.g. a good performance on efficiency would offset a very bad one on effectiveness or vice versa. It has to be noted that CBA always allows the highest degree of compensability since it is explicitly based on the Kaldor-Hicks compensation principle and costs and benefits are aggregated linearly (and thus in a compensatory fashion).

[13]This relates to the famous bald paradox in Greek philosophy (how many hairs one has to cut off to transform a person with hairs to a bald one?), later on Poincaré (1935, p. 69) and finally Luce (1956) made the point that the transitivity of indifference relation is incompatible with the existence of a sensibility threshold below which an agent either does not sense the difference between two objects, or refuses to declare a preference for one or the other. Luce was the first one to discuss this issue formally in the framework of preference modelling. Mathematical characterizations of preference modelling with thresholds can be found in Roubens and Vincke (1985).

[14]This of course applies to discrete methods only and implies that the aggregation rules belong to the family of non-frontier methods.

[15]In social choice, the reaction to Arrow's theorem has been the search for less ambitious voting structures; there is a need to keep a few basic requirements only. These basic requirements are generally three:

1. Anonymity: all criteria must be treated equally.
2. Neutrality: all alternatives must be treated equally.
3. Monotonicity: more support for an alternative cannot jeopardize its success.

One should note that, while anonymity is clearly essential in the case of voters, it is not so in the multi-criterion problem since criterion weights can be normally introduced.

5 Conclusions

Cost-benefit analysis is grounded on market mechanisms; this implies that only the behaviour of individuals as consumers is considered. Is this fully acceptable in public policy? CBA and MCE can be considered as competitive methods only if all consequences of a policy decision can be correctly transformed into monetary values and efficiency is the only relevant policy objective. Obviously, when a plurality of policy objectives exists, CBA can be used as a criterion in a MCE framework dealing with the objective of efficiency in a consistent way. We can thus conclude that CBA and MCE are complementary in nature (MCE being the most comprehensive one).

In CBA complete compensability and preference independence are always assumed; in MCE, various mathematical aggregation rules exist. This makes MCE more flexible but also more confusing since a method has to be selected and policy option rankings may be very sensitive to this step. On the other hand this problem of "method uncertainty" is also present in CBA, since numerous valuation techniques, time horizons, discount rates and aggregation rules (e.g. NPV or IRR) may be chosen.

Social multi-criteria evaluation seems to be an appropriate public policy framework to integrate different scientific and social languages, when concerns about civil society and future generations have to be considered along with policy objectives and market conditions. In this framework, ex-ante policy assessment can be defined as the combination of representation (social actors, criteria, weights and actions considered), valuation (construction of criterion scores), mathematical aggregation (formal properties of the algorithms used) and quality check (transparency of the steps by which a multi-criterion model is built) connected to a given policy problem.

We may conclude that the SMCE approach is fully consistent with the recent research directions in the field of welfare economics and public policy, which are characterised by the attempt of introducing political constraints, interest groups and collusion effects into the analysis explicitly (see e.g. Laffont 2000). In this context, transparency becomes an essential feature of public policy processes (Stiglitz 2002).

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Perspectives on Multi-criteria Decision Analysis and Life-Cycle Assessment

Luis C. Dias, Fausto Freire and Jutta Geldermann

Abstract This chapter covers the combined use of Multi-Criteria Decision Analysis (MCDA) and Life-Cycle Assessment methodologies. It first reviews environmental Life-Cycle Assessment (LCA), introduces the main challenges and perspectives, including how to extend LCA towards Life Cycle Sustainability Assessment (LCSA), and discusses how LCAs might be useful for the MCDA practitioner. Then, it discusses how MCDA can complement LCA. Challenges and perspectives are presented concerning LCSA, relative versus absolute evaluation, criteria weighting, and criteria selection.

1 Introduction

Environmental Life-Cycle Assessment (LCA) is a well-known methodology in the fields of industrial ecology and environmental management. It aims at quantifying the environmental impacts of a product or service in a holistic and integrated manner, over its life cycle, on different dimensions called impact categories. This is fundamental to avoid shifting burdens between environmental impacts or from one part of the product life cycle to another (e.g., from production to consumption). The standardized LCA methodology (ISO 2006a, b) addresses only environmental aspects, usually giving rise to multiple impact indicators (e.g., depletion of resources, impacts of

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emissions on the environment and on human health). Over time, however, LCA-based approaches have emerged that focus on Life-Cycle Costing (LCC), Social Life Cycle Assessment (SLCA) and, more recently, in a multi-dimensional approach to sustainability, Life Cycle Sustainability Assessment (LCSA = LCA + LCC + SLCA) (Kloepffer 2008).

Multi-criteria decision analysis (MCDA) is an approach to evaluate alternatives (policies, projects, etc.) in the context of selection, ranking and classification problems. MCDA recognizes that most decisions involve the need to compromise between conflicting objectives. It explicitly acknowledges multiple evaluation criteria, which allows one to incorporate the concerns of multiple stakeholders. The performance of each alternative on each criterion is assessed, and these performances are then aggregated to derive a recommendation. Typically, aggregation involves criteria weighting.

Many authors have proposed joining LCA and MCDA for a combined assessment. Pioneering work in the period 1995–2005 includes applications (Bloemhof-Ruwaard et al. 1995; Spengler et al. 1998; Azapagic and Clift 1999; Geldermann and Rentz 2005) and some of the first frameworks (Miettinen and Hämäläinen 1997; Hertwich and Hammitt 2001; Seppala et al. 2002). In this chapter, we focus on discrete MCDA methods for brevity's sake, but we should also mention the potential of combining LCA with multi-objective optimization (Azapagic and Clift 1999) and data envelopment analysis (Thore and Freire 2002; Martín-Gamboa et al. 2017).

The number of publications reporting work that combines LCA and MCDA has been growing steadily. A recent review of work combining MCDA and LCA appears in (Zanghelini et al. 2018), who found 12 articles in 1995–2005, 18 articles in 2006–2010, and 61 articles in 2011–2015. They also reported 17 articles in 2016 alone, and replicating their methodology we have found 29 applications in 2017. This number was obtained by searching for "(multicriteria OR multi-criteria) AND (lif*cycle OR lca OR lcia)", a search that might miss articles using the expression "multiattribute", for instance, but which nonetheless indicates the growing popularity of LCA-MCDA applications. Applications can be roughly divided in two groups: one consists of MCDA applications where some of the criteria correspond to LCA categories, so that the measurement of the performance on those criteria follows a life-cycle perspective; the other consists of LCA studies that are complemented a posteriori by an MCDA aimed at synthetizing the LCA results to recommend a choice, a ranking, or a classification of the assessed alternatives. Besides these uses to support, interpret, or integrate LCIA results, MCDA can also be used to support decisions on how to conduct the LCA, for example, when selecting impact categories or defining the allocation approach (Zanghelini et al. 2018).

LCA and MCDA share the perspective that multiple dimensions of assessment are required to inform decision making. Each field offers something to complement the other. LCA can be helpful for the MCDA practitioner, since it aids in defining the set of criteria and how performance on these criteria can be measured. This is presented on Sect. 2, which reviews LCA and related methodologies. Conversely, MCDA can be helpful for the LCA practitioner, since it assists Decision Makers (DMs) in making sense of the results without inadvertently biasing them (Dias and Domingues 2014).

This is discussed on Sect. 3, which briefly reviews the main characteristics of MCDA. Section 4 discusses challenges and offers some perspectives concerning LCA-MCDA applications.

2 Life Cycle Assessment

Environmental Life Cycle Assessment (LCA) is enjoying increasing international recognition in the scientific community (high number of articles published in prestigious international journals, e.g., Poore and Nemecek (2018)), in industry (numerous private sector LCA studies), and in environmental policy. LCA and "Life Cycle thinking" are increasingly important for the development of key environmental policies, such as the European Union Integrated Product Policy. This targets environmental improvements and better product performance to support long-term industrial competitiveness and contribute to sustainable development (European Comission 2003). In the past, product-related environmental policies tended to focus on industrial emissions or waste management issues. However, the environmental impacts throughout product life-cycles must be addressed in an integrated way, not least to avoid shifting from one part of the life cycle to another.

The first studies addressing product life cycles are from the late 1960s. At that time, the focus was on energy and raw materials. In the early 1990s, LCA emerged in an organized form, addressing various categories of environmental impacts. The first LCA guide was published in 1992 by the Institute of Environmental Sciences of the University of Leiden (Heijungs et al. 1992). A few years later, the International Organization for Standardization (ISO) published the first LCA standards (ISO 14040: 1997—"Environmental management—Life cycle assessment—Principles and framework", etc.). In 2006, the four original LCA standards were replaced by two: ISO 14040 and 14044 (ISO 2006a, b). According to the ISO standards, LCA addresses the environmental aspects and potential environmental impacts throughout a product life cycle from the extraction of raw materials, through production, use, end-of-life treatment, recycling and final disposal, that is, from "cradle-to-grave". The LCA methodology is organized into four phases, as represented in Fig. 1 (left part).

The goal and scope definition includes the system boundary, functional unit, and level of detail, which depend on the intended use of the study. Figure 2 shows an example for an LCA of soybean-based biodiesel, addressing three alternative pathways: biodiesel totally produced in Brazil and exported to Portugal (A); biodiesel produced in Portugal using soybean oil (B); and soybean imported from Brazil (C). It illustrates the definition of a system boundary (the unit processes accounted for by the LCA) and the functional unit (which provides a reference for calculating the life cycle impacts, in this case, 1 MJ of biodiesel energy content). The functional unit is a key and unique element of the LCA methodology. It ensures the comparability of LCA results, which is particularly critical when different systems are being compared. The life cycle inventory analysis (LCI) involves the compilation and quantification

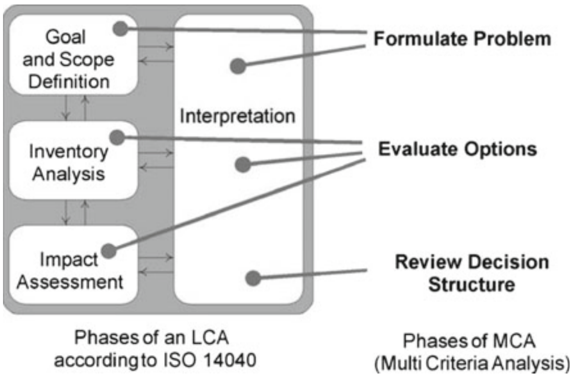

Fig. 1 The phases of LCA and their correspondence to MCDA phases (Geldermann and Rentz 2005)

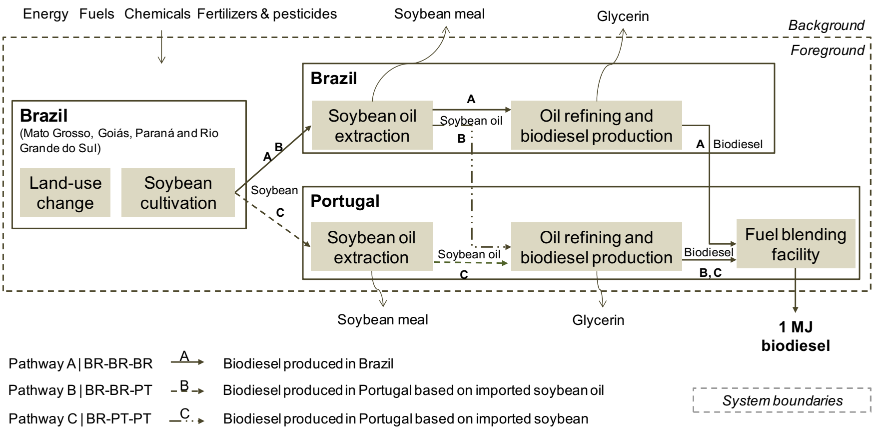

Fig. 2 Identification of a system boundary and a functional unit: the example of a life-cycle assessment of soybean-based biodiesel in Europe (functional unit = 1 MJ), comparing different pathways (Castanheira et al. 2015)

of the input/output data of the product system. The life cycle impact assessment (LCIA) involves associating LCI data with specific environmental impact categories and category indicators. It uses factors calculated by impact assessment models on the basis of impact pathways, generally considering three areas of protection: human health, natural environment, and natural resource use.

LCIA has mandatory elements, such as selection, classification, and characterization, which lead to the calculation of category indicator results, as well as optional elements, such as normalization, grouping and weighting. Normalization—the calculation of the magnitude of the category indicator results relative to some reference information—serves to highlight the relative magnitude of each indicator. It can use external references (e.g., the total impacts for a given area: global, regional, or national) or internal references (e.g., a baseline scenario, such as a given alternative product system). Grouping is the assignment of impact categories to one or more sets (ISO 2006b). Weighting aggregates different impact category results into a single score based on weights allocated to each impact category. This is very subjective—and hence, controversial—and it implies a value judgement, which may influence the results and conclusions of an LCA. As stated in ISO (2006b), "weighting shall not be used in LCA studies intended to be used in comparative assertions intended to be disclosed to the public". However, weighting is commonly used in studies due to its practicality for comparing impacts of different products or scenarios, supporting decision-making and communication of results (Pizzol et al. 2017). There are several LCIA methods (CML, ReCiPe, IMPACT World+, etc.), which can be organized into two main groups according to the level of the cause-effect chain: (i) midpoint methods (also known as problem-oriented methods), which provide indicators at a level of the cause-effect chain between emissions/resource consumption before the endpoint for environmental problems (climate change, ozone depletion, eutrophication, acidification, etc.); and (ii) endpoint methods (also known as damage-oriented methods), which provide indicators at the level of areas of protection against environmental damage. Endpoint methods permit straightforward communication of the LCIA results, but with considerably higher uncertainty than midpoint methods. It should be also noted that some LCIA methods, such as ReCiPe, have both midpoint and endpoint indicators and some impact categories do not have a natural midpoint (e.g., water or land use) (UNEP SETAC 2016).

Interpretation is the final phase of the LCA, in which results are summarized and discussed as a basis for conclusions, recommendations, and decision-making. LCA is iterative (as shown in Fig. 1) and as data are collected or LCIA is performed, various aspects may require modification, including the goal and scope definition.

3 Aggregation of LCA Results

Choosing between environmental profiles involves balancing different types of impact and is typical of multi-criteria decision problems, in which explicit or implicit trade-offs are needed to construct an overall judgment.

Generally, MCDA methods are applied to provide decision support to one or more DMs in choosing an alternative based on the consideration of multiple criteria. Since the preferences of DMs are also considered, their participation in the process is crucial (Belton and Stewart 2002). Besides comparing alternatives via a multi-criteria assessment, it is also the goal to offer DMs a structured decision process. As a result, MCDA methods increase the transparency of the decision process and make complex decision problems easier to understand (Belton and Stewart 2002; Greco et al. 2016).

The process of conducting an MCDA comprises three high-level steps with a fluid transition between them: problem formulation, evaluation of options, and review of

Table 1 A taxonomy of MCDA methods (adapted from (Dias et al. 2015))

		Does the evaluation of one alternative depend on other alternatives belonging to *A*?	
		No (evaluation independent of other alternatives)	*Yes (evaluation relative to other alternatives)*
Underlying approach	*Value*	Global value aggregating individual performances, e.g.: • Weighted sum • MAVT/MAUT	Global value synthetizing comparisons of alternatives in *A*, e.g.: • AHP/ANP • PROMETHEE II
	Distance	Distance to an externally defined reference, e.g.: • Euclidean distance • Chebyshev distance	Distance to a reference defined from *A*, e.g.: • TOPSIS • DEA
	Binary relations	Binary relation between alternative and external references, e.g.: • ELECTRE TRI	Binary relation on the alternatives in *A*, e.g.: • ELECTRE I–IV • PROMETHEE I • NAIADE
	If-then rules	Rules based on thresholds, e.g.: • Dominance based rough set approach (DRSA)	Rules based on binary relations on *A*, e.g.: • DRSA

the decision structure (Belton and Stewart 2002; French and Geldermann 2005). These steps are presented in Fig. 1 (right side) alongside the phases of an LCA study.

Several aggregation methods (for an overview see, e.g., Greco et al. 2016) are available to formally evaluate the options (Table 1). Depending on the underlying decision context, some methods are more suitable than others (Roy and Słowiński 2013). Naturally, different decision methods may generate different results from the same data (Lahdelma et al. 2000). Therefore, the choice of a particular method or combination of methods (Marttunen et al. 2017) should be matched to the application (Baudry et al. 2018).

The MCDA method and the decision process are guided by an analyst (or facilitator), who gathers the information needed for problem structuring and supports the required methodological competence (Ormerod 2014). Sometimes, a decision is to be made by a group, which means that there are probably conflicting interests to be considered. In this case, MCDA provides a way to structure the dialogue between DMs (Slotte and Hämäläinen 2015).

MCDA methods thus permit DMs to consider personal preferences (e.g., in the form of weights) and witness the impacts of their choices. The discussions that take place among stakeholders with diverging positions also increase the acceptance of the ultimately chosen alternative (Renn et al. 1997; Belton and Stewart 2002; Stirling 2006; Munda 2008; Lerche et al. 2017).

4 Challenges and Perspectives

This section discusses several issues that confront the actors (LCA experts, MCDA experts, and other) involved in LCA-MCDA applications.

4.1 Towards LCSA

The standardized LCA methodology (ISO 2006a, b) addresses only environmental aspects, usually giving rise to multiple impact indicators (e.g., depletion of resources, impacts of emissions on the environment and on human health). Over time, however, LCA-based approaches have emerged that focus on Life-Cycle Costing (LCC), Social Life Cycle Assessment (SLCA) and, more recently, on a multi-dimensional approach to sustainability (Life Cycle Sustainability Assessment; LCSA = LCA + LCC + SLCA) (Kloepffer 2008).

Guinée (2016) distinguished three dimensions along which LCSA is expanding when compared to LCA: (i) broadening impacts by including social and economic indicators, (ii) broadening level of analysis from predominantly product-related questions to sector-wide and economy-wide questions and analyses, and (iii) deepening analysis to add physical, economic, and behavioral relations to the existing technological relations, and to include more mechanisms to account for interrelations among the system elements, uncertainty analysis, and stakeholder involvement. Application of LCSA requires integration of various methods, tools, and disciplines. According to Guinée et al. (2011), structuring, selecting, and making the plethora of models practically available for different types of life cycle sustainability questions is the main challenge. The challenges associated with an increasing number of indicators from LCSA studies include how to communicate results to DMs and how to evaluate and aggregate the indicator results. Here, the application of MCDA can be very helpful.

4.2 Criteria Selection

MCDA applications that involve LCA or SLCA may also consider other criteria, such as security, convenience, and aesthetics. All these applications entail making some choices about the criteria that are used. In the simplest case, MCDA is used exclusively to aggregate environmental LCIA indicators (according to CML, ReCiPe, or other LCIA methods). Special care should be taken when weighting the criteria.

Redundancies (double-counting) might arise if LCIA indicators from different methods are used. Moreover, some impacts are included as a single indicator in some LCIA methods (e.g., eutrophication in TRACI), but as multiple indicators in other methods (e.g., maritime eutrophication and freshwater eutrophication in ReCiPe). This affects results when the analysis considers all criteria on an equal basis rather than eliciting weights. Even if weights are elicited, however, the splitting bias might cause the total weight to increase when an indicator is decomposed (Jacobi and Hobbs 2007).

The selection of indicators coming from a method such as ReCiPe can be done at the midpoint or endpoint level. Eliciting weights might be simpler at the endpoint level, since there are fewer criteria at that point. On the other hand, however, these are possibly harder to trade-off then. For the same DM, eliciting weights at these two levels might even lead to different conclusions when comparing alternatives (Du 2017).

Besides environmental indicators, a more comprehensive LCSA assessment will also incorporate economic and social indicators, as mentioned in the previous section. In such cases, a choice must be made between considering a hierarchy of criteria vs. a flat structure. In the first case, there are three main criteria (environmental, cost, and social impact), each one decomposed into lower-level criteria. In the second case, the criteria are all at the same level (no hierarchy). Again, this means that the analyst must be concerned with effects caused by decomposition bias. When assessing products, productive processes, etc., there may also be other dimensions to account for that do not derive from a life-cycle perspective, such as how user-friendly or appealing a product is to its consumers.

To address these issues, MCDA has a rich literature on problem structuring that can be useful in guiding criteria selection (e.g., (Keeney 1992; Neves et al. 2009)) and on weighting biases that might derive from these choices (Jacobi and Hobbs 2007). Adequate communication between analysts and DMs is essential to ensure that the meaning of the indicators is well understood in weight elicitation processes. Lastly, when in doubt, trying out different analyses (e.g., at the midpoint and at the endpoint level) may yield additional insights.

4.3 Actors to Be Involved

The majority of environmental decision problems involve uncertainty and risk. By their very nature, the estimates and long-term forecasts required in LCA are uncertain. For reviews discussing different types of uncertainty, variability, and risk, see (French 1995; Huijbregts 2001). The scale of the impacts and when they are incurred is also an important differentiator. In particular, there is little agreement on how to evaluate options with very long term impacts (Atherton and French 1999). In the context of LCA, cultural differences can be easily identified: e.g., the German scientific literature on technique assessment is fairly concentrated on risk assessment, whereas, in the UK, there is a wide recognition of the need to include socio-political issues more explicitly into the decision making (French and Geldermann 2005).

There are many parties to such decisions. DMs are responsible for making the decision; they ‘own the problem’. They are accountable to some, but not necessarily to all the stakeholders in the problem. Stakeholders share, or perceive that they share, the impacts arising from a decision. They have a claim, therefore, that their perceptions and values should be taken into account. Experts provide economic, engineering, scientific, environmental, and other professional advice used to model and assess the likelihood of the impacts. The DMs may have technical advisors who are undoubtedly experts in this sense, but they are unlikely to be the only experts involved. Other experts may advise some of the stakeholders, thus influencing the stakeholders’ perceptions and hence shaping their decision making. Analysts develop and conduct the analyses, both quantitative and qualitative, which draw together empirical evidence and expert advice to assess the likelihood of the outcomes. They will also be concerned with a synthesis of the DMs’ and stakeholders’ value judgements. These analyses are used to inform the DMs and guide them towards a balanced decision. Whereas experts support decision making by providing information on the content of the decision, analysts provide process skills, thus helping to structure the analysis and interpret the conclusions. This separation of roles is much idealized; some of those involved may take on several roles. Clearly, DMs are necessarily stakeholders because of their accountabilities; but they may also be content experts and may conduct their own analyses. Similarly, experts may be stakeholders and vice versa.

4.4 Criteria Weighting

MCDA typically elicits preferences from a DM or a group of DMs, acknowledging the legitimacy of considering their subjective preferences. An MCDA analyst’s job is to support the decision process of the DMs so that they obtain recommendations as compatible as possible with their value system. A company performing MCDA on LCA indicators can also proceed in this manner according to its policies and preferences. To select suppliers or evaluate potential changes to its product range or productive processes, for instance, a company may conclude that option x is better than option y. Similarly, a government department can proceed in this manner following its policies and priorities, for instance, to sort products into categories for taxing purposes. Here, concluding that x is better than y thus reflects the policies and priorities of the company or the government, and not an objective truth.

In LCA, however, there is often no DM involved in the analysis, and the implicit perspective is that the alternatives are being objectively evaluated according to the best scientific state of the art. This is probably why the LCA standard ISO 14044:2006 states that weighting LCIA indicators is an optional step in the methodology and should not be used for comparative assertions intended to be disclosed to the public.

There are attempts to circumvent subjectivity by deriving weighting vectors backed by science. LCIA endpoint indicators, for instance, already aggregate multiple LCIA midpoint impact indicators considering more generic dimensions (the so-called areas of protection), such as “Damage to human health”, with weights that attempt to capture the relative damage pathways caused by each different impact (climate change, particulate matter, ionizing radiation, etc.). Soares et al. (2006) suggested obtaining weights by using a panel to score the importance of LCIA indicators on attributes such as scale, duration, reversibility, etc. Another proposal to derive weights backed by science is to associate the weight of an indicator with the seriousness of the impacts with regards to planetary boundaries (Tuomisto et al. 2012): if the impacts in a given category have gone beyond the limits that our planet can stand as a "safe operating space" for humanity, then it should have a high weight; if the impacts are far away from this boundary, the category could be assigned a lower weight. Nevertheless, all these proposals are still subject to large uncertainties due to lack of consensus in the scientific community about how midpoint indicators translate into higher order consequences.

Given the concern about the subjectivity of weighting, many LCA studies simply assume all indicators have the same weight, sometimes considering other "scenarios" (i.e., weight vectors) that place more weight in different groups of criteria. From an MCDA perspective, however, the concept of equal weights is meaningless in some methods (e.g., when a normalization or a value function is used) and setting all weights to the same value is still a subjective choice. Ultimately, one might simply accept that obtaining a purely objective result is an impossible goal, since there is subjectivity in the choice of alternatives that are evaluated, the choice of what criteria are considered, and even the choice of an MCDA method. One might even argue that LCA itself already brings subjective choices when defining system boundaries, allocation method, etc. (Myllyviita et al. 2014).

If the subjectivity of weighting is acknowledged, then the main concern should be that weights are adequately elicited from the DMs (or panels of experts or citizens on their behalf) and made transparent. First, it should be acknowledged that different MCDA methods are associated with different meanings for the criteria weights. Therefore, weights cannot be elicited without defining beforehand what MCDA method is being used, including the possible definition of normalization processes (Myllyviita et al. 2014), and following elicitation protocols adequate for the chosen method (e.g., (Dias and Mousseau 2018; Morton 2018)). The choice of the MCDA approach should reflect considerations of the study's purposes and needs, in particular, the issue of compensatory versus non-compensatory aggregation (Guitouni and Martel 1998).

Regardless of the process used to define weights, the concerns about choosing a vector of weights can be mitigated if one adopts an incomplete/partial information perspective. This means acknowledging multiple and equally acceptable criteria weight vectors $w \in W$ (W being a set of weights large enough to accommodate the analyst's concerns). A "robustness analysis" can then be used to determine the worst possible result for each alternative (a cautionary perspective), along with the best possible result (a benefit-of-doubt perspective), as proposed by (Domingues et al. 2015). Stochastic analysis is another way to study a problem according to an SMAA-type approach, simulating results for randomly sampled weights, as suggested by (Prado-Lopez et al. 2014). Robustness and stochastic analysis can be used together to inform decision making with complementary results (Dias et al. 2016).

4.5 Relative Versus Absolute Evaluation

MCDA usually compares several alternatives, which is not the case in many LCA studies. Indeed, some LCAs are devoted to assessing the impacts of a single product or service, for instance, with the aim of learning which stages of the life cycle have the greatest impacts. Often an LCA study is performed to compare a new or modified product with an existing one. Clearly, MCDA methods that base their recommendations on a competition among alternatives, assessing how each one compares to each other one (e.g., AHP, PROMETHEE and most ELECTRE methods), cannot be used if there is a single alternative to be evaluated.

A possible solution to this issue is to use MCDA methods that evaluate one alternative at a time, independently of any other alternatives (Table 1). Such methods assign a global value or category, respectively, to each alternative according to predefined parameters (value functions, category profiles) without comparing it to other alternatives being considered. Nevertheless, they still require setting parameters or fictitious alternatives that often depend on the anticipated range of performance scores.

Another solution might be to add more alternatives, possibly fictitious or irrelevant, to allow a richer comparative analysis. However, this raises another concern in the relative vs. absolute evaluation debate, which is the independence with regard to irrelevant alternatives. Indeed, methods based on pairwise comparisons (AHP, PROMETHEE, most ELECTRE methods, etc.) do not provide this independence. If their recommendation is that A is preferred to B, and B is preferred to C, then removing C or adding a new alternative D might lead to the conclusion that B is preferred to A (the rank-reversal problem) (Millet and Saaty 2000; Wang and Luo 2009).

It should be noted that even methods not based on pairwise comparisons can be affected by rank-reversal issues (Wang and Luo 2009). One possible reason is that alternatives are compared with an ideal and/or anti-ideal solution (as occurs in TOPSIS and similar methods), which can change when adding or removing an alternative. Another reason is that many methods (e.g., the weighted-sum method) require normalization approaches, and some of these approaches are based on the performances of the best (and sometimes also the worst) alternative regarding each criterion. Again, this can cause reversals when adding or removing an alternative (Dias and Domingues 2014). To address this issue, a "status quo" normalization (Domingues et al. 2015) can be used instead. Avoiding the need for normalization is an advantage of some relative evaluation methods (Prado-Lopez et al. 2014),

5 Conclusions

LCA and MCDA communities can benefit from each other by mutual learning and exchange of ideas. To begin with, LCA is already multi-criteria by its very nature. The impact categories are assessed separately in incommensurable units of measurement and are usually in conflict with each other. Therefore, LCA and MCDA share the perspective that the consideration of multiple criteria is in general the most adequate way of supporting decision making.

Increasingly, DMs in engineering and business settings are required to select the “most sustainable” alternative, or to at least consider environmental and social responsibility concerns. MCDA practitioners involved in such decision problems might easily forget important issues. They might omit life cycle stages, impact categories, or impacts in other geographies, for example, or they might lack consistency in their assessments. In such settings, the LCA or LCSA framework can be extremely helpful for the MCDA when structuring the set of criteria. In particular, LCSA directs the MCDA practitioner to consider environmental, social, and economic criteria, thus broadening and deepening the level of analysis. It therefore contributes to a more comprehensive evaluation and helps ensure that all the concerns of DMs and stakeholders are included in the analyses. Moreover, LCSA aims at measuring the performance of the alternatives on many environmental and social criteria where a life cycle perspective is in order. The existence of standards and software facilitating the computation of results is another advantage the analysts can appreciate. DMs and analysts can thus understand that finding the “most sustainable solution” is an elusive goal, observing how alternatives compare to each other on multiple impact categories, and possibly also how they compare with external references.

On the other hand, MCDA theory and methods are needed to make adequate use of LCA or LCSA results for decision aiding purposes. This applies not only to the aggregation of impact categories, but also to all other problems (probably most of them) where additional criteria not encompassed by LCA are important (e.g., reliability, ease of maintenance, throughput time, comfort, etc.). MCDA is a field of knowledge that offers methods to define and structure a set of evaluation criteria, to guide the dialogue between analysts and DMs, to set parameters that reflect preferences (namely criteria weights), and to aggregate all the information in a logical manner. Moreover, MCDA makes decisions transparent and auditable, which is especially important if there is no absolute truth.

As a consequence, we expect that the already large number of MCDA-LCA/LCSA applications will continue to grow, and that LCA practitioners will become increasingly knowledgeable about MCDA methods, and vice versa. LCA practitioners will tend to use a reduced number of MCDA approaches that will become increasingly popular in this area. We thus expect that proper application of LCA and MCDA will become state of the art both in science and in practice. Yet, many more studies are needed regarding the acceptability of different approaches and their adequacy to inform decision making in real-world situations.

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The Monitoring of Social Innovation Projects: An Integrated Approach

M. F. Norese, F. Barbiero, L. Corazza and L. Sacco

Abstract When the Municipality of Turin first decided to invest in social innovation, a public program and a network of partners were created, and a procedure to support social innovation start-ups was developed, and applied for the first time in 2014. After selection and funding of several young social entrepreneur projects, the Municipality activated a monitoring process. Different methodological approaches, including cognitive mapping, actor network analysis and multicriteria analysis, have been combined to analyse the behaviour of these start-ups and to evaluate whether they would address the social needs of their specific fields, and develop business projects as part of an inclusive and sustainable economy. Each element of this analysis has been proposed and discussed in relation to the monitoring and decision processes. The adopted multi-methodology and its results are here presented as a proposal for new models, metrics and methods for the social economy.

Keywords Multicriteria models and methods · Cognitive mapping · Actor network analysis · Social innovation

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1 Introduction

A multicriteria (MC) model can be used to easily express multiple visions of the same problem and synthesize knowledge elements that include quantitative data and intangible values. Moreover, MC methods can deal with a high heterogeneity of model components, without reducing their richness, and can facilitate an easy and direct comprehension of the people who are involved in any way, as decision makers and stakeholders, or at least as proponents of specific visions or of detailed knowledge of some problem elements or domain expertise. MC models and methods can be used in a communication context, even when the problem is not well defined and the main aim is to acquire and structure knowledge, rather than to choose a solution or implement a clear problem statement of ranking or sorting (Norese 2016a). They can be used to identify specific new points of view or to modify already expressed ones, and even to better formulate a decision problem.

MC applications to Public Administrations (PA) have been proposed in literature in relation to different possible decision and problem contexts. A structured visualization, which distinguishes the main complexities MC applications have to cope with, was proposed in (Norese 2016b) by means of a theoretical framework. A specific typology was described, in terms of a *new and unstructured decision problem situation, internal to the PA decision system* (i.e., a system that includes decision makers and decision structures, with rules and formal relationships with other actors in the decision process), with participants from the involved organization units or with specific expertise in relation to the decision problem situation. Decision aiding activities, in relation to these situations, are often oriented toward defining, activating or improving a new policy or internal procedure (see, for instance, Bana e Costa 2001; Norese 2009; Merad et al. 2013). The Multicriteria Decision Aiding (MCDA) methodology (see the EURO Working Group MCDA website "http://www.cs.put.poznan.pl/ewgmcda/") adopts a constructivist approach, where the model as constructed, the concepts and the procedures constitute a communication and reflection tool that allows the participants in the decision process to carry forward a process of thinking and to talk about the problem (Genard and Pirlot 2002).

In the policy analysis field, the process is characterized by a cycle of design, testing, implementation, evaluation and review of public policies (Tsoukias et al. 2013). In the 1990s, the British Government defined policy making as a learning process that should be studied, analysed and monitored in order to obtain new evidence that could be used to build future policies. MCDA plays an important role in policy making processes that allocate tangible or intangible public resources. In general, these processes involve a single organization, with different institutional levels and sometimes with different departments. In rare cases, the organizational nature of the decision system is more complex (Norese and Torta 2014).

When a problem situation is new and unstructured, a monitoring action should be associated to each action implementation, but the aims of the monitoring and future use of the acquired data cannot be clearly defined, because of a total lack of previous experience or well tested reference procedures (Norese 2010).

In these situations, a structuring and a visualization of the main aspects facilitate communication between knowledge sources and the involved actors. Methodologies in the field of Problem Structuring Methods (Rosenhead and Mingers 2001) can be used to actively support public authorities during the preliminary phases of complex decision processes, when their uncertainties have to be analysed and reduced, the decision problem has to better formulated and structured, and the feasibility of each action has to be verified. The structuring and visualization potentialities of these methodologies can also play an important role in the monitoring processes of new and unstructured decision problems. Moreover, their potentialities can be multiplied through an integration with MCDA. MC models can be used to transparently synthesize knowledge and allow possible decisions to be formulated and analysed. MC methods can be used not only to aid decision making, but also to describe how a decision system could deal with a problem and which elements of a preference system could be elicited and used to understand the consequences of a policy implementation.

A new and unstructured decision problem, and the monitoring context in which it was formulated, are proposed in the first section, while the second section describes the knowledge acquisition process and the adopted methodological approach. The third section presents the development and the use of two models in the evaluation process, and the last section deals with the applications of MC methods, which were proposed in the monitoring context as examples of a formal and transparent use of acquired knowledge and information elements.

2 The Context

Urban communities and cities in Europe are currently the focus of an intense debate, at both a political and an institutional level, which has identified them as protagonists of a process of redesigning strategic development toward sustainable, smart and inclusive growth models. Cities create "a great combination of new business types of cooperation and employment opportunities with a strong social dimension" (European Commission 2013). The concentration of social and environmental problems and pressure on local welfare systems and on economy are problems that can easily be recognized in urban areas, but, at the same time, the potential of the cities as fields of transformation and laboratories of technological and social innovation can also be recognized. In this context, the city of Turin is making an effort to disseminate a culture of social interaction, aimed at co-designing development policies, in order to stimulate new forms of entrepreneurship in the citizens to respond effectively to local needs. The goal is to transform innovative ideas into new services, products and solutions which, at the same time, create economic and social value for the region and the community.

When the Municipality of Turin decided to invest in social innovation, it involved several organizations from the social economy and non-profit contexts, as well as public and private incubators, in a Public Program and in a network (Turin Social

Innovation—TSI) that had the aim of connecting people, organizations and ideas in the field of social innovation. TSI was created in 2013 to promote and disseminate a social innovation culture, as a stimulus to explore new markets, and to promote and support new forms of entrepreneurial projects in a synergic and collaborative economy environment. One of the initiatives of the Public Program, FaciliTO Giovani,[1] which was elaborated and applied for the first time in January 2014, was to aid social innovation start-ups, through financial support and accompanying measures, in the development of the technical, economic and financial feasibility aspects of their projects. In 2015, the Municipality activated a monitoring process, and the Social Economy Office (SEO) of the Chamber of Commerce, a TSI member, was asked to participate in the process and, in particular, to evaluate the social impact of the funded start-ups. SEO set up a team to analyse several aspects of the monitoring process and to participate in meetings with the municipality. The invitation to evaluate the social impact was discussed and criticized and, eventually, it was refused, above all because the team felt that only some months of project implementation were not sufficient to produce a social impact. The team underlined that the definition of social innovation dynamics cannot be generalized easily, but the presence of some specific elements could indicate a tendency of the projects and the social entrepreneurs of going in the direction of an effective social innovation.

A different kind of involvement of SEO in the monitoring process was proposed: acquiring and using not only financial and other quantitative data, but also knowledge elements and intangible values, in order to evaluate the different attitudes of the start-ups to produce social innovation in the first steps of project implementation. The proposal was accepted, and a working group was created. The group involved the authors: Fabrizio Barbiero, who represented the Municipality of Turin and the FaciliTO Giovani Council; Laura Sacco, who represented SEO as the coordinator of its activities; Laura Corazza and Maria Franca Norese, who contributed with different competences (studies on the development of a shared economy, experiences in social innovation, studies and applications in the MC evaluation, decision aid and problem structuring fields) and specific technical and methodological support from two different University Departments.

Different methodological approaches were adopted and integrated to analyse the behaviour of the start-ups and to evaluate whether they were able to address social needs, in their specific fields, and develop business projects for an inclusive and sustainable economy.

The documentation about each funded project and start-up (above all referring to their initial business plans) was analysed with the aim of organizing a set of interviews with the members of the start-ups, but first with the members of four incubators which, as TSI partners, had accompanied and oriented each start-up to obtain funds. The acquired elements of knowledge were structured and discussed in the working group and then oriented toward two different aims: (i) to help the TSI

[1]The name of the project, FaciliTO, combines the word facilitation with TO, the acronym of Turin, while Giovani (Italian word that means the young) identifies the young social entrepreneurs who have been the subjects and targets of the project.

promoters to better understand whether they had moved and were still moving well toward the promulgation and fulfilment of specifications that characterize the idea of promoting social innovation, or whether it was necessary and possible to introduce some modifications and improvements; (ii) to construct a pilot project that should be general enough to be applied to other situations.

Some cognitive maps were created to include all the acquired knowledge elements, in relation to the FaciliTO Giovani project (above all criticisms, positive judgements and improvement proposals). These elements were illustrated and discussed to facilitate the Municipality monitoring and decision processes. The other elements, in relation to the behavior of the start-ups, were used to evaluate their propensity to produce social innovation.

Logical graphs were elaborated to synthesize and visualize information about the social innovation network each start-up had created (Hermans and Thissen 2009). A pilot multicriteria model was then structured to evaluate the social innovation comprehension of each start-up and the ability of each start-up to implement its social innovation project. The results of the working group were then proposed and discussed with the FaciliTO Giovani committee, in relation to the Municipality monitoring and decision processes.

The adopted approach and its results are presented in the next sections, starting from the inquiry and its main results, which are presented in the second section. The different methodological approaches and their structuring of the acquired elements of knowledge are dealt with in the third section. The work is concluded with some remarks on the different possible uses of the results, in this decision process and for future use.

3 The Knowledge Acquisition Process

The analysis started with an examination of the FaciliTO procedure documentation, and above all of the evaluation criteria of the procedure steps; the former was used to select projects and their access to an initial entrepreneurial support, in terms of an accompanying action and a small quantity of money, and the latter to decide on their access to the financial facilitation process. The role of the FaciliTO committee, which had initially been created to include all the involved actors, was analysed and directly observed by means of working group participation in some committee meetings.

Documentation and data about each funded project and start-up company were then acquired and analysed, to obtain more detailed information about who the companies were and what the history of their ideas was, as well as to organize a set of interviews. The analysed documentation included the situation of the companies when had been accepted for the financial support in the first year of FaciliTO, their business ideas and the business plans the companies had prepared, together with the incubators, in order to obtain financial support at the end of the second step. The main elements of the business plans (the nature of the project, the social and innovation aspects, the positioning of the new idea on the market) were schematized in order to use them to start the interviews. When there were any confused elements in the business plans, they were underlined, for each project, in order to clarify them during the interviews.

Before these interviews, the four incubators that had given entrepreneurial support to the companies after the first project selection step were contacted to describe their involvement in the FaciliTO procedure, and to establish their approach to helping the companies better define their business idea, in terms of social innovation. At that point, a general framework was created for the interviews, which were oral and conducted without a tape recorder in order to create a friendly environment in which the interviewees could express their opinions freely. Moreover, the original framework was adapted each time to the attitude of the interviewees, in order to enlarge specific aspects of interest and allow them to give more details. Each interview was conducted by two people in order to follow the lines of discussion without losing any important concepts the interviewees were proposing or explaining, and each interview lasted about one hour and a half. The results of the interviews were accurately written down and sent, by e-mail, to the start-ups so that they could check the content. In some cases, some parts of the text were changed and/or integrated by the start-ups, and some of the interviewers' doubts were clarified.

3.1 A Cognitive Mapping Approach to Knowledge Structuring

The texts of the interviews were analysed, structured by means of a cognitive mapping approach (see Norese and Salassa 2014) and used to understand the visions and actions adopted by the start-ups to produce social innovation. The analysis of each interview included a coding of each expressed concept in information cells. A clustering approach was then activated on the coded sentences of all the interviews to identify a possible structure of themes (or topics or main concepts) which, in some cases, were deliberately introduced during the interview, but in other cases often emerged freely, without prompting from the interviewer.

Five main themes were identified: definition of the *perceived social needs,* of the *updated business plan* elements, positive *opinions or criticisms of the FaciliTO procedure, FaciliTO improvement proposals*, descriptions of *their social networks* (the subjects who could be influenced by the new idea or who could influence the idea and the project) or hypotheses on how their social networks could be created.

All the collected opinions and proposals about the FaciliTO procedure were organized in cognitive maps that can be defined as logical graphs, in which groups of concepts are connected on the basis of relationships of a different kind and can be used to identify specific aspects that require attention and processing or better explanations. The maps were analysed by the working group and then described to and discussed with the FaciliTO committee. The first two clusters were used to revise and complete the project description schemata and were used, together with the components of the last cluster, to start the evaluation process that is described in the next section.

4 The Evaluation Process

After the first interviews, it became evident that the projects were very different, in terms of content, aims and implementation procedures. The main differences between them not only concerned the nature of the innovative idea and the complexity of its implementation, but also the perception of the importance that should be given to the actors who could facilitate a social innovation project to be developed and social needs to be satisfied. The European Commission (2013) guide to social innovation states that "social innovation can be defined as the development and implementation of new ideas (products, services and models) to meet social needs and create new social relationships or collaborations" (page 6). For this reason, the first stage of the evaluation process was oriented toward analysing the completeness and quality of the *social innovation network* that each start-up created in the first steps of the project development.

Only at that point, was the second stage activated to formally use the knowledge acquired during the interviews and synthesized to a great extent in the social innovation networks, in order to analytically evaluate the different attitudes of the social entrepreneurs to produce effective social innovations.

4.1 The Social Innovation Networks

During the first interviews, the different descriptions of the relationships activated by the social entrepreneurs with possible actors of their social innovation projects were synthesized and visualized in very simple graphs, which became richer and clearer whenever their structure was proposed in a new interview, to obtain information on the networking of each specific social innovation. The general framework of this logical and visual representation of the social innovation networks was defined step by step, and when the structure of this logical graph became stable, each network that resulted from an interview with a funded start-up was sent, by e-mail, to the appropriate company to test the visualization effect of this tool and to check the quality of the working group interpretation of their network descriptions. The reactions of the start-ups were positive, and in just a few cases did they propose a change to include new relationships or new actors.

The structure of this logical graph includes nodes, which denote the actors and their roles in the social innovation network, and arcs that explain the nature of the different relationships between an enterprise and the actors involved in the social innovation project.

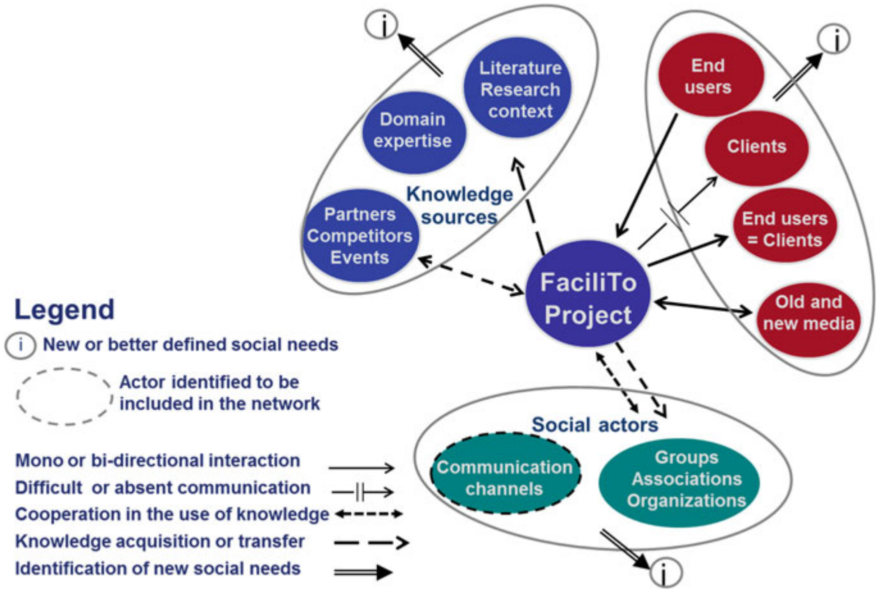


Fig. 1 General framework of the social innovation network

Individual or organizational actors may be *human* or *non-human* (the terminology that was proposed in the Actor Network Theory (Law 2007) to define and analyse the role of technologies or events in the processes). The actors' roles in these social innovation projects were indicated by the entrepreneurs during the interviews.

The general framework divides these roles into three categories (see Fig. 1). The first includes human and non-human knowledge sources that can help the start-ups to better define the features of their project ideas. These sources may be taken from literature, research institutes, incubators or from people with professional competences in the specific ambit of the project, but also competitors and production or distribution partners, international events, such as fairs and exhibitions, or events that TSI proposes with a knowledge mobilization aim. The second category comprises potential clients and/or end users who have a direct relationship with the start up, but also commercial agents, or old and new media, which become communication channels that enable the diffusion of a new idea. The last category is composed of "social actors" (in general associations or organizations that express social needs) that may be essential for a better definition of the social needs and the generation of a market for the specific social innovation idea. In some cases, they are directly involved in the innovation project, in others they are included in the social innovation network to bridge the gap between an innovation project and the social needs that have to be satisfied.

The arcs that explain the nature of the different relationships can indicate mono or bi-directional interactions, which may become more specific (cooperation in the

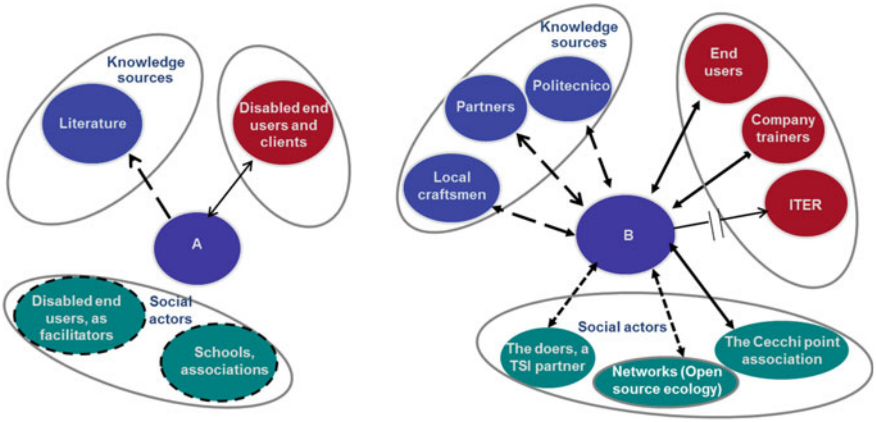


Fig. 2 Two different social innovation networks

use of knowledge, knowledge acquisition or knowledge transfer, partnership activation, identification of new social needs and so on). In some cases, communication difficulties or interruptions of these relationships can be underlined in the logical graph, together with an absence of communication and of the relationships that are considered essential for the project.

The social actor networks that were elaborated for each start-up (two of which are proposed in Fig. 2), were analysed by the working group and then presented to the FaciliTO committee. The committee appreciated the clear visualization of the differences between the funded start-ups, in terms of their behaviours during the first steps of their project implementations.

4.2 The Multicriteria Evaluation Model

The logical structure of an MC model includes the strategic aspects of the problem (or model dimensions) and their analytical formalization in criteria pertaining to the different related factors. Two main aspects were proposed during the interviews and they could be used to describe the propensity of a funded start-up to generate social innovation. The first is a cognitive aspect, that is, the start-up's *comprehension of the complex concept of social innovation*, and the second is an operational aspect, namely, the start-up's *capability to implement a social innovation project*. Several different knowledge elements were proposed, during the interviews, in relation to these two model dimensions. They were expressions of specific points of view and attitudes or descriptions of implementation actions and their consequences. Each proposal was analysed and the structured whole analysis was synthesized and formalized as criteria.

The logical *Comprehension of the complex concept of social innovation* dimension is dealt with in the MC model by means of two criteria. The first is *Awareness*, which proposes the idea that comprehension starts from the perception of the relational characteristics of each social innovation. However, this perception is easier in some situations than in others where the complexity of the project implementation is high and the enterprise that implements this project is new and does not have a clear vision or knowledge of the specific complexity characteristics. For this reason, the need for new relation activation, to reduce complexity and uncertainty and facilitate social innovation, may not be clearly perceived in these situations. Instead, awareness should be evaluated as being very poor when the same limited perception and comprehension of the basic elements of social innovation are present in enterprises with sufficient expertise that deal with less complex project implementations. The second criterion of the first dimension is *Knowledge mobilization*. Knowledge and expertise are resources that may be present in a funded start-up and need to be improved in relation to the new project, or have to be acquired and used by means of oriented actions. This criterion evaluates how well knowledge is mobilized and used to improve the comprehension of the complex concept of social innovation.

Two criteria are included in the model in relation to the second logical dimension, that is, Capability to implement a social innovation project. The first criterion is *Quality of the social innovation network* that the start-up has created, in terms of multiplicity and nature of the activated relations and presence of social actors. The second criterion is *Quality of the results* that can be generated from the activated relationships, in relation to the definition of the social needs and the verification of the validity and feasibility of the project idea.

The criteria are associated with different scales. In two cases, the evaluation states of the ordinal scales result from documented combinations of values (see Tables 1 and 2). The ordinal scale of the *Knowledge mobilization* criterion includes only three evaluation states, whose meanings are described hereafter, together with the criterion. The evaluations of the last criterion, *Quality of the results*, could be expressed in terms of the different levels of importance of the possible results and of the time available to attain them. However, the differences between the analysed implementation processes, in terms of time, were fund to be minimal, and the definition of the different levels of importance of each result was considered a topic that needed to be defined in later phases of the monitoring process. Therefore, the adopted evaluations were only linked to the different kinds of achieved results. The scales and evaluation states of the four criteria are described hereafter in detail.

Table 1 Ordinal scale of the awareness criterion

Complexity of the project implementation	Perception of the relational nature		
	VL	L	G
New product/new enterprise	5	7	10
New service/new enterprise or new product/old enterprise	4	6	9
New service/old enterprise or product evolution/old enterprise	2	4	7
Service evolution/old enterprise	1	3	6

Table 2 Ordinal scale of the quality of the network criterion

Relations	Actors			
	M	O	F	A
EK	8	7	–	–
2K	6	5	4	3
1K	5	4	3	2
NO	–	–	2	1

Awareness

The evaluations of the *Awareness* criterion are the result of a combination of two aspects (Complexity of the project implementation and Perception of the relational nature of the social innovation) and their values.

Complexity of the project implementation is related to the nature of the project and to the experience of the enterprise, which could either be "new", that is, created specifically for the FaciliTO funding project, or "old", i.e. created, and sometimes incubated, before FaciliTO had been set up. Each funded project was different, but the nature of the project, in terms of implementation complexity, could be divided into four different kinds of social innovation project: New service (in general activated by means of Internet technology), New product, Evolution of an existing Service or Evolution of an existing Product. The four situations were ordered, in terms of decreasing complexity, in New product, New service, Product evolution and Service evolution. A logical combination of the different kinds of project with the conditions of new or old enterprise generated four ordered project complexity states that satisfied the conditions of the projects when funded by FaciliTO. These states are described in Table 1.

A good perception of the relational nature of social innovation is underlined in the social innovation network by the presence of social actors in relation with the start-up and of cooperation relations with possible clients and/or final users in the use of the acquired knowledge (Good-G). If the need for relations with certain identified social actors is recognized, but no relationship has been activated with them, perception is Limited (L), and becomes Very Limited (VL) if no social actors are present or have been identified. The combination of these aspects generated an ordinal scale of ten Awareness evaluation states (see Table 1) ranging from 1 (VL perception in conditions of relatively simple project implementation) to 10 (G perception in conditions of very complex project implementation).

Each combinatorial approach generates a scale, whose values/evaluation states have to be defined in relation to the specific decision problem and together with the decision makers. In this case, there was a maximum number of different states of twelve, and they were defined by the number of input components (four situations of complexity combined with three levels of perception), while the final number was the result of a shared analysis of the problem situation.

Knowledge Mobilization

The knowledge sources can be of a different nature, and their identification and a cooperative relationship with them could have been used to mobilize knowledge. The *Knowledge mobilization* criterion distinguished three levels of mobilization, which were expressed by means of three evaluation states. Mobilization is classified as *Reach* (R) when multiple knowledge sources are identified and a cooperative action with them is activated to acquire, use and improve knowledge. Mobilization is classified as *Limited* (L) when a cooperative knowledge acquisition and use action is only oriented toward a single source. Knowledge mobilization is classified as *Minimal* (M) when it is only oriented toward the analysis of literature and/or the competitor operations.

Quality of the Network

The quality of the social innovation network that each start-up had organized can be evaluated in terms of the presence and, if possible, multiplicity of involved social actors, an aspect that was combined with the nature of the relations that were activated. Each network is different, because each one had to be created in relation to a specific social innovation idea.

Four clearly different situations were considered in the model, in relation to the funded projects: social actors are Absent (A) in the network; some possible social actors have been identified, but no relationships have been activated (F, for Future involvement of identified actors); only One typology of social actors has been activated in the network (O) and Multiple typologies of social actors are involved (M).

Three kinds of non-generic relations were recognized in the analysed networks: Knowledge acquisition or transfer, Cooperation in the use of knowledge and Identification of new social needs. Some of the relations were found to almost always be activated, while others were activated more rarely, but all the kinds of relations were activated in each reach network. Four different network completeness levels were distinguished in the model: each kind of relation (EK), only two kinds (2 K), only one kind (1 K), and no kind (NO). When the two aspects and their characteristics were combined, four combinations were found to be impossible, while the others generated an ordinal scale that included values ranging from 1 to 8 (see Table 2).

Quality of the Results

The activated relationships generated different results during the project implementation process. The interviews identified the following results: (a) identification of new social needs; (a′) improved definition of social needs and the requirements; (b) validity and/or feasibility verification of a project idea; (b’) verification without results;

Table 3 Structural elements of the MC model

Main aspects	The start-up's comprehension of the complex concept of social innovation		The start-up's capability to implement its social innovation project		
Criteria	Awareness	Knowledge mobilization	Network quality	Result quality	
Scales	[1–10]	[M, L, R]	[1–8]	[0–5]	
Start-up					
a1	9	Reach	6	1	(a′)
a2	3	Minimal	5	0	–
a3	6	Limited	8	4	(a + a′ + b + d)
a4	7	Reach	8	1	(a′)
a5	6	Limited	4	0	–
a6	7	Reach	8	2	(a + a′)
a7	5	Minimal	2	0	–
a8	6	Limited	7	4	(a + a′b + c)

(c) development and management of the relationships with possible end users. Some results may have been more consistent with the aims of the FaciliTO project, but it was not possible to distinguish their different levels of importance, and therefore the number of achieved results was used to evaluate the Quality of the results criterion.

5 An Application

The model was tested in relation to a small group of enterprises and their social innovation projects. Table 3 synthesizes the evaluations of eight start-ups that had been funded in the first year of the FaciliTO project. The same incubator had been involved during the first phase of accompanying measures for the development of the technical, economic and financial feasibility of these projects. The evaluations, in relation to the four criteria, arose from the elements of knowledge that were acquired in the interviews and were used to describe the social innovation networks.

This application was developed above all to demonstrate how a visualization of the network characteristics can be translated into an evaluation model and how an MC model can facilitate a transparent visualization of the differences between the propensities of start-ups to generate social innovation.

Table 3 facilitates a first reading, which underlines how start-ups a1 and a6 show a clearly better propensity than start-ups a2 and a7, because the first group presents the best values in almost all the criteria and the second group the worst ones. The other four start-ups are in intermediate positions. Another reading of the evaluation model can divide the set of start-ups into two groups, the efficient group (or Pareto optimal solutions) and the non-efficient group. Start-ups a2, a4, a5, a7 and a8 are not efficient, because they are dominated by at least one other start-up, i.e. another start-up is equal or better than the analysed start-up as far as each criterion is concerned. The only efficient start-ups are a1, a3 and a6. Therefore, a monitoring process could lead to the activation of actions to improve the limited propensity of some start-ups, above all that of start-ups a2 and a7, and/or to analyse the possible reasons for their limited propensity in the accompanying activities phase and/or in the selection process.

A different approach could be adopted in relation to a problem situation that requires a ranking of the different start-ups (classification problem statement), for example to identify which accompanying actions produced the best propensities to produce social innovation. Another situation could require the assignment of each start-up to a pre-defined category (sorting problem statement), which is associated with a specific management and control action in a monitoring process, to maximize the results when a new procedure has to be activated.

In these situations, the Table 3 model should include other parameters that the problem situation and its actors can propose: *weights*, which distinguish the criteria in terms of relative importance, and *parameters*, which translate the nature and risks of a specific decision, for the decision makers, into formal terms, or reduce a negative impact on the result when uncertainty is associated with data and/or evaluations. Structure, components and parameters allow specific MC methods to be applied to an MC model, in order to produce rankings or assignments to ordered categories (Roy 1996).

The limited dimensions of the analysed case (only four criteria that could have almost the same importance and eight start-ups that were evaluated in relation to scales that present a limited uncertainty) can be used to demonstrate how two MC outranking methods, ELECTRE II and ELECTRE Tri, can be used to facilitate decisions (Roy 1990, 1996).

ELECTRE II (Roy and Bertier 1973) was the first ELECTRE method designed specifically to deal with ranking problems. It is now only used in rare situations (to rank actions when no uncertainty is associated with the evaluations), but it is still an interesting option because the complete development of a method application can be described, without the aid of a SW tool, and used to explain the logic of an outranking method. ELECTRE Tri (Roy and Bouyssou 1993; Yu 1992) is a sorting method that is used for many different decision problems and which may easily be associated with different visions of how a problem can be dealt with.

5.1 ELECTRE II

The ELECTRE II method is an outranking method that can be used to deal with the problem of ranking a set of actions from the best option to the worst (Figueira et al. 2005) in the classification problem statement. Like the other ELECTRE methods, ELECTRE II includes two phases: construction of an outranking relation, S, whose meaning is *at least as good as*, followed by a procedure that applies a decision rule that is consistent with the specific decision problem and is used to elaborate recommendations from the results obtained in the first phase.

The ELECTRE II method is applied to an MC model whose components are: A, a complete set of actions ai ∈ A; a family J of consistent criteria gj ∈ J, which associates, to each ai ∈ A, its evaluation, gj(ai) ∈ E, in relation to a specific criterion gj and its scale E, and inter-criterion parameters.

5.1.1 First Phase of ELECTRE II

The outranking relation S is a binary relation that is used to model preferences between couples of actions. Considering two actions, a and a′, four situations may occur: aSa′ and not a′Sa, i.e., aPa′ (a is strictly preferred to a′); a′Sa and not aSa′, i.e., a′Pa (a′ is strictly preferred to a); aSa′ and a′Sa, i.e., aIa′ (a is indifferent to a′); not aSa′ and not a′Sa, i.e., aRa′ (a is incomparable to a′). If one of the P or I situations is verified, there is outranking. If neither P nor I are verified, there is incomparabily, R, a preference relation that is useful to account for situations in which the decision maker is not able to compare two actions. The ELECTRE II method can only be applied if each criterion is a true-criterion, for which there is strict, or net, Preference for each difference between evaluations and Indifference for the same evaluations. The outranking relation is based on the concordance-discordance principle, which involves declaring that an action is at least as good as another if a "majority" of the criteria supports this assertion (concordance condition) and if the opposition of the other criteria does not generate "too strong" reasons (non-discordance condition). An outranking relation is constructed with the aim of comparing, in a comprehensive way, each pair of actions (a, a′), and the concordance—discordance principle is implemented in ELECTRE II by means of two tests that verify concordance and non-discordance conditions.

Concordance Test

An action a can outrank an action a′, aSa′, if a sufficient majority of criteria are in favor of this assertion. The concordance condition can be defined as follows: the concordance index C(aSa′) has to be at least equal to a concordance level c, and C(aSa′) has to be at least equal to C(a′Sa), in order to consider only conditions of preference and not of indifference. In order to make this definition operational, the criteria are partitioned into J+, which includes the criteria in favour of the first element of the couple (a, a′), J = (when the evaluations of a and a′ are equal) and J−, the criteria in favour of the second element of the couple (a, a′). The weights pj of the criteria included in J^+, $J^=$ and J^- are synthesized in P^+, $P^=$ and P^-.

$$\mathbf{P^+(a, a') = \sum pj_{j \in J^+}}$$

$$\mathbf{P^=(a, a') = \sum pj_{j \in J^=}}$$

$$\mathbf{P^-(a, a') = \sum pj_{j \in J^-}}$$

These weights are used in the concordance test:

$$\mathbf{C}(\mathbf{a}, \mathbf{a}') = \frac{\mathbf{P}^{+}(\mathbf{a}, \mathbf{a}') + \mathbf{P}^{=}(\mathbf{a}, \mathbf{a}')}{\sum \mathbf{P_j}} \geq \mathbf{c}(\textbf{level of concordance})$$

$$\mathbf{P}^{+}(\mathbf{a}, \mathbf{a}') \geq \mathbf{P}^{-}(\mathbf{a}, \mathbf{a}')$$

Non Discordance (or veto) Test

When the concordance condition holds, none of the criteria in the minority should oppose the assertion aSa′ too much. In order to make this definition operational, a set of discordance Dj* is created to include couples of values (e, e') that are considered too discordant (e is "too much" worse than e') in relation to the J* criteria, which can activate the discordance test (the test can be activated in relation to all the criteria, but also in relation to just some of them). If (a, a′) is a couple of actions and their evaluations are

$$\mathbf{g_{j*}}(\mathbf{a}) = \mathbf{e} \quad \textbf{and} \quad \mathbf{g_{j*}}(\mathbf{a}') = \mathbf{e}'$$

for at least one of the J* criteria, a does not outrank a′, even though the concordance test for the couple (a, a′) has been passed.

5.1.2 Application of the Two Tests to an MC Model

The two model dimensions shown in Table 3, that is, Comprehension of the complex concept of social innovation and Capability to implement its project of social innovation, may have a different importance that indicates Capability as the most important (55% of the total importance) and Comprehension as strategic but less important (45%). Therefore, the relative importance p_j of the four criteria shown in Table 3 is linked to the different importance of the model dimensions. These parameters are essential to apply the concordance test. Other parameters have to be defined to activate the non-discordance test: a set of discordance Dj*, which includes couples of values logically in discordance, in relation to situations, and criteria J*, where a very bad evaluation of an "interesting" action can generate a risky decision, when another action presents a very good evaluation. In this case, there are three J* criteria, while the discordance test is not activated in relation to the Knowledge mobilization criterion, because the logic distance between the three evaluation states is not so high (Table 4).

The last parameter that has to be defined is the concordance level. The Concordance condition is modelled in ELECTRE II in order to take into account the notion of embedded outranking relations. There are two embedded relations: a strong outranking relation, which is used in the first phase of the method and generates the input for the second phase, and a weak outranking relation, which is used only in the second phase of the method, when there are actions with the same merit. The strong and weak relations are built thanks to the definition of two concordance levels, c^s

Table 4 MC model

Criteria	g1 Awareness	g2 Know. mobilization	g3 Network quality	g4 Result quality	
Weights	0.20	0.25	0.30	0.25	
Scales	[1–10]	[M, L, R]	[1–8]	[0–5]	
Start-ups					
a1	9	Reach	6	1	(a′)
a2	3	Minimal	5	0	–
a3	6	Limited	8	4	(a + a′ + b + d)
a4	7	Reach	8	1	(a′)
a5	6	Limited	4	0	–
a6	7	Reach	8	2	(a + a′)
a7	5	Minimal	2	0	–
a8	6	Limited	7	4	(a + a′ + b + c)
D_{j*}	(1–10, 1–9, 2–10)		(1–8, 2–8)	(0–5, 0–4)	

and c^w, where $c^s > c^w$. The suggested values for c^s and c^w are $c^s = 3/4$ and $c^w = 2/3$, and both have to be included in the [0.5; 1-min pj] interval.

The results of the first phase of ELECRE II are synthesized in Table 5, where the eight start-ups are compared (56 comparisons), and the columns J^+, $J^=$ and J^- indicate the criteria (or more precisely their identification numbers) that are partitioned in the three groups. The concordance test is expressed in the two columns ($P^+ \geq P^-$) and ($P^+ + P^=$), and when P^+ is less than P^-, the second part of the test is not useful (the concordance test is not verified) and is therefore not activated. The $P^+ + P^=$ values are expressed and compared with the concordance level c^S, which in this case is 0.76, that is, slightly more than ¾, because the concordance indices are very high for several couples of actions. The c^w concordance level, which is used in the second phase, is 2/3.

5.1.3 Second Phase of ELECTRE II

The outranking relation S, which is constructed in the first phase, can be represented by an outranking graph, where the actions are the nodes and the oriented arcs indicate the presence of an outranking relation between two nodes (see Fig. 3). The second phase activates two iterative procedures on the graph to produce two preorders (i.e. orders that accept an element in joint position with others in some classes). The first procedure is oriented toward identifying, at each iteration, a sub-set of actions that follow the "the best actions are not outranked" rule (ascending procedure), and the second procedure actions that follow the "the worst actions do not outrank any other action" rule (ascending procedure).

Table 5 First phase of the ELECTRE II applicatiom

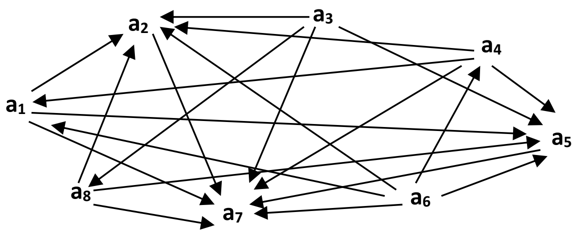
(a, a′)	J^+	$J^=$	J^-	$P^+ \geq P^-$	$P^+ + P^=$	Veto	S
a_1 a_2	1, 2, 3, 4	/	/	Yes	1		S
a_1 a_3	1, 2	/	3, 4	No			
a_1 a_4	1	2, 4	3	No			
a_1 a_5	1, 2, 3, 4	/	/	Yes	1		S
a_1 a_6	1	2	3, 4	No			
a_1 a_7	1, 2, 3, 4	/	/	Yes	1		S
a_1 a_8	1, 2	/	3, 4	No			
a_2 a_1	/	/	1, 2, 3, 4	No			
a_2 a_3	/	/	1, 2, 3, 4	No		Yes	
a_2 a_4	/	/	1, 2, 3, 4	No			
a_2 a_5	3	4	1, 2	No			
a_2 a_6	/	/	1, 2, 3, 4	No			
a_2 a_7	3	2, 4	1	Yes	0.80		S
a_2 a_8	/	/	1, 2, 3, 4	No		Yes	
a_3 a_1	3, 4	/	1, 2	Yes	0.55		
a_3 a_2	1, 2, 3, 4	/	/	Yes	1		S
a_3 a_4	4	3	1, 2	No			
a_3 a_5	3, 4	1, 2	/	Yes	1		S
a_3 a_6	4	3	1, 2	No			
a_3 a_7	1, 2, 3, 4	/	/	Yes	1		S
a_3 a_8	3	1, 2, 4	/	Yes	1		S
a_4 a_1	3	2, 4	1	Yes	0.80		S
a_4 a_2	1, 2, 3, 4	/	/	Yes	1		S
a_4 a_3	1, 2	3	4	Yes	0.75		
a_4 a_5	1, 2, 3, 4	/	/	Yes	1		S
a_4 a_6	/	1, 2, 3	4	No			
a_4 a_7	1, 2, 3, 4	/	/	Yes	1		S
a_4 a_8	1, 2, 3	/	4	Yes	0,75		
a_5 a_1	/	/	1, 2, 3, 4	No			
a_5 a_2	1, 2	4	3	Yes	0,70		
a_5 a_3	/	1, 2	3, 4	No		Yes	
a_5 a_4	/	/	1, 2, 3, 4	No			
a_5 a_6	/	/	1, 2, 3, 4	No			
a_5 a_7	1, 2, 3	4	/	Yes	1		S
a_5 a_8	/	1, 2	3, 4	No		Yes	
a_6 a_1	3, 4	2	1	Yes	0,80		S
a_6 a_2	1, 2, 3, 4	/	/	Yes	1		S

(continued)

Table 5 (continued)

(a, a′)	J^+	$J^=$	J^-	$P^+ \geq P^-$	$P^+ + P^=$	Veto	S
a_6 a_3	1, 2	3	4	Yes	0,75		
a_6 a_4	4	1, 2, 3	/	Yes	1		S
a_6 a_5	1, 2, 3, 4	/	/	Yes	1		S
a_6 a_7	1, 2, 3, 4	/	/	Yes	1		S
a_6 a_8	1, 2, 3	/	4	Yes	0.75		
a_7 a_1	/	/	1, 2, 3, 4	No			
a_7 a_2	1	2, 4	3	No			
a_7 a_3	/	/	1, 2, 3, 4	No		Yes	
a_7 a_4	/	/	1, 2, 3, 4	No		Yes	
a_7 a_5	/	4	1, 2, 3	No			
a_7 a_6	/	/	1, 2, 3, 4	No		Yes	
a_7 a_8	/	/	1, 2, 3, 4	No		Yes	
a_8 a_1	3, 4	/	1, 2	Yes	0.55		
a_8 a_2	1, 2, 3, 4	/	/	Yes	1		S
a_8 a_3	/	1, 2, 4	3	No			
a_8 a_4	4	/	1, 2, 3	No			
a_8 a_5	3, 4	1, 2	/	Yes	1		S
a_8 a_6	4	/	1, 2, 3	No			
a_8 a_7	1, 2, 3, 4	/	/	Yes	1		S

Fig. 3 Outranking graph



If the graph does not include circuits, at least one action is consistent with the procedure rule at each iteration. When only one action is consistent with the rule, it is assigned to a preorder class and eliminated from the graph. When more than one action is identified by the rule, a weak outranking relation is applied, by means of a weak concordance level, c^w, to the sub graph that includes the identified actions. The same rule is then applied to the sub graph.

At the end of the second phase, the intersection of the two preorders produces the result, that is, a final partial graph (some remarks on the analysis of these graphs have been proposed in Norese et al. 2016)

Application to the Second Phase of ELECTRE II

In the second phase, the descending ($P(A)^+$) and ascending ($P(A)^-$) procedures are applied to the outranking graph shown in Fig. 3 (which is without circuits). Each arc represents one of the outranking relations that were modelled in the first phase, in relation to the concordance level $c^S = 0.76$.

P(A) + (descending procedure, to create a ranking from the best to the worst)

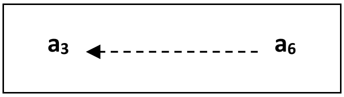
The actions that are not outranked are identified at each iteration.

Iteration 1: $A1 = A$

$D1 = \{a3, a6\}$

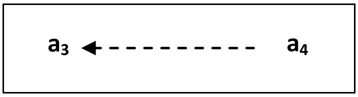
D1 includes the two actions that are not outranked. The weak outranking relation is activated in order to distinguish between the actions. It adopts the weak concordance level $c^W = 0.67$ in the concordance test, in relation to the sub-graph which only includes the actions of D1. The weak outranking relation can distinguish between the actions: a6 is the only action that is not outranked, and only this action is therefore assigned to the first class, C^{1+}, of the descending pre-order.

$C^{1+} = \{a6\}$



Iteration 2: $A^2 = A^1 \backslash C^{1+} = \{a1, a2, a3, a4, a5, a7, a8\}$

$D_2 = \{a3, a4\}$



$C^{2+} = \{a4\}$

Iteration 3: $A^3 = A^2 \backslash C^{2+} = \{a1, a2, a3, a5, a7, a8\}$

$D_3 = \{a3, a1\}$

$C^{3+} = \{a3, a1\}$

In this case, the weak outranking relation cannot distinguish between the two actions, which are assigned to the same class together. After the fourth iteration, the outranking graph is completely changed (see Fig. 4) and only includes four actions.

Iteration 4: $A^4 = A^3 \backslash C^{3+} = \{ a2, a5, a7, a8\}$

$C^{4+} = \{a8\}$

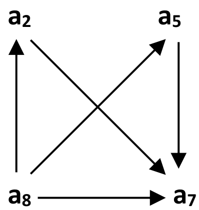
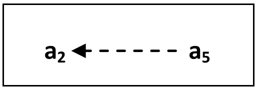


Fig. 4 The outranking graph after the fourth iteration

Iteration 5: $A^5 = A^4 \backslash C^{4+} = \{a2, a5, a7\}$
$D_3 = \{a2, a5\}$



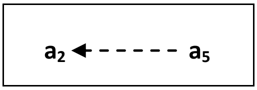
$C^{3+} = \{a5\}$

Iteration 6: $A^6 = A^5 \backslash C^{5+} = \{a2, a7\}$
$C^{6+} = \{a2\}$

Iteration 7: $A^7 = A^6 \backslash C^{6+} = \{a7\}$
$C^{7+} = \{a7\}$
$A^8 = A^7 \backslash C^{7+} = \acute{\emptyset} \rightarrow |A^8| = 0$ STOP
P(A)⁺ (sequence of the classes from the best to the worst) = {a6}, {a4}, {a1, a3}, {a8}, {a5}, {a2}, {a7}
P(A)⁻ (ascending procedure, to construct a ranking from the worst to the best)
The actions that cannot outrank any other action are identified at each iteration.

Iteration 1: A1 = A
$C^{1-} = \{a7\}$

Iteration 2: $A^2 = A^1 \backslash C^{1-} = \{a1, a2, a3, a4, a5, a6, a8\}$
$D_2 = \{a2, a5\}$



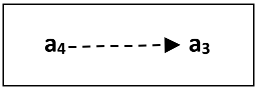
$C^{2-} = \{a2\}$

Iteration 3: $A^3 = A^2 \backslash C^{2-} = \{a1, a3, a4, a5, a6, a8\}$
$C^{3-} = \{a5\}$

Iteration 4: $A^4 = A^3 \backslash C^{3-} = \{a1, a3, a4, a6, a8\}$
$D_4 = \{a1, a8\}$

$C^{4-} = \{a1, a8\}$

Iteration 5: $A^5 = A^4 \backslash C^{4-} = \{a3, a4, a6\}$
$D_5 = \{a3, a4\}$



$C^{5-} = \{a3\}$

Iteration 6: $A^6 = A^5 \backslash C^{5-} = \{ a4, a6\}$
$C^{6-} = \{a4\}$

Iteration 7: $A^7 = A^6 \backslash C^{6-} = \{ a6\}$
$C^{7-} = \{a6\}$
$A^8 = A^7 \backslash C^{7+} = \acute{\emptyset} \rightarrow |A^8| = 0$ STOP
P(A)⁻ (sequence of the classes from the worst to the best) = {a7}, {a2}, {a5}, {a1, a8}, {a3}, {a4}, {a6}
The two preorders are similar and their intersection proposes, as final result, a ranking in which the sequence is

{a6}, {a4}, {a3}, {a1}, {a8}, {a5}, {a2}, {a7}

5.2 *Sorting Problem Statement and ELECTRE Tri*

In a sorting problem, each element of a set A (or an evolving set A(t)) of candidate actions is considered independently from the others, in order to determine its intrinsic value, an absolute judgement that is not influenced by the performance of the other candidates (Figueira et al. 2005).

Each candidate has to be assigned to one of the pre-existing categories, whose typical elements can be defined by levels of adequacy/urgency/priority/risk/…, or by reference profiles that express local/general norms/standards, or management and control activities that have to be arranged. The assignment results are expressed using the absolute notion of "assigned" or "not assigned" to a category, "adequate" or "not adequate" to some norms, and "similar" or "not similar" to a reference profile that represents a quality level, an activation level of a plan or a control action.

Each category (or segment or class) is conceived in order to receive certain potential actions that conform with the *assignment norms*—which include reference actions and assignment procedures—that characterize the category. These assignment norms are not always made explicit or completely formalized in the decision systems. In this sense, a sorting problem has to be faced through a three-step procedure: the first step includes modelling/validation activities of the assignment norms, the second includes the exploitation of the outranking relation procedure and the third the assignment to categories procedure.

Each action is evaluated in relation to a family J of consistent criteria and compared with a set of reference actions, or profiles, that have been evaluated on the same criteria. These reference actions, which can be typical elements of the categories or bounds that distinguish the categories, have to be indicated in the first step of each sorting procedure, and defined in relation to the problem and therefore to the chosen method.

The ELECTRE Tri method was specifically designed to sort a set of actions A, evaluated on the basis of criteria J, into a set of predefined and ordered categories (classes or groups), denoted here by C_h. The assignment of a given action, a, to a certain category, C_h, results from the comparison of the action, a, to the profiles b_{h-1} and b_h that define the (lower and upper) limits of the categories. The outranking relation is built in order to enable a comparison of an action a with a profile b.

The model (Table 3) and the ELECTRE Tri method could be used in the monitoring process, in relation to the problem described in the second section, to assign each start-up to a different "need of control" category. In this case, the application of the method has not been described, because the aim of the paper is only to underline the different problem vision that this approach can make explicit.

The situation is described logically in Fig. 5, where one action (continuous line) is included completely within category C_1 (need for an immediate control action) and another action (dotted line) is included in category C_3 (control action is not required) for a most of the criteria, but with an evident discordance (a bad performance in relation to the last criterion), which could require an investigation action to better understand the strange and perhaps risky situation. This methodological approach is particularly consistent with the aims of a monitoring process, in terms of both easy visualizations of local policies and of an analytic assignment of each action to a category, an assignment that is absolute, i.e. independent of the other action assignments, and directly connected to the formal expression of a policy.

6 Conclusions

MCDA proposes tools that facilitate communication in decision processes and activate a process of thinking, in relation to the several components of a problem situation.

These tools are models, procedures and methods, but also concepts, which can be used with different meanings in debates involving opposing viewpoints that have to be clarified and shared, or which present different meanings, in the involved knowledge fields, that have to be harmonized.

MCDA facilitates a shared definition of concepts in models of a different nature and can integrate different models in a unified, formal and procedural approach.

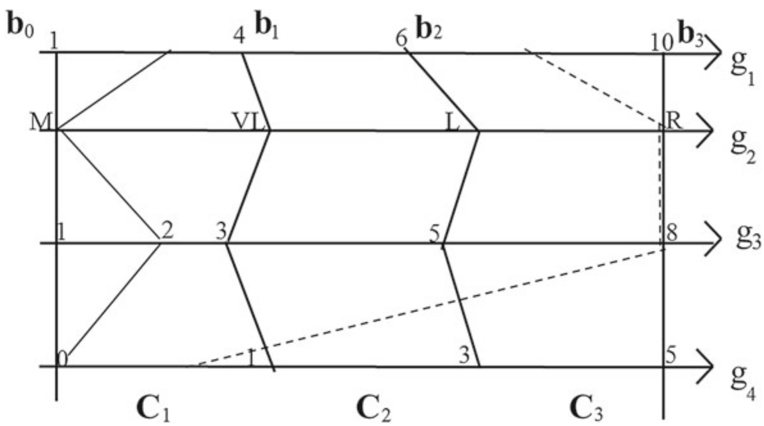


Fig. 5 Categories and action assignment

Intangible values can be explicitly included in MC models and used to facilitate decisions. PA in general and social innovation, in particular, can benefit from these MCDA features.

In this case, cognitive maps and actor networks were built and integrated in an MCDA approach that involved the actors of Turin Social Innovation in relation to the monitoring of a Public Program.

The immediate and easy visualization of these tools was appreciated by both the Program Committee and the evaluated start-ups. The possibility of analysing activities and events, in the social innovation context, and of expressing values, in terms that are analytical but neither quantitative nor financial, were judged positively by the involved organizations.

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Multiobjective Optimization in the Energy Sector: Selected Problems and Challenges

Carlos Henggeler Antunes

Abstract The application of multiobjective models and methods in a vast range of problems in the energy sector has been a consolidated practice in the last four decades. The need to consider explicitly multiple axes of evaluation of the merits of solutions in decision processes, generally involving large investments as well as social and environmental impacts, has led to the recognition of the potential benefits of multiobjective optimization approaches. Trends such as the increasing share of renewable sources in the energy generation matrix, the evolution towards smart grids involving the deployment of information and communication technologies, the dissemination of electric mobility and the consumer empowerment by means of the utilization of demand-side resources introduce challenging problems for which the capability of multiobjective models and methods to explore and provide assistance in the appraisal of well-balanced solutions is of utmost importance. This chapter aims to offer a broad view of some of the most challenging problems concerning the application of multiobjective optimization models and methods in the energy sector, with focus on electricity smart grids, outlining promising research avenues in problems of planning and operational nature.

Keywords Energy sector · Multiobjective optimization · Smart grids

1 Introduction

The energy sector is of central importance for the satisfaction of societal needs in modern societies and the utilization of some form of energy is pervasive in all aspects of everyday life. Models and methods of operational research have had a relevant role in supporting decisions in the energy sector, from long-term strategic planning (e.g., power generation expansion planning) to short-term operational scheduling (e.g.,

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unit commitment and dispatch of a power system with renewable generation). Until the late 1970s, objective functions in mathematical models for energy planning were generally cost functions, even occasionally encompassing aspects of different nature whose effects were monetized, with constraints referring to demand satisfaction, quality of service and operational issues. Energy companies were mostly vertically integrated, operating in the entire supply chain (generation, transmission, distribution and retail), which promoted a cohesive view of optimization models. This context started to change with the energy crises in the 1970s and the growing awareness of the environmental impacts of burning fossil fuels for power generation. In this setting, evaluation dimensions such as foreign dependency, pollutant emissions, effects on ecosystems, diversification of supply sources, coordination of hydropower plants with multiple water uses in river basins, etc., began to emerge in optimization models devoted to several problems in the energy sector. In general, the trend to unbundling the once vertical energy business, having as motivation to foster economic efficiency with respect to the operation of energy monopolies, created competitive wholesale and retail markets, while the (transmission and distribution) network businesses are considered natural monopolies and are regulated. Moreover, the conflicting interests of multiple stakeholders and sources of uncertainty have been considered in decision support models to offer compromise and robust recommendations. All these factors have had a decisive impact on the growing applications of multiobjective models and methods in problems arising in the energy sector (Antunes and Henriques 2016).

New and challenging problems continue to emerge in which multiobjective models and methods can provide sound decision support in the evaluation of the merits of different courses of action (technology adoption, facility and equipment sizing and location, market design, policy development, etc.) according to economic, environmental, technical and social axes of evaluation. The impact of decisions can range for several decades in strategic problems (as the construction of a new power plant) to short timeframes in operational problems. Some of these challenging problems are associated with the evolution to smart grids, encompassing high shares of dispersed generation, generally based on renewable sources, information and communication technologies (ICT) enabling bi-directional data flows, capability of using demand-side resources including consumer's flexibility regarding energy usage patterns to assist grid operation, introduction of electric vehicles both as a load (grid-to-vehicle) but also capable of providing grid services (operating in vehicle-to-grid mode), etc. Moreover, uncertainty is pervasive regarding, for instance, demand dynamics or renewable generation output.

Due to the vastness of problems and models of case studies in the energy sector, this chapter is mainly focused on the application of multiobjective models and methods, i.e. in which mathematical programming models include multiple objective functions to be optimized in a feasible region defined by a set of constraints, with different types of decision variables (binary, integer, continuous, etc.). Solutions are then obtained using mathematical programming approaches, in particular resorting to the optimization of some type of scalarizing function (based on weighted-sums, reference points, etc.), or meta-heuristics, in general population-based ones (as evolutionary algorithms, particle swarm optimization, differential evolution, etc.) with the aim to approximate the nondominated front, which is generally unknown. It should be highlighted that most papers reported in the scientific literature unfortunately fail to deliver a thorough characterization of the nondominated front and a sound analysis of the trade-offs at stake between the competing objective functions. Moreover, in general, information on the involvement of actual decision makers is also scarce, which impairs having useful insights on relevant issues as the acceptance by decision makers of decision processes based on multiobjective approaches or the degree of satisfaction with solutions recommended for implementation as the outcome of those processes.

This chapter aims to offer a broad view of some of the potential most challenging applications of multiobjective optimization models and methods in the energy sector, with focus on electricity smart grids, outlining promising research avenues in planning and operational problems. I.e., the aim is not providing an overview (for this purpose the reader may refer to (Antunes and Henriques 2016), which offers a comprehensive overview of multiobjective optimization and multicriteria analysis models and methods for a vast range of problems in the energy sector, (Cui et al. 2017) for an overview about applications in energy saving and emissions reduction or (Oliveira et al. 2016) for a review of the study of economy–energy–environment–social interactions) but rather unveiling the main characteristics and objectives at stake in emerging problems and the value-added of multiobjective approaches to tackle them. In the general framework of evolution towards smart grids, these problems include, among others: • the penetration of dispersed renewable generation and its coordination with conventional (thermal, nuclear, large hydro) generation in face of their intermittent nature (being based on wind and solar radiation); • the emergence of new market designs and entities such as aggregators; • the consumer empowerment through the use of the flexibility in the usage of appliances (and possibly local microgeneration and storage) and willingness to respond to price signals (dynamic tariffs) to assist grid operation; • the charging/discharging strategies of electric vehicles seen simultaneously as significant new loads and having the capability to sell energy to the grid; • the planning of grid expansion/reinforcement also considering sizing and location of electric vehicle charging stations and storage systems; • the operation of grid-connected microgrids.

The number of papers published in operations research and energy journals and conferences reporting the application of multiobjective models and methods to problems in the energy sector is immense. Therefore, a selection of very recent references (no more than five years old, published in journals) is provided with the aim of revealing up-to-date problems and research trends, which frame the most relevant challenges in which multiobjective models and methods are expected to have a value-added role by enabling not just the exploration of different trade-offs between the competing axes of evaluation but also a deeper critical analysis of potential solutions thus leading to better and more balanced solutions.

The aims and scope of this chapter are expressed in this section. Section 2 is devoted to generation capacity and grid expansion. Section 3 presents unit commitment and dispatch problems with focus on microgrids. Section 4 addresses resilient systems, involving protection, restoration and reconfiguration issues. Section 5 deals with the usage of demand-side resources as an untapped potential to improve the overall efficiency of the system, with mutual benefits to consumers and the grid. Section 6 describes problems associated with electric vehicles in the perspectives of grid-to-vehicle (a load) and vehicle-to-grid (a battery that can assist the grid). Section 7 is devoted to problems concerning different types of markets. Finally, Sect. 8 discusses perspectives of development for the application of multiobjective models and methods in the energy sector.

2 Generation Capacity and Grid Expansion

Power generation capacity expansion planning was one of the first problems addressed using multiobjective optimization models. In addition to the cost dimension, models began to include environmental impacts (hazardous emissions, land-use, effects on ecosystems) as explicit objective functions rather than incorporating them in an overall cost function. The aim is determining the power to be installed (technology, number and location of generation units) and output (energy to be produced by new and already installed units) throughout a planning period. The planning timeframe for these problems is, in general, a few decades. The increasing deployment of dispersed generation units based on renewable energy sources introduces further challenges, namely due to their intermittent nature. Further objectives include technical issues such as the maximization of the system reliability/safety (or minimization of outage cost or energy not delivered) as well as broader economic/social issues such as minimization of the external energy dependency (or maximization of the use of endogenous resources) and the maximization of employment at national or regional level (Antunes and Henriques 2016).

The transmission and distribution network infrastructures, which are natural monopolies in general regulated by an independent regulatory entity, need steady reinforcement, modernization and maintenance investment plans to offer the required quality of service in delivering power from generation plants to customers. Dispersed generation may be connected to transmission or distribution grids, which imposes additional evaluation issues, including economic, environmental and technical perspectives. Furthermore, transmission and distribution operators should promote nondiscriminatory access to networks according to the regulatory framework.

Different institutional and regulatory landscapes exist in distinct geographies to frame the operation of transmission and distribution networks. Transmission and mainly distribution companies generally operate in a given region (which requires appropriate inter-regional balances to be established) or they may operate in an entire country. Transmission companies may own the network assets or be just independent system operators. With specificities associated with, for instance, voltage levels and length of lines, transmission and distribution network planning models should lead to solutions offering a reliable operation complying with technical and quality of service parameters, considering the demand dynamics. These models encompass objectives of distinct nature: economic (infrastructure construction/reinforcement costs regarding e.g. substations and feeders, equipment upgrade costs, congestion costs, energy loss costs, regional or national economic growth induced by projects, improving competition in wholesale and retail markets), environmental (impacts of line corridors, effects on location of power plants, need to account for remote dispersed renewable generation, avoided emissions with the deployment of renewable generation), and reliability/quality of service (losses, line thermal, voltage profile and frequency stability requirements, system/customer average interruption frequency/duration indices, momentary average interruption frequency index). Further aspects may need to be encompassed in models such as public health concerns, for instance related to population exposure to electromagnetic fields, or landscape protection.

Important challenges refer to the ongoing evolution to smart grids, providing the technological basis using advanced Information and Communication Technologies (ICT) to accommodate responsive demand, storage, and local generation, in a framework of integrated optimization of all supply and demand resources. The increasing technical sophistication of network operation in the realm of smart grids involves the need to make sound decisions regarding the sizing and location of measurement and data acquisition equipment. The deployment of smart meters at the customer's premises provide very frequent and precise consumption measurements enabling, for instance, to cut costs of billing and offering a wealth of information to consumers that can be used, for instance by ESCOs—energy service companies—to derive consumption patterns and influence them to lower bills in face of dynamic prices also with a positive impact in decreasing the grid peak power (and therefore to postpone or avoid grid infrastructure reinforcements). With the development of wide area measurement systems, phasor measurement units (PMUs) have been deployed in power grids, providing voltage and current phasor measurements much faster than existing supervisory control and data acquisition (SCADA) systems. The optimal PMU location involves trade-offs between the minimization of cost and the maximization of network observability and state estimation performance.

Planning and operation decisions are sometimes combined by means of bi/multilevel optimization models assuming a hierarchical decision framework, i.e. multiobjective and multilevel models. For instance, in distribution systems expansion planning with significant assets of renewable generation and energy storage systems, a three-level optimization model can address different perspectives of multiple stakeholders (Li et al. 2018): the upper level aims to minimize investment costs, operation and maintenance costs, minimize the expected energy not supplied (as a surrogate for reliability) and maximize the penetration of renewable generation; in the middle level the objectives are to maximize the revenues of the sale of electricity by renewable generators and the arbitrage profit from the storage system; the lower level considers the optimal scheduling of the storage system to maximize arbitrage revenue by shifting consumption from peak to valley load periods. Bilevel optimization can also be helpful to design coordinated planning models for the interaction of dispersed generation and the distribution network: the objective functions at the upper level are the minimization of the cost of integrated investment and operation maintenance, active power losses, power purchasing and power failure; the objective functions at the lower level are the minimization of the cost of distributed generation investment and operation maintenance, the minimization of the expected deviation rate of voltage vis-à-vis nominal values, and the maximization of loss reduction and power generation of distributed generation (Gao et al. 2017).

3 Unit Commitment and Dispatch with Focus on Microgrids

The unit commitment problem may be defined as scheduling generating power plants to be on, off, or in stand-by mode, within a planning period to meet demand load. When the power system is vertically integrated, unit commitment is carried out in a centralized manner and the objective function is minimizing overall costs (the generation cost function is generally approximated as a quadratic function of the power output) subject to meeting demand and reserve margins. Under competition, companies decide their unit commitment plan to maximize profit considering power contracts to be fulfilled and the energy estimated to be sold in wholesale markets. Technical restrictions include capacity constraints, stable operating levels, minimum time period the unit is up and/or down, maximum rate of ramping up or down. Economic dispatch consists in determining the optimal combination of power output of online generating power plants to minimize the total cost (e.g. fuel costs of thermal plants and penalty/reserve costs for under/over estimating available wind power) while satisfying load demand and operational constraints (e.g., generator capacities and prohibited operating zones for the thermal units, transmission line capacities, bus voltage limits). Dispatch solutions should be able to react to load demand variations, guaranteeing adequate cost or profit levels, and considering technical issues as voltage control, congestion, transmission losses, line overloading, voltage profile, deviations of technical indicators from standard values. Models should encompass different market or regulatory structures; for instance, a generation company operating under competition generally aims to maximize profits, while entities such as an independent system operator aim to maximize social welfare. In economic-environmental dispatch, cost minimization, or profit maximization, and environmental impact minimization (namely harmful emissions originated at fossil-fuel power plants, such as sulfur dioxide and nitrogen oxides) are explicitly considered (Zhang et al. 2018). The minimization of the network transmission losses may be also considered in this context. Reliability concerns can be captured by minimizing the worst-case costs of energy outage.

Microgrids encompass distinct generation sources, a range of heterogeneous loads including electric vehicles, energy storage systems and power flows as sub-systems generally operating in a grid-connected mode (i.e., power can be imported from or exported to the main grid) allowing for local control of dispersed generation and thereby reducing or eliminating the need for central dispatch). Objective functions of economic, environmental and technical nature are generally considered such as minimizing the investment, operation and maintenance costs, minimizing power losses, minimizing pollutant emissions, minimizing energy not supplied in both connected and islanded modes of microgrids (reliability), minimizing the voltage deviation from its nominal value at grid nodes (power quality), minimizing the cost of the energy imported from the upstream grid, optimizing the security margin, maximizing the provision of services to the main grid (leveling the active power at the interconnection bus), minimizing the energy level required for emergency demand response programs at interconnections, maximizing the overall utility of microgrids (net value derived from energy consumption) (Chiu et al. 2015; Carpinelli et al. 2017). In some models, some of these aspects (e.g., losses, environmental impacts, load not served) are monetized and incorporated in an overall cost objective function.

In disturbance or fault conditions in the main grid, the microgrid can be isolated (islanded) from the distribution system to minimize the amount of load not supplied, maximize power quality and maximize quality of service locally. Future power grids are expected to be constituted by interconnected microgrids.

Economic, environmental and system security/stability objectives are broadly at stake in the optimization of microgrid operation performed by a central controller. Operational costs in normal conditions and a load curtailment index, or other customer satisfaction indicator, in case of unscheduled islanding events are objectives to be optimized in day-ahead scheduling of microgrids with energy storage systems (Farzin et al. 2017). The maximization of battery lifetime is also a concern, which depends on factors as operating temperature, depth of discharge, and levels of the charging/discharging currents. When power and heat demands are at stake, microgrids may include a combined heat and power (CHP) plant. The energy management of these systems involve the minimization of cost and emissions, possibly including also demand-side resources through demand response programs (which may be physically detailed to consider heat transfer and thermal dynamics of buildings) and energy storage systems. More complex systems may include, for instance, boilers and heat recovery systems to supply heating load to buildings, whereas the cooling demand can be met by absorption chillers and compression chillers, which should be designed for minimum cost and maximum efficiency. District energy networks can be designed to transfer surplus energy between buildings and heat/cold storage systems can also be envisaged.

In this general setting, the most realistic multiobjective models are constrained mixed integer nonlinear models, although in some cases bilevel optimization is also used in which, in general, the upper level deals with design objectives and the lower level with operational objectives. In the optimization of the operation of a distribution network with grid-connected microgrids, the upper (distribution network) level determines the optimal dispatch to optimize technical requirements (power losses and voltage profile) and the lower (microgrid) level uses the dispatch plan to minimize the operation cost of distributed generators, possibly including the operation of energy storage systems and incentive/compensation to consumers to encourage the modification of consumption profiles in a favorable way to the grid. The interplay between energy-saving and emission-reduction potentials of thermal generation units and demand-side can also be captured using multiobjective bilevel optimization models. The upper level objective functions are the minimization of generation cost and hazardous emissions, whereas the minimization of compensation and incentive costs of consumers are considered at the lower level framed by demand response regulations (Liu and Li 2015).

Several sources of uncertainty arise in these problems. Uncertainty can derive from power outputs of wind turbines and photovoltaic cells in microgrids or demand patterns, as well as the duration of disconnection from the main grid. Uncertainty is dealt with using stochastic, interval or fuzzy programming approaches, often jointly with scenario modeling which often requires scenario reduction techniques.

In addition to mathematical programming algorithms and meta-heuristic approaches to characterize the nondominated fronts, model predictive control (MPC) is also utilized for solving the optimization problem and develop (near) real-time implementations in a closed-loop framework.

4 Resilient Systems—Protection, Restoration and Reconfiguration

Power distribution networks require adequate protection systems to ensure continuous energy delivery to all consumer loads. The increasing deployment of dispersed generation in the distribution grid, generally based on renewable sources, require the installation of protection devices (e.g., reclosers) to allow islanded operation to reduce the amount of energy not supplied. The placement of those devices in the network is evaluated according to economic and technical dimensions, these latter generally encompassing the SAIDI (System Average Interruption Duration Index) and SAIFI (System Average Interruption Frequency Index) indexes.

After system failure, the distribution system restoration problem consists of determining the combination of (automatic and manual) maneuvering switch devices to be activated to maximize the number of consumers with restored supply in the post-fault period, or minimize de-energized loads, and to minimize the time required by the maneuvers, or minimize the number of switching operations, to recover from outage. Different categories of consumers may have different levels of priority in supply restoration and switching sequences may be defined.

The reconfiguration of a distribution system, which is normally operated in a radial configuration, is an important action for achieving high levels of operational system performance, which involve changing the functional links. According to network conditions (e.g., congestion), reconfiguration schemes aim at minimizing active power losses and maximizing reliability indicators.

The power system resilience, e.g. anticipating extreme events, can be considered in the planning phase deriving, for instance, from decisions of siting and sizing battery storage and photovoltaic generation. The objectives at stake in these models are the minimization of investment and operation and maintenance costs, the maximization of the capacity to supply demand and the capacity to support non-black-start generating units (black-start is the process of restoring a power plant or part of a grid to operation without relying on external energy sources) (Zhang et al. 2019).

The dynamic stability of the power network can also be studied in interdependence with the natural gas network in smart grids, since these are the main energy carriers. The coordinated operation of natural gas and electricity networks, in which natural gas-fired generation units may have a significant share of the generation matrix, involves economic, dynamic stability of electricity network and security of the natural gas network objectives.

5 Using Demand-Side Resources

Demand response, through incentive- or price-based schemes, is currently seen as a relevant Demand-Side Management (DSM) strategy involving technologies and schemes to make an integrated usage of demand-side resources considering the flexibility consumers generally have in the operation of their loads (including appliances, electric vehicles and storage systems). In face of time-differentiated energy prices, more accurately reflecting power generation and delivery costs, demand response typically consists of mechanisms, supported by technologies and algorithms integrated in energy management systems, to make the most of the consumer's flexibility to shift the operation cycle of some loads (e.g. dish or cloth washers in the residential sector) or change the settings of thermostatic-controlled loads (e.g., air conditioning systems) to decrease peak demand. Demand response actions are mutually beneficial for consumers, who can shift consumption to lower priced periods, and the grid, enabling to release network congestion and decrease losses. The peak management enabled by demand response has a potential impact on postponing investments in network reinforcement and peak generation capacity expansion. The reduction of load during more critical peak periods lowers the use of power plants with higher variable costs (and generally higher emissions) thus decreasing energy marginal prices (and environmental impacts). Demand management also enables to adjust the demand profile to the intermittent nature of renewable energy sources thus promoting a better utilization (i.e., adopting "load follows supply" strategies). Moreover, as renewable surplus generation often happens in periods of lower demand, reshaping consumption profiles according to generation capacity also lower the chance to export energy generated by renewable sources at very low (or even zero) prices.

The models to optimize the usage of demand-side resources, which may include also local microgeneration (typically a photovoltaic installation) and storage systems (a static battery or an electric vehicle also capable to deliver the energy stored in the battery when not necessary for transportation), generally consider cost and comfort objective functions and exploit the corresponding trade-offs according to the consumers' flexibility in the operation of loads in face of their routines and preferences as well as willingness to accept a device to control energy usage (even parameterized with their preferences). The cost objective function may include an income term if selling back energy to the grid is allowed, for instance the energy locally produced that is not used or the energy stored in a battery (profiting from time differentiated tariffs, which may display significant energy price changes in short periods). The comfort objective function (or in some cases a dissatisfaction function associated with rescheduling loads outside the most preferred periods) can be constructed in different manners, e.g. a penalty coefficient associated with the most/least preferred time periods for load operation, thermal comfort measured by the deviation of the indoor temperature with respect to a reference temperature, the time a certain service (e.g., laundry) is completed beyond the desired time, average delay for all appliances (possibly considering operation priorities), indoor air quality comfort, etc. (Soares et al. 2017; Muralitharan et al. 2016).

The deployment of energy storage systems at the consumer's premises is expected to contribute to mitigate the adverse effects of uncertain generation based on renewable sources. Planning decisions involve the sizing of hybrid renewable generation-storage systems to minimize investment costs whereas operational decisions pertain to determining the charge/discharge sequence to minimize peak load, decrease energy losses, improve voltage stability, and maximize revenue due to offering grid support services (e.g., providing reserve supply and power quality support, according to the regulatory framework).

The consumer's flexibility can be used by aggregator entities to respond to grid requests, which may involve temporarily decreasing, or even increasing, load. Assets such as electric vehicles, air conditioning systems and electric water heaters are the loads most used for this purpose. Aggregators can also contribute to coordinate the response of individual energy management systems, which would react in the same way to periods of lower energy prices, thus creating a severe peak rebound with a negative impact on the distribution network. This problem can be framed as a bilevel optimization problem to flatten the total load profile (minimizing the aggregate peak) considering that consumers seek minimum cost and maximum comfort (Safdarian et al. 2014).

The definition of demand response programs can also consider the network nodes where they are more necessary from a technical point of view, namely regarding line congestion and security enhancement, although nodal prices and market efficiency may also be aspects of concern. In this setting, objective functions generally considered are active power losses, available transmission capacity and demand response program capacity.

6 Electric Vehicles

Electric vehicles are gaining increasing acceptance as an important means of (private and public) transportation contributing to reduce local hazardous emissions, namely in cities. Vehicle prices, the existence of charging points, in particular fast charging, and government incentives (for instance, in the replacement of internal combustion engine vehicles) are the factors that will trigger a larger adoption of electric vehicles. Due to the power involved in the charging process, namely in fast charging, the large dissemination of electric vehicles is expected to impose a significant challenge to the stability of the electrical grid, including the need of expansion/reinforcement of distribution networks.

The problems associated with electric vehicles involve issues such as the charging infrastructure (namely siting and sizing of charging stations) and its impact on the power distribution network (which requires computing the power flow to assess technical indicators as line losses, which are higher for fast charging stations, and voltage deviations), the scheduling of charging associated with mobility patterns, etc.

From the electric vehicle owner perspective, objective functions include the minimization of costs and the maximization of battery state-of-charge (SoC), considering time-differentiated charging prices. From the charging station operator perspective, objective functions comprise the minimization of investment and operation and maintenance costs (associated with sizing and location), the maximization of the number of electric vehicles departing with required SoC (consumer satisfaction), the minimization of peak-to-average ratio (technical concern), the maximization of market share (traffic captured) and the maximization of profits. This economic objective function may also derive from performing an aggregator role, that is, being able to use the energy stored in the electric vehicle batteries in vehicle to grid mode to provide ancillary services to the grid. The design of appropriate pricing policies is of utmost importance to cope with demand variability and energy price fluctuations to keep adequate levels of profitability, customer satisfaction and impact on the grid. The optimization models also depend on the purpose of the application (e.g., the controlled setting of a parking lot vs. a street station) and mobility patterns (e.g., private users vs. fleets of goods delivery or public transportation).

Uncertainty is associated with the charging patterns of private electric vehicles, although some commuting patterns may be forecasted. The uncertainty associated with demand is generally tackled by stochastic programming or fuzzy approaches, whereas bilevel optimization is used to model pricing decisions in which there is a hierarchical game between agents. Population meta-heuristics that make the most of the specificities of the problem, with some prominence to NSGA-II based approaches, are mostly used to characterize the nondominated front due to the combinatorial and nonlinear characteristics of mathematical models.

Regarding the operation of electric vehicles, the management of multiple sources is a relevant problem whenever hybrid energy storage systems consisting of rechargeable batteries and ultracapacitors should be jointly managed by exploiting their complementary characteristics (high energy density vs. high power density). The objectives herein at stake are to reduce costs, extend battery life and improve mileage, while not jeopardizing driving performance.

7 Markets

Energy markets aim to curb economic inefficiencies associated with the operation of energy monopolies through the introduction of competition in wholesale (generation) and retailing (commercial offers to different types of consumers), while transmission and distribution grids remain regulated natural monopolies. The aim is to guarantee lower energy prices to residential and commercial/industrial consumers and lower production costs thus contributing to social policies and economic competitiveness. Issues such as security of supply and environmental protection should in some way be internalized in prices. In the power sector, markets include not just energy, but also reserve and ancillary services markets.

The combined energy, reserve and reactive power dispatch in electricity markets require the consideration of objectives such as minimizing expected total market payment, transmission congestion, emissions, as well as technical issues to guarantee reliable operation such as maximizing expected voltage security margin and reactive power reserve.

The increasing share of dispersed generation, mainly based on renewable sources connected to the distribution grid, creates further challenges regarding the corresponding power injections, also considering additional issues, namely the minimization of power losses in the distribution network which in turn increases the grid hosting capacity. Market models should be designed to address these challenges, explicitly considering objectives such as maximizing the profits of distributed generation companies, minimizing the cost of serving loads and maximizing the grid overall efficiency.

In pool-based market clearing, including day-ahead joint energy and reserve markets and also balancing settlements, the independent system operator typically validates energy and reserve bids and corresponding payments. In addition to the economic/welfare dimension, joint energy and reserve market clearing models also include the system security evaluation. In this setting, the impact of energy storage systems, which aim to maximize its expected profit, on the clearing process of multiple markets introduces further complexity in the modeling process.

In competitive retail markets, the design of tariff schemes can be modelled using bilevel optimization models in which the retailing company defines time differentiated prices to maximize profits and market share, and consumers reschedule the usage of appliances to minimize the energy bill and the discomfort associated of changing their routines (Alves and Antunes 2018).

8 Conclusions and Perspectives

Multiobjective models and methods have had a significant contribution to deal with relevant problems arising in the energy sector, involving decisions ranging from strategic to operational. Decision support approaches should explicitly encompass multiple, conflicting and incommensurate aspects of evaluation of the merits of distinct courses of action pertaining to economic, environmental, reliability and quality of service concerns.

In planning and operational problems arising with the evolution of power systems to smart grids, multiobjective models and methods have the capability to offer the tools for a thorough analysis and balanced recommendations regarding issues such as accommodating the growing share of intermittent renewable generation without jeopardizing grid stability, optimizing the interaction of electric vehicles with the grid in grid-to-vehicle and vehicle-to-grid modes, exploiting the flexibility of demand-side resources by empowering the role of prosumers (simultaneously producers and consumers of energy). Such decisions should be made in a context of technological advancements, new market designs, and increasing pressure for the decarbonization of the economy in face of the need to curb climate change.

This chapter selected a set of consolidated problems (but gaining new features) and emerging challenges in the areas of generation capacity and grid expansion (paying special attention to renewable sources); unit commitment and dispatch with focus on grid-connected microgrids; resilient systems, involving protection, restoration and reconfiguration issues; integrated management of demand-side resources unveiling their potential to improve the overall system efficiency benefiting consumers and the grid; the growing penetration of electric vehicles, seen in the perspectives of a load and a battery that can assist the grid; design of different types of energy, reserve and ancillary services markets.

The range of problems in the energy sector for which multiobjective models and methods can contribute to support sounder decisions is immense, often displaying complex technical specificities. In general, a mix of integer and continuous variables is necessary and models present nonlinear and/or combinatorial characteristics, which justifies employing mathematical programming approaches, dealt with specific algorithms or commercial solvers to solve scalarizing instances, and meta-heuristics, namely population-based ones aimed at characterizing the nondominated front.

Multiobjective models and methods go well beyond the "realistic" argument that real-world problems intrinsically possess multiple evaluation axes; in addition to capturing the complexity of these problems, they enable the comprehensive exploration of distinct potential solutions meaningfully including in the decision process the interests of stakeholders and coping with uncertainty. Moreover, the novel features of emerging problems will require innovative models and (computationally feasible) methods thus leading to an enriching cross-fertilization between the methodological and application components.

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Optimization and Multicriteria Evaluation of District Heat Production and Storage

Risto Lahdelma, Genku Kayo, Elnaz Abdollahi and Pekka Salminen

Abstract Climate change mitigation policy requires reducing dependence on fossil fuels and transition to low carbon energy production in district heating (DH). We study here inclusion of two kinds of renewable energy to a CHP based DH system in Finland: solar heat and ground source heat. In addition, we apply heat storages to balance the gap between production and fluctuating demand. The optimal operation of the extended systems is determined by a simulation and optimization model to minimize the operating costs. We evaluate the different possible extensions in terms of multiple economic, technical and environmental criteria using Stochastic Multicriteria Acceptability Analysis (SMAA). The results show that under Finnish conditions, ground source heat is more favourable than solar heat for DH.

Keywords Carbon-neutral · District heating · Heat-only production · Multicriteria decision analysis · SMAA

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1 Introduction

District heating (DH) is an energy efficient and environmentally friendly heating system for cities due to its good energy efficiency and centralized management of emissions. The heating network enables use of any available heat source including waste and surplus heat from other industrial processes (Lund et al. 2014). Also, combined heat and power (CHP) production technologies can yield very high, over 90% overall energy efficiency. In Finland, DH covers nearly half of space heating, and almost 80% of the DH is produced efficiently using CHP technology. However, many current DH systems still use fossil-based fuels causing CO_2 emissions. The CO_2 emissions of existing DH systems can in many situations be reduced cost-efficiently by integrating them with renewable energy production (Lund et al. 2010; Ghafghazi et al. 2010b). Popular renewable sources for DH include solar heat, ground source heat, and biomass combustion.

In this study we evaluate and compare different renewable heat production alternatives to be integrated in a fossil fuel based DH system in Finland. The original system contains a coal-fired CHP plant for base load and an oil-fueled heat only boiler (HOB) for peak load. A production simulation and optimization model is developed for determining the cost-optimal operation of the different alternative extended systems. The alternatives are then evaluated in terms of multiple economic, technical and environmental criteria. Measurements for several criteria are based on the optimized production plans. Other assessment methods, such as expert evaluation on ordinal scale is also used.

Multiple criteria decision analysis (MCDA) methods have been widely employed in different energy planning problems, but with emphasis on power systems. Mixed heat and power system analyses include, for example, an MCDA model for evaluating renewable energy technologies for the island of Crete by Tsoutsos et al. (2009), definition of a general sustainability index for an urban energy system by Jovanovic et al. (2010), sustainability ranking of renewable power and heat generation technologies by Dombi et al. (2014), and MCDA evaluation of multi-source energy systems by Catalina et al. (2011). CHP system analyses include evolutionary multicriteria optimization of fuel cell–gas turbine combined cycle by Burer et al. (2003), evaluation of CHP technologies in terms of energy, economy and environmental criteria by Wang et al. (2015a), and selection of residential energy supply system by Alanne et al. (2007). MCDA for heating systems include both building level studies (Chinese et al. 2011; Wang 2015; Loikkanen et al. 2017) and community level analyses (Mroz 2008; Ghafghazi et al. 2010a; Kontu et al. 2015; Jung et al. 2016; Kirppu et al. 2018). For reviews about MCDA for sustainable energy planning, see Pohekar and Ramachandran (2004), Wang et al. (2009), Si et al. (2016), Kumar et al. (2017) and Mardani et al. (2017).

The Stochastic Multicriteria Acceptability Analysis (SMAA) method is used for comparing the different alternatives. SMAA was selected because it can handle mixed ordinal and uncertain cardinal criteria measurements, and it can also be used with absent preference information. SMAA was introduced by Lahdelma et al. (1998) and extended by Lahdelma and Salminen (2001), Lahdelma et al. (2003), and Lahdelma and Salminen (2010). SMAA was developed for decision problems where criteria measurements and preference information can be uncertain, inaccurate and even partly missing. Different kinds of uncertain information are represented by probability distributions. SMAA computes by using stochastic Monte-Carlo simulation the probabilities for each alternative to obtain any particular rank. During the simulation, values for the uncertain variables are sampled from their distributions and alternatives are evaluated by applying the decision model (Tervonen and Lahdelma 2007). The recommended solution is typically the alternative with highest probability for the first rank. However, the probabilities for other possible solutions are also provided for the decision makers (DMs). This means that SMAA describes how robust the model is subject to different uncertainties in the input data (Lahdelma and Salminen 2012, 2016). For a survey on SMAA methods, see Tervonen and Figueira (2008). Recent developments of SMAA include the SMAA-PROMETHEE method by Corrente et al. (2014), SMAA with Choquet Integral by Angilella et al. (2015), and extensions for pairwise comparison methods such as the analytic hierarchy process (AHP) by Durbach et al. (2014) and the Complementary Judgement Matrix (CJM) method by Wang et al. (2015a).

SMAA has been applied to many problems in the areas of municipal planning (Hokkanen et al. 1998), harbor development (Hokkanen et al. 1999), polluted soil remediation (Hokkanen et al. 2000; Lahdelma et al. 2001; Lahdelma and Salminen 2008a), waste treatment plant siting (Lahdelma et al. 2002), forest management (Kangas et al. 2003, 2005, 2006), waste storage area siting (Lahdelma and Salminen 2008b), risk-based classification of nanomaterials (Tervonen et al. 2009); multimodal cargo hub development (Menou et al. 2010), strategic environmental assessment (Rocchi 2012), rural electrification in developing countries (Rahman et al. 2013), energy policy assessment (Rahman et al. 2016), benefit-risk analysis of drugs (Tervonen et al. 2011; van Valkenhoef et al. 2012; Okul et al. 2014), energy monitoring systems selection (Pesola et al. 2014), dredged material management (Scheffler et al. 2014), peak heating plant siting in DH system (Wang et al. 2015b), residential heating alternative evaluation (Kontu et al. 2015; Jung et al. 2016; Loikkanen et al. 2017; Kirppu et al. 2018), and public sector facility selection (Karabay et al. 2016).

2 Decision Problem

In the following sections, we describe the overall problem, the alternative extended DH systems, and the evaluation criteria.

2.1 Problem Description

International climate change mitigation policy requires reducing CO_2 emissions of energy production. The target of this study is a typical community in Southern Finland with majority of buildings heated with DH. Current production is based on fossil fuels. The problem is to find the most suitable ways to reduce the CO_2 emissions of this kind of communities.

Various economic, technical and environmental criteria must be considered when choosing the most suitable extension alternative for the DH system. Therefore we apply MCDA for comparing the alternatives. The criteria should meet the general requirements listed by Keeney and Raiffa (1976):

- Completeness: all the important points of view of the problem are covered.
- Operationality: the set of criteria can be measured and used meaningfully in the analysis.
- Non-redundancy: two or more criteria should not measure the same thing.
- Minimality: the dimension of the problem should be kept to a minimum.

All criteria are measured in relation to the original DH system without renewable energy. We assume that each extension alternative (including the original DH system) is operated cost-optimally, i.e. minimizing the production costs while satisfying the heat demand. The optimal operation of each alternative is determined by a production planning and optimization model.

The target system consists of a coal-fired CHP plant for base load and an oil-fueled HOB for peak load located. We consider integrating the system with two alternative forms of renewable energy: solar heat (SH), and ground source heat (GSH). The production of the DH system needs to be adjustable to meet the demand that fluctuates greatly both within the day and between the seasons. Figure 1 shows the hourly heat demand of the target community for one year. The yearly heat demand is about

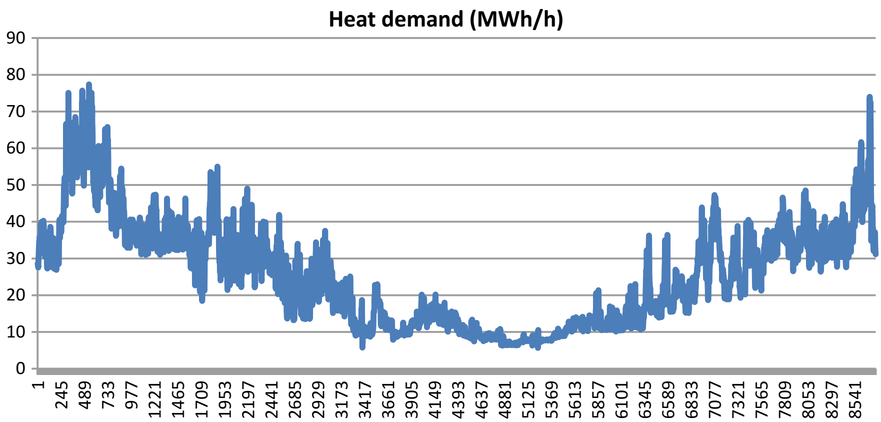

Fig. 1 Hourly DH consumption for one year

225 GWh and the peak heat demand is about 75 MW. The demand in the winter is almost ten times higher than in the summer. The main factor affecting DH demand is outdoor temperature. Figure 2 shows the outdoor temperature for the same year. In the summer, DH is not needed for space heating, but in the Finnish DH system, DH is still used for heating up the household water. This means that the DH system must operate year around. Figure 3 shows the heat demand for a sample day in March. Daily variation in heat demand is much smaller, but still significant.

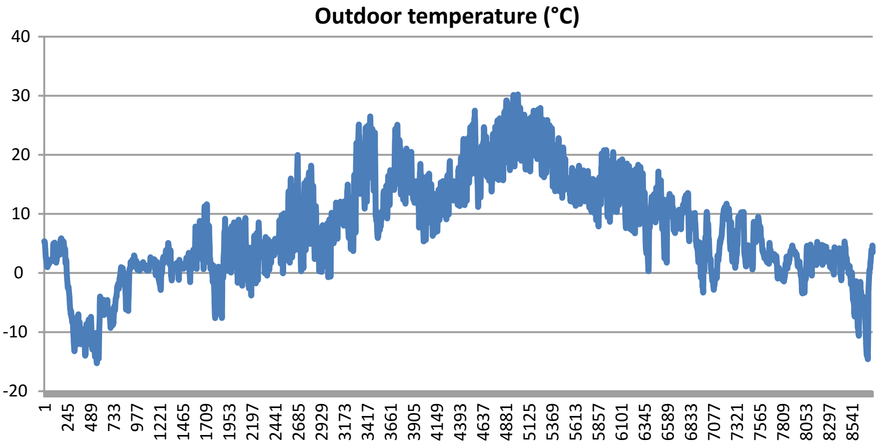

Fig. 2 Hourly outdoor temperature for one year

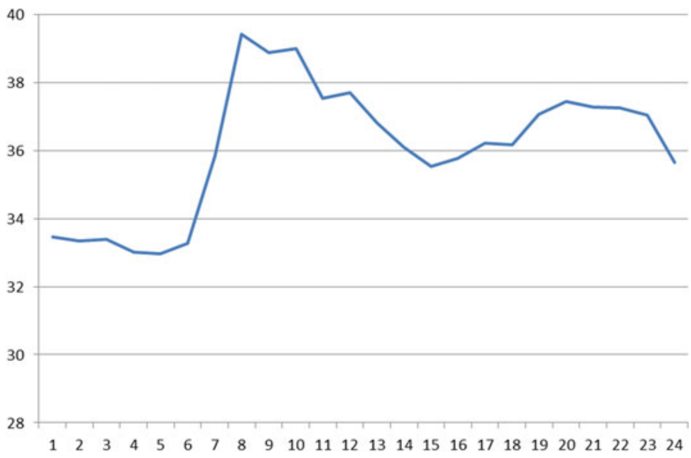

Fig. 3 Hourly DH consumption for a sample day

2.2 Alternatives

The set of alternatives was formed based on most promising available renewable energy technologies. The original DH system without extensions is called the BASIC alternative. The different extension alternatives based on SH and GSH are then compared with the BASIC alternative. Because SH production is not adjustable according to fluctuating demand, different size storages can be combined with the SH plant in order balance SH production better with the demand. Also GSH can benefit from heat storage. GSH is typically dimensioned at a fraction of the peak demand and is ideally run at maximum level while the demand fluctuates. The alternatives are listed in Table 1 and described in detail in the subsequent sections.

Table 1 Renewable DH production alternatives

Alternative	Description
BASIC	Original system without renewables or storage
SH	SH plant without heat storage
SH-SS	SH plant with short-term heat storage
SH-MS	SH plant with medium-term heat storage
SH-LS	SH plant with long-term heat storage
GSH	Ground source heat without heat storage
GSH-SS	Ground source heat with short-term storage
GSH-MS	Ground source heat with medium-term storage
GSH-LS	Ground source heat with long-term storage

2.2.1 Solar Heat

A SH plant consists of a field of solar collectors, circulation pumps, and pipes to transfer the collected heat via heat exchangers to the DH system. We assume that the SH collectors are installed south-facing and with constant tilt angle $\phi \in [0, \pi/2]$. The intensity of the solar radiation G (W/m^2) that hits the collectors in a given time consists of two components: direct radiation and ambient (diffuse) radiation. The meteorological institute reports the direct radiation G_{DIR} measured on a surface orthogonal to the sun rays and the ambient radiation G_{AMB} on a horizontal surface. We only consider radiation for heat production when the sun is above the horizon. The radiation that hits the collectors is a combination of the two components according to

$$G = \cos(\theta)G_{DIR} + (1 - \phi/\pi)G_{AMB}.$$

The direct radiation is projected to the collector using the cos of the angle θ between the sun ray and the normal to the collector surface. When the sun is behind the collector, cos of the angle becomes negative and the direct radiation term is excluded from the formula. The ambient radiation is proportional to the part of the sky that the tilted collector surface faces. The yearly average of the direct radiation component is maximized by choosing a tilt angle close to the latitude of the collector location. However, the ambient component is maximized by horizontal collectors. Higher tilt angles also makes multiple rows of collectors shadow each other unless they are installed far apart. For this reason the optimal tilt angle for the combined radiation is smaller than the latitude. In this study we have applied 45°(=π/4) tilt angle.

The efficiency ratio η_{SH} of the solar collectors is a time-dependent factor, because it depends heavily on the solar irradiation, the outdoor temperature, and the operating temperature of the collector. The operating temperature of the collector is the average of water temperature entering and leaving the collector and it depends on how the solar collectors are connected to and operated with the DH system.

The efficiency ratio of SH collectors can be written as a second degree polynomial function

$$\eta_{SH} = \eta_0 - \frac{a_1 \Delta T}{G} - \frac{a_2 \Delta T^2}{G}.$$

Figure 4 illustrates the efficiency ratio. We can see that the efficiency drops rapidly when the irradiation is low and temperature difference is high.

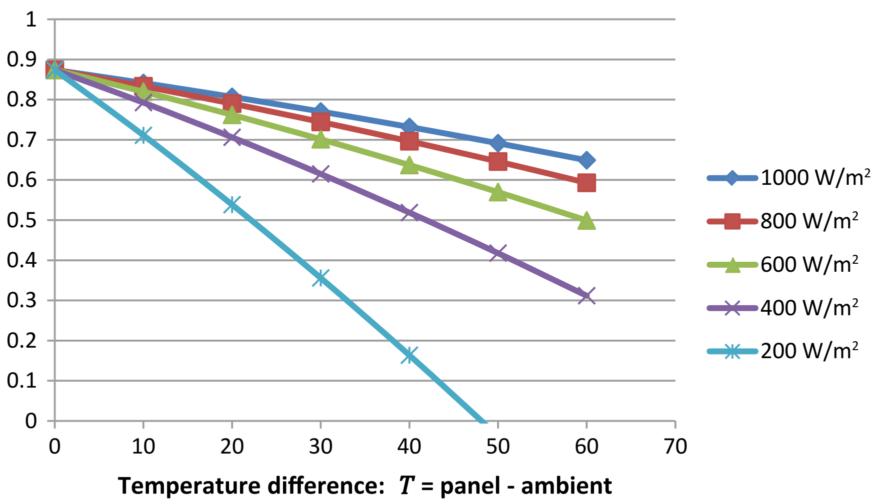

Fig. 4 Efficiency η_{SH} of solar panel as function of temperature difference and radiation intensity

SH can be connected to the DH system in different ways. In this study we assume the RR-assisted connection, where SH is used to heat up part of the DH water in the return pipe, and the heated water is fed back into the return pipe. The RR-assisted connection is favorable in Finland because a large part of the year outdoor temperature is cold and the sun radiation is low. The RR connection allows a low operating temperature for the collector, minimizing the temperature difference and maximizing the panel efficiency. In practice, the maximal efficiency that can be obtained in southern Finland is about 0.8 corresponding to 1200 W/m^2 irradiation and a 40° temperature difference on a sunny summer day.

The SH plant can be configured to operate without any specific storage by utilizing only the storage capacity of the district heating network itself (SH alternative). Alternatively, the SH plant can be provided with a short-term (SH-SS) or medium-term (SH-MS) storage such as a hot water tank. The system can also include a long-term seasonal pit storage (SH-LS) to balance the seasonal variations in the heat supply.

In this study, we consider a SH plant with 2 ha panel area. This corresponds to about 18.6 MW peak power and 10,300 MWh annual production.

2.2.2 Ground Source Heat

GSH is extracted from a field of deep (about 300 m) heat wells drilled into to the ground. The underground temperature is year-around about 12 °C in Southern Finland. The GSH production from boreholes in Finland is about 70 W/m. The heat is extracted by circulating water in the wells and upgrading the heat to required temperature using a heat pump. The ratio between produced heat and consumed power is the COP (coefficient of performance) factor of the heat pump. Typical COP factor for GSH in Finland is 3 which means that one third of the produced heat is from electricity and two thirds from the ground. GSH is considered renewable only if the electricity the pumps use is renewable. Otherwise the emissions of GSH are reduced into 1/COP part of the emissions caused by electricity production. Also operating costs are reduced in same proportion compared to electricity price.

Because a GSH system has high investment costs and relatively low operating costs, it should operate at as high utilization ratio as possible. To achieve this, GSH capacity is dimensioned to satisfy only a fraction of the peak demand which occurs only during a small number hours per year. The utilization ratio of GSH can be further improved with a heat storage which is charged during low heat demand and discharged during higher demand. We consider four alternatives, GSH without storage, GSH-SS with short term storage, GSH-MS with medium term storage, and GSH-LS with long term storage.

In this study, we consider a GSH plant with 3 MW peak power. This means that about 43 km of boreholes are needed. In principle the plant can operate at this power year-around, yielding 26,280 MWh annual production (about 2.5 times the SH plant production).

2.2.3 Storages

Different techniques for heat storages are available. The simplest techniques are based on storing hot water. The water storage can be either pressurized or unpressurized. Pressurized storages allow storing hot water in higher than 100 °C temperature. Unpressurized storages allow max 100 °C temperature before the water starts to boil. Depending on the operating temperature of the DH network, under 100 °C water can be used directly as DH supply water, or may need heated further. In this study we assume that unpressurized storages can be used in the DH system. Assuming a 40 °C temperature difference, 1 m^3 water storage can store 167 MJ = 46.5 kWh heat. The short-term (SS) and medium-term heat storages (MS) are implemented as hot water tanks with capacities 500 MWh (10,750 m^3) and 1000 MWh (21,500 m^3). The long-term heat storage (LS) is implemented as large pit storage with capacity 9000 MWh (193,500 m^3).

2.3 Criteria

Choosing the most suitable alternative for integrating renewable energy into a DH system is not only an economic decision. From the technical point of view, it is important that the production can be easily adjusted to meet the variable heat demand in different operating situations. Also impact on the surrounding community, such as space requirement and transportation logistics must be considered. Environmental points of view are the driving force in the problem, which means that reduction in CO_2 and other emissions is important. The criteria applied in this study are listed in Table 2. In the following, we describe each criterion in detail.

Two economy-related criteria were included, the investment costs (**InvC**) and annual operating costs (**OperC**). The investment costs are calculated per year using

Table 2 Set of criteria

Criterion	Unit	Description
InvC	k€	Additional investment costs annuity
OperC	k€	Reduction in (net) operating costs per year
Space	ha	Additional space requirement for plant and storage
Logist	tonne	Reduction in fuel transportation logistics per year
CO_2	tonne	Reduction in CO_2 emissions per year
Partic	tonne	Reduction in particulates emissions per year
Adjust	(rank)	Adjustability of production technology on ordinal scale

the annuity method with 5% interest rate. The lifetime for the different production alternatives is 20 years (annuity factor = 12.46) and for the heat storages 50 years (annuity factor = 18.26). The total investment costs for the alternatives are estimated based on reported investment cost per capacity in recent projects.

Investment costs for solar district heating in recent projects have been reported by IEA (2017). They published a regression line with 4 million DKK + 1.30 DKK/m^2 per solar collector area. With 1 DKK ≈ 0.134 € the investment costs for 2 ha panel area is about 200 €/m^2.

Investment costs for different size GSH systems in France were analysed by Boissavy (2015). Boissavy divides the costs into two parts, for the heat pump system on ground and for the vertical system (boreholes) underground. To make the investment costs based on French conditions applicable for southern Finland, we replace COP factor 4 with 3 and heat yield from boreholes from 50 to 70 W/m. For large scale GSH in Finland, this gives investment costs 1.3 M€/MW.

Investment costs for different size heat storages are shown in Fig. 5 (IEA 2012). The storage investment costs are related to the water equivalent storage volume (m^3). The investment cost for large storage tanks are about 120 €/m^3 (Friedrichshafen). The investment cost for large pit storages is about 25 €/m^3 (Marstal-2). Assuming 40 °C temperature difference, the investment costs per stored energy capacity are about 5376 €/MWh and 535.6 €/MWh, correspondingly.

The annual (net) operating costs for the alternatives are computed as the sum of production costs and other operating cost (operations and maintenance). Because DH business is a natural monopoly whose operation is regulated, pricing of heat must follow cost correlation and be uniform for all customers. This means that revenues

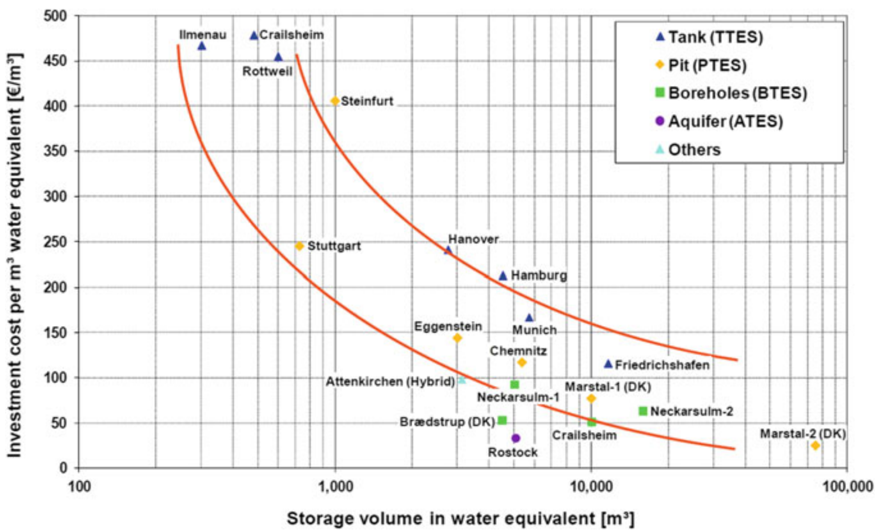

Fig. 5 Specific storage costs of demonstration plants (IEA 2012)

from heat sales are not considered in the problem. However, power produced by CHP can be sold on the free market, and these revenues should be subtracted from the operating costs. Thus, the production costs are computed from consumed fuel subtracted by the revenue from selling CHP power to the market. The fuel and power consumption and power sales are determined by using a production simulation/optimization model. The model is described in detail in Sect. 3.

Space requirement (**Space**) includes space for the new production technology and possible heat storage. The goal is to minimize the space requirement. The GSH alternatives require a little space for the heat pump, more for the locations where the boreholes are drilled, and for piping to circulate liquid from the boreholes to the heat pump. SH requires large area for the solar collectors. Also the heat storages require considerable space depending on storage size and how it is implemented.

Logistics criterion (**Logist**) is measured in terms of fuel transportation needed in the different alternatives. In comparison to the BASIC alternative, the GSH and SH alternatives replace fossil-based production and therefore reduce the overall fuel transports needed. The production planning model determines how much coal and oil (in MWh) is annually needed for operating the CHP plant and HOB in each alternative. To make the two fuels comparable in terms of transportation, the weight of 1 MWh of coal is about 125 kg and weight of 1 MWh of oil is about 91 kg.

Reduction in CO_2 emissions (**CO_2**) is computed based on change in fuel consumption and electricity balance compared to the BASIC alternative. The production planning model determines how much coal, and oil is consumed and how much electricity is produced and consumed in each alternative. For computing the CO_2 emissions in each alternative, the specific CO_2 emission factor in Table 3 are applied for coal, oil and electricity on the grid.

Reduction in particulates emissions (**Partic**) is computed based on change in combustion of fuels compared to the BASIC alternative. Each fuel has a specific particulates emission factor, which also depends on the combustion technique. However, most significant for the particulates emissions is the flue gas filtering technique. In this study we apply for coal and oil the maximum limits for particulate emissions (dust) in large combustion plants set by the Directive 2010/75/EY (EU 2010).

The adjustability criterion (**Adjust**) measures how easily the production of DH can be adjusted to meet the variable heat demand. This depends on how easily energy production can be started, adjusted and operated on partial load in different operating situations—now and in the future. In particular, intermittent SH production

Table 3 Emission factors for different fuels and electricity in the grid

Fuel	Mass/energy (kg/MWh)	CO_2 (kg/MWh)	Particulates (g/MWh)
Coal	125	335.5 kg CO_2/MWh	72
Heating oil	91	264.6 kg CO_2/MWh	54
Electricity (grid average)	–	181 kg CO_2/MWh	–

improves adjustability of the DH system marginally. Very little SH is available in Finland during the cold season when DH is needed the most. However, storages improve the adjustability. Adjustability is measured by ranking the alternatives from the most adjustable alternative (1) to the least adjustable (9).

3 Production Planning Model

A production planning model is developed for determining the cost-optimal operation of original system and the different alternative extensions. The model is based on hourly heat demand for the community. Because the system includes combined heat and power (CHP) production, also the hourly power price on the market affects the optimal system operation. Optimization minimizes the production costs over the time horizon, which is one year. The optimal solution determines how different production units are run each hour in terms of fuel consumption and energy production. Also the optimal operation of the heat storage is determined in terms of hourly charges and discharges. The caused emissions (CO_2, particulates) and fuel transportation needs can be computed based on the optimal solution. The model is solved for the BASIC alternative and for all extension alternatives. This allows comparing the performance of the extension alternatives against the BASIC alternative.

The production planning model is formulated as a minimization problem over the hours in one year.

$$\begin{aligned} &\text{Min} \quad \sum_{t=1}^{T} c^t\left(\boldsymbol{x}^t\right) \\ &\text{s.t} \quad \boldsymbol{x} \in \boldsymbol{X}. \end{aligned}$$

Here $\boldsymbol{x}^t$ is the vector of decision variables for hour t, $\boldsymbol{x}$ is the vector of all decision variables, and $\boldsymbol{X}$ represents the production technology constraints. The hourly production costs $c^t\left(\boldsymbol{x}^t\right)$ are the sum of the production costs for different production forms

$$c^t\left(\boldsymbol{x}^t\right) = c^t_{CHP} + c^t_{HOB} + c^t_{GSH} + c^t_{SH}.$$

The hourly heat production is similarly the sum of production using different production forms. The hourly heat production plus possible discharge from heat storage minus possible charge of storage must be greater than or equal to the hourly heat demand.

$$x^t_{CHP} + x^t_{HOB} + x^t_{GSH} + x^t_{SH} + x^t_{SOUT} - x^t_{SIN} \geq Q^t.$$

All terms in the cost function and the heat supply constraint are defined in terms of linear constraints. This means that the overall model can be solved as a linear programming (LP) model. In the following, we present the models for different production technologies. If a technology is absent from the system configuration, corresponding costs and production amount are set to zero.

CHP model: A CHP plant produces simultaneously heat and power. Depending on the technology, heat and power production may be tightly coupled, or certain freedom to adjust the production amounts may exist. In all cases, the CHP plant can be modelled as a convex combination of extreme characteristic operating points in 3 dimensions. The dimensions are fuel consumption, power output, and heat output (x_F, x_P, x_Q). A CHP plant with n characteristic operating points (F_j, P_j, Q_j) is modelled using constraints.

$$c^t_{CHP} = C_{FUEL}x^t_F - C^t_P x^t_P, \text{ (net production costs)}$$
$$x^t_F = \sum_1^n x^t_j F_j, \text{ (fuel consumption)}$$
$$x^t_P = \sum_1^n x^t_j P_j, \text{ (power production)}$$
$$x^t_Q = \sum_1^n x^t_j Q_j, \text{ (heat production} = x^t_{CHP})$$
$$\sum_1^n x^t_j = 1, \text{ (convexity constraints)}$$
$$x^t_j \geq 0.$$

The hourly production costs consist of the fuel costs subtracted by revenue from selling the produced electric power to the market at hourly varying price C^t_P. The x^t_j variables are used to form the convex combination of the characteristic points. The maximum heat production capacity of the CHP plant is determined by the operating point with maximal Q_j.

HOB model: A heat-only boiler (HOB) converts fuel into heat with specific efficiency ratio and fuel cost. The HOB is modelled using two constraints

$$c^t_{HOB} = C_{FUEL}x^t_{HOB}/\eta_{HOB,FUEL},$$

$$0 \leq x^t_{HOB} \leq x^{MAX}_{HOB}.$$

Here C_{FUEL} is the fuel price (€/MWh), $\eta_{HOB,FUEL}$ is the efficiency ratio of the HOB (depending both on HOB type and the fuel), and x^{MAX}_{HOB} is the production capacity of the HOB.

SH model: SH production costs is the electric power demand for circulation pumps times the power price C^t_P. Power demand for circulation pumps is computed as consumption ratio α_{SH} times the SH production,

$$c^t_{SH} = C^t_P \alpha_{SH} x^t_{SH}.$$

SH production depends on the area of the SH collectors A_{SH}, the efficiency ratio of the collectors ${}^t_{SH}$, and the intensity of solar radiation per square meter G^t that hits the solar panels during hour t.

$$x_{SH}^{t} = G_{SP}^{t} A_{SH}.$$

Both the efficiency ratio and the solar radiation intensity are time-dependent factors as described earlier.

GSH model: GSH is obtained from deep bore holes in the ground by an electrically driven heat pump. The ratio between produced heat and consumed electricity is the COP-factor (coefficient of performance) of the heat pump η_{COP}.

$$c_{GSH}^{t} = C_{P}^{t} x_{GSH}^{t} / \eta_{COP}$$

$$0 \leq x_{GSH}^{t} \leq x_{GSH}^{MAX}$$

Storage model: The heat storage level at end of each period t is represented by variable x_{STOR}^{t}. The storage level for each period depends on the previous storage level plus charged minus discharged amount. Storing, charging and discharging efficiencies are defined by specific efficiency ratios.

$$x_{STOR}^{t} = \eta_{STOR} x_{STOR}^{t-1} + \eta_{SIN} x_{SIN}^{t} - x_{SOUT}^{t} / \eta_{SOUT}$$

$$0 \leq x_{STOR}^{t} \leq x_{STOR}^{MAX}$$

$$0 \leq x_{SIN}^{t} \leq x_{SIN}^{MAX}$$

$$0 \leq x_{SOUT}^{t} \leq x_{SOUT}^{MAX}$$

The initial storage level x_{STOR}^{0} is a parameter set to zero. For charging and discharging efficiency we use 95% and for hourly storage efficiency we use 99.99%.

4 Multicriteria Modelling

4.1 SMAA Method

The Stochastic Multicriteria Acceptability Analysis (SMAA) method (Lahdelma and Salminen 2001; Lahdelma et al. 2003) was applied for evaluating the DH production alternatives. In SMAA the problem is defined as a set of m alternatives that are measured in terms of n criteria forming a matrix $\mathbf{x} = [x_{ij}]$ where i identifies the alternative and j the criterion. SMAA can be used with any decision model. In this application we use the additive value function

$$u(x_i, \boldsymbol{w}) = w_1 \cdot u_{i1} + w_2 \cdot u_{i2} + \cdots + w_n \cdot u_{in}. \quad (1)$$

The partial values u_{ij} are obtained by mapping criteria measurements x_{ij} into the range [0, 1] so that 0 corresponds to the worst and 1 to the best outcome. The importance weights w_j represent preference information. The weights are non-negative and normalized, i.e. the feasible weight space is defined as

$$W = \{\boldsymbol{w} \mid w_j \geq 0 \text{ and } w_1 + w_2 + \cdots + w_n = 1\}. \quad (2)$$

Figure 6a illustrates the feasible weight space in case of 3 criteria. Both criteria and weight information can be imprecise, uncertain or partially missing. The incomplete information is represented by suitable (joint) probability distributions:

- $f_X(\boldsymbol{x})$, density function for stochastic criteria measurements.
- $f_W(\boldsymbol{w})$, density function for stochastic importance weights.

Because all information is represented as distributions, Monte-Carlo simulation can be used for analyzing the problem efficiently. The simulation is implemented by drawing simultaneously criteria measurements and weights from their distributions, applying the value function to rank the alternatives. The calculation is repeated K times (about 10,000 iterations are sufficient) and the following statistics about the performance of the alternatives is collected:

B_{ir} The number of times alternative x_i obtained rank r.
C_{ik} The number of times alternative x_i was more preferred than x_k.
$\mathbf{W}_i$ Sum of the weight vectors that made alternative x_i most preferred.

The basic SMAA measures computed from the statistics are *rank acceptability indices*, *pairwise winning indices*, *central weight vectors*, and *confidence factors*.

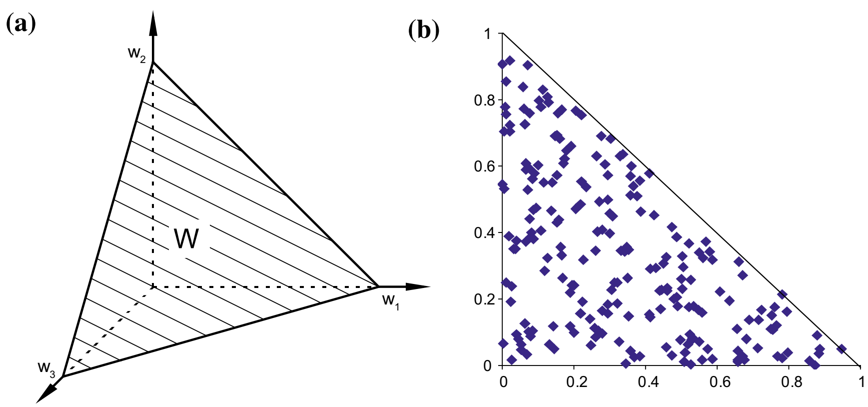

Fig. 6 **a** Feasible weight space in the 3-criterion case. **b** Sampling uniformly distributed weights in the 3-criterion case projected on the (w_1, w_2) plane

The *rank acceptability index* b_i^r measures the variety of different preferences that place alternative $\mathbf{x}_i$ on rank r. It is the probability that the alternative obtains a certain rank. The first rank acceptability index b_i^1 is the probability that the alternative is most preferred. For inefficient alternatives the first rank acceptability index is zero. The rank acceptability indices are estimated from the simulation statistics by

$$b_i^r \approx \mathrm{B}_{ir}/\mathrm{K}. \tag{3}$$

Alternatives with high acceptability for the best ranks are candidates for the most acceptable solution.

The *pairwise winning index* c_{ik} is the probability for alternative x_i being more preferred than x_k, considering the uncertainty in criteria and preferences (Leskinen et al. 2006). The pairwise winning indices are estimated by

$$c_{ik} \approx \mathrm{C}_{ik}/\mathrm{K}. \tag{4}$$

The pairwise winning indices are useful when comparing the mutual performance of two alternatives. Unlike the rank acceptability index, the pairwise winning index between each pair of alternatives is independent on all other alternatives. This means that the pairwise winning index can be used to form a ranking among the alternatives.

The *central weight vector* $\mathbf{w}_i^c$ is the expected center of gravity of the weights that make an alternative most preferred. The central weight vector represents the preferences of a DM supporting the alternative. The central weight vectors are estimated by

$$\mathbf{w}_i^c \approx \mathbf{W}_i/\mathrm{B}_{i1}. \tag{5}$$

The *confidence factor* p_i^c is the probability for an alternative to obtain the first rank when its central weight vector is chosen. The confidence factors can be used to determine if the criteria measurements are accurate enough for choosing the most preferred alternative. Because central weights are available only after SMAA simulation, a second simulation is needed to compute the confidence factors. Additional statistics P_i representing the number of times alternative x_i was most preferred using weights $\mathbf{w}_i^c$ is collected and the confidence factor is estimated as

$$p_i^c \approx \mathrm{P}_i/\mathrm{K}. \tag{6}$$

4.2 Modelling Uncertainty

In this application, both ordinal and cardinal criteria measurements are applied. In both cases, suitable distributions are used for modelling the uncertainty. For cardinal criteria, the partial values u_{ij} are computed from the actual criteria measurements x_{ij}

by linear scaling $u_{ij} = (x_{ij} - x_j^{worst})/(x_j^{best} - x_j^{worst})$ so that the best outcome x_j^{best} on that criterion corresponds to 1 and the worst outcome x_j^{worst} corresponds to 0. When the actual criteria measurements x_{ij} are stochastic quantities, so are the partial values u_{ij}. Because no analytic treatment of the distributions is needed in the simulation, arbitrary independent or dependent distributions can be applied for the uncertain parameters. In this application, independent normal (Gaussian) distributions were applied for simplicity.

Ordinal criteria are measured by ranking the alternatives from the best (1) to the worst (m). An ordinal scale does not carry information about the intervals between the steps of the scale, i.e. it is uncertain how much better each rank r is to rank r + 1. Ordinal uncertainty is handled by defining for the different ranks $r = 1, \ldots, m$ stochastic cardinal values $(s_1, \ldots, s_r, \ldots, s_m)$ in the range [0, 1] so that they are consistent with the ranking information. That means that $s_1 = 1$, $s_r > s_{r+1}$, and $s_m = 0$. Also, the stochastic cardinal values follow a uniform distribution subject to these constraints.

No preference information was available in this application. In the absence of weight information, any feasible weights are equally possible, which is represented by a uniform distribution $f_W(\boldsymbol{w})$ in the feasible weight space W. A special technique for generating uniformly distributed normalized weights is presented by Tervonen and Lahdelma (2007). Sampling uniformly distributed weights in the feasible weight space is illustrated in Fig. 6b.

5 Results

5.1 Optimal Production Plan for Alternatives

The production planning model determines the cost-optimal operation for the different alternative configurations. In particular, the model determines how much energy is produced using the each available technology, and how much fuels and electricity is needed. Table 4 lists for each alternative the operative net costs (production costs subtracted by power sales revenue), coal consumption, oil consumption, net electricity production (CHP power production subtracted by electricity consumption for SH or GSH), CHP heat production, HOB heat production, SH production and GSH production. Note that the net operative costs are negative in all alternatives, which positive operative margin, but without considering any fixed costs. These data are basis for measurements of several criteria.

Table 4 Optimal production in different alternatives

Alternative	NetCost (M€)	Coal (GWh)	Oil (GWh)	Net el. (GWh)	CHP (GWh)	HOB (GWh)	SH (GWh)	GSH (GWh)
BASIC	−6.87	419	47	188	193	42	0	0
SH	−7.01	408	44	183	188	40	11	0
SH-SS	−7.35	401	40	181	184	36	11	0
SH-MS	−7.41	403	38	182	185	34	11	0
SH-LS	−7.80	433	26	195	198	24	11	0
GSH	−7.28	405	34	176	186	30	0	19
GSH-SS	−7.60	404	29	177	184	26	0	18
GSH-MS	−7.65	406	28	178	185	25	0	18
GSH-LS	−7.96	430	19	189	196	17	0	19

5.2 *Criteria Measurements*

Criteria measurements are summarized in Table 5 and explained in the following. A 10% standard uncertainty is assumed for the cardinal criteria, except 5% for the Space criterion. Space requirement can be estimated quite accurately.

The **InvC** criterion is measured as annuity based on past investment costs per production and storage capacity in recent projects as described in Sect. 2.3. The investment cost for SH is about 200 €/m^2 or 4 M€ for 2 ha panel area. Annuity (20 years, 5%) for this is 321 k€. For GSH, the investment cost is 1.3 M€/MW or 3.9 M€ for 3 MW capacity. Annuity for this is 313 k€. The investment cost for tank storages is 120 €/m^3. For the SS an MS storages this means 1.3 M€ and 2.6 M€

Table 5 Criteria measurements

Alt\crit	InvC (k€) min	OperC (k€) max	Space (ha) min	Logist (tonne) max	CO_2 (tonne) max	Partic (tonne) max	Adjust min
BASIC	0	0	0	0	0	0	9
SH	321	138	3	1633	3489	0.94	8
SH-SS	392	485	3.2	2865	6580	1.66	6
SH-MS	463	538	3.4	2755	6440	1.60	5
SH-LS	584	930	4.3	212	2236	0.16	4
GSH	313	416	0.3	3008	6060	1.75	7
GSH-SS	384	735	0.5	3510	7747	2.05	3
GSH-MS	455	779	0.7	3369	7560	1.97	2
GSH-LS	576	1088	1.6	1122	3723	0.69	1
Uncertainty	±10%	±10%	±5%	±10%	±10%	±10%	–

investment cost, or 71 k€ and 142 k€ annuity (50 years), correspondingly. For the LS pit storage, investment cost is 25 €/m^3, or 4.8 M€. Annuity for this is 363 k€.

The **OperC** criterion is computed directly from the operative planning model results as improvement compared to the BASIC alternative.

The **Space** criterion is measured as follows: For the SH alternative, to avoid shading, the rows of collector panels are installed at distance 1.5 times the panel width. This means that the space requirement for 2 ha SH plant is 3 ha. For the GSH alterative with 43 km of boreholes, 144 heat wells of 300 m depth are needed. The holes should be at least 20 m apart. To reduce needed surface space, it is possible to drill at each location 4 holes at a small angle apart. Then 36 quadruples of holes will take corridor of 1.4 km length, 2 m width, and area of 0.28 ha. With additional space for the heat pump plant, the space requirement if GSH is 0.3 ha. For the SS and MS tank storages the space requirement is 0.2 ha and 0.4 ha, correspondingly. For the LS pit storage 1.3 ha space is needed.

The **Logist** criterion depends on the consumption of coal and oil (Table 5). In the SH and GSH alternatives, fossil fuels are replaced with renewables, which reduces transportation logistics.

The **CO_2** criterion depends on emissions caused by combusting coal and oil, and net consumption of electricity from the grid (Table 5). Table 3 lists the specific CO_2 emission factors.

The **Partic** criterion is evaluated based on coal and oil combustion (Table 5) by applying emission factors for particulates (Table 3).

The **Adjust** criterion is measured on an ordinal scale, i.e. experts ranked the alternative DH system configurations with respect to this criterion. All extensions to the original system improve in principle the adjustability. However, the SH alternative (without storage) causes very little improvement, because SH is intermittent and cannot be adjusted at all when there is not enough sun radiation. GSH alternatives are better for adjustability, because GSH is practically always available. Most improvement in adjustability is obtained by storages, the larger the better.

5.3 SMAA Results

We evaluated the alternatives without preference information using SMAA based on the criteria measurements in Table 5. Figure 7 shows the rank acceptability indices for the alternatives. The rank acceptability indices describe the probabilities for each alternative to obtain a certain rank when considering uncertain criteria measurements and uniform distribution to represent absent preference information. To make reading the figure easier, the alternatives have been sorted lexicographically, i.e. in decreasing order by their first, second, etc. rank acceptability index.

Based on the results, the GSH-SS and GSH-MS alternatives are the most acceptable alternatives. They obtain about 58 versus 37% acceptability for the first rank, and 37 versus 53% acceptability for the second rank. The confidence factors of these alternatives are 68 versus 43%. This means that they are so similar that even with per-

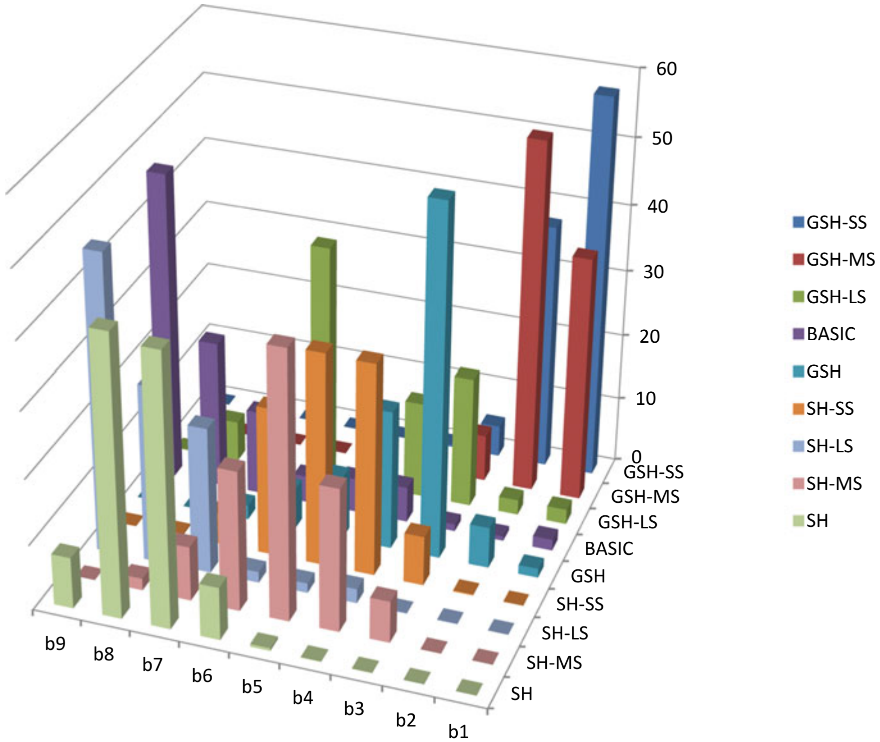

Fig. 7 Rank acceptability indices (%) for alternatives sorted lexicographically

fect preference information they cannot be discerned reliably without more accurate criteria measurements.

The following alternatives are GSH-LS, BASIC, and GSH with 2.3, 1.7, and 1.4% acceptability for the first rank. Out of these, only the BASIC alternative has a high confidence factor (99.5%), which means that under favorable preferences, that could be the most preferred alternative. The other alternatives have very low acceptability for the first rank, less than 0.01%, so they can be eliminated from further analysis.

Table 6 shows the pairwise winning indices between alternatives sorted topologically so that each alternative in the list has at least 50% winning index compared to the later alternatives. This way the pairwise winning indices form a rough ranking of the alternatives. We can see that this ranking is a little different from the lexicographical order based on rank acceptability indices. The two top alternatives are the same, but for example GSH-LS has moved down to place 6 after the SH-SS and SH-MS alternatives. BASIC is now on last place.

Figure 8 shows the central weight vectors for the efficient alternatives. The central weight vectors describe typical (average) preferences that make an alternative most preferred. (SH is inefficient, i.e. no weights can make it most preferred, thus it has no central weight vector). Based on central weights for different criteria we can see

Table 6 Pairwise winning indices (%) between alternatives sorted topologically

Alt	GSH-MS	GSH	SH-SS	SH-MS	GSH-LS	SH	SH-LS	BASIC
GSH-SS	61	97	100	100	96	100	100	98
GSH-MS		92	99	100	97	100	100	97
GSH			82	82	73	100	96	97
SH-SS				59	56	99	95	87
SH-MS					53	97	96	85
GSH-LS						88	100	85
SH							69	66
SH-LS								51
BASIC								

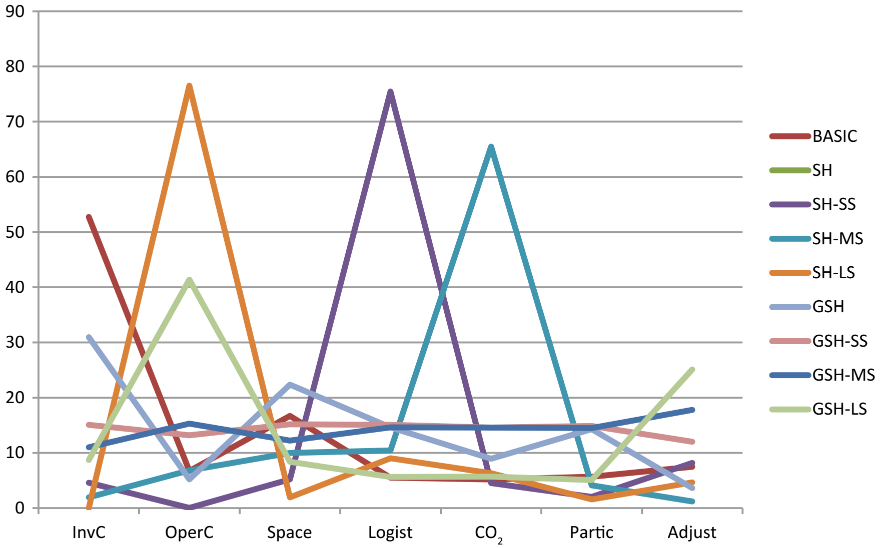

Fig. 8 Central weight vectors for efficient alternatives

that the GSH-MS and GSH-SS alternatives are favored by almost uniform weights for all criteria. In contrast, the BASIC alternative is favored by placing much weight (53%) on InvC criterion. Alternative SH-LS is favored by placing 77% weight on OperC, SH-SS by 75% weight on Logist, and SH-MS by 65% weight on CO_2.

6 Discussion

Based on the analysis, GSH seems to be superior to SH in Finland. A common misconception is that due to the long days in the summer season, there is almost as much sunshine in Finland than in northern parts of the European continent, and this would make the conditions for solar energy nearly as good. The conditions are significantly worse in Finland due to the non-coincidence between supply and demand. There is very little demand for heat and power in the summer when the availability of sun energy is good. In contrast, the demand for heat and power is much higher during the cold winter, when solar energy is almost non-existent.

Based on the criteria measurements (Table 5) it is interesting to observe, that while the InvC criterion (the objective function in the production planning model) improves monotonically by storage size (as it should), CO_2 and Partic criteria do not. Instead, the reductions in CO_2 and particulates are maximized both with SH and GSH with the short-term storage (SS). The reason for this is that there are losses involved with storage. For short term storage these losses are small, but for medium term and seasonal storage they become significant. While seasonal storage does improve OperC most, the involved losses mean that a smaller amount of fossil fuels are replaced by SH or GSH, and this disproves the CO_2 and Partic criteria for larger storages. Better insulation for the long-term storage could alleviate this problem and make the long-term storage more competitive.

Some simplifications were made in the uncertainty modelling. Independent normal distributions were applied for the cardinal criteria measurements. However, some of the uncertainties should be dependent. For example, the investment costs for alternatives is the sum of investment for the production technology (SH, GSH) and the storage (SS, MS, LS), and this makes the uncertainties of the investment costs of the combined alternatives dependent. Also, the criteria measured based on the simulation model would have similar dependencies. Considering such dependencies would make the results more accurate and discern the alternatives better. See, e.g., Lahdelma et al. (2006, 2009). However, estimating dependencies from the simulation model would require hundreds of simulation runs for each alternative.

7 Conclusion

Inclusion of renewable solar heat and ground source heat with heat storages into a fossil-based district heating (DH) system in Finland was studied. A combination of optimization and multicriteria analysis was applied. The alternatives were compared in terms of multiple economic, technical and environmental criteria using Stochastic Multicriteria Acceptability Analysis (SMAA).

The results show that under Finnish conditions, ground source heat is more favourable than solar heat for DH. The short-term storage seems more favourable than medium or long-term storage, both with SH and GSH.

The applied methodology can be easily applied to similar problems where renewable energy integrated with fossil-based energy systems.

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Comparison of Routing Methods in Telecommunication Networks—An Overview and a New Proposal Using a Multi-criteria Approach Dealing with Imprecise Information

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Abstract The performance evaluation and comparison of routing models in telecommunication networks, normally imply the necessity of evaluating them through multidimensional, potentially conflicting, often incommensurate criteria, frequently involving imprecise information regarding the relative importance of the various network performance criteria. As we will show, this is particularly relevant for flow-oriented, decentralized routing optimization methods, having in mind their inherent limitations. Therefore, we formulate a decision problem focused on the comparison and selection of flow-oriented routing models, evaluated through multiple global network performance measures. A proposal of a multi-criteria/multi-attribute approach for tackling this decision problem, based on the VIP (Variable Interdependent Parameter) software, will be described. The adequacy of the features of the multi-attribute decision analysis model, which uses additive aggregation of criteria with variable interdependent importance parameters, coping with imprecise information, will be discussed. A detailed formulation of the application of the proposed approach to a specific problem involving the choice of a point-to-point routing method in a modern transport telecom network, from a set of height routing models, by considering their performance evaluated in terms of nine global network performance measures, will be presented. Moreover, the extension of the decision analysis model, based on the VIP decision support tool, for dealing with this problem, in the case of face-to-face cooperative group decision, will be addressed. A case study concerning the application of this approach to the aforementioned decision problem, in a setting involving three decision makers, including a facilitator, will be presented. Finally, some conclusions, both from a methodological and practical nature, founded on the application study, will be put forward, highlighting the interest of this type of approach in this important area of telecom-network design.

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1 Introduction and Motivation

1.1 Introduction and Background Concepts

In the general context of network design activities, routing is a fundamental issue that may be envisaged as part of the network operational planning decision process, strongly related to other network design steps, namely network structure design (which includes topological design and capacity facility dimensioning) and traffic network management. Routing methods may have different natures and a great multiplicity of formulations, depending fundamentally on the following factors: mode of information transfer and possibly other key technological features, the type of service(s) associated with the routed traffic flows, the level(s) of representation of the network, the features of the routing principle, associated with the objective(s) (to be optimized), the constraints to be satisfied, the time dependence of the execution of the routing algorithm and the information for the routing calculation. Routing methods are, in practice, technically implemented, in a given network environment, involving multiple technical factors, trough 'routing protocols'. Concerning the levels of network representation, for example, at least two levels are considered when resilience objectives are considered in the network design: the physical or transmission network and the logical or functional network. The physical network includes the transmission systems (such as optical fiber cables or microwave links and associated transmitter/receiver equipment), the switching and/or routing devices and their physical interconnections. The logical network is an upper level simplified representation of the network, of mathematical nature, over which the routing functionalities can be specified through logical rules and includes, as basic elements, a capacitated graph and a matrix of node to node offered demand, where the nodes typically represent switches or routers and the arcs transmission capabilities between the end nodes. As for the concept of 'routing principle' we mean the fundamental features of the routing method, for example whether it is concerned with node-to-node or with node-to-multiple nodes connections, or whether it is static or dynamic, i.e. if routes are time varying according to traffic fluctuations or network conditions, in a given time scale.

The extremely fast pace of the evolution in basic network technologies and architectures has led, in recent years, to a sharp increase in the proposals of routing models for different types of networks. Note that different routing methods often lead to different routing solutions for each node-to-node flow or VC (Virtual Connection, i.e. a specific node-to-node logical connection, with a given bandwidth requirement) in a given network environment, for given traffic offered to the network, solutions which are specified by a sequence of network resources, topologically defined by loopless paths in the network representation, in the case of point-to-point communications. Similar comments apply to point-multipoint or multipoint-multipoint communications, in which case Steiner trees or spanning trees have to be calculated in the network representation, according to the nature of the communication service. Reference monographs on routing models for telecommunication networks, including key mathematical formulations and algorithms can be seen in (Pióro and Medhi 2004; Medhi and Ramasamy 2018).

Most routing models, proposed in recent years, may be included in the category of 'QoS (Quality of Service) routing models', a type of routing that involves the selection of a chain of network resources along a feasible path explicitly satisfying certain requirements (dependent on traffic features associated with service types) and seeking to optimize some relevant metric such as delay, cost, number of edges of a path or loss probability. Therefore, in this context, routing algorithms need to consider as objective function or constraint(s), distinct path metrics. Comprehensive reviews on classical constrained-based QoS routing models and algorithms are provided in (Kuipers et al. 2002a, b) In various modern network design situations there are advantages in formulating routing problems, as explicit multi-criteria optimization problems, where two or more metrics or cost functions are considered as objective functions to be optimized. Note that this type of formulations enable the trade-offs among distinct path metrics and other network cost function(s), to be pursued in a mathematically consistent manner (Clímaco et al. 2007). An overview on multi-criteria routing models in telecommunication networks including the discussion of a case study can be seen in (Clímaco et al. 2007). A recent state of art review on applications of multi-criteria analysis in telecommunication network planning and design problems, including a section on multi-criteria routing models is in (Clímaco et al. 2016).

For scalability considerations, most proposals of routing methods, from classical single-criterion optimisation models and QoS routing methods (based on multiple variants of constrained shortest path exact algorithms or heuristics) to explicitly multi-criteria routing optimization models, are decentralised *flow-oriented routing* models. Alongside other authors (see, for example Mitra et al. 1999) we make the important distinction (see the analysis in Craveirinha et al. 2008), between flow-oriented routing optimization models, for which routing calculation is performed for each node-to-node flow separately, in terms of a path optimization problem (with one or multiple objective functions and constraints), and network-wide optimization routing models. That is, *flow oriented optimization* routing approaches are characterized by objective function(s) which are defined on a per connection demand basis, i.e. for each offered end to end traffic flow—this means that the routing optimization model is solved separately every time a new end to end traffic flow is considered in the network. In contrast, *network-wide optimization* routing approaches consider the objective function(s) defined at network level, i.e. specified as global network performance measures that depend, explicitly and simultaneously, on all traffic flows present in the network, this implying that the optimization model is solved considering explicitly the combined effect of all network flows in all links/edges of the network—for example, through global network flow programming formulations.

As noted in (Craveirinha et al. 2008), in flow oriented optimization routing approaches the objective functions, although closely related to the global network performance measures which we are "really" seeking to optimize, are unable to encompass consistently all the interactions among all network flows, the reason why the specified objective functions for any end-to-end flow can be designated as "surrogate" objective functions with respect to the ones in the associated network-wide optimization routing models. For example, the optimization of a given path metric (such as mean delay, blocking probability or load cost) for an end-to-end flow, seeks to optimize the corresponding global network metric. This is an *inherent limitation* of flow oriented optimization routing models in comparison with network-wide optimization routing models, which often leads to routing solutions that are poor in terms of the global network performance metrics.

For these reasons it is necessary to evaluate the performance of flow oriented routing models through relevant global network performance measures, to be specified by the network designer in a given network environment. Furthermore, QoS and multi-criteria routing models should be compared, among themselves and with more conventional single-criterion routing optimisation models, by considering various relevant network performance measures. In the vast literature on routing methods, their performance, including performance comparisons among routing methods in a given network context, has typically been carried out through experimental studies involving simulation of multiple network scenarios followed by an empirical comparison of the network performance results obtained with the compared routing solutions.

A few examples of typical performance comparisons studies on flow-oriented routing methods, in various types of network environments, may now be referred to, in order to illustrate the analysis above. In (Meghanathan et al. 2009) a performance comparison study of three different types of routing protocols for mobile wireless ad hoc networks (or "MANETS"), is presented. A simulation test-bed is used and these routing methods are compared and ranked in terms of average packet delivery ratio, number of route transitions, average hop count and end-to-end packet delay, considering these criteria separately, by using multiple graphics for these metrics, in different conditions. The paper (Iyer et al. 2013) presents a performance comparison of three routing protocols for wireless "smart utility networks", a specific type of wireless ad hoc networks. The heuristic routing methods implemented by such protocols are compared, using a discrete event stochastic simulation, in terms of transmission resource usage (measured by the hop-count), average packet delay and average packet delivery ratio, considering three types of services and also taking into account route reliability. The performance criteria are directly compared through graphical representations. The work (Sllame et al. 2015) presents a performance comparison of a Voice over Internet Protocol (VoIP) communication service working over wireless ad hoc networks, considering multiple alternatives defined in terms of three different routing protocols and three different queuing techniques. The performance comparison uses four technical criteria, concerning jitter, packet delays and packet delivery ratio. Multiple graphics, obtained by real-time simulation, are used for empirical comparison of the alternatives, where, for each routing protocol, the criteria values are displayed, considering the various packet queuing techniques that may be used at the nodes.

The results of these, and of similar studies involving the comparison of routing methods, are typically expressed through statements of the type: "*The method A has better performance (or a certain % improvement in performance) than method B with respect to network metric X, in load condition L, …*". In 'robust' and more technically sound studies, multiple network performance metrics/costs should be considered for comparison purposes, also having in mind that most flow-oriented routing models—and also, possibly, network-wide optimisation routing models unable of encompassing explicitly some relevant network performance metrics—tend to perform asymmetrically with respect to different network metrics. This makes that the comparison and ultimately the choice—by a network design expert, working for a given network operator—of "a routing method with better overall performance" in a given network context, may easily become a difficult task, involving what is, in fact, a complex decision problem.

1.2 Motivation and Contents

As explained above, the inherent limitations of flow-oriented optimization routing models, make that the global effect of the interactions between traffic flows is not fully represented in their objective functions, justifying why we focused our study on the application of multi-attribute decision analysis to the comparison and selection of routing models of this type. In fact, these routing models may treat in quite an unbalanced manner, various relevant network performance metrics and may lead to poor performance in some of those metrics. The interest and necessity, in our view, of developing a decision support model, with sound methodological foundations, for tackling the mentioned type of evaluation and decision problem, of great practical interest in this specific area of network design, laid a major motivation for this work.

The decision problem at stake, is, from an Operational Research point of view, a problem involving the ranking of decision alternatives according to multiple criteria/attributes, where the alternatives are in a small number, are known explicitly, and the attributes correspond to the network global performance parameters chosen for evaluation purposes, often conflicting and incommensurate.

A major contribution of our study is to show the usefulness and potential, both from a methodological and practical point of view, of using a certain multi-attribute (MA) analysis model dealing with incomplete information. A reference monograph on key concepts concerning multi-attribute models can be seen in (Keeney and Raiffa 1976).

The used MA model will consider an additive value function (see Keeney and Raiffa 1976) which is constructed through a weighted sum of the attribute values. Remember that the attribute values have to be previously normalized. Furthermore, at each evaluation exercise, the fixing of the values of the scaling constants/weights, corresponding to *criteria importance parameters* assigned to the various attributes, has to be tackled. Note that this is a most difficult task, since the relative importance, particularly in quantitative terms, among the different attributes, is normally imprecise. For example, the question "how more important is mean total carried traffic than total residual bandwidth?" may have a variety of answers in terms of network routing design. That is why the imprecise nature of the scaling constants is a fundamental feature of the addressed decision problem, that has an important role in the specification of the MA analysis model. Moreover, note that various attributes may be conflicting and incommensurate, as the two exemplified above (mean total carried traffic and total residual bandwidth).

Furthermore, the interest in considering a group decision process in this application environment has to do with several aspects. Firstly, although some general assumptions on relative importance of the network performance metrics may be common to most network designers, the specification of relations between the criteria IPs (importance parameters) may vary significantly from one expert to another, even when those differences of perspective, in terms of systems of preferences, are not assumed explicitly, but just tacitly. This is also reflected in the literature in this area where some authors give more relevance to certain performance measure than others, when analyzing and comparing the performance of routing methods in a given network environment (see examples, in the overview of routing models in Craveirinha et al. 2008; Clímaco et al. 2016). The need for the elicitation of such differences and the analysis of their consequences in the network design process, is one of the advantages of considering a setting of group decision in the context of this particular multi-criteria decision problem. Secondly, this may be a realistic decision scenario in the context of a major network operator, since more than one engineer/network designer is often involved in a decision which has decisive impact in terms of network performance, cost and expected revenues. This can be a typical situation when new, more advanced, routing methods are to be implemented in a transport network, for example provided by the development by the operator or by a company of a related group, of more sophisticated routers, in relation to the deployment of modern telecommunication protocol technologies, for example the MPLS-TP (Multiprotocol Label Switching-Transport Profile) stack (Niven-Jenkins et al. 2009). The authors themselves were involved in research contracts of collaboration of their research institute (INESC-Coimbra) with Portugal Telecom-Innovation, where this type of decision environments and issues were at stake.

In this study, we consider, as alternatives of the formulated decision problem, several variants of flow-oriented routing models, namely, in the presented case study, different variants of a bi-criteria flow oriented routing model, as well the two associated single-criterion routing models. Hereafter, we designate as bi-criteria routing model, a routing model characterized by two specific objective functions. We consider, in our case study, two different forms of normalizing the objective functions and two forms of aggregation of preferences, leading to multiple *variants* of the routing model. Each of these variants corresponds to a *routing method*, in the sense defined in (Clímaco and Craveirinha 2005). In the considered routing methods, for technical requirements, the aggregation of preferences is performed automatically, as explained later on. The attributes of the decision problem are various network performance metrics that enable the evaluation of the global effect, at network level, of using the various routing methods, when incremental traffic is offered to a given transport network. These performance metric values were obtained in the context of a previous study (Martins et al. 2013) on network performance improvement through evaluation of bi-criteria/single-criterion flow oriented routing methods in transport networks, focused on applications in Carrier-Ethernet (MEF 10.3 2013) and MPLS-TP networks. This study was focused on bandwidth allocation and traffic performance, having in mind its paramount importance in packet transport networks, in relation with the design of adequate routing methods, capable of obtaining improved network performance and prevent degradation of the QoE (Quality of Experience) as perceived by the users, specially in overload conditions.

As analyzed above, the imprecise information feature of the proposed MA model stems from the fact that the scaling constants, associated with the considered attributes are not fixed a priori, although various constraints between them can be set a priori as agreed among possible decision makers, for a given network operator. Of course, this is more realistic and flexible than requiring an a priori fixation of the scaling constants. Furthermore, although some of those inequality relations are consensual, for technical reasons, different decision agents may assign different relative importance to the scaling constants associated with some attributes, or even consider different inequalities among them. Congruently, we will consider a multi-attribute analysis tool (Dias and Clímaco 2000), the *VIP Analysis package*, which will enable the achievement of a compatibility of the incomplete information supplied by different scenarios of relative importance among the attributes (network performance metrics). The aim is that, as a final result of an interactive analysis process, some robust conclusions may be achieved, hence helping a well founded evaluation and choice of a routing method alternative, to be implemented in a particular network environment.

The main contributions of our work are the following:

- specification of a multi-attribute (MA) analysis model, based on the *VIP* analysis software, for comparing and selecting routing methods, in terms of multiple network performance measures, enabling the consideration by the network designer (decision maker) of multiple scenarios concerning different forms of valuation of the relative importance of the network performance measures, i.e. in a context of imprecise information on the importance parameters assigned by the DM to the network performance measures; this includes the extension of the decision analysis model, based on the VIP *software*, for face-to-face cooperative group decision;
- development of a case study of application of the (MA) analysis model to the comparison and selection of flow-oriented routing methods, namely methods based on bi-criterion shortest paths and single-criterion shortest paths algorithms, using as path metrics load costs and hop count, in the context of transport networks with incremental traffic; this application study, highlighting the capabilities of the used MA model, involves experiments with one DM and with three DMs, considering a realistic cooperative group decision setting.
- outline of relevant conclusions, of methodological and practical nature, founded on the MA analysis case study, concerning the relative performance of those types

of flow-oriented routing methods, in various decision scenarios and putting in evidence the interest of this type of approach in this important area of telecommunication network design.

The contents of the paper are as follows. The next section describes the decision problem concerning the selection of flow-oriented routing methods, in a given transport network, and specifies the features of the multi-attribute analysis model, based on the VIP analysis package, used for tackling this problem. Section 3 presents the case study of application of the (MA) analysis model to the comparison and selection of flow-oriented routing methods, in the context of transport networks with incremental traffic, taking as inputs (attribute values) nine network performance metrics, the values of which were obtained from discrete event simulations. Multiple scenarios for the imprecise information on the importance parameters, as possibly assigned by the decision maker(s), were considered. Also some relevant conclusions, founded on the MA analysis case study, will be outlined in the second part of this section. Finally, Sect. 4 presents the conclusions of this study and outlines further work on this research theme.

2 Outline of the Decision Problem and of the Multi-attribute Analysis Model

2.1 The Decision Problem

The alternatives

The first six *alternatives* a_i (i = 1, …, 6) or solutions of the proposed *decision problem*, are variants of a bi-criteria flow-oriented routing model, in a transport telecommunication network considering incremental traffic, all using as path metrics, to be optimized, the load cost and the number of arcs (or "hop count"), the features of which are briefly reviewed next and in Appendix A. The other two alternatives are the two single criterion routing models that use, as path metric to be minimized, either the load cost or the hop count.

In order to briefly review these routing alternatives, let us consider a directed capacitated network (N, L, C) where N is the node set, L the arc set and C the set of the capacities (total bandwidths) C_k of the arcs l_k ($k = 1, \ldots /L/$). Let f_s denote a node to node traffic flow from node v_i to node v_j of service type 's'. A flow f_s is associated with a virtual connection request (VC) requiring a certain bandwidth, d_s, in the used arcs, and may use a feasible loopless path or route r_s—i.e. a loopless path from v_i to v_j such that every arc in r_s has, at the moment of the arrival of the VC request, an available bandwidth $b_k \geq d_s$. The current set of feasible routes for flow f_s is designated as $D(f_s)$. Let m_k^i designate an additive metric, corresponding to the *path metric* $m^i(r_s)$ for any route r_s, (i = 1, 2), associated with every arc l_k.

Then, the general basic bi-criteria flow oriented routing optimization model is formulated as:

$$\boldsymbol{min}_{r_s}\, \boldsymbol{m}^{\boldsymbol{i}}(\boldsymbol{r_s}) = \sum_{\boldsymbol{l_k} \in \boldsymbol{r_s}} \left(\boldsymbol{m}_{\boldsymbol{k}}^{\boldsymbol{i}}\right) \quad (i = 1, 2) \wedge r_s \in D(f_s) \tag{1}$$

In the considered model, the first path metric, m^1, is the load cost that is expressed through a piecewise linear function in terms of the relative bandwidth occupation in the arcs, based on the model in (Fortz and Thorup 2002) thenceforth depending, for arc l_k, on b_k and C_k (see details on this piecewise linear function in (Martins et al. 2013). The minimization of $m^1(r_s)$ seeks to obtain a balanced traffic load distribution in the network arcs (or 'links'), preventing the use of excessively loaded links, while less loaded links are available, hence favoring traffic carrying capability for future VCs. The second metric is the number of arcs of the path (or hop count), i.e. $m_k^2 = 1$, and its minimization seeks to minimize the number of network resources used by any VC. The bi-criteria flow oriented routing optimization model seeks to obtain a compromise solution in terms of these two, often conflicting objective functions. The final compromise solution, to be settled for each VC, has to be chosen in the set of non-dominated (or Pareto optimal) solutions. A solution r' is said to dominate the solution r iff $m^1\left(r'\right) \leq m^1(r)$ and $m^2\left(r'\right) \leq m^2(r)$ and at least one of inequalities is strict. A path r* is said to be non-dominated (or Pareto optimal) iff there is no other feasible path which dominates r*. The non-dominated paths were obtained by calculating k-shortest paths and by using a non-dominance test on the calculated solutions, as proposed in (Clímaco and Martins 1982). Note that this procedure enables the exact calculation of all non-dominated solutions of the bi-criteria shortest path problem. The variants of this basic bi-criteria routing model are related to different ways of automated selection of a "good" trade-off solution to the bi-criteria routing model (1), in the non-dominated solution set, for each offered flow f_s, and to different forms of normalizing the two path cost functions. The parameters involved in the selection of a route depend not only on the current flow being offered to the network but also on the global network states, so they vary dynamically as more flows are offered to the network, in a scenario of stochastic incremental traffic. Note that the final routing solution selected for each VC offered to the network, is calculated in an automated manner by the routing management system. As for the way in which this selection is carried out, two types of methods were considered. The first type uses priority regions in the objective function space, defined from preference thresholds for the two objective functions, namely required and acceptable values m_{req}^i, m_{ac}^i $(i = 1, 2)$. This leads to four priority regions, as specified in (Martins et al. 2013), so that the first non-dominated solution found in the highest priority region, is the one selected. The second type of method chooses the non-dominated solution which minimizes, either a Chebychev or a Euclidian distance to the optimal ideal point (op^1, op^2), in the objective function space. The second factor that influences the selected solution is the choice of the normalizing coefficients of the objective functions, as explained

Table 1 alternatives of the decision problem-routing methods

Alternative designation	Routing optimization approach	Solution selection
a_1	Bi-criteria	Normalization by WA and priority regions
a_2	Bi-criteria	Normalization by WB and priority regions
a_3	Bi-criteria	Normalization by WA and Euclidian distance
a_4	Bi-criteria	Normalization by WA and Chebychev distance
a_5	Bi-criteria	Normalization by WB and Euclidian distance
a_6	Bi-criteria	Normalization by WB and Chebychev distance
a_7	Single-criterion: load cost	
a_8	Single-criterion: hop count	

in the Appendix A. Further details on the traffic modelling and calculation aspects, can be seen in (Martins et al. 2013).

The other two alternatives, a_7, a_8, of the decision problem are, naturally, the single criterion routing optimization methods that use, as path metric to be minimized, either $m^1(r_s)$, the load cost, or $m^2(r_s)$, the hop count. The alternatives (routing methods) of our decision problem, are summarized in Table 1.

The attributes

Concerning the *attributes* of the decision model, these are global network performance metrics involving three fundamental types of metrics: mean total carried bandwidth (TCB), mean total residual bandwidth (TRB) and mean number of accepted node-to-node VCs (TAC). Each of these fundamental metrics is decomposed into three attributes corresponding to the associated performance values obtained while the blocking probability of a connection request remains in zero ($Br1 = 0\%$) or attains the thresholds of $Br2 = 5\%$ or $Br2 = 10\%$, leading to a total of nine attributes.

The values for these performance measures in the network case study were estimated through stochastic discrete event simulations, considering incremental traffic and using the method of batch means for sample mean and confidence interval estimation.

2.2 The Multi-attribute VIP Analysis

The multi-attribute analysis model for tackling the considered decision problem is based on a decision support tool, the VIP Analysis software (Dias and Clímaco 2000, 2005) having in mind the adequacy of its features to the to the nature of our decision problem. This adequacy results, in first place, from the fact that the DM (the network designer) is not able or does not wish to establish a priori precise values for the importance parameters, or scaling constants, associated with the attributes—this is the imprecise information feature of the decision model. Note that the DM not only may find very difficult (or questionable under technical-economic reasons) to fix precise values for the importance parameters, but also may wish to consider and test various scenarios for the relative values of those parameters. This will become very clear, in the explanation of the MA package features, as well as in the case study in Sect. 3, specially taking in account the way in which the scaling constants can be treated, namely as variable interdependent parameters, enabling, at the same time, to draw well founded conclusions concerning the ranking of alternatives.

Hence, to help the DM in a process of finding a most preferred alternative, the MA model considers an *additive value function* under imprecise information:

$$V(\mathrm{a}_i, k) = \sum_{j=1}^{n} k_j v_j(\mathrm{a}_i). \tag{2}$$

where a_i and v_j represent the ith alternative and the jth normalized global network performance measure, of one of the types described above, k_j is the importance parameter (IP) or scaling constant of v_j and k represents the vector of scaling constants, $k = (k_1, k_2, \ldots, k_n)$.

Let us briefly review the most relevant concepts/definitions used in the multi-attribute analysis package (see Dias and Clímaco 2000 for further details).

Let T denote the set of acceptable values of the vector k of scaling constants.

The *regret*(a_i, a_j) associated with alternative a_j, when compared with a_i, and here denoted as reg_{ij}, is the maximal difference:

$$reg_{ij} = \max_{k \in T}\left\{V(\mathrm{a}_i, k) - V(\mathrm{a}_j, k)\right\}. \tag{3}$$

If reg_{ij} is negative then $V\left(\mathrm{a}_j, k\right) \geq V(\mathrm{a}_i, k)\ \forall k \in T \wedge \exists k \in T{:}V\left(\mathrm{a}_j, k\right) > V(\mathrm{a}_i, k)$ and a_j is said to dominate a_i, (i.e. a_i is *dominated* by a_j). An even more demanding situation of dominance, corresponds to the case:

$$V(\mathrm{a}_j, k) \geq V(\mathrm{a}_i, k')\ \forall \mathrm{k}, \mathrm{k}' \in \mathrm{T} \wedge \exists \mathrm{k}, \mathrm{k}' \in \mathrm{T}{:}V(\mathrm{a}_j, k) > V(\mathrm{a}_i, k') \tag{4}$$

In this case we say that a_i *is absolutely dominated by* a_j.

A relaxation to the dominance relation, by a tolerance parameter ε, means that:

$$V\left(\mathrm{a}_j, k\right) \geq V(\mathrm{a}_i, k) - \varepsilon\ \forall k \in T \wedge \exists k \in T{:}V\left(\mathrm{a}_j, k\right) > V(\mathrm{a}_i, k) - \varepsilon \tag{5}$$

In this case we say that a_j *quasi dominates* a_i with tolerance ε.

For every alternative a_i the *maximal regret* associated with it, when it is compared with all other alternatives which may have a higher additive value for given T, is:

$$reg_{\max}(a_i) = \max_{j \neq i}\{reg_{ji}\} = \max_{k \in T}\left\{\max_{j \neq i}\{V(a_j, k)\} - V(a_i, k)\right\}. \quad (6)$$

If $reg_{max}(a_i)$ is negative or null then a_i is *optimal*; if $reg_{max}(a_i) - \varepsilon$ is negative or null then a_i is *quasi-optimal.* If this is true only for a subset K* of T then we can say that a_i is *k-optimal* (or quasi optimal) at K*.

VIP Analysis (Dias and Clímaco 2000) is an interactive software package dedicated to the choice problematic regarding the evaluation of a discrete set of alternatives according to a multi-attribute additive value function. The principal characteristic of this tool is that no precise values, for the scaling constants, are required. Instead, it can accept imprecise information (typically, intervals and/or linear constraints) on these values, usually identified by indirect ways, as for example by comparing swings. The major objectives are the discovering of robust conclusions—that may be shown mathematically to hold for every feasible combination of the scaling constants—and, secondly, identifying what is the variability of the results due to the imprecision in the parameter values. Furthermore, by considering multiple specifications of the set T, the DM may address the choice of a best compromise alternative, in different scenarios, that he/she finds more relevant on technical-economic grounds or more consistent with his/her experience as network designer.

In a first phase of the MA procedure some tools enable filtering the alternatives. In particular, the VIP module calculates the *range of values* of each alternative:

$$\left[\min_{k \in T}\{V(a_i, k)\},\ \max_{k \in T}\{V(a_i, k)\}\right]. \quad (7)$$

This enables, for example, the elimination of the alternatives with a minimum value below a certain threshold (fixed by the decision makers) regarding: the maximum regret concerning each alternative (enabling the elimination of those with a max regret beyond a threshold fixed by the decision makers); possibly the elimination of absolutely dominated or dominated solutions. It must be remarked that the identification of the alternative(s) with "min-max regret" constitutes an indicator regarding equity and the identification of the alternative(s) with "max min value" is an indicator regarding "best case". The alternatives passing the filtering phase are analysed using the *matrix of regrets,* $[reg_{ij}]$ (i, j = 1, …, m), defined from (3), designated as *pairwise confrontation table*. In conjunction with the relaxation of the concepts of optimality and dominance, this enables the exploration of the concepts of quasi-optimality and quasi-dominance, enabling the decision maker to identify robust conclusions, in order to help in the search for the best alternative. Also note that, in those cases for which the DM is able of fixing some trade-offs between pairs of criteria, it is possible to reduce the number of independent variable scaling constants. When just three independent variables remain, the system provides a graphical representation of the space of the scaling constants, allowing a user-friendly representation of the optimality and quasi-optimality domains.

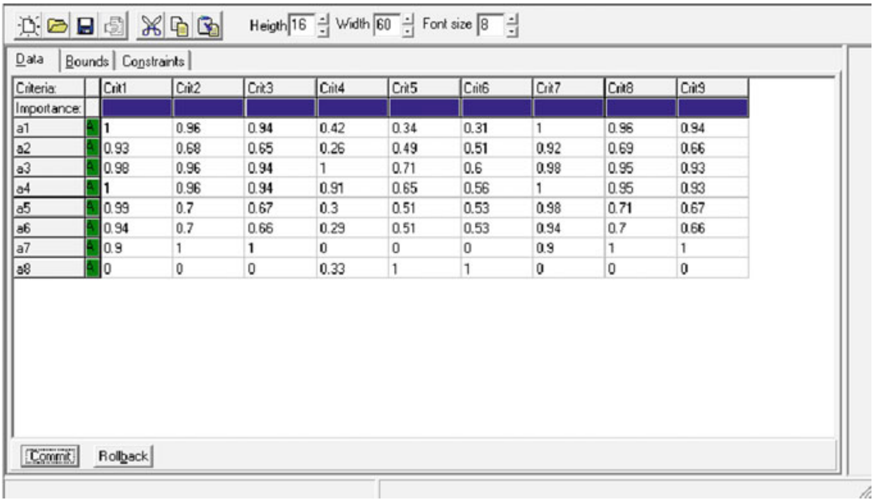
In the addressed network design decision problem, a final routing method alternative must be chosen. In theory two situations may occur at the end of the analysis process: the most common, where one alternative becomes the one clearly preferred by the DM, in all admitted scenarios for T, as a result of its inherent merits in a given network environment, or a situation in which the DM finds that at least two alternatives are worthy being considered, since neither of them is clearly preferred to, in the admitted scenarios. In this case the DM may have two courses of action: either to consider a 'narrowing' of the conditions which specify the set T, leading, through a complementary process of analysis with the VIP package, trying to clarify the situation, or to present those best alternatives to a 'higher hierarchy' DM of the network operator, and confront him/her with the *pros* and *cons* of those alternatives, so that he/she makes an ultimate selection in this final short list of alternatives.

As for the upgrading of these concepts to Group VIP Analysis (see Dias and Clímaco 2005), for instance, concerning the concept of "quasi-domination of alternatives", the DMs tolerance ε may vary jointly with a "α-majority operator". In the case study dealt with in this paper, we consider three decision makers, so $\alpha \in \{1/3, 2/3, 1\}$. In fact, the DM's may be interested in analyzing which alternatives are quasi-dominated, in terms of an "α-majority rule". Note that the obtainment of a quasi-dominance condition with a high value of α may imply a high value for ε—an example is shown in the case study described in the next section. Of course this type of approach can also be used for other types of decision issues.

3 Case Study

3.1 Main Features of the MA Application Model

A reference network based on the France telecommunication transport network, described in (Martins et al. 2013) and with topology given in (Orlowski et al. "SNDLIB" 2010), considering that all links (arcs of the network topology) have 10 Gb/s capacity and three connection service types, was considered. The three service types were assumed to have effective bandwidths of 20, 50 and 100 Mbit/s. In the stochastic simulation study, point to point VC requests were randomly generated over all origin-destination pairs and service types, considering uniform distributions. The accepted VCs are assumed to be maintained 'indefinitely' in the network (that is the established routes correspond to physical paths which are held during the routing study time scale), which corresponds to the usual assumption of offered traffic of incremental type, typical of routing studies in transport networks. The study in (Martins et al. 2013), was carried out in collaboration with experts of Portugal Telecom Inovação and was focused on the evaluation of the bandwidth allocation and traffic performance when various point to point routing methods, as the ones described in



Criteria:	Crit1	Crit2	Crit3	Crit4	Crit5	Crit6	Crit7	Crit8	Crit9
Importance:									
a1	1	0.96	0.94	0.42	0.34	0.31	1	0.96	0.94
a2	0.93	0.68	0.65	0.26	0.49	0.51	0.92	0.69	0.66
a3	0.98	0.96	0.94	1	0.71	0.6	0.98	0.95	0.93
a4	1	0.96	0.94	0.91	0.65	0.56	1	0.95	0.93
a5	0.99	0.7	0.67	0.3	0.51	0.53	0.98	0.71	0.67
a6	0.94	0.7	0.66	0.29	0.51	0.53	0.94	0.7	0.66
a7	0.9	1	1	0	0	0	0.9	1	1
a8	0	0	0	0.33	1	1	0	0	0

Fig. 1 Normalized performance matrix

Table 1, were applied in the context of Carrier-Ethernet and MPLS-TP (Multiprotocol Label Switching-Transport Profile) transport networks. The estimated values for the network performance metrics, used as attributes in the present MA model, were obtained by considering 100 simulation replicas of incremental traffic and their average values were calculated, from raw data collected during the experiments in (Martins et al. 2013).

As mentioned above, a total of nine attributes were considered. The corresponding values were normalized by using difference ratios, i.e. the estimate $y_j(a_i)$ of the jth network performance metric, for a given alternative a_i, is normalized into the attribute value in [0, 1]:

$$v_j(a_i) = \frac{y_j(a_i) - y_{j\min}}{y_{j\max} - y_{j\min}};\ y_{j\min} = \min_{a_k}\{y_j(a_k)\},\ y_{j\max} = \max_{a_k}\{y_j(a_k)\} \quad (8)$$

The first three attributes (or criteria) v_j (j = 1, 2, 3), are the *mean total carried bandwidths* (TCB) while the blocking probability of a VC request remains in zero (Br1 = 0%) or attains the thresholds of Br2 = 5% or Br2 = 10%, respectively, and the corresponding scaling constants are denoted by $k_j = k_{(j)TCB}$ j = (1, 2, 3). The attributes v_j (j = 4, 5, 6) are the *mean total residual bandwidths* (TRB) defined in the same conditions as above, and the corresponding scaling constants are denoted by $k_{j+3} = k_{(j)TRB}$ j = (1, 2, 3). Finally the attributes v_j (j = 7, 8, 9) are the *mean numbers of accepted node-to-node* VCs (TAC), defined in the same conditions, and the corresponding scaling constants are denoted by $k_{j+6} = k_{(j)TAC}$ j = (1, 2, 3).

The obtained values $v_j(a_i)$ are shown in the form of the normalized performance table of the VIP package, in Fig. 1.

3.2 Experimental Study with a Single Decision Maker

We begin by presenting two sets of experiments involving two separate decision makers, DM1 and DM2, whose constraints on the IPs (importance parameters) reflect common assumptions of network designers regarding the relative importance of the performance metrics. This will enable the potentialities of the proposed MA analysis model to be illustrated, in our decision environment.

First Set of Experiments (DM1)

The first set of experiments, regarding the first decision maker, DM1, was carried out considering a total of 15 constraints on the scaling constants, which are either inequality relations (corresponding to 13 constraints) or equality relations (2 constraints). These constraints correspond to the general assumptions of most network designers, making it explicit the relative importance of some pairs of network metrics, when evaluating routing methods, namely considering that: (i) the total carried bandwidth (TCB) and the total number of accepted connections (TAC) measures are more relevant than total residual bandwidth (TRB) measures, for the same level of blocking probability; (ii) for a given type of network metric (namely associated with TCB or TAC) the measure for blocking probability Br1 (0%) is more important than the measure for Br2 (5%) and similarly for measures for Br2 and Br3 (10%), excepting in the case of the measures for TRB, a situation in which those preferences are the reverse. The equality relations concern the measures TCB and TAC, for Br1 and Br2. In this first scenario of relations between scaling constants it is further assumed that the performance in terms of TCB is at least as important than TAC for blocking probability Br3 (10%). This set of constraints (denoted as constraint set scenario S1) is expressed, according to the defined notation:

$$\begin{array}{ll} k_{(j)TCB} \succ k_{(j+1)TCB}\ (j=1,2) & (c1) \\ k_{(j)TRB} \prec k_{(j+1)TRB}\ (j=1,2) & (c2) \\ k_{(j)TAC} \succ k_{(j+1)TAC}\ (j=1,2) & (c3) \\ k_{(j)TCB} \succ k_{(j)TRB}\ (j=1,2,3) & (c4)\ \text{Constraint set S1} \\ k_{(j)TAC} \succ k_{(j)TRB}\ (j=1,2,3) & (c5) \\ k_{(j)TCB} \equiv k_{(j)TAC}\ (j=1,2) & (c6) \\ k_{(j)TCB} \succcurlyeq k_{(j)TAC}\ (j=3) & (c7) \end{array} \qquad (9)$$

Here, "$\succ$" denotes "more important than", "$\succcurlyeq$" means "at least as important as" and "$\equiv$" denotes "as important as". Note that the VIP software uses the simplex method, therefore strict inequality relations cannot be used, so that we have to use, to implement such relations, a small perturbation of "$\geq$" or "$\leq$" relations.

The summary of the main features of the alternatives, in this experiment, is shown in Fig. 2, where the corresponding VIP table is reproduced as actually seen in the computer screen. The left hand side of this picture is just a part of the table, used in VIP software, to introduce the constraints S1, the remainder of that table is not shown since it does not add any significant information. This first experiment enabled to identify one alternative absolutely dominated (a_8), i.e. its more favorable value

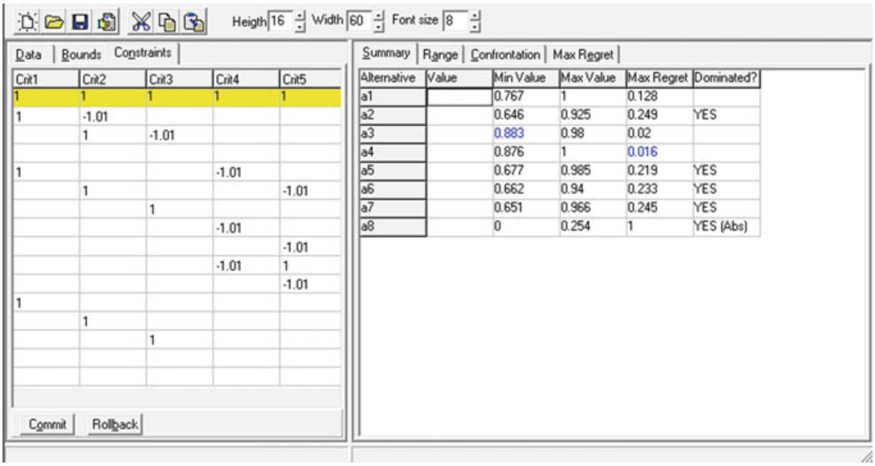


Fig. 2 Main features of alternatives, for S1

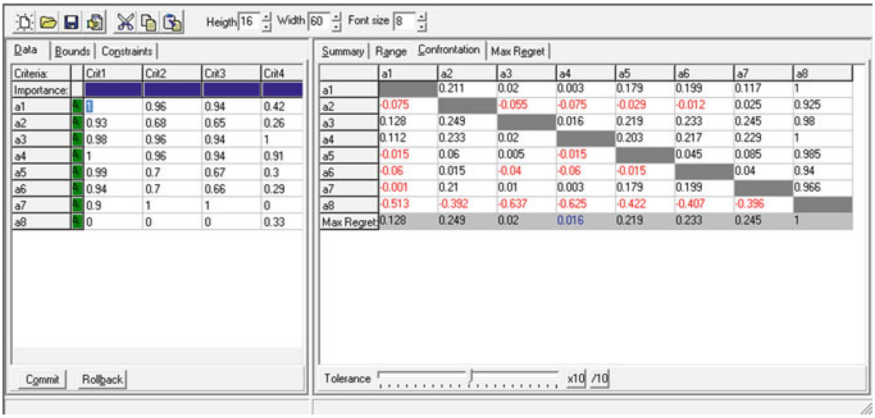


Fig. 3 The pairwise confrontation table, for S1

is lower than the worst value of other alternative(s), and with very poor relative values in almost all attributes, and four alternatives which were dominated by a_3 or by a_4. Moreover, a_3 was the solution with maximal minimal value and a_4 the solution with minimal maximal regret. The pairwise confrontation table is shown in Fig. 3 and graphics with the min-max ranges and the maximal regrets for the eight alternatives, are shown in Figs. 4 and 5, respectively, enabling a visual assessment of the relative performances of the alternatives concerning these important features of the MA model results.

In this set of experiments various runs of the VIP software were carried out varying the level of dominance relation relaxation, i.e. quasi-dominance, specified by the relative tolerance ε (different ε values were considered for testing quasi-dominance

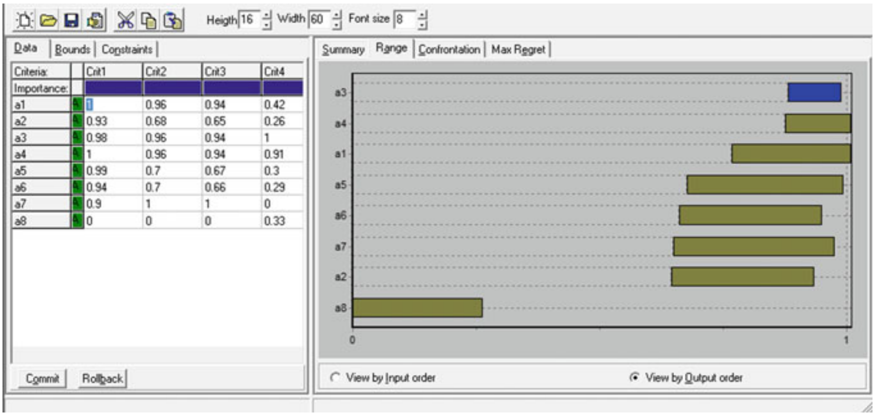


Fig. 4 The min-max ranges for S1

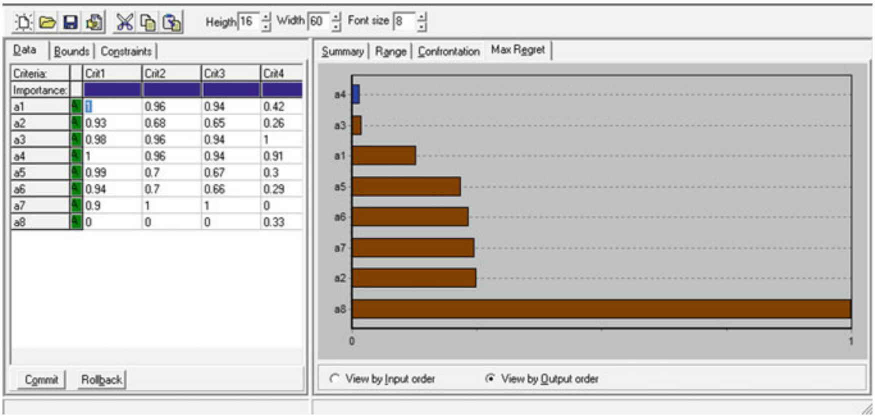
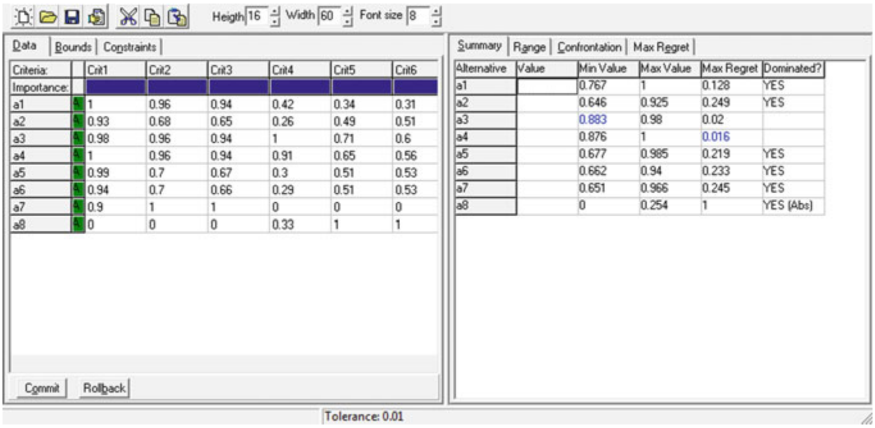


Fig. 5 The maximal regrets for S1

or "reinforced" dominance—negative ε). This was done by sliding in a horizontal screen bar, provided by the software.

The consideration of a small relaxation to dominance revealed, by examining the confrontation table, that all alternatives (different from a_3, a_4) were quasi-dominated by a_3 or a_4, for $\varepsilon \geq 0.01$, as illustrated in the summary of Fig. 6 and through the confrontation table in Fig. 7, obtained for $\varepsilon = 0.01$.

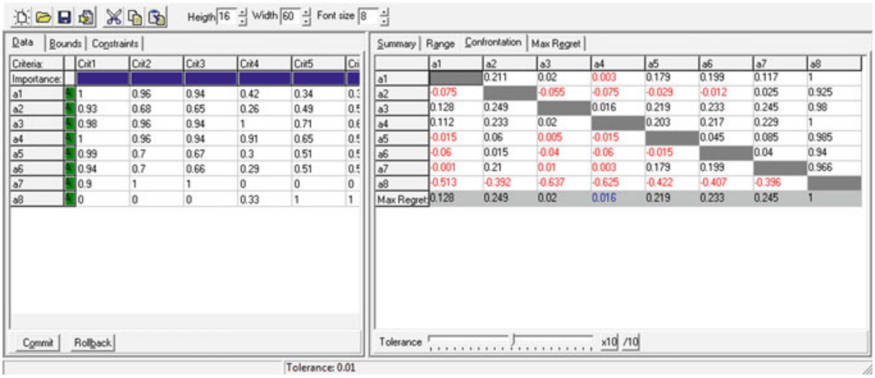
In a second experiment, a_8 was eliminated (a "filtering" procedure), taking into account that this alternative is clearly the worst in terms of all the major evaluation properties used by the VIP analysis, namely "max-regret", "max-min value" and dominance features (note, from Fig. 4, that it is absolutely dominated by all other alternatives). For the remaining alternatives one could conclude that for $\varepsilon > 0.01$ all alternatives, other than a_3 and a_4, were quasi-dominated by one of these two.



Criteria:	Crit1	Crit2	Crit3	Crit4	Crit5	Crit6
Importance:						
a1	1	0.96	0.94	0.42	0.34	0.31
a2	0.93	0.68	0.65	0.26	0.49	0.51
a3	0.98	0.96	0.94	1	0.71	0.6
a4	1	0.96	0.94	0.91	0.65	0.56
a5	0.99	0.7	0.67	0.3	0.51	0.53
a6	0.94	0.7	0.66	0.29	0.51	0.53
a7	0.9	1	1	0	0	0
a8	0	0	0	0.33	1	1

Alternative	Value	Min Value	Max Value	Max Regret	Dominated?
a1		0.767	1	0.128	YES
a2		0.646	0.925	0.249	YES
a3		0.883	0.98	0.02	
a4		0.876	1	0.016	
a5		0.677	0.985	0.219	YES
a6		0.662	0.94	0.233	YES
a7		0.651	0.966	0.245	YES
a8		0	0.254	1	YES (Abs)

Fig. 6 Main features of alternatives, for S1 and ε = 0.01



Criteria:	Crit1	Crit2	Crit3	Crit4	Crit5	Cri
Importance:						
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a2	0.93	0.68	0.65	0.26	0.49	0.5
a3	0.98	0.96	0.94	1	0.71	0.6
a4	1	0.96	0.94	0.91	0.65	0.5
a5	0.99	0.7	0.67	0.3	0.51	0.5
a6	0.94	0.7	0.66	0.29	0.51	0.5
a7	0.9	1	1	0	0	0
a8	0	0	0	0.33	1	1

	a1	a2	a3	a4	a5	a6	a7	a8
a1		0.211	0.02	0.003	0.179	0.199	0.117	1
a2	-0.075		-0.055	-0.075	-0.029	-0.012	0.025	0.925
a3	0.128	0.249		0.016	0.219	0.233	0.245	0.98
a4	0.112	0.233	0.02		0.203	0.217	0.229	1
a5	-0.015	0.06	0.005	-0.015		0.045	0.085	0.985
a6	-0.06	0.015	-0.04	-0.06	-0.015		0.04	0.94
a7	-0.001	0.21	0.01	0.003	0.179	0.199		0.966
a8	-0.513	-0.392	-0.637	-0.625	-0.422	-0.407	-0.396	
Max Regret	0.128	0.249	0.02	0.016	0.219	0.233	0.245	1

Fig. 7 Confrontation table, for S1 and ε = 0.01

Note that the elimination of one or more alternatives from the initial set of solutions, the so called "filtering" procedure, may alter the dominance characteristics of the other alternatives, but this comparison is useful for further exploration of dominance properties of the alternatives which are not eliminated.

In order to analyse, in separate, the relative performance of those two most promising alternatives (a_3 and a_4,) with respect to the remaining ones, two more "filtering" experiments were performed, by further elimination of either a_3 or a_4 (together with a_8). This enabled the conclusion that, in isolation, a_3 and a_4 are *quasi-optimal* with respect to a_1, a_2, a_5, a_6, a_7 for $\varepsilon > 0.02$ and $\varepsilon > 0.003$, respectively. Finally, a separate comparison of the third more promising solution (a_1), regarding the "min value" and the "max-regret", with a_3 and a_4 alone (by filtering all the other alternatives), showed that a_1 is quasi-dominated by a_4, for $\varepsilon > 0.003$ and also by a_3 for $\varepsilon \geq 0.05$. Moreover,

a_4 *quasi-dominates* a_3 for $\varepsilon = 0.02$, that is it becomes *quasi-optimal* in this particular experiment.

Second Set of Experiments (DM2)

This set of experiments concerns the second decision maker, DM2.

This DM admits that the set of three attributes considered as the more important, under the same basic inequality assumptions as DM1, for Br = 0%, correspond to importance parameters (IPs)—namely, k_1 (TCB, total carried bandwidth, for Br1 = 0%), k_4 (TRB, total residual bandwidth, for Br1 = 0%), k_7 (TAC, total accepted connections, for Br1 = 0%)—the sum of which has, in relative terms, a fixed value, Mip %, and are all variable. Furthermore, he/she assumes that specific values are assigned to the remaining IPs, k_2, k_3, k_5, k_6, k_8, k_9, while respecting the above constraints (and of course the normalization equation of the IPs).

So, this DM2, although respecting the same basic inequality relations in S1, concerning the attribute IPs for Br = 0%, admits that the IPs corresponding to less important attributes, (that is those associated with network conditions with standard blocking probabilities greater than 0%) are numerically specified by the DM *a anteriori*, also leading to a significant simplification of the constraints. This significantly reduces the number of variable IPs and the number of constraints on the IPs. In this case, since six of the IPs are fixed, *k-optimality regions* in the space (k_1, k_4, k_7), which can be calculated by the VIP software, may be obtained and explored, as referred to in Sect. 2.

Therefore, the constraints for the constant IPs are now:

$$k_2, k_3, k_5, k_6, k_8, k_9 \text{ have fixed values such that}$$
$$k_2 + k_3 + k_5 + k_6 + k_8 + k_9 = 1 - \mathrm{Mip}/100 \tag{10}$$

and, for the variable IPs:

$$k_1 + k_4 + k_7 = \mathrm{Mip}/100 \tag{11a}$$

$$k_1 > k_4 \tag{11b}$$

$$k_7 > k_4 \tag{11c}$$

In the set of experiments for DM2 we considered Mip = 80% and $k_2 = 0.054$, $k_3 = 0.026$, $k_5 = 0.02$, $k_6 = 0.03$, $k_8 = 0.05$, $k_9 = 0.02$. This parametrization of (10), (11a–11c) corresponds to the constraint set denoted by S2.

The summary of the main features of the alternatives, in this experiment, is shown in Fig. 8, while Figs. 9 and 10 show the confrontation table and the max-min range graphics for S2, respectively. The first major conclusion is that a_2, a_5, a_6, a_7, a_8 are absolutely dominated by a_4 and a_3; a_1 is dominated by a_4 but not by a_3. In this case it is the alternative a_3 which has, simultaneously, max min value and min max regret, and both a_3, a_4 are, in isolation, ε-optimal, for $\varepsilon > 0.014$ and $\varepsilon > 0.016$, respectively. These alternatives are also k-optimal, that is k-optimality regions, associated with a_3

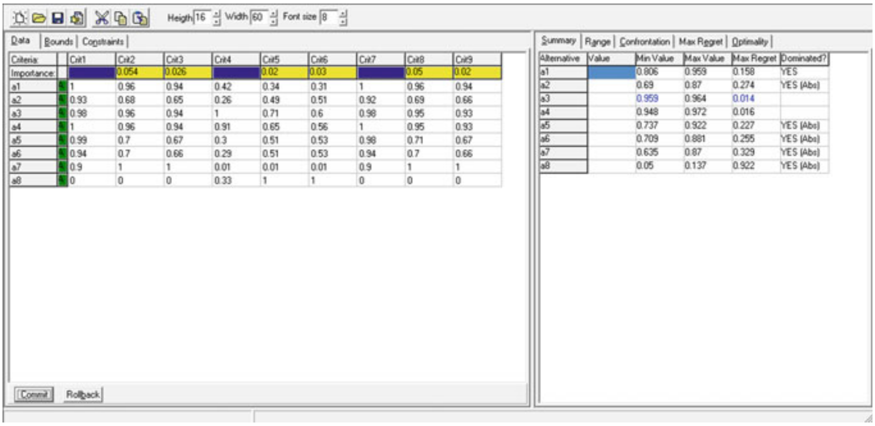


Fig. 8 Main features of alternatives, for S2

Fig. 9 Confrontation table, for S2

and a_4 can be defined in the space (k_1, k_4, k_7). The execution of the VIP functionality concerning the analysis of the k-optimality properties of a_3 and a_4 in the space (k_1, k_4, k_7), enabled to obtain the results in Fig. 11, where the triangle and the trapezium in color represent the projections of the k-optimality regions of a_3 and a_4, respectively. The table on the right hand-side of the optimality region graphics indicates that the k-optimality region of a_3 represents 28.4% of relative volume in the feasible space (k_1, k_4, k_7), while the k-optimality region of a_4 represents 71.6% of that volume (Fig. 11).

Two other sets of experiments, corresponding to variants of the above decision scenario for DM1, where specific proportion relations between some pairs of scaling constants were defined, were carried out. For example instead of $k_1 > k_2$, he/she considers $k_1 = b_{12}k_2$ with a specific value $b_{12} > 1$ and similarly for three other constraints on (k_7, k_8), (k_5, k_4) and (k_3, k_9). That is, four of the inequality constraints in S1 were replaced by proportion relations of the general type (while the other equality/inequality relations still hold):

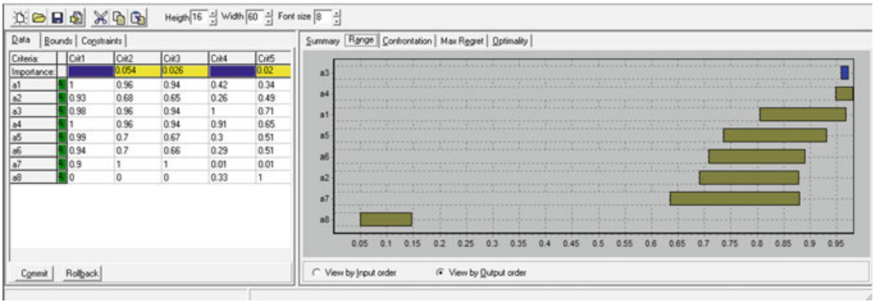


Fig. 10 Min-max ranges for S2

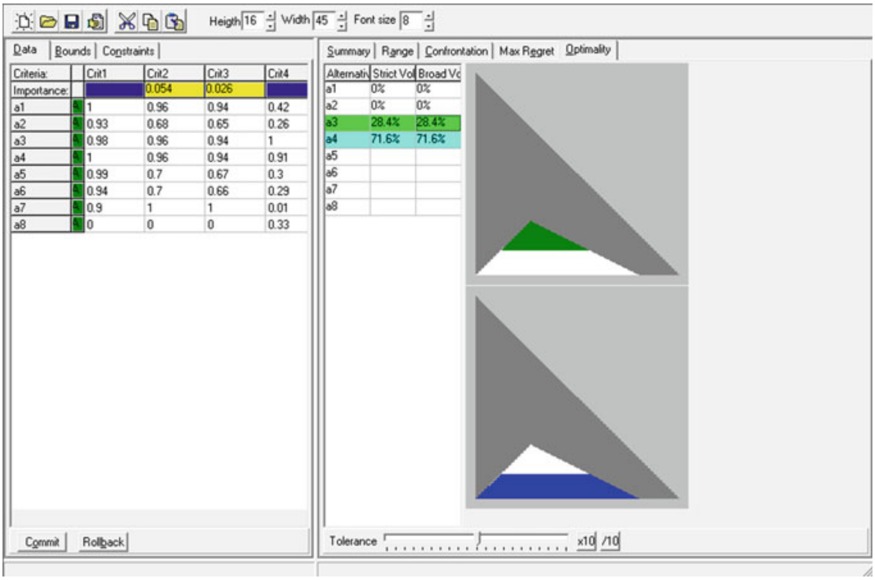


Fig. 11 k-optimality regions of a_3 and a_4 for S2. The shaded areas represent projections of volumes in the space (k_1, k_4, k_7), onto a two dimensional space

$$k_i = \beta_{ij} k_j \tag{12}$$

These experiments enabled this DM to fix, in quantitative terms, possible relative importance values, between certain pairs of attributes and evaluate the effect of such choices. The results of these experiments are not presented here since they were not significantly different, in qualitative terms, from the base scenario for DM1, namely that five alternatives are dominated by a_3 or a_4, four being absolutely dominated, and a_3 (with max-min value) and a_4 (with min-max regret) are still the most promising solutions. Details of these results can be seen in a short report (Craveirinha et al. 2018).

Synopsis of Results with One DM

The two sets of separate experiments, for DM1 and DM2, showed two variants of decision scenarios, considering the same basic underlying assumptions regarding the relative importance of the criteria, as assigned, by typical network designers, to TCB, TAC and TRB network performance measures. Two other sets of experiments, in this same framework of assumptions, considering one DM, were carried out, as mentioned above (see details in the research report (Craveirinha et al. 2018).

The major conclusions, from all these experiments, were:

(i) the routing method (a_8) using minimal hop paths is absolutely dominated by all the other methods, so it is consistently a very poor solution to be avoided, thence confirming, in a systematic and mathematically consistent manner (in a scenario of incremental traffic) all previous empirical results in the telecommunication literature concerning the evaluation of this routing method;

(ii) the two 'best' alternatives were the bi-criteria routing methods, using either the Euclidian or the Chebyschev distances (both using the normalization coefficients WA) to the ideal optimum in terms of load cost and hop count, namely a_3 and a_4.

(iii) these two more promising alternatives, a_3 and a_4 either dominated, absolutely dominated or quasi-dominated the other routing methods, with slight variations of these features in the different scenarios, therefore they are quasi-optimal *vis a vis* the other six remaining methods;

(iv) the next method with reasonable behaviour in terms of max-min value and maximal regret, in all scenarios, was the bi-criteria method a_1, which uses the normalization coefficients WA and priority regions; nevertheless, this alternative is quasi-dominated by a_3 and by a_4 for relatively low values of ε, in all these scenarios.

Furthermore, one can say that a_4 is the one alternative which has, overall, stronger dominance properties in these sets of experiments. In fact, a_4 quasi-dominates a_3 except for DM2, a case in which a_3 has the advantage of being the alternative with max min value and min max regret (although at a very short distance from a_4 in this respect), but a_4 has a significantly better k-optimality feature, namely a k-optimality region with a relative volume in the feasible space (k_1, k_4, k_7), more than 2.5 times greater than the corresponding volume for a_3. One may conclude that, although a_3 should not be disregarded as routing method with very good overall network performance, overall, in this application context, a_4 would be the selected bi-criteria routing method.

3.3 Study for Cooperative Group Decision

Next we considered the extension of the experimental study to the case of face-to-face cooperative group decision with a facilitator. Although some general assumptions on relative importance of the network performance metrics may be common to most network designers, the specification of relations between the criteria IPs may, in some few cases, vary significantly from one DM to another. Furthermore, as noted above, this may be a realistic decision setting in the context of a major network operator, since more than one engineer/network designer is often involved in a decision which has decisive impact in terms of network performance and cost. As stressed above, the group decision support model, based on the VIP software (see Dias and Clímaco 2005), was designed in order to reflect to each DM the consequences of the other DMs inputs. Hence, each DM is confronted with analogous images of the decision group elements' inputs. However, in the addressed decision problem, a routing method has to be chosen. Two situations may occur in the MA analysis process: either one alternative becomes accepted by all the DMs, as the more favourable, or two or more alternatives should be considered, in equal standing, by the DMs. In the latter case, the facilitator will be the 'head of the team', i.e. the DM who will be accountable to the operator management for the implementation of the routing solution, whom we also may designate as 'last resort DM'. In this case, he/she will have to make a final choice of a method among a final short list of alternatives, as suggested in the short paper (Clímaco et al. 2015).

Hereafter, we consider a cooperative group decision setting comprising the decision makers DM1, DM2 whose system of preferences concerning the relations between IPs, was described in the previous section, plus a third decision maker, DM3, who has some distinct qualitative feature, concerning the relative importance of major performance criteria. This DM3, is, in a sense, out of the 'main stream', in terms of common preferences, *vis a vis* the other DMs, namely by considering that some of the inequality relations between IPs, assumed by DM1 and DM2, should be reversed. Namely, DM3 considers that TRB (Total Residual Bandwidth) is more important than TCB (Total Carried Bandwidth) and TAC (Total Accepted Connections), for the same level of blocking probability, that is he/she favours more short term minimisation of the usage of networks resources than total mean carried bandwidth or mean total accepted connections. This type of preferences may favour other types of routing solutions as compared with the ones favoured by the analysis of DM1 and DM2.

The new constraints for the IPs are:

$$K_{(i)TCB} \prec K_{(i)TBR} \; (i = 1, 2, 3) \tag{13a}$$

$$K_{(i)TAC} \prec K_{(i)TBR} \; (i = 1, 2, 3) \tag{13b}$$

These constraints replace (c4) and (c5) in the constraint set S1, the remaining constraints of which remain unchanged. The new constraint set, characterizing DM3, will be denoted by S3. The major results of the analysis of alternatives by this DM are shown in the Fig. 13 (summary of results), Fig. 14 (confrontation table) and Fig. 15 (max-min ranges) of Appendix B.

The main results from this analysis are:

(i) Six alternatives (a_1, a_2, a_4, a_5, a_6, a_7) are dominated by a_3 and five (a_1, a_2, a_5, a_6, a_7) are dominated by a_4;
(ii) Alternative a_3 has 'max-min value' and 'min max regret' and dominates a_4, this last feature being in contrast with the analysis of DM1 and DM2, but still a_4 is the second more favorable in terms of 'max-min value' and 'max-min regret';
(iii) The only alternative not dominated by a_3 is a_8, but a_8 still has very poor performance in terms of 'min value' and 'max regret'.

Therefore, the major conclusion is that, for DM3, a_3 is overall the best compromise alternative, since it dominates the second more favourable, a_4.

Remember that we are assuming a cooperative group decision analysis environment, where the three DMs are confronted with the preference choices and results of the other decision group elements' inputs. Subsequently, we now address the interplay between the tolerance ε, defining quasi-dominance relations between two alternatives, and α-majority relations, which may be analysed according to the concepts in (Dias and Clímaco 2005), highlighted in Sect. 2. For this purpose, exemplifying in our case study, we will consider the aggregation of preferences at the output level and identify two major conclusions concerning ε dominance properties of a_3, a_4, with respect to five alternatives (a_2, a_5, a_6, a_7, a_8). This assumes that, after a preliminary discussion, all DMs agreed that the two more promising alternatives are a_3 and a_4, consistently with the results previously presented for the three DMs. Let us apply an extension of the quasi-dominance concept to multiple DMs. Given DMd (i.e. a DM specified by "d"), characterized by a set T_d of admissible importance parameter values, we denote the results/propositions, assuming an α majority rule: "a_i *quasi-dominates* a_{j1}, a_{j2}, …, a_{jm} with tolerance ε, for a majority of α decision makers" by:

$$a_i \Delta_{\varepsilon(\alpha)} a_{j1}, a_{j2}, \ldots, a_{jm} \tag{14}$$

We considered this type of proposition in the context of our experiment, in order to examine and compare the quasi-dominance properties of a_3 and a_4 with respect to a_2, a_5, a_6 a_7, a_8, in terms of the α majority rule. These results can be represented by the graphics of Fig. 12.

For example, in Fig. 12a, we have:

$$a_3 \Delta_{0.01(2/3))}(a_2, a_5, a_6, a_7, a_8) \tag{15}$$

This means that a_3 dominates a_2, a_5, a_6 a_7, a_8 for two of the three DMs, in our case study, with a tolerance of 0.01. It also holds, in Fig. 12a:

$$a_3 \Delta_{0.4(1)}(a_2, a_5, a_6, a_7, a_8) \tag{16}$$

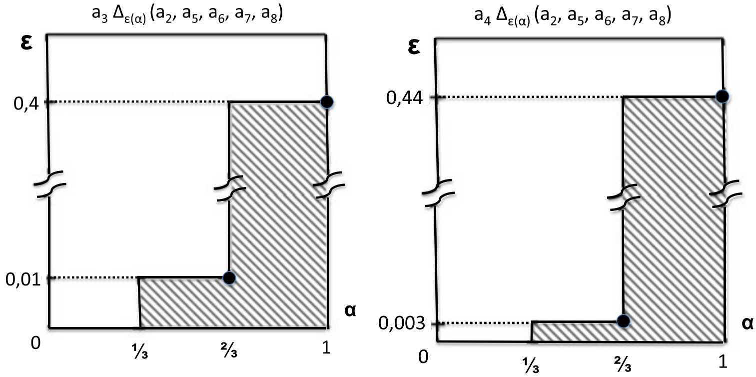


Fig. 12 **a** ε(α)-majority relations for a_3. **b** ε(α)-majority relations for a_4

This means that a_3 dominates a_2, a_5, a_6 a_7, a_8 for all the three DMs, in our case study, with a tolerance of 0.4.

Similar information can be seen in the graphic of Fig. 12b, in this case concerning the dominance properties of a_4 with respect to a_2, a_5, a_6 a_7, a_8.

We can conclude that the interplay between the tolerance ε, defining quasi-dominance relations, and α-majority relations, suggests that, overall, a_4 has somehow better ε-dominance properties than a_3 for a 2/3 majority of DMs. Moreover, as seen in the previous experiments, the ε-optimality properties of these two alternatives (when analysed separately in confrontation with the other alternatives) for DM1 and DM2 and the k-optimality properties for DM2, discussed above, also suggest that this conclusion is defensible for our group decision process, although we can say that a_3 is still a very good routing alternative. If we considered the third more favourable alternative, $a_{1,}$ as an additional argument in the properties of type (14) above, we might further conclude that, for even stronger reasons, a_4 outperforms a_3 in terms of a ε-2/3 majority rule.

4 Conclusions and Further Work

After analyzing the reasons why it is necessary to evaluate the performance of flow oriented routing models through relevant global network performance measures, we concluded that the comparison and ultimately the choice—by a network design expert—of "a routing method with better overall performance" in a given network context, may easily become a difficult task, involving what is, in fact, a complex decision problem. Such decision problem should, in our view, be tackled with methodologically sound OR-decision support techniques, in the framework of multi-attribute decision analysis.

A major conclusion of our study was to show the practical usefulness and great potential, from a methodological point of view, of using a *multi-attribute analysis* model, dealing with *incomplete information*, in this network design decision process. This has to do with the fact that flow oriented routing models, due to their inherent limitations, have to be evaluated through global network performance parameters, corresponding to the attributes of our decision problem, that are often conflicting and incommensurate. Moreover, the incomplete and imprecise information features of the proposed MA model stems from the fact that the scaling constants (or importance parameters), associated with the considered attributes are not fixed a priori, although various inequality, proportion relations, or specific values for some of these IPs, can be set a priori by the decision maker(s), assuming possible different scenarios for such relations. We have specified a multi-attribute (MA) analysis model, based on the *VIP Analysis* software (which considers variable interdependent importance parameters), for comparing and selecting routing methods, in terms of multiple network performance measures, enabling the consideration by the network designer (decision maker) of multiple scenarios concerning different forms of valuation of the relative importance of the network performance measures. Furthermore, we considered the application of the extension of this decision analysis model, based on the VIP decision support tool, for dealing with this problem, in the case of face-to-face cooperative group decision.

The application of this model to a case study involving the comparison and selection of one of various flow-oriented routing methods, based on bi-criterion shortest paths and single-criterion shortest paths algorithms (using as path metrics load cost and hop count) in the context of transport networks with incremental traffic, showed, as second important conclusion, the total adequacy of this VIP based multi-attribute analysis model, that assumes an additive value function under imprecise information, to tackle the formulated decision problem. Furthermore, in this type of study, we could make the most of this learning oriented interactive tool.

Moreover, from a network design point of view, a number of relevant conclusions could be drawn from our extensive experimental case study, both in the case of a single decision maker and in the case of three cooperative decision makers. In the case study we considered nine attributes associated with total carried bandwidth, total residual bandwidth and total number of accepted connections, in different load/overload conditions, in a transport network with incremental traffic. Firstly, the routing method (a_1) using minimal hop paths is absolutely dominated by all the other methods, so it is consistently a very poor solution to be avoided, thence confirming, in a systematic and mathematically consistent manner all previous experimental results in the literature, concerning the evaluation of this routing method. Secondly, the two "best" routing procedures, were two specific bi-criteria routing methods, with a certain set of normalization coefficients depending on the network conditions, which either dominated, absolutely dominated or quasi-dominated the other routing methods. Thirdly, taking into account that one of these two had the strongest dominance properties, we may conclude that—although the other bi-criterion method should not be disregarded as routing method with very good overall network performance—the recommended bi-criteria routing method, in this particular network and decision environment, was the bi-criteria routing method which used a Chebyschev distance to the ideal optimum. This conclusion was still valid, in a cooperative group decision environment with three decision makers, in particular, by applying a 2/3 majority rule, following the concepts in (Dias and Clímaco 2005), and also taking into account the quasi dominance and k-optimality features of the two most promising solutions.

Further work on this research line might involve the application of the developed MA model to other type of network environments and to other problems of telecommunication network design.

Appendix A—Variants of the Bi-criteria Routing Model

Concerning the bi-criteria routing model, in two of the considered variants, a_1, a_2, the aggregation of the bi-criteria preferences was performed by using preference regions in the objective function space. These regions were obtained by defining required and acceptable values m^i_{req}, m^i_{ac} ($i = 1, 2$), specified by the coordinates corresponding to points at distances, taken from the optimal point coordinates, at 1/3 and 2/3 of the variation range of the corresponding function $m^i(r_s)$, as shown below (cf. Eq. A3). In these variants, the non-dominated paths were obtained by calculating k-shortest paths, using the additive path cost function (A1), and choosing the first solution in the highest non-empty priority region:

$$min_{r_s} m(r_s) = \sum_{l_k \in r_s} (m_k^{*1} + m_k^{*2}) \tag{A1}$$

where $m_k^{*i} = \varepsilon_i m_k^i$ is the normalized value of the cost function m^i at arc l_k, ($i = 1, 2$). The normalizing coefficients ε_i (such that $\varepsilon_1 + \varepsilon_2 = 1$) were calculated in two different forms, in the various variants of the model. Let op^i de the minimal value of the two path metrics $m^i(r_s)$ ($i = 1, 2$), and Δ^i the range of values of $m^i(r_s)$ defined in terms of the Nadir point (M^1, M^2), in the objective function space:

$$\Delta^i = M^i - op^i;\ M^i = m^i\left(\arg\left\{\min m^j(r_s)\right\}\right)\ (i = 1, 2)\ \wedge j \neq i \tag{A2}$$

Therefore, the required and acceptable values m^i_{req}, m^i_{ac} ($i = 1, 2$) were calculated as follows:

$$m^i_{ac} = op^i + \frac{2}{3}\Delta^i;\ m^i_{req} = op^i + \frac{1}{3}\Delta^i \tag{A3}$$

Concerning the *first set of normalizing coefficients*, *WA*, it is obtained by equalizing the two ranges $\varepsilon_i \Delta_i$ $(i = 1, 2)$, leading to:

$$\varepsilon_i = \frac{1}{\Delta^i}\left(\sum_{k=1}^{2}\frac{1}{\Delta^k}\right)^{-1} \quad i = (1,\ 2) \tag{A4}$$

Note that these coefficients are calculated each time a VC (node to node virtual connection) is established.

The *second set of normalizing coefficients*, *WB*, is calculated (cf. Martins et al. 2013) by considering the average of each path metric, for the current state of the network links l_k:

$$\overline{m_k^i} = \frac{\sum_{l_k \in L} m_k^i}{|L|} \tag{A5}$$

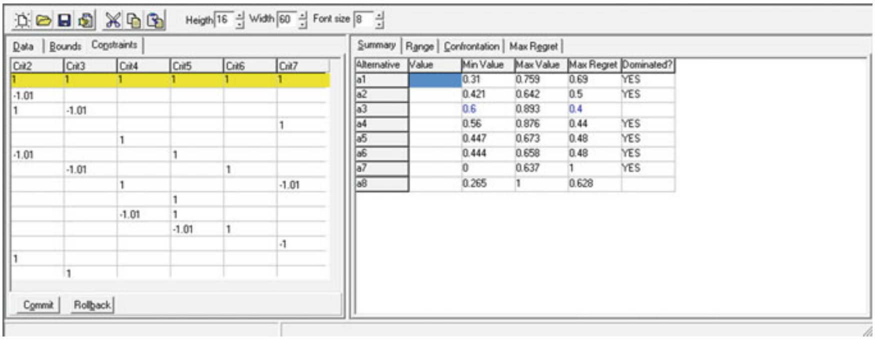
where L is the set of network links, $|L|$ denotes the cardinal of L and m_k^i is the cost associated with metric m^i, considering all current occupations in link l_k. The equalization of the variation ranges, considering these averages $\overline{m_k^i}$ leads to:

$$\varepsilon_1 = \frac{1}{1 + \overline{m_k^1}}; \varepsilon_2 = 1 - \varepsilon_1$$

Note that, in this case, the coefficients don´t have to be calculated for each VC since they depend on the average metric values.

The other variants of the routing model seek non-dominated solutions which minimize either the Euclidian or the Chebyschev distance to the ideal optimum, also considering the two different sets of normalizing coefficients WA, WB, determined as explained above. This leads two four variants of the routing model, a_3, a_4, a_5, a_6. Further details can be seen in (Martins et al. 2013).

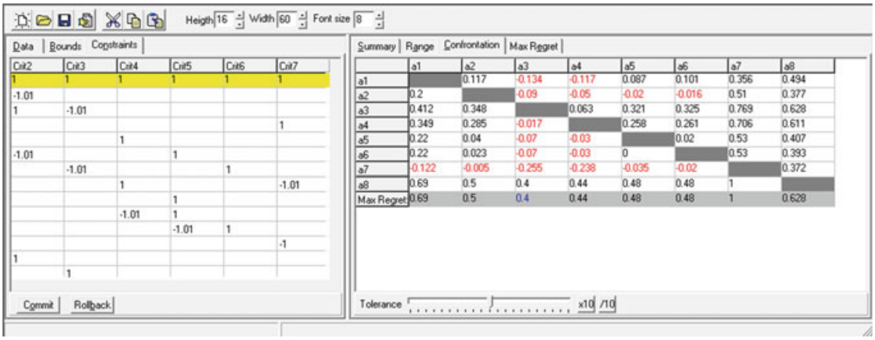
Appendix B—Results for Decision Maker DM3



Crit2	Crit3	Crit4	Crit5	Crit6	Crit7
1	1	1	1	1	1
-1.01					
1	-1.01				
					1
		1			
-1.01			1		
	-1.01			1	
		1			-1.01
			1		
		-1.01	1		
			-1.01	1	
					-1
1					
	1				

Alternative	Value	Min Value	Max Value	Max Regret	Dominated?
a1		0.31	0.759	0.69	YES
a2		0.421	0.642	0.5	YES
a3		0.6	0.893	0.4	
a4		0.56	0.876	0.44	YES
a5		0.447	0.673	0.48	YES
a6		0.444	0.658	0.48	YES
a7		0	0.637	1	YES
a8		0.265	1	0.628	

Fig. 13 Summary of results for S3



	a1	a2	a3	a4	a5	a6	a7	a8
a1		0.117	-0.134	-0.117	0.087	0.101	0.356	0.494
a2	0.2		-0.09	-0.05	-0.02	-0.016	0.51	0.377
a3	0.412	0.348		0.063	0.321	0.325	0.769	0.628
a4	0.349	0.285	-0.017		0.258	0.261	0.706	0.611
a5	0.22	0.04	-0.07	-0.03		0.02	0.53	0.407
a6	0.22	0.023	-0.07	-0.03	0		0.53	0.393
a7	-0.122	-0.005	-0.255	-0.238	-0.035	-0.02		0.372
a8	0.69	0.5	0.4	0.44	0.48	0.48	1	
Max Regret	0.69	0.5	0.4	0.44	0.48	0.48	1	0.628

Fig. 14 Confrontation table for S3

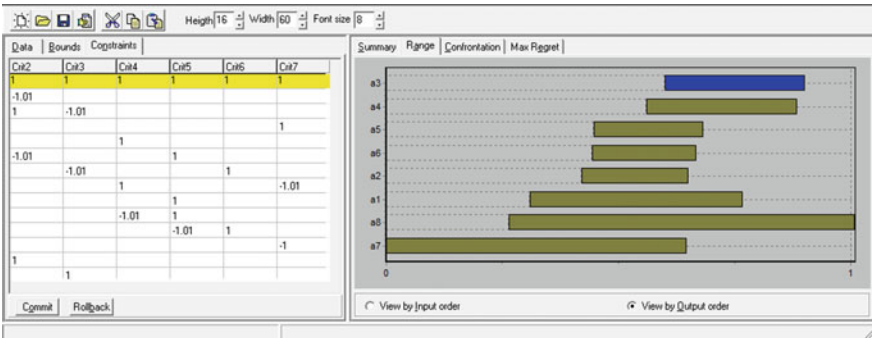


Fig. 15 Min-max ranges for S3

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