



# Non-stationary Operating Conditions of Rotating Machines: Assumptions and Their Consequences

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**Abstract.** The growing use of rotating machines operating in non-stationary conditions gave rise to a greater need to a higher precision for describing their dynamic behavior. The latter has always been based on a certain number of simplifying assumptions. In particular, the spinning speed is considered either constant or following a given law of variation as a function of time, resulting in a dynamic model that is limited to specific operating conditions. The aim of this work is to present a more general dynamic model of rotating machines, which accurately reflects its behavior in real working conditions. No assumption is made on the speed of rotation; it is included as an unknown of the dynamic problem by introducing a degree of freedom combining both the free body rotation and the torsional deformation. The instantaneous angular speed (IAS) is then deduced not only from the induced torque, but also from the whole dynamic behavior of the structure taking into account the periodic geometry (e.g.: gears, bearings) as well as the operating conditions (e.g.: going through the critical speeds). Making no assumption on the angular speed leads to a new formulation of the gyroscopic effect strongly present at very high speeds. This new formulation shows a coupling between the different degrees of freedom as well as a nonlinear behavior of the structure. The results of both classic and new formulations are compared for an architecture of a rotating machine to highlight the utility of the innovative approach in non-stationary operating conditions.

**Keywords:** Rotor dynamics · Gyroscopic effect ·  
Non-stationary conditions · Very-high speed ·  
Instantaneous angular speed (IAS)

## 1 Introduction

Generally, the assumption of constant speed is made in rotordynamic studies for the sake of simplicity. This assumption is however very constraining since it allows the understanding of the behavior of rotors only when the permanent regime is

achieved. Since the understanding of the rotors behavior in non-stationary working conditions is important, especially at start-up and shut-down periods, when the rotor goes through its critical speeds. Many studies have been carried out to understand the behavior of the rotating machineries in such conditions.

On the one hand, some of those studies were made by imposing an angular speed following a function of time which is often taken linear or exponential. Lalanne et al. [1] considered a dynamic model accounting only for the flexural behavior of the rotor and showed that the more important the acceleration through a critical speed is, the lower is the lateral displacement amplitude. Yamamoto [2] focused more on the gyroscopic term present in a rotor, whose magnitude is proportional to the angular speed. He considered a rotating speed varying periodically resulting on a rotor governed by differential equations having variable coefficients with time. He showed that contrary to all expectations, there occurs no unstable vibrations in such working conditions. Al-Bedoor [3] studied the coupled torsional and lateral vibrations of unbalanced Jeffcott-rotor in the case of rotor-to-stator rubbing during start-up period. But he only dealt with angular speeds below the second critical one in order to simplify the gyroscopic effect term.

On the other hand, some other studies were carried with an imposed driving torque and not with an imposed angular speed. In this case a problem called 'unsufficient torque acceleration' [4] may be encountered. If not enough torque is induced to the rotating machinery, the rotor stalls at the first critical speed and the angular speed gets trapped at resonance. Mastuura [5] was interested in this kind of working conditions. He considered a rigid rotor with a mass unbalance and investigated conditions under which the rigid rotor easily passes through critical speeds without stalling. Li et al. [6] proposed a new method in the instantaneous frequency domain to find the closed-form solutions of displacement, velocity and acceleration envelopes for passage through resonance. Their studies were however made under the assumption of constant speed acceleration rates for run-up and run-down processes in order to ensure tractable solutions. Srinivasan [7] focused on modeling the phenomenon of limited-torque acceleration through critical speed of a Jeffcott rotor, with a non-linear torque profile.

In this work, a most complete as possible dynamic model based on the finite element method is developed to describe the rotor dynamic behavior at non-stationary working conditions. It accounts for traction-compression, bending and torsion. An imposed driving torque is induced to the system and no assumption is made on the angular speed of rotation. This requires the introduction of degree of freedom combining both the torsional deformation and the rigid body angular displacement as an unknown of the dynamic problem. The angular speed will then be deduced, not only from the imposed time-varying torque, but also from the working conditions, namely, the passing through the critical speeds.

## 2 A New Dynamic Model Formulation for Rotors Working at Non Stationary Conditions

### 2.1 System Description

The fixed reference frame is denoted (XYZ) while the inertial reference frame is denoted (UVW). To describe the general orientation of the cross-section of the shaft element one first rotates by an angle  $\Phi$  about the Z-axis then by an angle  $\theta$  about the new X'-axis and finally by an angle  $\Psi$  about the final W'-axis. The Finite Element Method is used to model the shaft. The relationships between Euler angles and the degrees of freedom of rotation at any point of the shaft in the global coordinate system are the following:

$$\theta_x = \theta \cdot \cos(\phi) \quad (1a)$$

$$\theta_y = \theta \cdot \sin(\phi) \quad (1b)$$

$$\theta_z = \phi + \psi \quad (1c)$$

### 2.2 Governing Equations of the Rotor

The governing equations are obtained by the application of Lagrange equations on the expression of the kinetic energy of the different components of the rotor.

**The Disk**, it is assumed to be rigid, and it is fully characterized by its kinetic energy associated to the displacement of its center of mass C ( $u_c, v_c, w_c$ ) and to the rotational motion of its section. The associated displacement vector is then:

$$\{\delta_c\}^t = \{u_c, v_c, w_c, \theta_{x_c}, \theta_{y_c}, \theta_{z_c}\}_{\{1,6\}} \quad (2)$$

The application of the Lagrange equations on the kinetic energy related to the disk  $T_D$  results on the following matrix equation:

$$\frac{d}{dt} \left( \frac{\partial T_D}{\partial \dot{\delta}} \right) - \frac{\partial T_D}{\partial \delta} = [M_D(\{\delta_c\})] \{\ddot{\delta}\} + [C_D(\dot{\theta}_{z_c})] \{\dot{\delta}\} \quad (3)$$

such as:

$$[M_D(\{\delta_c\})]_{(6,6)} = \begin{bmatrix} m_D & 0 & 0 & 0 & 0 & 0 \\ 0 & m_D & 0 & 0 & 0 & 0 \\ 0 & 0 & m_D & 0 & 0 & 0 \\ 0 & 0 & 0 & I_d & 0 & \frac{I_p}{2} \theta_{y_c} \\ 0 & 0 & 0 & 0 & I_d & -\frac{I_p}{2} \theta_{x_c} \\ 0 & 0 & 0 & \frac{I_p}{2} \theta_{y_c} & -\frac{I_p}{2} \theta_{x_c} & I_p \end{bmatrix}$$

and  $[C_D(\dot{\theta}_{z_c})]$  is the classical skew-matrix related to the gyroscopic effects.

The matrix  $[M_D]$  is no longer a diagonal matrix as it is the case when studying the stationary regime, it has extra-diagonal terms variable with time due to the assumption of non-stationary regime inducing coupling between flexural and torsional behavior. This matrix is dissociated to a constant diagonal matrix and a time varying matrix with only time-varying extra-diagonal terms as follows:

$$[M_D(\{\delta_c\})]_{\{6,6\}} = [M_{Dconst}]_{\{6,6\}} + [M_{Dvar}(\{\delta_c\})]_{\{6,6\}} \quad (4)$$

**The Mass Unbalance.** The mass unbalance is defined by its mass  $m_u$  situated at a distance  $d$  from the geometric center of the shaft  $C(u_c, v_c)$ . The application of the Lagrange equations on its kinetic energy  $T_u$  results on the following matrix equation:

$$\frac{d}{dt} \left( \frac{\partial T_u}{\partial \dot{\delta}} \right) - \frac{\partial T_u}{\partial \delta} = [M_u(\theta_{z_c})] \{ \ddot{\delta} \} + \{ F_{nl_u}(\theta_{z_c}) \} \quad (5)$$

where

$$[M_u(\theta_{z_c})] = d m_u \begin{bmatrix} 0 & 0 & 0 & 0 & \cos(\theta_{z_c}) \\ 0 & 0 & 0 & 0 & -\sin(\theta_{z_c}) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cos(\theta_{z_c}) & -\sin(\theta_{z_c}) & 0 & 0 & d \end{bmatrix}_{\{6,6\}}$$

$$\{ F_{NL_u}(\theta_{z_c}) \}^t = -d m_u \dot{\theta}_{z_c}^2 \{ \sin(\theta_{z_c}) \quad \cos(\theta_{z_c}) \quad 0 \quad 0 \quad 0 \quad 0 \}^t_{\{6,1\}}$$

In addition to the classical centrifugal force, due to the non-stationary regime working condition, an additional mass matrix with time varying components is also obtained. Again, this matrix reflects a coupling between the flexural behavior and the torsional one.

**The Shaft.** Using the finite element method, we obtain the following expression for the kinetic energy over a shaft element:

$$T_{s_e} = \frac{1}{2} \{ \dot{\delta}_e \}^t [M_{s_e}] \{ \dot{\delta}_e \} + T_{gyr_e} \quad (6)$$

where  $[M_{s_e}]$  is the elementary mass matrix including both the classical mass matrix and the secondary effects of the rotatory inertia matrix.  $T_{gyr_e}$  is the gyroscopic effect expression such as:

$$\begin{aligned} T_{gyr_e} = & \rho I_p \int_0^l \langle N_3 \rangle \{ \delta \dot{\theta}_z \} - \left\langle \frac{\partial N_2}{\partial z} \right\rangle \{ \delta \dot{v} \} \left\langle \frac{\partial N_1}{\partial z} \right\rangle \{ \delta u \} dz \\ & + \rho I_p \int_0^l \langle N_3 \rangle \{ \delta \dot{\theta}_z \} \left\langle \frac{\partial N_1}{\partial z} \right\rangle \{ \delta \dot{u} \} \left\langle \frac{\partial N_2}{\partial z} \right\rangle \{ \delta v \} dz \end{aligned} \quad (7)$$

where  $\{\delta u\}^t = \langle u_1, \theta_{y1}, u_2, \theta_{y2} \rangle$  and  $\{\delta v\}^t = \langle v_1, \theta_{x1}, v_2, \theta_{x2} \rangle$  are vectors used to describe the shaft element flexural behavior in both of the lateral directions. Whereas  $\{\delta w\}^t = \langle w_1, w_2 \rangle$  and  $\{\delta \theta_z\}^t = \langle \theta_{z1}, \theta_{z2} \rangle$  are used to describe the axial and torsional displacements.

$N_1$ ,  $N_2$  and  $N_3$  are shape functions vectors such as:

$$\langle N_1(z) \rangle = \left\langle 1 - \frac{3z^2}{l^2} + \frac{2z^3}{l^3}, 0, 0, 0, z - \frac{2z^2}{l} + \frac{z^3}{l^2}, 0, \frac{3z^2}{l^2} - \frac{2z^3}{l^3}, 0, 0, 0, -\frac{z^2}{l} + \frac{z^3}{l^2}, 0 \right\rangle \quad (8a)$$

$$\langle N_2(z) \rangle = \left\langle 0, 1 - \frac{3z^2}{l^2} + \frac{2z^3}{l^3}, 0, -z + \frac{2z^2}{l} - \frac{z^3}{l^2}, 0, 0, 0, \frac{3z^2}{l^2} - \frac{2z^3}{l^3}, 0, \frac{z^2}{l} - \frac{z^3}{l^2}, 0, 0, 0 \right\rangle \quad (8b)$$

$$\langle N_3(z) \rangle = \left\langle 0, 0, 0, 0, 0, 1 - \frac{z}{l}, 0, 0, 0, 0, 0, 0, \frac{z}{l} \right\rangle \quad (8c)$$

In order to be able to write the kinetic energy related to the gyroscopic effect under a matrix form, we proceed by an integration by parts, which leads to the following formulation for the gyroscopic effects:

$$T_{gyr_e} = [S_{gyr_e}(\{\delta_e\})]\{\ddot{\delta}_e\} + \{Fnl_{gyr_e}(\{\dot{\delta}_e\})\} \quad (9)$$

where:

$$[S_{gyr_e}(\{\delta_e\})] = -\langle N_3(l) \rangle^t \{\delta_e\}^t [M_{67}^g(l)]^t - [M_{67}^g(l)] \{\delta_e\} \langle N_3(l) \rangle + \left\langle \frac{\partial N_3}{\partial z} \right\rangle^t \{\delta_e\}^t [M_{67}^{g*}(l)]^t + [M_{67}^{g*}(l)] \{\delta_e\} \left\langle \frac{\partial N_3}{\partial z} \right\rangle \quad (10)$$

$$\begin{aligned} \{Fnl_{gyr_e}(\{\dot{\delta}_e\}, \{\dot{\delta}_e\})\} &= -\langle N_3(l) \rangle^t \left( \{\dot{\delta}_e\}^t [M_{67}^g(l)] \{\dot{\delta}_e\} \right) \\ &\quad - 2 \left( \langle N_3(l) \rangle \{\dot{\delta}_e\} \right) \left( [M_{67}^g(l)] \{\dot{\delta}_e\} \right) \\ &\quad + \left\langle \frac{\partial N_3}{\partial z} \right\rangle^t \left( \{\dot{\delta}_e\}^t [M_{67}^{g*}(l)] \{\dot{\delta}_e\} \right) \\ &\quad + 2 \left( \left\langle \frac{\partial N_3}{\partial z} \right\rangle \{\dot{\delta}_e\} \right) \left( [M_{67}^{g*}(l)] \{\dot{\delta}_e\} \right) \end{aligned} \quad (11)$$

such as

$$[M_{67}^g(l)]_e = \rho I_p \int_0^l \left( \left\langle \frac{\partial N_2}{\partial z} \right\rangle^t \left\langle \frac{\partial N_1}{\partial z} \right\rangle - \left\langle \frac{\partial N_1}{\partial z} \right\rangle^t \left\langle \frac{\partial N_2}{\partial z} \right\rangle \right) dz \quad (12a)$$

$$[M_{67}^{g*}(l)]_e = \rho I_p \int_0^l \left( \int_0^z \left( \left\langle \frac{\partial N_2}{\partial z} \right\rangle^t \left\langle \frac{\partial N_1}{\partial z} \right\rangle - \left\langle \frac{\partial N_1}{\partial z} \right\rangle^t \left\langle \frac{\partial N_2}{\partial z} \right\rangle \right) dz \right) dz \quad (12b)$$

The application of the Lagrange equations on the kinetic energy of the shaft leads to:

$$\frac{d}{dt} \left( \frac{\partial T_{s_e}}{\partial \dot{\delta}_e} \right) - \frac{\partial T_{s_e}}{\partial \delta_e} = [M_{s_e}] \{\ddot{\delta}_e\} + [S_{gyr_e}(\{\delta_e\})] \{\ddot{\delta}_e\} + \{Fnl_{gyr_e}(\{\dot{\delta}_e\})\} \quad (13)$$

$[S_{gyre}(\{\delta_e\})]$  and  $\{Fnl_{gyre}(\{\dot{\delta}_e\})\}$  induce a strong non linearity to the system and couplig between the lateral and torsional degrees of freedom.

The calculation of the strain energy of the shaft and the virtual works of the bearings, assumed to be with linear stiffness and damping, leads to the classical stiffness matrix and we finally write the equation of the dynamics of rotating shaft element such as:

$$[M_{s_e}]\{\ddot{\delta}_e\} + [C_{s_e}]\{\dot{\delta}_e\} + [K_{c_e}]\{\delta_e\} = -[S_{gyre}(\{\delta_e\})]\{\ddot{\delta}_e\} - \{Fnl_{gyre}(\{\dot{\delta}_e\})\} \quad (14)$$

### 2.3 Equations of the Rotor

Once the contribution of each of the constitutive elements of the rotor calculated, we consider them thouroughly to write the equation of the dynamic behavior of the rotor. As we have seen previously, the vector and matrix related to the gyroscopic effect depends explicitly on the displacement and velocity vectors  $\{\delta_e\}$  and  $\{\dot{\delta}_e\}$ . We need then, to evaluate those vectors at each time step before doing the assembly over all the shaft elements. The choice of an explicit integration scheme is made. At each time step  $t_{i+1}$ , vectors  $\{\delta\}_{t_i}$  and  $\{\dot{\delta}\}_{t_i}$  are assumed to be known and then the gyroscopic effect terms over an element, namely  $[S_{gyre}(\{\delta_e\})]$  and  $\{Fnl_{gyre}(\{\dot{\delta}_e\})\}$  are calculated and assembled over the whole shaft.

Finally, the equation of motion, in the absence of axial efforts and in the presence of external ones can be written as follows:

$$\begin{aligned} & ([M_s] + [M_{D_{const}}])\{\ddot{\delta}\}_{i+1} + [C_s]\{\dot{\delta}\}_i + [K_c]\{\delta\}_i = \\ & \quad - ([S_{gyr}(\{\delta\}_i)] + [M_{D_{var}}(\{\delta\}_i)] + [M_u(\theta_{z_c})]_i)\{\ddot{\delta}\}_{i+1} \\ & \quad - [C_D(\dot{\theta}_{z_c})]_i\{\dot{\delta}\}_i \\ & \quad - (\{Fnl_{gyr}(\{\dot{\delta}\}_i)\} + \{Fnl_u(\theta_{z_c})\}_i) \\ & \quad + \{F_{ext}\}_i \end{aligned} \quad (15)$$

where in the left side of the equation, the terms which are constant at each time step and, in the right side, the ones which need to be updated at every time step.

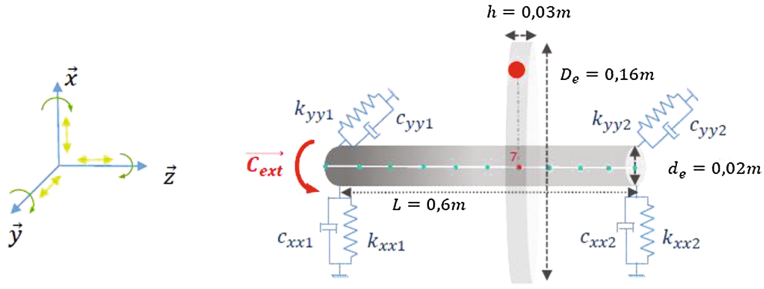
## 3 Results and Discussions

### Flexible Rotor with Mass Unbalance

We consider a rotor made of a shaft and a disk with a mass unbalance (Fig. 1) with the following properties:

*Shaft properties*

length  $L = 0.6$  m; cross-sectional radius  $R_1 = 0.02$  m;  $\rho = 7800$  kg.m<sup>-3</sup>;  $E = 2.10^{11}$  N.m<sup>-2</sup>



**Fig. 1.** Rotor model

The shaft is descrittized into 10 finite elemets.

*Disk properties*

Inner radius  $R_1 = 0.02m$ ; Outer radius  $R_2 = 0.16m$ ; Thikness  $h = 0.03m$ ;  
 $\rho = 7800 \text{ kg.m}^{-3}$   
 Position of the disk: node number 7 of the shaft.

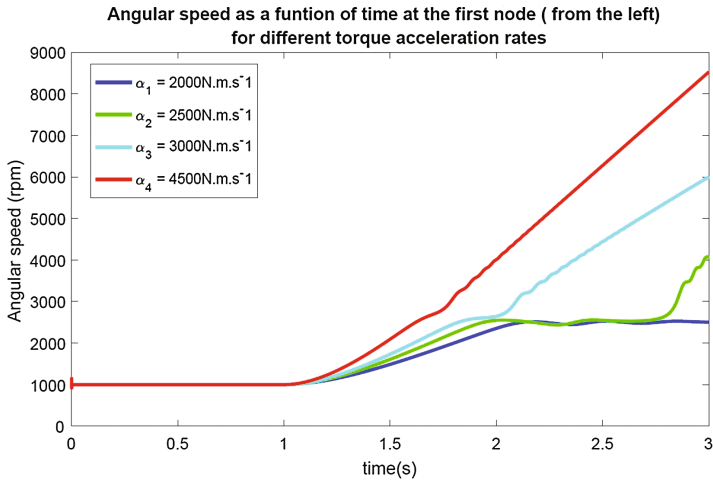
*Mass unbalance*

Mass  $m_u = 1\%$  de la masse du disque; Eccentricity  $d = 0.1 m$

*Bearings*

$$k_{xx1} = k_{xx2} = k_{yy1} = k_{yy2} = 10^8 \text{ N.m}^{-1}$$

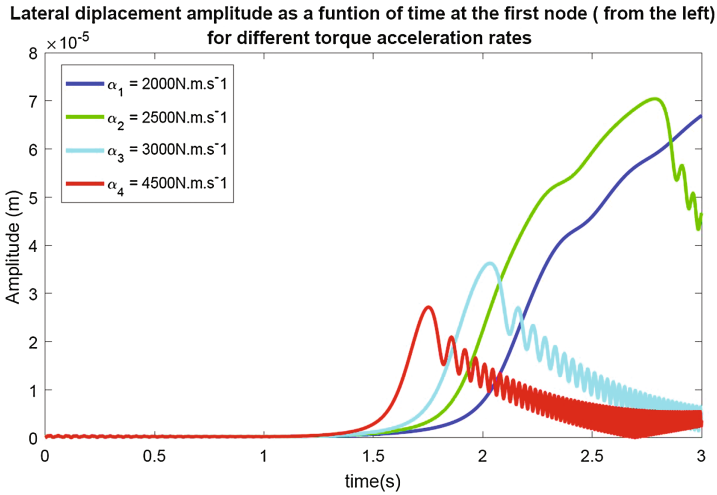
$$c_{xx1} = c_{xx2} = c_{yy1} = c_{yy2} = 0.01 \text{ N.s.m}^{-1}$$



**Fig. 2.** Plot of the angular velocity for successful and unsuccessful acceleration through critical speeds

In this example, the torque is constant at first, it brings the rotor to a speed of 1000 rpm and afterwards it follows a linear law of variation as a function of time. The problem of 'limited torque acceleration' introduced by Gluse [4] for rigid rotors was encountered in our case. The first critical speed of the rotor is at  $\omega_{c1} \simeq 2513$  rpm.

Below some torque acceleration rate or damping values the rotor stalls in the vicinity of the first critical speed. In this case, the rotor generates large deflection displacements which results in important resistive torque. Which explains the dependency of the success or failure of the passing through the critical speeds on the applied driving torque. Figure 2 presents the passing through the first critical speed of the same rotor for different torque acceleration rates  $\alpha_i, i \in \{1, 2, 3, 4\}$ . It confirms that the more the torque acceleration rate is important, the easier the passing through critical speeds is.



**Fig. 3.** Plot of the lateral displacement amplitude for successful and unsuccessful acceleration through critical speeds

It is also shown that, in accordance to the previous works, the higher the torque acceleration rate is, the lower is the lateral displacement amplitude (Fig. 3).

## 4 Conclusion

The rotor dynamic model presented in this paper based on the finite element methods is a complete model taking into account the shear deformation, the secondary effects of rotatory inertia and the gyroscopic effects. It accounts for traction-compression, deflexion and torsion which makes it strongly non-linear



with different coupling between lateral and torsional degrees of freedom. The new proposed formulation for the rotor dynamics makes no assumption on the instantaneous angular speed which implies the possibility to simulate the rotor response for any non-stationary working condition with linear or non-linear driving torques.

If the maximum of the lateral vibrations when going through critical speeds was investigated in many previous works, no focus was made on the IAS considered in our case as unknown of the dynamic problem. The next step of the present work will be the analysis, in non stationary working conditions, of the IAS which is believed to provide a rich signal containing information not only about the torsional behavior but also about the lateral one due to the coupling between the different degrees of freedom as it is shown in the dynamic model. The rotor model will also be used as an identification tool at non-stationary working conditions.

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