Chapter 6
Allgemeinbildung, Mathematical Literacy, and Competence Orientation

Rolf Biehler with a reaction by Mogens Niss

Abstract The first part of this chapter has been written by Rolf Biehler on the basis of his presentation at ICME 13. Mogens Niss was invited to react to this presentation at ICME 13 and elaborated his reaction as the second part of this chapter. Although the authors are only responsible for their respective sections, they both belong together and are therefore published here as a joint chapter. The first part gives a sketch of the discussion on 'Allgemeinbildung' (general education for all) and mathematical literacy in Germany from the late 1960s to today. In the 1970s, educational goals for Allgemeinbildung were condensed in different visions, for example, a 'scientifically educated human being', a 'reflected citizen', an 'emancipated individual being able to critique society', and a person 'well educated for the needs of the economic system'. In the early 1990s, a book by H. W. Heymann on Allgemeinbildung and mathematics education initiated a controversial discussion, which will be critically examined and related to other conceptions. Due to bad results in TIMSS (Third International Mathematics and Science Study) and PISA (Programme for International Student Assessment) starting in the late 1990s, a new discussion on educational goals in mathematics arose and made PISA's conception of mathematical literacy popular in Germany. However, the idea of mathematical literacy was modified and extended by the German debate, some traits of which can be traced back to Humboldt and the 19th century. In his reaction "Allgemeinbildung, mathematical competencies and mathematical literacy: Conflict or compatibility?" Mogens Niss relates the German discussions to the international development on competence orientation, featuring the KOM project (Competencies and Mathematical Learning), including the various conceptualisations in the PISA frameworks.

Keywords Bildung · Allgemeinbildung · General education · Mathematical literacy · Mathematical competencies · Mathematical literacy

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6.1 Allgemeinbildung, Mathematical Literacy, and Competence Orientation in Mathematics Education in Germany (Rolf Biehler)

6.1.1 Introduction: Bildung and Allgemeinbildung in Germany

Since the 19th century in German history, Bildung and Allgemeinbildung have repeatedly been reinterpreted depending on changes in society and the school system, and on changes of views about the function of the school system. Niels Jahnke discusses the approaches to Bildung in the times of Humboldt in Chap. 5 of this book. Horlacher (2016) provides an informative overview reaching from the origins to modern contemporary conceptions, including the relation of Bildung to PISA's conceptions of mathematical literacy.

> In a comprehensive study of Bildung und Schule, Dohmen concludes in 1964 that the concept of "Bildung" is one of the most ambiguous and vague fundamental concepts of German pedagogy (Dohmen 1964, p. 15). Indeed, it is typical of the lack of clarity of the concept that in the discussion on school reform, it is used by conservatives and reformers alike. Alternatives, like Erziehung (education) or Unterricht (instruction, teaching), do not really catch on, as they cannot rival the grandness and splendor that lies in concept of Bildung. When related to the concept of Bildung, says Dohmen, the school becomes elevated into the high winds of the spiritual, so to speak (ibid., p. 16), whereby this ideal concept generally refers to perfecting the person's "true nature," or "higher self". (Horlacher 2004, pp. 410–411)

'Allgemeinbildung', which can also be characterised as 'general Bildung' is more related to the school system, meaning goals of general education (grades 1–9(10)) for all students with the connotation of holistic self-enculturation. It is contrasted to vocational 'Bildung' (vocational education) that prepares for specific vocations.

In German-speaking countries, after grade 9 or 10, students can either go to a *Gymnasium* or a vocational school. Vocational schools usually mean part-time courses, as students are being educated in craft businesses or companies in parallel (this system is called 'dual system'). This description is a simplified model, and the actual system is much more complicated. Grades 11–13 (or 10–12, depending on the *Land* and the type of school) which are meant to prepare for university and finish with the *Abitur* (school-leaving certificate allowing access to academic education) are also said to provide Allgemeinbildung. As a distinction, the specific Allgemeinbildung at this level is referred to with attributes such as 'higher' (Fischer 2001), 'academic' (Huber 2009) or 'deepened' (KMK 2012) as it does concern only those students who have chosen to attend these grades. In 2017, nearly half of each student cohort in Germany was enrolled in a *Gymnasium*. In the 1960s this was 10% and in the 19th century even less.

Several years ago, German-speaking countries introduced national standards. In Germany, they were published for grades 5–10 in 2004 (KMK 2004) and for the final level of grade 12(13) in 2012 (KMK 2012). Although Allgemeinbildung or deepened Allgemeinbildung are mentioned, the main focus of the standards—which

are called 'Bildungsstandards' is on (mathematical) competencies, a notion that was influenced by its use in PISA and international discussions on competence orientation (Niss and Højgaard 2011). This development has raised concerns that the national standards do not take the more general understanding of Bildung into sufficient consideration. This concern was not only raised in mathematics education but also in educational theory on education until grade 12/13. The title of the book "Bildung at the Gymnasium between competence orientation and cultural work" edited by Bosse (2009) is symptomatic. The debates in educational philosophy are reflected in the discussions on the goals of mathematics education at school level in the community of mathematics educators too (Neubrand 2015).

Despite the variability in the meaning of Allgemeinbildung, there are some common elements in these debates. Conceptions of Allgemeinbildung emphasise the relative autonomy of students and schools and are critical against too direct transpositions of societal and economic needs into the school system and against the direct transposition of mathematical content into schools. The selection of mathematical content for school education has to be justified from a theory of goals of general education respectively from a conception of Allgemeinbildung. Conceptions of Allgemeinbildung in principle have to take a specific view of the relationships depicted in Fig. 6.1.

Conceptions of Allgemeinbildung always deal with the 'justification problem' in the sense of Niss (1994), they provide an argumentation basis from which the selec-

Fig. 6.1 Conceptions of Allgemeinbildung
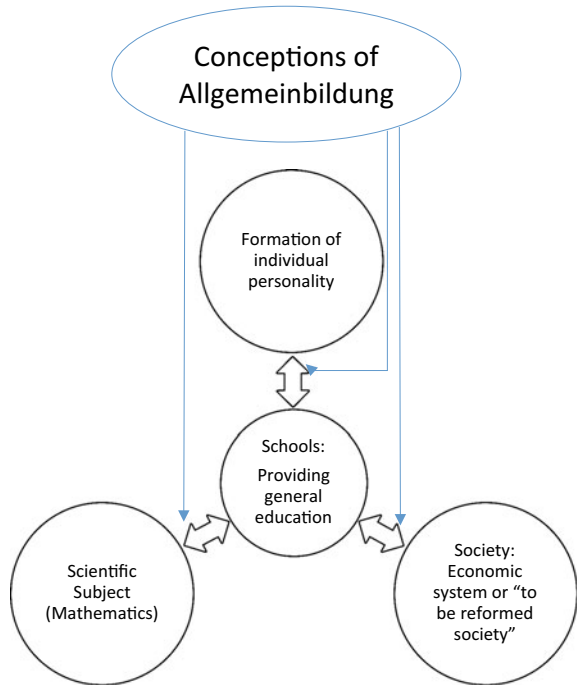
tion of content and goals of teaching can be justified. However, they do not always deal sufficiently with what Niss (1994, p. 373) calls the 'possibility problem' and the 'implementation problem'. Is it possible to teach the selected and justified curricular content to the concrete students in a specific society at a specific point in time, and is it possible to achieve a large implementation of such curricular goals? Actually, there is a dialectic interrelation and tension between the justification, the possibility, and the implementation problem. As Horlacher's (2004) quote mentions, discussions on Bildung sometimes taste of idealistic debates and are somewhat removed from the social reality of societies and real classrooms.

Allgemeinbildung comes with the additional connotation of general in the sense of 'for all students'. In this respect, there is a relation to international discussions (Gates and Vistro-Yu 2003). Beyond gender, ethnicity, language etc., the specific meaning of Allgemeinbildung in German-speaking countries has to be put into context, namely the streaming in the school system, which is less common in other countries. In general, at least since the beginning of the 20th century, students are streamed after grade 4 already into three different types of schools representing three levels of education: *Hauptschule* (formerly *Volksschule.* low), *Realschule* (medium) and *Gymnasium* (high). The Hauptschule und Realschule prepare for different types of vocations (blue- or white-collar workers or business employees, whereas Gymnasium lasts three years longer than the other schools and is preparing for university studies. This streaming in Germany before World War II was resumed in West Germany after the war. Comprehensive schools entered the scene only in the late 1960s in West Germany. They never became the regular standard in West Germany, but about 16 % of the students in grades 5–10 went to a comprehensive school in 2014/15 (Malecki 2016, p. 12). In contrast, the German Democratic Republic (East Germany) implemented a comprehensive school system during its existence, from 1949 to 1989. After the reunification, the new federal states from East Germany adopted the West German system to a large extent, some of them implementing two streams instead of three. This streaming into three school types goes back to the 19th century (see Sect. 6.1 of this chapter), and the Humboldtian spirit of Bildung influenced mostly the *Gymnasium* with its pronounced function of preparing for university, whereas the other school types provided education in a much more utilitarian way.

Thus, Allgemeinbildung is not just a term of educational philosophy but a concept that is used and filled with different meanings in the spheres of general politics, educational politics and school administration, in the educational sciences and philosophy of education, and in didactics of mathematics itself. This adds a level of complexity to this notion.

In this paper, we will deal with this complexity by focusing on three periods, which provide interesting insights into the development in (West) Germany.

- The late-1960s to the 1980s, where significant reforms of the whole educational system in West Germany were partially undertaken in parallel to considering 'new math' in schools.
- The mid-1990s with a specific sudden public concern of 'Allgemeinbildung and mathematics education'.

- The late-1990s where Germany showed unsatisfactory results in TIMSS and PISA, which among other aspects gave rise to the first national standards in mathematics education since World War II and lead to the introduction of competence orientation.

The paper focuses on West Germany from the 1960s and will integrate the developments in other German-speaking countries (German Democratic Republic, Austria, Switzerland).

6.1.2   Looking Back into the Late 1960s and 1970s: New Math and Educational Reforms

We will focus on the situation in West Germany, where, as mentioned above, the school system consisted of three streams in grade 5–10. All of these schools were supposed to provide Allgemeinbildung but had very different goals, and mathematics education in these three school types differed considerably. In a first approximation, *Haupt-* and *Realschule* provided more utilitarian types of mathematics education, whereas the *Gymnasium* started its preparation for university studies from grade 5 onwards with a view towards mathematics as a scientific discipline. Grades 5 and 6 still had elements of utilitarian mathematical content, stemming from the curricular reforms in the 19th century, as was analysed by Jahnke (in this volume).

Moreover, the type of mathematics taught in these schools was largely influenced by teacher education: mathematics teachers for the *Gymnasium* were educated at universities practically along the same curriculum as mathematics majors, whereas teachers of *Haupt-* and *Realschule* were educated at special institutions of higher education for teachers (*Pädagogische Hochschulen*), learning different types of mathematics more remote from the 'real academic mathematics'.

Beginning in the 1960s, the West German school system came under pressure because it was considered to be dysfunctional for the economic system. Strengthening the role of the sciences and mathematics was seen as an essential contribution to the economic development of the country. Processes to revise syllabi were initiated on a large scale. However, fundamental changes also affected the school system and the university system itself. Many new universities were created, and the *Gymnasium* was opened up for a broader range of students, both measures aiming at increasing the number of university students. The *Gymnasium* was and still is the primary course of education for future university students. The proportion of students attending a *Gymnasium* has risen from about 10% in the 1960s to more than 50% today. Moreover, new ways were then opened for students of vocational schools to pass the *Abitur* and enter Universities of Applied Science (*Fachhochschulen*). At the same time—as in other western countries—a broad political movement began, including the 1968 students' movements, aiming at more equity, political participation, individual emancipation, and democratisation in western societies. This progressive movement obviously had different views on the needs of society and what it meant to cultivate and develop a

holistic personality within the school system. The view ranged from being an emanci-
pated personality able to criticise society over a 'reflected citizen' to a 'scientifically
educated human being'.

From this perspective, the educational system was supposed to contribute to the
social change and not primarily to the economic transformation. The three streams
school system was attacked in favour of establishing comprehensive schools for
compulsory education until grade 9 or 10. The three different educational goal sys-
tems of the three school types were put into question in favour of *one* conception of
Allgemeinbildung stating that three—for each school type different—conceptions
of Allgemeinbildung are a contradiction in terms. However, comprehensive schools
were implemented in only some of the federal states, and only some states developed
a common syllabus for all school types in the first place, making some differentia-
tion according to the three levels, but still based on common ground. Education was
the responsibility of the federal states in West Germany, which largely explains this
variability.

During this period, we observe contradictory factors influencing mathematics
education (Fig. 6.2). On the one hand, societal pressure was put on the school system
to revise the curricula, to break with traditions, and to question all curricular content
with regard to its contribution to the education of students. In its extreme end, cur-
ricular content has to come from analysing how students have to act competently in
societal or vocational situations (Robinsohn 1969a, b). On the other hand, and this
thinking was more influential in mathematics education worldwide, the orientation
towards the fundamental ideas of a scientific discipline (Bruner 1960) was seen as
the principle from which mathematics education had to be revised from primary
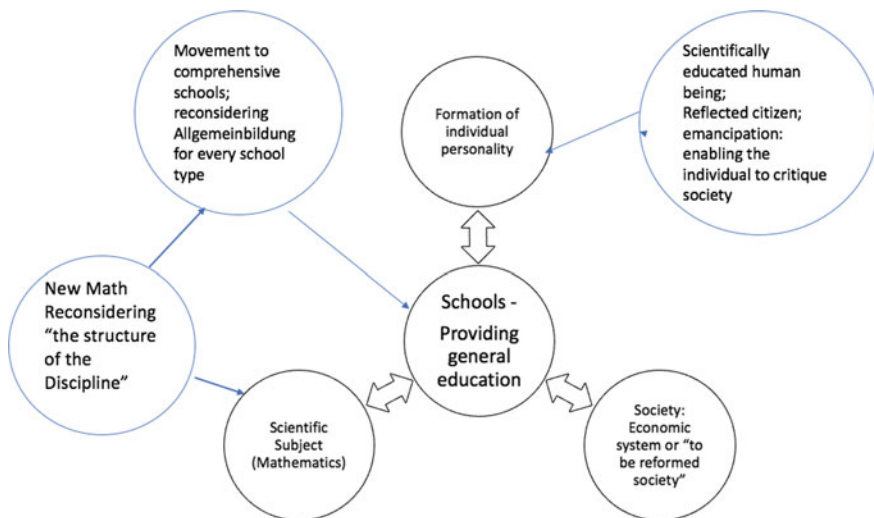education up to the *Abitur*, concerning all types of schools. This approach is based

Fig. 6.2 Factors influencing the role of mathematics in the educational system

on the conviction that orientation towards the sciences and scientific thinking is *the* way to achieve economic and societal growth.

A common ground for these curriculum revisions in mathematics, therefore, was, for a while, the orientation towards mathematics as a scientific discipline, interpreted in the Bourbakian view of mathematics as a science of structures in accordance with international developments. The *Gymnasium* curricula were criticised as being oriented towards an old-fashioned view of mathematics, which was to be replaced by new and modern mathematics closer to the current state of the discipline. The curricula of the secondary schools *Haupt-* and *Realschule*, as well as those of primary schools, were criticised as being insufficiently rooted in mathematics as a discipline at all but pursuing much more practical utilitarian goals. Introducing the principle of orientation towards mathematics as a scientific discipline into these schools was also seen as an act of emancipation for the pupils there, allowing them access to scientific knowledge instead of treating them as second- or third-class pupils, thereby contributing to increased social mobility.

As in other countries, these reforms were not very successful in either school type for many reasons. One reason for this failure was assigned to a rather naïve transposition didactics. Although this was the established approach in the emerging field of mathematics education, there were some different approaches as well.

The call for curriculum revision also created the need to identify better ways to express learning goals for students. Conceptions from the educational sciences ('learning goal orientation') entered German mathematics education and were deployed regularly (Bloom 1956; Gagné 1970; Gagné and Briggs 1974), but approaches rooted in a conception of 'Bildung' were also put forward. One of the leading and influential educational scientists, who aimed at re-defining Bildung and Allgemeinbildung based on the German tradition and the new challenges from society and science, was Wolfgang Klafki, who also inspired real reform projects in various school subjects (Klafki 1963, 1974). The 1963 book received many new editions in the 1970s and 1980s [see Klafki (1995, 2000) for basic ideas in the English language]. He attempted to integrate the conceptions of material and formal Bildung (see the section by Jahnke) and established an analytical framework with which teachers and educators can analyse the educational value (Bildungswert) of a certain topic, reflecting on the current and future meaning of this topic for students and on the 'exemplary character' of a topic to be taught. However, he did not relate his framework to the analyses of a scientific discipline.

The debate on curriculum revision included a critique of educational practices in the Gymnasium from the perspective of Bildung. This perspective was also taken up in mathematics education and is best represented by Wittenberg (1963) and Wagenschein (1965)—with relations to Klafki's approach. Compared to different early versions of New Math in German-speaking countries, these approaches were profoundly critically analysed by Lenné (1969). A more recent account of these historical developments in the English language, focusing on the contribution of Hans-Georg Steiner, is the paper by Vollrath (2007). Among other things, it was to Lenné's merit that he had already pointed out the large discrepancy between the ideals of general education formulated within mathematics education (including neohumanistic

aspects) in the preambles of syllabi and the reality of its catalogues of topics, and not to speak of the realities in the classroom.

In the 1970s, the debate about the goals of mathematics education continued, and the approaches of 'new math' were regarded more critically from both practical and theoretical perspectives. In this context, Winter (1975) tried to specify how mathematics education could and should contribute to "general educational goals" (*Allgemeine Lernziele*) in schools. Heinrich Winter strongly influenced the debate on Allgemeinbildung in mathematics education in the 1990s (Winter 1990, 1995), which we will refer to in the next section.

First, we will look at his influential 1975 paper. Winter asks the question of how mathematics education can contribute to Allgemeinbildung, using the then modern terminology of general educational goals to express his view of Allgemeinbildung for mathematics education. He relates general goals for schools to general goals for mathematics education, which in turn correspond to features of mathematics as a scientific discipline and to general characteristics of human beings. This approach is more anthropological than sociological or political.

Winter's Table 6.1 provides an overview and summary; the article itself gives examples for mathematics education that illustrate and further interpret the different facets, and that would help realise these general goals in everyday mathematics teaching. Two aspects are remarkable. First, his view of mathematics is different from a perspective focusing on ready-made mathematics without applications, whereas this view was partly underlying the new math approach. Second, general goals of mathematics education are not 'deduced' from general goals of schooling, but rather from a broad philosophy of mathematics interacting with requirements from education and anthropological aspects.

A different contemporary contribution was the book by Damerow et al. (1974) on "Elementary mathematics: Learning for the practice", which today we consider as a contribution to critical mathematics education (Skovsmose 1994). Christine Keitel made important contributions to this field later on. The authors' project had the aim

Table 6.1 General learning goals for mathematics education (Winter 1975, p. 116; transl., R.B.)

Human being	Mathematics	General goals: school	General goals: mathematics education
Creative, playful	As a creative science	Unfolding the creative potential	Learning heuristic strategies
Thinking, searching for reasons and insight	As a proving, deducting science	Supporting rational thinking	Learning to prove
Designing, economising, using technology	As an applicable science	Supporting understanding reality and its usage	Learning to mathematise
Speaking	As a formal science	Supporting the use of language faculty	Learning to formalise, learning technical skills

to determine interdisciplinary goals for mathematics education that qualify students for competent and autonomous actions in future situations of practice determined by heteronomy. For accountants that means not only doing the necessary calculations but understanding the whole mathematical system (the real abstraction) behind the limited calculations they are supposed to focus on. The analysis is rooted in the approach by Robinsohn (1969a, b), who worked at the same institution in Berlin (Max Planck Institute for Educational Research in Berlin) as Damerow et al. (1974). However, their emphasis is on autonomous action, besides competent action, providing the specific character of this approach.

In Winter's 1990 paper on citizens and mathematics (Winter 1990), he took elements from this politically critical tradition and combined them with aspects in the Humboldtian tradition. We will discuss it as an intermediate step in this section before we enter into the debate of the mid-1990s.

In this paper, Winter indicated 'enlightenment in the Kantian sense' as an essential goal of education at public schools. Accordingly, we have to think about the question of how to educate our children to become free and responsible citizens of society, be it in local communities, the state, or the world in general.

> Can the teaching of mathematics - and how could this be afforded - help to develop the faculty of judgement in matters of public life? In short: can it contribute to *enlightenment*? (Winter 1990, 131)

According to Winter, the idea that all human beings are equal, have equal votes in elections, and should have equal chances in life does make sense only under the assumption of responsible citizens. Democracy can only be imagined as a society of responsible persons.

Winter considers the participation of citizens in public affairs under the perspective of the tension between experts and non-specialists. Most political decisions require highly specialised knowledge which is provided by experts, whereas in principle all members of society are supposed to decide upon political and social matters and at the same time are laymen regarding most questions. Consequently, in a democratic society, non-specialists should be qualified to understand how experts arrive at their specialised knowledge, how safe this knowledge is and to ask critical questions, in short: citizens should develop a faculty of judgement.

Taking these principles and notions as a starting point, in his (1990) paper Winter concentrated on those parts of mathematics that might contribute to furthering an understanding of society, politics and economy. Only in the last part of the paper examples from science are discussed under the heading of 'Public mathematics'.

It is no wonder that talking about society and economy leads among others to those subjects that, in the first part of our paper, we called everyday applications. We shall keep to this term though Winter uses the German word 'Bürgerliches Rechnen' ('Civil calculations'). Concerning these everyday applications, he sees a tension between 'mathematical systematics' and the 'reality of life' (1990, 134) that implies a twofold danger of trivialisation. There is, first, the danger of trivialising reality to apply predefined mathematical procedures; then there is the danger of trivialising mathematics, abandoning the elaboration of sophisticated mathematical algorithms

and concepts to increase concreteness. According to Winter, a way out consists of examples of 'authentic and reflective modelling' (l.c.).

Winter warned against trivialising 'everyday applications' not only from a mathematical point of view but hinted at the danger of reducing applications to innocent and harmless subjects without providing insights of a more general nature (l.c., 135). In doing so, the aim of enlightenment would be missed. Thus, we see here an anti-utilitarian argument in a similar vein to early 19th century thinkers. 'Bildung' should open the way to critically thinking in alternatives and not to stick to existing conditions. No wonder, exactly at this point of his argument, Winter quoted F. Diesterweg whom we discussed above in connection to Humboldt's disregard for learning, "Carolins, Ducats, and the like".

The essence of Winter (1990) consists of carefully elaborated examples of how he understood 'authentic and reflective modelling'. As noted above, most of these examples are chosen from economy and social sciences—seemingly a consequence of his approach to view Allgemeinbildung as education for free and responsible citizens. In fact, this is a restriction of the very idea of Bildung in a twofold sense. First, we do not know whether the participation of citizens in social life will be organised in the same way in the future as it is done today, nor do we know whether economic conditions will remain the same as they are at present. Second, a human being is more than a citizen of a society, and reducing human beings to this role would deprive them of many potentialities (see for a similar remark Vohns 2017b).

6.1.3 **Allgemeinbildung** *and Mathematics Education in the Mid-1990s—A Snapshot*

Starting in the 1970s, there was a growing concern among some mathematics educators that a too narrow orientation towards mathematics as a scientific discipline was insufficient or even misleading if interpreted as in the new math reform movement. The need for foundational research in this domain was one reason to create the Institute for Didactics of Mathematics (IDM) in 1973, with the aim of conducting research in the didactics of mathematics, thus providing foundations for more well-grounded and successful educational reforms in the future. One of its research programs was involved with studies on the history and epistemology of mathematics to develop a broader and more in-depth view of what constitutes mathematics as a scientific discipline as part of the broader culture and society (Biehler 1994). Another program was concerned with analysing the relation of mathematics education to conceptions of Bildung and Allgemeinbildung (Biehler et al. 1995; Heymann 1996, 2003; Jahnke 1990). Niels Jahnke, Hans-Werner Heymann and I were colleagues at the IDM from the late 1970s to the mid-1990s.

Starting in the early 1990s, a working group at the IDM in Bielefeld was founded to bring together several perspectives on the topic 'Allgemeinbildung and mathematics education' (Biehler et al. 1995). A specific view on this topic was developed in the

habilitation theses of Heymann (1996), which initiated a large public controversy. A symposium in Bielefeld in 1996 again tried to bring together different views (Biehler and Jahnke 1997). Heymann' s book was later published in English, unfortunately without the chapter on the theory of Bildung (Heymann 2003). The paper by Winter (1995) is a reaction to the 'Heymann controversy' and is still today a standard reference, whenever general goals of mathematics education are being discussed in German-speaking countries.

The public controversy was initiated by some newspaper articles that interpreted Heymann's work in an extremely reduced way distorting the original meaning. For example, the *Süddeutsche Zeitung* wrote on 8 October 1995: "Seven years of mathematics is enough. What adults need in mathematics, they can learn in the first seven school years. Everything taught to pupils in mathematics beyond this, plays practically no role in their future lives." (my translation R.B.) Accompanying texts claim that this 'was shown' in the habilitation thesis of the Bielefeld 'mathematician' Hans-Werner Heymann. Heymann received massive criticism from mathematicians who pointed out that he was not a mathematician but a mathematics educator, and that his habilitation thesis was submitted at the faculty of education and not at the faculty of mathematics. However, Heymann received many letters from readers expressing opinions such as: "It was time that somebody told the truth"; "I always suffered from my mathematics lessons: senseless calculation with no meaning to me - and much pressure from the teachers"; "Again and again I am dreaming of my horrible matriculation examination (*Abitur*)". He also received support from educationalists and politicians who had been critical about the amount and type of mathematics education within general education, sometimes suspecting that the hidden aim of mathematics education simply was the selection of students.

The reaction of the wider public indicates that many did not consider mathematics as a valuable enrichment of their Allgemeinbildung and their development as fully-educated human beings. Many wish for a reduction of mathematical content in school (or maybe a different type of mathematics education). The reaction of mathematicians indicated that they did not consider (parts of) didactics of mathematics as a partner in fostering mathematics education at school.

However, summarising Heymann's thesis as "seven years of mathematics education are enough" was a gross misunderstanding of his work. The main reason for this public misunderstanding was, among others, the following passage in his book:

> Concluding Remarks. In their private and professional everyday lives, adults who are not involved in mathematics-intensive careers make use of relatively little mathematics. Everything beyond the content of what is normally taught up to the 7th grade (computing percentages, computing interest rates, rule of three) is practically insignificant in later life. (Heymann 2003, p. 104)

However, this was just a summary of one chapter of his book with the heading "Mathematics Instruction and Preparation for Later Life (in a strict sense)". The rest of the book puts forward arguments from a multitude of perspectives for justifying a much larger role for mathematics education within general education, based on Heymann's elaborated theory of Bildung, that specifies a framework for goals for general

education that he then interprets for mathematics education. Heymann's argument was essentially a hypothetical one "if only we justified mathematics education from the perspective of everyday applications then this would imply …"

The complete framework has further dimensions that are formulated as chapter headings (Heymann 2003, p. v):

- Preparation for Later Life
- Promoting Cultural Competence
- Developing an Understanding of the World
- Development of Critical Thinking
- Developing a Willingness to Assume Responsibility
- Practice in Communication and Cooperation
- Enhancing Students' Self-Esteem.

All dimensions are concerned with preparing students for their future lives (in a broader sense). The first chapter only covers the narrow range of everyday skills, but it features a title that supports misinterpretations. Positively speaking, Heymann distinguishes narrow utilitarian arguments (in his first chapter) from other justifications and contributions to the education of students. For instance, 'developing critical thinking' is akin to requirements in the era of Humboldt to develop thinking skills, and 'developing an understanding of the world' contains what Niels Jahnke termed 'theoretical applications' in his contributions to this volume.

These seven dimensions were constructed and elaborated upon the basis the German history of the theory of Bildung and structured the goals of general education for all subjects. In this sense, Heymann understands his contribution as a contribution to general educational science critically synthesising the theories of Bildung.

The second half of Heymann's book is devoted to how these dimensions can be used for a critical reflection and new determination of the goals, as well as the teaching and learning methods of school mathematics education that are adequate to realise the general goals.

Before we go into the details of his specification, we should understand that scientific disciplines as such are not explicitly mentioned in the framework. This observation is in contrast to other approaches to a theory of Bildung and in particular the theories of Bildung in mathematics education mentioned above, for example by Winter (1975), which are based on an at least rudimentary philosophy of mathematics in bidirectional interaction with philosophies of education.

If we are convinced that Thom's famous dictum, "In fact, whether one wishes it or not, all mathematical pedagogy even if scarcely coherent, rests on a philosophy of mathematics" (Thom 1973, p. 204) is right, we will regard it as a deficiency that we do not find an elaborated view on a philosophy of mathematics in his book.

Another critical point in the debate hit on Heymann's distinction of mathematics-intensive careers and careers that are not mathematics-intensive (see quote above) that was not well founded in an analysis of the needs of various vocations and future careers in tertiary education, the latter presents a relevant point for a growing proportion of students.

These deficiencies, the partly irrational debate in public spheres, and discussions between mathematicians and mathematics educators all initiated a reaction by Winter that was simultaneously published in the newsletter of the German Mathematics Education Society (GDM) and the newsletter of the German Mathematics Union (DMV).

Here, he defined 'Allgemeinbildung' as competencies and knowledge that are essential to every human being as an individual and as a member of society independent of his/her gender, religion, (future) profession, etc. This definition could have been stated in similar terms by Humboldt and neohumanist authors two centuries ago.

Supposing this definition, the teaching of mathematics can and should provide 'three basic experiences'. The term 'experience' is important since it says that something is offered to the students, but it is up to the student what she or he makes of it. According to Winter, the learning of mathematics is more than simply storing knowledge. Thus, Allgemeinbildung has to be seen in terms of the self-development of the individual.

The three basic experiences are (all quotes from Winter are translated by the author of this chapter):

(1) to perceive and understand the phenomena of the world around us in nature, society and culture in a specific way,

(2) to get to know and to apprehend mathematical objects and facts represented using language, symbols, images or formulae as intellectual creations and as a deductively organised world of its own

(3) to acquire by working on tasks capabilities of problem-solving which go beyond mathematics (heuristic competencies). (Winter 1995, p. 37)

The term 'problem solving (experience 3) refers to activities that in the 19th century were discussed under the heading of 'formale Bildung'; the contribution of a discipline to developing the very capability of thinking. However, while at the beginning of the 19th century the core of productive thinking was seen in the faculty of judgement with all the connotations that Kant's fundamental book *Critique of Judgement* provided, the later understanding of 'formale Bildung' degenerated to simply meaning 'logical thinking'.

Experience (2) refers to mathematics as an autonomous subject and in a world of its own. Students should become aware that human beings are capable of both creating concepts and building whole architectures with them. Thus, experience (2) aims at pure mathematics as a deductive science, and we take explicit notice of the fact that Winter thinks this an indispensable dimension of mathematics in relation to Allgemeinbildung. Experience (2) is a significant point of departure from Heymann's approach.

Experience (1) refers to mathematics as a useful discipline, and this immediately leads to the question of an utilitarian or anti-utilitarian view of education. According to Winter, mathematics shows an almost infinite wealth of applications. However, he was quick to state that the utility of an application does not per se qualify it as a subject of Allgemeinbildung.

> Applications of mathematics are interesting and really indispensable for education only when a student can experience by them how mathematical modelling works. (Winter 1995, 38)

Thus, he returns to the concept of modelling as the core for qualifying applications as belonging to Allgemeinbildung. As expected, he adds the remark,

> Even the everyday applications, in spite of their realism, fail to contribute to Allgemeinbildung when their status of being a model is concealed and their context remains vague. (l.c., p. 38)

He exemplifies this latter statement by discussing that and how the topic 'calculation of interest' should be treated as a case of modelling.

> A desirable and actually necessary conception of "Civil Arithmetic" should also contain fundamental questions of population growth, pensions, insurance, and taxation, as a component of politically enlightening arithmetic (and not the arithmetic of an insurance salesman or a taxman). (Winter 1995, p. 38)

He then continues,

> Allgemeinbildung also comprises descriptive models of phenomena of the physical world insofar they are relevant to life and let experience in an exemplary way how mathematisations in technology and physics work and have been significant in the history of humankind. Above all, one should think of simple physical movements (throw, free fall, rotation, oscillation) including their causes and consequences. For example, the discovery of the law of falling bodies by Galileo in its historical context allows to experience paradigmatically: from a plausible hypothesis (velocity grows proportional to time) can be derived purely mathematical consequences whose interpretation illuminates phenomena which one would not observe with the naked eye and without mathematics. In general, successful mathematisations of a real phenomenon allows to look beyond the surface and substantially extends the everyday experience. (l.c., pp. 38/39)

Thus, it is mathematics that allows human beings to see beneath the surface of phenomena and detect deep structures which could not be uncovered otherwise. But to do so researchers must be able to play the game of hypotheses and deductions and to handle mathematics as a deductive science. The latter, however, is basic experience (2). Thus, both experiences (1) and (2) are inseparably intertwined.

In Winter's conception of Allgemeinbildung an application is not interesting, per se. Instead, the interactive and creative process of inventing hypotheses and drawing mathematical conclusions from them is the essential value for the intellectual development of young people. This, however, is only possible when students experience mathematics as an argumentative and deductive science. In Winter's word, students should experience that "rigorous science is possible". This is the essence of basic experience (2). Without a basic competence in pure mathematics, modelling cannot be understood and, of course, vice versa.

This general approach is different from Heymann, although we find similarities to Heymann's dimension that mathematics education has to contribute to the understanding of the world (and not 'just prepare for future life'). The relevance paradox (Niss 1994) is a major obstacle to overcome: mathematics contributes to many aspects of modern technology and society but remains hidden from the user who does not

need any advanced mathematical knowledge to operate cars, computers and smart-phones, for instance. So the relevance argument of mathematical content has to be much more sophisticated.

The examples that Winter and Heymann put forward include 'theoretical applications' (see Jahnke's contribution) from the sciences (modelling motions in physics, particularly planetary motions), and also 'hot societal topics' such as an understanding of environmental problems, quantitative aspects in equity issues, etc.

There is a consensus that widespread textbook problems that are solely made up for teaching mathematics and not for understanding the world outside mathematics, do not adequately contribute to this general goal. Mathematical modelling is considered a central notion in making the contribution of mathematics for the real world understandable, however, neither Winter nor Heymann advocates mathematical modelling only as a formal competence that can be developed at any example whatsoever. Authentic examples of 'the world' and reflection on the achievements and limitations of mathematical modelling are part of their approach. The theories of Allgemeinbildung, however, remain incomplete in that we cannot deduce which part of the world we should make more accessible to students by using mathematics.

The above judgment that Heymann's approach is not based on a philosophy of mathematics has to be relativised. We find analyses of mathematics under the heading of 'promoting cultural competence' (a verbatim translation from German would be 'maintaining cultural coherence'). Heymann discusses the 'cultural meaning of mathematics', the meaning of mathematics in wider culture and elaborates:

> Yet, in which way can mathematics teaching actually contribute at all to promoting cultural competence? My reply to this question is to be substantiated and explicated in this and the following sections: The decisive contribution of mathematics instruction to the promotion of cultural competence *is to allow for the specific universal features of mathematics and their significance for culture as a whole to be vividly experienced in an exemplary fashion on the basis of main ideas.* (Heymann 2003, p. 108, emphasis as original)

Heymann relates to the discussion on "fundamental ideas of a discipline" (Bruner 1960). The critical reception of New Math, which interpreted fundamental ideas in the sense of the Bourbakian view of mathematics as the science of structures, lead to various new attempts, especially in Germany, to identify and base fundamental ideas on a broader view of mathematics. This is an ongoing program. A most recent account and critical analysis of the development in German-speaking countries is the paper by Vohns (2016), who also puts Heymann's approach into perspective. We will now concentrate on Heymann's specific focus.

After a broad review of approaches to fundamental, central, or main ideas, Heymann develops the following criteria for selecting what he calls 'main ideas'. He wants to point out a difference to the 'fundamental ideas of a discipline', because he postulates that the main ideas cannot be determined exclusively by the discipline itself. He develops the following selection criteria for main ideas:

– with the main ideas for instruction oriented to general education, the universal features of mathematics should be expressed in a way comprehensible to students;
– they should be meaningful for various individual mathematical topics;

– they should be something other than simply basic mathematical concepts, i.e. they should not exclusively have a significance internal to mathematics;
– above all, they should demonstrate how mathematics is interrelated to other aspects of the culture of our society. (Heymann 2003, p. 109)

Applying these criteria to list of such ideas developed by others, he filtered out the following main ideas:

– the idea of number
– the idea of measuring
– the idea of structuring space
– the idea of functional relationship
– the idea of an algorithm
– the idea of mathematical modeling. (Heymann 2003, p. 124)

Heymann sketches what he regards as the meaning of these ideas in the broader culture. The approach stresses that a main idea cannot just be considered as a concept inside mathematics. The concept of function, for example, can be defined as a mapping between sets (especially sets of numbers). However, in the real world, we deal with and model functional relations of magnitudes. Magnitudes are no longer concepts of modern mathematics (which abandoned magnitudes in the 19th century). It indeed poses a challenge for mathematics education to deal with this difference, which is still a prevailing problem in mathematics education. Which concept of function and functional relationships should be developed in school? How can consistency with mathematics as a discipline be developed on one hand and the use of functions for modelling functional relationships (of magnitudes) on the other? An elaboration is necessary, albeit this is not the focus of Heymann's book.

Such an elaboration would have to take into account other approaches to this fundamental or main idea. For example, 'functional thinking' has been a main or fundamental idea in the history of mathematics education in German-speaking countries, associated with the Meran reform of mathematics education in 1905 and the name of Felix Klein (Krüger 2000), and it is not clear how this is taken into account. Klein stressed that functional thinking could relate mathematics to the broader culture and bridge between secondary and tertiary mathematics education. The following introduction of calculus into the Gymnasium curriculum was a lasting achievement.

The identified need for further elaboration applies to other main ideas as well, as, for example, 'structuring space' can be done on various levels, and the specific role of Euclidean deductive theory in this context remains unclear.

An obvious deficiency of Heymann's list is that probability and statistics, or data and chance, are not mentioned at all, although many other authors assign a most prominent role to these domains in any conception of mathematics education that intends to contribute to general education (but see Burrill and Biehler 2011 for an approach to fundamental ideas in statistics).

The list can also be criticised from a different perspective, namely with regard to the symbol systems that are characteristic of mathematics. When Winter (1975) analysed the role of mathematics in general education he pointed to mathematics as

a formal science. This is repeated by the second point in his 1995 paper, emphasising the specific sign systems that mathematics has developed to deal with its ideas and concepts, and that are relevant for general education. The role of symbol systems is characteristic of mathematics (Dörfler 2016). However, we can turn it around and regard this aspect of mathematics as part of its contribution to the culture at large. A good example of this emphasis is provided by Whitehead (1929) book where he justifies why students (in general) should learn to solve quadratic equations:

> Quadratic equations are part of algebra, and algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world. (ibid., p. 7)

Despite these critical remarks, we have to put Heymann's book into perspective. From his analysis of the normative function of mathematics for general education, he concluded that contemporary mathematics education had deficiencies in many respects. For instance, he argued for a more significant role of aspects of estimation and approximation of magnitudes, of interpretations with graphs and tables with data, simple forms of mathematical modelling, and the use of mathematics as a means of communication instead of using it only as a calculation tool. This is necessary because of changes in society and the living environment of students. Contributing to the understanding of the world requires mathematical modelling with authentic examples and reflection about the role of mathematics. The cultural coherence should be achieved by focusing on the main ideas of mathematics. The development of critical thinking, entailing a willingness to assume responsibility, practice in communication and cooperation requires a new, student-oriented, culture of teaching and learning.

Heymann became a consultant of the state government of North Rhine-Westphalia, where he influenced the emerging new curricula. In general, the public debate on mathematics education soon became weaker. It was the TIMSS and the PISA shock in the late 1990s and the beginning of the twenty-first century that had a more lasting effect on mathematics at school level.

6.1.4 Allgemeinbildung and Mathematical Literacy, Competence Orientation Since the Late 1990s Due to the TIMSS and PISA 'Shocks'

Mathematical Literacy and Competence Orientation
Since the German results in TIMSS and PISA, which were considered to be too much below average, mathematics and science education have become a broad political and public concern again, similarly to the big educational reforms in the 1960s. One of the differences to the late 1960s and the beginning of the 1970s was the specific focus on mathematics and science education, whereas, in the earlier 'crisis of education', the focus was the educational system in general.

The diagnosis that mathematics and science education had to be improved led to countless initiatives, development projects, and efforts in the professional development of mathematics and science teachers, which cannot be discussed here. 'Output orientation' became an influential concept, and since that time the expected outcome of school mathematics education regarding students' knowledge became more specified, and students' achievements vis-à-vis the newly specified output goals are being checked.

The notion of 'competencies' then became the royal way to specify the required outputs of the school system. Moreover, PISA also introduced its particular notions of 'mathematical literacy' and the notion of 'big ideas' into the German debate on mathematics education and initiated a discussion on how these notions are related to German conceptions of Allgemeinbildung and to traditional general goals for mathematics education at school level.

PISA is based on a conception of mathematical literacy. The following quotes provide the definition for the PISA 2000 and the PISA 2012 framework:

> Mathematical literacy is an individual's capacity to identify and understand the role mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen. (OECD 1999, p. 41)

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD 2013, p. 25)

On an abstract level, the PISA approach to mathematics in general education is surprisingly similar to that of Allgemeinbildung in that it stresses the function of mathematics education for the future life of students. In a first approximation, Allgemeinbildung is more general than mathematical literacy because it does not view the individual as just a citizen. A conception of Allgemeinbildung would probably include mathematical literacy. However, what exactly is a constructive, concerned, and reflective citizen in the sense of PISA? The answer to this question determines how the relation to conceptions of Allgemeinbildung with their different views of the subject can be analysed.

The concept of mathematical literacy is similarly complex and variable as Allgemeinbildung. Jablonka (2003) distinguishes different interpretations of mathematical literacy. Mathematical literacy may aim at developing the following aspects that she uses as section headings:

- Human Capital
- Cultural Identity
- Social Change
- Environmental Awareness
- Evaluating Mathematics.

She mainly identifies PISA's conception with the 'human capital interpretation', but being very general and abstract at the same time.

> Thus a conception of mathematical literacy as behaving mathematically - a definition not intrinsically related to the social community in which this behaviour is to be performed - may equally be underpinned by educational arguments advocating critical citizenship for participation in the public life of an economically advanced society as well as by workforce demands in underdeveloped countries. (Jablonka 2003, p. 81)

In PISA's meaning of the notion of mathematical literacy, two further aspects are relevant—mathematical competencies and big ideas:

> *Mathematical competencies* are general skills and competencies such as problem-solving, the use of mathematical language and mathematical modelling.

> *Mathematical big ideas* represent clusters of relevant, connected mathematical concepts that appear in real situations and contexts. Some of these big ideas are well established, such as chance, change and growth, dependency and relationships and shape. "Big ideas" are chosen because they do not result in the artificial approach of separating mathematics into different topics. (OECD 1999, p. 42)

The eight competencies are mathematical thinking skills, mathematical argumentation skills, modelling skills, problem posing and solving skills, representation skills, symbolic, formal and technical skills, communication skills, and aids and tool skills (ibid., p. 43). The six big ideas are chance, change and growth, space and shape, quantitative reasoning, uncertainty, dependency and relationships (ibid., p. 48). These aspects are important; of course, though the definition alone does not cover the meaning of 'mathematical literacy'.

If we compare this perspective to Heymann, we note that the *approach to identify big ideas* is similar to Heymann's identification of main ideas in that their contextual and cultural role is taken into account and they also cross boundaries of traditional curricular topics. However, PISA's resulting ideas are different. The analysis of competencies is based on a much broader and differentiated view of what constitutes mathematics as a scientific discipline in terms of mathematical activities than in Heymann's approach.

The aspect to "understand the role mathematics plays in the world" in the PISA definition is a reflective dimension. The phrasing in the PISA 2012 framework is weaker in that mathematical literacy should only 'assist' in judging the role mathematics plays in the world. PISA's test items also do not assess this understanding directly.

It is very interesting to compare the PISA approach to the KOM project (Niss and Højgaard 2011) that significantly influenced the competencies of PISA. In addition to the competencies, the authors state:

> The above-mentioned competencies are all characterised by being action orientated in that they are directed towards handling different types of challenging mathematical situations. Besides the mathematical mastery we have tried to capture with these competencies, we have also found it desirable to operate with types of "active insights" into the nature and role of mathematics in the world, and which are not directly behavioural in nature. (ibid., p. 74)

The authors distinguish and elaborate on three forms of 'overview and judgment': "The actual application of mathematics in other subject and practice areas"; "The historical development of mathematics, both internally and from a social point of view"; and "The nature of mathematics as a subject" (ibid., p. 75).

These aspects constitute one way to specify the meaning of "understanding the role mathematics plays in the world", but they are not systematically developed in the PISA framework. Competencies focus on behavioural aspects ('mathematical mastery') and do not explicitly cover reflective knowledge about mathematics and its cultural and societal role. The holistic view of developing individuals' personalities as is expressed in most conceptions of Allgemeinbildung is also more general than mathematical mastery.

On the other hand, it is unclear how these more general desirable outputs of education can be assessed. Advocates of competence orientation would argue that the general educational goals found in preambles of syllabi often do not succeed in effectively influencing the practice of mathematics education, which often focuses on technical mathematical skills. Competence orientation aims at a much broader spectrum of mathematical behaviours to be assessed, which is moving mathematics education in the direction of important behavioural parts of mathematical Allgemeinbildung without exhausting this notion.

6.1.5 Mathematical Literacy, Allgemeinbildung and National Standards for Mathematics in Germany

The designers of PISA never claimed that national curricula are validly assessed in their totality and regard their assessments as a kind of 'partial assessment'.

> The term literacy has been chosen to emphasise that mathematical knowledge and skills as defined within the traditional school mathematics curriculum do not constitute the primary focus of OECD/PISA. Instead, the emphasis is on mathematical knowledge put to functional use in a multitude of different contexts and a variety of ways that call for reflection and insight. (OECD 1999, p. 41)

It should also not be forgotten that the designers of PISA deliberately did not include important components of general mathematics education and mathematical Allgemeinbildung:

> Attitudes and emotions, such as self-confidence, curiosity, a feeling of interest and relevance, and a desire to do or understand things, to name but a few, are not components of the OECD/PISA definition of mathematical literacy but nevertheless are important prerequisites for it. (OECD 1999, p. 42)

However, the PISA results supported advocates who argued for strengthening the mathematical literacy aspect in German curricula, campaigning for a more prominent place for applications and mathematical modelling in the curriculum than before, but the influence was more general.

From its very beginnings, however—internationally, but especially in Germany—PISA also pursued a kind of meta-goal: to stimulate thinking about the objectives of the tested domains within an education system. This meta-goal was made more or less explicit, at least in the domain of mathematics, where the conceptualisation of the domain as "mathematical literacy" was a signal to the community of mathematics educators to restructure their thinking about how mathematics is addressed in schools, and how the outcomes of mathematics education should be evaluated. (Neubrand 2013, p. 39)

The influence PISA had on the German mathematics education on various levels is complex, and of course, some interactions with specific German conceptions of mathematical Allgemeinbildung and traditional ways of expressing curricular goals were sparked off.

It is remarkable that from the beginning of PISA testing in Germany, even in the German PISA team, it was clear that PISA's mathematical literacy covers only a part of the goals of mathematics education in general education. Thus, the advisory board of German mathematics educators for the PISA 2000 project created a supplementary German test, which was based on the notion of 'Mathematische Grundbildung', from which mathematical literacy is only a proper subset (Neubrand et al. 2001, 2004).

The paper by Neubrand (2003) characterises the relations and differences between mathematical literacy and 'Mathematische Grundbildung'. 'Mathematische Grundbildung' gives an independent value to mathematical techniques and conceptual thinking in mathematics. Neubrand points out that PISA's framework was influenced by Freudenthal, as quoted in OECD (1999, p. 41): "Our mathematical concepts, structures and ideas have been invented as tools to organise the phenomena of the physical, social and mental world" (Freudenthal 1983, p. ix). This also influenced the selection of the conceptual modelling in PISA. However, Freudenthal is also arguing for the reorganisation and constitution/construction of mathematical concepts as mental objects that give an independent value to mathematics itself. Neubrand gives credit to these features as well as to the notion of 'mathematical proficiency' (Kilpatrick 2001) and Schoenfeld's interpretation of quantitative literacy (Schoenfeld 2001) as akin to the broader German approach to 'Mathematische Grundbildung'. "The PISA definition for 'mathematical literacy'—in contrast—is more specific in that it explicitly includes the role of the citizen in (Western, developed) societies in its definition and in that it gives less emphasis to the abilities to structure and restructure within mathematics itself." (Neubrand 2003, p. 344. transl. R.B.)

Neubrand characterises the ambitions expressed in the notion of Allgemeinbildung as going far beyond 'Mathematische Grundbildung', referring to Winter (1995) and the educational philosophy of mathematics that was published in BLK (1997), the latter an expertise reaction to the poor TIMSS results in Germany. He summarises:

Applications are interesting and really indispensable only if it is experienced, how mathematical modelling is functioning, and which kind of elucidation can be achieved thereby. The model character of mathematical problem solutions should not be disguised or remain obscure […]

Each student should experience that human beings are capable of constructing concepts and whole architectures of them, or, differently put, that stringent science is possible. (Neubrand 2003, p. 345. transl. R.B.)

These aspects are of course very akin to the 'overview and judgment' aspects of the KOM project that were cited above. Fischer (2001) provides a specific elaboration and interpretation of these reflective dimensions for education up to the final grade 12/13 that concludes with the *Abitur* (see Vohns 2017a for more details). He postulates that students should have been educated as well-informed laypeople who can communicate with experts in modern society rather than doing expert work on a small scale themselves. This requires mathematical mastery and mathematical competencies, but this is not sufficient on its own and has to be supplemented by reflective knowledge rather than more 'operative knowledge'. Instead of gaining accessible modelling competencies in oversimplified situations, students should be able to critically question assumptions made in societal applications of mathematics in dialogues with experts.

Beyond this conceptual debate, the discussions among administrators, politicians, and mathematics educators following the PISA shock had enormous consequences on the level of curriculum standards. One of the most important results was the creation of national standards in mathematics for grade 9/10 in 2004 and for grade 12/13 (*Abitur*) in 2012 for the first time in the history of West Germany (KMK 2012, 2004). The standards specify the function of mathematics education in schools by specifying competencies that should be achieved by students. According to these standards, the mathematical content should be organised and structured around 'leading ideas', the latter approach constituting an influence by PISA, the NCTM standards, and the German debate on fundamental or central ideas for mathematics, which were discussed, among others, by Heymann.

The new national standards are called Bildungsstandards although the notion of Bildung or Allgemeinbildung as such is hardly mentioned in them (only the standards of 2012 quote Winter's (1995) three basic experiences). They define the goals of secondary mathematics education by a three-dimensional framework with the dimensions of *Leitideen* (leading ideas), (process oriented) mathematical competencies, and the level of complexity. The leading ideas are number (and algorithm), measurement, space and shape, functional relations, data and chance. The competencies are:

- Arguing mathematically
- Mathematical problem solving
- Mathematical modelling
- Use of mathematical representations
- Working with symbolic, formal and technical elements of mathematics
- Communicating (mathematically).

The standards are a result of a complex process of negotiations, which were of course influenced by PISA, the NCTM standards, and the German discussion on Allgemeinbildung.

Quoting Winter does not mean that the standards also share the detailed elaboration of his three experiences. A more concrete notion of Allgemeinbildung in the German tradition is not elaborated in the standards, nor can we identify a clear educational philosophy of mathematics. Competence orientation is focusing on the behavioural

side and not on the 'three forms of overview and judgment' that can be detected in the KOM project, for instance. This has been the object of criticism in the German community itself, arguing from various perspectives. We are unable to go into further details here.

6.1.6 Further Developments

The theoretical debate on Allgemeinbildung and mathematics education in Germany continues. The historical and actual discussion on fundamental ideas is well analysed in the paper by Vohns (2016). Moreover, Neubrand (2015) contributed a chapter in the German handbook on mathematics education on the foundations of mathematics education rooted in a theory of Bildung. He also argues for elaborating an educational philosophy of mathematics and compares the different approaches: Freudenthal's mathematics as an educational task, the approach from identifying fundamental ideas, general learning goals in the sense of Winter (1975), as well as the notions of mathematical literacy and mathematical proficiency. Winter's (1995) three experiences are considered as a synthesis, a challenge for future research that has to elaborate and fill these ideas. Using them as a superficial justification of current curricula and standards by just quoting the three experiences is something different and does not convey the critical stance that notions of Allgemeinbildung have always had in the further development of mathematics education.

6.2 Allgemeinbildung, Mathematical Competencies and Mathematical Literacy: Conflict or Compatibility? (Reaction by Mogens Niss)

6.2.1 The Concept of Allgemeinbildung

Let us begin by noting that the German word 'Allgemeinbildung' hardly has any suitable counterpart in English (neither 'general formation' nor 'general education' carries quite the same meaning but may serve as a first approximation). Moreover, the term is certainly used in Scandinavian languages ('almendannelse' in Danish and Norwegian, 'allmän bildning' in Swedish), but it seems that a corresponding notion doesn't really exist in other European languages. This should not be taken to mean that other languages and cultures do not nurture similar ideas, only that these haven't been coined into one short term with all the connotations of Allgemeinbildung, which, as Rolf Biehler and Hans Niels Jahnke have convincingly shown in their presentations, is a very rich and complex concept.

As I see it, three important dualities—not to be mistaken for dichotomies—generated by the notion of Allgemeinbildung, deserve further attention.

The first duality emerges from the fact that in educational contexts, the word 'allgemein' ('general' in English) can have two different targets. Either it can refer to the 'general population' in a given society, so that, in principle, all citizens constitute the intended subjects of formation or education. Or it can refer to the general nature of the 'substance' of the formation or education to be received by the members of the intended recipient group. Or—of course—allgemein might refer both to the population addressed and to the substance of formation/education at issue. In either case, any sensible discussion of the nature and role of Allgemeinbildung requires clarification of which of the possible targets are in focus. When it comes to the recipient population, it is not usually that clear whether this population is, in fact, meant to encompass literally all 'normal' citizens in society, and if not, who should then be included or excluded, respectively, as recipients? If instead, we are focusing on the generality of the substance of formation/education, many issues need further clarification. What exactly is it that is meant to be allgemein regarding substance? Is it substance that is supposed to be common to all domains of knowledge? Is it substance that is considered universally useful or valuable in the lives of every member of the intended category of recipients? Is it substance that underpins our understanding of the fundamentals of the world? Is it substance of an overarching (meta) nature, above and beyond scientific and scholarly disciplines? Is it substance that deals with the formation of the moral, mental, intellectual and aesthetic capacities of the individual, his or her character? Or is it…?

The second duality occurs in the case where the target of Allgemeinbildung is substance pertaining to formation/education. The question then arises of whether this substance is primarily defined in terms of *content*, i.e. what people should *know* and understand and the ways in which they should *do* so, or whether it primarily involves *processes*, i.e. what people should be able to *do* with their knowledge, and in what contexts, circumstances and situations.

The third duality has to do with the ultimate purpose of Allgemeinbildung. For whose sake should it be pursued? For the personal benefit of the individual, so that he or she can thrive and develop as a person in the world and surroundings in which he or she lives? Or for the sake of the community or society at large, which is supposed to benefit from having several knowledgeable, thoughtful, as well as intellectually, morally and aesthetically cultured citizens?

As I see it, we haven't really specified what we mean by Allgemeinbildung before we have specified how to take a stance related to each of these dualities. My own position—however, I certainly realise that others are possible and defendable—is that Allgemeinbildung should have the vast majority of citizens as its population target, not just a small elite, 'the happy few', and that its substance should be focused on the fundamentals of our understanding of nature, culture and society and on the ways this understanding has come into being and has developed and grown, whilst involving analytic and critical perspectives on this understanding and its outcomes, especially with regard to what it means and takes to know something. I am more sceptical about the possibility of generating general—content and context-free—intellectual and moral faculties that go beyond the basics of logical reasoning and appreciation of universal human rights. Finally, as regards the purpose of Allgemeinbildung, I

emphasise the need for society to consist of *allgemeingebildete* citizens, who are able and willing to engage in discussions and activities that can foster the development of a just, equal, free, humanistic, sustainable and democratic society, in which it is not the case that "few have too much and fewer too little" (Grundtvig 1820).

Now, in most definitions and conceptualisations of Allgemeinbildung it is a point in itself that no disciplines or school subjects, mathematics included, are referred to in the conceptualisation. What then, does Allgemeinbildung have to do with mathematics education? Well, as I perceive it, mathematics does play a crucial role in several of the aspects mentioned above, simply because mathematics permeates the fundamentals of our understanding of nature, culture and society, as it does with logical and formal reasoning. Hence, in my view, mathematics should enjoy 'civil rights' as a key component of Allgemeinbildung. Conversely, Allgemeinbildung is indeed of relevance in the context of mathematics education by offering general formative and educational perspectives to its pursuit.

6.2.2 False or Genuine Dichotomies?

I fully share Rolf Biehler's and Niels Jahnke's insistence (Chap. 5 of this book) on the relevance, value and necessity of Allgemeinbildung also in today's societies and education systems. It goes without saying—even though Rolf Biehler actually says it—that this requires continuous updating of our understanding of the concept in order to relate it to the economic, technological, cultural, ideological and political developments that our societies undergo. The original point of departure of Allgemeinbildung in humanistic ideals based on classical languages, literature, philosophy and art in antiquity, as the prototypical point at infinity setting the standards for our formative and educational endeavors, is no longer adequate or sufficient, despite the indisputable value of the intellectual, societal and artistic accomplishments of antiquity.

Today, it seems that some modern defenders and active supporters of Allgemeinbildung (amongst whom I count myself) see that there are antagonistic relationships between Allgemeinbildung and a number of other ideas and notions that have been put forward and have gained momentum during the last two to three decades. More specifically, some establish a contradiction between Allgemeinbildung and utilitarianism, others between Allgemeinbildung and (mathematical) literacy, and still others between Allgemeinbildung and (mathematical) competencies.

In what follows, I shall argue that there is indeed a dichotomy between Allgemeinbildung and utilitarianism, at least if utilitarianism is understood in its traditional—rather narrow—sense. I shall further argue that there is no contradiction between Allgemeinbildung and competencies and literacy, respectively. On the contrary, they are highly compatible, albeit not identical.

The everyday, non-philosophical (as with Mill 1863) understanding of utilitarianism focuses on the practical utility (usefulness) of objects, processes and undertakings for life, work, occupations and professions, business and industry, technology, econ-

omy, infrastructure, and the running of society, etc. More often than not, such utility is required to be rather direct (i.e. displaying clear causality) and effective within a relative short time span. It was on the basis of this narrow understanding of utility that Heymann (1996) in Germany was misinterpreted by the public when he made his famous claim that seven years of school mathematics would be enough if we only considered direct applications in everyday life (see Sect. 6.1.3).

Irrespective of which specific interpretation of Allgemeinbildung one adheres to, it is pretty obvious that it cannot be reduced to utilitarianism in the sense just outlined. By focusing on complex insights and reflectiveness going far beyond the needs of the day, both in scope and in time, the perspectives offered by Allgemeinbildung are entirely different from those of utilitarianism. To be sure, Allgemeinbildung implies no discarding or downgrading of everyday utility—that would be insane—it just insists that there is much more to be said about and done for individual and communal life in culture and society than just pursuing direct and short-term usefulness.

If, however, utilitarianism is given a much broader meaning than everyday utility, such as to comprise the fostering and furthering of a balanced and inclusive development of culture, science, art, technology, society, and democracy, Allgemeinbildung is eminently compatible with utilitarianism. As a matter of fact, one might go as far as to say that the ultimate purpose of Allgemeinbildung is to be utilitarian in this much wider sense.

Mathematical *competencies* are to do with the enactment, practice and exercise of mathematics, i.e. doing mathematics. Even if this does indeed presuppose a lot of content knowledge and theoretical understanding of the edifice of mathematics, mathematical competencies go beyond such knowledge by being action-orientated. Since Allgemeinbildung in almost any conceptualisation of it places emphasis on knowledge and understanding in their own right, it follows that mathematical competencies and Allgemeinbildung are not identical, nor is one a subset of the other. They have different foci. However, they are by no means incompatible let alone contradictory; on the contrary, they complement each other. Moreover, as mathematical competencies were conceived as a way of liberating the enactment of mathematics from specific mathematical topics and specific educational levels or settings, mathematical competencies are meant to be of a general nature in analogy with the way in which Allgemeinbildung is intended to be of a general nature transgressing specific disciplines, educational levels and contexts, vocations and professions. So, the fact that there are indeed significant distinctions between competencies and Allgemeinbildung does not at all imply that these notions are antagonistic. That is simply a false dichotomy.

In much the same way, there is certainly no antagonistic relationship between Allgemeinbildung and mathematical literacy. Mathematical literacy is to do with individuals' ability to put mathematics to functional use in extra- and intra-mathematical contexts and situations that are of significance to the individuals' actual and future lives as active, concerned and reflective citizens. One might well claim that this ought to be an element of Allgemeinbildung, but even if it isn't accepted as such an element, there is indeed no contradiction between the two. They will then simply have different foci and emphases. Moreover, by its very definition, mathematical literacy

goes far beyond narrow utilitarianism as outlined above. By the way, mathematical competencies and mathematical literacy are not identical either. Mathematical competencies do in fact underpin mathematical literacy, but mathematical competencies are much more than mathematical literacy (Niss 2015). Once again, a dichotomy between Allgemeinbildung and mathematical literacy is yet another false dichotomy.

6.2.3   Conclusion

I very much agree with those—including Rolf Biehler and Niels Jahnke—who are making a strong case for the importance of Allgemeinbildung, both in general and in the context of mathematics education. This, however, requires a clear conceptualisation of the notion of Allgemeinbildung. As I see it, dichotomies between Allgemeinbildung and mathematical competencies, respectively, are nothing but false dichotomies and hence should be abandoned.

References

Biehler, R. (1994). History and epistemology of mathematics and mathematics education. Introduction. In R. Biehler, R. W. Scholz, R. Straesser, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 327–333). Dordrecht: Kluwer.

Biehler, R., Heymann, H. W., & Winkelmann, B. (Eds.). (1995). *Mathematik allgemeinbildend unterrichten: Impulse für Lehrerbildung und Schule*. Köln: Aulis.

Biehler, R., & Jahnke, H. N. (Eds.). (1997). *Mathematische Allgemeinbildung in der Kontroverse—Materialien eines Symposiums am 24. Juni 1996 im ZiF der Universität Bielefeld*. IDM Occasional Paper 163. Bielefeld: Universität Bielefeld.

BLK. (1997). Gutachten zur Vorbereitung des Programms „Steigerung der Effizienz desmathematischnaturwissenschaftlichen Unterrichts". Bonn: Bund-Länder-Kommission für Bildungsplanung und Forschungsförderung.

Bloom, B. S. (1956). *Taxonomy of educational objectives; The classification of educational goals* (1st ed.). New York: Longmans, Green.

Bosse, D. (Ed.). (2009). *Gymnasiale Bildung zwischen Kompetenzorientierung und Kulturarbeit*. Wiesbaden: VS Verlag für Sozialwissenschaften.

Bruner, J. S. (1960). *The process of education*. Cambridge: Harvard University Press.

Burrill, G., & Biehler, R. (2011). Fundamental statistical ideas in the school curriculum and in training teachers. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics-challenges for teaching and teacher education—A joint ICMI/IASE study: The 18th ICMI study* (pp. 57–69). Dordrecht: Springer.

Damerow, P., Elwitz, U., Keitel, C., & Zimmer, J. (1974). *Elementarmathematik Lernen für die Praxis? Ein exemplarischer Versuch zur Bestimmung fachüberschreitender Curriculumziele*. Stuttgart: Klett.

Dohmen, G. (1964). *Bildung und Schule. Die Entstehung des deutschen Bildungsbegriffs und die Entwicklung seines Verhältnisses zur Schule. Bd. 1: Der religiöse und der organologische Bildungsbegriff*. Weinheim: Beltz.

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, R. Bruder, O. Schmidt, & W. Dörfler (Eds.), *Theories in and of mathematics education* (pp. 21–31). Springer.

Fischer, R. (2001). Höhere Allgemeinbildung. In A. Fischer, A. Fischer-Buck, K.-H. Schäfer, & D. Zöllner (Eds.), *Situation—Ursprung der Bildung. Franz-Fischer-Jahrbuch der Philosophie und Pädagogik* (Vol. 6, pp. 151–161). Leipzig: Universitätsverlag.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Reidel.

Gagné, R. M. (1970). *The conditions of learning* (2nd ed.). New York: Holt.

Gagné, R. M., & Briggs, L. J. (1974). *Principles of instructional design*. New York: Holt.

Gates, P., & Vistro-Yu, C. P. (2003). Is mathematics for all? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 31–73). Dordrecht: Springer, The Netherlands.

Grundtvig, N. F. S. (1820). Lang højere bjerge så vide på jord [poem]. In R. Nyerup (Ed.) Sange til den 10de April 1820, til Trykken befordrede af R. Nyerup. Copenhagen.

Heymann, H. W. (1996). *Allgemeinbildung und Mathematik*. Weinheim: Beltz.

Heymann, H. W. (2003). *Why teach mathematics? A focus on general education*. Dordrecht, Boston: Kluwer Academic Publishers.

Horlacher, R. (2004). Bildung—A construction of a history of philosophy of education. *Studies in Philosophy and Education, 23*(5), 409–426. https://doi.org/10.1007/s11217-004-4452-1.

Horlacher, R. (2016). *The educated subject and the German concept of Bildung. A comparative cultural history*. New York: Routledge Taylor & Francis Group.

Huber, L. (2009). Von „basalen Fähigkeiten" bis „vertiefte Allgemeinbildung": Was sollen Abiturientinnen und Abiturienten für das Studium mitbringen? In D. Bosse (Ed.), *Gymnasiale Bildung zwischen Kompetenzorientierung und Kulturarbeit* (pp. 107–124). Wiesbaden: VS Verlag für Sozialwissenschaften.

Jablonka, E. (2003). Mathematical literacy. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 75–102). Dordrecht: Kluwer.

Jahnke, H. N. (1990). *Mathematik und Bildung in der Humboldtschen Reform*. Göttingen: Vandenhoeck & Ruprecht.

Kilpatrick, J. (2001). Understanding mathematical literacy: The contribution of research. *Educational Studies in Mathematics, 47*(1), 101–116. https://doi.org/10.1023/a:1017973827514.

Klafki, W. (1963). *Studien zur Bildungstheorie und Didaktik*. Weinheim: Beltz.

Klafki, W. (1974). *Didaktische Analyse* (11th ed.). Hannover: Schroedel.

Klafki, W. (1995). Didactic analysis as the core of preparation of instruction. *Journal of Curriculum Studies, 27*(1), 13–30.

Klafki, W. (2000). The significance of classical theories of Bildung for a contemporary concept of Allgemeinbildung. In I. Westbury, S. Hopmann, & K. Riquarts (Eds.), *Teaching as a reflective practice. The German Didaktik tradition* (pp. 85–107). Mahwah, NJ: Erlbaum.

KMK. (2012). *Bildungsstandards im Fach Mathematik für die Allgemeine Hochschulreife* (*Beschluss der Kultusministerkonferenz vom 18.10.2012*). https://www.kmk.org/fileadmin/veroeffentlichungen_beschluesse/2012/2012_10_18-Bildungsstandards-Mathe-Abi.pdf.

KMK. (2004). *Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss—Beschluss der Kultusministerkonferenz vom 4.12.2003*. München: Wolters Kluwer.

Krüger, K. (2000). *Erziehung zum funktionalen Denken—Zur Begriffsgeschichte eines didaktischen Prinzips*. Berlin: Logos Verlag.

Lenné, H. (1969). *Analyse der Mathematikdidaktik in Deutschland*. Stuttgart: Klett.

Malecki, A. (2016). *Schulen auf einen Blick. Ausgabe 2016*. Wiesbaden: Statistisches Bundesamt.

Mill, J. S. (1863). *Utilitarianism* (1st ed.). London: Parker, Son & Bourn, West Strand.

Neubrand, M. (2003). "Mathematical literacy"/„Mathematische Grundbildung". *Zeitschrift für Erziehungswissenschaft, 6*(3), 338–356. https://doi.org/10.1007/s11618-003-0039-9.

Neubrand, M. (2013). PISA mathematics in Germany: Extending the conceptual framework to enable a more differentiated assessment. In M. Prenzel, M. Kobarg, K. Schöps, & Rönnebeck (Eds.), *Research on PISA: Research Outcomes of the PISA Research Conference 2009* (pp. 39–49). Dordrecht: Springer.

Neubrand, M. (2015). Bildungstheoretische Grundlagen des Mathematikunterrichts. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch der Mathematikdidaktik* (pp. 51–73). Berlin, Heidelberg: Springer.

Neubrand, M., Biehler, R., Blum, W., Cohors-Fresenborg, Flade, L., Knoche, N., et al. (2001). Grundlagen der Ergänzung des internationalen PISA-Mathematik-Tests in der deutschen Zusatzerhebung. *ZDM, 33*(2), 45–59.

Neubrand, M., Biehler, R., Blum, W., Cohors-Fresenborg, E., Flade, L., Knoche, N., et al. (2004). Der Prozess der Itementwicklung bei der nationalen Ergänzungsuntersuchung von PISA 2000: Vom theoretischen Rahmen zu den konkreten Aufgaben. In M. Neubrand (Ed.), *Mathematische Kompetenzen von Schülerinnen und Schülern in Deutschland: Vertiefende Analysen im Rahmen von PISA-2000* (pp. 31–50). Wiesbaden: VS—Verlag für Sozialwissenschaften.

Niss, M. (1994). Mathematics in Society. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 367–378). Dordrecht: Kluwer.

Niss, M. (2015). Mathematical literacy. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 409–414). Heidelberg, New York, Dordrecht, London: Springer

Niss, M., & Højgaard, T. (Eds.). (2011). *Competencies and mathematical learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark*. Roskilde: Roskilde University, IMFUFA.

OECD. (1999). *Measuring student knowledge and skills a new framework for assessment*. Paris: OECD Publishing.

OECD. (2013). *PISA 2012 assessment and analytical framework. Mathematics, reading, science, problem solving and financial literacy.* OECD Publishing.

Robinsohn, S. B. (1969a). *Bildungsreform als revision des curriculum* (2nd ed.). Neuwied: Luchterhand.

Robinsohn, S. B. (1969b). Educational reform through curriculum revision. *Western European Education, 1*(1), 20–29. https://doi.org/10.2753/eue1056-4934010120.

Schoenfeld, A. H. (2001). Reflections on an impoverished education. In L. A. Steen (Ed.), *Mathematics and democracy: The case for quantitative literacy* (pp. 49–54). Princeton: NCED.

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer.

Thom, R. (1973). Modern mathematics: Does it exist? In A. G. Howson (Ed.), *Developments in mathematical education* (pp. 194–209). Cambridge: Cambridge University Press.

Vohns, A. (2016). Fundamental ideas as a guiding category in mathematics education—Early understandings, developments in German-speaking countries and relations to subject matter didactics. *Journal für Mathematik-Didaktik, 37*(Suppl 1), 193–223. https://doi.org/10.1007/s13138-016-0086-4.

Vohns, A. (2017a). Bildung, mathematical literacy and civic education: The (strange?) case of contemporary Austria and Germany. In A. Chronaki (Ed.), *Mathematics Education and Life at Times of Crisis. Proceedings of the Ninth International Mathematics Education and Society Conference* (Vol. 2, pp. 968–978). Volos: University of Thessaly Press.

Vohns, A. (2017b). Einkommensteuertarife mit Schulmathematik erkunden—Ein Zugang aus bildungstheoretischer. Perspektive. *Der Mathematikunterricht, 63*(3), 25–35.

Vollrath, H.-J. (2007). Towards an authentic teaching of mathematics: Hans-Georg Steiner's contribution to the reform of mathematics teaching. *ZDM Mathematics Education, 39*(1), 39–50. https://doi.org/10.1007/s11858-006-0013-0.

Wagenschein, M. (1965). *Ursprüngliches Verstehen und exaktes Denken* (1st ed.). Stuttgart: Klett.

Whitehead, A. N. (1929). *The aims of education & other essays*. New York: The Macmillan Company.

Winter, H. (1975). Allgemeine Lernziele für den Mathematikunterricht? *Zentralblatt für Didaktik der Mathematik, 7,* 106–116.

Winter, H. (1990). Bürger und Mathematik. *Zentralblatt der Didaktik der Mathematik (ZDM), 22*(4), 106–116.

Winter, H. (1995). Mathematikunterricht und Allgemeinbildung. *Mitteilungen der Gesellschaft für Didaktik der Mathematik, 61,* 37–46.

Wittenberg, A. I. (1963). *Bildung und Mathematik Mathematik als exemplarisches Gymnasialfach* (2nd ed.). Stuttgart: Klett.