On Optimal Selection of Coefficients of Path Following Controller for a Wheeled Robot with Constrained Control

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Abstract. Stabilization of motion of a wheeled robot with constrained control resource by means of a continuous feedback linearizing the closed-loop system in a neighborhood of the target path is considered. The problem of selection of the feedback coefficients is set and discussed. In the case of a straight target path, the desired feedback coefficients are defined to be those that result in the partition of the phase plane into two invariant sets of the nonlinear closed-loop system while ensuring the greatest asymptotic rate of the deviation decrease. A hybrid control law is proposed that ensures the desired properties of the phase portrait and minimal overshooting and is stable to noise. The proposed techniques are extended to the case of circular target paths.

Keywords: Wheeled robot · Path following problem
Saturated control · Optimal feedback coefficients

1 Introduction

There exist many applications (for example, in agriculture [1,2]) where a vehicle is to be driven along some target path with a high level of accuracy. Such tasks are performed by automatic vehicles (further referred to as *wheeled robots* (WRs), or simply *robots*) equipped with navigational and inertial tools and satellite antennas. The problem of bringing the robot from an initial state to a preassigned target path and stabilizing its motion along the path is called *path stabilization problem* (or *path following problem*); it was discussed in a great number of publications. Various models (e.g., monocycle, simple car, car-like model with and without drive actuator, tractor with trailers, etc.) and target curves (e.g., straight lines, circles, general-form curvilinear paths) were considered (see, e.g., [1–7] and references therein).

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One of the commonly accepted approaches frequently and successfully used for solving the path stabilization problem is based on the exact linearization of the equations of motion in a part of variables by applying an appropriate nonlinear feedback (partial feedback linearization) [1–5]. In the framework of this approach, the original affine system is transformed into a normal form, from which the desired feedback linearizing the system in stabilizable variables is easily found. If the vehicle speed varies in time, in order to transform to the normal form, it is required first to change the independent variable. By applying different time scales, one can obtain different normal forms and, accordingly, different feedbacks. A detailed discussion of this issue, as well as comparison of different linearizing feedbacks can be found in [3].

The presence of control constraints makes the path stabilization problem much more complicated since it cannot be linearized in the entire state space. One of the commonly accepted techniques, in this case, is to use the saturated linearizing feedback, which yields linearity only in the vicinity of the target path. The goal of this study is to find out how to select coefficients of the saturated linearizing feedback to improve the efficiency of stabilization when the control resource is bounded.

2 Problem Statement

We consider the *kinematic model* of a wheeled robot, which describes the simplest vehicle moving without lateral slippage with two rear driving wheels and front wheels responsible for steering the platform. In the planar case, the robot position is described by two coordinates (x_c, y_c) of some point of the platform, the so-called *target point,* and one angle describing the orientation of the platform with respect to a fixed reference system Oxy. For the target point, the point located in the middle of the rear axle is taken, and for the angle, the angle θ between the central line of the platform (which coincides with the direction of the velocity vector) and the x-axis. The kinematic equations of such a robot are well known to be (see, for example, [1–7])

$$\begin{aligned} \dot{x}_c &= v\cos\theta, \\ \dot{y}_c &= v\sin\theta, \\ \dot{\theta} &= v\tan\phi/L. \end{aligned} \tag{1}$$

Here, the dot over a symbol denotes differentiation with respect to time, $v \equiv v(t)$ is a scalar linear velocity of the target point, and L is the distance between the front and rear axles. The vehicle is controlled by turning the front wheels through an angle ϕ, $|\phi| \le \phi_{\max} < \pi/2$. Since the angle ϕ in the above range and the instant curvature u of the curve described by the target point are related by the one-to-one relationship $u = \tan\phi/L$, it is convenient to take u to be the control variable, which satisfies the two-sided constraints

$$-\bar{u} \le u \le \bar{u}, \tag{2}$$

where $\bar{u} = \tan\phi_{\max}/L$ is the maximal possible curvature of the actual trajectory described by the target point.

In the path stabilization problem, it is required to synthesize a control law u that brings the robot to a given target path and stabilizes its motion along the curve. The target path is given in a parametric form by a pair of functions $(X(s), Y(s))$, where s is a natural parameter (arc length), and is assumed to be feasible. The latter means that the functions $X(s)$ and $Y(s)$ are twice differentiable [4] everywhere except for a finite number of points (and, hence, curvature $k(s)$ of the target curve is a piecewise continuous function), and the maximum curvature $\bar{k} = \max_s k(s)$ of the target path satisfies the constraint $\bar{k} < \bar{u}$.

It has been shown in [5] that, by changing state variables and applying time scaling, the path following problem can be written in the canonical form [3], as the problem of finding a feedback that stabilizes the zero solution of the system

$$z_1' = z_2,\ z_2' = (1+z_2^2)^{3/2}u - \frac{k(1+z_2^2)}{1-kz_1}. \tag{3}$$

In (3), z_1 is the deviation of the robot from the target path, $z_2 = \tan\psi$, where ψ is the angle between the direction of the velocity vector and the tangent line to the target curve at the point closest to the robot, and the prime denotes differentiation with respect to the new independent variable ξ, which satisfies the equation $\dot{\xi} = v\cos\psi$.

If the control resource is not bounded ($\bar{u} = \infty$), then closing system (3) by the feedback

$$u = -\frac{\sigma(z)}{(1+z_2^2)^{3/2}} + \frac{k}{\sqrt{1+z_2^2}(1-kz_1)}, \tag{4}$$

where $z = [z_1, z_2]^{\mathrm{T}}$ and $\sigma(z)$ is a linear function with positive coefficients, we obtain the linear system

$$z_1' = z_2,\ z_2' = -\sigma(z), \tag{5}$$

the zero solution of which is globally asymptotically stable. For convenience of calculations, without loss of generality, we will represent function $\sigma(z)$ in the form

$$\sigma(z) = \lambda^2 z_1 + 2\lambda\gamma z_2,\ \lambda > 0, \gamma > 0. \tag{6}$$

In the case of the constrained control resource, applying the saturation function to the linearizing control law, i.e., selecting the feedback in the form

$$u = -\mathrm{sat}_{\bar{u}}\left(\frac{\sigma(z)}{(1+z_2^2)^{3/2}} - \frac{k}{\sqrt{1+z_2^2}(1-kz_1)}\right), \tag{7}$$

we get a hybrid system given by the linear Eq. (5) in the set where $|u| < \bar{u}$ and by the nonlinear equations

$$z_1' = z_2,\ z_2' = -\frac{k(1+z_2^2)}{1-kz_1} - \mathrm{sign}(\sigma(z))(1+z_2^2)^{3/2}\bar{u} \tag{8}$$

in the set where the control reaches saturation.

As can be seen, the properties of the system under study are determined by the four parameters: the feedback coefficients λ and γ, path curvature k, and the control resource $\bar{u}$. Let us turn to an equivalent system of equations in dimensionless variables, which will allow us to get rid of one parameter. It is easy to verify that all above equations are invariant with respect to the transformation

$$\tilde{u} = u/\bar{u},\ \tilde{\xi} = \xi\bar{u},\ \tilde{z}_1 = z_1\bar{u},\ \tilde{z}_2 = z_2,\ \tilde{\lambda} = \lambda/\bar{u},\ \tilde{\gamma} = \gamma,\ \tilde{k} = k/\bar{u}. \tag{9}$$

Thus, study of the behavior of an arbitrary WR reduces to studying a dimensionless WR with the control resource equal to one. To simplify subsequent calculations and formulas, we will use the same notation (without tilde) as in the dimensional case to denote dimensionless quantities and parameters. Thus, all above equations remain valid for the dimensionless variables by setting $\bar{u} = 1$ in Eqs. (7) and (8).

Clearly, the efficiency of the stabilization directly depends on the coefficients of the linear function $\sigma(z)$. However, the author failed to find any publications on wheeled robots where the problem of finding optimal (in one or another sense) feedback coefficients is solved or even posed. In works [1,2,5], when discussing the practical implementation of the proposed linearizing feedbacks, the values of the coefficients are either set arbitrary (in numerical experiments with WR models) or selected experimentally (when the case in point are real automated vehicles). For example, in [1,2], to control a farm tractor, the authors used a linearizing feedback depending on one parameter, the value of which was selected experimentally from the condition that stabilization in a strip of width 15 m is achieved without reaching the control constraint (without getting into the saturation mode). Such an approach is not only badly justified but also is too cumbersome, since the results obtained for one WR cannot be used for another WR with different geometric characteristics and/or different control resource, which brings us to the necessity of development of a theoretically justified approach to solving the problem of selection of the feedback coefficients.

In the particular case of a straight target path and one-parameter family of the coefficients with the fixed $\gamma = 1$ (in this case, system (5) has one repeated pole $-\lambda$), this problem was studied in [8]. The desired value of the parameter was defined in [8] as the greatest λ for which there exists a partition of the phase plane into two invariant half-planes (i.e., any trajectory of the closed-loop system completely lies in one of the half-planes). The existence of two invariant half-spaces implies that the phase portrait of the nonlinear system (3), (7) is topologically equivalent to the phase portrait of the linear system (5) in the entire plane R^2 (rather than in a neighborhood of the origin). It was proved that the optimal in this sense value is $\lambda_{\text{opt}} = 3\sqrt{3}\bar{u}/2$ and that the phase plane is partitioned into the desired half-planes by the asymptote $z_2 = -\lambda_{\text{opt}} z_1$.

In this paper, we study the more general case where the roots of the characteristic polynomial of the linear system (5) are different. First, we pose the same problem as in [8]:

Problem 1. Determine feedback coefficients for which (i) there exists a partition of the phase plane into two invariant sets and (ii) the asymptotic rate of approaching the target path is as high as possible.

We will also study how the results obtained for the straight paths can be extended to the case of circular paths.

It should be emphasized that the above criterion of the selection of the feedback coefficients is quite natural. The fulfillment of this criterion means that the WR approaches the target path in a non-oscillatory way: the trajectory of the WR intersects the target curve at most once.

To get an idea of how the feedback coefficients affect system behavior, we first consider the phase portrait of the linear system (5), which governs the closed-loop system behavior when the control resource is unbounded.

3 Phase Portrait of the Linear System

The roots of the characteristic equation $\mu^2 + 2\gamma\lambda\mu + \lambda^2 = 0$ of the linear system (5) are easily found to be

$$\mu_{1,2} = -\lambda_{1,2},\ \lambda_1 = \gamma\lambda(1 - \sqrt{1 - 1/\gamma^2}),\ \lambda_2 = \gamma\lambda(1 + \sqrt{1 - 1/\gamma^2}). \tag{10}$$

For $\gamma \geq 1$, the roots are real negative numbers and the linear system has a stable node at the origin. If $\gamma < 1$, μ_1 and μ_2 are complex conjugate numbers, and $z = 0$ is a focus of the linear system. Clearly, in the latter case, no entire trajectory of the system can lie from the one side of a straight line passing through the origin; i.e., the above-formulated criterion certainly cannot be satisfied. Therefore, in what follows, we assume that $\gamma \geq 1$. Note that, for $\gamma = 1$, we arrive at the case of a degenerate node (repeated root $\mu_{1,2} = -\lambda$) considered in [8].

A typical phase portrait of a system with a stable node is shown in Fig. 1. Here, $\gamma = 1.1$, $\lambda \approx 4.0$, $\lambda_1 \approx 2.6$, and $\lambda_2 \approx 6.3$. The system has two eigenvectors collinear to the straight lines $z_2 = -\lambda_1 z_1$ and $z_2 = -\lambda_2 z_1$. Any system trajectory, except those beginning at points on the straight line corresponding to the larger eigenvalue (λ_2), touches the asymptote $z_2 = -\lambda_1 z_1$ at the origin. With regard to (10), the equation of the asymptote can be written as $z_2 + \gamma\lambda(1 - \sqrt{1 - 1/\gamma^2})z_1 = 0$. Multiplying this equation by $\gamma\lambda(1 + \sqrt{1 - 1/\gamma^2})$, we obtain the asymptote equation in the form

$$\gamma\lambda(1 + \sqrt{1 - 1/\gamma^2})z_2 + \lambda^2 z_1 = 0. \tag{11}$$

The asymptote divides the phase plane into two half-planes A_- (below the asymptote) and A_+ (above the asymptote), where the left-hand side of (11) is less or greater than zero, respectively. Clearly, A_- and A_+ are invariant sets of system (5), i.e., any trajectory completely lies in one of these half-planes and may intersect the target path not more than once. The deviation decreases exponentially with the exponent equal to the lesser eigenvalue $\mu_1 = -\lambda_1$.

4 Stabilization of a Robot with Constrained Control Along a Straight Path

The system with a constrained control ceases being linear when it comes to the "saturation" region, the set where the inequality

$$|\sigma(z)| \geq (1 + z_2^2)^{3/2} \tag{12}$$

holds. Clearly, the saturation region is a disconnected set consisting of two non-intersecting sets lying from both sides of the straight line $\sigma(z) = 0$. It is easy to see that the system moves along an integral curve in the direction of increasing (decreasing) variable z_2 in the left (right) saturation region.

Any trajectory of the nonlinear system (3), (7) completely lies in the domain A_- or A_+ if and only if the asymptote $z_2 = -\lambda z_1$ does not intersect the saturation regions, since a system trajectory can intersect the asymptote only in the saturation region (where the system is nonlinear). Let us find conditions the fulfillment of which guarantees that the asymptote does not intersect the saturation region. In view of symmetry, it will suffice to consider one (say, left) component of the saturation region. Let us rewrite inequality (12) holding in the saturation region as

$$\lambda^2 z_1 + \lambda\gamma z_2 \leq -(1 + z_2^2)^{3/2} - \lambda\gamma z_2.$$

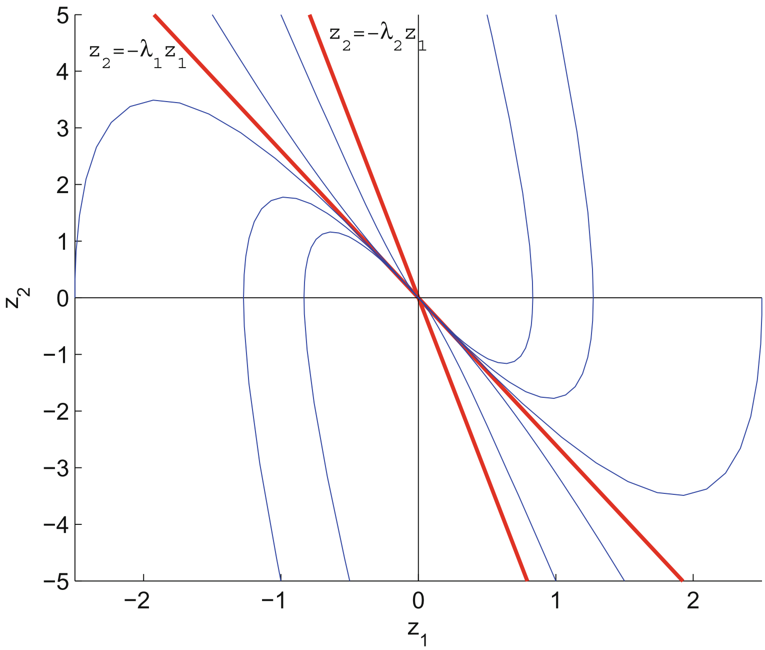

Fig. 1. Phase portrait of a linear system with stable node at the origin.

Adding $\gamma\lambda\sqrt{1-1/\gamma^2}z_2$ to the both sides of this inequality, we obtain

$$\lambda^2 z_1 + \lambda\gamma(1+\sqrt{1-1/\gamma^2})z_2 \le -(1+z_2^2)^{3/2} - \lambda\gamma(1-\sqrt{1-1/\gamma^2})z_2. \quad (13)$$

In order that the entire component of the saturation region lies under the asymptote, the right-hand side of the inequality must be negative in it. Indeed, in this case, the left-hand side of the inequality is also negative in the saturation region, i.e., in view of (11), belongs to the set A_-. The condition of negativeness of the right-hand side of the inequality can be written with regard to (10) as $(1+z_2^2)^{3/2} \ge -\lambda_1 z_2$. Replacing the inequality sign by the equality sign, squaring both sides of the equality obtained and introducing the notation $x = z_2^2$, we arrive at the equation $(1+x)^3 = \lambda_1^2 x$. This equation can be viewed as the equation in the unknown point of tangency of the cubic and linear (with an unknown coefficient) functions of x. Equating derivatives of both functions at the point of tangency, $\lambda_1^2 = 3(1+x)^2$, and substituting the right-hand side of the last equation for λ_1^2 into the previous equation, we obtain $x = 1/2$. Substituting this into the last equation, we find that the asymptote touches the saturation region when $\lambda_1 = 3\sqrt{3}/2$ and that the ordinate of the point of tangency is $z_2 = -\sqrt{1/2}$. Thus, for the smaller eigenvalue λ_1, we obtained the same value $\lambda_{\rm opt}$ that was obtained in [8] for λ in the case of the multiple roots. The corresponding λ is easily found from the relations (10): $\lambda = \lambda_{\rm opt}/(\gamma(1-\sqrt{1-1/\gamma^2}))$. Thus, we have proved the following assertion valid for a straight target path.

Theorem 1. *Let the coefficients of feedback (7), (6) satisfy the condition*

$$\gamma \ge 1, \quad \lambda(\gamma) = \frac{\lambda_{opt}}{\gamma(1-\sqrt{1-1/\gamma^2})}, \quad \lambda_{opt} = \frac{3\sqrt{3}}{2}. \quad (14)$$

Then, the half-planes A_- or A_+ lying from the two sides of the straight line $a(z) = 0$, where

$$a(z) = \lambda_{opt} z_1 + z_2, \quad (15)$$

are invariant sets of the closed-loop system (3), (7), (6), and any solution of the system asymptotically tends to the origin with the exponential rate $(-\lambda_{opt})$.

Theorem 1 implies that there exist an infinite number of pairs of the parameters γ and $\lambda(\gamma)$ related by the condition (14) for which we have the same partition of the phase plane and the same asymptotic rate of the deviation decrease. On the asymptote, the closed-loop system (3), (7), (6) is linear for any $\gamma \ge 1$, and any solution of the system tends to zero exponentially with the exponent $(-\lambda_{\rm opt})$ not depending on γ. Indeed, for any γ, we have

$$\sigma(z) = \lambda^2 z_1 + 2\gamma\lambda z_2 = \lambda_1\lambda_2 z_1 + (\lambda_1+\lambda_2)z_2 = \lambda_2(\lambda_1 z_1 + z_2) + \lambda_1 z_2,$$

where $\lambda_1 = \lambda_{\rm opt}$. On the asymptote, $\lambda_1 z_1 + z_2 = 0$, and the linearizing control (4) takes the form $u = \lambda_1 z_2/(1+z_2^2)^{3/2}$. It is easy to check that the right-hand side of the last expression does not exceed one, with the extreme values $u = \pm 1$

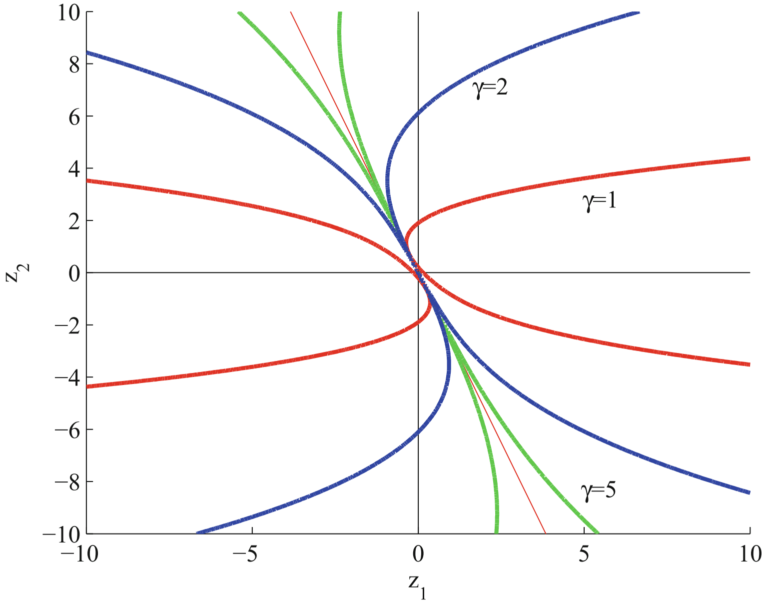

Fig. 2. Saturation regions of system (3) with $k = 0$ closed by feedback (7), (6) for $\gamma = 1$, $\gamma = 2$, and $\gamma = 5$.

being achieved only at $z_2 = \pm 1/\sqrt{2}$. Hence, on the asymptote, the closed-loop system (3), (7) takes the form $z_1' = z_2$, $z_2' = -\lambda_{\text{opt}} z_2$.

Let us find out in what way the selection of the value of γ affects the behavior of the closed-loop system. First, the greater the value of γ, the larger the saturation region. Figure 2 shows the saturation sets for three values of γ: $\gamma = 1$, $\gamma = 2$, and $\gamma = 5$. The thin line depicts the asymptote, which separates the two components of the saturation region for any $\gamma \geq 1$ and touches them at the points with the coordinates $z_2 = \pm 1/\sqrt{2}$.

Let $z(\xi, z^0, \gamma)$ denote the trajectory of the system (3) closed by the feedback (7), (6) with the initial condition $z(0) = z^0$. From Theorem 1, it follows that deviation $z_1(\xi, z^0, \gamma)$ either monotonically tends to zero or has one local extremum at the point of intersection with the axis z_1. In the latter case, the quality of the control can additionally be characterized by the magnitude of this extremum, which will we referred to as "overshooting" and denoted as $M(z^0, \gamma)$,

$$M(z^0, \gamma) = \max_{\xi} [z_1(\xi) \text{sign}(z_2(0))]. \tag{16}$$

If the deviation tends to zero monotonically, the overshooting is zero.

It can be shown that the minimum of the overshooting is achieved on the limit trajectories, the trajectories to which trajectories $z(\xi, z^0, \gamma)$ tend as $\gamma \to \infty$. The corresponding limit feedback (7), (6) is given by the discontinuous function

$$u(z) = \begin{cases} -1, \; z \in A_+, \\ \;\; 1, \;\; z \in A_-. \end{cases} \tag{17}$$

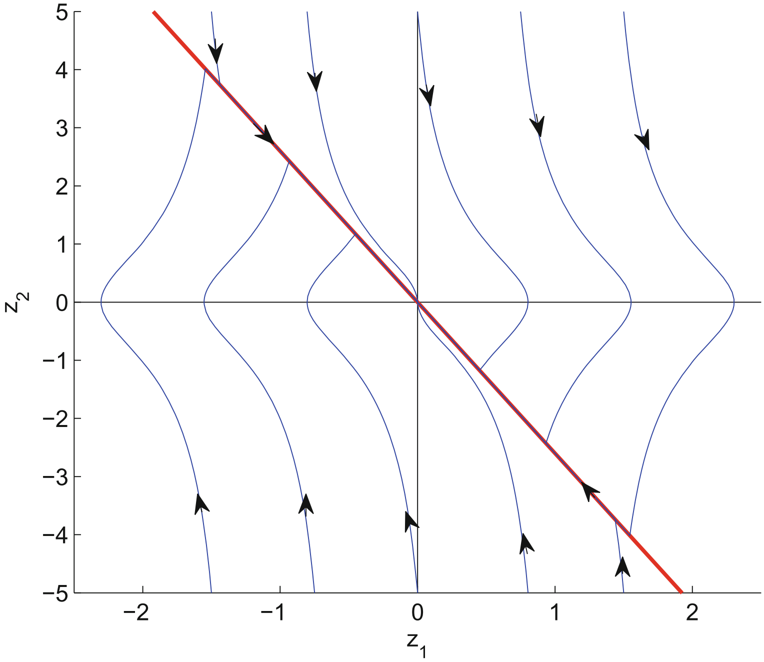

Fig. 3. Phase portrait of the closed-loop system (3), (17).

Figure 3 shows the phase portrait of system (3) closed by the limit control law (17). In the sets A_+ and A_-, the system moves along integral curves of equation (8) given by

$$z_1 = z_1(0) \mp \left(\frac{1}{\sqrt{1 + z_2^2(0)}} - \frac{1}{\sqrt{1 + z_2^2}} \right). \tag{18}$$

Having reached the asymptote, the system moves ("slides") along it to the origin.

However, in practice, the discontinuous control law (17) is not applicable because of the chattering arising when the system moves along the asymptote. Indeed, since the set of points belonging to the asymptote has zero measure, the control will alternately take limit values ± 1 when moving along it.

Is it possible to get rid of the above drawback and still preserve minimality of the overshooting? To answer this question, let us analyze the character of the trajectories of the closed-loop system. It can be seen from Fig. 3 that motion of the system closed by the limit control law (17) consists of the following two stages: motion along an integral curve (18) with the limit control $+1$ or (-1)

and motion along the asymptote. In terms of the original system, on the first stage, the robot makes a turn moving with the front wheels turned through the maximal angle until the angle ψ takes the value $\psi = -\arctan(\lambda_{\mathrm{opt}} z_1)$, after which the second stage starts when the system "slides" along the asymptote. It is on the second stage where chattering arises. Note also that, if the overshooting $M(z^0)$ is positive, then it is achieved on the first stage, when the system moves with the limit control.

The above observation makes us think of a combined (hybrid) two-stage control law: to apply the limit (saturated) control on the first stage, like in the discontinuous control law (17), and, after hitting the asymptote, to use the continuous feedback (7), (6) with the minimal $\gamma = 1$ on the second stage. Such a strategy makes it possible to combine advantages of the limit discontinuous control law and the continuous feedback with small γ and to get rid of disadvantages of both. The discontinuous law (17) brings the system from an initial state z^0 to the asymptote in a minimal time and with the minimal overshooting $M(z^0)$. The control on this stage takes only one value $u = 1$ or $u = -1$ and is insensitive to noise. Switching to the continuous feedback on the second stage allows the system to avoid chattering. When moving along the asymptote, the system is linear, and the deviation decreases exponentially. Since the rate of convergence does not depend on γ, the use of the minimal $\gamma = 1$ ensures the least sensitivity to measurement noise without sacrificing the convergence rate.

In practice, the exact hit of the asymptote is impossible because of measurement noise and approximation errors. Therefore, switching from the discontinuous to continuous feedback should occur upon entering some neighborhood of the asymptote, the so-called *control switching set Π*, so that the control law takes the form

$$u(z) = \begin{cases} -1, & z \notin \Pi,\ z \in A_+, \\ 1, & z \notin \Pi,\ z \in A_-, \\ -\mathrm{sat}_1[(\lambda_{\mathrm{opt}}^2 z_1 + 2\lambda_{\mathrm{opt}} z_2)/(1 + z_2^2)^{3/2}], & z \in \Pi. \end{cases} \tag{19}$$

How to select the asymptote neighborhood depends on a particular implementation of the control law (19). The answer to this question may depend on many factors, such as the accuracy of measurements of the state variables, digitization frequency, robot's velocity, and so on. Here, we would only like to emphasize that it is important that the switching set be invariant. This property guarantees that have occurred in the set, the system will never leave it.

5 Stabilization of a Robot with Constrained Control Along a Circular Path

The case of curvilinear target paths is much more complicated, and the optimality criterion adopted in the case of a straight path cannot be satisfied. In this section, we confine our consideration to only paths with constant curvature, i.e., circles, and will show, first, why this criterion is not applicable and, second, how to modify it.

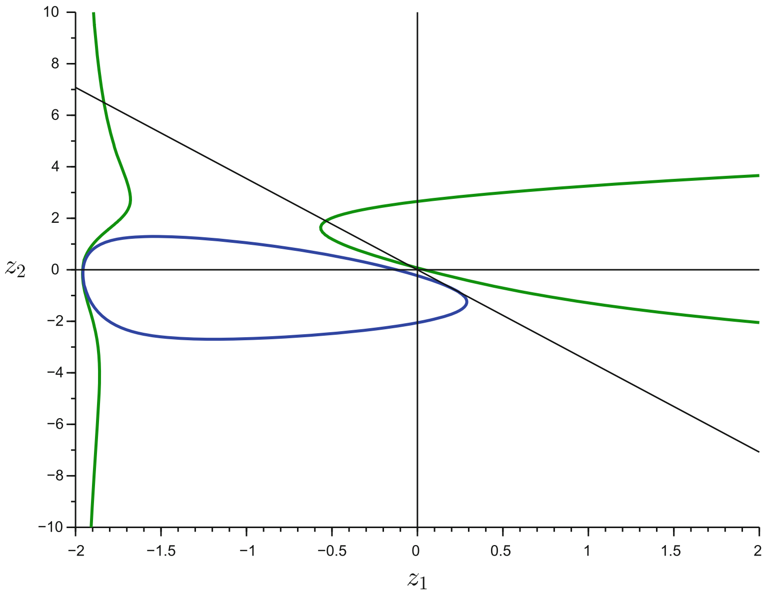

Fig. 4. Saturation regions U_+ (blue boundary) and U_- (green boundary) of system (3) with $k = -0.5$ closed by feedback (7), (6) with $\gamma = 1$ and $\lambda = 3.54$. (Color figure online)

For definiteness, we assume that the robot moves in the clockwise direction, which implies that the path curvature is negative, $k = \text{const} < 0$. Unlike in the previous case, the domain of the system is a half-plane $z_1 > 1/k$ rather than R^2, and the saturation regions depend not only on the feedback coefficients but also on the path curvature k. Let us denote the sets where the control takes values $+1$ and -1 as $U_+(k)$ and $U_-(k)$, respectively,

$$U_+(k) = \left\{ z : -\frac{\sigma(z)}{(1+z_2^2)^{3/2}} + \frac{k}{\sqrt{1+z_2^2}(1-kz_1)} \geq 1 \right\}, \tag{20}$$

$$U_-(k) = \left\{ z : -\frac{\sigma(z)}{(1+z_2^2)^{3/2}} + \frac{k}{\sqrt{1+z_2^2}(1-kz_1)} \leq -1 \right\}. \tag{21}$$

Taking into account the negativeness of k, it is not difficult to see that $U_+(k)$ is a bounded set and $U_+(k) \subset U_+(0)$, whereas $U_-(k) \supset U_-(0)$ is unbounded and may consist of two disconnected subsets (see, e.g. Fig. 4), where $U_+(0)$ and $U_-(0)$ are the saturation regions for a straight target path. A typical picture of the saturation regions for a circular path is shown in Fig. 4. Here, $k = -0.5$, $\lambda = 3.54$, and $\gamma = 1$. The boundary of the bounded set U_+ is shown by the thick blue line, while the boundaries of the two unbounded components of U_- are depicted by the thick green lines.

Since the term $k/\sqrt{1+z_2^2}(1-kz_1)$ tends to $-\infty$ when $z_1 \to 1/k$ for any z_2, any straight line passing through the origin (including the asymptote for any feedback coefficients), necessarily intersects the set $U_-(k)$. Let, similar to the straight-path case, A_+ and A_- denote the intersection of the system domain $z_1 > 1/k$ with the half-planes $z_2 + \lambda z_1 > 0$ and $z_2 + \lambda z_1 < 0$, respectively. Taking into account that trajectories can intersect the asymptote from the lower set A_- to the upper one A_+ in the region $U_-(k)$, the both sets cannot be invariant sets of the system whatever feedback coefficients are; i.e., the criterion formulated in Sect. 2 cannot be satisfied.

On the other hand, a trajectory beginning in the upper half-plane A_+ can intersect the asymptote and enter the lower half-plane A_- only in the set $U_+(k)$. Hence, if the asymptote does not intersect the set $U_+(k)$, no trajectories can come from A_+ to A_-; i.e., A_+ is a *positive invariant set* of the system in this case. Taking into account that our goal is to obtain the closed-loop system with the phase portrait similar to that of the corresponding linear system and the greatest convergence rate, this brings us at the following optimization problem statement.

Problem 2. Find feedback coefficients that guarantee the existence of a positive invariant half-plane while ensuring the greatest rate of deviation decrease.

Similar to the case of a straight path, given the curvature k, we seek for λ_1 such that the line $z_2 = -\lambda_1 z_1$ touches the boundary of the set $U_+(k)$, which is found similar to that in the case of a straight target path. To this end, like in Sect. 4, we rewrite the inequality defining the set $U_+(k)$ such that its left-hand side coincide with the left-hand side of the asymptote Eq. (11) (cf. (13))

$$\lambda^2 z_1 + \lambda\gamma(1 + \sqrt{1 - 1/\gamma^2})z_2 \leq -f(z, k, \lambda_1)z_2, \tag{22}$$

where

$$f(z, k, \lambda_1) = (1 + z_2^2)^{3/2} - \frac{k(1 + z_2^2)}{1 - kz_1} + \lambda_1 z_2. \tag{23}$$

From (22), it follows that the set $U_+(k)$ completely lies under the asymptote when $f(z, k, \lambda_1) \geq 0 \ \forall z \in U_+$.

To find the value of λ_1 for which the asymptote touches the boundary of the set $U_+(k)$, we consider the restriction

$$\bar{f}(z_2, k, \lambda_1) = (1 + z_2^2)^{3/2} + \frac{\lambda_1(\lambda_1 z_2 - k)}{\lambda_1 + kz_2} \tag{24}$$

of the function $f(z, k, \lambda_1)$ to the asymptote, which is obtained by substituting $z_1 = -z_2/\lambda_1$ into (23). The desired $\lambda_1(k)$ is a solution to the nonlinear equation

$$F(k, \lambda_1) = 0, \tag{25}$$

where

$$F(k, \lambda_1) = \min_{z_2 < 0} \bar{f}(z_2, k, \lambda_1). \tag{26}$$

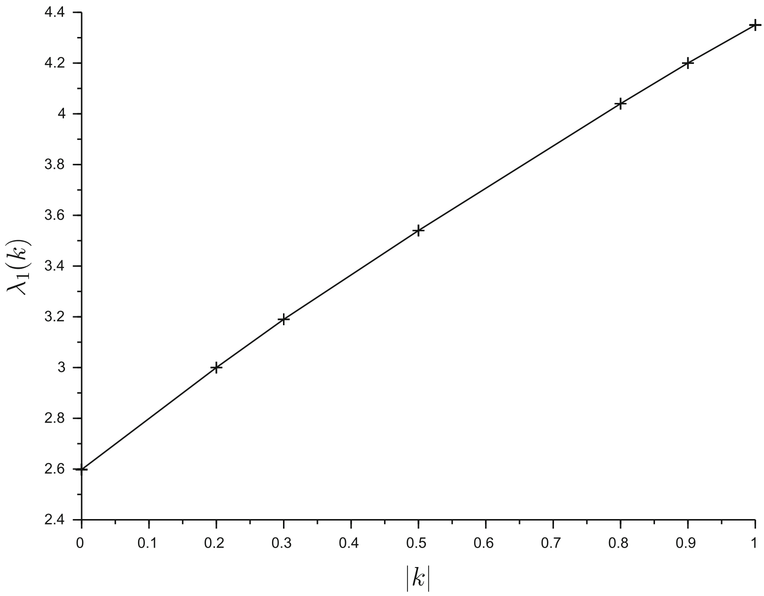

Fig. 5. Dependence of the asymptotic convergence rate λ_1 on the circular path curvature.

For a given k, the solution to problem (24)–(26) is easily found numerically. Simple analysis shows that $\lambda_1(k)$ is a monotonically increasing function of $|k|$. Figure 5 shows dependence of the convergence rate λ_1 on the path curvature obtained by numerical solving problem (24)–(26). For example, for the case of the circular path with $k = -0.5$, the saturation regions for which are depicted in Fig. 4, $\lambda_1 \approx 3.54$. As can be seen, the corresponding asymptote $z_2 = -3.54z_1$ shown in Fig. 4 by the inclined thin line actually touches the set U_+.

The results of this section are summarized in the following theorem.

Theorem 2. *Let $k =$ const and the coefficients of feedback (7), (6) satisfy the condition*

$$\gamma \geq 1, \quad \lambda(\gamma) = \frac{\lambda_1(k)}{\gamma(1 - \sqrt{1 - 1/\gamma^2})}, \tag{27}$$

where $\lambda_1(k)$ is the solution of problem (24)–(26). Then, the set $A_+ = \{(z_1, z_2) : z_1 > 1/k, z_2 > -\lambda_1(k)z_1\}$ is a positive invariant set of the closed-loop system (3), (7), (6), and any solution of the system asymptotically tends to the origin with the exponential rate $-\lambda_1(k)$.

Thus, like in the case of a straight target path, there exist an infinite number of pairs $(\gamma, \lambda(\gamma))$ resulting in the same partition of the phase plane and the same convergence rate. All reasonings regarding the selection of a particular pair presented in Sect. 4 remain valid for the circular paths. The best option is the combined (hybrid) strategy: to apply the limit (saturated) control to bring the system from an initial state $z(0)$ to the asymptote (which is equivalent to using infinitely large γ and λ) and, then, to switch to the saturated linearizing feedback with the minimal $\gamma = 1$ to slide to the origin along the asymptote. The only difference compared to the straight-path case is that in the region of negative z_1 (inside the circle), where the asymptote intersects the set U_- (see Fig. 4), the system can leave the asymptote staying in the positive invariant set A_+. For any initial condition, the WR can intersect the target path at most twice.

6 Conclusions

In the paper, stabilization of a wheeled robot along a target path has been discussed. In the case of an unlimited control resource, the problem is easily solved by applying the feedback linearization technique. If the control is bounded, the application of the saturated linearizing feedback results in a nonlinear closed-loop system and brings one to the problem of selecting the feedback coefficients to optimize the performance of the stabilization. For a straight target path, the desired feedback coefficients are defined to be those that result in the partition of the phase plane into two invariant sets of the nonlinear closed-loop system while ensuring the greatest asymptotic rate of the deviation decrease. The use of the feedback law with such coefficients guarantees that the robot intersects the target path not more than once. It has been proved that there exists a family of the optimal coefficients. A hybrid control law has been proposed that ensures the desired properties of the phase portrait and minimal overshooting and is stable to noise. Such a partition has been shown to be impossible for circular target paths. In this case, optimal feedback coefficients are defined to be those that guarantee the existence of a positive invariant half-plane while ensuring the greatest rate of deviation decrease. The problem of numerical finding the optimal coefficients has been solved.

In the future, we plan to study the problem of finding optimal feedback coefficients for the problem of stabilizing robot's motion along general-form target paths.

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