Numerical Damping of Forced Oscillations of an Elastic Beams

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Abstract. The beam oscillations are modeled by the fourth-order hyperbolic partial differential equation. The minimized functional is the energy integral of an oscillating beam. Control is implemented via certain function appearing in the right side of the equation. It was shown that the solution of the problem exists for any given damping time, but with decreasing this time, finding the optimal control becomes more complicated. In this work, numerical damping of beam oscillations is implemented via several fixed point actuators. Computational algorithms have been developed on the basis of the matrix sweep method and the second order Marquardt minimization method. To find a good initial approximation empirical functions with a smaller number of variables are used. Examples of damping the oscillations via a different number of actuators are given. It is shown that the amplitude of the oscillations of any control functions increases with the reduction of the given damping time. Examples of damping the oscillations in the presence of constraints on control functions are given; in this case, the minimum damping time exists. The damping of oscillations is considered also in the case when different combinations of actuators are switched on at different time intervals of oscillation damping.

Keywords: Marquardt minimization method · Oscillations damping
Fixed point actuators · Matrix sweep method

1 Introduction

Methods of damping of oscillations of elements of complex mechanical systems began to develop intensively in the 70s of the XX century. The most significant were the works of Lagness [1], Russell [2], Butkovskiy [3,4], in which the problem of damping of string oscillations was considered and conditions for the existence of a solution to the problem were obtained. In particular, Butkovskiy proposed to use a point actuator for damping the oscillations of the string. However, later it was shown that in the case if a solution appears in the form of standing waves, if the actuator is in a node of standing waves, then the solution of the problem

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may not exist. To avoid such a situation, Muravey [5,6] suggested using a point actuator moving along a small section of a string, but the practical implementation of such actuator is very difficult. In [7,8] it was shown that the solution of the problem exists for any positive time T, however, as T decreases, finding the optimal control becomes more complicated. In this paper, we consider the possibility of damping the beam oscillations using several fixed point actuators.

2 Problem Statement

2.1 Oscillations Damping

The purpose of this work is to develop numerical method for damping forced transverse oscillations of a beam via multiple fixed point actuators. The transverse oscillations of a beam are described by the Petrovsky-hyperbolic equation

$$u_{tt} = -a^2 u_{xxxx} + g(x,t), (x,t) \in \prod = \{0 \le x \le l, 0 \le t \le T\}. \tag{1}$$

Here, the time t and the linear dimension x are related to the characteristic values t^* and x^*. We will consider the initial displacement and the velocity of the beam movement

$$u|_{t=0} = h_0(x), u_t|_{t=0} = h_1(x), 0 \le x \le l \tag{2}$$

as initial perturbations. At the ends of the beam, the conditions of articulation are superimposed.

$$u|_{x=0} = u_{xx}|_{x=0} = 0, u|_{x=l} = u_{xx}|_{x=l} = 0, 0 \le t \le T. \tag{3}$$

The energy of the oscillating beam is

$$E(t) = \int_0^l \left[u_t^2(x,t) + a^4 u_{xx}^2(x,t)\right] \mathrm{d}x. \tag{4}$$

The problem of damping is to find the control function $g(x,t)$, which transfers the beam from the initial state (2) to the state

$$u|_{t=T} = 0, u_t|_{t=T} = 0, 0 \le x \le l \tag{5}$$

in time T. According to Lions [9], this property of the system is called strict controllability.

Thus, the problem of damping of oscillations consists in finding the optimal control function $g(x,t) \in L_2((0,T) \times (0,l))$ such, that for any initial perturbations $h_0(x), h_1(x)$

$$E(T) = 0. \tag{6}$$

As a control function, we consider p fixed point actuators

$$g(x,t) = \sum_{i=1}^{p} w_i(t)\,\delta(x - x_i), \tag{7}$$

where $w_i(t), i = 1, \ldots, p$ - control functions, δ - Dirac delta-function, x_i - points in which the actuators are placed. We will assume that $w_i(t) \in L_2(0,T), i = 1, \ldots, p$.

2.2 Numerical Solution

Equation (1) can be reduced to a system of two equations of the second order

$$\begin{cases} u_t = av_{xx}, \\ v_t = -au_{xx} + \sum_{i=1}^{p} f_i(x,t); \end{cases} \tag{8}$$

where

$$f_i(x,t) = \begin{cases} w_i(t)\left(-\frac{x}{al}(l - x_i)\right), x < x_i, \\ w_i(t)\left(\frac{1}{a}(x - x_i) - \frac{x}{al}(l - x_i)\right), x \geq x_i. \end{cases} \tag{9}$$

We solve it by the matrix sweep method [10]. We approximate the control functions $w_i(t)$, $i = 1, \ldots, p$ with piecewise-constant functions: $\forall t \in [t_i, t_{i+1})$ assume $w_i(t)$ where w_j^i - const, $i = 1, \ldots, p, j = 0, \ldots, N_{T-1}$. Then the integral of the beam energy will be a function of the variables w_j^i

$$\begin{gathered} E(T) = L\left(w_0^1, \ldots, w_{N_T}^p\right) \\ = \int_0^l \left[u_t^2\left(w_0^1, \ldots, w_{N_T}^p, x, T\right) + a^4 u_{xx}^2\left(w_0^1, \ldots, w_{N_T}^p, x, T\right)\right] \mathrm{d}x \end{gathered} \tag{10}$$

For the numerical computation of the energy integral (10) we use the Simpson method.

2.3 Minimization

The optimal values $w_0^1, \ldots, w_{N_T}^p$, which minimize (10) with a specified accuracy ε, are the required solution of the problem. To solve the oscillation damping problem, we use the Marquardt method [11].

For large sizes of a finite-difference grid or when using a sufficiently large number of actuators, the numerical computation of control functions by using second-order minimization methods can be a computationally complex problem that requires a lot of computation time. However, it is possible to significantly reduce the computation time by finding a good initial approximation from minimizing some empirical function that depends on a small number of parameters.

The basic idea of using empirical functions is to replace the initial minimizable function with another continuous function $w(t)_{\mathrm{EMP}}$ which depends on a small number of parameters. Suppose that each of the control functions has the following form

$$w(e_1, \ldots, e_7, t)_{\mathrm{EMP}} = e_1 \sin(e_2 t + e_3) + e_4 \sin(e_5 t + e_6) \sin(e_7 t), \tag{11}$$

where constant values $e_1, \ldots, e_7$ are not yet known. We introduce a special transformation function

$$L_{\mathrm{EMP}}(e) = L\left(w(e)_{\mathrm{EMP}}, \ldots, w(e)_{\mathrm{EMP}}\right). \tag{12}$$

To find the empirical coefficients $e_1^1, \ldots, e_7^1, \ldots, e_1^p, \ldots, e_7^p$, we will solve the problem of finding the minimum of the function (12) using the Marquardt minimization method. The resulting control functions are used as the initial approximation for minimizing (10) with a specified accuracy ε.

Empirical formulas are best used as an initial approximation or in tasks where precision is not a high priority.

3 Use of Multiple Actuators

3.1 Solution Existence Problem

Consider the following example. The initial conditions are $h_0(x) = 0.1\sin(2\pi x)$, $h_1(x) = 0$. The input parameters are $a = 1, l = 1$, and we set the required damping time equal to $T = 0.5$, the size of the finite-difference grid will be $N \times K = 20 \times 250$, so $h_x = 0.05, h_t = 0.002$. We will assume that the oscillation damping task is solved if $L(w(t)) \le \varepsilon$, where $\varepsilon = 10^{-4}$.

The actuator placed at the point $x_0 = 0.5$ can not dampen the beam oscillations, since the point $x_0 = 0.5$ is a node of standing waves. This is clearly seen in the Fig. 1.

The model suggested by Butkovsky has the following drawback: in the case of the appearance of a solution (1) in the form of standing waves, if x_0 falls into a node of standing waves, then the solution of the problem may not exist. Let us consider the same conditions of the example, but for damping the oscillations we use two fixed point actuators at the points $x_1 = 0.25, x_2 = 0.75$, respectively. We rewrite the condition for solving the task in the form $L(w_1(t), w_2(t)) \le 10^{-4}$. The size and steps of the finite-difference grid are the same.

We solve the task for two cases: using the minimization of the function (12) with initial control of the form $w_1(t) \equiv 0, w_2(t) \equiv 0$ and using empirical functions (12) to obtain the initial approximation. In the first case, the task was solved with the error $L(w_1(t), w_2(t)) = 5.0701 \cdot 10^{-13}$.

In the second case, using the Marquardt minimization method, we find the following empirical coefficients e:

$$e = \begin{pmatrix} 1000 & -0.1196 & 0.0317 & 1000 & -6.6844 & 2.9977 & -6.5086 \\ 1000 & -2.7326 & 3.2003 & 1000 & -2.5081 & -3.1785 & 9.9958 \end{pmatrix} \quad (13)$$

Substituting them into (12), we obtain the control functions $w_1(t)_{\mathrm{EMP}}$, $w_2(t)_{\mathrm{EMP}}$, allowing to solve the system with an error of $L_{\mathrm{EMP}}\left(e_1^1, \ldots, e_7^2\right) = 0.32495$. Next, we take them as the initial approximation and use the Marquardt method again for the final determination of the control functions $w_1(t), w_2(t)$. As a result, we minimized the value of the beam energy integral (12) with the error $L(w_1(t), w_2(t)) = 4.8804 \cdot 10^{-13}$.

The graphs of the values of the function $u(x,t)$, illustrating the process of damping the beam oscillations, and the final form of the control functions $w_1(t)$ and $w_2(t)$ for both cases are shown on Figs. 2 and 3 respectively.

Thus, the task is solved for a time $T = 0.5$. It is noticeable that, despite the same initial conditions and grid parameters, the form of the control functions differs depending on the initial approximation, and the damping proceeds in different ways.

3.2 Alternation of Actuators

To dampen oscillations in the case of certain initial conditions, it may be necessary to use different groups of actuators at different time intervals. Let the initial

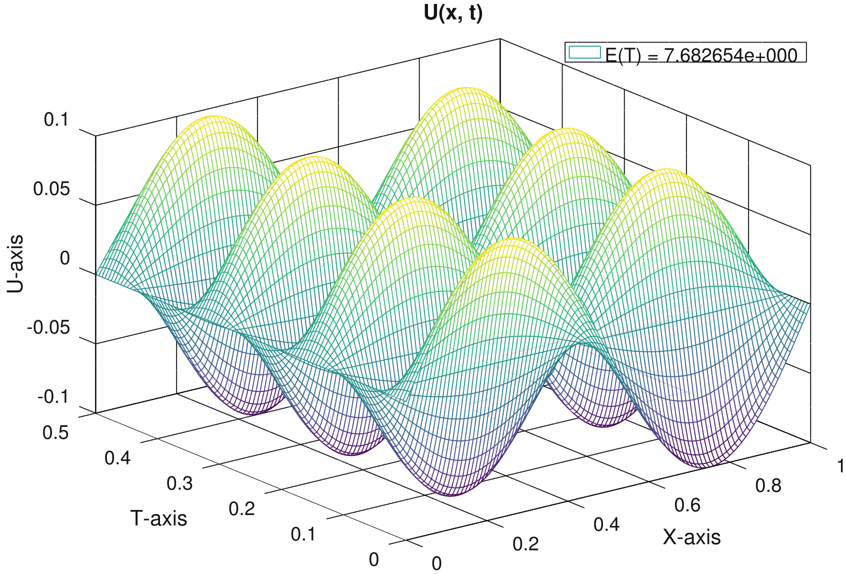

Fig. 1. The process of damping of oscillations via an actuator at the point $x_0 = 0.5$ (damping does not occur)

conditions be $h_0(x) = 0.25 \exp(x) \sin(2\pi x), h_1(x) = 0$. The input parameters are $a = 1, l = 1$, the required damping time is $T = 0.2$, the size of the finite-difference grid will be $N \times K = 20 \times 250$, so $h_x = 0.05, h_t = 0.0008$. We assume, that the task of damping the oscillations is solved if

$$L(w_1(t), \ldots, w_p(t))) \leq \varepsilon, \tag{14}$$

where $\varepsilon = 10^{-4}$.

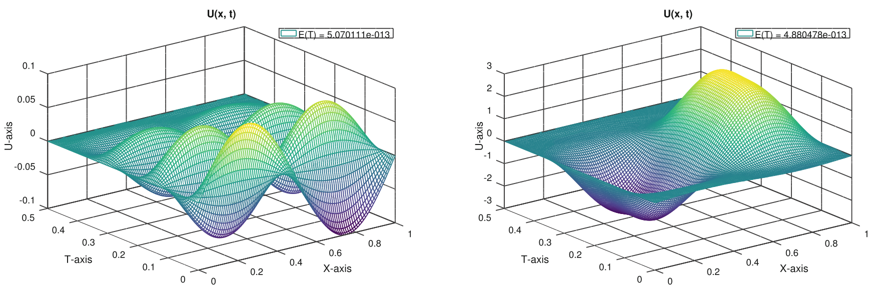

Fig. 2. The oscillation damping process via two actuators at the points $x_1 = 0.25, x_2 = 0.75$ (a) with the initial values $w_0^1, \ldots, w_{N_T}^2 = 0$, (b) with the empirical approximation (3.2)

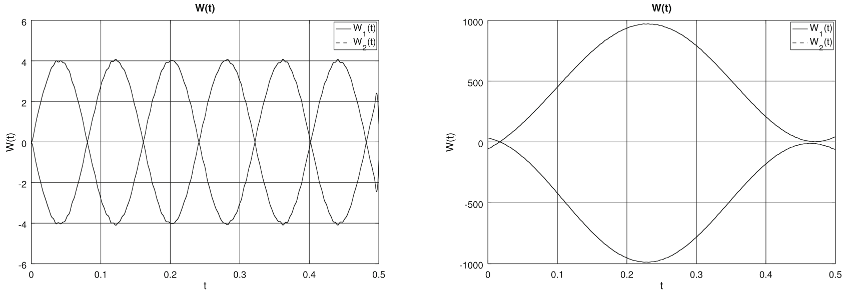

Fig. 3. The control functions $w_1(t)$ and $w_2(t)$, obtained (a) with the initial values $w_1, \ldots, w_2 = 0$, (b) with the empirical approximation (3.2)

On the time slice $T = 0.2$ (Fig. 4(b)), it is noticeable that the two actuators placed at the points $x_1 = 0.25$ and $x_2 = 0.75$ can not dampen the initial oscillations (Fig. 4(a)).

To dampen the oscillations, we divide the task into two time intervals. We will use 4 actuators placed at the points $x_1 = 0.15, x_2 = 0.25, x_3 = 0.65, x_4 = 0.75$, but in the first interval $T \in [0; 0.1]$ we will use only two of them at the points x_2 and x_4. The other two actuators at the points x_1 and x_3 on this interval will be left inactive.

Minimizing the function (12), we obtain the empirical coefficients e:

$$e = \begin{pmatrix} 1000 & -3.2196 & 3.0779 & 1000 & -5.9374 & -0.0239 & 7.5418 \\ 1000 & 30.536 & 3.0844 & 1000 & 25.405 & 0.1781 & 29.208 \end{pmatrix} \quad (15)$$

Further we get $w_2(t)_{\mathrm{EMP}}, w_4(t)_{\mathrm{EMP}}$ and use them as the initial approximation for $w_2(t), w_4(t)$. We solve the task with the error $L(w_2(t), w_4(t)) = 1.1797$. In Fig. 5 the process of partial damping of oscillations in the time interval $T = [0; 0.1]$ and a time slice of $T = 0.1$ is shown.

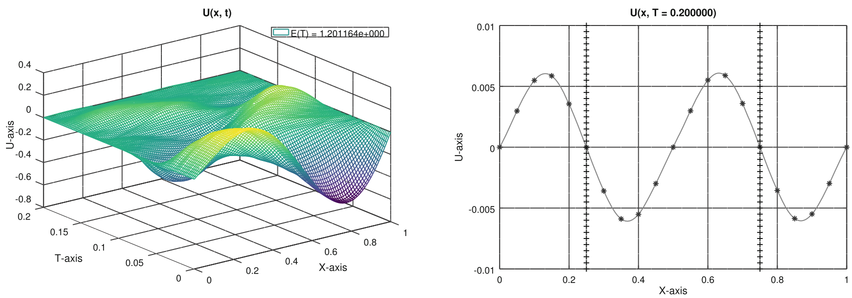

Fig. 4. The process of oscillation damping via two actuators at the points $x_1 = 0.25, x_2 = 0.75$ (partial damping), (b) the cut of the values of the function $u(x,t), t = T = 0.2$ (further damping is not possible)

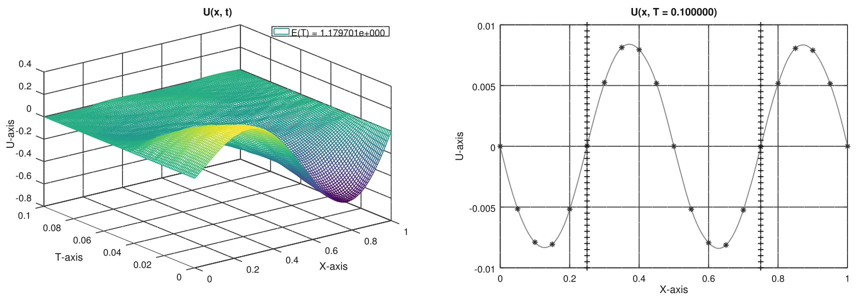

Fig. 5. The oscillation damping process via two actuators at the points $x_2 = 0.25, x_4 = 0.75$ on the interval $T\ in[0;0.1]$ (partial damping), (b) the cut of the values of the function $u\left(x,t\right), t = T = 0.1$ (further damping is not possible)

In the second time interval $T \in [0.1; 0.2]$ to dampen the oscillations we use two remaining actuators at the points $x_1 = 0.15, x_3 = 0.65$. The previous two actuators at the points x_2 and x_4 are left inactive. We will use the solution of the previous subtask as a new initial perturbation. For the initial velocity, in this case we put $v\left(l,t\right)$ of the previous subtask as $v\left(0,t\right)$ of the second subtask.

By minimizing (12), we obtain e:

$$e = \begin{pmatrix} 1000 & 2.8741 & 3.1415 & 1000 & 10.572 & 0.9895 & 3.0815 \\ 1000 & 6.8311 & 3.1421 & 1000 & 13.532 & -2.3983 & -7.1904 \end{pmatrix} \quad (16)$$

By using $w_1\left(t\right)_{\text{EMP}}, w_3\left(t\right)_{\text{EMP}}$ as the initial approximation for $w_1\left(t\right)$ and $w_3\left(t\right)$ we get the error $L\left(w_1\left(t\right), w_3\left(t\right)\right) = 2.2163{\cdot}10^{-13}$ for the second subtask. Figure 6 shows the process of damping of oscillations in the time interval $T \in [0.1; 0.2]$ (a scale is used for $u\left(x,t\right)$, which is 20 times smaller than in Fig. 5) and a time slice of $T = 0.2$.

Thus, the task is solved in time $T = 0.2$ via two actuators $x_2 = 0.25$ and $x_4 = 0.75$ on $T \in [0; 0.1]$ and two actuators $x_1 = 0.15$ and $x_3 = 0.65$ on $T \in [0.1; 0.2]$ with the resulting error $L\left(w_1\left(t\right), \ldots, w_p\left(t\right)\right) = 2.2163 \cdot 10^{-13}$. Combining $u_1\left(x,t\right)$ and $u_2\left(x,t\right)$ into $u\left(x,t\right)$, we illustrate on Fig. 7 a complete process of damping the oscillations in this task.

Thus, a numerical method for damping the beam oscillations is developed via several fixed point actuators. It makes it possible to research the process of damping the oscillations for a different time T.

3.3 Dependence of the Damping Process on Time

Suppose that the initial conditions have the following form $h_0 = 0.1\sin\left(2\pi x\right), h_1\left(x\right) = 0$. The input parameters are $a = 1, l = 1$, the size of the finite-difference grid is $N \times K = 20 \times 50$, so $h_x = 0.05, h_t = 0.002$. To dampen the oscillations, we use 4 actuators placed at the points $x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$, respectively. The condition for damping the oscillations, as before, will be assumed

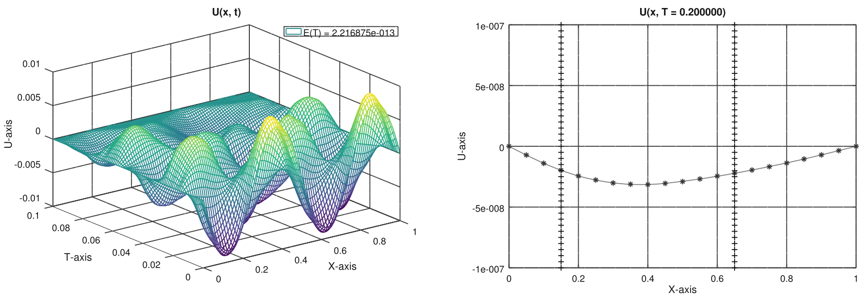

Fig. 6. (a) The oscillations damping process via two actuators at the points $x_1 = 0.15, x_3 = 0.65$, on the interval $T \in [0.1; 0.2]$, (b) the cut of the values of the function $u(x,t), t = T = 0.2$

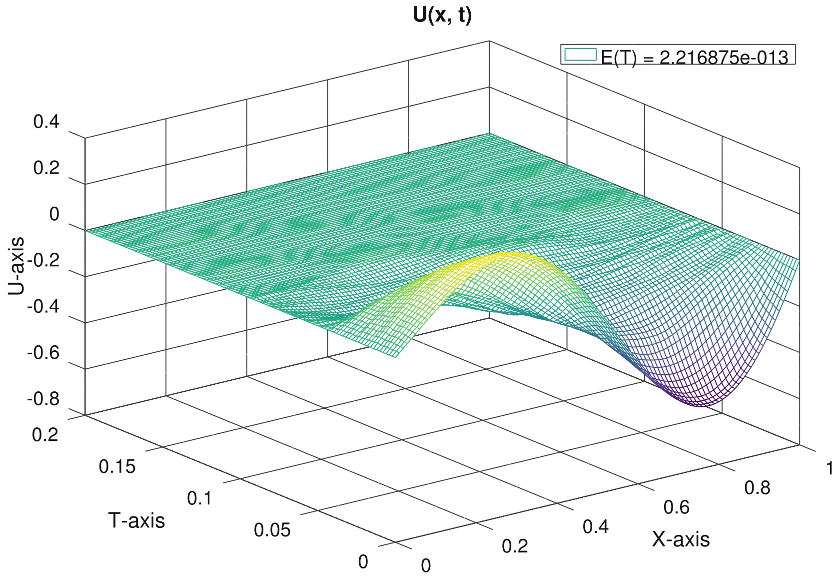

Fig. 7. The complete process of damping oscillations via four actuators

$$L(w_1(t), w_2(t), w_3(t), w_4(t)) \leq \varepsilon, \tag{17}$$

where $\varepsilon = 10^{-4}$. We set the damping time to $T = 0.1$. By default, the initial approximation for all control functions will be assumed to be zero.

On Fig. 8 the process of damping of the oscillations $u(x,t)$ and the control functions $w_i(t), i = 1, \ldots, 4$ are shown.

Consider the same conditions of the example, but put $T = 0.01$. On Fig. 9 the damping process and control functions are shown.

Drawing attention to Figs. 8(a) and 9(a), it is possible to clearly notice the difference in the process of damping of the oscillations depending on the prescribed damping time T. Thus, if we set a sufficiently long time, the damping

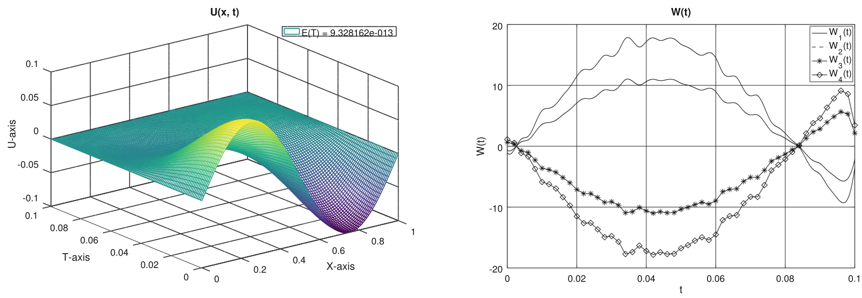

Fig. 8. (a) The oscillations damping process via four actuators at the points $x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$ for the time $T = 0.1$, (b) the control functions $w_1(t), w_2(t), w_3(t)$, and $w_4(t)$

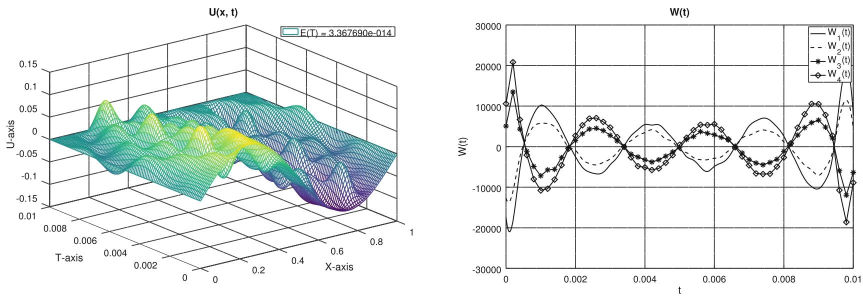

Fig. 9. (a) The oscillations damping process via four actuators at the points $x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$ for the time $T = 0.01$, (b) the control functions $w_1(t), w_2(t), w_3(t)$, and $w_4(t)$

proceeds more smoothly. Conversely, if a small time is set, a multitude of micro-oscillations on the beam arise, which are then smoothed by the control functions. The amplitude of the oscillations of any control functions increases almost exponentially along with a decrease in the prescribed damping time.

3.4 Constrained Control Functions

Let us consider the case when constraints are imposed on control functions. This case is more approximate to the practical implementation since in the design of actuator mechanisms it is necessary to lay the maximum permissible power of the drives. To find constrained control functions, it is necessary to use methods of finding a conditional minimum. In this work, the penalty minimization method is used using the Marquardt method to solve the corresponding problem of finding an unconditional minimum.

Since the maximum amplitude of each of the control functions begins to increase as the damping time approaches zero, it is necessary to select a damping time for which $L(w_1(t), \ldots, w_p(t)) \leq \varepsilon$ and $w_i(t) \in [a; b]$. We call the minimal time T, under which both conditions are satisfied, by the optimal damping time. The damping time can be reduced either by expanding the admissible boundaries of the control function or by increasing the number of actuators.

Let the initial conditions be $h_0(x) = 0.2x(1-x), h_1(x) = 0$. The input parameters are $a = 1, l = 1$, the size of the grid is $N \times K = 40 \times 120$. We assume that the oscillation damping task is solved if $L(w_1(t), \ldots, w_p(t)) \leq \varepsilon$, where $\varepsilon = 10^{-4}$.

We show that it is possible to reduce the minimum damping time by increasing the number of actuators with constant constraints. Initially, we will solve the task using a single fixed point actuator, placed in $x_1 = 0.5$. On the control function, we impose the constraints $w(t) \in [-2; 2]$. The minimum time required for damping is $T = 0.2265$.

The control function $w(t)$ is shown on Fig. 10(a).

Now we solve the same task via of two actuators placed in $x_1 = 0.25$ and $x_2 = 0.75$ respectively. The constraints imposed on the control functions remain the same. In this case, it is possible to reduce the minimum time required for damping to $T = 0.1825$.

The control functions $w_1(t), w_2(t)$ are shown on Fig. 10(b).

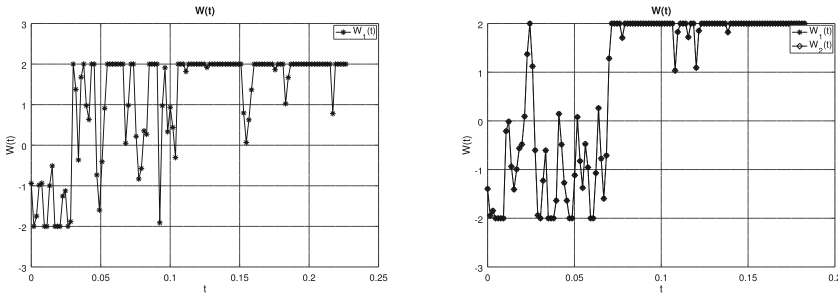

Fig. 10. Control functions $w_i\ left(t\ right)$ with constraints $w_i(t) \in [-2; 2], i = 1, \ldots, p$ for (a) $p = 1, T = 0.2265$, (b) $p = 2, T = 0.1825$

And, finally, we solve the task with the use of 4 actuators at the points $x_1 = 0.25, x_2 = 0.4, x_3 = 0.6, x_4 = 0.75$. We impose the previous constraints on all control functions. In this case, the minimum time was reduced to $T = 0.1237$.

The oscillation damping process and control functions are shown on Fig. 11.

3.5 Realtime Oscillations Damping

In the previous examples, the problem of searching for some "ideal" control is considered, which is assumed a priori known before the direct beginning of oscillations damping. However, in fact, at the moment of the beginning of the damping of the oscillations, we do not know anything about how the actuators should behave, and the task of finding the control arises. Of course, we could

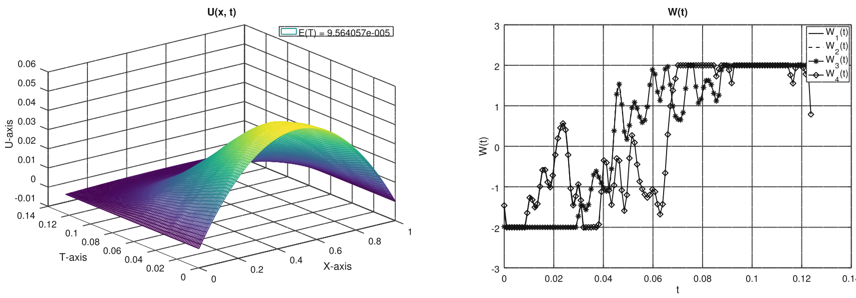

Fig. 11. (a) Control functions $w_1(t), w_2(t), w_3(t)$ and $w_4(t)$ with constraints $w_1(t), w_2(t), w_3(t), w_4(t) \in [-2; 2]$ for $T = 0.1237$, (b) the oscillations damping process via four actuators at the points $x_1 = 0.25, x_2 = 0.4, x_3 = 0.6, x_4 = 0.75$ for the time $T = 0.1237$ with constraints $w_1(t), w_2(t), w_3(t), w_4(t) \in [-2; 2]$

set the required damping time as a constant and start the process of finding control functions using the minimization methods right when registering the displacement, but it is obvious that this should take some time (which can be very, very impressive, up to several hours for some conditions). Moreover, even if we spend some time after the oscillations begin to look for control functions and try to apply the resulting control, it is likely that the displacement of the beam at this point in time will be completely different, and our control will, at best, not have the required smoothing effect on the beam, or even worse, will intensify the oscillations. We show that using the method of numerical damping described above it is possible to obtain the required control directly during the damping process. In this task, we made the transition from the dimensionless T to the dimensional damping time.

Let the parameters of the beam be $a = 1, l = 1$. On the beam are set $N + 1 = 21$ sensors at the same distance $h_x = l/N = 0.05$ from each other, which register the oscillations. Consider the case when sensors has detected the oscillations of the beam shown in the Fig. 12.

Now we will search the damping time T in seconds.

Numerous computations have shown that the time in which one iteration of the Marquardt method is calculated is on the average 90 ms. Under the iteration calculation time, here is meant the total execution time of the OpenCL kernels of calculating the integral of the energy, the gradient and the Hessian matrix of the function (10), obtained using the runtime profiler. Suppose that the response time of the actuators is 10 ms. Thus, the total response time of the system will be $h_t = 90\,\text{ms} + 10\,\text{ms} = 0.1\,\text{s}$.

By the condition, the time spent by the actuators for damping the beam oscillations on each individual time interval $t_i \in [0, T], t_i - t_{i-1} = h_t$ is exactly equal the real program execution time on calculation the iteration of the numerical damping method corresponding to this time interval. In other words, the time of damping T, which was considered in all previous tasks, will be equal to the time of the program execution in seconds.

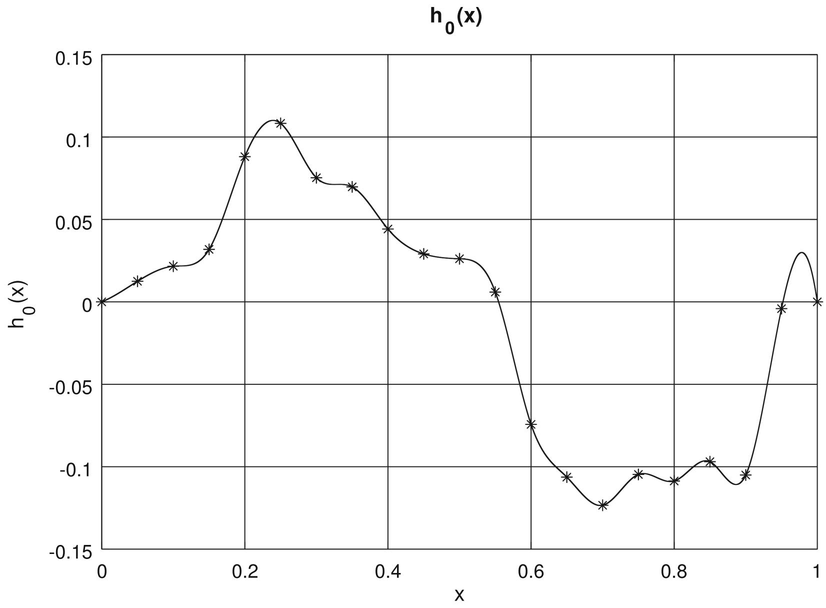

Fig. 12. The initial displacement of $h_0(x)$

The calculation process ends when $L(w_1(t), w_2(t)) \leq \varepsilon$, where $\varepsilon = 10^{-4}$. In this case, oscillations damping was performed with an allowable error of $L(w_1(t),\ w_2(t)) = 7.7879 \cdot 10^{-5}$ for 30 iterations of minimization method. Multiplying the number of iterations by the response time, we get that the damping time of the oscillations was $T = 3\,\mathrm{s}$.

The process of damping of the oscillations $u(x,t)$ and the form of the control functions $w_1(t)$, $w_2(t)$ are shown on Fig. 13.

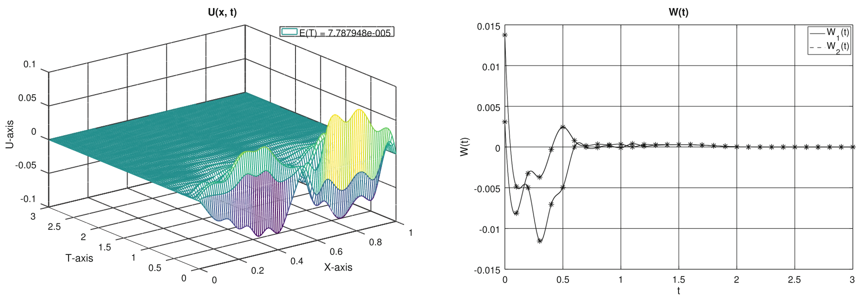

Fig. 13. (a) The process of damping the oscillations in real time via two actuators at the points $x_1 = 0.25, x_2 = 0.75$ for the time $T = 3\,\mathrm{s}$, (b) the control functions $w_1(t)$ and $w_2(t)$

4 Conclusion

The problem of damping of forced transverse oscillations of an elastic beam for an arbitrary predetermined time T is considered. A computational algorithm with second-order convergence is developed for approximating the calculation of the oscillations of a freely supported beam with a given control. Control is considered via several fixed point actuators. Finding the control that performs oscillations damping is accomplished via minimization of a certain function of multiple variables by the Marquardt method.

For sufficiently large sizes of a finite-difference grid, the process of finding a numerical solution can be computationally complex and time-consuming. Empirical formulas are proposed, which make it possible to significantly reduce the calculation time. Empirical formulas are best used as an initial approximation or in tasks where precision is not a high priority.

The case is considered when different groups of actuators are used for different periods of time to dampen oscillations.

The oscillations damping is considered in the presence of constraints imposed on the control functions. In this case, the minimum time for damping the oscillations exists. It is shown that this time can be reduced by increasing the number of actuators.

In conclusion, an estimate was made of the real-time of finding the control of oscillations damping and, as a consequence, the possibility of practical implementation of the proposed algorithm.

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