Chapter 5
The Spreadsheet Affordances in Solving Complex Word Problems

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5.1 Introduction

Solving word problems has long been considered a very important context for the use and development of students' algebra knowledge, from elementary to higher education. Problem solving has also been advocated as a rich learning context for engaging students in the learning of algebraic methods as well as in promoting algebraic reasoning (Blanton & Kaput, 2005; Kaput, 1999; NCTM, 2000; Yerushalmy, 2006). However, research has revealed in several studies (e.g. Stacey & MacGregor, 2000) that many students, while knowing the formal methods of algebra avoid their use in solving word problems and choose numerical methods instead, that is, they take on arithmetic reasoning rather than algebraic reasoning. The value ascribed to the arithmetic methods that students use to find the solutions to problems that could be solved through the formulation of an equation or set of equations is still a subject of contention among educators and researchers. For some, the so-called informal (or non-algebraic) methods, as opposed to the formal algebraic methods (symbol use and techniques) that students generate for solving word problems, becomes a barrier to the learning of powerful methods for solving a large class of problems (Stacey & MacGregor, 2000). Others have claimed that algebraic thinking should not be reduced to the use of symbols and formal methods, suggesting that informal methods are relevant for making sense of problems and represent a means of reaching conceptual understanding of the algebraic methods (Johanning, 2004).

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In our research, we are privileging problem solving as an activity that can foster and anticipate the learning of algebraic methods by emphasizing its meaning and applicability. In addition, we want to know how the spreadsheet affordances can help students in dealing with complex word problems as a springboard for their learning of formal algebra methods. Our focus is on students' understanding of the algebraic formal methods supported by the use of the spreadsheet and its specific form of operation, which many authors consider useful in making the transition between arithmetic and algebraic language (e.g. Friedlander, 1998; Haspekian, 2005; Kieran & Yerushalmy, 2004; Nobre, 2016; Wilson, 2007). For example, the process of developing relational thinking through the construction of columns that dependent on other columns may encourage the understanding of algebraic conditions; likewise, solving an equation has its numerical counterpart in the action of inspecting the values of a table obtained from formulas that represent algebraic conditions. The same holds for the solution of simultaneous equations, except for the number of formulas and the type of functional relations that obviously get more complex.

In the existing research (e.g. Ainley, Bills, & Wilson, 2004; Calder, 2010; Dettori, Garuti, & Lemut, 2001; Rojano, 2002), the spreadsheet has shown to be an appropriate instrument to establish functional relations as well as a vehicle to promote pattern recognition and rule generalization, which may lead students to a deeper understanding of the algebraic language and methods. In our study, we intend to know more about the ways in which the affordances of the spreadsheet are significant to the success in solving problems entailing many algebraic conditions. One of the reasons for our interest is the perception that the range of spreadsheet affordances that are relevant to develop a sound understanding of the fundamentals of solving systems of simultaneous equations remains to be investigated. For example, we may note that solving systems of linear equations can be treated as a way of obtaining the coordinates of a point in which two or more functions intersect. Hence, it results the importance of bringing the spreadsheet to the dialogue between equations and functions as both are key concepts in school algebra.

The problems here labeled as 'complex word problems' are situations that require a clear identification of the multiple conditions and variables involved and of the ways the variables depend on each other; so being, these are not straightforward start-unknown problems.

We have collected data on how such problems were successfully solved by means of the spreadsheet by students who had not yet learned the algebraic methods of solving simultaneous linear equations. Taking into account the focus of our study, we set the following research question: in what ways are the affordances of the spreadsheet, particularly its representational possibilities, significant in fostering students' models of the complex structure of the problem situation?

5.2 Theoretical Background

5.2.1 Informal Versus Formal Methods in Problem Solving

Some word problems may be categorized as algebraic word problems, in the sense that they imply finding one or more unknowns and can be solved through its translation into algebraic equations and subsequent resolution by means of algebraic methods. If the student has already learned specific formal algebraic methods (such as solving equations and inequalities, or solving systems of equations, etc.) then much of the algebraic word problem is (immediately) solved and therefore it becomes a routine task. However, the research has shown that the success in solving algebraic word problems does not rely exclusively on the mastery of formal methods, especially if such methods were learned without conceptual understanding.

The formal methods of algebra are undoubtedly powerful and effective in solving various problems, leading students directly to the solution and freeing them from pursuing alternative strategies. However, moving from the informal to the formal methods is not easy for most students. If little time is spent in the use of informal strategies and if procedures are rapidly imposed and routinely performed, then students are more likely to make mistakes, which they will not be able to identify and/or correct (Wagner, 1983). In addition, students who typically perform well in formal procedures often reveal a limited understanding of their meaning and are unable to deal with problem situations other than the standard ones.

Kieran (2006), alongside the results of other researchers, acknowledges that students often prefer to resort to arithmetic methods and reveal difficulty in using algebraic equations when solving algebraic word problems. Although, at first glance, arithmetical thinking may seem to be an obstacle to the development of algebraic thinking, the fact is that it can also be taken as a valuable source of that development. In this sense, having problem solving as a learning context to engage students in the learning of algebraic methods appears as a legitimate option, mainly because word problems usually allow a variety of approaches and strategies, ranging from informal to algebraic.

Koedinger, Alibali, and Nathan (2008) have found that in an initial stage of algebra learning students perform better in solving word problems than in solving equations. This is because they can use their reasoning with quantities and numbers, drawing on their knowledge of arithmetic without the obligation to manipulate symbolic language. That form of reasoning, which is more independent of the use of algebraic symbolism, can be seen as an opportunity for the emergence of multiple representations to work on the problem structure and can contribute to a better subsequent learning of algebraic methods.

Hiebert and Carpenter (1992) also argue that the use of symbolic representations should not be forced when students are at an early stage of solving a certain type of algebraic word problems. From their point of view, the work with symbols and algebraic procedures may naturally follow from experience in solving problems. Otherwise there is the risk that students develop an incomplete understanding of algebra and algebraic methods. Although there are students who are more proficient with symbolic representations and procedural algebra and who are able to apply symbols and rules in solving a problem, the learning of any mathematical procedure must be supported by conceptual knowledge including an understanding of what symbols and rules mean.

5.2.2 Problem Solving in the Development of Algebraic Thinking

Algebra involves a specific form of thinking that goes beyond the simple manipulation of symbols. Thus, working with symbols must be enriched so that the study of algebra is not reduced to rote learning. Problem solving is a context for facilitating the assigning of meaning to symbols and to the work with symbols. Several researchers have highlighted the role of problem solving in the development of algebraic thinking (e.g. Bell, 1996; Mason, 2008). Among other advantages, it has been suggested: the possibility of leading the students to mentally work on one or more as-yet-unknown quantities and to focus on the relations between the mathematical objects rather than on the objects themselves (Windsor, 2010).

We may consider that algebraic thinking is a developing form of reasoning, which instead of requiring a cut with arithmetic thinking, entails a progress through different stages. In a first phase, students learn to describe relations in natural language and begin to deal with generalization. Later, they will be encouraged to use diagrams, abbreviations, and symbolic notations to express their reasoning. Finally, they will begin to use algebraic expressions, such as equations, together with tables of values, graphs, and other formal representations.

Non-routine problems that present real challenges to the students (rather than a task for the application of a previously learned method) can be seen as opportunities for the construction of new algebraic knowledge. The problem solving activity allows the emergence of several strategies that are born from the knowledge they already have but it may also be an opportunity to give meaning to the subsequent formalization of the initial processes (Rojano, 1996, Slavitt, 1999). As suggested by Kieran (1996) and others, algebraic thinking has its roots in the establishing of relations between quantities and it progresses as different tools, and not only the symbols, are introduced as a form of structuring a discourse that is inherent to algebra. As such, thinking algebraically involves knowing various forms of representation, namely symbolic. But, it also implies flexibility in the transition between modes of representation, as well as the ability to operate with symbols, in a given context and when appropriate (Schoenfeld, 2008). From focusing on concrete objects, to the relations between them and to the ways of representing them, algebraic thinking evolves towards reasoning about those relations in a general and abstract way.

Recognizing the givens, the unknowns, and the conditions that make the structure of a problem and representing them appropriately is a key step in using algebra for solving problems. As suggested by Dettori et al. (2001), the spreadsheet can be helpful for algebra learning in that it can support students' understanding of what means to solve an equation or a system of equations, even before the formal learning of those methods.

5.2.3 The Spreadsheet in Algebraic Problem Solving

Using the spreadsheet allows students to explore and obtain a solution to an algebraic word problem in an informal way. This environment emphasizes the need to identify all the relevant variables and, in addition, stimulates the search for functional relations between variables. Translating the conditions of the problem into variable-columns under appropriate labels enables a tabular representation, which gives a suitable and clear image of the variables and functional relations involved (Ainley et al., 2004; Calder, 2010; Dettori et al., 2001).

Solving problems with the spreadsheet favors the establishment of a connection between the language of formulas, which is distinctive of this digital environment, and the symbolic algebraic language with pencil and paper. The use of the spreadsheet is also a means to bridge the gap between informal algebraic thinking and the ability to use algebraic notation to express such thinking, as highlighted in the study by Carreira, Jones, Amado, Jacinto, and Nobre (2016).

As emphasized by several authors (e.g. Ainley et al., 2004), the spreadsheet is a powerful tool in mathematical problem solving and particularly in algebraic problem solving. This digital tool allows using and combining different types of representations, such as words, numbers and formulas, and the creation of tables and graphs, besides the insertion of objects produced with other tools, namely an image editor. One feature that distinguishes the spreadsheet from other digital environments is the fact that it supports the connection between different registers (numeric, symbolic and graphical).

When handling a spreadsheet, students have the opportunity to discover and understand the meaning of a cell, a column, and a formula, what it means to drag down the handle of a cell with a formula, as they automatically receive numerical feedback from the computer. According to Haspekian (2005), "communicating with a spreadsheet requires that pupils use an interactive algebra-like language, which focuses their attention on a rigorous syntax. This is why it is said that spreadsheets help to translate a problem by means of an algebraic code" (p. 113).

We claim that in the case of algebraic word problems the spreadsheet can help students find and express relationships between the givens and the unknowns in a given problem. In addition, it provides forms of control based on instantaneous and constant numerical feedback, which encourages exploration and prediction (Carreira et al., 2016; Nobre, 2016).

In solving and expressing the solution of a mathematical problem, the ability to record and organize information, the clarity in the expression of ideas and the production of solid explanations are important abilities. The use of different representations is fundamental in the expression of mathematical thinking within problem solving. However, any representation may be transparent or opaque. This distinction, made some time ago by Lesh, Behr, and Post (1987), means that a representation may be more or less obviously attached to the idea that it is meant to stand for, as it tends to underline a few aspects of the idea while fading others. Zazkis and Liljedahl (2004) further developed the transparency/opacity of the representations in suggesting that there is a certain degree of opacity in any mathematical representation. In the case of the representational register of the spreadsheet, the apparent opacity tends to dissipate as students gain familiarity with the specific syntax of the tool and greater flexibility in making a connection between their algebraic thinking and their actions with the spreadsheet.

Moreno-Armella, Hegedus and Kaput (2008) have put forward the idea of co-action to explain and describe the changes that the use of digital technology brings into students' mathematical activity. The idea of co-action is related to the fact that students are at the same time guiding and being guided by the dynamic and interactive digital environments. In solving a problem with the spreadsheet, the co-action between the student and the tool begins with the need for structuring the conditions of the problem in columns or cells that are assigned particular roles. This procedure allows connecting a set of numbers (in a column, for example) with a single label (or column heading), which is consistent with an idea of variable, and that action pushes students' reflection on the conditions involved in complex algebraic word problems and helps them to understand the mathematical meaning of variable and function (Wilson, 2007). The introduction of numerical data in different cells, which may include the use of formulas, becomes part of establishing the relationships described in a problem situation. In addition, students can analyze the immediate numerical feedback provided by the spreadsheet and redirect their actions in a permanent flow of interactions with the computer. This work, based on the identification and implementation of functional relations, induces an algebraic organization in the way of addressing a problem that apparently has a numerical form (Haspekian, 2005). Students are then able to inspect their numerical tables in search for the solution to the given problem.

The affordances of a digital medium are related to the opportunities that the environment offers to the learning process (Gibson, 1986). We might consider them as perceived opportunities in line with the intentions of the user. This indicates a feature of complementarity between the learner and the environment. Therefore, in the activity of problem solving with the spreadsheet, a very important aspect is the knowledge that the student has of the tool.

Previous studies about the use of the spreadsheet in the learning of mathematics (e.g. Ainley et al., 2004; Calder, 2010; Haspekian, 2005) show that this digital tool stimulates students' mathematical reasoning. In dealing with a word problem, the students can make various experiences in a short time. This will enable a stronger focus on the underlying mathematical ideas rather than on the routine mathematical calculations. One of the advantages offered by the spreadsheet is the possibility, in a quick way and whenever necessary, of visualizing the representation of a relation between cells. This supports students in the recognition, understanding, and expression of algebraic relationships they have formulated through the spreadsheet syntax. Students use spreadsheet-specific calculations in building general rules and often check their general rule with reference to numbers. In this way, links between algebraic symbols and general expressions are established. This idea is corroborated by the work of Abramovich (1998), which shows that the use of the spreadsheet supports the transition from computations to algebraic formal language.

The immediate feedback that students receive from the tool gives them freedom to explore different trials and encourages them to make conjectures. This permanent reflection about the results obtained leads students into new conjectures and new questions. We argue that such distinctive processes carried out by the students with the spreadsheet help them to refine their strategies, broaden their knowledge about the variables and about the relationships involved, therefore influencing the nature of their conceptual understanding of the problem structure.

In this chapter, our aim is to look at how the affordances of the spreadsheet allow the students to use different conceptual models that correspond to several ways of formulating and solving systems of simultaneous equations.

5.3 Context and Method

In the following, we describe and analyze how different middle school students express their mathematical thinking when solving a complex algebraic word problem (Fig. 5.1) with the use of the spreadsheet. We will focus on their digital representations in solving the problem by considering their spreadsheet-based models in relation to their algebraic thinking and to their problem-driven algebraic models.

The empirical context was a class of 8th grade students (13–14 years old), in a Portuguese middle-school located in the southern region of the country where the economic activity is strongly rooted in the primary sector (agriculture and fisheries). The class had a total of twenty-four students, 10 boys and 14 girls, two of whom were migrant children from Ukraine. During the school year, problem solving was implemented as the context for mathematical activity, aiming for the development of

The restaurant *Sombrero Style* was opened yesterday and I went there for dinner with three friends. The maximum capacity of customers – said the manager – is 100 people. Luckily, I had booked a table for four as when I got there several tables were already full with four people and one table with three people. While I was waiting for the waiter to take us to the table, I counted the women and men who were in the restaurant and the number of women was exactly twice the number of men.
What could be the maximum number of people who were already at the restaurant when I came in?

Fig. 5.1 The opening of the Restaurant "Sombrero Style"

algebraic thinking and the learning of formal algebraic methods. In each problem-solving lesson, the students were given the freedom to choose whether they wanted to work in groups or individually as part of the established didactical contract. Most of the students worked in pairs and only very few worked individually. In the class, the teacher regularly engaged in dialog with the students and asked questions whenever necessary to monitor, support or challenge students' reasoning and approaches. The students had previously gained some experience in solving word problems with a spreadsheet, in their mathematics classes, from which they acquired the basics of the spreadsheet functioning. Many of the problems given to the students were chosen among the problems proposed in the web-based competition SUB14 promoted by the University of Algarve. It is a mathematical problem solving competition of inclusive nature, addressing 7th and 8th graders and running through the Internet, which launches a new mathematical problem every two weeks at the competition website. Some of those problems were solved by the students of the class during class time and after that each of them had the choice of sending their answers to the competition through e-mail, if they wished to.

In the classroom, the detailed recording of the students' processes in the computer was achieved with the use of the software Camtasia Studio. This software allows the simultaneous recording of the students' dialogues and of the computer screen, therefore capturing all the actions performed on the computer.

All the solutions developed by the students were examined on the basis of (i) the organization of the tables created in the spreadsheet, and (ii) the choice of independent variables and consequent decision on dependent variables. This screening of the solutions led to the identification of three different types. They are discernible by the kind of strategy adopted in organizing the spreadsheet and the corresponding way of modelling the simultaneous conditions that were given in the problem.

The analysis of the data was developed in two phases. In the first phase, the models and the representations developed by the students in the spreadsheet to obtain the solution were identified. At this stage two main approaches could be distinguished: (i) one in which the students constructed two independent tables (each one translating a set of conditions of the given problem) and which were later compared for the search of the intersection point (value that was a simultaneous solution of the conditions translated in each table); (ii) the other in which the students constructed only one table where all the conditions mentioned in the problem were contemplated and where the search for the simultaneous solution was made in a row of that table. In a second phase, students' models were analyzed, considering, in particular, their choices for an independent variable, which made the remaining unknowns depending on that one, through formulas that represented given conditions. With this analysis, it was possible to create schematic models of their interpretations of the given situation. Finally, those models were algebraically expressed as systems of simultaneous equations, with the aim of detecting the different alternatives involved in the students' approaches from the point of view of the use of formal algebraic methods.

Here we will discuss three specific solutions that are, to a certain extent, representative of the variety of solutions that have emerged in the classroom.

w - number of women, m - number of men, y - total of persons,
x - number of tables of 4 persons

$$\begin{cases} w = 2m \\ y = w + m \\ y = 4x + 3 \\ 0 < y \le 100 \end{cases} \quad (\max\ y)$$

Fig. 5.2 An algebraic formulation with a system of equations

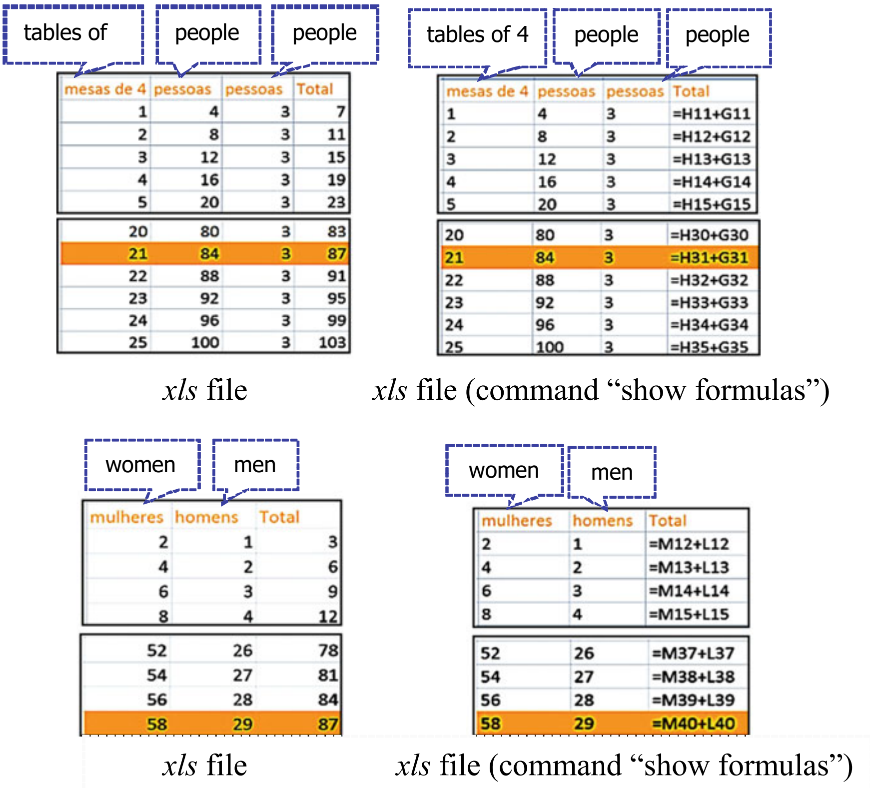
5.4 Results

As mentioned before, the students were not yet aware of the formal methods of solving systems of simultaneous equations. Besides, in middle school, students do not work with more than two equations and two unknowns. As such, from a symbolic algebra perspective, the given problem was beyond the students' knowledge of formal algebraic methods in terms of solving a system of three equations with three unknowns, as the following (Fig. 5.2).

Without the formal algebra method, students began to use the spreadsheet as a means for structuring the problem conditions in successive interconnected columns. They went through the translation, in numbers or formulas, of the relations between variables, thus obtaining numerical tables. Finally, the students controlled the data produced by searching the solution that satisfied the conditions imposed as those were displayed in the spreadsheet. To a certain extent, the representations provided by the spreadsheet were also a means of verifying the solution to the problem.

5.4.1 Solution 1

A pair of students, Maria and Jessica, started by addressing the condition of the problem on the number of people seated at tables of 4 (Fig. 5.3). In their model, the number of tables of 4 people is treated as a variable that changes within the set of whole numbers (represented as variable-column). Moreover, in the students' model of the situation it plays the role of an independent variable, in the sense that other variables are dependent on this one, such as the number of people sitting at the tables (therefore the multiples of four). That second set of values was generated in a second column using the Autofill to produce a linear sequence increasing by 4. Then Jessica entered a constant-valued column filled with the constant 3, which referred to the three people that were sat at a particular table of the restaurant. This third column mostly plays the role of a numerical parameter in the sense that it is a fixed value regardless of the number of tables in the restaurant or the number of people sitting in groups of four. Finally, the students created a fourth column to compute the total



mesas de 4	pessoas	pessoas	Total
1	4	3	7
2	8	3	11
3	12	3	15
4	16	3	19
5	20	3	23
20	80	3	83
21	84	3	87
22	88	3	91
23	92	3	95
24	96	3	99
25	100	3	103

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mesas de 4	pessoas	pessoas	Total
1	4	3	=H11+G11
2	8	3	=H12+G12
3	12	3	=H13+G13
4	16	3	=H14+G14
5	20	3	=H15+G15
20	80	3	=H30+G30
21	84	3	=H31+G31
22	88	3	=H32+G32
23	92	3	=H33+G33
24	96	3	=H34+G34
25	100	3	=H35+G35

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mulheres	homens	Total
2	1	3
4	2	6
6	3	9
8	4	12
52	26	78
54	27	81
56	28	84
58	29	87

xls file

mulheres	homens	Total
2	1	=M12+L12
4	2	=M13+L13
6	3	=M14+L14
8	4	=M15+L15
52	26	=M37+L37
54	27	=M38+L38
56	28	=M39+L39
58	29	=M40+L40

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Fig. 5.3 Maria and Jessica’s spreadsheet model

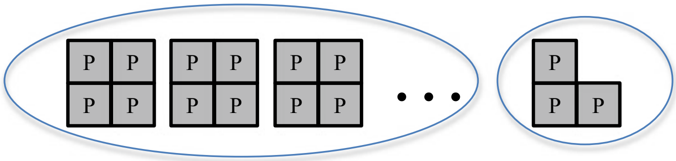

Fig. 5.4 First part of the model—People organized by tables

number of people sitting at the restaurant by adding the variable “people in groups of 4” and the parameter “3 more persons”.

The following diagram (Fig. 5.4) illustrates the reasoning undertaken by the students in constructing the first table on the spreadsheet (the total of people is represented by the multiples of 4 plus 3).

Then, the students represented, in a separate spreadsheet table, the condition for the ratio of men and women in the total of clients, by creating new variable-columns (number of men, women, and total of individuals). They realized that the number

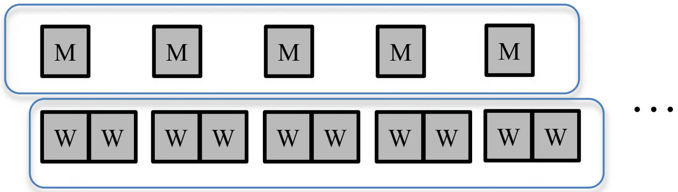

Fig. 5.5 Second part of the model—People grouped by gender

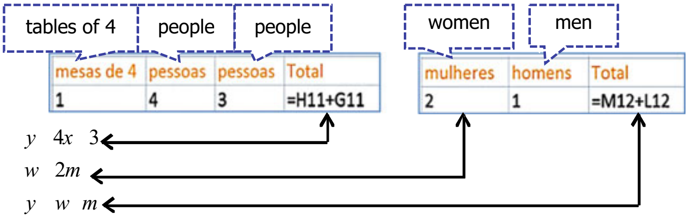

Fig. 5.6 The corresponding algebraic model (Maria and Jessica's model)

of women had to be an even number, as it was twice the number of men in the restaurant. This led them to the idea of creating a variable-column filled with the consecutive even numbers. A column for computing the number of men was then obtained by dividing the previous one by 2, which yielded the consecutive whole numbers. Finally, by adding the values of the two columns, a new column generated the total number of clients in the restaurant. The resulting total is obviously given by the multiples of 3. This line of reasoning is illustrated by the diagram of Fig. 5.5.

With this approach, the students developed a model based on the separation of two sets of conditions, each of them generating an output of values that would have to match at some point. The shaded rows (Fig. 5.5) show that the students carried out the inspection of the solution by comparing the columns of totals in the two separate tables (their answer is 87 people in the restaurant, not counting the remaining 4 people that booked a table). The solution of the problem, as their conceptual model highlights, is therefore the sum of a multiple of 12 with 3.

We may notice an interesting relationship between Maria and Jessica's model on the spreadsheet and the corresponding symbolic model (Fig. 5.6). The students initially separated the conditions in two unconnected tables (two separate equations) but later, by inspecting the results in the two columns of totals, they made the necessary connection (two simultaneous equations). As a result of the way they expressed the conditions on the spreadsheet, these students were also able to obtain additional information about the situation of the problem, namely the number of women and men already in the restaurant.

1/3 Men + 2/3 Women | Men | Women

1/3 Homens + 2/3 Mulheres	TOTAL	Homens	Mulheres
3	98	32,667	65,3333
3	97	32,333	64,6667
3	96	32	64
3	95	31,667	63,3333
3	94	31,333	62,6667
3	93	31	62
3	92	30,667	61,3333
3	91	30,333	60,6667
3	90	30	60
3	89	29,667	59,3333
3	88	29,333	58,6667
3	87	29	58
3	86	28,667	57,3333

xls file

Men | Women

1/3 Homens + 2/3 Mulheres	TOTAL	Homens	Mulheres
3	98	=C2/B2	=D2*2
3	97	=C3/B3	=D3*2
3	96	=C4/B4	=D4*2
3	95	=C5/B5	=D5*2
3	94	=C6/B6	=D6*2
3	93	=C7/B7	=D7*2
3	92	=C8/B8	=D8*2
3	91	=C9/B9	=D9*2
3	90	=C10/B10	=D10*2
3	89	=C11/B11	=D11*2
3	88	=C12/B12	=D12*2
3	87	=C13/B13	=D13*2
3	86	=C14/B14	=D14*2

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Tables of 4 | Tables of 3

Mesas 4	Mesas 3	Total
4	3	7
8	3	11
12	3	15
16	3	19
80	3	83
84	3	87
88	3	91
92	3	95
96	3	99

xls file

Tables of 4 | Tables of 3

Mesas 4	Mesas 3	Total
4	3	=H2+G2
8	3	=H3+G3
12	3	=H4+G4
16	3	=H5+G5
80	3	=H21+G21
84	3	=H22+G22
88	3	=H23+G23
92	3	=H24+G24
96	3	=H25+G25

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Fig. 5.7 Excerpt of Carolina’s solution in the spreadsheet

5.4.2 Solution 2

Carolina is another student who also organized the conditions by separating them in two distinct tables, as shown in Fig. 5.7.

In her resolution, the student used simultaneous increasing and descending sequences. She starts by separating the customers by gender. In a column, she generates a sequence of whole numbers accounting for the total number of people in the restaurant. By starting with 98, she takes into account some of the problem givens although she does not consider the fact that one table for four persons would be still available. In the next column, she calculates the division of the totals by 3, in order to get one-third of the totals (the number of men). In another column, she calculates twice the previous results (the number of women). The diagram of Fig. 5.8 can illustrate this first part of the model used by Carolina.

In the last three columns, she makes the distribution of customers by tables of 3 and of 4. She used increasing sequences where the successive multiples of 4 represent the varying number of people in tables of 4 and since there was only one table with 3 people, the number 3 is repeated along another column. Then she adds the values in the previous two columns, which yields a column for the total of people. This

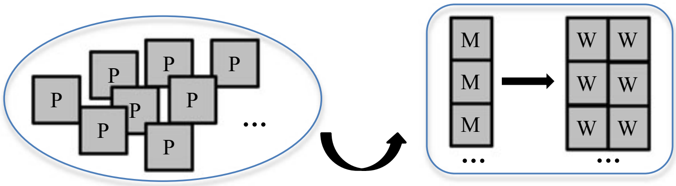


Fig. 5.8 First part of the Carolina's model—People organized by gender

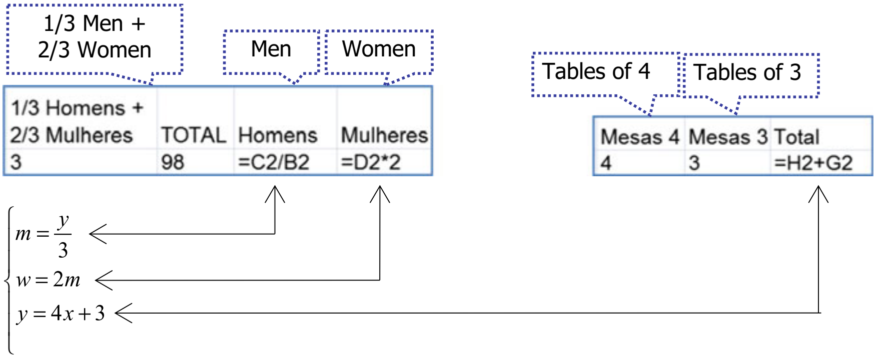


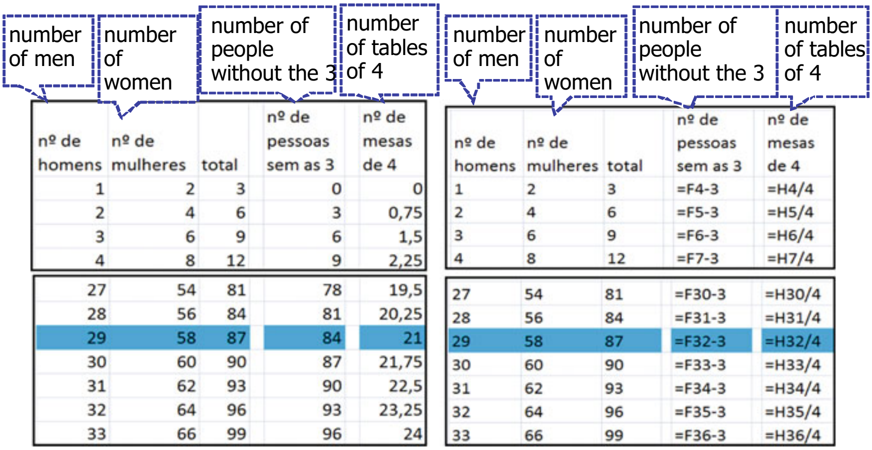
Fig. 5.9 Translation into algebraic language of Carolina's model on the spreadsheet

second part of Carolina's model is similar to the first one used by Maria and Jessica (Fig. 5.4).

To get the solution, Carolina compares the two columns with the totals and she finds the same number appearing in both columns (the shaded rows), which gives her the number of people in the restaurant.

In solving the problem, the student uses the idea of proportion to 'separate' the customers by gender, as mentioned in her answer: "Since the number of women is exactly twice the number of men, it can be said that the total of persons is represented as 3-thirds, being one third of men, and two thirds of women". She also uses the notion of multiples of four to define the number of people sitting at tables of four and a column with the number 3 to act as a constant standing for the three people seating in one table.

It is apparent that using the spreadsheet pushed her to identify all the relevant variables and constants and encouraged the search for dependency relations. In addition, it led to a strategy that allowed addressing the two conditions involved in the problem separately, and later making their connection by finding equal outputs in the two independent tables created. Her reasoning may be translated into algebraic symbolic language through the system of equations presented in Fig. 5.9.



nº de homens	nº de mulheres	total	nº de pessoas sem as 3	nº de mesas de 4
1	2	3	0	0
2	4	6	3	0,75
3	6	9	6	1,5
4	8	12	9	2,25
27	54	81	78	19,5
28	56	84	81	20,25
29	58	87	84	21
30	60	90	87	21,75
31	62	93	90	22,5
32	64	96	93	23,25
33	66	99	96	24

nº de homens	nº de mulheres	total	nº de pessoas sem as 3	nº de mesas de 4
1	2	3	=F4-3	=H4/4
2	4	6	=F5-3	=H5/4
3	6	9	=F6-3	=H6/4
4	8	12	=F7-3	=H7/4
27	54	81	=F30-3	=H30/4
28	56	84	=F31-3	=H31/4
29	58	87	=F32-3	=H32/4
30	60	90	=F33-3	=H33/4
31	62	93	=F34-3	=H34/4
32	64	96	=F35-3	=H35/4
33	66	99	=F36-3	=H36/4

xls file *xls* file (command "show formulas")

Fig. 5.10 Ana's spreadsheet model

5.4.3 Solution 3

The approach of another student in the class reveals a similar start but it progressed in a way that prevented the separation of the problem conditions in two tables. In fact, Ana started by representing the relationship between the number of men and the number of women and then obtained a column for the total of people (Fig. 5.10). The student soon concluded that the totals in that column were multiples of 3. Then she subtracted 3 to the total of persons to account for the fact that only one table had 3 individuals, and then divided the result by 4. Her idea was to have the remaining people arranged in groups of 4 due to the fact that they were all at tables of 4. Therefore, in a new cell, Ana entered the title "Number of tables of 4" and bellow she created a formula that made the division by four of the total number of people and then dragged the fill handle. This way, she was aiming to find the number of tables of 4 that were taken in the restaurant. To obtain the solution, she just had to inspect the values in that column in search for the whole numbers and for the highest number lower than 100. In her answer to the problem, Ana wrote: "The maximum number must be 87 before the 4 friends came in; if I had considered the number 99 and added the 4 friends I would get 103 as the total, but the capacity of the restaurant is 100 people, which means that it is not the solution".

We may also describe Ana's conceptual model schematically to better see the similarities and differences from the previous cases (Fig. 5.11). Although hers is also a model that involved two steps, it did not require two separate tables and instead she managed to combine the various problem conditions in a single table. In fact, Ana expressed the conditions on the spreadsheet by chaining them in successive compositions.

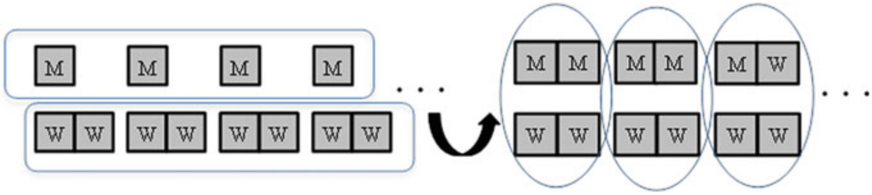


Fig. 5.11 People grouped by gender rearranged in groups of four

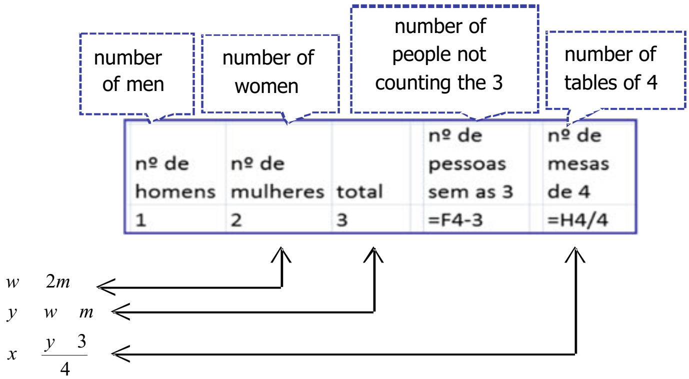


Fig. 5.12 Translation into algebraic language of Ana's model on the spreadsheet

Next, we propose a representation of Ana's spreadsheet-based model with the corresponding algebraic language (Fig. 5.12). The symbolic model shows that in this case the number of persons at the restaurant (minus 3) has to be a multiple of 3; moreover, the test to reach the solution means finding out the multiples of 3 that are also divisible by 4. Therefore, the solution belongs to the set of multiples of 12.

5.5 Discussion and Conclusion

Although the students had not yet learned the algebraic representations, namely a system of equations with several unknowns, the spreadsheet allowed them to undertake and explore other approaches, which reflected a variety of representations involving formulas and tables and consequently a diversity of conceptual models of the given problem.

The three solutions presented above indicate that the spreadsheet proved to be useful to solve the problem and that the students recognized it as an appropriate tool for solving this problem. They also took different approaches; some considered all the conditions of the problem jointly represented, while others used a strategy of separating two sets of conditions and then comparing the numerical values to find a common value to obtain the solution.

In general, the spreadsheet helped students dealing with a complex word algebraic problem that was beyond their mathematical knowledge of algebraic methods, namely of solving systems of simultaneous equations. One important aspect emerging from the set of solutions was the fact that students showed a clear notion of the solution as a value that would need to satisfy simultaneously a set of equations and conditions. In this sense, we may consider that students gave meaning to the concept of a solution of simultaneous equations, which is often a difficult idea in initial stages of the learning of the algebraic method.

By means of sophisticated numerical approaches based on defining numerical sequences that described functional relations, they were able to make use of variable-columns, using the specific syntax of the spreadsheet, and create chained relations between variables. We claim that those affordances of the spreadsheet were fundamental in structuring the students' problem solving approaches and in providing them a representational system to express the conditions given in the problem. This conclusion resonates with other studies such as Ainley et al. (2004) and it provides a clear indicator of how students interpreted the problems in light of their mathematical knowledge and their knowledge of the tool. For example, decisions such as the ways of defining variable-columns are intrinsically connected to the students' model on how some variables depend on other variables and to their choice of independent and dependent variables. Therefore, the different spreadsheet organizations appearing in a particular problem are a consequence, among other things, of conceptual choices and constitute a powerful mirror into students' successful ways of conceiving the structure of the problem situation. Such decisions can be seen as originating from the ongoing interaction between the subject and the tool, in a way that makes it impossible to separate the two. Evidences of such interdependence are quite clear in students' problem-solving activity, particularly in their ways of reporting the reasoning developed while working in the digital medium.

The work with the spreadsheet transforms the nature of students' mathematical representations to the extent that those become encapsulated in a medium with very specific characteristics. The solution of the problem solved with the spreadsheet arises from the student's ongoing interaction with the tool; both the student and the spreadsheet act and react to each other throughout the activity (Moreno-Armella & Hegedus, 2009). This type of work has significant consequences for the expression of students' mathematical thinking, particularly of algebraic thinking, during their problem solving-and-expressing (Carreira et al., 2016).

The three solutions here analyzed, in terms of their symbolic algebraic counterpart (the systems of equations that they mirror) highlight that the individual's interactions with the spreadsheet can lead to solutions with distinct algebraic features. Algebraic problems of some complex level seem to be especially useful for bringing out different conceptual models and simultaneously different forms of expression of these models based on the spreadsheet.

We suggest, following the results by Calder (2010), that different solutions may take place depending on the level of experience of the students with the spreadsheet, on their understandings of the ideas involved, and on their perceptions about the ways to model the conditions of the problem with the digital tool. The participants in this study took advantage of the spreadsheet in different ways, using different mathematical concepts and simultaneously different affordances of the digital tool.

Hegedus (2013) has underlined the idea that technological affordances must become mathematical affordances and argued that meaningful integration of technology in the learning environment should be developed through mathematization of technological affordances. Thus, he pointed out a set of future design principles (executable representations, co-action, navigation, manipulation and interaction, variance/invariance, mathematically meaningful shapes and attributes, magnetism, pulse/vibration, construction and aggregation) that must deserve further attention in the upcoming research and development efforts. The co-action, one of the characteristics in the list, is one that we find clearly important to the study of solving algebraic word problems with the use of a spreadsheet. Examples of co-action, as those presented in the solutions above, illustrate ways in which the spreadsheet affordances offer routes to obtain the solution to a problem.

Based on the data collected, we may state that the idea of transforming unknowns into variables and creating tables that translate functional relations is a mathematization of specific affordances of the spreadsheet. One of them is the use of formulas to create dependency relations between variable-columns. Another is the possibility of decomposing the set of conditions of a given problem into several tables representing pieces of the same problem. The comparison between those tables is one of the ways to obtain the solution, by searching for the value that verifies all the subcomponents simultaneously (subsystems of equations).

Finally, the many different possibilities of translating mathematical conditions into relations between variable-columns suggest that the spreadsheet favors the production of diverse conceptual models that are interesting and mathematically powerful. Such diverse models, when represented by means of the spreadsheet syntax, provide a rich image not only of the many ways of having a word problem translated into a set of equations but also of the many ways in which the solution of the simultaneous equations may be algebraically obtained, depending on the transformations and operations you perform in solving it.

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