Chapter 4
Mathematical Problem Solving and the Use of Digital Technologies

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4.1 Introduction

4.1.1 Technology in Society and Its Importance in Education

The irruption of digital technologies in society is transforming the way individuals interact, communicate, and carry out daily activities. People rely on digital technologies to get access to several online services and information to make daily decisions. Likewise, the use of technology is also opening new routes for students to learn disciplinary knowledge. In dealing with mathematical tasks, students, with the use of technology, have an opportunity of relying on technology affordances to represent and explore ways to understand mathematical concepts and solve mathematical problems. Mason (2016) argues that "…something or some situation is a problem only when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in some sense-making activity" (p. 263). Asking and pursuing questions, checking examples or considering and exploring some special cases, making conjectures, looking for counterexamples, and supporting mathematical relations are problem solving strategies and actions that are important for learners to work on mathematical tasks (Santos-Trigo & Moreno-Armella, 2016). What then could the use of digital technologies offer to learners in terms of implementing these types of strategies during the process of understanding mathematical concepts and solving problems? Leung (2011) points out that "a pedagogic reason for using technology is to empower learners with extended or amplified abilities to acquire knowledge…technology can empower their cognitive abilities to reason in novice ways" (p. 327). That is, learners, with the use of technology, can engage in dynamic explorations of mathematical ideas and enhance their ways of reasoning to

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formulate and support conjectures. Similarly, the use of communication technologies can facilitate and enrich mathematical discussions within an extended learning community. Walling (2014) argues that "learning design must be flexible not only because students are diverse in their needs, interests, aspirations, and abilities but also because the very nature of the modern world demands it" (p. 14).

Gros (2016) point out that, the use of technology is changing and shaping what we learn, how we learn, where and when we learn. What types of digital technologies are helpful and how can students use them to understand mathematics and develop problem solving competencies? In a technological environment, teachers and students might rely on different digital technologies and online developments such as Internet, a Dynamic Geometry System (DGS), mobile applications, tablets, Wikipedia, etc. to represent, explore, expand, analyze, explain, and share their mathematical ideas, concepts or problems solving approaches.

Using digital technologies in learning environments implies addressing issues regarding what new pedagogies are needed to frame mathematical working and learning spaces in which learners participate in the construction and use mathematical knowledge. Gros (2016) states that "technology must enable and accelerate learning relationships between teachers and students and between students and other "learning partners" such as peers, mentors and others with similar learning interests" (p. 18). That is, technologies might expand and enhance students' ways to share and discuss mathematical ideas as a part of an extended learning community.

> Mathematical tasks and ways to implement them are essential ingredients in structuring a learning environment for students to engage in mathematical activities. "A challenge in digital task design is to conceive tasks that can extend and amplify pedagogical features present in non-digital environments". (Leung & Baccaglini-Frank, 2017b, p. x)

Gros, Kinshuk, and Maina (2016) argue that for students to deal with the complexity involved in this technological society, they need to develop and exhibit strategies to solve problems collaboratively, communicate results, and to constantly interact with peers and other experts. Indeed, with the use of communication tools or mobile applications students expand individual and self-directed problem solving behaviors to include collaborative learning through direct and continuous interaction with peers and group experts. Gros (2016) point out that the incorporation of digital technologies in learning scenarios involves discussing the design of mathematical tasks, the role of teachers and students, and the educational context or learning scenarios to implement the tasks.

The goal of this chapter is to analyze and discuss ways in which the coordinated and systematic use of several digital technologies provides affordances for teachers/students to represent dynamically concepts, explore and solve mathematical problems.

To delve into the representations, strategies and ways of reasoning that emerge in technology problem solving approaches, four types of tasks are identified and analyzed in terms of characterizing how technology affordances shape their solution processes.

(a) Focusing on figures. This group emphasizes the importance of using DGS in reconstructing figures that are embedded in problem statements;
(b) Investigation tasks. This group deals with investigation tasks where students are encouraged to transform initial routine problems that appear in textbooks into a series of mathematical activities;
(c) A variation task. This group addresses ways to represent and analyze tasks that involve some variation phenomena in which a graphic model is achieved without having an explicit algebraic model of the situation; and
(d) Dynamic configurations. This group refers to the construction of dynamic configurations that aim to foster problem posing activities and ways to validate mathematical relationships.

4.1.2 Learning Environments and the Coordinated Use of Digital Technologies

In framing and characterizing a digital learning environment it is important to address and discuss ways in which the coordinated use of several digital technologies not only offer affordances to represent and explore mathematical tasks; but to also enhance students' interaction to continuously share and discuss ways to solve problems. Leung (2017) states that "teachers must experience for themselves, as learners, the potentials and pitfalls of digital tool in the learning of mathematics, thus gain knowledge about how students can learn mathematics in various digital environments" (p. 6). Thus, teachers need to work on problems and discuss ways in which technology help them restructure their teaching practices that pay attention to the type of reasoning that emerges throughout the problem-solving process.

> In the eyes of many digital natives, learning is more than just going to lectures and relying on textbooks; rather, learning involves engaging in technology-mediated learning activities such as doing research on the Internet, searching, finding, and analyzing a variety of resources available in the virtual world and bringing into their own lives (p. x). (Kinshuk & Spector, 2013)

Leung (2011) point out that "when one is faced with a new tool, one has to learn how to use it and in doing so, gradually realizes the "knowledge potential" that is embedded in it" (p. 327).

The use of digital technologies, such as a DGS (GeoGebra) and communication applications, provides the learners with a set of affordances to continuously engage in exploration, reconstruction, explanation, and communication activities to make sense of concepts and to solve mathematical problems. Thus, multiple purposes technologies such as Internet, tablets or smart phones play an important role in extending learners' mathematical discussions beyond formal settings. That is, students can access online materials, consult encyclopedias (Wikipedia) or share mathematical ideas via a digital wall (Padlet) and discuss their ideas (through email or online forums) within a learning community that includes peers, experts and teachers.

Santos-Trigo, Moreno-Armella, and Camacho-Machín (2016) argue that:

> …Representing and exploring mathematical tasks mediated by digital technologies bring in new challenges for teachers that include the appropriation of the instruments afforded by these technologies in order to identify and analyze what changes to mathematical contents and teaching practice are fostered through its use (p. 829).

In addition, Moreno-Armella and Santos-Trigo (2016) state that "the use of mediating instruments, in particular, digital technologies, are never epistemologically neutral. The ways of approaching a problem depend upon the resources we have at our reach" (p. 829). That is, the subject's experience or expertise in using the tool shapes and permeates how it is used in problem solving approaches. The transit in learners' initial use of empirical or visual approaches (via the use of technology) to eventually construct and present geometric and analytic arguments to support results appears important throughout all problem-solving activities. Freiman et al. (2009, p. 128) state that "…the most important advantage of using technology is the diversification of teaching and learning approaches, rediscovery of dynamic aspects of mathematics, and, especially, learning through communication with others".

It is argued that the use of technology demands that teachers and students analyze and discuss what problem solving strategies, concepts, resources and ways of reasoning appear important during the construction and exploration of dynamic models of problems via technology affordances. To this end, it is relevant to discuss how problem representations and strategies such as moving orderly objects within the model, quantification and exploration of objects' attributes, finding and analyzing objects' loci; using sliders, and arranging data in tables become important throughout the learners' problem solving process.

4.2 A Focus on Problem-Solving Activities

Curriculum and teaching proposals worldwide recognize that problem-solving activities are essential to frame mathematical learning environments (Törner, Schoenfeld, & Reiss, 2007). Likewise, the mathematical problem solving research agenda has provided relevant results and information regarding the importance of tasks or problems, the research methods to elicit and analyze both cognitive and metacognitive processes involved in learners' construction of mathematical knowledge, and the development of conceptual frameworks to analyze and document the students' problem solving competencies (Santos-Trigo, 2014; Silver, 1990). Although mathematical contents that appear in curriculum proposals might be the same in different countries, the ways to structure and implement a problem-solving approach to learn those contents might differ since such implementation is shaped by countries' cultural and social or educational traditions. Indeed, Stanic and Kilpatrick (1988) point out that "problem solving has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is, and of why we should teach mathematics in general and problem solving in particular" (p. 1). Similarly, research agendas in the field not only include distinct themes and interpretations of what a problem-solving approach entails; but have also evolved in terms of the use of research methodologies (Santos-Trigo, 2014).

> The term [problem solving] has served as an umbrella under which radically different types of research have been conducted. At minimum there should be a facto requirement (now the exception rather than the rule) that every study or discussion of problem solving be accompanied by an operational definition of the term and examples of what the author means. …Great confusion arises when the same term refers to a multitude of sometimes contradictory and typically underspecified behaviors. (Schoenfeld, 1992, p. 364)

What does it distinguish, then a problem-solving approach to frame a learning environment for students to construct and use mathematical knowledge? A key principle in any problem-solving approach to learn mathematics and to solve problems is that learners need to conceptualize the discipline as a set of dilemmas that are important and need to be represented, explored, analyzed, and explained in terms of mathematical resources (Santos-Trigo, 2014). Mason (2016) recognizes the importance for students to experience problematicity in dealing with mathematical tasks and to make use of their own powers and to engage in problem solving approaches. To this end, learners need to develop and value an inquiring approach to understand concepts and to solve problems. Santos-Trigo and Camacho-Machín (2016) point out that an underlying principle in problem solving activities is "to conceptualize learning as an inquiring process to delve into concepts and problems in order to identify and explore mathematical relations" (p. 45). Mason, Burton, and Stacy (2010) stated that an atmosphere of questioning, challenging and reflection is crucial for students to develop mathematical thinking. Leikin, Koichu, Berman, and Dinur (2017) states that "The construction of questions is an important way for learners to build conceptual conflict, and the search for answers may begin the process of resolving that conflict" (p. 67). Thus, posing questions and looking for different ways to pursue those questions are key activities for learners to learn and use mathematical knowledge. Barbeau (2009) refers to a challenge for learners to delve into mathematical tasks:

> …we will regard a challenge as a question posed deliberately to entice its recipient to attempt a resolution while at the same time stretching their understanding and knowledge of some topic… A good challenge will often involve explanation, questioning and conjecturing, multiple approaches, evaluation of solutions for effectiveness and elegance, and construction and evaluation of examples" (p. 5).

4.2.1 On the Use of Technology to Construct and Explore Dynamic Models

Within a technological learning environment, students might rely on different digital tools' affordances as a means, to represent, make sense, analyze and solve mathematical tasks. In this process, it is important to characterize what type of reasoning learners construct and exhibit throughout their problem-solving approaches. How could I construct a dynamic model of a problem? What parameters involved in the problem representation can be quantified or measured? How can I orderly move some parameters within the model? How can I determine or visualize the loci of specific objects when I move some elements within the model? The discussion of these questions sheds lights on what type of reasoning students might get engaged with the use of a DGS in problem solving activities.

> The long-term commitment students need to make is a willingness to engage in problem-solving activities and to form habits of mind such as thinking about word meanings, justifying claims and conjectures, analyzing answers and solution strategies, using alternative representations, and acquiring a toolkit of problem-solving strategies. (Lester & Cai, 2016, p. 121)

Likewise, during the students' development of problem solving experiences it is important that they share, analyze, and discuss concepts, ideas, solutions as a part of a learning community and the use of digital technologies allows them to continuously discuss their ideas with peers and experts in and out of formal settings. Similarly, learners can consult online materials or learning platforms to recall or extend conceptual information or to watch an expert presentation via an online video of the topic in study. As Mishra and Koehler (2006) stated:

> … there is no single technological solution that applies for every teacher, every course, or every view of teaching. Quality teaching requires developing a nuanced understanding of the complex relationships between technology, content, and pedagogy, and using this understanding to develop appropriate, context-specific strategies and representations (p. 1029).

In this perspective, during the process of working on a mathematical task, learners should always look for different ways to represent and solve a problem and to examine the extent to which the methods used in solving it can be used in other tasks. In this context, a task is conceived of as departure point to engage learners in mathematical reflection and thinking. Santos-Trigo and Reyes-Rodríguez (2016) discusses the importance for students to think of and discuss several ways to solve a task that involves an equilateral triangle. The multiple approaches to represent and solve the task became important for students not only to consider and analyze different concepts and results associated with the equilateral triangle; but to also make connections among contents that often are studied separately. Lester and Cai (2016) mention that teachers should provide learning conditions for students to engage in a variety of problem-solving activities that include: "(1) finding multiple solution strategies for a given problem, (2) engaging in problem posing and mathematical explorations, (3) giving reasons for their solutions, and (4) making generalizations" (p. 13). That is, looking for different ways to solve a task, discussing what concepts are used, and exploring ways to extend mathematical tasks become an important goal for learners to pursue in the process of development their problem-solving competencies. This goal is achieved as a part of a learning community that demands that each member shares and constantly reflects on what he/she contributes to task's solution. Blaschke and Hase (2016) pointed out that:

> Working together toward a common goal, learners are able to solve problems and reinforce their knowledge by sharing information and experiences, continuously practicing, and exper-

imenting by trial and error. They simply help each other along the way. The teacher serves as coach during the collaboration process, letting learners forge forward together and stepping in only when absolutely necessary (p. 33).

Santos-Trigo and Moreno-Armella (2016) argue that "[s]earching for alternative ways to represent and solve problems is a powerful strategy for students to identify and contrast the role played by concepts and their representations across the whole problem-solving process" (p. 192). The development of Geometry Dynamic Systems such as GeoGebra represents a milestone in the study and development of mathematical knowledge. Leung and Bolite-Frant (2015) pointed out that GDS "can be used in task design to cover a large epistemic spectrum from drawing precise robust geometrical figures to exploration of new geometric theorems and development of argumentation discourse" (p. 195). That is, its use provides affordances for learners to both finding objects' relationships and properties and arguments to support or validate them.

4.2.2 Technology Affordances and Mathematical Explorations

Some problems that involve paper and pencil approaches can be explored and extended with the use of technology. Schoenfeld (1985, p. 16) asks some college students to divide a given triangle in two parts of equal area (using a straightedge and compass) by drawing a parallel line to one of the triangle side. What about if we remove the parallel line and the use of a straightedge and compass conditions and approach the problem with the use of GeoGebra? That is, we ask: divide a given triangle in two regions with same area. The goal is to look for different ways to find two regions with the same area. In Fig. 4.1, M is constructed as a midpoint of side AB, point E lies on segment AM and side EG is constructed to be a half of side AB. Thus, students can see that for any position of point E on segment AM the area of triangle ECG is always half of the area of triangle ABC. Properties of the construction validate the solution since triangles ABC and ECG share the same height with respect to sides AB and EG respectively. Therefore, for any position of point E on segment AM, then the area of triangle ECG is the same as the sum of areas of triangles ACE and BCG.

Another way to divide the given triangle is shown in Fig. 4.2, segment ED is perpendicular to AB and segment DH is parallel to AB, the coordinates of point Q are the x-coordinate of point E and the area of polygon EBHD as y-coordinate. Line $y = 3.22$ (half of the area of triangle ABC) intersects the locus of point Q that results when point E is moved along side AB at points O and P. Then, when point Q coincides with point O and P the area of quadrilateral EBHD will be the same as the sum of the areas of triangles ADE & DHC. The latter approach involves describing graphically the area variation of polygon EDHB when point E moves along side AB

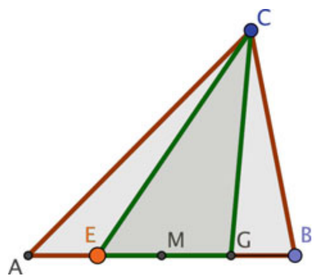


Fig. 4.1 Point E moves along segment AM (M midpoint of AB) and side EG is half of side AB, then area of triangle ECG is half of area of triangle ABC

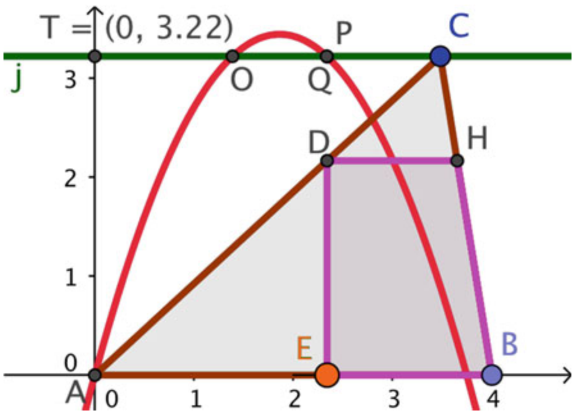


Fig. 4.2 Finding the locus of point Q when point E moves along segment AB

and to determine the position where polygon EBHD has half of the area of triangle ABC.

The tool affordances that include drawing a precise model, generating a family of objects (triangles and polygons), and finding loci of the polygon's area variation become important not only to solve the task; but also are essential to identify properties and arguments to support results. In this case, an open question that involves dividing a triangle represents an opportunity for students to think of and explore different ways to divide the figure, and so, several concepts and problem solving strategies appear during the solution process.

Thus, the use of digital technologies seems to provide a context and an opportunity for students to activate a variety of concepts and resources during the process of constructing and exploring different approaches to the task. Freiman, Kadijevich, Kuntz, Pozdnyakov, and Stedoy (2009) summarizes what the use of technology might bring to the learning community in terms of extending learning mathematical discussion beyond classrooms:

- Technology can give access to the resources that cannot be otherwise accessed.
- Technology can provide a free choice of resources based upon the level and the particular needs.
- Technology can provide dynamic tools of mathematical investigation giving a chance to modify parameters of an activity in an interactive way.

- Technology is a valuable tool of communication about mathematics with other people.

- Technology empowers the people with the instruments, facilitating routine operations and more sophisticated mindtools (p. 129).

In the same vein, Liljedahl, Santos-Trigo, Malaspina, and Bruder (2016) argue that the use of technologies demands that students engage in a tool's appropriation process to develop an expertise in representing and exploring concepts and problems.

> …learners not only need to develop skills and strategies to construct dynamic configuration of problems; but also ways of relying on the tool's affordances (quantifying parameters or objects attributes, generating loci, graphing objects behaviors, using sliders, or dragging particular elements within the configuration) in order to identify and support mathematical relations (p. 23).

4.3 Problems as a Departure Point to Engage Students in Mathematical Thinking

Mathematical problems play an important role in fostering students' learning and guiding the development of mathematical knowledge (Leung & Baccaglini-Frank, 2017a). Silver (2016) pointed out that "mathematics problems form the foundation of students' opportunities to learn mathematics". Similarly, Lester and Cai (2016, p. 122) stated that:

> Mathematical tasks provide intellectual environments for students' learning and the development of their mathematical thinking…Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites speculation and hard work. Most importantly, worthwhile mathematical tasks should direct students to investigate important mathematical ideas and ways of thinking toward the learning goals.

Teachers design, select, adjust and implement mathematical tasks to foster their students' development of mathematical thinking. Margolinas (2013) stated that:

> Tasks…are the mediating tools for teaching and learning mathematics…Tasks generate activity which affords opportunity to encounter mathematical concepts, ideas, strategies, and also to use and develop mathematical thinking and modes of enquiry (p. 12).

What types of problems are important for students to work and discuss in problem solving environment? What does the process of designing or selecting mathematical tasks entail? How does the use of digital technologies influence the design and selection of mathematical problems? The discussion of these types of questions implies also addressing issues regarding choosing, designing and implementing mathematical tasks in learning scenarios. Selden, Selden, Hauk, and Mason (2000) pointed out the importance for students to deal with non-routine problems to develop a robust understanding of mathematical concepts. Working on non-routine problems requires that students figure out mathematical features associated with the structure of the problem, to identify key concepts involved in the problem statement, and to select and search for resources and strategies needed to explore and eventually solve the problems. In a technological environment, learners could engage in exploration activities that involve moving objects, exploring their behaviors, looking for invariance and properties to support conjectures. In this process, they examine concepts to grasp features associated with the deep structure of the problem (Santos-Trigo & Camacho-Machín, 2016).

In this context, working on tasks or problems represents an opportunity for learners to get involved in a continuous investigation that lead them to look for patterns, to make connections, and to extend initial problems. That is, problems are conceived of as a departure point for students to engage in mathematical discussions. Likewise, the way teachers implement the tasks in learning scenarios plays an important role in the students' learning of concepts and solving problems. Thus, the type of questions and the mathematical reflection that students engage in while working on the task are essential for student to focus on what is important during the solution process. Lester and Cai (2016) pointed out that:

> The learning environment of teaching through problem solving provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Such activities help students clarify their ideas and acquire different perspectives on the concept or idea they are learning. Empirically, teaching mathematics through problem solving helps students go beyond acquiring isolated ideas toward developing increasingly connected and complex system of knowledge (pp. 119–120).

What should students pay attention to or look at while using technology to solve mathematical problems? An initial categorization of groups of problems is proposed in terms of identifying how the tool's affordances shape the ways of reasoning and approaching each group solution. This categorization comes from analyzing and discussing ways in which we have used several digital technologies in problem solving approaches (Santos-Trigo & Camacho-Machín, 2016). Thus, the presentation of each category shows, representations, strategies, concepts, and resources that appear relevant in approaching the problem.

4.4 Towards a Categorization of Mathematical Problems and the Use of Technology

4.4.1 Problem Statements and Embedded Figures

In paper and pencil approach, some problems or tasks statements often include figures that show objects and data that are important to identify properties or relations to solve the problem. For example, in the statement: *Let ABC be an equilateral triangle and let P be any point on its circumcircle, for instance, on the shorter arc AB, as shown in* Fig. 4.3*… [show] that AP + BP = CP* (Melzak, 1983, p. 13), the figure

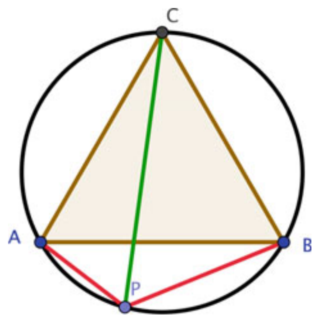


Fig. 4.3 Triangle ABC is equilateral, show that AP + BP = CP

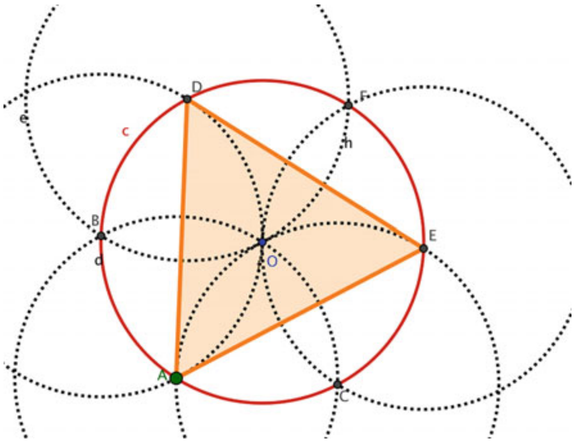


Fig. 4.4 Inscribing an equilateral triangle in a given circle

becomes a referent to identify possible properties and results (similar triangles, cyclic quadrilateral, etc.) to support or validate the involved relation.

By looking at the figure, one might ask: How can I reconstruct the figure? How should one draw an equilateral triangle and its circumcircle? Or given a circle, how should one inscribe an equilateral triangle? With the use of GeoGebra, these questions become relevant not only to identify and explore concepts needed to draw the figure, but also provide an opportunity for learners to connect the problem goal with a series of mathematical ideas and resources to solve and extend the initial statement. How can I inscribe an equilateral triangle into a given circle? Figure 4.4 shows a way that involves choosing a mobile point A on the circle and drawing a circle with center at A and radius AO (O the center of the given circle). This circle intersects the given circle at point B and C, then two other circles are drawn with centers at B and C and radius BO and CO, etc. Then, points A, D & E are the vertices of the inscribed triangle (Fig. 4.4).

Another approach (Fig. 4.5) to inscribe an equilateral triangle involves selecting any point A on the circle and drawing line AO (O the center of the circle). Line AO intersects the circle *c* at point D. Then, a circle *d* with center at D and radius DO is drawn. Points C and B are the intersection points of the circles and then triangle

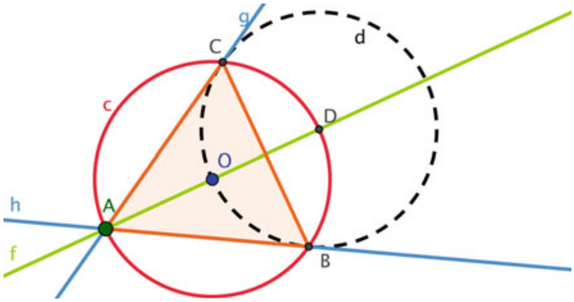


Fig. 4.5 inscribing an equilateral triangle in a given circle

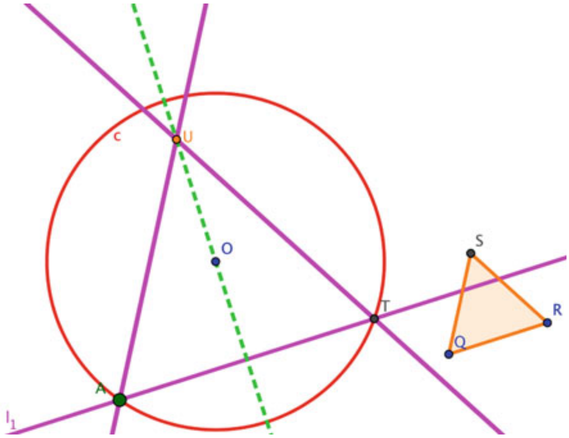


Fig. 4.6 Focusing on any equilateral triangle as a referent to inscribe a similar triangle in a given circle

ABC is equilateral (Fig. 4.5). This is because line AO is the perpendicular bisector of CB.

Another approach might focus on drawing any equilateral triangle QRS as a reference to inscribe a similar triangle into the given circle. Figure 4.6 shows an equilateral triangle QRS as a reference one, then point A is any point of the given circle c. From point A two parallel lines to side QR and QS are drawn and from point T (the intersection of the parallel to QR and the circle) also a parallel to side RS is drawn. Triangle ATU is equilateral and the locus of point U when point A moves along the circle is a line. So, the position of point A at which the locus of point U intersects the circle, is the third needed vertex to determine the inscribed equilateral triangle.

Yet, another approach to inscribe an equilateral triangle focuses on examining a simpler case and analyzing the area variation of the inscribed circle. In Fig. 4.7a, point A is a mobile point on the given circle and triangle ABC is equilateral (its inscribed circle is a circle with center at the intersection of two perpendicular bisectors and radius the distance between the center and any vertex). Line m is the graph of y = area of circle c and point Q has coordinates the x-coordinate of point A and as y-coordinate the area of the circle that inscribes the equilateral triangle ABC. What

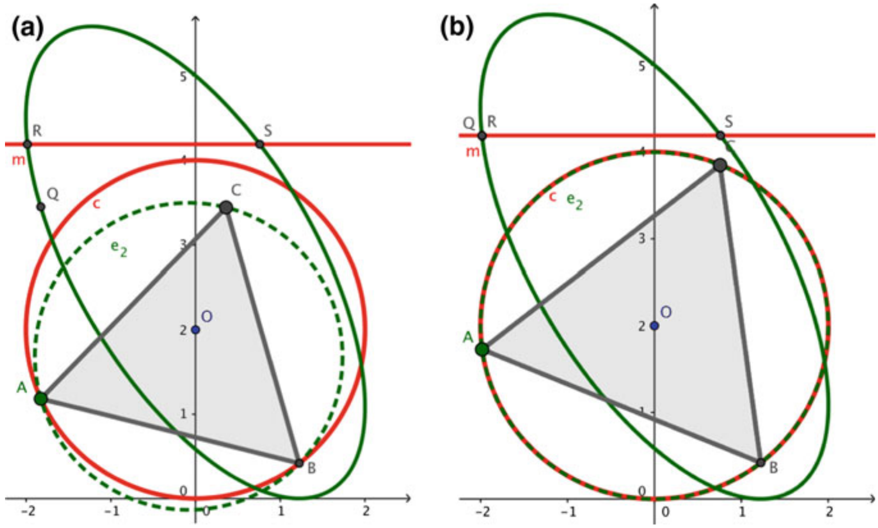


Fig. 4.7 **a** What is the locus of point Q when point A is moved along circle c? **b** When point Q coincides with point R, the inscribed triangle is the solution

is the locus of point Q when point A is moved along circle c? Figure 4.7a shows that the locus (that seems to be an ellipse) intersects line m at point R and S. That is, when point Q coincides with point R, then triangle ABC is the inscribed equilateral triangle (Fig. 4.7b).

Comment: Pólya (1945) argues that understanding the problem statement is a crucial stage in the process of solving the problem and it involves identifying relevant concepts and possible relations. The use of technology can help students connect concepts and delve into the problem understanding process by focusing on the reconstruction of figures. Indeed, this phase becomes a problem posing activity where students begin reflecting on how, what order, and properties are important to draw the figure. In this case, asking about how to inscribe an equilateral triangle becomes important to think of the use of different concepts and strategies to reconstruct the figure given in the problem.

4.4.2 Investigations Tasks

A problem statement is conceived of as a departure point for students to look for mathematical relations and to extend the task. That is, the learners' goal while interacting with a mathematical task is not only to find its solution; but it is also important to look for ways in which the initial task can be extended or connected with other problems. How can a routine or a textbook task be transformed into an investigation task? To delve into this question, an adjusted version of a problem that appears in *Con-*

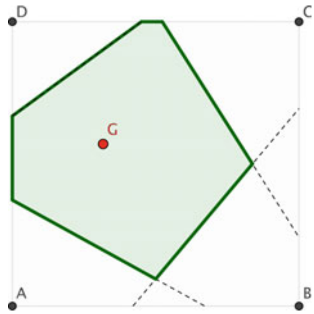


Fig. 4.8 Folding a square sheet

nected Geometry (2000, p. 76) is discussed in terms of identifying problem solving episodes in which the systematic and coordinated use of digital technologies offers affordances for learners engage learners in mathematical experiences. Santos-Trigo and Reyes-Martínez (2018) present what prospective high school teachers exhibited during the problem solving episodes that involved working on this investigation task. A complete analysis of the implementation of this task appears in Reyes-Martínez (2016).

The initial task. Draw a square ABCD and choose an interior point G. Fold each vertex or corner into make it coincide with point G. Figure 4.8 shows the position of point G, the folding lines (creases) and a polygonal region that appear when all four vertices coincides with point G. What happens to the number of sides of the polygonal region when point P moves inside the square?

(a) **A dynamic representation**. At the understanding and making sense stage of the statement, it is always important to ask about properties, relations and ways to represent objects involved in the task (Schoenfeld, 1992; Santos-Trigo, 2007). What mathematical concepts are important to represent the folding line (segment)? Is there any type of symmetry involved in the folding process? What concepts can be used to draw the figure? Is it possible to construct a dynamic model of the task? These questions might lead the students to identify that the creases (folding lines) are the perpendicular bisectors of segments that join the interior point (G) with each square vertex. Indeed, with the use of a slider (Fig. 4.9), it is possible to identify steps involved in moving each vertex to point G and to explore what type of polygonal region is formed for different positions of point G. Likewise, this dynamic representation requires that the problem solver thinks of the task in terms of mathematical concepts and properties that can be expressed or represented through the tool' affordances.

(b) **A robust model**. Looking at the intersection of two perpendicular bisectors LO and QN (Fig. 4.10) provides important information to construct a robust model of the problem. Thus, when segment UI is longer than half of the side of the square, then the intersection point I is outside of the square and the sides of the polygonal region would be ON and PQ respectively. This information leads to

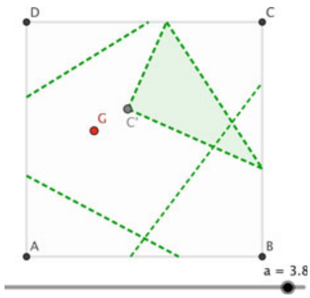


Fig. 4.9 An animated representation of the task that shows the creases movement when each vertex approaches to point P

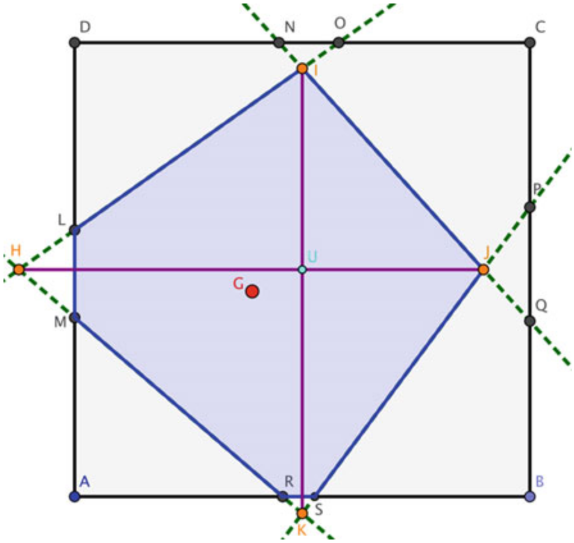


Fig. 4.10 Identifying conditions to construct a robust model of the task

relate the position of point G and the number of sides of the formed polygonal regions (Fig. 4.10).

The construction of a robust model of the problem means that point G can freely be moved inside of square ABCE and for any position of point G there will always be a be well-defined polygonal region. Figure 4.11 shows that when G is located outside the "petal region" (the intersection of semicircles with center at midpoint of each side of the square and radius half of the length of the side) then the polygonal region will have five sides.

(c) **A characterization of the polygonal regions**. The exploration of the robust model of the task provides important information and clues to visualize and relate the position of point G to the number of sides of the generated polygonal region. With the use GeoGebra, it is possible to reveal, through coloring, what polygonal regions share the same numbers of sides. Leung (2008) call *spectral dragging* to a heuristic that allows to trace and assign colors to properties of

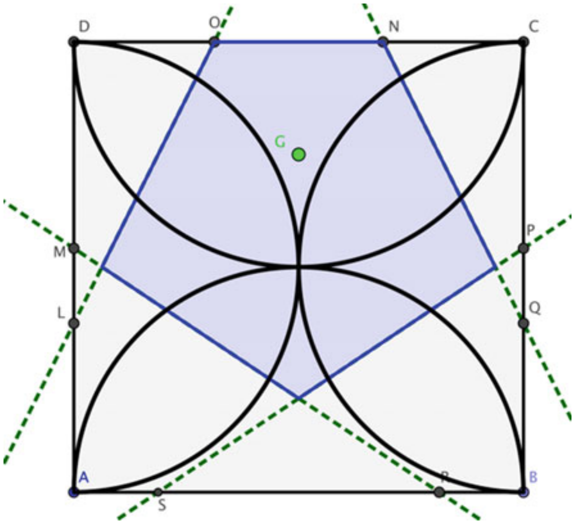


Fig. 4.11 The construction of the robust polygonal region and identifying regions where the polygonal regions hold specific properties

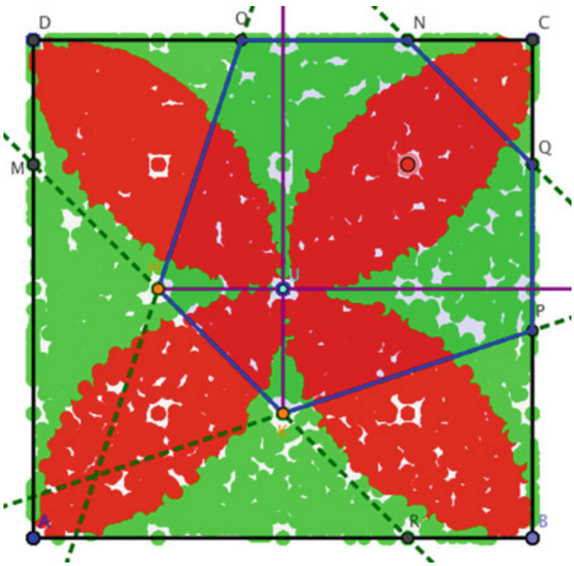


Fig. 4.12 A relationship between the position of point G and the number of sides of the generated polygonal region

involved objects. In this case, the colored region identifies the family of polygons that shares the same number of sides. Figure 4.12 shows that when point G lies on the red part then the polygonal family that appear on that region will have six sides and when point G lies on the green part then the polygonal family on that region will have five sides. Likewise, when point G coincides with the center of the square polygon becomes a square.

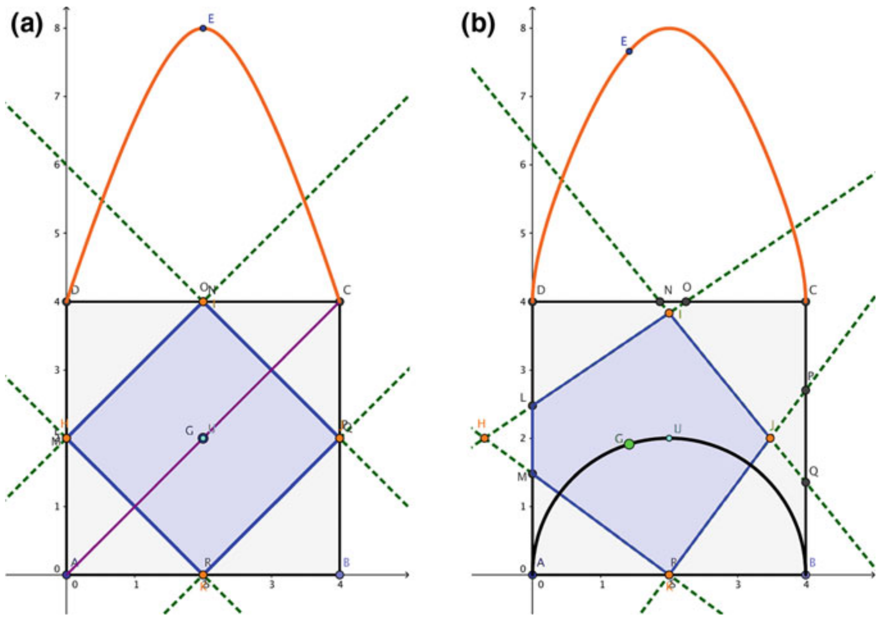


Fig. 4.13 **a** Exploring the area variation of the polygonal region when point G moves along diagonal AC. **b** Exploring the area variation of the polygonal region when point G moves on a semicircle AB

In addition, the robust model of the task provides information regarding the area variation of the family of polygons generated for different position of point G. Figure 4.13a, b show that when point G is moved along diagonal AC or the semicircle AOB the maximum area is reached when point G coincides with the center of the square in which the region becomes a square.

(d) **Extension and generalization**. Can the method used to construct the robust model be extended to explore what happens to the polygonal region when the initial square becomes others regular polygons? Figure 4.14 shows polygons with different number of sides and the corresponding generated polygonal regions. Based on the exploration for regular polygons with different number of sides, some conjectures emerge:

1. When the position of the interior point coincides with the center of polygon, then the generated polygonal region is a regular polygon that has the same number of sides as the initial regular polygon.
2. When the number of sides of the initial regular polygon increases (Fig. 4.14 shows a polygon with 200 sides), then the intersection of the corresponding perpendicular bisectors (red points) seems to form and ellipse and when point G is outside of the circle the intersection points generate a hyperbola.
3. When the number of side of the regular polygon tends to infinity, the polygon tends to be a circle. Figure 4.15 shows a circle with center at point A, D is any point on the circle, f is the perpendicular bisector of segment DG that intersects

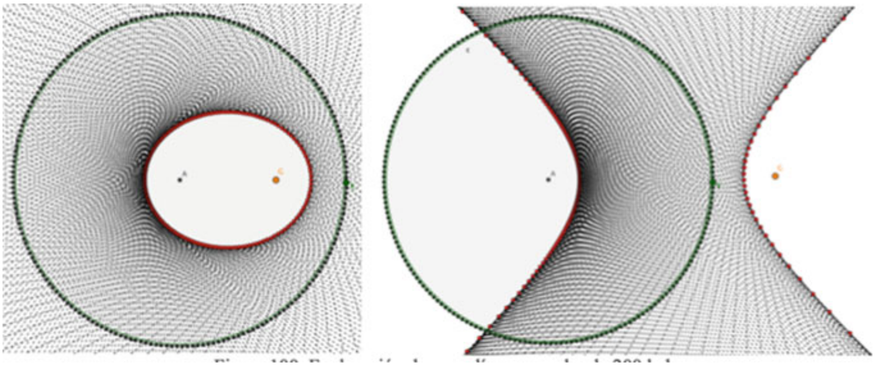
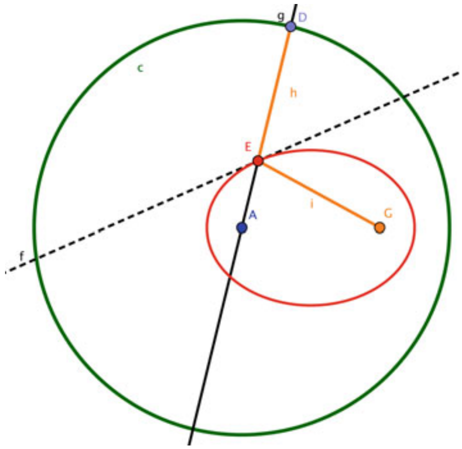


Fig. 4.14 For a polygon with 200 sides, the intersection of the corresponding perpendicular bisector seems to form an ellipse when point G is inside and a hyperbola when point G is outside of the polygon

Fig. 4.15 For a polygon in which its number of sides tends to an infinite number of sides the intersection of the perpendicular bisector of segment GD and AD generates an ellipse



line AD at point E. Then the locus of point E when point D is moved along the circle is an ellipse. This is true because segment ED and GE are congruent (E is on the perpendicular bisector) the radius AD is constant. Then it holds that $d(A, E) + d(E, G)$ is always constant (definition of an ellipse).

Comment: What concepts are embedded in the task' representation? How can they be represented via the DGS affordances? These types of questions are important to think of the problem in terms of the tool affordances. Thus, connecting the folding lines (creases) with the perpendicular bisector was essential to construct the dynamic model of the task. The exploration of this model provided clues and information regarding the polygonal regions behavior. Can the robust model for the square be extended to other regular polygons? This question leads to focus on how the intersection points of the corresponding perpendicular bisectors behaves and to find a

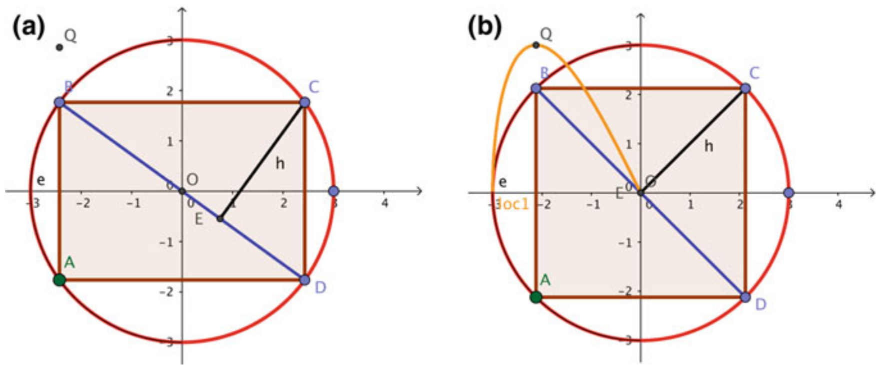


Fig. 4.16 **a** A is a movable point and h changes as point A moves. **b** The locus of point Q describes the behavior of the length of segment EC (height of triangle BCD)

serendipitous result: The intersection point of the perpendicular bisectors forms or determines an ellipse when the interior point G lies in the interior of the polygon or circle and a hyperbola when G is outside the circle.

4.4.3 A Variation Task

With the use of a dynamic geometry system, some problems that involve a phenomenon variation (optimization calculus problems) can be modelled without constructing explicitly its algebraic model. That is, the tool's affordances can help students construct a graph representation of the variation phenomenon without expressing the involved algebraic model. For example, the task that focuses on examining a family of inscribed rectangles and asks to identify which element of that family has a maximum area can be represented and analyzed through a dynamic model.

In Fig. 4.16a, point A is a mobile point on the circle and a rectangle ABCD is drawn. One way to inscribe the rectangle is to reflect point A with respect to the x-axis to determine point B, then point B is reflected with respect the y-axis, etc. At what position of point A does the rectangle ABCD reach the maximum area? It is observed that when point A is moved along the circle, the diagonal BD has a constant length (this is because its length is always the diameter of the circle). In triangle BCD, h is its height and the maximum area of the family of triangles BCD that is generated when point A is moved is obtained when h gets is maximum value. Point Q has coordinates the x-coordinate of point A and as a y-coordinate the length of h. Figure 4.16b shows the locus of point Q when point A moves along the circle. This leads to conclude that the inscribed rectangle with maximum area is when the rectangle becomes a square.

Similarly, Fig. 4.17 shows directly the area variation of the rectangle which is the locus of point M (whose x-coordinate is the length of side AD and as the y-coordinate

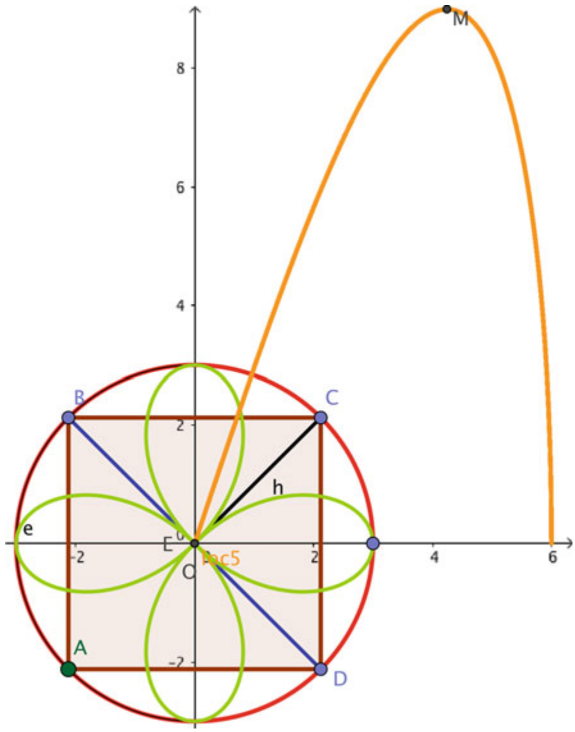


Fig. 4.17 The locus of point M that describes the area variation of the family of rectangles when point A is moved on the circle

the area of rectangle ABCD) when point A is moved along the circle. In addition, the locus of point E (one extreme of segment h) is another interesting curve that learners might be interested in exploring its properties.

Comment: The study of Calculus problems, that involve analyzing variation phenomena, emphasizes and focuses mainly on constructing and dealing with an algebraic model. With the use of a DGS is possible to generate the locus that describes the variation phenomenon without making explicit the algebraic model. The idea is to relate the variation of one element of the dynamic model with the variation of the phenomenon in study. In this process, issues regarding the domain to move elements within the model and the analysis and interpretation of what is generated (loci properties) become crucial to make sense of relationships and mathematical results or solutions. This method of visualizing the behavior of an object attribute relation is also shown in Figs. 4.2, 4.7a, b and 4.13a, b.

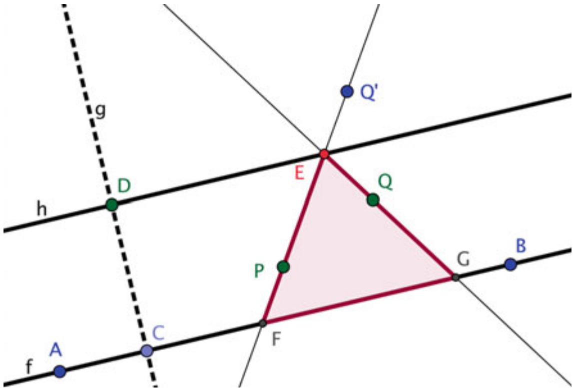


Fig. 4.18 Constructing a dynamic configuration to pose questions

4.4.4 The Construction of a Dynamic Configuration and Problem Posing Activities

In these tasks, the idea is to construct a dynamic configuration based on putting together some simple mathematical objects such as points, lines, triangles, etc. Then, the controlled movement of specific elements, within the configuration, becomes susceptible of being explored and analyzed in terms of properties and mathematical resources. As a result, some questions or conjectures regarding the behavior of some objects attributes emerge and the goal is to look for arguments to support and validate those conjectures or mathematical relations. Figure 4.18 shows a configuration that includes a line AB, a point C on line AB and the perpendicular *g* to AB that passes through C. Point D is any point on the perpendicular g and line *h* is the parallel to line AB that passes through point D. Point P and Q are any points on the plane and Q′ is the symmetric or reflected point of Q with respect to line h. Line Q′P intersects lines h and AB at E and F respectively and line EQ intersect line AB at G. Based on this initial configuration, some questions might be posed: What type of triangles are formed when points P or Q are moved?

The goal is to explore properties and invariants of embedded objects when some elements are moved within the model. For example, since lines h and f are parallel, then angles GFQ′ and Q′EH are congruent, similarly, angles HEQ and EGF are congruent and for symmetry properties angles Q′EH and HEQ are also congruent; therefore, the family of triangles EFG is always isosceles. What is the locus of point E when point D is moved along line g? Figure 4.19 shows that the locus seems to be a hyperbola.

Another variant of this type of task involves using the tool to get information regarding the objects’ attributes embedded in the task. For example, a dynamic representation of a task that involves determining the area of the triangle with vertices at the orthocentre, the circumcentre, and the centroid of a given triangle ABC leads to conclude that these three points are collinear and therefore, such area is always zero. Figure 4.20 shows a dynamic representation where point O, P, and Q are the

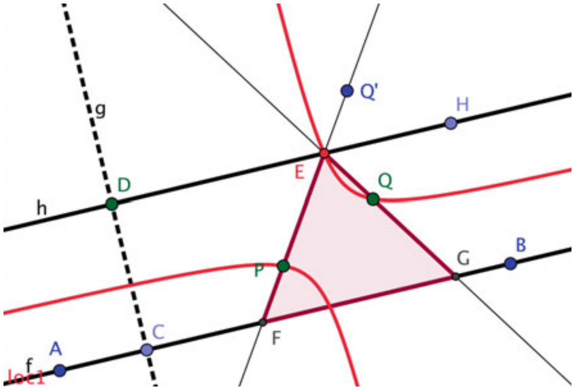


Fig. 4.19 What is the locus of point Q when point D is moved along line g?

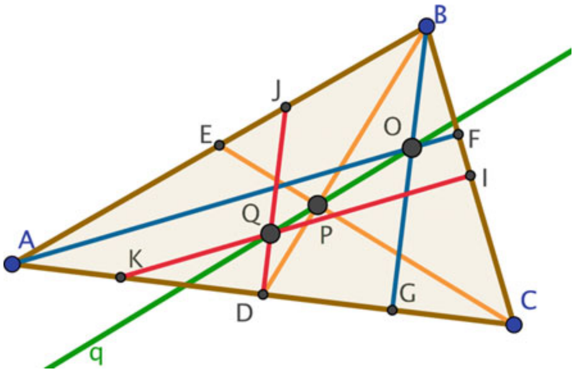


Fig. 4.20 The orthocentre, circumcentre and centroid are collinear

orthocentre, the centroid, and the circumcentre of triangle ABC. It is observed that O, P and Q are collinear. To prove that points O, P, and Q are collinear it is sufficient to show that $d(O, P) + d(P, Q) = d(O, Q)$.

Comment: In this type of tasks there is no initial problem or question to solve, the goal is to assemble a dynamic configuration in which the movement of some of its elements will lead the student to observe invariance or relation among some objects' attributes. In this process, students have an opportunity to engage in problem posing activities that involve the formulation of conjectures and to look for different arguments to validate them. Similarly, with the tool's affordances, learners can identify patterns and properties of objects' attributes and to explore the pertinence or conditions to define and represent the objects. In this case, the collinearity property of points O, P and Q leads to conclude there is a degenerated triangle or segment with area zero.

4.5 Reflection and Closing Remarks

In the last ten years, the use of technologies has been transforming not only how people communicate and interact in both their daily life and professional environments; but individuals also rely on tools and digital developments to access and use online information. Recently, there have been several publications (Gros et al., 2016; Hokanson & Gibbons, 2014; Liljedahl et al., 2016; Singer, Ellerton, & Cai, 2015) that address the need and importance of analyzing what the use of technologies brings to both the subject content learning and the structure and dynamics of learning scenarios. Walling (2014) presents and discusses an instructional design model called ADDIE (Analyze, Design, Develop, Implement, and Evaluate) to incorporate tablets (iPads) in learning environment. Throughout this chapter, it is argued that mathematical tasks are the vehicle and a key ingredient to identify, discuss, and analyze what representations, explorations and ways of reasoning emerge in approaches that involve the systematic and coordinated use of digital technologies.

Grouping mathematical tasks in terms of identifying questions and strategies that problem solvers can explore during their interaction with the tasks could help teachers focus on ways to rely on technology affordances to foster mathematical thinking. In the first group, a question (how can we inscribe an equilateral triangle in a given circle?) becomes important to think of concepts and relationships needed to reconstruct the figure and explore and analyze different solutions. How can I draw the figure? and in which order should the elements or involved objects be drawn? These generic questions are important to identify concepts representation and relations to reconstruct the figure. In addition, the process of reconstructing a figure might lead the problem solver to engage in problem posing activities that include looking for different ways to draw the figure.

The second group (investigation tasks) refers to the process of transforming routine or textbook tasks in a series of activities that can foster students' problem solving experiences. To this end, the initial analysis of the task involves identifying key concepts that can be represented dynamically. In this process, it is possible to analyze how embedded objects in the dynamic model behave and use this information to construct a robust model. The robustness of the model implies analyzing the domain of movable points to always generate a consistent representation. For instance, the animated model (Fig. 4.9) was adjusted to leave out non-convex polygons (Fig. 4.21). The robust model is examined to detect patterns or invariants associated with the objects' attributes behaviors. Then, it is important to discuss the extent to which the construction of a robust model can be applied to explore what happens to the generated polygonal regions that are formed when considering other regular polygons (pentagon, hexagon, etc.).

The third group focuses on ways to represent and explore tasks that involve analyzing variation phenomena. The tool' affordances offer a possibility of generating a graphic representation of the variation phenomenon parameters without making explicit an algebraic model. In general terms, the main idea is to construct a point that represents a relationship between two parameters, one that describes the position

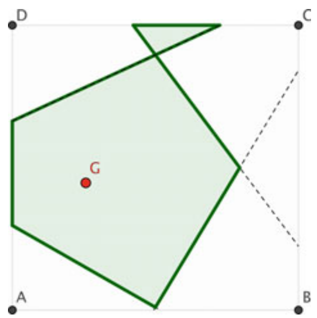


Fig. 4.21 Implementing conditions to always generate a concave polygon

of movable point (independent variable) and other that represents the variation of the attribute associated with the variation phenomenon. For example, point M (Fig. 4.17) represent a relationship between the length of side AD (x-coordinate) and the area of rectangle ABCD (y-coordinate). Thus, using the locus command is possible to generate the graphic variation of the phenomenon (area in this case) that can be analyzed to explore increasing/decreasing intervals, optimization points, and other locus' properties. This approach is important for students to focus on interpreting involved concepts and later understand meaning and properties of the corresponding algebraic model.

The fourth group emphasizes the importance of using GeoGebra's affordances in problem posing activities. The goal is to rely on simple mathematical objects such as points, segments, perpendicular bisectors, circles, tangents, triangles, rectangles, etc. to construct a dynamic configuration and to move some objects within the configuration to observe and analyze the mathematical behaviors of attributes and properties associated with those objects. What is invariant? What does it change? Is there any pattern or does the area of a certain family of polygon reaches a maximum value? etc. are questions that might lead the problem solver to identification of conjectures and look for ways to support or validate them.

In dealing with the tasks, the use of technology provides affordances for students to pay attention to activities that includes reconstructing and examining figures associated with problem statements, the construction of dynamic models of tasks, the formulation of conjectures, the quantification of objects or parameters behaviors, the search for mathematical arguments and the communication of results. In this context, learners have an opportunity to expand or enhance not only important problem solving heuristics (that include the construction of dynamic models, finding and examining loci of points or objects, using slicers, quantification of parameters, exploring simpler cases, or assuming the problem as solved, etc.) but also to construct and incorporate ways of reasoning associated with the use of the tool.

Finally, communication technologies provide affordances to extend mathematical discussions beyond formal settings (Santos-Trigo, Reyes-Martínez, & Aguilar Magallón, 2016). In this context, learners can focus their attention to how the use of GeoGebra expand and introduce new ways to represent, explore and find mathematical relations. Reyes-Martínez (2016) uses a digital wall in which students can share and exhibit their ideas with peers or experts and each participant, as a part of the group or learning community, can react, analyze, critique or extend other's ideas. As a result, students' initial ideas and contributions are constantly refined and they eventually recognize that learning mathematics and developing problem solving competencies is a constant process that involves both individual and group participation. In using technologies, an important goal is that learners rely transparently on technology affordances to work on representing and exploring mathematical tasks and in discussing with peers and others their mathematical ideas and problem solving approaches. As Weiser (1991, p. 94) points out "[t]he most profound technologies are those that disappear. They weave themselves into the fabric of everyday life until they are indistinguishable from it".

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