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Peter Liljedahl
Manuel Santos-Trigo *Editors*

Mathematical Problem Solving

Current Themes, Trends, and Research

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Contents

Introduction

Mathematical problem solving has long been seen as an important aspect of mathematics, the teaching of mathematics, and the learning of mathematics. It has infused mathematics curricula around the world with calls for the teaching of problem solving as well as the teaching of mathematics through problem solving. And as such, it has been of interest to mathematics education researchers for as long as our field has existed. In July of 2016, over 80 researchers gathered at ICME-13 to expand on this important topic through the presentation of research, critical reflection, and discourse. The coming together of researchers within TSG 19: Problem Solving in Mathematics Education resulted in the presentation of 13 extended papers, 28 oral communications, and 18 posters organized on a wide variety of topic focused on, and stemming from, research into the problem solving. From the richness of the interaction over those 7 days in Hamburg emerged this book—consisting of the extended versions of 15 invited papers on a wide variety of topics, results, and perspectives on mathematical problem solving.

In Part I “Problem Solving Heuristics”, Tjoe revisited Pólya’s framework, characterizing problem solving phases that appear in individuals’ attempts to solve problems and focuses on looking backstage as an approach to encourage high school students to look for different ways to solve problems. Using a mathematics test as well as interviews, he explores and discusses the extent to which students were familiar, fluent and flexible in using multiple solution methods. An interesting finding in Tjoe’s study was that students showed little interest in finding other solution methods in addition to those that they reported in the test. Tjoe recommends that students explicitly discuss throughout instruction the importance of finding multiple solution methods to approach any type of problems and teachers should value and encourage their students to looking back and find different approaches to solve the same problem.

Likewise, Maciejewski’s contribution invites us to reconceptualise the mathematical problem solving processes to include, what he calls, mathematical foresight and the importance of future thinking when approaching a problem solving situation. Maciejewski grounds these ideas in the literature, where he illustrates the relationships between mathematicians’ work, problem solving (Schoenfeld, Pólya), and select psychological work and posits mathematical foresight as a possible lens to analyse students' future-oriented thinking and actions to deal with mathematical situations. In contrasting the mathematicians' foresight models and that of students approaches, Maciejewski reports that while mathematicians see two interrelated components—the sphere of finding the solution and the resolution process or trajectory—students often only see one of these components. That is, students either see a possible solution to a task without seeing the process or path necessary to reach that solution, or they see the beginnings of a trajectory without seeing where this will lead them.

Part II "Problem Solving and Technology" begins with Carreira and Jacinto who investigate how a middle-grade student engages in a web-based mathematics competition. Drawing on the notion of humans-with-media they emphasize the interaction between the solver and the tool in problem solving activities. To document the student's processes, they use a blending framework that accounts for the problem solving phases (read, analysis, exploration, planning and implementation, and verification) as well as the explicit students' use of technology affordances throughout all phases. Based on the analysis of one case the authors report that the use of technology affords the student the possibility to engage in different forms of reasoning, including exploration, manipulation, observation, conjecture, formulation, explanation, and validation.

Similarly, Santos-Trigo also presents a framework for characterizing reasoning that a problem solver might develop as a result of using digital technology to solve mathematical problems. In so doing, he illustrates how the affordances of technology can shape the reconstruction of figures that often are embedded in problem statements, the transformation of textbook or routine problems into an investigation task, the graphical representation and exploration of a variation phenomenon or problem, and the construction and exploration of dynamic configurations to formulate conjectures and ways to support them. Santos-Trigo uses these four problem types to discuss the importance of building dynamic models of problems, the role of controlled movement of certain objects, the search and exploration of loci of points to analyse some variation phenomena, and the use of sliders to visualize patterns and relationships.

Finally, Amado, Carreira, and Nobre look at ways in which the use of spreadsheets provides affordances for students to represent and solve word problems. The chapter begins by addressing both the difficulties that students experience with algebraic representation and the affordances of spreadsheets to make sense and represent key information associated with problem statements. The cases presented in the chapter illustrate different models that students used to solve a word problem. They conclude that the use of spreadsheets allowed middle school students to think of a variety of approaches that involves formulas and tables to identify and explore relations between variables.

Part III "Inquiry and Problem Posing in Mathematics Education" includes two chapters and begins with Hersant and Choquet's use of inquiry-based approaches to engage elementary students in problem solving activities. The chapter includes a review of how inquiry-based learning and teaching has been interpreted and used in both science and mathematics in Europe and elsewhere. They argue that this approach can be characterized as a student-centred way of teaching in which students are encouraged to formulate questions as a way to delve into concepts and solve problems. In the chapter, they present two case studies, framed through an inquiry-based approach, that encourages elementary students to pose and discuss questions during the process of solving specific tasks. Through these cases the authors point out that the role of the teacher in such an environment will either foster or limit what students can achieve in this type of approach.

The second paper, by Malaspina, Torres, and Rubio, presents results from a study that looks closely at problem posing activities during a workshop with 15 high school teachers. The participants were asked to pose a problem at two different stages of the workshop (pre-problem and post-problem) and these were used to analyse the teachers' didactic and problem posing competencies. The authors relied on what they call an onto-semiotic framework to analyse the posed problems via epistemic and cognitive configurations. This analysis led the authors to characterize the participants' didactic competencies by contrasting the mathematical structures between the given problem and those they proposed and discussed. The authors also report on the difficulties participants experienced during the development of the problem-posing sessions.

Part IV "Assessment of and Through Problem Solving" is comprised of four chapters beginning with Loh and Lee's study on grade 7 students use of metacognitive strategies while solving mathematical tasks. The research design involves the use of both quantitative and qualitative methods to gather information about the participants' metacognitive behaviours. Results identify different students' frequency use of metacognitive strategies with an emphasis on surface strategies. However, the analysis of the students' written self-report and interview led the authors to identify students' robust use of metacognitive strategies. The authors suggest that the use of both quantitative and qualitative instruments provided important insights into the students' metacognitive behaviours.

Chanudet's chapter looks at the use of an assessment tool in a problem solving course that fosters an inquiry approach to learn mathematics. It includes a review of what an inquiry and problem solving approach might entail and the importance of designing a tool to assess problem solving competencies. The first part of the study focuses on the nature of the tasks that participant teachers use to assess students' problem solving. The second part of the study delves deeper into assessment and involves first working collaboratively with teachers to design an assessment tool that involved both summative and formative assessment, and then testing this tool through an exploratory study into one of these teacher's practice. Results indicate that this teacher relied on classroom conversations to assess her students throughout the course.

Meanwhile, Di Martino and Signorini look at assessment of problem solving through the use of standardized assessments such as PISA or national tests. The authors discuss several cases in which students' answers to specific test items, although well-supported within the students' reasoning, do not necessarily lead them to choose the *right* answer. The authors also showed that the time limitation to complete the test becomes an obstacle for students to show what is behind their answers and they argue that teachers and researchers should pay attention to the students' process involved in working on these types of questions.

The final chapter in this part, by Mendoza Álvarez, Rhoads, and Campbell, is centred on a quest to develop an efficient tool to assess the mathematical problem solving abilities necessary for a student to leverage pre-requisite knowledge to be successful in the STEM fields. Grounded in literature, the authors develop and test Likert items that link a student's mathematical problem solving capacity to five key problem solving domains (sense-making, representing and connecting, reviewing, justifying, and challenge) and do not require content knowledge beyond secondary school level algebra.

Part V "The Problem Solving Environment" begins with Koichu and Keller's report on the development of online forums to engage students in problem solving activities. In their chapter, they include examples of problems, the interaction among three communities (two classroom communities and the research group), and a narrative on how these communities behave and interact throughout the development of the forums. The authors characterize how online problem solving discussions became a routine practice in one community in which its members valued and engaged in meaningful discussions beyond classroom problem solving activities. The second community did not activate the use of the forum; but the interaction of this community with the research group led the participants to enhance their peer's interaction within the classroom. The authors also argued that all three communities evolved, and they characterize stages on how this evolution took place including the identification of boundaries that appear during the community interactions.

Meanwhile, Liljedahl's chapter aims to characterize what a thinking classroom involves in terms of the type of tasks used to engage students in problem solving activities, the way teachers give and structure the tasks development, how the students work in groups including work surfaces (vertical non-permanent surfaces), how questions are answered, and the assessment of students' problem solving performances. Throughout the chapter, the author describes a series of studies that led him to identify and categorize students learning behaviours in different classroom environments. He proposes an inventory of classroom norms and practices to examine how classroom activities are developed; indeed, the inventory is expressed in terms of 11 questions that researchers/teachers can use to analyse not only what and how students learn, but also the quality of that learning. Those questions include: What type of tasks are used, and when and how they are used? Where, and on what surfaces, do students work on tasks? How the room is organized, both in general and when students work on tasks? When and how is assessment carried out, both in general and when students work on tasks? etc. Addressing these questions provides useful information for researchers/teachers to construct powerful and cohesive learning environments that foster students' thinking as well as powerful and cohesive professional development environments for teachers to explore and question their practice.

In the same part, Felmer, Perdomo-Díaz and Reyes present initial results from a research and professional development program (Activating Problem Solving in Classrooms, known as ARPA in Spanish) that aims to introduce a problem solving approach into regular teachers' instructional practices. The chapter provides a context to explain the project rationale to focus on problem solving approaches to help teachers improve their practices and their students' mathematical competencies. The program includes a series of workshops in which teachers have an opportunity to work on problems and to think of ways to implement them into regular classrooms. After 3 years of implementation, the authors report that teachers have begun to question their practices, to change their beliefs about teaching and ways to introduce a problem solving approach in their classrooms.

Finally, Ho, Yap, Tay, Leong, Toh, Quek, Toh, and Dindyal present and discuss results from a project whose aim is to implement a mathematical problem solving approach in all classrooms in Singapore. They identify the factors that contribute to, and explain, the success or failure of a school to implement the project. To this end, they focus on analysing factors such as programs and school levels in terms of outcomes, inputs, resources, constrains, strategies, and feedback and evaluations. The authors argue that the sustainability of introducing and maintaining a problem solving approach in schools can be achieved through the infusion and diffusion of a school culture that fosters integration between curriculum and school problem solving practices.

Manuel Santos Trigo
Peter Liljedahl

Part I
Problem Solving Heuristics

Chapter 1
"Looking Back" to Solve Differently: Familiarity, Fluency, and Flexibility

Hartono Tjoe

Problem solving clearly plays an important role in mathematics (Duncker, 1945; Kaiser & Schwarz, 2006; Lesh, 1985; Mason, Burton, & Stacey, 1982), and its role in mathematics education is equally prominent (Common Core State Standards Initiative, 2010; NCTM, 2000). Apart from solving unsolved problems, the professional practice of research mathematicians also often involves solving, through different approaches, problems that have been previously solved (Davis & Hersh, 1981; Liljedahl & Sriraman, 2006; Thurston, 1994). A comparable pursuit of multiple solutions in the classroom experience of K-12 students, however, has seldom been researched (Santos-Trigo, 1996; Silver, Ghousseini, Gosen, Charalambous, & Font Strawhun, 2005).

The present study focuses on the second part of Pólya's (1945) fourth step of problem solving, namely, "looking back" in order to solve a problem differently. In particular, it examines the extent to which the practice of "looking back" to solve differently has been integrated into mathematics instruction in the United States, and thus, whether this practice is familiar to American students. Mathematical interconnectedness was analyzed through student fluency and flexibility in supplying different solution methods. An assessment involving multiple mathematics concepts was utilized to explore the relationship between students' mathematical understanding and their awareness of mathematical interconnections.

The following three research questions guided the present study: (a) Based on a mathematics problem-solving test and interview results, to what extent were students familiar with the practice of problem solving using multiple solution methods? (b) Given their familiarity or unfamiliarity with the practice of solving problems using multiple solution methods, to what extent were the students fluent in understanding, reproducing, and identifying a particular mathematics topic related to the various solution methods? and (c) Given their fluency or non-fluency in such a range of

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mathematics topics, to what extent were the students flexible in making mathematical connections among the different solution methods and in adjusting to these different methods?

In the context of the present study, "familiarity" refers to the quality of a topic being well-known or generally recognizable based on prior mathematical experience; "fluency" refers to the ability to formulate, demonstrate, and communicate strong mathematical ideas effortlessly and articulately; and "flexibility" refers to the willingness to forgo one's familiar solution method in favor of a novel or unfamiliar method either generated by oneself or presented by others (Leikin, 2009; Silver, 1997; Sriraman, 2009; Star & Rittle-Johnson, 2008; Torrance, 1966).

1.1 Conceptual Framework

1.1.1 Problem-Solving Process

Literature in mathematics education indicates that problem solving was one of the most highly researched topics in the field for several decades (Kilpatrick, 1985; Lester, 1994; Schoenfeld, 1985). More recently, many issues regarding problem solving have been discussed in connection with other emerging topics in mathematics education (Felmer, Pehkonen, & Kilpatrick, 2016; Schoenfeld, 2008; Singer, Ellerton, & Cai, 2015).

The important place of problem solving in school mathematics is natural given its strategic role in teaching and learning mathematics (Liljedahl, 2016; Owen & Sweller, 1985). A number of pedagogical approaches have been proposed to incorporate the problem-solving experience into everyday mathematics classrooms (Pressley, Forrest-Pressley, Elliott-Faust, & Miller, 1985). The topic draws considerable interest and attention not only from school teachers and educators, but also from research mathematicians.

Pólya (1945) enumerated four distinct steps in the process of mathematical problem solving: (a) understanding the problem, (b) devising a plan, (c) carrying out the plan, and (d) "looking back."

The first step, understanding the problem, begins with the identification of what is posed by the problem; that is, problem solvers must determine the nature of the question being asked (Michener, 1978). To this end, it is important to recognize all available data in the problem, and to determine and differentiate necessary, sufficient, relevant, redundant, and contradictory conditions amongst the given information. Additional facts may be further derived from drawing appropriate figures or introducing suitable notation.

The second step is devising a plan. A well-devised plan makes the most straightforward connection between the data and the unknowns. In addition, it builds on comparable problem-solving experiences from the past. It is therefore important to consider analogous problems, some of which may vary in appearance from the problem under consideration in several ways, from the structure of the data they present to the construction of the unknowns (Gick & Holyoak, 1980). Particular techniques and established results employed in the course of past problem solving may inform the restatement of problems presently at hand.

Pólya discussed many heuristic strategies for solving mathematics problems (Schoenfeld, 1979a, b), including drawing pictures, solving simpler, analogous problems, considering special cases to find general patterns, working backward, and adopting different points of view.

The third step is to carry out the plan. It is critical to execute each step of the plan carefully (Garofalo & Lester, 1985), and to verify that each step follows logically.

The fourth step is "looking back." Arrival at a solution does not necessarily mean that the process of problem solving has ended. In the first part of Pólya's fourth step, problem solvers examine the obtained solution of a problem by checking the argument along the way, ascertaining in particular an absence of errors in reasoning (Silver, Leung, & Cai, 1995).

In the second part of Pólya's fourth step, problem solvers review the solution to find alternative approaches to solving the same problem. Deriving the obtained result through the use of alternative approaches can be valuable for future problem solving (Silver et al., 2005).

Pólya devoted much time to illustrating his model of problem solving with concrete exemplars. The model, as a result, gained many enthusiasts from a large audience. He convinced his readers that the problem-solving processes he analyzed were not only accessible to research mathematicians, but could also be utilized by broader audiences.

1.1.2 Problem Solving Using Multiple Solution Methods

Many researchers in mathematics education have comprehensively and systematically examined Pólya's model. A review of prior literature reveals, however, that much of this attention has focused specifically on the first three steps. In fact, many researchers were particularly attracted by the second step, devising a plan (Schoenfeld, 1985)—and understandably so, as this is what most classroom practitioners expect their students to develop and implement while learning mathematics. This was, after all, the principal reason the model was constructed in the first place. Nonetheless, Pólya's (1945) model of problem solving does not end at the third step.

Only a limited number of studies in mathematics education have examined students' use of alternative approaches in problem solving. Despite its importance, Pólya's fourth step has received less attention in mathematics education community than the other three steps from the empirical point of view (Schoenfeld, 1985; Silver, 1985; Tjoe, 2014).

Some researchers in this field have been particularly successful in exploring the use of mathematical tasks requiring students to solve a single problem via several different approaches. These researchers investigated the presence of multiple mathematics concepts through the solution of non-standard problems via different but related solution methods (Leikin & Lev, 2007), through the transformation of standard problems into non-standard problems (Santos-Trigo, 1998), and through the recognition of specific attributes within standard problems (Tjoe & de la Torre, 2014).

An understanding of interconnections among different mathematical concepts is recognized by many mathematicians as a driving force in the appreciation of mathematical beauty (Borwein, Liljedahl, & Zhai, 2014; Davis & Hersh, 1981; Hadamard, 1945; Poincare, 1946). In turn, mathematics teachers, educators, and practitioners in general agree that knowing how and why mathematics works—and in understanding in particular the connections among many different solutions to a problem as opposed to superficial memorization of solution procedures—should be viewed as fundamental to students' development of mathematical reasoning (Eisenhart et al., 1993; Hiebert & Lefevre, 1986; Rittle-Johnson & Siegler, 1998).

Clearly, the fourth step of Pólya's (1945) problem-solving process plays a critical role in prompting the discovery of a variety of different solution methods. In particular, the idea of "looking back" to solve differently is closely related to the qualities of familiarity, fluency, and flexibility.

In the absence of familiarity with problem solving using multiple solution methods, problem solvers may be less inclined to reflect on the solution process and to seek more than a single solution method. Without considerable fluency in a range of mathematical subjects, "looking back" to solve differently is far less likely to be effective or successful. Similarly, lack of flexibility in switching between different solution methods may lead to an unfavorable attitude toward finding alternative approaches to solve the same problem. The analysis of familiarity, fluency and flexibility might therefore be considered necessary for the fourth step of Pólya's (1945) problem-solving process to materialize in an optimal manner.

Many earlier discussions of problem solving via multiple solution methods focus on a variety of potential benefits of the practice. Silver et al. (2005), for instance, maintain that students "can learn more from solving one problem in many different ways than [they] can from solving many different problems, each in only one way" (p. 288). They particularly advise students interested in mathematics to obtain more experience in solving problems via multiple solution methods. Silver and colleagues regard such experience as having "the potential advantage of providing students with access to a range of representations and solution strategies in a particular instance that can be useful in future problem-solving encounters" (p. 288). They also consider the use of multiple solution methods in order to "facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas" (p. 288).

Leikin and Levav-Waynberg (2007) were interested in surveying teachers for their thoughts about alternative solution methods in problem solving. They interviewed several high school mathematics teachers in a comparative study of teachers' beliefs. Their findings reveal positive attitudes toward the use of multiple solution methods in problem solving. Most teachers in the study by Leikin and Levav-Waynberg considered the use of these methods beneficial to fostering student success in problem solving. They believed that working with many different approaches accommodated the learning experiences of students who had pronounced preferences in learning style. In turn, they reasoned that struggling students could benefit from the presentation of various approaches, especially with regard to problems having a high level of difficulty. Such presentations should be applied to problems with complex approaches requiring sophisticated mathematical knowledge yet which are solvable using elementary techniques. As one teacher remarked, when presented with different solution methods, students should be able to choose the solution method "that is easiest [for them] to understand" (Leikin & Levav-Waynberg, 2007, p. 363).

Other teachers in the study by Leikin and Levav-Waynberg (2007) valued in particular the students' development of mathematical thinking and reasoning as integral to the establishment of a solid foundation for future academic success. Several teachers acknowledged the significance of students' awareness of connections between mathematics topics. Mathematics should be viewed "as a whole"—that is, as a collection of connected, rather than disjoint, ideas (Leikin & Levav-Waynberg, 2007, p. 363). In general, Leikin and Levav-Waynberg (2007) concluded that these teachers evinced favorable views of the use of multiple solution methods.

In addition to mathematics education researchers, a number of cognitive psychologists interested in educational psychology with applications to learning and cognition have also endorsed employing multiple solution strategies in problem solving. Collins, Brown, and Newman (1989) discuss the use of multiple perspectives by means of their "cognitive apprenticeship" approach to instructional method. In their model, students' learning processes were considered in light of five teaching methods: modeling, coaching, scaffolding, reflection, and articulation. The role of the teacher in supporting the students' learning experience gradually decreased as the students felt more confident in communicating their understanding of the problem-solving solutions.

Collins et al. (1989) argue that the more approaches and perspectives students explore, the more effective the implementation of this cognitive-based learning method will be. Some benefits of this method they found included improved "apprenticeship" through the use of real-world activities and assessments (Collins et al., 1989). The method also enhanced students' motivation and engagement in overall learning (Collins, 1991), greater transfer and retention rates (Resnick, 1989), and higher-order reasoning (Hogan & Tudge, 1999).

Spiro, Feltovich, Jacobson, and Coulson originated the "cognitive flexibility theory" (Spiro, Feltovich, Jacobson, & Coulson, 1991). Spiro et al. (1991) maintain that restructuring knowledge through changes in approach makes learning new concepts more effective. Such adaptations are based on the notion that the human mind can be trained to be flexible enough to accommodate different situations (Spiro & Jehng, 1990). New information and experiences are processed via the transfer of knowledge and skills, and further constructed to develop new meaning and understanding. In other words, Spiro and Jehng (1990) assert that learning through different perspectives associated with different situations deepens students' understanding and learning experiences.

Tabachneck, Koedinger, and Nathan (1994) also recognized the purpose of adopting many different solution methods in problem solving. They argue that on its own, each solution method entails certain disadvantages and weaknesses. In order to overcome these, Tabachneck et al. (1994) recommend students employ a combination of different solution methods instead of relying on only one. More specifically, they emphasize that students could benefit from employing this learning style in mathematical problem solving.

In addition to advocating the use of many different solution methods, many cognitive psychologists encourage teaching a coherent interrelation among those solution methods (Bodemer, Plötzner, Feuerlein, & Spada, 2004; de Jong et al., 1998; Skemp, 1987; Van Someren, Boshuizen, de Jong, & Reimann, 1998). Equally important, Reeves and Weisberg (1994) suggest showing students many analogical problems or examples concurrently.

On the whole, cognitive psychologists have taken a positive stance on problem solving using multiple solution methods, as have mathematics education researchers. Despite the benefits of implementing this learning style, some of these discussions were not without uncertainties.

A few teachers in the study by Silver et al. (2005) discussed issues and concerns in teaching problem solving via multiple approaches. They included the constraints of instructional time, limitations involving instructors' perceptions of student abilities, the selection and presentation order of solution methods, and uncertainty about the advantages and disadvantages of reviewing incorrect approaches to problems.

Some teachers in the study by Leikin and Levav-Waynberg (2007) showed genuine concern about students' learning experiences. They worried that students might confuse "whether the object of study is to solve the problem, the fact that there is more than one solution to the problem, or the principles behind the solutions and the connections between them" (p. 366).

Despite these constraints and concerns, many researchers still felt firmly confident in their recommendations for teaching problem solving using many different approaches. Silver et al. (2005) nonetheless point out the possibility that teachers may possess inadequate mathematical knowledge to effectively employ this instruction technique. They hypothesized that this might constitute a significant limiting factor in its overall success as an instructional strategy.

Several empirical findings have been presented to demonstrate students' learning outcomes as a result of approaches teaching multiple solution strategies. Große and Renkl (2006) examined the effects of teaching problem solving using many different solution methods presented in the form of worked-out examples. Their experiment involved combinatorics lessons for university-level students. The authors found that exposing students to the presentation of many different solution methods did in fact improve their procedural and conceptual understanding.

Rittle-Johnson and Star (2007) analyzed the effect of comparing many different solution methods upon students' learning experience. Their experiment involved algebra lessons for seventh grade students. The researchers found that exposing students to the practice of comparing and contrasting different solution methods in a simultaneous manner improved their procedural understanding more than their conceptual understanding.

In general, experimental studies, along with their pedagogical recommendations described earlier, showed that the benefits and potential opportunities of problem solving using multiple solution methods outweigh the concerns and challenges associated with the actual teaching of these methods. The present study examines the extent to which the practice of problem solving using multiple solution methods might be effectively presented in an existing classroom routine.

1.2 Methodology

The present study involved nine students (4 female, 5 male, aged 16–18, in grades 11–12) in a highly regarded urban northeastern American high school which has graduated notable scientists in the past. It is one of the highest ranking among public high schools with an academic specialization in mathematics and sciences (Vogeli, 2015).

The nine students who participated in the present study received strong recommendations from their mathematics teachers. These students were carefully selected to be part of the present study with an expectation that they might be significantly more capable than their peers of not only solving the problems involved in the study, but also of supplying more than one solution method for each problem.

At the time of the study, these students were enrolled in an Advanced Placement (AP) Calculus course, a university-level calculus course with topics in differential and integral calculus typically taken by high school students in the United States seeking university credit or placement in a university calculus course. These students volunteered to take a paper-and-pencil test consisting of three non-standard mathematics problems (Problems 1, 2, and 3; Tjoe, 2015). The researcher identified beforehand, as part of the careful selection process of the problems included in the test, 15 different solution methods associated with the three non-standard mathematics problems: four solution methods for Problem 1 (P1S1, P1S2, P1S3, and P1S4), eight solution methods for Problem 2 (P2S1, P2S2, P2S3, P2S4, P2S5, P2S6, P2S7, P2S8), and three solution methods for Problem 3 (P3S1, P3S2, P3S3; Tjoe, 2015).

On the surface, these three problems appear to depend only on the three most common elementary mathematics topics, namely arithmetic, algebra, and geometry. At a deeper level, they incorporate multiple access points to more advanced mathematics topics such as trigonometry, calculus, linear algebra, and real analysis. Overall, the three problems were carefully selected to allow accessibility for average students in a typical American high school that has adopted the national curriculum in mathematics (Common Core State Standards Initiative, 2010; NCTM, 2000). For instance, the approaches involved in P1S4, P2S1, and P3S1 can be readily comprehended by students in regular high school arithmetic, algebra, and geometry courses, respectively, and not exclusively by more advanced students in the specialized high schools as described by Vogeli (2015).

The nine students were instructed to creatively solve the three problems using as many different solution methods as they could without the aid of a calculator and without any time limitations, and they were specifically instructed to solve the problems using multiple methods. While this methodology was deliberately and specifically adopted in order to assess students' familiarity with the practice of problem solving using multiple solution methods (Leikin & Lev, 2007), it was well noted that it departed from the normal assessment procedure with respect to the role of didactical contract (Hersant, 2011).

After their written responses were checked for accuracy, the students were presented with their work and the 15 solution methods, and were interviewed individually. A video recorder was utilized to capture the students' problem solving processes as presented in written responses as well as during the individual interviews.

Students' solution methods were evaluated on the basis of a simple acceptability scoring system. An acceptability score of 1 indicated that a student successfully supplied a correct answer by using an approach in a logical manner to solve the problem; otherwise, an acceptability score of 0 was given. Students' solution methods were also classified based upon the list of 15 different solution methods identified by the researcher beforehand.

Follow-up interviews were conducted with the nine students who had previously taken the paper-and-pencil test. The interview was designed to elicit the students' explanations for their particular solution methods. In addition to questions about their mathematical background, each of the nine students was asked (a) whether they were familiar with the practice of "looking back" to solve differently, (b) whether they understood each of the 15 solution methods, (c) whether they had learned the content involved in each of the 15 solution methods in their previous mathematics coursework, and (d) whether in the future they might solve similar problems to the three tested using any of the 15 solution methods they had considered in reviewing the test.

The first question assessed students' familiarity with the practice of "looking back" to solve problems differently. The second and third questions assessed the students' fluency in diverse mathematical knowledge. The fourth question assessed students' flexibility in accepting solution methods other than their own. In analyzing these four questions, the researcher coded the nine students' responses with the following scoring system: a score of 1 indicating familiarity with the practice of "looking back" to solve differently, understanding of a particular solution method, recognition of the relation of a particular solution method to mathematics courses previously taken, and likelihood of supplying a different solution method in the future; otherwise, a score of 0 was assessed.

The results of the test and the student interviews were analyzed to detect similarities or differences in the justifications provided by the other students regarding their supply of particular solution methods. The responses to the interview questions were analyzed to determine the students' familiarity, fluency and flexibility regarding problem solving using many different solutions.

1.3 Findings

Nine students participated in the present study. Eight of these students were enrolled in Grade 12, and one was enrolled in Grade 11. The nine students reported an average SAT Math Section score of 754, SAT Subject-Math I score of 750, and SAT Subject-Math II score of 790. The national average scores of SAT Math Section, SAT Subject-Math Level I, and SAT Subject-Math Level II were 516, 605, and 649, all of which were out of a possible maximum score of 800 (*The College Board*, 2011a). One student reported an American Mathematics Contest 12 (AMC-12) score of 94.5. The SAT is a standardized test that universities in the United States generally use in admission criteria to measure college readiness of prospective students (*The College Board*, 2011b), whereas the AMC is a series of mathematics competitions generally used to determine participants' eligibility for the International Mathematical Olympiad (*Mathematical Association of America*, 2011).

Because they were all recruited from the same high school and because the school utilized a relatively uniform mathematics curriculum (with the exception of honors courses), all of the nine students were found to have received formal courses in algebra, geometry, trigonometry, pre-calculus, calculus, and linear algebra throughout their mathematics education in this particular, specialized high school.

Although they were reminded several times of the unlimited time to solve the problems using numerous methods, the students generally finished the test in less than one hour. Six, three, and seven students successfully solved Problems 1, 2, and 3, respectively.

Table 1.1 summarizes the mathematical background of the nine students as well as the problems each successfully solved. (If a student did not report taking the SAT Math Section, SAT Subject-Math I, SAT Subject-Math II, or AMC-12, "n/a" is recorded in Table 1.1 to indicate that the score is not available.)

Table 1.1 Summary of students' mathematical background and test results

Student	Grade level	SAT Math Section	SAT Subject-Math I	SAT Subject-Math II	AMC-12	Solved problems
1	12	770	n/a	800	94.5	1, 2, 3
2	12	780	n/a	800	n/a	1, 2
3	12	770	750	770	n/a	1, 3
4	12	740	n/a	770	n/a	1, 3
5	12	640	n/a	n/a	n/a	1, 3
6	11	n/a	n/a	n/a	n/a	1, 3
7	12	770	n/a	800	n/a	2
8	12	800	n/a	800	n/a	3
9	12	760	n/a	n/a	n/a	3

1.3.1 Familiarity

Only one student (Student 1) solved a problem (Problem 3) using more than one solution method; the other eight students either failed to solve certain problems entirely or solved them using only one solution method. Based upon the interview responses, the nine students were not at all familiar with the practice of “looking back” to solve differently. There were nine scores of 0 for the first question in the interview.

The impulsive manner in which the nine students were eager to find the answers to the three problems suggests, to a certain degree, that they were more accustomed to contently solving problems using a single, familiar method than they were to persistently and purposefully looking for alternative solutions. Obtaining a correct answer to a problem appeared more important to these students than searching for more efficient or enlightening solution methods. It did not appear to occur to most of the nine students that problem solving in mathematics might be a recurrent process, or that exploring alternative solution methods might be beneficial.

When asked whether they could relate the practice of “looking back” to solve problems differently to their past experiences in learning mathematics, many of them highlighted their algebra class. Specifically, they referred to the topic of solving systems of simultaneous linear equations using graphical, substitution, and elimination methods, among others approaches. Yet, they expressed that tests in this topic, like any other tests in their mathematics classes, specified explicitly which solution methods were expected in addressing particular problems. There was not much liberty provided by their instructors with regard to choosing any viable solution method, including those that students might devise on their own, in solving test problems. That being said, some students mentioned that their mathematics teachers were generally more amenable to student-invented solutions in a classroom discussion than during formal examinations.

Other students offered their impressions that mathematical concepts were supposed to be learned sequentially; that is, they felt that topics in mathematics were properly viewed as preconditions to further study rather than as interrelated ideas. They described, for example, the belief that the techniques of algebra are only applicable to classes such as coordinate geometry or calculus when employed in the process of manipulating variables. They did not recall many classroom discussions about connecting topics from different mathematics courses, such as how one might approach calculus problems using concepts from elementary algebra. Essentially, the nine students in the present study considered their mathematics courses as disconnected subjects under the single label of “mathematics.”

1.3.2 Fluency

The nine students in the present study had achieved top percentiles in standardized tests and had received a more rigorous mathematics curriculum—including classes in trigonometry, pre-calculus, and calculus—than one could find in typical public high schools in the United States. They were also among the students most highly recommended by their mathematics teachers. As such, they might be considered to have acquired a high level of mathematical training.

Based on the interview responses, the nine students understood all 15 solution methods and recognized all 15 solution methods as being related to specific mathematics courses they had previously taken. There were nine scores of 1 for both the second and third questions in the interview.

In fact, after being presented with the 15 solution methods for the three problems, within a relatively short period of time, all of the students immediately acknowledged that they understood all of the methods. They could each replicate the different solution methods without difficulty during the interview.

They were also able to spontaneously and accurately identify specific mathematics courses in which they were taught content associated with each of the 15 solution methods. Moreover, they mentioned with confidence that there were no concepts involved in the 15 solution methods that they had not previously encountered in their mathematics courses.

They described, for example, how the geometric and algebraic solutions (P1S1 and P1S2, respectively) to Problem 1 were accessible based on the material they learned in their algebra course, how the limit-definition-of-derivative solution (P1S3) was accessible based on material learned in their pre-calculus course, and how the arithmetic solution (P1S4) was accessible based on material learned in their middle school mathematics course. For Problem 2, the students confidently related the geometric solution (P2S1) to their coordinate geometry course, the Cauchy-Schwartz-inequality solution (P2S2) to their pre-calculus course, the contradiction-via-symmetry and quadratic-equation solutions (P2S3 and P2S5, respectively) to their algebra course, the vector-dot-product solution (P2S4) to their linear algebra course, the calculus-in-polar-coordinate and single-variable-calculus solutions (P2S6 and P2S8) to their calculus course, and the angle-sum-trigonometric-identity solution (P2S7) to their trigonometry course. In Problem 3, as in the previous two problems (Problems 1 and 2), the nine students easily recognized distinct mathematics concepts from their geometry course (such as the congruent-diagonals property of a parallelogram, the characteristics of inscribed angles of a circle, and the sum of internal angles of a circle) in the three solutions (P3S1, P3S2, and P3S3, respectively).

It was clear that the students were relatively fluent as regards their knowledge of mathematical content. The results of the interview particularly substantiated the mathematics background they had reported prior to the interview as well as their perceptions of their own mathematics skills. Overall, the nine students in the present study demonstrated an uncommon level of mathematics proficiency compared to typical high school students in the United States.

1.3.3 Flexibility

Despite their fluency, the nine students for the most part failed to supply more than one solution method for each problem contrary to the instructions for the test. Based on the interview responses, the nine students were not at all likely to supply a different solution method aside from their own preferred solution method. There were nine scores of 0 for the fourth question in the interview. One clear indicator was observed in the students' written work for Problem 2 (which is in essence an algebra problem but was perceived by the nine students as being a calculus problem).

All nine students in fact identified Problem 2 as a calculus problem: they immediately operated the differentiation technique to arrive at an answer. One might expect that the students' past mathematical experience (especially given that they were enrolled in an AP Calculus course at the time of the study) had directly influenced their focus on certain solution methods.

Their fixation on a single solution method became more apparent after they were presented with the 15 solution methods for the three problems. The calculus approach that most students supplied was only one of the eight possible solution methods for Problem 2. (The other seven solution methods included topics involving elementary algebra, geometry, trigonometry, and linear algebra.)

The nine students maintained that they would not solve problems similar to Problem 2 in the future using any of the other seven solutions, even though they had no difficulty grasping those seven other solution methods. They argued that their calculus solution was more practical than other solution methods in obtaining the correct answer. This result demonstrates the fixation effect students revealed in their rigid association between particular problems and particular solution methods.

Nevertheless, the one student who solved one of the three problems using more than one solution method might be analyzed differently than the other eight students. Compared to the others, Student 1 had a greater past mathematical experience: he had taken the AMC-12 test, he was an active member of the mathematics team in that particular high school, and he mentioned having seen a mathematical fact similar to that in Problem 3 in the course of reading a number of mathematics books outside the confines of his course requirements.

Furthermore, the test results of Student 1 differed substantially from those of the other eight students both in terms of quantity and quality. Student 1 was the only student who was able to solve all three problems correctly, and he was the only student able to produce more than one solution method to a problem.

Student 1 solved Problem 1, an arithmetic problem, using an algebraic solution (P1S2), whereas the other five students who solved the same problem successfully did so using an arithmetic solution (P1S4). Student 1 solved Problem 2, an algebra problem, using a polar coordinate substitution approach from calculus (P2S6), whereas the other two students who solved the same problem successfully did so using a single variable substitution approach from calculus (P2S8).

Furthermore, Student 1 solved Problem 3, a geometry problem, using two different solution methods: one used the given facts from the sum of internal angles (P3S3), and the other used an extension of the inscribed angle of a circle (P3S2). The former was the only solution method supplied by the other six students who successfully solved Problem 3. Student 1 discussed in the interview how he simply attempted to prove a known fact that he recalled from a mathematics book as he was solving Problem 3, instead of formulating an answer anew.

To the extent that Student 1 demonstrated the capacity to transform his mathematical background into a unique test result, such a positive correlation between fluency and flexibility was nonetheless rather unclear in his consideration of solution methods beyond those he presented in his written responses. Despite his clear understanding of all of the 15 solution methods for the three problems, Student 1 maintained that if he were to take the test again, he would still supply the same solution methods he did previously.

As Student 1 asserted that his solution methods resulted in correct answers and that there was no need for him to consider the other methods, it was clear that the same fixation effect observed in the case of the other eight students emerged in spite of Student 1's distinct combination of mathematical background and test results. In summary, the emphasis on doing well on mathematics assessments, and on ensuring that each problem was solved correctly irrespective of how it might have been solved differently appeared, to a certain extent, pervasive and persistent. Despite how capable the students involved in the present study may be, they nevertheless became desensitized to the directive to use multiple solution methods. It was evident that the nine students somehow overlooked the relationship between their mathematical understanding and their realization of mathematical interconnectedness in the pursuit of academic success in mathematics.

1.4 Conclusions and Discussions

The present study reveals, to some extent, that based on a mathematics problem-solving test and subsequent interview results, the nine students were less familiar with the practice of problem solving using multiple solution methods at the assessment level than in the classroom discussion environment. It suggests for the most part that despite their fluency in understanding, reproducing, and identifying a particular mathematics topic or course related to specific solution methods, the nine students were unfamiliar with the practice of "looking back" to solve problems differently. It also indicates that, regardless of their fluency with a variety of mathematics topics, the nine students were not flexible in making mathematical connections among different solutions or in adjusting to the different solution methods.

The nine students' perceived mastery of particular methods and disinterest in others indicates, to some extent, that pedagogical recommendations or educational policies that underscore fluency in acquired mathematical concepts and procedures might not guarantee flexibility in accepting different solution methods. This condition appears to be exacerbated by the unfamiliar ways in which problem-solving processes might encourage, or even necessitate, students to “look back” to find alternative approaches to solve the same problem.

Given that it is not generally required or part of any curriculum, mathematics teachers cannot expect students to demonstrate the importance of the fourth step of Pólya’s (1945) problem solving process on their own or without additional prompts. Students in the present study pointed out that student-invented strategies usually only make their appearance during classroom discussions, not at the assessment level where it may be more valuable to invite elements such as surprise and creativity. It is evident from the interviews that, regardless of their mathematical background, students need early exposure to and constant opportunities to cultivate the practice of “looking back” to find different solution methods to previously solved problems.

The present study not only identifies that the practice of “looking back” has not been effectively integrated into mathematics classroom instruction in one of the most highly-regarded high schools in the United States, but also demonstrates that non-standard problems have the potential to offer students an appreciation for mathematical interconnections. In relation to earlier studies (Leikin & Lev, 2007; Silver et al., 2005), the present findings show that a more concrete pedagogical framework (Collins et al., 1989; Skemp, 1987; Spiro et al., 1991) is necessary to effectively integrate the practice of “looking back” into the current curriculum and classroom practice in mathematics. Changing the didactical approach to assessing problem solving in the mathematics classroom consequently requires careful consideration of different pedagogical frameworks, from one assessment which did not require multiple solution methods to another that did (Douady & Perrin-Glorian, 1989).

The present study also demonstrates the value of mathematics teachers adept at, and adaptive to, the identification and examination of the appropriateness and effectiveness of student-invented strategies relating to the solution methods introduced in their classroom, and to other related mathematics topics outside their classroom. To this end, it calls attention to the need to train, equip and enable future classroom instructors teaching rigorous and advanced mathematics courses to place an emphasis on illustrating connections between various topics in mathematics. Ill-equipped classroom instructors may be more liable to dismiss student-invented strategies when faced with unfamiliar solution methods (Silver et al., 2005). By accepting accurate solution methods that they did not explicitly teach in class, and by making connections between students’ mathematical backgrounds and the content they are currently teaching, teachers can nurture students’ deeper understanding of mathematics (Michener, 1978).

This understanding should therefore be carefully evaluated not only in terms of how well students might retain their acquired mathematical knowledge, but also in terms of how far students might form mental connections between new knowledge and past knowledge. Students need to appreciate that the whole field of mathematics was not developed in isolation of its parts (Davis & Hersh, 1981) the way it is presently studied in the elementary and secondary schools, but rather presented as a gradual progression of ideas that built one result upon another in a consciously connected manner.

Furthermore, by revealing many different solution methods, teachers can open up the possibility for students to consider the idea that topics in mathematics courses might be viewed on a coherent and interrelated continuum. What happens in algebra class, for instance, does not have to stay in algebra class; what happens in algebra class can and should be carried forward to other mathematics classes such as geometry and calculus. Further studies might be considered to examine a pedagogical framework integrating the need for problem solving using different solution methods within mathematics instruction, especially one incorporating students possessing a wider range of mathematical abilities.

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Chapter 2
Future-Oriented Thinking and Activity in Mathematical Problem Solving

Wes Maciejewski

2.1 Introduction

Problem solving is a central focus of mathematics education and has been, arguably, since before mathematics education was a legitimate field of study. The roots of problem solving as a topic of inquiry are traced to expert mathematicians writing reflections on their own mathematical work; see, for example (Hadamard, 1945; Poincaré, 1910). However, it was Polyá's *How to Solve It* (1945) that brought problem solving into education. Although Polyá did intend for How to Solve It to be an educational work, it may be of little educational value beyond inspiring teachers to engage students in problem solving; indeed, How to Solve It predates any rigorous, modern educational theory. It was subsequent work by other authors that attempted to operationalize Polyá's groundwork; see Kilpatrick (1985) and references therein. Most notable is Schoenfeld's *Mathematical Problem Solving* (1985).

The strength of Schoenfeld (1985) is two-fold: (1) it reports the first serious attempt to teach Polya's heuristics—to really infuse his students with them; and, (2) it acknowledges the complexity of authentic problem-solving behaviour and attempts to incorporate a succinct, empirically-grounded perspective to the field. The importance of this second point cannot be understated—little subsequent work has attempted to articulate authentic problem solving activity the same way Schoenfeld (1985) did. Rather, contemporary problem solving research tends to take Pólya (1945) and Schoenfeld (1985) as a basis, as if they bookend the entirety of problem solving. Relying too heavily on these works and not broadening and strengthening the foundations of problem solving education with further empirical work may be contributing to an impoverishing of the field; the cause of the conclusion of some authors that traditional, heuristic and strategy-type problem solving education isolated from the rest of a mathematical education, has largely been ineffective (English

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& Sriraman, 2010; Lester & Cai, 2017; Schoenfeld, 1992; Silver, 1985). In the words of Lester and Kehle (2003), "Teaching students about problem-solving strategies and heuristics and phases of problem solving … does little to improve students' ability to solve general mathematical problems" (508).

There have been notable attempts to dislodge the field from this equilibrium of isolated problem solving instruction towards a more authentic mathematical education. The two I mention here stem from the acknowledgement that *problems* are somehow artificial, connoting a pre-defined starting and ending state, manufactured for students as applications of material learned separately. The first, *mathematical modelling* (Lesh & Zawojewski, 2007), addresses the artificiality of problem solving. This approach takes the stance that the world is complex and dynamic and can be, at least partly, understood through mathematical representations. Unlike in problem-solving situations, the salient mathematics is not predefined in mathematical modelling; it is as idiosyncratic as the situation and the modeller (Hamilton, 2007).

The second notable attempt at shifting the focus of problem solving is *problem posing* (Silver, 1985). This is the practice of having students generate their own problems from, for example, given data or situations, or as refinements of existing problems. The reasons for encouraging problem posing in education are manifold (Silver, 1985): through posing, students "own" the problems and are therefore invested in them; the posed problems shed light on a student's knowledge, understanding, and creativity; and problem posing improves dispositions towards mathematics and students' mathematical problem-solving ability. Problem posing is therefore more than just word play; posing problems deepens students' mathematical knowledge.

Other authors provide more detailed accounts of the historical development of mathematical problem solving as a both an educational and research topic (English & Sriraman, 2010; Kilpatrick, 1985; Lester & Cai, 2017). My intention with the brief chronology here is to partly highlight that problem solving research has been shaped by its roots and partly that advances in the field have and will come from departures from those roots. Much of the effort in attempting to improve students' problem solving ability has been on teaching them what mathematicians had identified through self-reflection as characteristics of effective problem solving. Despite the level of research activity in problem solving, and the numerous efforts to improve problem solving throughout K-16 education spanning decades, students seem no better equipped to solve problems, at least those who are explicitly taught problem solving strategies and heuristics (Lesh & Zawojewski, 2007; Schoenfeld, 1992). More contemporary results highlight the need to integrate problem solving throughout the mathematics curriculum, making it a primary focus, which seems to have a stronger effect; see the review (Lester & Cai, 2017).

In light of the origins of problem solving research and the equivocal results on teaching problem solving to students, I suggest that the problem solving researchers return to observing problem solving in situ and attend to its development. Observing mathematicians, of all stripes, including learners of mathematics, solving problems may result in a deeper understanding of authentic problem solving processes in-action. From this a richer ontological description of mathematical problem solving abilities will emerge. A deepened understanding of the development of problem solving will lead to a richer theory of problem solving, from which implications for practice may be drawn. That is to say, I claim that mathematics education researchers do not yet understand authentic mathematical problem solving activity well enough to make tenable claims for educational practice.

In this spirit, I present here some theoretical and empirically-grounded work that details a complimentary perspective on mathematical problem solving. Many authors recognize that knowledge of mathematics or heuristics or strategies is not sufficient for effective problem solving. In the words of English and Sriraman (2010), "… knowing when, where, why, and how to use heuristics, strategies, and metacognitive actions lies at the heart of what it means to understand them. … students need to know which tools to apply, when to apply them, and how to apply them" (p. 265). The question arises, how might a user of mathematics know when/where/why/how to use these tools? There is not a singular answer to this question. Metacognition, schema acquisition and activation strategic knowledge, algorithmic knowledge, belief systems, control, and cognitive resources, for example, are ways in which students might leverage their mathematical experience to solve problems (Mayer, 1982; Schoenfeld, 1985, 1992). I propose another possible answer is that reflecting on a possible future state of a problem may inform the solver's current action. This future-thinking guides the present action and brings about the eventual solution.

The purpose of this chapter is to present a complementary characterization of problem solving processes. Whereas traditional approaches—the heuristics of Pólya (1945) and the resource/heuristics/belief systems/control framework of Schoenfeld (1985), for example—take a problem in its current state forward into a future state, the approach presented and employed here, *mathematical foresight*, casts problem solving as imagining a future state of a problem and letting this image pull the current state forward. Imagining *what might be* may allow a solver to choose how best to act in the present. Such an idea was present in the writings of Polyá—"What are we required to do? Let us visualize the final solution we aim at as clearly as possible. Let us imagine the solution." (Pólya, 1945, p. 227)—but the subsequent use and emphasis from his work was on heuristics.

Thus, I present a shift in perspective: Problem solving may not exclusively be about choosing actions/heuristics/strategies, but may also be about recognizing and choosing possible future states of the problem and, subsequently, actions to reach those states. The shift is temporal—this approach to problem solving works in the future to bring present actions forward.

This chapter has two foci. The first is a review of my recent attempts at characterizing future-thinking in mathematical situations. This draws on previous empirical and theoretical work and attempts to position the characterization relative to constructs in the psychology literature. The second focus is on presenting results from a small-scale study that help flesh out the future-thinking model and provide directions for further investigations.

2.1.1 A Digression on Terminology

The terms I use here are intentionally departing from more well-established terminology. A *mathematical situation* is an event in which mathematics is deemed to be required. This could be a "real world" situation—building a Norman window for a house, for example—or one that might not exist in the tangible, sensuous world. It may be encountered by one or more individuals who may or may not all determine that mathematics is needed. The *resolution* of such a situation is the use of mathematics to change the state of the situation and/or how it relates to those who encountered it to the point that it no longer is deemed to require changing. So, for example, a "word problem" given to a student in primary school is a mathematical situation. The event is the student reading the problem; they deem mathematics is required because they are told it is; and they attempt to resolve the situation by, as is known to occur, writing out all possible operations relating to the words present in the problem. A group of mathematicians devising a conjecture over a coffee at a conference is another example. I choose to work with the terms *situation* and *resolution* to aid myself and the reader in detaching the ideas present here from the connotations that come with *problem* and *solving/solution*—"problem" has become understood as artificial, somehow manufactured for torturing students, whereas "solving/solver/solution" suggests a neat, textbook like denouement to the problem. In reality, mathematical situations are seldom like that. They are often quite messy, with many failed, incomplete, or non-rigorous attempts. Additionally, the *situation/resolution* terms and their detachment from *problem/solving* align well with the recommendations to pervade mathematics education with problem solving (Lesh & Zawojewski, 2007; Lester & Cai, 2017).

2.2 Foresight and Episodic Future Thinking

As previous authors have acknowledged (English & Sriraman, 2010), little is known in the mathematics education literature about how students solve (mathematics) problems outside the mathematics classroom. I argue that we need not restrict ourselves to considering "mathematical" problems outside of the classroom; mathematical problem solving may have analogues in general problem solving domains. Therefore, I now turn to the question, how do people solve (non-mathematical) problems encountered in life? This is just as open as the analogous question in mathematics, but psychologists are gaining traction on it nonetheless. There are, of course, many possible answers. One approach to solving authentic problems in particular has recently appeared in the literature and has attracted a high degree of interest: *episodic future thinking* (Atance & O'Neill, 2001). To articulate this construct, I first review results in the psychology of memory.

Memory, according to the prevailing model in psychology (Tulving, 1983), is of two types: (1) procedural, and (2) declarative. Procedural memories are those of actions that the actor is unable to consciously explain or verbalize or otherwise communicate. For example, being able to walk or ride a bicycle or read a book chapter. Declarative memories are those that can be recalled and actively communicated. Mathematical knowledge exists in declarative memory—at least I hope yours does! Declarative memory is further partitioned into *semantic* and *episodic* memories. Semantic memories are those that do not have a personal component—factual knowledge of the world, for example, like its period of orbit around the sun—whereas episodic memories do. For example, memories of Tulving's (1983) model of memory may be semantic, as simple recollection of the types of memory and how they relate, but they also may involve an episodic component, such as a memory of reading Tulving's work on a back porch in New Zealand.

When planning a resolution to a task, mathematical or otherwise, memory is drawn upon. Concerning mathematical problem solving, this is implicitly acknowledged in, for example, Schoenfeld's (1985) notion of "control" and "resources" or Pólya's (1945) question, "do you know a related problem?", or the suggestion of Mason, Burton, and Stacey (2010) to draw on what "I know." Thinking of a resolution to an authentic—for now, non-mathematical—task similarly draws on memory and we are led to ask, "what type of memory?" Consider the task of going from your office back home. You know where your house is in relation to your office and solving the task could be as straight-forward as recalling the sequence of left/right turns you'll need to make—a resolution using semantic memory. Or, you might remember travelling home last week and recall the frustration you felt when sitting in traffic on Queen Street and decide you'd rather take King Street—a resolution employing an episodic memory. Further still, you may need to stop at the grocery on your way and you might imagine the experience of driving to the store, navigating the isles, standing in the checkout, and then driving home. This third way of resolving a task is somehow different than the first two and requires further elaboration.

What is happening in imagining the grocery trip is a mental simulation of how an event might unfold. Episodic memories, of driving and shopping and paying, are drawn upon to create an episodic memory of an event that has yet to occur. This fabricated episodic memory then acts to inform actions just as episodic memories of events actually experienced do (Schacter, 2012; Schacter & Addis, 2007; Schacter et al., 2012). In a sense, a possible future event is experienced before it occurs. This is the process of episodic future thinking (Atance & O'Neill, 2001). Episodic memories of the simulated event are formed and these inform current actions. A key result to highlight is that there is a common neural structure involved in both remembering the past and imagining the future; episodic future thinking relies on the same cognitive structure as remembering past events, the so-called *core network* (Buckner & Carroll, 2007; Raichle et al., 2001; Schacter, Addis, & Buckner, 2007). When imagining future events, it is therefore reasonable to suspect the involvement of past events in the process of imagining.

The most current results indicate that episodic future thoughts are associated with more successful task resolution and greater coping ability; see Schacter, Addis, and Buckner (2008) for a review. These corroborate results from studies on outcome and process simulations. For example, in Taylor, Pham, Rivkin, and Armor (1998), students in a psychology course were assigned to one of three conditions: students were either trained in and practised (1) imagining achieving a good grade in the course, (2) imagining what concrete actions would be needed—studying, reading the text—to achieve a good grade, or (3) a control with no training. Those that imagined what they would need to do to succeed outperformed the control group. Moreover, those that only imagined succeeding underperformed both other conditions. Results like this suggest that future-thinking can be taught and be leveraged for the benefit of learning.

Most, if not all, of the psychology literature on episodic future thinking considers only people solving tasks encountered in daily life. The possibility that people might engage in episodic future thinking in discipline-specific tasks, such as mathematical problem solving, has not been explored. Only now is it being considered and early results are favourable (Maciejewski, 2017; Maciejewski & Barton, 2016; Maciejewski, Roberts, & Addis, 2016); mathematicians, students and professional mathematicians alike, employ future-thinking, including episodic future thinking, in their mathematical work. It seems, then, that the frameworks being developed in the psychology literature are equally applicable to mathematics education contexts.

Future-thinking aside, extant results and constructs in the mathematics education literature strongly suggest that mathematicians do engage in episodic future thinking when working with mathematics. Indeed, if mathematicians have only their *concept images*—defined as the "total cognitive structure associated with a mathematical concept" (Tall & Vinner, 1981)—to rely on when engaging in mathematics, and if these concept images contain episodic memories, which recent work identifies that they often do (Maciejewski, 2017), then mathematical activity likely contains an episodic component. Planning in a mathematical situation, therefore, may be analogous to planning in general, non-mathematical situations, and may involve episodic future thinking. The model presented in the next section is an attempt to understand future-oriented thinking processes in mathematics in this light.

2.2.1 Mathematical Foresight

Upon encountering a mathematical situation, a mathematician—a user of mathematics—may form an image in their mind about a possible resolution to the situation and a course of action likely to bring the situation closer to the resolution. In this way the mathematician is imagining an event that has yet to occur; they are imagining the future and allowing that image to inform their present actions.

The notion of working a mathematical problem by seeing how it might unfold into the future seems to sit well with mathematicians. For example,

> One phenomenon is certain and I can vouch for its absolute certainty: the sudden and immediate appearance of a solution at the very moment of a sudden awakening. On being very abruptly awakened by an external noise, a solution long searched for appeared to me at once without the slightest instant of reflection on my part—the fact was remarkable enough to have struck me unforgettably—and in a quite different direction from any of those which I had previously tried to follow. (Hadamard, 1945, p. 8)

The image that stuck Hadamard was not in the form of so-called "rigorous" mathematics. Rather, it was a rough image of what could be. This possibility guided Hadamard's mathematical work to its ultimate resolution.

Another comes from Poincaré (1910) in his reflections on his processes of discovery in his own mathematical work. The oft-quoted passage is of Poincare receiving a sudden flash of insight into a problem he had struggled with for over a decade. This all occurs as Poincare steps onto a bus during a geological expedition. It is the passage after this that is especially relevant to the current work:

> Returned to Caen, I meditated on this result and deduced the consequences. The example of quadratic forms showed me that they were Fuchsian functions other than those corresponding to the hypergeometric series; I saw that I could apply to them the theory of theta-Fuchsian functions other than those from the hypergeometric series, the ones I then knew. Naturally I set myself to form all these functions. I made a systematic attack upon them and carried all the outworks, one after another. There was one however that still held out, whose fall would involve that of the whole place. But all my efforts only served at first the better to show me the difficulty, which was indeed something. All this work was perfectly conscious. (Poincaré, 1910, p. 327)

What Poincaré acknowledges here is that his imagined future state of his research program informed the actions he could take in the present to make progress. Moreover, he recalls how he could recognize that there were certain keystone problems to overcome to develop a complete theory.

Though the quotes above are likely embellished, they get at the phenomenon I am attempting to describe. They also align well with observations of my own mathematical work. Much of my earlier mathematical work was on understanding certain stochastic processes on graphs—so-called *evolutionary graph theory* (Nowak, 2006). In a particular instance, I had been working on calculating the genetic similarity of individuals thought of as residing on a social graph. This turns out to be a cumbersome calculation, primarily accessible through simulation. After working on the problem for some time—and devoting large computational resources to it—I realized that the calculation ought to be simplified via other, more well-established graph processes. In particular, thinking of the social graph as an electrical network seemed to make sense—a loose analogy formed in my mind between genes and electrons. After a slow, preliminary calculation of the graph depicted in Maciejewski (2012), I knew a general result would follow—it did (Maciejewski, 2012). Which begs the question, how did I know that? Some would say "experience", but *what about experience* allows some users of mathematics to see into the future of a mathematical problem and not others? Taking a step back, what actually is happening when one sees into the mathematical future?

These observations and questions have led me and coauthors to model future-thinking processes in mathematics. The notion of *mathematical foresight* (Maciejewski & Barton, 2016) emerged as a viable model, though it is currently evolving. The strength of mathematical foresight seems to be as a construct that may aid researchers in understanding mathematicians' future thinking processes in a way that avoids the imprecision and mysticism of notions such as intuition, creativity, and insight, as discussed previously in Maciejewski and Barton (2016).

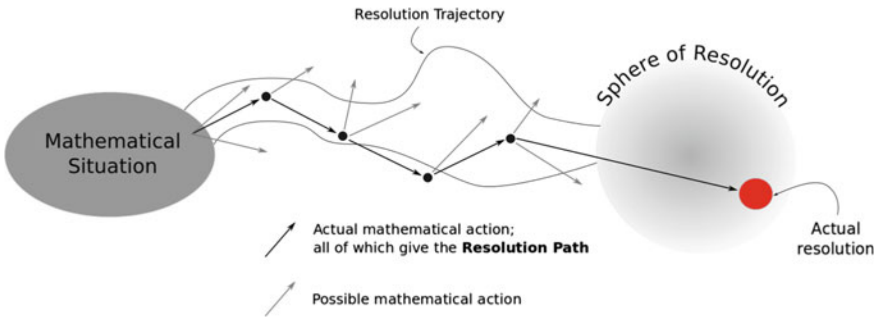


Fig. 2.1 The mathematical foresight model of Maciejewski and Barton (2016)

The mathematical foresight model consists of two main components: the *sphere of resolution* and *resolution trajectory*. The sphere of resolution is the space in which the resolution to a mathematical situation might reside. It characterizes the type of resolution and the form it might take. For example, a statement in number theory may be deemed by the one who encounters it to require a proof. Another example comes from mathematical biology where a model is envisioned. The *resolution trajectory* is a set of possible actions intended to bring the situation closer to its resolution. This may not be an exact sequence of steps to follow; often, only an idea of the *types* of steps is generated.

These two components of mathematical foresight are summarized in Fig. 2.1. The sphere of resolution is the, possibly hazy, shape of where the resolution is likely to reside. The resolution trajectory is a, possibly windy path, narrow at points and wider in others, connecting the starting state of the situation to the sphere of resolution. As the mathematician sets out on this trajectory, they encounter possibly many stopping points at which an action must be decided upon. Seldom is there one and the mathematician may use their mathematical foresight to aid in choosing. This choice may lead the mathematician outside of their initially-imagined resolution trajectory and subsequent choices may lead them back in. Ultimately, the mathematician arrives at the resolution, which may or may not reside in the initial sphere of resolution.

The key contribution of this model to improving understanding of the problem solving process is that the initial image of how a mathematical situation might be resolved can guide the mathematician in their choices made toward a resolution. In this way, the mathematician is using an imagined future state of the situation to guide their current actions.

Having established a workable model of mathematical foresight that seems to align well with mathematicians' future-thinking in mathematical situations, we are left questioning about the genesis of such thinking processes. The mathematicians we discussed this model with identified it clearly in their own work; indeed, it often featured centrally. But what might the ontogeny of such thinking be? To address this, I turn to students—mathematicians in their academic adolescence. It is hoped that analysing the work of students through a mathematical foresight lens will elaborate the model and provide for a richer description of future-thinking processes in mathematical activity.

2.2.2 Students Engaging with Mathematical Foresight

To initiate an investigation of the development of mathematical foresight, Bill Barton and I conducted a study with university students enrolled in a first-year mathematics course covering basic calculus and linear algebra. The results of that study appeared initially in unpublished form, as referenced in Maciejewski and Barton (2016). This section elaborates that report.

2.3 Methods

The data used in this study was gathered in two independent sessions. In the first, students in a third-year undergraduate mathematics course intended for prospective teachers were presented with tasks one to three in Fig. 2.2 and asked to write their approaches to solving each. The second round consisted of a set of interviews with 11 student volunteers, each enrolled in a first-year mathematics course covering calculus and linear algebra and who did not participate in the first round. These students were given two of Task 1, 2, 4, or 5—task three was excluded from this round of data gathering, based on the poor responses to the task given by the students in round one—asked to think about how they would solve the task, and interviewed about their imagined approaches to a solution. Tasks four and five were created for use in the second round based on the researchers' perceived need for tasks that appeared familiar to the students. The interviews were recorded and transcribed.

The tasks in Fig. 2.2 were created with the intention of encompassing a variety of mathematical situations, from (1) an applied modelling task where the student is asked to describe the growth of algae with mathematics, (2) one in which the student is asked to generate a graph of a function given only properties of the function, (3) a task from game theory where the student must choose and justify a strategy, (4) one on the eigenvalues of an inverse linear transformations, and (5) another asking for the volume of an uncommon shape. Initial student participants were given a few minutes to describe what they would do with each problem, or how they saw it, without actually attempting to solve the problem. From this first round of participant responses we generated a draft of the framework below. This was achieved by each author individually identifying features of each solution that were indicative of foresighting behaviour, and generating a classification for these features. We then met and created the framework from our separate observations.

Next, we returned to the data and separately re-interpreted it in terms of categories of the framework. We then met to compare the framework categories we individually assigned to each student response to check for agreement and to make any necessary

Task 1: A Modelling Task

We are becoming familiar with the sight of algal blooms on our beaches in the Summer. We are told that these are more frequent as a result of warmer sea temperatures and nutrient concentrations. We observe that the bloom congregates in certain places on our beaches, and comes and goes over a period of days and weeks.

How might we construct a mathematical model of the algal bloom? What possible forms might the model take? Given a certain type of model, what might the variables be and how would they be represented? What mathematical characteristics is the model likely to have?

In what ways might the model be a useful tool for beach-goers or environmentalists?

Task 2: A Graphing Task

Graph a function $h(x)$ that satisfies the following conditions.

- $h(0) = 2$
- $h'(x) > 0$ when $x < -1$, $h'(x) < 0$ when $x > -1$
- $h''(x) > 0$ when $x < -2$ and when $x > 0$, $h''(x) < 0$ when $-2 < x < 0$
- $\lim_{x \to 0} h'(x) = \infty$

Draw a graph without performing too much written work. Justify your choice of this graph by writing your thoughts in as much detail as possible.

Task 3: A Mathematical Game

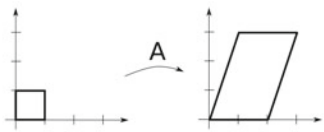
Suppose you are playing a mathematical game with a group of people. The game is as follows. Everyone writes down a number between 0 and 100, inclusive. The average is calculated. Each person's score is calculated to be the distance between their score and two-thirds the average. Mathematically,

$$Score = \left| Number - \frac{2}{3}(Average) \right|$$

Your goal is to get as low a score as possible. What number do you write down? Justify your choice of this number by writing your thoughts in as much detail as possible.

Task 4: A Linear Algebra Task

Suppose the matrix A maps the unit square in $\mathbb{R}^2$ to the parallelogram in the right half of the figure below.



What are the eigenvalues of A^{-1}?

Task 5: A Geometry Task

Suppose you work on a road construction site where fuel for all the machines is stored in a large cylindrical container. This cylinder is lying on its side. There is a small opening halfway along the tank facing upwards in which a measuring stick can be inserted to find the distance between the surface of fuel and the opening. What is an expression for the amount of fuel in the cylinder?

Fig. 2.2 The tasks used in this study revisions. The focus of our attention in the analysis of the first round of data was the anticipated solution trajectory rather than the shape of the final solution, although some students did comment on the solution space. The second round of data aided in supporting our initial framework and generated a more detailed classification of students' imagining of the sphere of resolution.

Below is the framework describing students' initial problem-solving thoughts that emerged from an analysis of the students' utterances and inscriptions through the lens of mathematical foresight. Recall that there are two components of mathematical foresight: forming an image of (i) the resolution to the mathematical situation (the sphere of resolution), and (ii) a likely path to that resolution (the resolution trajectory). Both of these can be imagined to varying degrees of clarity. In addition, clarity with one of these does not necessitate clarity in the other: a student may see the form of the resolution but be no closer to reaching the resolution. Both of these components were considered during the analysis of the participant inscriptions and utterances. Utterances are followed by participant identification marks of the form (Pxxx).

2.4 Results

I consider the sphere of resolution and resolution trajectory separately, in turn. Though I recognize that these two may not exist as separate entities in the students' minds, I find I am in need of a way of analyzing these two simultaneously. Until the methodology has been developed to allow me to do so, I keep the analyses separate. As discussed subsequently, there is evidence that analyzing these two components is a valid approach—one can exist less or more developed than the other. After this initial analysis, possible interactions between students' images of the sphere of resolution and resolution trajectory are considered.

I note that the participants' utterances often contained episodic components, lending credence to the inclusion of an episodic future thinking perspective. However, reporting explicitly on those utterances is not the focus of the current work. Rather, I focus on a refinement of the mathematical foresight model, introducing hierarchical categories for both the sphere of resolution and resolution trajectory. A focus on the episodic nature of the participants' utterances are the focus of Maciejewski et al. (2016).

2.4.1 Sphere of Resolution

Responses, both written and spoken, indicated that students were often able to imagine a resolution to the mathematical task. We characterize these responses as four qualitatively distinct categories, arranged according to increasing clarity of the imagined resolution.

1. *No image of the resolution*. Student responses in this category had no indication of a resolution to the mathematical task.
2. *A generic image*. Responses of this type were general statements concerning the nature of the resolution to the given task.

For example, some students indicated the resolution to Task 1 is a system of differential equations but were unable to elaborate. Another participant interpreted Task 1 as a statistical problem: "Like in Stats or something, we did…I forget what it's called but there was like a graph of a variable here, variable there, and you see the connection between these two factors" (P104). The participant discusses which factors would be included in the resolution, but does so generically—they say only that the resolution is a graph with axes labelled, but do not propose a particular shape of the graph.

Another speculated the solution to Task 1 to be an expression:

> So, if I was to like, flesh this out I would write it in terms of 'n'…it would be an expression and it would be in terms of initially how many there are. And it would be some kind of multiple or power maybe. (P101)

Interestingly, one participant determined the solution to Task 5 is a differential equation: "It's the one thing I can't do, writing differential equations. I know what I have to do, but I don't know how to do it" (P102). They verbalized that they picked this form based on how the top of the fluid changes as the fluid drains from the tank, revealing a dynamic understanding of the task.

3. *An incomplete image*. Some students indicated particular features of their imagined resolution, which was not completely well-formed.

For example, in addition to imagining a resolution to Task 1 as being a system of differential equations, some students indicated the system must be periodic.

4. *A particular image*. Responses in this category are explicit forms for a resolution.

For example, one participant wrote "b(x) = x^(1 + (t x c x n))" as a particular resolution to Task 1. They verbalize their reasoning: "So from my understanding of the question, what comes to mind is a kind of rate of change over time. So this rate of change over time is given by something called differentiation. This is also the rate of change. So, all you do is…I form this equation…then to make a solution, I will differentiate it" (P108). They then go on to describe the variables present in their equation and what contribution each makes. This response is worth noting because the participant appeared to have a vivid image of the resolution to this task, but did not have a clear idea for the resolution trajectory; they saw where they would end in the task, but not a clear way to get there. Many of their statements regarding the inclusion of the variables and constants in their equation were vague and not necessarily grounded in the task statement.

2.4.2 Resolution Trajectory

Students' imagined resolution trajectories were more nuanced than their images of the spheres of resolution. This is due to resolution trajectories having more degrees of freedom; trajectories can be imagined with varying degrees of clarity, as was the case above, but there is more freedom in how the trajectories are imagined. We were nevertheless able to construct a descriptive framework of the students' images of the resolution trajectories, presented below. The levels of the framework each have varying degrees of clarity and we take higher levels as subsuming lower levels.

1. *No indication of a resolution trajectory*. Responses of this type had little relevance to the problem with no indication of anticipated progress.
2. *Identifying factors relevant to the resolution*. Students at this level are able to identify information that is given either explicitly or implicitly in the problem statement that will aid in its solution.

In the Graphing task, many students identified what properties of the function affect the shape of the graph; e.g. the first derivative conditions lead to where the graph is increasing/decreasing. Another example comes from a participant engaging with Task 5. The participant recognizes relevant variables and formulae, but does not anticipate anything other than a straight-forward calculation: "I'd chuck in the equation … the volume of the cylinder equation, which is related to the circumference and length … I'd see how you get the volume as a function of that distance."

This level is distinguished from level 1 in at least one important regard: the control exhibited by the student, as in the sense of Schoenfeld (1985). Students identify relevant factors but also identify factors not relevant either explicitly or by refraining from writing them.

3. *Creating/identifying (mathematical) relationships, between the relevant factors*. At this level, students are able to recognize how relevant factors (ought to) interact to contribute to a resolution. These interactions may or may not be explicitly mathematical.

In the Mathematical Game task, many students identified that a strategy must consider possible actions of the other players.

Another participant—the one mentioned previously—verbalized how they saw the resolution to Task 5 as a differential equation:

> OK. The fuel is the total cylinder volume minus the rate of the volume change with the given unit of measurement, with said measurement substituted in there. I think. Which I'm pretty sure that's what it is, because you have to take into account the total capacity of the cylinder…And then, because it is a cylinder, it's not going to be one on one, or a simple ratio, but it's changing, with every measurement you go down. So say you're measuring in centimetres…one centimetre doesn't mean that you're gonna have one litre less fuel unless the tank has been specifically designed for…like that…but you can't assume that. So you actually need to find the rate of the volume change with respect to the unit of measurement in order to find the total fuel in there. I think. (P102)

4. *Recognizing consequences of the relationships*. Having established how the factors relate, a student at this level identifies the mathematical consequences of these relationships.

For example, one student identified algal growth as a relevant factor in the modelling task and chose to represent the relationship between algal concentration and time as exponential. They then write, “would likely see a curve as conditions approach the ideal.” The connection between this statement and the exponential relationship is not entirely clear, but we suspect the student is anticipating a sigmoidal, logistic relationship, which can involve an exponential function, between algal concentration and time, with concentration levelling out as saturation is approached.

5. *Identifying limitations/strengths/generalizations of the chosen approach*. Responses at this level were exhibited by only one of the participants.

The participant wrote a complete expression for the volume requested in Task 5, which involved the radius r of the cylinder as a parameter and the length l of the measuring stick as a variable. He verbalized his process of generating the formula:

> The first thing I did was visualize it…because it’s lying on it’s side, it’ll be a uniform height all the way through so I had to just think of a circle and then multiply it by the length oat the end. And then I though, how would I work out anything like this? Because, it’s like kinda annoying. So then I thought, okay, I can think of it as a sector minus a triangle making the segment. So, it’s in two cases… (P111)

The student then goes into detail on these two cases and how there ought to be two formulas, one for $l < r$ and the other for $l > r$. He then realized that this was irrelevant if the equation was set up in a certain way: “And here it doesn’t matter, ‘cause it’s all squared…so it doesn’t make a difference.” This data sample reveals the participant had recognized relevant factors for a solution, created mathematical relationships between them, recognized how these relationships might interact to generate a formula, and then acknowledged how the formula might be refined. In this way, this sample demonstrates the hierarchical nature of this categorization.

This participant also responded to Task 4 at the same level of the hierarchy. They verbalized their entire thinking process:

> First, again, I thought about the process that was needed to solve the problem. So…first of all I had to find the matrix A … then invert it, and then find the eigenvalues of that inverted matrix…so first I thought, OK, I haven’t done transformations ever, so this is going to be a bit weird… But then I realized that if I think of it as just a function…[a lengthy and detailed recollection of their thinking]…I thought, OK, if one goes to two, like this, then a’s going to be two, so I’ll just call that like 2, b, c, d. And for some reason I assumed that b was zero at this point, And then I thought about another point, this one here … and then I noticed it was a triangular matrix so the eigenvalues of this matrix were easy to find. (P111)

This is just excerpt from a longer monologue of the student recalling what they had thought while considering the task. This participant is a bit of an outlier in that they were able to construct complete solutions in their minds, replete with details, upon initial contact with the tasks.

2.4.3 Discussion of Results from Student Interviews

The results presented here from the two data sources support mathematical foresight as a viable lens for analyzing students' future-oriented thinking and actions in mathematical situations. Further, the student data shifted my perspective on how the mathematical foresight model might be used. The original model (Maciejewski and Barton, 2016) was informed by reflections of professional mathematicians on their own practice and considered the sphere of resolution and resolution trajectory as coexisting, each in a fairly refined state. The student data presented here suggests three revisions to the model: the sphere of resolution and resolution trajectory (i) may exist to varying degrees of lucidity, including not existing at all; (ii) may be decoupled in their development; (iii) are likely dynamic, varying in development as the relationship between mathematician and mathematical situation evolves.

Point (i) above was the impetus for the creation of the hierarchies presented in the results section; further studies may help to refine these categories. Concerning point (ii), there is some indication from the data that a student's ability to imagine the resolution or trajectory is related to their ability to imagine the other. So, the more vivid a participant's sphere of resolution, the more vivid the resolution trajectory, and vice versa. Such a relationship is expected, given what is known of the mathematical foresight of working mathematicians (Maciejewski & Barton, 2016). It should be noted that even vague images of resolutions to the situation often suggested a way forward for the participants. The detail may have been lacking, but such images do seem beneficial. In addition, incomplete images of a sphere of resolution were the most common among the participants; only one participant demonstrated strength in imagining both the sphere of resolution and trajectory. The bi-directional relationship mentioned above did not consistently exist in the student responses. Some could imagine a particular form of a resolution but had no clear indication of a trajectory. Others could imagine a trajectory without a clear sphere of resolution. For example, one student solving Task 5 could not see a possible form for the volume expression, but suspected it could be arrived at by using the formula for the volume of a cylinder: "The general form … is an equation … Yep, it's blank. How I would go about finding the solution, I'd chuck in the cylinder equation." They continue by identifying the length and radius of the tank as being important but are unable to incorporate the height from the top of the fuel to the top of the cylinder. This is an important point to make—though a participant's image of a sphere of resolution often co-emerges with their image of a resolution trajectory, they may not exist or develop simultaneously. This lends credence to the characterization of sphere of resolution and resolution trajectory as separate, yet linked constructs in the mathematical foresight model. The point here is that the possible vividness of the two components of mathematical foresight is far more nuanced then originally cast from interviews with practising mathematicians (Maciejewski & Barton, 2016).

Point (iii) above emerged as a conjecture from the current study. The results presented here are from snapshots of students' problem solving activity. What needs to be explored further is a mathematicians' evolving relationship with a given mathematical situation. With each thought and action, the situation changes for the mathematician—they uncover new information and develop insights, culminating in a resolution. In problem solving terms, there comes a point in developing the solution that the problem ceases to be a *problem* in the sense of Schoenfeld (1985). This dynamic process remains largely undocumented; an exploration of it seems a necessary next step in deepening the mathematics education community's understanding of authentic problem solving practices.

2.5 Overall Discussion

Data from participants in the two-round study reported here supported the conjecture that students engage in mathematical foresight when encountering a novel mathematical situation. The data gathered has informed the creation of a framework that describes students' initial thinking about a mathematical situation through the lens of mathematical foresight. This framework has elaborated the initial model for mathematical foresight as presented in Maciejewski and Barton (2016). The original mathematical foresight model, as exhibited by mathematicians, sees the two components—the sphere of resolution and the resolution trajectory—as coupled: one does not exist without the other. This was not true for the students who participated in this present study. Some students were able to see a likely form for a resolution to a mathematical task but were unable to see a trajectory leading to that resolution. Others could see how to "set out" but did not necessarily see where they were headed.

The purpose of this work is not to highlight yet another novelty in the complex enterprise of problem solving. Indeed, other authors have recently introduced constructs similar to aspects of mathematical foresight, such as *implemented anticipation* (Niss, 2010) and *anticipatory metacognition* (Galbraith, Stillman, & Brown, 2015). Rather, I wish to argue that the mathematics education community can enrich their understanding of mathematical problem solving by suspending their reliance on the de facto foundational texts, returning to the field to further an empirical program, and to turn to the broader literature on, non-mathematical, problem solving. Such a shift in perspective might further insights into why previous attempts to teaching problem solving have gone largely unsuccessful.

The branches of *mathematical modelling* and *problem posing* that have stemmed from mathematical problem solving are very promising for an advancement of the field. I argue that they should not be viewed as somehow distinct from their roots. All have common features that ought to be articulated and brought to light so as to be studied further. Mathematical foresight is one such commonality. In choosing a model, the mathematician *sees* the salient features of the situation and chooses what aspects to include in the model to further insights. When posing a problem, a mathematician *sees* which might be interesting for others to consider. And in solving a problem, the mathematician *sees* a possible resolution and a way to that resolution. Having identified this phenomenon of future thinking, we researchers are tasked to further articulate and understand it—might we *see* how it pulls forward our understanding of mathematical problem solving?

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Part II
Problem Solving and Technology

Chapter 3
A Model of Mathematical Problem Solving with Technology: The Case of Marco Solving-and-Expressing Two Geometry Problems

Susana Carreira and Hélia Jacinto

3.1 Introduction

Innovative and increasingly powerful technological tools are introducing new kinds of problem-solving situations where mathematics is useful, thus changing the mathematical abilities needed outside school. So 21st century youths need to have access to and develop the skills to use these tools for mathematical learning and, particularly, in problem solving activities (Forgasz, Vale, & Ursini, 2010). While little is still known about the problem solving that occurs beyond school (English & Sriraman, 2010), further research is recommended to understand the role of digital tools in such activity (Santos-Trigo & Barrera-Mora, 2007).

This study brings new knowledge about the spontaneous use of digital tools in solving non-routine mathematical problems by youngsters engaged in an online problem solving competition. The context in which the research was conducted is that of the mathematical problem solving competition SUB14®, which is aimed at middle graders (12–14 years-old) of the southern regions of Portugal. The Qualifying stage of the competition consists of answering a problem every two weeks, either through e-mail or an online text editor available on the competition website. Participants may solve the problems using their preferred methods and tools but are explicitly required to report on their solving process and must offer a complete explanation of their reasoning. The inclusive character of this competition makes it accessible to average-ability students and its rules permit and encourage help seeking from relevant others at this stage. This context offers the opportunity to study how youngsters are

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using their mathematical knowledge in problem solving beyond the classroom, where they are also allowed to choose any technological tool at their disposal to solve the problems and express their solutions.

Our research aim is to understand the ways in which the processes of mathematical problem solving are reshaped when these youngsters spontaneously resort to digital technologies. In addressing this purpose, we intend to develop our understanding of how the use of digital technologies, including everyday and general purpose tools, is embedded in the process of solving and expressing a solution to a non-routine mathematical problem. Here, we will limit ourselves to one of the cases selected in the course of the beyond-school competition SUB14 that served as the basis for the construction and application of an analytical model of mathematical problem solving with technology. We assume that the case of Marco, when solving a geometry problem, offers a valuable report on this model and on its strength in providing new insights into young students' use of digital tools in mathematical problem solving.

3.2 Theoretical Background

The prevailing theoretical models on solving mathematical problems, which conceive paper and pencil as the predominant tools, do not account for the role of digital technology. Thus, they do not provide the tools to explain the interaction between individuals' technological and mathematical knowledge in their problem solving activity (Santos-Trigo & Camacho-Machín, 2013).

Our theoretical framework, aiming to address mathematical problem solving with technology, is built upon the notion of *humans-with-media*, acknowledging the inseparability between the solver and the technological tool whilst solving the problems and expressing their solutions. The youngsters' interaction with digital media is seen from the point of view of placing *affordances* in the tools. Furthermore, we address *mathematical problem solving with technology* by combining two analytical tools: one accounting for the processes involved in mathematical problem solving, and the other for the processes taking place with the use of digital tools in digitally-framed tasks (Jacinto & Carreira, 2017; Jacinto, Carreira, & Mariotti, 2016).

3.2.1 Solving-and-Expressing: An Overall Concept

In addressing the students' ways of tackling mathematical problems with digital tools, we consider several theoretical notions and perspectives that offer a theoretical frame for such activity. Problem solving is here understood as the development of a productive way of thinking about a challenging situation (Lesh & Zawojewski, 2007) where the solver must adopt a mathematical point of view in order to carry out mathematization processes. Moreover, it is regarded as a synchronous process of mathematization and expression of mathematical thinking (Carreira, Jones, Amado, Jacinto, & Nobre, 2016), which means that obtaining a solution to a problem is to find the required answer *and* to create an explanation for it. Hence the solution phase and the reporting stage are closely linked aspects of this problem solving-and-expressing activity. This means that all the material incorporated in the final product, and not merely the result, is actually part of the solution process (Lesh & Doerr, 2003), such as the use of color, diagrams, tables, images, along with textual explanations or descriptions. These descriptive elements carry new knowledge about the situation, which is fundamental in solving-and-expressing each problem.

In this study, we also adopt the notion of humans-with-media (Borba & Villarreal, 2005) as a core conceptual unit that postulates the inseparability between the subject and the tool, thus leading to interlace mathematical thinking and expressing with the representational power of digital technologies. In fact, the introduction of a specific tool in the system of humans-with-media impels relevant changes in the activity, according to the type of media that it encloses, thereby resulting that different collectives originate different ways of thinking and knowing (Jacinto & Carreira, 2013, 2017; Villarreal & Borba, 2010).

The interactions taking place within this conceptual unit, i.e., between the individual and the digital media whilst solving-and-expressing mathematical problems, is seen from the point of view of placing affordances in the tools (Chemero, 2003) in the sense that affordances are both relative to the object and to the subject who realizes its advantages. The affordances emerge from the interaction between the agent and the object, insofar as the perception of the possibility for action and the ability of the agent are not "specifiable in the absence of specifying the other" (Greeno, 1994, p. 338). Hence, the recognition of particular features in the tools that are potentially useful support the individual in solving-and-expressing the problem, thus leading us to consider the impossibility of separating the solver's mathematical and technological skills (Jacinto et al., 2016).

3.2.2 Developing a Model of Mathematical Problem Solving with Technology

The development of a new conceptual framework that aims to account for both components of the problem solving process encompasses the redesign and expansion of well-known theoretical models in order to suggest more efficient ways to describe the connection between mathematical knowledge and the affordances of digital tools that solvers bring to their problem solving-and-expressing activity. This lead us in bringing together two frameworks: one addressing the activity of an individual while dealing with a technological task or problem (Martin & Grudziecki, 2006), and another one particularly focused on describing the processes involved in mathematical problem solving (Schoenfeld, 1985).

The DigEuLit Project (Martin, 2006) proposed a framework on Digital Literacy in which a set of processes performed in the context of solving a task or problem that requires the use of a digital resource were stated. These processes comprise: *statement*—clearly state the problem and the actions likely to be required; *identification*—identify the digital resources required to achieve the solution; *accession*—locate and obtain those digital resources; *evaluation*—assess the accuracy and reliability, and relevance of the digital resources; *interpretation—understand* the meaning they convey; *organization*—organize them in ways that may enable the solution; *integration*—bring these resources together in relevant combinations; *analysis*—examine them using concepts and models that will enable the solution; *synthesis*—recombine them in new ways to achieve the solution; *creation*—create new knowledge objects, units of information or digital outputs that contribute to achieve the solution; *communication*—interact with others while solving the problem; *dissemination*—present the solution to others; *reflection*—consider the success of the task performed (Martin & Grudziecki, 2006, p. 257).

Although this set of processes resembles well-known models in mathematics education, it is necessary to account for the mathematical thinking developed in this activity. Being successful in mathematical problem solving, as Schoenfeld (1985) suggests, requires considering essential facts and procedures, effective use of resources, strategies, and actively engaging in mathematical thinking.

Aiming to describe students' mathematical problem solving performance, Schoenfeld (1985, pp. 297–298) proposed a model that comprises five stages: *read*—time spent "ingesting the problems conditions"; *analysis*—attempt to fully understand the problem "sticking rather closely to the conditions or goals" that may include a selection of ways of approaching the solution; *exploration*—a "search for relevant information" that moves away from the context of the problem; *planning and implementation*—defining a sequence of actions and carrying them out orderly; *verification*—the solver reviews and assesses the solution.

By comparing and relating the processes proposed by Martin and Grudziecki and the stages identified by Schoenfeld, and by selecting the most prominent actions in the two frameworks, we reached the following model by merging some of the processes of digital problem solving and also decomposing some of the stages of mathematical problem solving (see Table 3.1). Even though these processes are clearly defined and have distinct boundaries, as acknowledged by the original models, in this combined model they are flexible enough to be considered in different phases.

3.3 Research Method

As stated above, the overall goal of our research is to understand the processes of mathematical problem solving by acknowledging the role of digital tools, considering the problem solving activity carried out by young students within the context of the competition SUB14.

Following an interpretative stance that involved qualitative techniques for data collection and analysis (Merriam, 2009), we developed several cases of participants who usually resort to a variety of technological tools to solve the problems of the com-

Table 3.1 Processes underlying mathematical problem solving with technology

	Mathematical problem solving with technology (MPST)	
Grasp	Appropriation of the situation and the conditions in the problem, and early ideas on what it involves. (Read[a]; Statement[b]).	**Communicate**—Interact with relevant others whilst dealing with the problem or task. (Communicaton[b]).
Notice	Initial attempt to comprehend what is at stake, namely the mathematics that may be relevant and the digital tools that may be necessary. (Analysis[a]; Identification[b], Accession[b]).	
Interpret	Placing affordances in the technological resources in pondering mathematical ways of approaching the solution. (Analysis[a]; Evaluation[b], Interpretation[b]).	
Integrate	Combining technological and mathematical resources within an exploratory approach. (Exploration[a]; Organisation[b], Integration[b]).	
Explore	Using technological and mathematical resources to explore conceptual models that may enable the solution. (Exploration[a]; Analysis[b]).	
Plan	Outlining an approach to achieve the solution based on the analysis of the conjectures explored. (Planning and Implementation[a]; Synthesis[b]).	
Create	Carrying out the outlined approach, recombining resources in new ways which will enable the solution and create new knowledge objects, units of information or other outputs which will contribute to solve-and-express the problem. (Planning and Implementation[a]; Creation[b]).	
Verify	Engaging in activities to explain or justify the solution achieved based on the mathematical and technological resources. (Verification[a]).	
Disseminate	Present the solutions or outputs to relevant others and consider the success of the problem-solving process. (Verification[a]; Reflection[b], Dissemination[b]).	

[a]Stage of mathematical problem solving as proposed by Schoenfeld (1985)
[b]Process of digital technology problem solving as proposed by Martin & Grudziecki (2006)

petition and who present detailed explanations of their solutions (Jacinto, 2017). In this chapter, we confine ourselves to the case of one participant, under the pseudonym of Marco, who has a preference for geometrical problems in which he is able to use his digital skills in implementing visual methods (Jacinto & Carreira, 2015) and resorts to conventional and unconventional tools in developing his approaches to those geometry problems posed by SUB14 (Jacinto et al., 2016). The case serves the purpose of illustrating and substantiating some main results from the broader work that has spanned over several years of data analysis.

The collection of data initially consisted of gathering all the digital solutions produced by Marco along two yearly editions of the competition. This chapter deals initially with the analysis of Marco's solution to the problem "United and Cropped" (see Sect. 3.4.1), which he developed using GeoGebra. The GeoGebra file allows disclosing the sequence in which the constructions were performed by means of its Construction Protocol.

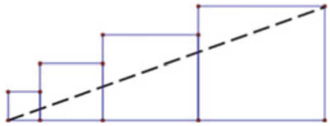
We proceeded to a second stage of our research by observing and video recording Marco's work while solving a problem in his home environment, with the consent of his parents. He was asked to choose one out of three problems posted for this purpose on the SUB14 website, then solve it by performing as closely as possible to his usual problem solving activity in the competition, and to explain out loud his actions and thinking. Marco chose to solve the problem "Decorative Drawing" (see Sect. 3.4.2) and resorted to several technological tools during the process.

The NVivo software was used in the organization process, for transcribing the interviews, segmenting and coding data. As for the data analysis we followed an interpretative perspective considering that providing a holistic description of the case would encompass the results in light of the proposed MPST model and the theoretical notions discussed. The following section illustrates the case of Marco-with-media solving-and-expressing problems within the competition SUB14.

3.4 Data Analysis and Results

Marco is a 13 year-old student enrolled in SUB14 for the second year, who is quite familiar with a diversity of digital tools. While studying geometric transformations at school, he learned to use GeoGebra. Marco enjoyed these lessons so much that, at home, he continued to explore GeoGebra on his own. However, he often uses a spreadsheet or editing tools in solving-and-expressing the problems of the competition. Below, we firstly analyze Marco's processes while solving a mathematical problem with GeoGebra based on a solution submitted during the qualifying stage of SUB14. We then report on the processes he engages in while solving another problem, based on the in-depth interview and observation.

Consider a sequence of squares of sides 1, 2, 3 ... 4 cm, arranged so as to be joined to each other, as shown in the figure. Once together, we cut up all the squares along a line from the lower left corner of the smaller square to the upper right corner of the larger square. What is the area above the cut-off line if the sequence has 8 squares?



Don't forget to explain your problem solving process!

Fig. 3.1 Statement of the problem 'United and Cropped' from the SUB14 competition

3.4.1 Marco's Processes in Solving a Mathematical Problem with Technology Based on the Digital Solution

Replicating the Complete Sequence of Squares

The problem 'United and Cropped' is one of the problems that Marco solved when participating in SUB14 and in which he resorted to GeoGebra (Fig. 3.1). The problem refers to a sequence of squares and presents a figure where only a few elements of the sequence are shown. It has to do with finding a way of extending the sequence and find a specific requested area.

Marco submitted a file containing his solution to the problem. He decided to use GeoGebra to obtain a figure like to the one presented in the problem (*grasp*) possibly realizing that he could obtain the sequence of 8 squares by marking their vertices, and later constructing their sides and, from there, find a way to obtain the requested area (*notice*). He seems to have recognized the advantages of combining two affordances of the GeoGebra graphical view—the axes and the grid. Those provided and supported a visual and orderly way for the construction of the sequence of squares, based on the pattern of increment of the sides (*interpret*).

Marco then plotted each vertex on the rectangular grid, considering its coordinates according to the dimensions of the sides of each square (Fig. 3.2). Some of the coordinates that are visible in the Construction Protocol (for example, E and F) (Fig. 3.3) suggest that Marco was just using the visual location of the point, based on the grid, to insert each point in an approximate position. Apparently he was convinced that he just needed a sketch of the figure rather than its exact geometrical construction in leading him to a path for the solution.

His next step is the construction of the sides of the squares, where he uses segments drawing. Next, he constructs a ray from the origin of the axis to the upper vertex of the sequence and, using the 'properties of objects', he changes the color of that ray to orange. While developing the construction of this element Marco is already combining technological and mathematical resources, which sets the beginning of an exploratory approach to the problem (*integrate*).

The Mathematization: Solving-and-Expressing the Solution

The conceptual model that is apparently starting to be developed will guide Marco to the solution. He realizes the relevance of the GeoGebra spreadsheet (Fig. 3.3) as

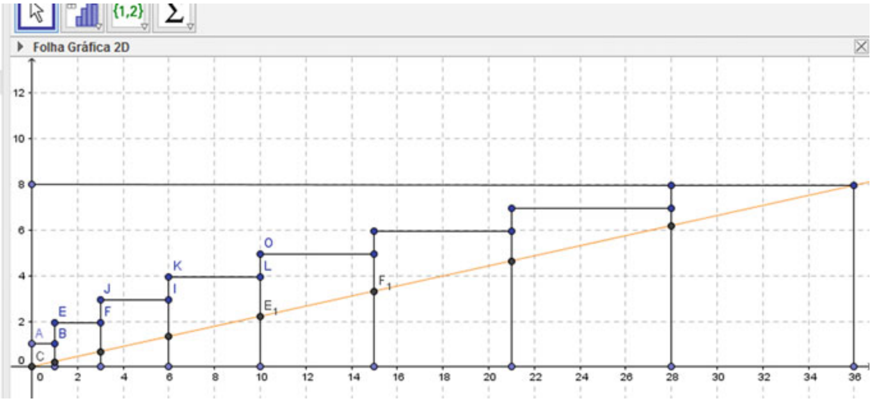


Fig. 3.2 Construction of the sequence of eight squares with GeoGebra

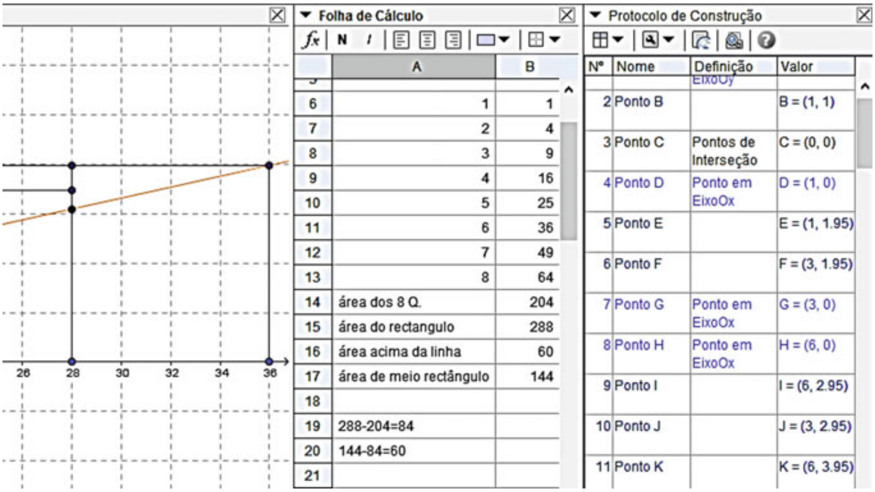


Fig. 3.3 GeoGebra's spreadsheet view and excerpt of the Construction Protocol

he chooses to use this tool to deal with the measurements involved in the figure. He creates a sequential list of the lengths of the sides and inserts them in column A, and another list containing the area of each corresponding square, which he organizes in column B (*explore*). Then he inserts the label "area of 8 Q" (abbreviation of 8 Squares) in cell A14, and turns to the figure to construct the upper side of the surrounding rectangle that contains the sequence of squares. He directly enters the total area of the 8 squares in the table and also the area of the surrounding rectangle. Although such rectangle is not mentioned in the problem, its construction reveals how Marco is developing his approach to the solution (*plan*) which is based on the realization that he can get the requested area by means of the difference between the area of the 8 squares and the area of a triangle (shown below the cut line), as the cut line is a diagonal of the rectangle. Thus the rectangle is a new object of knowledge and a key element in the mathematization of the situation that Marco uses to solve and express his solution, both drawing on his knowledge about GeoGebra and his knowledge about areas of polygons (*create*).

He proceeds by inserting the label "area above the line" in cell A16 and "area of half-rectangle" in cell A17. He then calculates the area of the half-rectangle and inserts it directly in cell B17. Below, he uses other cells to compute the difference between the area of the rectangle and the area of the 8 squares; he then subtracts this result to the area of the half-rectangle (*verify*). Only then he enters in cell B16 the value 60, which was the answer to the problem.

The file he submitted with the solution to the problem contains the construction of the sequence of squares and presents several calculations that are intended to explain and justify his answer, using the GeoGebra spreadsheet view (*disseminate*).

Although in the digital solution there is no evidence that Marco has sought other sources of information or help during the solving-and-expressing process, he mentioned in his electronic message that he counted on the help of family members during his activity (*communicate*). However, with the data available it is not possible to specify either the type of help that was provided or the stage at which such aid was relevant to the problem solving-and-expressing process.

The analysis of the Construction Protocol that supports this resolution shows that despite not having made a geometrically rigorous construction, Marco found the solution to the problem and presented it clearly. In addition, he identified a diversity of possibilities of action with GeoGebra although he has freely chosen to just make use of the indispensable tools to develop a feasible approach to the problem. This intentional choice of GeoGebra is based on an explicit knowledge of its affordances, its characteristic mode of multiple views, and embedded tools, but also on the students' own aptitude, i.e. on the things he knows, and can actually do with GeoGebra to solve the problem and express the solution.

The effective use of the tool appears to be related to the fact that the construction of the sequence of squares infused a visual approach that enabled to bring out an underlying conceptual model of the problem, which sustained the process of obtaining and presenting the solution. We could also say that the student mainly drew on the GeoGebra's affordances to create an acceptable sketch of the figure needed to represent the givens and goals of the problem. That sketch was then combined with new elements he inserted in the figure and with the visualization of the required area as a difference between specific areas that could be computed by means of the knowledge on polygons. In fact, there were other options that Marco's construction would have allowed to follow and explore, namely the GeoGebra capacity of constructing general and particular polygons and measuring their areas. This would enable, for example, making use of the points given by the intersection of the ray with each side of the squares (which he actually created in his construction). They would permit to obtain directly in GeoGebra the areas of the pieces of the squares above the cut line. Therefore, what seems to be more significant is that Marco develops his visual thinking through the use of the technology and combines it effectively with his knowledge related to finding areas.

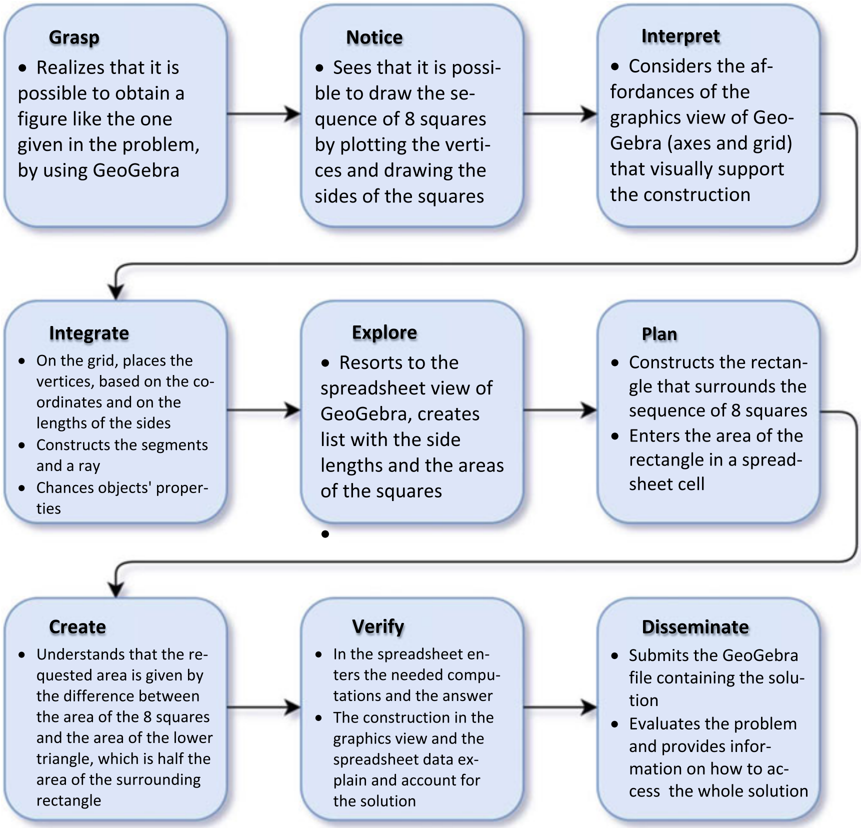


Fig. 3.4 Processes of solving-and-expressing the problem 'United and Cropped'

Summary of the Processes of Solving-and-Expressing with GeoGebra

The processes developed by Marco in solving-and-expressing this problem are summarized in the diagram presented in Fig. 3.4. For each of the processes considered in the MPST model, the key aspects that characterize them are identified. Those are then recorded in the diagram, although very succinctly. Since this solution was not subject to observation, the synthesis concerns the analysis of the file submitted by Marco, complemented by the analysis of the various stages of his work recorded in the construction protocol. Although Marco mentioned that he had the support of family members, it is not possible to specify when this exchange took place, so the communication process was not included in the diagram.

Another aspect depicted in the diagram above has to do with the flow along the various processes that took place. As it is apparent in the scheme, this flow is relatively straightforward and shows a linear progression from the initial appropriation of the conditions of the problem to the dissemination of its entire solution and attached products. In the following, we want to reconsider this apparent linearity as we will

The picture shows a decorative drawing that will be used in the construction of a stained glass window. The equilateral triangle has a height of 12 cm. The circles are all tangent to the triangle and also each small circle is tangent to the large circle.
Which is the radius of the smaller circle?

Don't forget to explain your problem solving process!

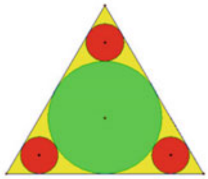


Fig. 3.5 Statement of the problem 'Decorative Drawing' chosen by Marco

be addressing data obtained from face-to-face activity and observation of the problem solving activity performed by Marco.

3.4.2 Marco's Processes in Solving a Mathematical Problem with Technology Based on the Observed Activity

In a face-to-face interview, Marco solved one of three problems that he was asked to choose from, with the request to recall and reproduce what he usually did during the participation in the competition, Marco began by carefully analyzing the three 'experimental problems' posted on the webpage of the competition for this purpose. He seemed hesitant but ended up choosing the one that he considered to be his favorite: the problem 'Decorative drawing' (Fig. 3.5). When asked about the reasons for his preference, Marco explained:

Marco: [This one] has more to do with triangles and stuff and it was in the seventh grade that I had 100 [%] in both tests.

Researcher: In geometry?

Marco: Yes, I studied congruence of triangles and such…

His choice is based on an initial identification of the mathematical topic and the notions that are apparently needed to solve the problem (geometry, triangles, congruence of triangles) and, at the same time, reflects his familiarity with those ideas and even a certain self-confidence to deal with those concepts since he had obtained excellent grades in this subject in the previous school year (*grasp*). Although Marco interacted with the researcher throughout his activity, following the request to verbalize his thoughts and procedures, at this initial stage he explicitly requested support for clarifying the meaning of the notion of tangency (*communicate*).

M: There's something that I don't understand. Tangents, the circles are tangent…

R: Tangent. Don't you know what tangent means? [Marco nods affirmatively] It means that they just touch in a single point. In this case, they just touch in this point [pointing to the screen].

Working on Attempts to Develop a Solution

Focusing on the reading of the problem and interacting with the figure presented

Fig. 3.6 Formulates and tests 2nd attempt

on the competition website, Marco begins to develop a series of attempts that lead him to conjecture about the solution. His first ideas were drawn on the fact that the triangle is equilateral (*notice*) and made him believe that he would be able to obtain the desired solution if he would focus on the central circle and from there obtaining the radius of the smaller circles (*interpret*).

M: I'm trying. I'm still trying … to see … how to do this. Hmm … since the triangle is equilateral … if you get to the middle circle maybe you can get to the others … [1st attempt].

Then, he silently stares at the screen for a while. The understanding of the situation begins to develop in close relation with his careful observation of the image. He rapidly sketches various visual decompositions of the equilateral triangle: sliding his finger across the screen, he 'draws' a bisector of the lower right angle of the triangle (Fig. 3.6) but continues to think aloud while 'drawing' also the bisector of the top angle (*notice/interpret*).

M: How shall I say this? It's like they are divided in halves. From each vertex to the midpoint of the opposite side, and then I could try to find out… If I could do it … But I'm still seeing how am I going to do it … [2nd attempt]

His attempts to find a visual method of approaching the problem continue, and after some time he proposes another analysis of the situation:

M: This measures 12 cm. The middle of the triangle is less than 12, for sure. It could be 4. If we divide in these parts … [with the forefinger and thumb sets a distance and slides it 3 times covering the height of the triangle]. Yeah, maybe. Because they are tangent… [Silence]… I can say they have the same length. [3rd attempt]

In spite of some imprecision in the language he uses, the student recognizes that the centroid of the triangle does not coincide with the midpoint of its height. In fact, he conjectures that the radius of the larger circle could be 4 cm, which is obtained from a visual intuition supported by a rudimentary measurement based on a fixed distance that he defines with the fingers (Fig. 3.7). Although he concludes that the radius of the larger circle corresponds to 1/3 of the height of the equilateral triangle, he realizes that this statement lacks clear justification, but he seems to find no information in the problem for that (*notice/ interpret*).

He knows that he has already attempted different approaches, which he feels that might work to solve the problem but is not totally confident with them. The

Fig. 3.7 Formulates and tests 3rd attempt

Fig. 3.8 Formulates and tests 4th attempt

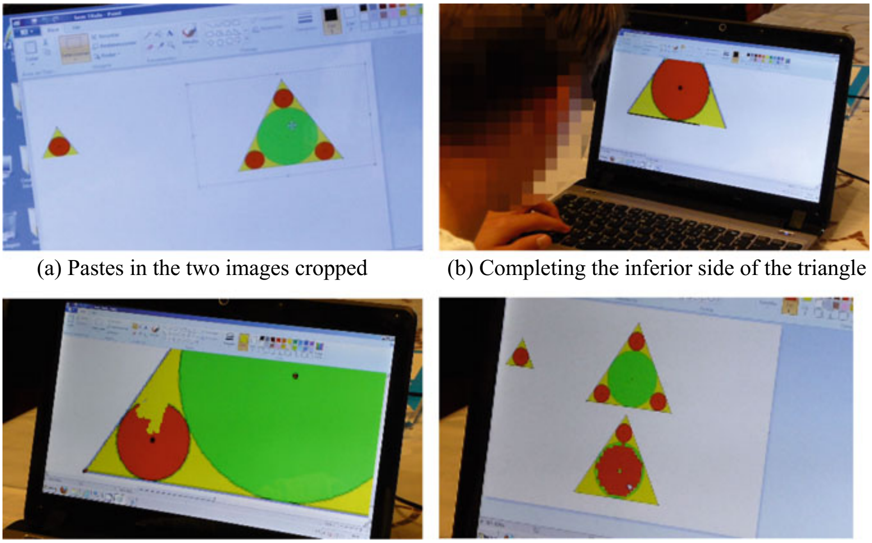
various approaches consisted of manipulations and mental transformations in the sense they have not actually been operationalized by Marco beyond the 'drawing' with the index finger on the screen. He finally decides to follow a strategy involving the decomposition of the equilateral triangle in two figures: a smaller triangle at the top and a trapezoid below (Fig. 3.8). He goes on explaining:

M: If we draw a triangle here… It's as if this one is an enlargement of that one. If this is 12, then 12 divided by 3, [equals] 4… It means that the radius is 2. Maybe the radius of the small circle is 2. [4th attempt]

Up to this point, Marco was trying to understand the main ideas involved in the problem (*notice*) and, in each hypothesis raised, he was considering the plausibility of a mathematical way of approaching the solution (*interpret*). Therefore, in the first minutes of his activity, there are cycles of *notice-interpret*, which are successively refined, and pave the way for the development of a conceptual model that will lead to the solution. While Marco is thinking aloud and developing a sequence of ideas, he 'interacts' with the figure on the screen by pointing, estimating distances, or by hiding areas with his hands. The development of a visual method to approach the solution starts to take shape, in analyzing the possibilities of decomposition of the figure while simulating transformations such as cut, reorganize or change colors. In this way, editing the figure looked as an indispensable action to get the solution.

A Visual Approach to Get the Solution

Marco then decides to pursue with his fourth attempt. Using the software Snipping Tool, he defines a rectangular area on the screen and crops the top of the large triangle given in the statement, thus obtaining a smaller triangle with a single red circle in its center. Using a similar process, he creates another file containing the original triangle, and then inserts the two images on a new window of the MS Paint (*integrate*). Once



(a) Pastes in the two images cropped
(b) Completing the inferior side of the triangle
(c) Covers the red circles with yellow
(d) Paints the central circle in red

Fig. 3.9 Four steps in the editing process with MS Paint

in the same window (Fig. 3.9a), Marco tries to overlap the two images but, as they had a solid white background, it was not possible to visually show that one was an enlargement of the other (*explore*).

This difficulty leads Marco into a slightly different approach: he decides to transform the large triangle so that it looks similar to the smaller triangle. He goes on, expanding the work area so that he can accurately draw a line that would make the bottom side of the smaller figure. In fact, since that figure resulted from a section of the original triangle, one of the sides was not visible, so he needed to complete it by drawing one missing segment. So, rather than just a matter of graphics, the need to draw new elements had a mathematical intentionality (Fig. 3.9b). Then he starts editing the original triangle by using the ‘eyedropper tool’ in MS Paint to identify the exact shade of yellow covering the background of the large triangle; he uses it to change the color of the smaller red circles into the background color so that they vanish from the figure (Fig. 3.9c). Again using the ‘eyedropper tool’ he captures the red shade and then paints the large central circle in that red color (Fig. 3.9d).

The editing of the images described above (*integrate*) is intended to show that the smaller triangle is, clearly, a reduction of the original triangle (same shape but different size). So Marco is developing and exploring a conceptual model to explain the similarity between those two triangles (*explore*), and this will guide him in producing the solution. As in the loop of processes *identify-interpret*, it was observed that *integration* and *exploration* also occurred in an iterative way, albeit in a short period of time. Marco studies the best way to demonstrate the similarity of the two triangles in close relation to the recognition of affordances of the image editing tools

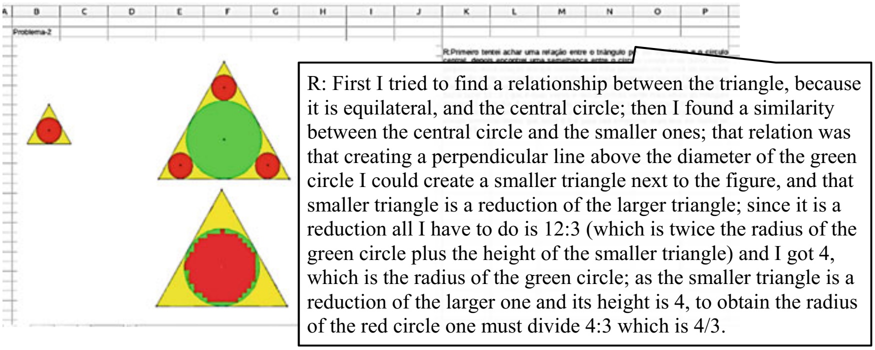


Fig. 3.10 Solution sent by Marco (print screen) with a translation of his written explanation

available and uses them to achieve a transformation that conveys the mathematical relationship in a visually convincing way. When asked about the reason to such a careful work on the graphic elements, he replies: “it’s to better show how you could see that one was an enlargement of the other”. Therefore the graphic treatment is of central importance in his approach to the problem. In addition to illustrating his way of thinking, in the most reliable way he finds, the images also become a visual mathematical argument that must convince those who will evaluate his solution.

Creating and Expressing the Solution

Later, Marco saves the file and opens the OpenOffice spreadsheet. Without resorting to a notebook or pencil, Marco continues to move between the competition website, where he has the problem statement, the image editing tools and the spreadsheet where he starts expressing his solution path (*plan*). He uses the original image and the two figures produced in MS Paint to compose his answer in the spreadsheet window (Fig. 3.10). The figures support his understanding of the problem and show how Marco visualized the similarity between the triangles. By incorporating mathematical ideas, such as similarity and triangle decomposition, Marco achieves a conceptual model of the problem situation (*create*).

As he usually does in the competition, he identifies the number of the problem on the upper left corner of the worksheet, and inserts or pastes the images he has created and explains in detail his resolution process on the right. Although he reports only a few of the attempts he actually made, he explains that he found “a similarity between the central circle and the smaller ones”, hence considering that the small triangle is a reduction of the larger triangle by a ratio of 12:3 although he does not prove that similarity. Thus, assuming that the radius of the larger circle is 1/3 of the height of the original triangle, Marco explains that the smaller circle has a radius that corresponds to 1/3 the height of the smaller triangle, that is, 1/3 of 4. It is, therefore, while producing a written explanation of the resolution process and making an analysis of the images he edited that Marco finds, effectively, the solution to the problem (*verify*). Contrary to his last hypothesis (“maybe the radius of the small circle is 2”), Marco now concludes that the radius of the small circle is actually 4/3.

When he considers his work to be finished, Marco saves the file. Then, he accesses the competition website to submit his answer using the online form available, where he uploads the file as an attachment; he fills in his personal data and adds the following sentence: “Here is the answer to the experimental problem 2” (*disseminate*). Marco also points out that ‘nobody’ helped him with the solution, that he enjoyed ‘very much’ the problem, and that he found it ‘easy’.

Initially, the technological tools assumed a hidden role in the problem solving activity, since Marco only interacted with the screen by visually inspecting the figure given in the statement. However, this visual approach is later developed through processes of transformation of the figure with the technological tools that he chose and with which he shows great familiarity with: he knows how to save the image from the website and knows how to edit it in a way that becomes relevant to find the solution to the problem—a new object of knowledge. His success seems to be anchored in his ability to recognize and make efficient use of various affordances of such tools to broaden his mathematical thinking and to develop a conceptual model for the similarity between the two triangles he seeks to compare.

Moreover, this initial activity appears to have a cyclic nature, in which each argument is formulated as Marco attempts, on the one hand, to assign meaning to the mathematics that may be useful or relevant to him (*notice*) and, on the other hand, to consider mathematical ways to approach the solution (*interpret*) while interacting with the figure on the screen. This cyclic activity leads Marco to a final conjecture—”the radius of the small circle is 2”—which is his first guess for the solution and will trigger subsequent exploration activity. Marco’s ability in finding the solution to the problem seems to be related to his aptitude in recognizing the affordances of the selected tools, which broadened his thinking process and ultimately influenced the expression of that thinking. As he starts to explore his guess, the elaboration of images in the graphic environment leads Marco to discover the correct similarity ratio. The use of the spreadsheet supports the combination of objects because it allows him an easy organization of visual and textual inscriptions, that is, he can move images freely and can easily format as well as merge cells.

Summary of the Processes of Solving-and-Expressing on the Screen

The processes of solving-and-expressing the problem ‘Decorative drawing’ are summarized and schematically presented in Fig. 3.11. Marco’s activity was entirely performed in the digital environment, moving only between the various programs that he used. In this second diagram, the flow has some salient differences from the previous one. Here, several loops or micro-cycles involving some specific processes are observed. Therefore, the apparent linearity that the first diagram seemed to indicate is now challenged by a result that is much more complex and less straightforward. In fact, the MPST model proves able to reveal and capture the processes carried out and also their linked and combined occurrence throughout the resolution, when the data available make known the particulars of the in situ and real-time solving-and-expressing of a mathematical problem with digital technologies.

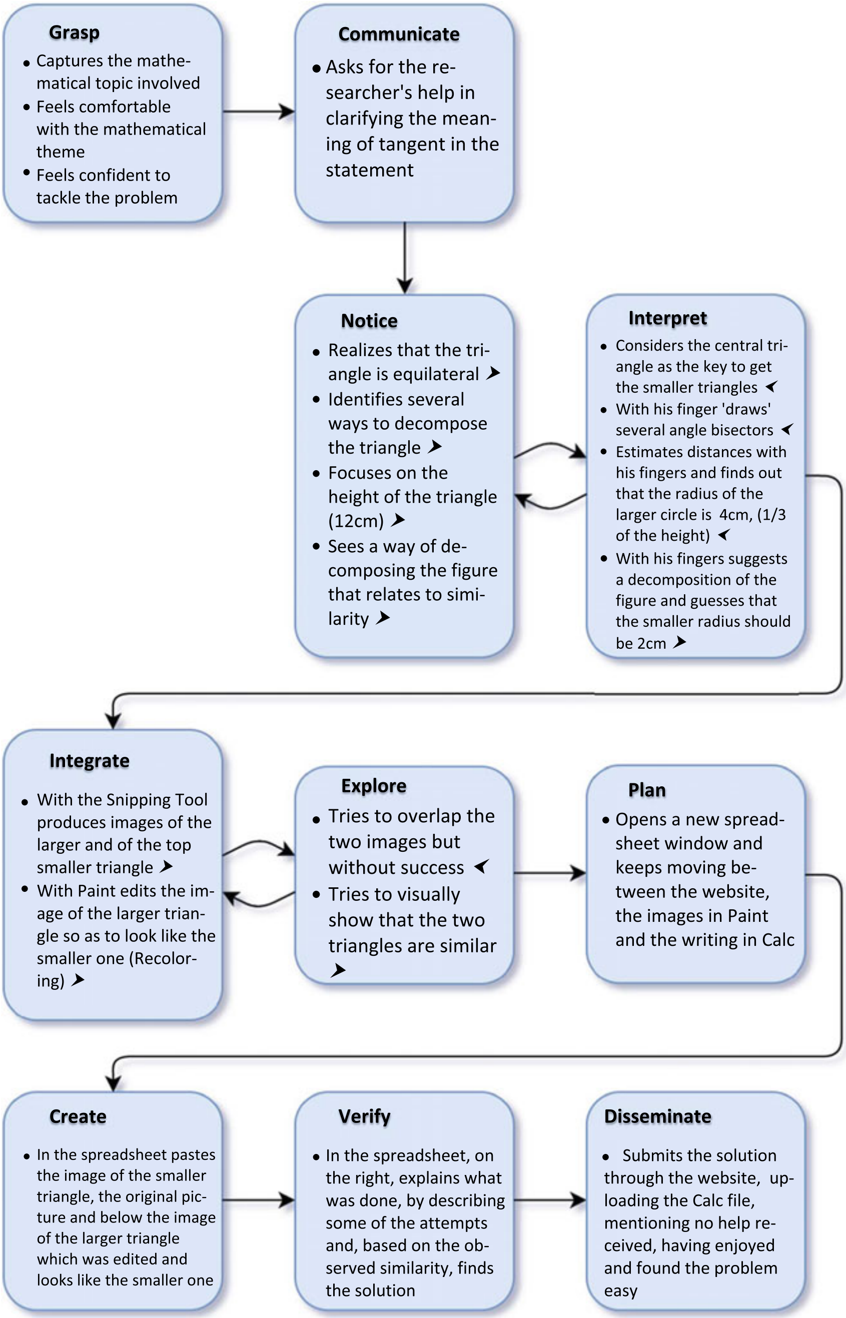


Fig. 3.11 Processes of solving-and-expressing the problem 'Decorative Drawing'

3.4.3 A Summary of the Processes Involved in Solving-and-Expressing Mathematical Problems with Technology

Our MPST model allowed a thorough and detailed description of Marco's processes while addressing two SUB14 geometry problems. As both the digital solution and the observed activity were analyzed, it is now possible to summarize the aspects that best characterize each of the processes involved in solving-and-expressing those problems with digital tools, also refining the descriptors presented previously in Table 3.1.

The student begins his approach to a new problem by reading the statement several times, in trying to get a sense of the mathematical notions and contents that may be involved, as well as by assessing his confidence on his ability to reach a solution based on the easiness he has with the subject or with possible ways of dealing with the problem (*grasp*). Sometimes he seeks support, at some point of his activity, by getting help from family members, from searching the Internet, from his teacher or, as it happens in the case of the activity observed, from the researcher (*communicating*). Then, there is a process of deepening the understanding of the conditions stated in the problems, either realizing that it is necessary and possible to construct the sequence of the 8 squares (in the 1st problem), or realizing some elements in the figures, such as the fact that the triangle is equilateral (in the 2nd problem) (*notice*).

While in the first solution the available data suggest that Marco proceeds to the recognition of certain affordances of the GeoGebra graphical view (*interpret*), the second solution offers evidence that this move can be much more complex. As it turned out, the production of a sequence of arguments and the several attempts initiated, that eventually led to the formulation of a conjecture about the unknown value, took place in a back-and-forth between two processes—*notice* and *interpret*. This means that the student realizes that the triangle is equilateral (*identify*) and analyzes the central circle so as to reach the smaller ones (*interpret*); then finds several possibilities of decomposition of the triangle (*identify*) and with the fingers draws imaginary bisectors and estimates distances (*interpret*); finally, he visualizes another way of decomposing the triangle into two that are similar (*identify*) and simulates this decomposition with the finger, formulating a conjecture about a possible solution to the problem (*interpret*).

The following processes are aimed at developing the formulated conjectures, which involves the use of digital tools with a mathematical sense: in the first solution, Marco uses the grid in GeoGebra's graphic view to build the extended sequence of squares, based on the coordinates of its vertices, constructs a ray, and also changes properties of some objects to highlight them; in the second solution, he uses the Snipping Tool to create files with the images of the original triangle and a top triangle resulting from decomposing the large one, then he draws the bottom side of this new one. In this problem, the *integrate* process is developed in association with the *explore*, i.e. an attempt is made to analyze the possibility of overlapping the two figures, but as it turned out to be unsuccessful, Marco graphically edits the images in order to transform the original triangle and recolor components of that figure (*integrate*), therefore visually showing that they are similar (*explore*). Otherwise, in the first problem, the analysis of a conceptual model occurs when Marco resorts to the spreadsheet in GeoGebra and inserts lists with the lengths of the sides of the squares and their areas (*explore*).

Then it follows the outline of an approach that leads to the solution from the conceptual models that were previously developed. In one case, completing the construction of a surrounding rectangle around the complete sequence of squares and recording its area in the spreadsheet indicates that Marco has found a way to examine his conjecture. In the other case, it is the abandonment of the editing tools and the move to the spreadsheet, where Marco normally composes the solutions, which indicates that the constructed figures already have a purpose (*plan*).

The next process concerns the development of the planned approach—in a case getting the difference between the calculated areas, and in the other through the insertion and arrangement of the edited images—during which Marco uses mathematical and technological knowledge to obtain the solution (*create*). In this process, certain elements intentionally created by Marco reveal a techno-mathematical understanding of the solutions, like the case of the surrounding rectangle or of the transformed triangle to exhibit its similarity to the smaller one. Those are new objects of knowledge created by Marco to solve-and-express the problem.

The following actions are directly related to the explanation of the solution or the justification of the reasoning through mathematical arguments supported by the technological resources (*verify*). In particular, Marco uses the GeoGebra spreadsheet to record the sequence of steps taken, so the combination of construction and organized calculations generates a techno-mathematical solution that 'self-explains' the problem solved. In his other solution, Marco describes in the spreadsheet some of the attempts he performed and explains how he got the solution, which occurred precisely when he articulated his mathematical thinking with the edited images.

Finally, the submission of the solutions is done through the online form of the SUB14 webpage and consists of sending the prepared files, which may contain some indications to the receivers on how to manage the information that he provides in his digital materials (*disseminate*). In the problems that he solved at distance and also in the case of the problem that Marco solved under the observation of the researcher, the young man made his report on the help he might had or not, about the degree of difficulty of the problem, and about whether he had enjoyed to solve it.

3.5 Discussion and Conclusions

The problem solving activity reported in this case illustrates how digital tools stimulate altogether the development of mathematical understanding that becomes crucial for finding and expressing the solution to the problems. It also shows that Marco's ability to perceive affordances in the tools is of significant relevance for achieving success in such activity.

In the first solution, this student-with-media uses GeoGebra in unconventional ways (Jacinto et al., 2016): although he recognizes a number of affordances, the construction is not built to be robust and the spreadsheet is not brought up to compute. Instead, the grid promotes an almost immediate 'materialization' of the squares' vertices and the construction prompts the development and exploration of a visual perception, while the spreadsheet allows recording every step of his strategy, which includes the reasoning and the procedures taken.

The second solution brings forth the relevant role of home-technologies which are often regarded as deprived of mathematical affordances, but were fundamental in the development of a mathematical way of approaching the problem.

At some point, in both solutions, Marco-with-media creates new objects not mentioned in the problems. The new mathematical meanings that he derives from them assist him in solving and in expressing the solutions: the enveloping rectangle, in the first solution, and the transformed triangles, in the second solution. Furthermore, the constructions, transformations, and the explanations Marco provides are not mere postscripts added after the solution is found. Those inscriptions are crucial elements within his work that assume a double role: they simultaneously support the finding and the reporting of the answer.

The MPST model provides analytical means to inspect and to account for the processes involved in Marco's activity, either based on digital documental data or on the observation of the activity itself. Solving-and-expressing accounts for the synchronous process of mathematization and expression of mathematical thinking (Carreira et al., 2016). Marco's activity reveals his purpose in producing a solution that is self-explainable, thus, solving-and-expressing-with-technology summarizes the whole process, from the beginning of his approach to the submission of his solutions.

Moreover, the MPST model reveals its potential as it accounts for the analysis of data stemming from multiple sources and characters. This is particularly relevant since the model allows identifying critical moments in the activity characterized by multiple sequences of processes, moving forth and back in an iterative way. For instance, the process of using editing tools to create similar triangles (*integrate*) lead to an attempt to overlap the figures (*explore*), while the analysis of this experience and the realization of its impossibility leads to using mathematical and technological resources (*integrate*) to look for a different way of demonstrating the similarity (*explore*).

While the integration of mathematical and technological resources aims to develop an exploratory approach, the analysis of such exploration (e.g., manipulation, conjecture, computation) may trigger the integration of new resources and, again engage in an exploration process. Thus, the *integration* is a key process in the simultaneous activity of mathematizing and expressing mathematical thinking by means of digital technologies.

This research may open new avenues on the kinds of mathematical thinking and problem solving skills that young students are capable of putting forth in challenging situations beyond school, entangling academic and informal knowledge. On the one hand, the results obtained demonstrate that technological resources and mathematical resources are equally indispensable to the problem-solving activity with technologies. On the other hand, they show that the nature of mathematical thinking developed with technology changes: technology opens up more ways of exploration, manipulation, observation, conjecture, and explanation.

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Chapter 4
Mathematical Problem Solving and the Use of Digital Technologies

Manuel Santos-Trigo

4.1 Introduction

4.1.1 Technology in Society and Its Importance in Education

The irruption of digital technologies in society is transforming the way individuals interact, communicate, and carry out daily activities. People rely on digital technologies to get access to several online services and information to make daily decisions. Likewise, the use of technology is also opening new routes for students to learn disciplinary knowledge. In dealing with mathematical tasks, students, with the use of technology, have an opportunity of relying on technology affordances to represent and explore ways to understand mathematical concepts and solve mathematical problems. Mason (2016) argues that "…something or some situation is a problem only when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in some sense-making activity" (p. 263). Asking and pursuing questions, checking examples or considering and exploring some special cases, making conjectures, looking for counterexamples, and supporting mathematical relations are problem solving strategies and actions that are important for learners to work on mathematical tasks (Santos-Trigo & Moreno-Armella, 2016). What then could the use of digital technologies offer to learners in terms of implementing these types of strategies during the process of understanding mathematical concepts and solving problems? Leung (2011) points out that "a pedagogic reason for using technology is to empower learners with extended or amplified abilities to acquire knowledge…technology can empower their cognitive abilities to reason in novice ways" (p. 327). That is, learners, with the use of technology, can engage in dynamic explorations of mathematical ideas and enhance their ways of reasoning to

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formulate and support conjectures. Similarly, the use of communication technologies can facilitate and enrich mathematical discussions within an extended learning community. Walling (2014) argues that "learning design must be flexible not only because students are diverse in their needs, interests, aspirations, and abilities but also because the very nature of the modern world demands it" (p. 14).

Gros (2016) point out that, the use of technology is changing and shaping what we learn, how we learn, where and when we learn. What types of digital technologies are helpful and how can students use them to understand mathematics and develop problem solving competencies? In a technological environment, teachers and students might rely on different digital technologies and online developments such as Internet, a Dynamic Geometry System (DGS), mobile applications, tablets, Wikipedia, etc. to represent, explore, expand, analyze, explain, and share their mathematical ideas, concepts or problems solving approaches.

Using digital technologies in learning environments implies addressing issues regarding what new pedagogies are needed to frame mathematical working and learning spaces in which learners participate in the construction and use mathematical knowledge. Gros (2016) states that "technology must enable and accelerate learning relationships between teachers and students and between students and other "learning partners" such as peers, mentors and others with similar learning interests" (p. 18). That is, technologies might expand and enhance students' ways to share and discuss mathematical ideas as a part of an extended learning community.

> Mathematical tasks and ways to implement them are essential ingredients in structuring a learning environment for students to engage in mathematical activities. "A challenge in digital task design is to conceive tasks that can extend and amplify pedagogical features present in non-digital environments". (Leung & Baccaglini-Frank, 2017b, p. x)

Gros, Kinshuk, and Maina (2016) argue that for students to deal with the complexity involved in this technological society, they need to develop and exhibit strategies to solve problems collaboratively, communicate results, and to constantly interact with peers and other experts. Indeed, with the use of communication tools or mobile applications students expand individual and self-directed problem solving behaviors to include collaborative learning through direct and continuous interaction with peers and group experts. Gros (2016) point out that the incorporation of digital technologies in learning scenarios involves discussing the design of mathematical tasks, the role of teachers and students, and the educational context or learning scenarios to implement the tasks.

The goal of this chapter is to analyze and discuss ways in which the coordinated and systematic use of several digital technologies provides affordances for teachers/students to represent dynamically concepts, explore and solve mathematical problems.

To delve into the representations, strategies and ways of reasoning that emerge in technology problem solving approaches, four types of tasks are identified and analyzed in terms of characterizing how technology affordances shape their solution processes.

(a) Focusing on figures. This group emphasizes the importance of using DGS in reconstructing figures that are embedded in problem statements;
(b) Investigation tasks. This group deals with investigation tasks where students are encouraged to transform initial routine problems that appear in textbooks into a series of mathematical activities;
(c) A variation task. This group addresses ways to represent and analyze tasks that involve some variation phenomena in which a graphic model is achieved without having an explicit algebraic model of the situation; and
(d) Dynamic configurations. This group refers to the construction of dynamic configurations that aim to foster problem posing activities and ways to validate mathematical relationships.

4.1.2 Learning Environments and the Coordinated Use of Digital Technologies

In framing and characterizing a digital learning environment it is important to address and discuss ways in which the coordinated use of several digital technologies not only offer affordances to represent and explore mathematical tasks; but to also enhance students' interaction to continuously share and discuss ways to solve problems. Leung (2017) states that "teachers must experience for themselves, as learners, the potentials and pitfalls of digital tool in the learning of mathematics, thus gain knowledge about how students can learn mathematics in various digital environments" (p. 6). Thus, teachers need to work on problems and discuss ways in which technology help them restructure their teaching practices that pay attention to the type of reasoning that emerges throughout the problem-solving process.

> In the eyes of many digital natives, learning is more than just going to lectures and relying on textbooks; rather, learning involves engaging in technology-mediated learning activities such as doing research on the Internet, searching, finding, and analyzing a variety of resources available in the virtual world and bringing into their own lives (p. x). (Kinshuk & Spector, 2013)

Leung (2011) point out that "when one is faced with a new tool, one has to learn how to use it and in doing so, gradually realizes the "knowledge potential" that is embedded in it" (p. 327).

The use of digital technologies, such as a DGS (GeoGebra) and communication applications, provides the learners with a set of affordances to continuously engage in exploration, reconstruction, explanation, and communication activities to make sense of concepts and to solve mathematical problems. Thus, multiple purposes technologies such as Internet, tablets or smart phones play an important role in extending learners' mathematical discussions beyond formal settings. That is, students can access online materials, consult encyclopedias (Wikipedia) or share mathematical ideas via a digital wall (Padlet) and discuss their ideas (through email or online forums) within a learning community that includes peers, experts and teachers.

Santos-Trigo, Moreno-Armella, and Camacho-Machín (2016) argue that:

> …Representing and exploring mathematical tasks mediated by digital technologies bring in new challenges for teachers that include the appropriation of the instruments afforded by these technologies in order to identify and analyze what changes to mathematical contents and teaching practice are fostered through its use (p. 829).

In addition, Moreno-Armella and Santos-Trigo (2016) state that "the use of mediating instruments, in particular, digital technologies, are never epistemologically neutral. The ways of approaching a problem depend upon the resources we have at our reach" (p. 829). That is, the subject's experience or expertise in using the tool shapes and permeates how it is used in problem solving approaches. The transit in learners' initial use of empirical or visual approaches (via the use of technology) to eventually construct and present geometric and analytic arguments to support results appears important throughout all problem-solving activities. Freiman et al. (2009, p. 128) state that "…the most important advantage of using technology is the diversification of teaching and learning approaches, rediscovery of dynamic aspects of mathematics, and, especially, learning through communication with others".

It is argued that the use of technology demands that teachers and students analyze and discuss what problem solving strategies, concepts, resources and ways of reasoning appear important during the construction and exploration of dynamic models of problems via technology affordances. To this end, it is relevant to discuss how problem representations and strategies such as moving orderly objects within the model, quantification and exploration of objects' attributes, finding and analyzing objects' loci; using sliders, and arranging data in tables become important throughout the learners' problem solving process.

4.2 A Focus on Problem-Solving Activities

Curriculum and teaching proposals worldwide recognize that problem-solving activities are essential to frame mathematical learning environments (Törner, Schoenfeld, & Reiss, 2007). Likewise, the mathematical problem solving research agenda has provided relevant results and information regarding the importance of tasks or problems, the research methods to elicit and analyze both cognitive and metacognitive processes involved in learners' construction of mathematical knowledge, and the development of conceptual frameworks to analyze and document the students' problem solving competencies (Santos-Trigo, 2014; Silver, 1990). Although mathematical contents that appear in curriculum proposals might be the same in different countries, the ways to structure and implement a problem-solving approach to learn those contents might differ since such implementation is shaped by countries' cultural and social or educational traditions. Indeed, Stanic and Kilpatrick (1988) point out that "problem solving has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is, and of why we should teach mathematics in general and problem solving in particular" (p. 1). Similarly, research agendas in the field not only include distinct themes and interpretations of what a problem-solving approach entails; but have also evolved in terms of the use of research methodologies (Santos-Trigo, 2014).

> The term [problem solving] has served as an umbrella under which radically different types of research have been conducted. At minimum there should be a facto requirement (now the exception rather than the rule) that every study or discussion of problem solving be accompanied by an operational definition of the term and examples of what the author means. …Great confusion arises when the same term refers to a multitude of sometimes contradictory and typically underspecified behaviors. (Schoenfeld, 1992, p. 364)

What does it distinguish, then a problem-solving approach to frame a learning environment for students to construct and use mathematical knowledge? A key principle in any problem-solving approach to learn mathematics and to solve problems is that learners need to conceptualize the discipline as a set of dilemmas that are important and need to be represented, explored, analyzed, and explained in terms of mathematical resources (Santos-Trigo, 2014). Mason (2016) recognizes the importance for students to experience problematicity in dealing with mathematical tasks and to make use of their own powers and to engage in problem solving approaches. To this end, learners need to develop and value an inquiring approach to understand concepts and to solve problems. Santos-Trigo and Camacho-Machín (2016) point out that an underlying principle in problem solving activities is "to conceptualize learning as an inquiring process to delve into concepts and problems in order to identify and explore mathematical relations" (p. 45). Mason, Burton, and Stacy (2010) stated that an atmosphere of questioning, challenging and reflection is crucial for students to develop mathematical thinking. Leikin, Koichu, Berman, and Dinur (2017) states that "The construction of questions is an important way for learners to build conceptual conflict, and the search for answers may begin the process of resolving that conflict" (p. 67). Thus, posing questions and looking for different ways to pursue those questions are key activities for learners to learn and use mathematical knowledge. Barbeau (2009) refers to a challenge for learners to delve into mathematical tasks:

> …we will regard a challenge as a question posed deliberately to entice its recipient to attempt a resolution while at the same time stretching their understanding and knowledge of some topic… A good challenge will often involve explanation, questioning and conjecturing, multiple approaches, evaluation of solutions for effectiveness and elegance, and construction and evaluation of examples" (p. 5).

4.2.1 On the Use of Technology to Construct and Explore Dynamic Models

Within a technological learning environment, students might rely on different digital tools' affordances as a means, to represent, make sense, analyze and solve mathematical tasks. In this process, it is important to characterize what type of reasoning learners construct and exhibit throughout their problem-solving approaches. How could I construct a dynamic model of a problem? What parameters involved in the problem representation can be quantified or measured? How can I orderly move some parameters within the model? How can I determine or visualize the loci of specific objects when I move some elements within the model? The discussion of these questions sheds lights on what type of reasoning students might get engaged with the use of a DGS in problem solving activities.

> The long-term commitment students need to make is a willingness to engage in problem-solving activities and to form habits of mind such as thinking about word meanings, justifying claims and conjectures, analyzing answers and solution strategies, using alternative representations, and acquiring a toolkit of problem-solving strategies. (Lester & Cai, 2016, p. 121)

Likewise, during the students' development of problem solving experiences it is important that they share, analyze, and discuss concepts, ideas, solutions as a part of a learning community and the use of digital technologies allows them to continuously discuss their ideas with peers and experts in and out of formal settings. Similarly, learners can consult online materials or learning platforms to recall or extend conceptual information or to watch an expert presentation via an online video of the topic in study. As Mishra and Koehler (2006) stated:

> … there is no single technological solution that applies for every teacher, every course, or every view of teaching. Quality teaching requires developing a nuanced understanding of the complex relationships between technology, content, and pedagogy, and using this understanding to develop appropriate, context-specific strategies and representations (p. 1029).

In this perspective, during the process of working on a mathematical task, learners should always look for different ways to represent and solve a problem and to examine the extent to which the methods used in solving it can be used in other tasks. In this context, a task is conceived of as departure point to engage learners in mathematical reflection and thinking. Santos-Trigo and Reyes-Rodríguez (2016) discusses the importance for students to think of and discuss several ways to solve a task that involves an equilateral triangle. The multiple approaches to represent and solve the task became important for students not only to consider and analyze different concepts and results associated with the equilateral triangle; but to also make connections among contents that often are studied separately. Lester and Cai (2016) mention that teachers should provide learning conditions for students to engage in a variety of problem-solving activities that include: "(1) finding multiple solution strategies for a given problem, (2) engaging in problem posing and mathematical explorations, (3) giving reasons for their solutions, and (4) making generalizations" (p. 13). That is, looking for different ways to solve a task, discussing what concepts are used, and exploring ways to extend mathematical tasks become an important goal for learners to pursue in the process of development their problem-solving competencies. This goal is achieved as a part of a learning community that demands that each member shares and constantly reflects on what he/she contributes to task's solution. Blaschke and Hase (2016) pointed out that:

> Working together toward a common goal, learners are able to solve problems and reinforce their knowledge by sharing information and experiences, continuously practicing, and exper-

imenting by trial and error. They simply help each other along the way. The teacher serves as coach during the collaboration process, letting learners forge forward together and stepping in only when absolutely necessary (p. 33).

Santos-Trigo and Moreno-Armella (2016) argue that "[s]earching for alternative ways to represent and solve problems is a powerful strategy for students to identify and contrast the role played by concepts and their representations across the whole problem-solving process" (p. 192). The development of Geometry Dynamic Systems such as GeoGebra represents a milestone in the study and development of mathematical knowledge. Leung and Bolite-Frant (2015) pointed out that GDS "can be used in task design to cover a large epistemic spectrum from drawing precise robust geometrical figures to exploration of new geometric theorems and development of argumentation discourse" (p. 195). That is, its use provides affordances for learners to both finding objects' relationships and properties and arguments to support or validate them.

4.2.2 *Technology Affordances and Mathematical Explorations*

Some problems that involve paper and pencil approaches can be explored and extended with the use of technology. Schoenfeld (1985, p. 16) asks some college students to divide a given triangle in two parts of equal area (using a straightedge and compass) by drawing a parallel line to one of the triangle side. What about if we remove the parallel line and the use of a straightedge and compass conditions and approach the problem with the use of GeoGebra? That is, we ask: divide a given triangle in two regions with same area. The goal is to look for different ways to find two regions with the same area. In Fig. 4.1, M is constructed as a midpoint of side AB, point E lies on segment AM and side EG is constructed to be a half of side AB. Thus, students can see that for any position of point E on segment AM the area of triangle ECG is always half of the area of triangle ABC. Properties of the construction validate the solution since triangles ABC and ECG share the same height with respect to sides AB and EG respectively. Therefore, for any position of point E on segment AM, then the area of triangle ECG is the same as the sum of areas of triangles ACE and BCG.

Another way to divide the given triangle is shown in Fig. 4.2, segment ED is perpendicular to AB and segment DH is parallel to AB, the coordinates of point Q are the x-coordinate of point E and the area of polygon EBHD as y-coordinate. Line $y = 3.22$ (half of the area of triangle ABC) intersects the locus of point Q that results when point E is moved along side AB at points O and P. Then, when point Q coincides with point O and P the area of quadrilateral EBHD will be the same as the sum of the areas of triangles ADE & DHC. The latter approach involves describing graphically the area variation of polygon EDHB when point E moves along side AB

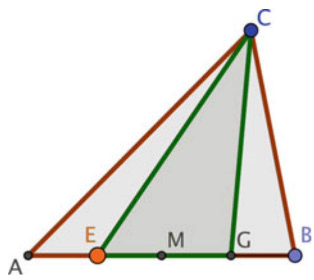


Fig. 4.1 Point E moves along segment AM (M midpoint of AB) and side EG is half of side AB, then area of triangle ECG is half of area of triangle ABC

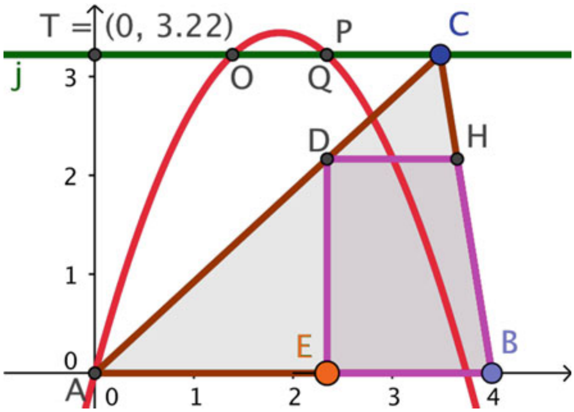


Fig. 4.2 Finding the locus of point Q when point E moves along segment AB

and to determine the position where polygon EBHD has half of the area of triangle ABC.

The tool affordances that include drawing a precise model, generating a family of objects (triangles and polygons), and finding loci of the polygon's area variation become important not only to solve the task; but also are essential to identify properties and arguments to support results. In this case, an open question that involves dividing a triangle represents an opportunity for students to think of and explore different ways to divide the figure, and so, several concepts and problem solving strategies appear during the solution process.

Thus, the use of digital technologies seems to provide a context and an opportunity for students to activate a variety of concepts and resources during the process of constructing and exploring different approaches to the task. Freiman, Kadijevich, Kuntz, Pozdnyakov, and Stedoy (2009) summarizes what the use of technology might bring to the learning community in terms of extending learning mathematical discussion beyond classrooms:

- Technology can give access to the resources that cannot be otherwise accessed.
- Technology can provide a free choice of resources based upon the level and the particular needs.
- Technology can provide dynamic tools of mathematical investigation giving a chance to modify parameters of an activity in an interactive way.

- Technology is a valuable tool of communication about mathematics with other people.

- Technology empowers the people with the instruments, facilitating routine operations and more sophisticated mindtools (p. 129).

In the same vein, Liljedahl, Santos-Trigo, Malaspina, and Bruder (2016) argue that the use of technologies demands that students engage in a tool's appropriation process to develop an expertise in representing and exploring concepts and problems.

> …learners not only need to develop skills and strategies to construct dynamic configuration of problems; but also ways of relying on the tool's affordances (quantifying parameters or objects attributes, generating loci, graphing objects behaviors, using sliders, or dragging particular elements within the configuration) in order to identify and support mathematical relations (p. 23).

4.3 Problems as a Departure Point to Engage Students in Mathematical Thinking

Mathematical problems play an important role in fostering students' learning and guiding the development of mathematical knowledge (Leung & Baccaglini-Frank, 2017a). Silver (2016) pointed out that "mathematics problems form the foundation of students' opportunities to learn mathematics". Similarly, Lester and Cai (2016, p. 122) stated that:

> Mathematical tasks provide intellectual environments for students' learning and the development of their mathematical thinking…Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites speculation and hard work. Most importantly, worthwhile mathematical tasks should direct students to investigate important mathematical ideas and ways of thinking toward the learning goals.

Teachers design, select, adjust and implement mathematical tasks to foster their students' development of mathematical thinking. Margolinas (2013) stated that:

> Tasks…are the mediating tools for teaching and learning mathematics…Tasks generate activity which affords opportunity to encounter mathematical concepts, ideas, strategies, and also to use and develop mathematical thinking and modes of enquiry (p. 12).

What types of problems are important for students to work and discuss in problem solving environment? What does the process of designing or selecting mathematical tasks entail? How does the use of digital technologies influence the design and selection of mathematical problems? The discussion of these types of questions implies also addressing issues regarding choosing, designing and implementing mathematical tasks in learning scenarios. Selden, Selden, Hauk, and Mason (2000) pointed out the importance for students to deal with non-routine problems to develop a robust understanding of mathematical concepts. Working on non-routine problems requires that students figure out mathematical features associated with the structure of the problem, to identify key concepts involved in the problem statement, and to select and search for resources and strategies needed to explore and eventually solve the problems. In a technological environment, learners could engage in exploration activities that involve moving objects, exploring their behaviors, looking for invariance and properties to support conjectures. In this process, they examine concepts to grasp features associated with the deep structure of the problem (Santos-Trigo & Camacho-Machín, 2016).

In this context, working on tasks or problems represents an opportunity for learners to get involved in a continuous investigation that lead them to look for patterns, to make connections, and to extend initial problems. That is, problems are conceived of as a departure point for students to engage in mathematical discussions. Likewise, the way teachers implement the tasks in learning scenarios plays an important role in the students' learning of concepts and solving problems. Thus, the type of questions and the mathematical reflection that students engage in while working on the task are essential for student to focus on what is important during the solution process. Lester and Cai (2016) pointed out that:

> The learning environment of teaching through problem solving provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Such activities help students clarify their ideas and acquire different perspectives on the concept or idea they are learning. Empirically, teaching mathematics through problem solving helps students go beyond acquiring isolated ideas toward developing increasingly connected and complex system of knowledge (pp. 119–120).

What should students pay attention to or look at while using technology to solve mathematical problems? An initial categorization of groups of problems is proposed in terms of identifying how the tool's affordances shape the ways of reasoning and approaching each group solution. This categorization comes from analyzing and discussing ways in which we have used several digital technologies in problem solving approaches (Santos-Trigo & Camacho-Machín, 2016). Thus, the presentation of each category shows, representations, strategies, concepts, and resources that appear relevant in approaching the problem.

4.4 Towards a Categorization of Mathematical Problems and the Use of Technology

4.4.1 Problem Statements and Embedded Figures

In paper and pencil approach, some problems or tasks statements often include figures that show objects and data that are important to identify properties or relations to solve the problem. For example, in the statement: *Let ABC be an equilateral triangle and let P be any point on its circumcircle, for instance, on the shorter arc AB, as shown in* Fig. 4.3*… [show] that AP + BP = CP* (Melzak, 1983, p. 13), the figure

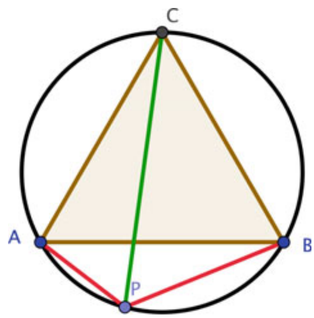


Fig. 4.3 Triangle ABC is equilateral, show that AP + BP = CP

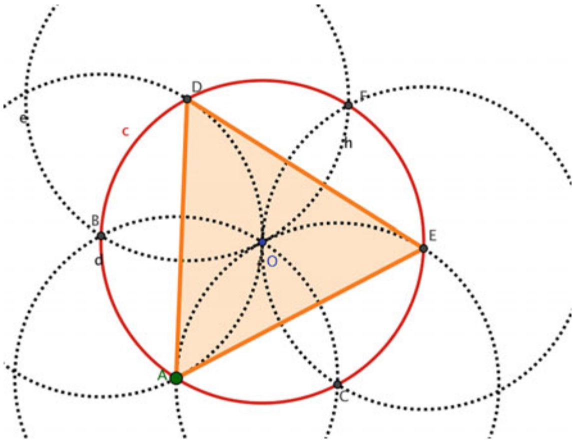


Fig. 4.4 Inscribing an equilateral triangle in a given circle

becomes a referent to identify possible properties and results (similar triangles, cyclic quadrilateral, etc.) to support or validate the involved relation.

By looking at the figure, one might ask: How can I reconstruct the figure? How should one draw an equilateral triangle and its circumcircle? Or given a circle, how should one inscribe an equilateral triangle? With the use of GeoGebra, these questions become relevant not only to identify and explore concepts needed to draw the figure, but also provide an opportunity for learners to connect the problem goal with a series of mathematical ideas and resources to solve and extend the initial statement. How can I inscribe an equilateral triangle into a given circle? Figure 4.4 shows a way that involves choosing a mobile point A on the circle and drawing a circle with center at A and radius AO (O the center of the given circle). This circle intersects the given circle at point B and C, then two other circles are drawn with centers at B and C and radius BO and CO, etc. Then, points A, D & E are the vertices of the inscribed triangle (Fig. 4.4).

Another approach (Fig. 4.5) to inscribe an equilateral triangle involves selecting any point A on the circle and drawing line AO (O the center of the circle). Line AO intersects the circle *c* at point D. Then, a circle *d* with center at D and radius DO is drawn. Points C and B are the intersection points of the circles and then triangle

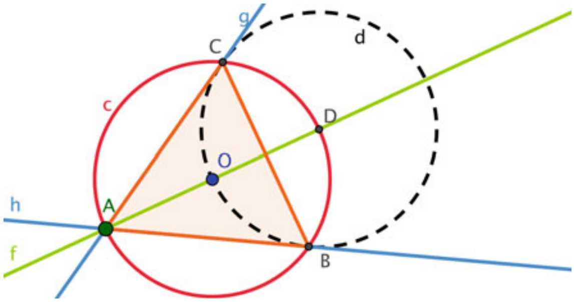


Fig. 4.5 inscribing an equilateral triangle in a given circle

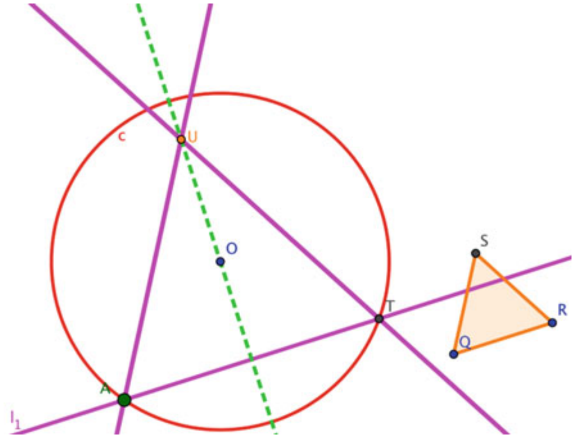


Fig. 4.6 Focusing on any equilateral triangle as a referent to inscribe a similar triangle in a given circle

ABC is equilateral (Fig. 4.5). This is because line AO is the perpendicular bisector of CB.

Another approach might focus on drawing any equilateral triangle QRS as a reference to inscribe a similar triangle into the given circle. Figure 4.6 shows an equilateral triangle QRS as a reference one, then point A is any point of the given circle c. From point A two parallel lines to side QR and QS are drawn and from point T (the intersection of the parallel to QR and the circle) also a parallel to side RS is drawn. Triangle ATU is equilateral and the locus of point U when point A moves along the circle is a line. So, the position of point A at which the locus of point U intersects the circle, is the third needed vertex to determine the inscribed equilateral triangle.

Yet, another approach to inscribe an equilateral triangle focuses on examining a simpler case and analyzing the area variation of the inscribed circle. In Fig. 4.7a, point A is a mobile point on the given circle and triangle ABC is equilateral (its inscribed circle is a circle with center at the intersection of two perpendicular bisectors and radius the distance between the center and any vertex). Line m is the graph of y = area of circle c and point Q has coordinates the x-coordinate of point A and as y-coordinate the area of the circle that inscribes the equilateral triangle ABC. What

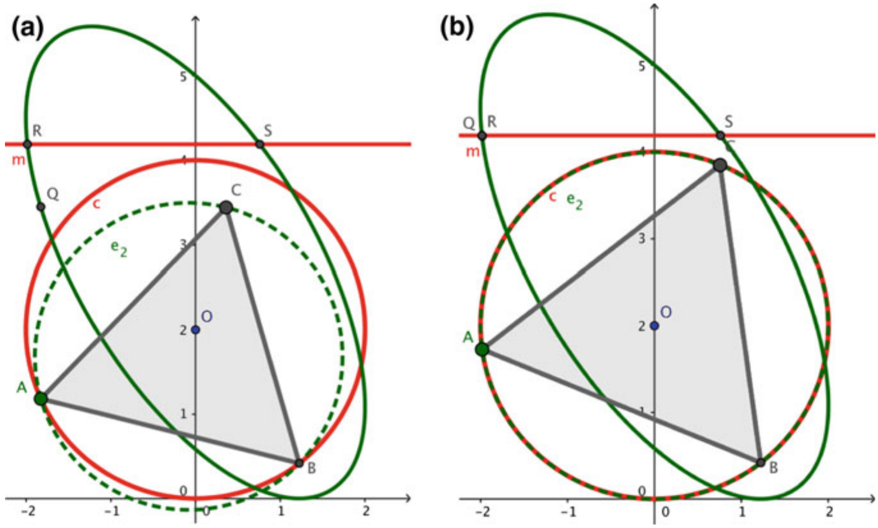


Fig. 4.7 **a** What is the locus of point Q when point A is moved along circle c? **b** When point Q coincides with point R, the inscribed triangle is the solution

is the locus of point Q when point A is moved along circle c? Figure 4.7a shows that the locus (that seems to be an ellipse) intersects line m at point R and S. That is, when point Q coincides with point R, then triangle ABC is the inscribed equilateral triangle (Fig. 4.7b).

Comment: Pólya (1945) argues that understanding the problem statement is a crucial stage in the process of solving the problem and it involves identifying relevant concepts and possible relations. The use of technology can help students connect concepts and delve into the problem understanding process by focusing on the reconstruction of figures. Indeed, this phase becomes a problem posing activity where students begin reflecting on how, what order, and properties are important to draw the figure. In this case, asking about how to inscribe an equilateral triangle becomes important to think of the use of different concepts and strategies to reconstruct the figure given in the problem.

4.4.2 Investigations Tasks

A problem statement is conceived of as a departure point for students to look for mathematical relations and to extend the task. That is, the learners' goal while interacting with a mathematical task is not only to find its solution; but it is also important to look for ways in which the initial task can be extended or connected with other problems. How can a routine or a textbook task be transformed into an investigation task? To delve into this question, an adjusted version of a problem that appears in *Con-*

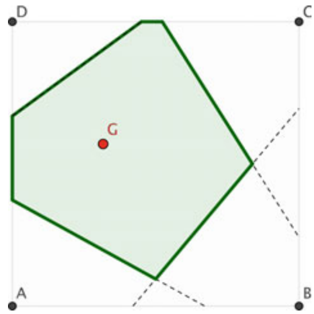


Fig. 4.8 Folding a square sheet

nected Geometry (2000, p. 76) is discussed in terms of identifying problem solving episodes in which the systematic and coordinated use of digital technologies offers affordances for learners engage learners in mathematical experiences. Santos-Trigo and Reyes-Martínez (2018) present what prospective high school teachers exhibited during the problem solving episodes that involved working on this investigation task. A complete analysis of the implementation of this task appears in Reyes-Martínez (2016).

The initial task. Draw a square ABCD and choose an interior point G. Fold each vertex or corner into make it coincide with point G. Figure 4.8 shows the position of point G, the folding lines (creases) and a polygonal region that appear when all four vertices coincides with point G. What happens to the number of sides of the polygonal region when point P moves inside the square?

(a) **A dynamic representation**. At the understanding and making sense stage of the statement, it is always important to ask about properties, relations and ways to represent objects involved in the task (Schoenfeld, 1992; Santos-Trigo, 2007). What mathematical concepts are important to represent the folding line (segment)? Is there any type of symmetry involved in the folding process? What concepts can be used to draw the figure? Is it possible to construct a dynamic model of the task? These questions might lead the students to identify that the creases (folding lines) are the perpendicular bisectors of segments that join the interior point (G) with each square vertex. Indeed, with the use of a slider (Fig. 4.9), it is possible to identify steps involved in moving each vertex to point G and to explore what type of polygonal region is formed for different positions of point G. Likewise, this dynamic representation requires that the problem solver thinks of the task in terms of mathematical concepts and properties that can be expressed or represented through the tool' affordances.

(b) **A robust model**. Looking at the intersection of two perpendicular bisectors LO and QN (Fig. 4.10) provides important information to construct a robust model of the problem. Thus, when segment UI is longer than half of the side of the square, then the intersection point I is outside of the square and the sides of the polygonal region would be ON and PQ respectively. This information leads to

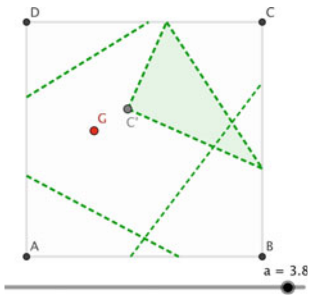


Fig. 4.9 An animated representation of the task that shows the creases movement when each vertex approaches to point P

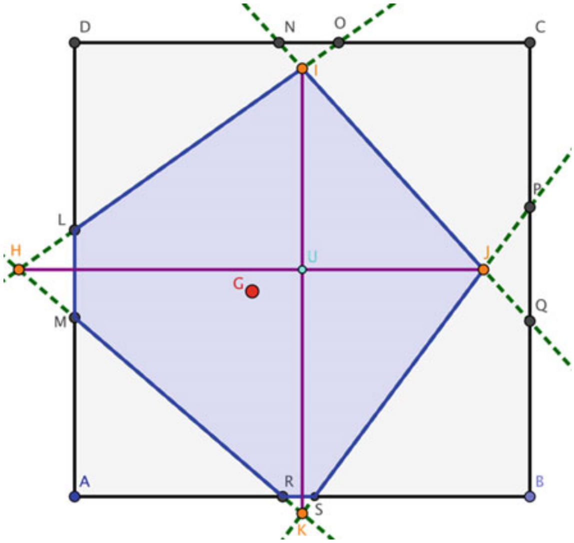


Fig. 4.10 Identifying conditions to construct a robust model of the task

relate the position of point G and the number of sides of the formed polygonal regions (Fig. 4.10).

The construction of a robust model of the problem means that point G can freely be moved inside of square ABCE and for any position of point G there will always be a be well-defined polygonal region. Figure 4.11 shows that when G is located outside the "petal region" (the intersection of semicircles with center at midpoint of each side of the square and radius half of the length of the side) then the polygonal region will have five sides.

(c) **A characterization of the polygonal regions**. The exploration of the robust model of the task provides important information and clues to visualize and relate the position of point G to the number of sides of the generated polygonal region. With the use GeoGebra, it is possible to reveal, through coloring, what polygonal regions share the same numbers of sides. Leung (2008) call *spectral dragging* to a heuristic that allows to trace and assign colors to properties of

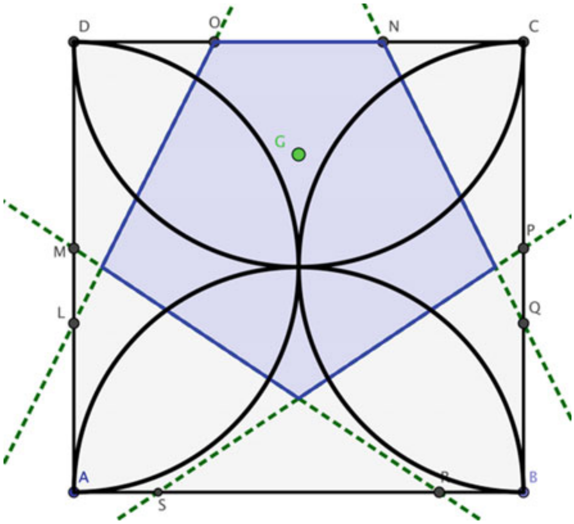


Fig. 4.11 The construction of the robust polygonal region and identifying regions where the polygonal regions hold specific properties

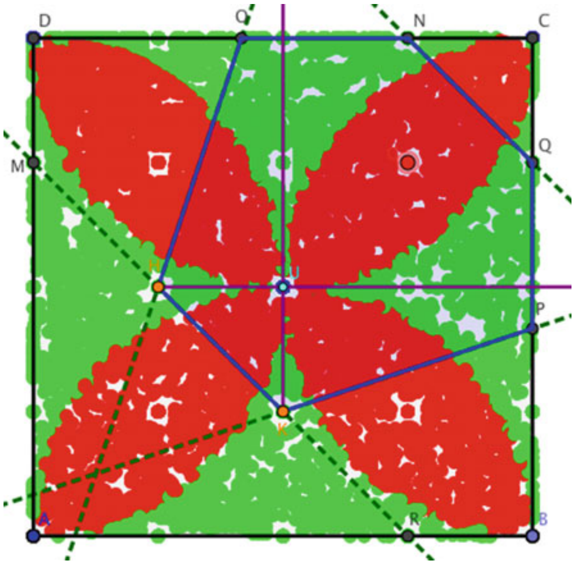


Fig. 4.12 A relationship between the position of point G and the number of sides of the generated polygonal region

involved objects. In this case, the colored region identifies the family of polygons that shares the same number of sides. Figure 4.12 shows that when point G lies on the red part then the polygonal family that appear on that region will have six sides and when point G lies on the green part then the polygonal family on that region will have five sides. Likewise, when point G coincides with the center of the square polygon becomes a square.

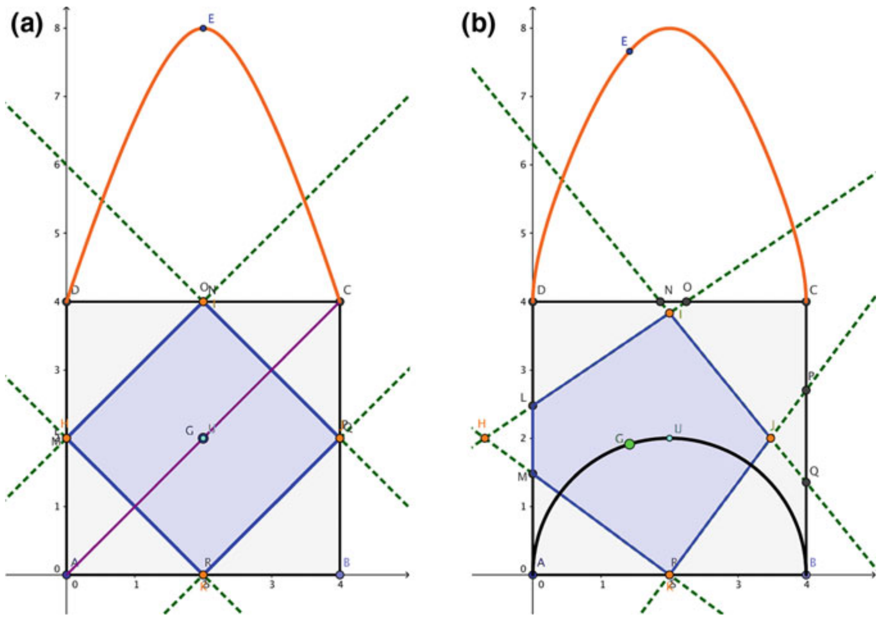


Fig. 4.13 **a** Exploring the area variation of the polygonal region when point G moves along diagonal AC. **b** Exploring the area variation of the polygonal region when point G moves on a semicircle AB

In addition, the robust model of the task provides information regarding the area variation of the family of polygons generated for different position of point G. Figure 4.13a, b show that when point G is moved along diagonal AC or the semicircle AOB the maximum area is reached when point G coincides with the center of the square in which the region becomes a square.

(d) **Extension and generalization**. Can the method used to construct the robust model be extended to explore what happens to the polygonal region when the initial square becomes others regular polygons? Figure 4.14 shows polygons with different number of sides and the corresponding generated polygonal regions. Based on the exploration for regular polygons with different number of sides, some conjectures emerge:

1. When the position of the interior point coincides with the center of polygon, then the generated polygonal region is a regular polygon that has the same number of sides as the initial regular polygon.
2. When the number of sides of the initial regular polygon increases (Fig. 4.14 shows a polygon with 200 sides), then the intersection of the corresponding perpendicular bisectors (red points) seems to form and ellipse and when point G is outside of the circle the intersection points generate a hyperbola.
3. When the number of side of the regular polygon tends to infinity, the polygon tends to be a circle. Figure 4.15 shows a circle with center at point A, D is any point on the circle, f is the perpendicular bisector of segment DG that intersects

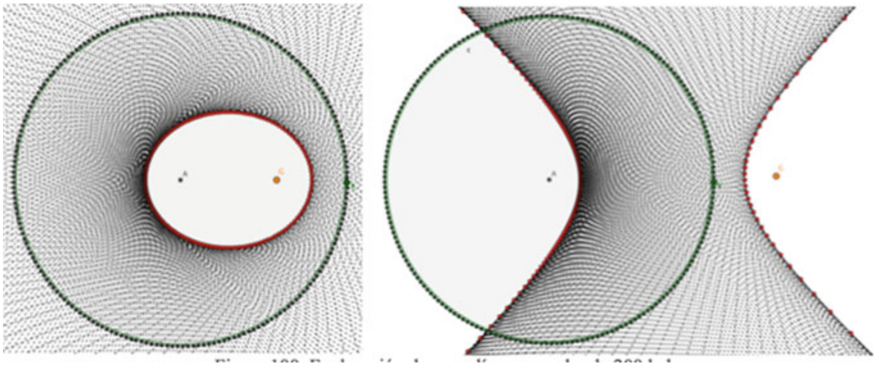
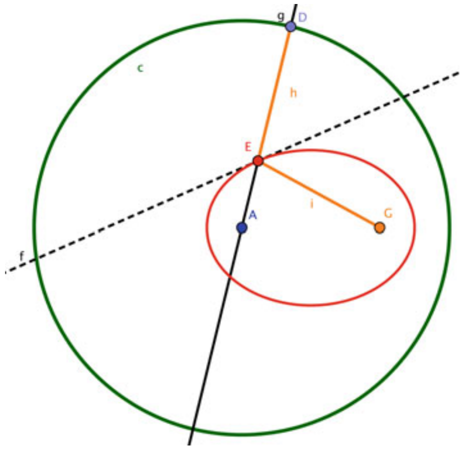


Fig. 4.14 For a polygon with 200 sides, the intersection of the corresponding perpendicular bisector seems to form an ellipse when point G is inside and a hyperbola when point G is outside of the polygon

Fig. 4.15 For a polygon in which its number of sides tends to an infinite number of sides the intersection of the perpendicular bisector of segment GD and AD generates an ellipse



line AD at point E. Then the locus of point E when point D is moved along the circle is an ellipse. This is true because segment ED and GE are congruent (E is on the perpendicular bisector) the radius AD is constant. Then it holds that $d(A, E) + d(E, G)$ is always constant (definition of an ellipse).

Comment: What concepts are embedded in the task' representation? How can they be represented via the DGS affordances? These types of questions are important to think of the problem in terms of the tool affordances. Thus, connecting the folding lines (creases) with the perpendicular bisector was essential to construct the dynamic model of the task. The exploration of this model provided clues and information regarding the polygonal regions behavior. Can the robust model for the square be extended to other regular polygons? This question leads to focus on how the intersection points of the corresponding perpendicular bisectors behaves and to find a

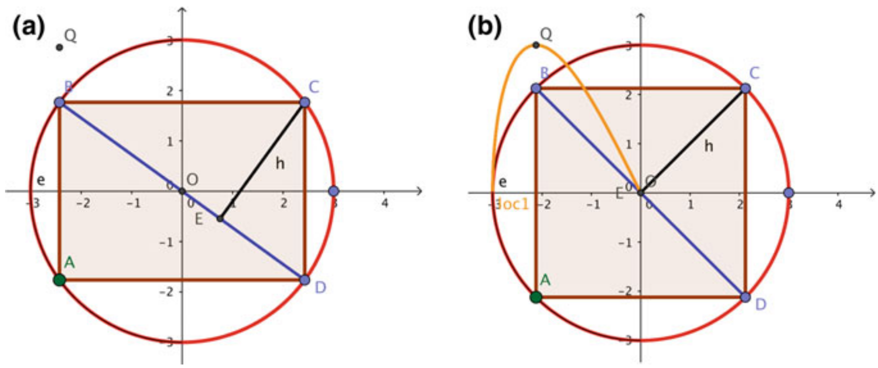


Fig. 4.16 **a** A is a movable point and h changes as point A moves. **b** The locus of point Q describes the behavior of the length of segment EC (height of triangle BCD)

serendipitous result: The intersection point of the perpendicular bisectors forms or determines an ellipse when the interior point G lies in the interior of the polygon or circle and a hyperbola when G is outside the circle.

4.4.3 A Variation Task

With the use of a dynamic geometry system, some problems that involve a phenomenon variation (optimization calculus problems) can be modelled without constructing explicitly its algebraic model. That is, the tool's affordances can help students construct a graph representation of the variation phenomenon without expressing the involved algebraic model. For example, the task that focuses on examining a family of inscribed rectangles and asks to identify which element of that family has a maximum area can be represented and analyzed through a dynamic model.

In Fig. 4.16a, point A is a mobile point on the circle and a rectangle ABCD is drawn. One way to inscribe the rectangle is to reflect point A with respect to the x-axis to determine point B, then point B is reflected with respect the y-axis, etc. At what position of point A does the rectangle ABCD reach the maximum area? It is observed that when point A is moved along the circle, the diagonal BD has a constant length (this is because its length is always the diameter of the circle). In triangle BCD, *h* is its height and the maximum area of the family of triangles BCD that is generated when point A is moved is obtained when *h* gets is maximum value. Point Q has coordinates the x-coordinate of point A and as a y-coordinate the length of h. Figure 4.16b shows the locus of point Q when point A moves along the circle. This leads to conclude that the inscribed rectangle with maximum area is when the rectangle becomes a square.

Similarly, Fig. 4.17 shows directly the area variation of the rectangle which is the locus of point M (whose x-coordinate is the length of side AD and as the y-coordinate

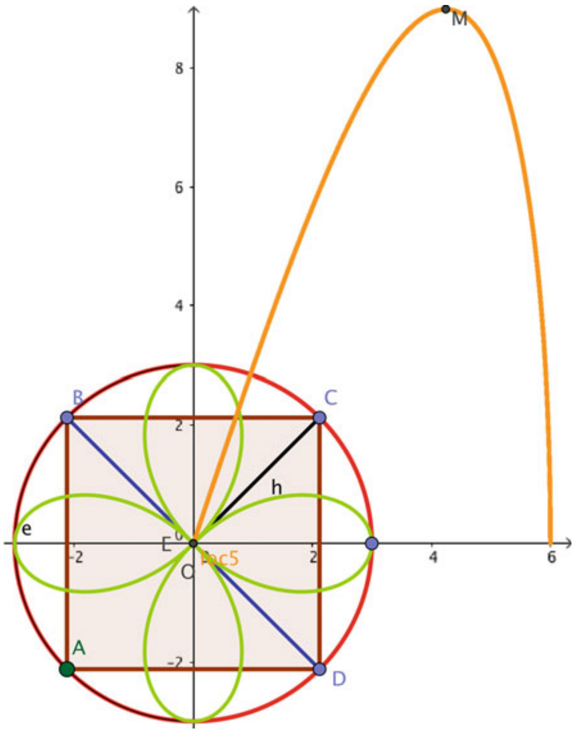


Fig. 4.17 The locus of point M that describes the area variation of the family of rectangles when point A is moved on the circle

the area of rectangle ABCD) when point A is moved along the circle. In addition, the locus of point E (one extreme of segment h) is another interesting curve that learners might be interested in exploring its properties.

Comment: The study of Calculus problems, that involve analyzing variation phenomena, emphasizes and focuses mainly on constructing and dealing with an algebraic model. With the use of a DGS is possible to generate the locus that describes the variation phenomenon without making explicit the algebraic model. The idea is to relate the variation of one element of the dynamic model with the variation of the phenomenon in study. In this process, issues regarding the domain to move elements within the model and the analysis and interpretation of what is generated (loci properties) become crucial to make sense of relationships and mathematical results or solutions. This method of visualizing the behavior of an object attribute relation is also shown in Figs. 4.2, 4.7a, b and 4.13a, b.

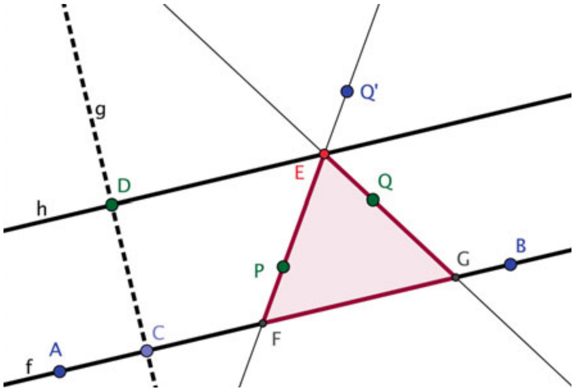


Fig. 4.18 Constructing a dynamic configuration to pose questions

4.4.4 The Construction of a Dynamic Configuration and Problem Posing Activities

In these tasks, the idea is to construct a dynamic configuration based on putting together some simple mathematical objects such as points, lines, triangles, etc. Then, the controlled movement of specific elements, within the configuration, becomes susceptible of being explored and analyzed in terms of properties and mathematical resources. As a result, some questions or conjectures regarding the behavior of some objects attributes emerge and the goal is to look for arguments to support and validate those conjectures or mathematical relations. Figure 4.18 shows a configuration that includes a line AB, a point C on line AB and the perpendicular *g* to AB that passes through C. Point D is any point on the perpendicular g and line *h* is the parallel to line AB that passes through point D. Point P and Q are any points on the plane and Q′ is the symmetric or reflected point of Q with respect to line h. Line Q′P intersects lines h and AB at E and F respectively and line EQ intersect line AB at G. Based on this initial configuration, some questions might be posed: What type of triangles are formed when points P or Q are moved?

The goal is to explore properties and invariants of embedded objects when some elements are moved within the model. For example, since lines h and f are parallel, then angles GFQ′ and Q′EH are congruent, similarly, angles HEQ and EGF are congruent and for symmetry properties angles Q′EH and HEQ are also congruent; therefore, the family of triangles EFG is always isosceles. What is the locus of point E when point D is moved along line g? Figure 4.19 shows that the locus seems to be a hyperbola.

Another variant of this type of task involves using the tool to get information regarding the objects’ attributes embedded in the task. For example, a dynamic representation of a task that involves determining the area of the triangle with vertices at the orthocentre, the circumcentre, and the centroid of a given triangle ABC leads to conclude that these three points are collinear and therefore, such area is always zero. Figure 4.20 shows a dynamic representation where point O, P, and Q are the

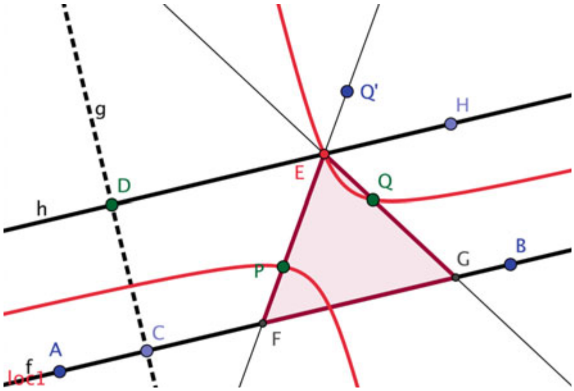


Fig. 4.19 What is the locus of point Q when point D is moved along line g?

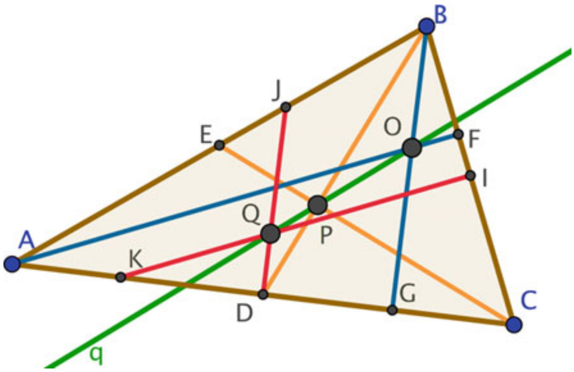


Fig. 4.20 The orthocentre, circumcentre and centroid are collinear

orthocentre, the centroid, and the circumcentre of triangle ABC. It is observed that O, P and Q are collinear. To prove that points O, P, and Q are collinear it is sufficient to show that $d(O, P) + d(P, Q) = d(O, Q)$.

Comment: In this type of tasks there is no initial problem or question to solve, the goal is to assemble a dynamic configuration in which the movement of some of its elements will lead the student to observe invariance or relation among some objects' attributes. In this process, students have an opportunity to engage in problem posing activities that involve the formulation of conjectures and to look for different arguments to validate them. Similarly, with the tool's affordances, learners can identify patterns and properties of objects' attributes and to explore the pertinence or conditions to define and represent the objects. In this case, the collinearity property of points O, P and Q leads to conclude there is a degenerated triangle or segment with area zero.

4.5 Reflection and Closing Remarks

In the last ten years, the use of technologies has been transforming not only how people communicate and interact in both their daily life and professional environments; but individuals also rely on tools and digital developments to access and use online information. Recently, there have been several publications (Gros et al., 2016; Hokanson & Gibbons, 2014; Liljedahl et al., 2016; Singer, Ellerton, & Cai, 2015) that address the need and importance of analyzing what the use of technologies brings to both the subject content learning and the structure and dynamics of learning scenarios. Walling (2014) presents and discusses an instructional design model called ADDIE (Analyze, Design, Develop, Implement, and Evaluate) to incorporate tablets (iPads) in learning environment. Throughout this chapter, it is argued that mathematical tasks are the vehicle and a key ingredient to identify, discuss, and analyze what representations, explorations and ways of reasoning emerge in approaches that involve the systematic and coordinated use of digital technologies.

Grouping mathematical tasks in terms of identifying questions and strategies that problem solvers can explore during their interaction with the tasks could help teachers focus on ways to rely on technology affordances to foster mathematical thinking. In the first group, a question (how can we inscribe an equilateral triangle in a given circle?) becomes important to think of concepts and relationships needed to reconstruct the figure and explore and analyze different solutions. How can I draw the figure? and in which order should the elements or involved objects be drawn? These generic questions are important to identify concepts representation and relations to reconstruct the figure. In addition, the process of reconstructing a figure might lead the problem solver to engage in problem posing activities that include looking for different ways to draw the figure.

The second group (investigation tasks) refers to the process of transforming routine or textbook tasks in a series of activities that can foster students' problem solving experiences. To this end, the initial analysis of the task involves identifying key concepts that can be represented dynamically. In this process, it is possible to analyze how embedded objects in the dynamic model behave and use this information to construct a robust model. The robustness of the model implies analyzing the domain of movable points to always generate a consistent representation. For instance, the animated model (Fig. 4.9) was adjusted to leave out non-convex polygons (Fig. 4.21). The robust model is examined to detect patterns or invariants associated with the objects' attributes behaviors. Then, it is important to discuss the extent to which the construction of a robust model can be applied to explore what happens to the generated polygonal regions that are formed when considering other regular polygons (pentagon, hexagon, etc.).

The third group focuses on ways to represent and explore tasks that involve analyzing variation phenomena. The tool' affordances offer a possibility of generating a graphic representation of the variation phenomenon parameters without making explicit an algebraic model. In general terms, the main idea is to construct a point that represents a relationship between two parameters, one that describes the position

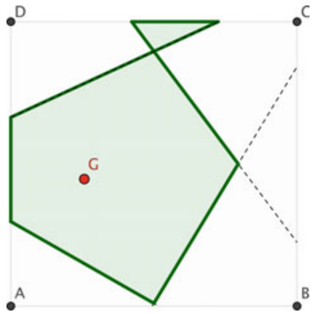


Fig. 4.21 Implementing conditions to always generate a concave polygon

of movable point (independent variable) and other that represents the variation of the attribute associated with the variation phenomenon. For example, point M (Fig. 4.17) represent a relationship between the length of side AD (x-coordinate) and the area of rectangle ABCD (y-coordinate). Thus, using the locus command is possible to generate the graphic variation of the phenomenon (area in this case) that can be analyzed to explore increasing/decreasing intervals, optimization points, and other locus' properties. This approach is important for students to focus on interpreting involved concepts and later understand meaning and properties of the corresponding algebraic model.

The fourth group emphasizes the importance of using GeoGebra's affordances in problem posing activities. The goal is to rely on simple mathematical objects such as points, segments, perpendicular bisectors, circles, tangents, triangles, rectangles, etc. to construct a dynamic configuration and to move some objects within the configuration to observe and analyze the mathematical behaviors of attributes and properties associated with those objects. What is invariant? What does it change? Is there any pattern or does the area of a certain family of polygon reaches a maximum value? etc. are questions that might lead the problem solver to identification of conjectures and look for ways to support or validate them.

In dealing with the tasks, the use of technology provides affordances for students to pay attention to activities that includes reconstructing and examining figures associated with problem statements, the construction of dynamic models of tasks, the formulation of conjectures, the quantification of objects or parameters behaviors, the search for mathematical arguments and the communication of results. In this context, learners have an opportunity to expand or enhance not only important problem solving heuristics (that include the construction of dynamic models, finding and examining loci of points or objects, using slicers, quantification of parameters, exploring simpler cases, or assuming the problem as solved, etc.) but also to construct and incorporate ways of reasoning associated with the use of the tool.

Finally, communication technologies provide affordances to extend mathematical discussions beyond formal settings (Santos-Trigo, Reyes-Martínez, & Aguilar Magallón, 2016). In this context, learners can focus their attention to how the use of GeoGebra expand and introduce new ways to represent, explore and find mathematical relations. Reyes-Martínez (2016) uses a digital wall in which students can share and exhibit their ideas with peers or experts and each participant, as a part of the group or learning community, can react, analyze, critique or extend other's ideas. As a result, students' initial ideas and contributions are constantly refined and they eventually recognize that learning mathematics and developing problem solving competencies is a constant process that involves both individual and group participation. In using technologies, an important goal is that learners rely transparently on technology affordances to work on representing and exploring mathematical tasks and in discussing with peers and others their mathematical ideas and problem solving approaches. As Weiser (1991, p. 94) points out "[t]he most profound technologies are those that disappear. They weave themselves into the fabric of everyday life until they are indistinguishable from it".

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Chapter 5
The Spreadsheet Affordances in Solving Complex Word Problems

Nélia Amado, Susana Carreira, and Sandra Nobre

5.1 Introduction

Solving word problems has long been considered a very important context for the use and development of students' algebra knowledge, from elementary to higher education. Problem solving has also been advocated as a rich learning context for engaging students in the learning of algebraic methods as well as in promoting algebraic reasoning (Blanton & Kaput, 2005; Kaput, 1999; NCTM, 2000; Yerushalmy, 2006). However, research has revealed in several studies (e.g. Stacey & MacGregor, 2000) that many students, while knowing the formal methods of algebra avoid their use in solving word problems and choose numerical methods instead, that is, they take on arithmetic reasoning rather than algebraic reasoning. The value ascribed to the arithmetic methods that students use to find the solutions to problems that could be solved through the formulation of an equation or set of equations is still a subject of contention among educators and researchers. For some, the so-called informal (or non-algebraic) methods, as opposed to the formal algebraic methods (symbol use and techniques) that students generate for solving word problems, becomes a barrier to the learning of powerful methods for solving a large class of problems (Stacey & MacGregor, 2000). Others have claimed that algebraic thinking should not be reduced to the use of symbols and formal methods, suggesting that informal methods are relevant for making sense of problems and represent a means of reaching conceptual understanding of the algebraic methods (Johanning, 2004).

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In our research, we are privileging problem solving as an activity that can foster and anticipate the learning of algebraic methods by emphasizing its meaning and applicability. In addition, we want to know how the spreadsheet affordances can help students in dealing with complex word problems as a springboard for their learning of formal algebra methods. Our focus is on students' understanding of the algebraic formal methods supported by the use of the spreadsheet and its specific form of operation, which many authors consider useful in making the transition between arithmetic and algebraic language (e.g. Friedlander, 1998; Haspekian, 2005; Kieran & Yerushalmy, 2004; Nobre, 2016; Wilson, 2007). For example, the process of developing relational thinking through the construction of columns that dependent on other columns may encourage the understanding of algebraic conditions; likewise, solving an equation has its numerical counterpart in the action of inspecting the values of a table obtained from formulas that represent algebraic conditions. The same holds for the solution of simultaneous equations, except for the number of formulas and the type of functional relations that obviously get more complex.

In the existing research (e.g. Ainley, Bills, & Wilson, 2004; Calder, 2010; Dettori, Garuti, & Lemut, 2001; Rojano, 2002), the spreadsheet has shown to be an appropriate instrument to establish functional relations as well as a vehicle to promote pattern recognition and rule generalization, which may lead students to a deeper understanding of the algebraic language and methods. In our study, we intend to know more about the ways in which the affordances of the spreadsheet are significant to the success in solving problems entailing many algebraic conditions. One of the reasons for our interest is the perception that the range of spreadsheet affordances that are relevant to develop a sound understanding of the fundamentals of solving systems of simultaneous equations remains to be investigated. For example, we may note that solving systems of linear equations can be treated as a way of obtaining the coordinates of a point in which two or more functions intersect. Hence, it results the importance of bringing the spreadsheet to the dialogue between equations and functions as both are key concepts in school algebra.

The problems here labeled as 'complex word problems' are situations that require a clear identification of the multiple conditions and variables involved and of the ways the variables depend on each other; so being, these are not straightforward start-unknown problems.

We have collected data on how such problems were successfully solved by means of the spreadsheet by students who had not yet learned the algebraic methods of solving simultaneous linear equations. Taking into account the focus of our study, we set the following research question: in what ways are the affordances of the spreadsheet, particularly its representational possibilities, significant in fostering students' models of the complex structure of the problem situation?

5.2 Theoretical Background

5.2.1 Informal Versus Formal Methods in Problem Solving

Some word problems may be categorized as algebraic word problems, in the sense that they imply finding one or more unknowns and can be solved through its translation into algebraic equations and subsequent resolution by means of algebraic methods. If the student has already learned specific formal algebraic methods (such as solving equations and inequalities, or solving systems of equations, etc.) then much of the algebraic word problem is (immediately) solved and therefore it becomes a routine task. However, the research has shown that the success in solving algebraic word problems does not rely exclusively on the mastery of formal methods, especially if such methods were learned without conceptual understanding.

The formal methods of algebra are undoubtedly powerful and effective in solving various problems, leading students directly to the solution and freeing them from pursuing alternative strategies. However, moving from the informal to the formal methods is not easy for most students. If little time is spent in the use of informal strategies and if procedures are rapidly imposed and routinely performed, then students are more likely to make mistakes, which they will not be able to identify and/or correct (Wagner, 1983). In addition, students who typically perform well in formal procedures often reveal a limited understanding of their meaning and are unable to deal with problem situations other than the standard ones.

Kieran (2006), alongside the results of other researchers, acknowledges that students often prefer to resort to arithmetic methods and reveal difficulty in using algebraic equations when solving algebraic word problems. Although, at first glance, arithmetical thinking may seem to be an obstacle to the development of algebraic thinking, the fact is that it can also be taken as a valuable source of that development. In this sense, having problem solving as a learning context to engage students in the learning of algebraic methods appears as a legitimate option, mainly because word problems usually allow a variety of approaches and strategies, ranging from informal to algebraic.

Koedinger, Alibali, and Nathan (2008) have found that in an initial stage of algebra learning students perform better in solving word problems than in solving equations. This is because they can use their reasoning with quantities and numbers, drawing on their knowledge of arithmetic without the obligation to manipulate symbolic language. That form of reasoning, which is more independent of the use of algebraic symbolism, can be seen as an opportunity for the emergence of multiple representations to work on the problem structure and can contribute to a better subsequent learning of algebraic methods.

Hiebert and Carpenter (1992) also argue that the use of symbolic representations should not be forced when students are at an early stage of solving a certain type of algebraic word problems. From their point of view, the work with symbols and algebraic procedures may naturally follow from experience in solving problems. Otherwise there is the risk that students develop an incomplete understanding of algebra and algebraic methods. Although there are students who are more proficient with symbolic representations and procedural algebra and who are able to apply symbols and rules in solving a problem, the learning of any mathematical procedure must be supported by conceptual knowledge including an understanding of what symbols and rules mean.

5.2.2 Problem Solving in the Development of Algebraic Thinking

Algebra involves a specific form of thinking that goes beyond the simple manipulation of symbols. Thus, working with symbols must be enriched so that the study of algebra is not reduced to rote learning. Problem solving is a context for facilitating the assigning of meaning to symbols and to the work with symbols. Several researchers have highlighted the role of problem solving in the development of algebraic thinking (e.g. Bell, 1996; Mason, 2008). Among other advantages, it has been suggested: the possibility of leading the students to mentally work on one or more as-yet-unknown quantities and to focus on the relations between the mathematical objects rather than on the objects themselves (Windsor, 2010).

We may consider that algebraic thinking is a developing form of reasoning, which instead of requiring a cut with arithmetic thinking, entails a progress through different stages. In a first phase, students learn to describe relations in natural language and begin to deal with generalization. Later, they will be encouraged to use diagrams, abbreviations, and symbolic notations to express their reasoning. Finally, they will begin to use algebraic expressions, such as equations, together with tables of values, graphs, and other formal representations.

Non-routine problems that present real challenges to the students (rather than a task for the application of a previously learned method) can be seen as opportunities for the construction of new algebraic knowledge. The problem solving activity allows the emergence of several strategies that are born from the knowledge they already have but it may also be an opportunity to give meaning to the subsequent formalization of the initial processes (Rojano, 1996, Slavitt, 1999). As suggested by Kieran (1996) and others, algebraic thinking has its roots in the establishing of relations between quantities and it progresses as different tools, and not only the symbols, are introduced as a form of structuring a discourse that is inherent to algebra. As such, thinking algebraically involves knowing various forms of representation, namely symbolic. But, it also implies flexibility in the transition between modes of representation, as well as the ability to operate with symbols, in a given context and when appropriate (Schoenfeld, 2008). From focusing on concrete objects, to the relations between them and to the ways of representing them, algebraic thinking evolves towards reasoning about those relations in a general and abstract way.

Recognizing the givens, the unknowns, and the conditions that make the structure of a problem and representing them appropriately is a key step in using algebra for solving problems. As suggested by Dettori et al. (2001), the spreadsheet can be helpful for algebra learning in that it can support students' understanding of what means to solve an equation or a system of equations, even before the formal learning of those methods.

5.2.3 The Spreadsheet in Algebraic Problem Solving

Using the spreadsheet allows students to explore and obtain a solution to an algebraic word problem in an informal way. This environment emphasizes the need to identify all the relevant variables and, in addition, stimulates the search for functional relations between variables. Translating the conditions of the problem into variable-columns under appropriate labels enables a tabular representation, which gives a suitable and clear image of the variables and functional relations involved (Ainley et al., 2004; Calder, 2010; Dettori et al., 2001).

Solving problems with the spreadsheet favors the establishment of a connection between the language of formulas, which is distinctive of this digital environment, and the symbolic algebraic language with pencil and paper. The use of the spreadsheet is also a means to bridge the gap between informal algebraic thinking and the ability to use algebraic notation to express such thinking, as highlighted in the study by Carreira, Jones, Amado, Jacinto, and Nobre (2016).

As emphasized by several authors (e.g. Ainley et al., 2004), the spreadsheet is a powerful tool in mathematical problem solving and particularly in algebraic problem solving. This digital tool allows using and combining different types of representations, such as words, numbers and formulas, and the creation of tables and graphs, besides the insertion of objects produced with other tools, namely an image editor. One feature that distinguishes the spreadsheet from other digital environments is the fact that it supports the connection between different registers (numeric, symbolic and graphical).

When handling a spreadsheet, students have the opportunity to discover and understand the meaning of a cell, a column, and a formula, what it means to drag down the handle of a cell with a formula, as they automatically receive numerical feedback from the computer. According to Haspekian (2005), "communicating with a spreadsheet requires that pupils use an interactive algebra-like language, which focuses their attention on a rigorous syntax. This is why it is said that spreadsheets help to translate a problem by means of an algebraic code" (p. 113).

We claim that in the case of algebraic word problems the spreadsheet can help students find and express relationships between the givens and the unknowns in a given problem. In addition, it provides forms of control based on instantaneous and constant numerical feedback, which encourages exploration and prediction (Carreira et al., 2016; Nobre, 2016).

In solving and expressing the solution of a mathematical problem, the ability to record and organize information, the clarity in the expression of ideas and the production of solid explanations are important abilities. The use of different representations is fundamental in the expression of mathematical thinking within problem solving. However, any representation may be transparent or opaque. This distinction, made some time ago by Lesh, Behr, and Post (1987), means that a representation may be more or less obviously attached to the idea that it is meant to stand for, as it tends to underline a few aspects of the idea while fading others. Zazkis and Liljedahl (2004) further developed the transparency/opacity of the representations in suggesting that there is a certain degree of opacity in any mathematical representation. In the case of the representational register of the spreadsheet, the apparent opacity tends to dissipate as students gain familiarity with the specific syntax of the tool and greater flexibility in making a connection between their algebraic thinking and their actions with the spreadsheet.

Moreno-Armella, Hegedus and Kaput (2008) have put forward the idea of co-action to explain and describe the changes that the use of digital technology brings into students' mathematical activity. The idea of co-action is related to the fact that students are at the same time guiding and being guided by the dynamic and interactive digital environments. In solving a problem with the spreadsheet, the co-action between the student and the tool begins with the need for structuring the conditions of the problem in columns or cells that are assigned particular roles. This procedure allows connecting a set of numbers (in a column, for example) with a single label (or column heading), which is consistent with an idea of variable, and that action pushes students' reflection on the conditions involved in complex algebraic word problems and helps them to understand the mathematical meaning of variable and function (Wilson, 2007). The introduction of numerical data in different cells, which may include the use of formulas, becomes part of establishing the relationships described in a problem situation. In addition, students can analyze the immediate numerical feedback provided by the spreadsheet and redirect their actions in a permanent flow of interactions with the computer. This work, based on the identification and implementation of functional relations, induces an algebraic organization in the way of addressing a problem that apparently has a numerical form (Haspekian, 2005). Students are then able to inspect their numerical tables in search for the solution to the given problem.

The affordances of a digital medium are related to the opportunities that the environment offers to the learning process (Gibson, 1986). We might consider them as perceived opportunities in line with the intentions of the user. This indicates a feature of complementarity between the learner and the environment. Therefore, in the activity of problem solving with the spreadsheet, a very important aspect is the knowledge that the student has of the tool.

Previous studies about the use of the spreadsheet in the learning of mathematics (e.g. Ainley et al., 2004; Calder, 2010; Haspekian, 2005) show that this digital tool stimulates students' mathematical reasoning. In dealing with a word problem, the students can make various experiences in a short time. This will enable a stronger focus on the underlying mathematical ideas rather than on the routine mathematical calculations. One of the advantages offered by the spreadsheet is the possibility, in a quick way and whenever necessary, of visualizing the representation of a relation between cells. This supports students in the recognition, understanding, and expression of algebraic relationships they have formulated through the spreadsheet syntax. Students use spreadsheet-specific calculations in building general rules and often check their general rule with reference to numbers. In this way, links between algebraic symbols and general expressions are established. This idea is corroborated by the work of Abramovich (1998), which shows that the use of the spreadsheet supports the transition from computations to algebraic formal language.

The immediate feedback that students receive from the tool gives them freedom to explore different trials and encourages them to make conjectures. This permanent reflection about the results obtained leads students into new conjectures and new questions. We argue that such distinctive processes carried out by the students with the spreadsheet help them to refine their strategies, broaden their knowledge about the variables and about the relationships involved, therefore influencing the nature of their conceptual understanding of the problem structure.

In this chapter, our aim is to look at how the affordances of the spreadsheet allow the students to use different conceptual models that correspond to several ways of formulating and solving systems of simultaneous equations.

5.3 Context and Method

In the following, we describe and analyze how different middle school students express their mathematical thinking when solving a complex algebraic word problem (Fig. 5.1) with the use of the spreadsheet. We will focus on their digital representations in solving the problem by considering their spreadsheet-based models in relation to their algebraic thinking and to their problem-driven algebraic models.

The empirical context was a class of 8th grade students (13–14 years old), in a Portuguese middle-school located in the southern region of the country where the economic activity is strongly rooted in the primary sector (agriculture and fisheries). The class had a total of twenty-four students, 10 boys and 14 girls, two of whom were migrant children from Ukraine. During the school year, problem solving was implemented as the context for mathematical activity, aiming for the development of

The restaurant *Sombrero Style* was opened yesterday and I went there for dinner with three friends. The maximum capacity of customers – said the manager – is 100 people. Luckily, I had booked a table for four as when I got there several tables were already full with four people and one table with three people. While I was waiting for the waiter to take us to the table, I counted the women and men who were in the restaurant and the number of women was exactly twice the number of men.
What could be the maximum number of people who were already at the restaurant when I came in?

Fig. 5.1 The opening of the Restaurant "Sombrero Style"

algebraic thinking and the learning of formal algebraic methods. In each problem-solving lesson, the students were given the freedom to choose whether they wanted to work in groups or individually as part of the established didactical contract. Most of the students worked in pairs and only very few worked individually. In the class, the teacher regularly engaged in dialog with the students and asked questions whenever necessary to monitor, support or challenge students' reasoning and approaches. The students had previously gained some experience in solving word problems with a spreadsheet, in their mathematics classes, from which they acquired the basics of the spreadsheet functioning. Many of the problems given to the students were chosen among the problems proposed in the web-based competition SUB14 promoted by the University of Algarve. It is a mathematical problem solving competition of inclusive nature, addressing 7th and 8th graders and running through the Internet, which launches a new mathematical problem every two weeks at the competition website. Some of those problems were solved by the students of the class during class time and after that each of them had the choice of sending their answers to the competition through e-mail, if they wished to.

In the classroom, the detailed recording of the students' processes in the computer was achieved with the use of the software Camtasia Studio. This software allows the simultaneous recording of the students' dialogues and of the computer screen, therefore capturing all the actions performed on the computer.

All the solutions developed by the students were examined on the basis of (i) the organization of the tables created in the spreadsheet, and (ii) the choice of independent variables and consequent decision on dependent variables. This screening of the solutions led to the identification of three different types. They are discernible by the kind of strategy adopted in organizing the spreadsheet and the corresponding way of modelling the simultaneous conditions that were given in the problem.

The analysis of the data was developed in two phases. In the first phase, the models and the representations developed by the students in the spreadsheet to obtain the solution were identified. At this stage two main approaches could be distinguished: (i) one in which the students constructed two independent tables (each one translating a set of conditions of the given problem) and which were later compared for the search of the intersection point (value that was a simultaneous solution of the conditions translated in each table); (ii) the other in which the students constructed only one table where all the conditions mentioned in the problem were contemplated and where the search for the simultaneous solution was made in a row of that table. In a second phase, students' models were analyzed, considering, in particular, their choices for an independent variable, which made the remaining unknowns depending on that one, through formulas that represented given conditions. With this analysis, it was possible to create schematic models of their interpretations of the given situation. Finally, those models were algebraically expressed as systems of simultaneous equations, with the aim of detecting the different alternatives involved in the students' approaches from the point of view of the use of formal algebraic methods.

Here we will discuss three specific solutions that are, to a certain extent, representative of the variety of solutions that have emerged in the classroom.

w - number of women, m - number of men, y - total of persons,
x - number of tables of 4 persons

$$\begin{cases} w = 2m \\ y = w + m \\ y = 4x + 3 \\ 0 < y \leq 100 \end{cases} \quad (\max\ y)$$

Fig. 5.2 An algebraic formulation with a system of equations

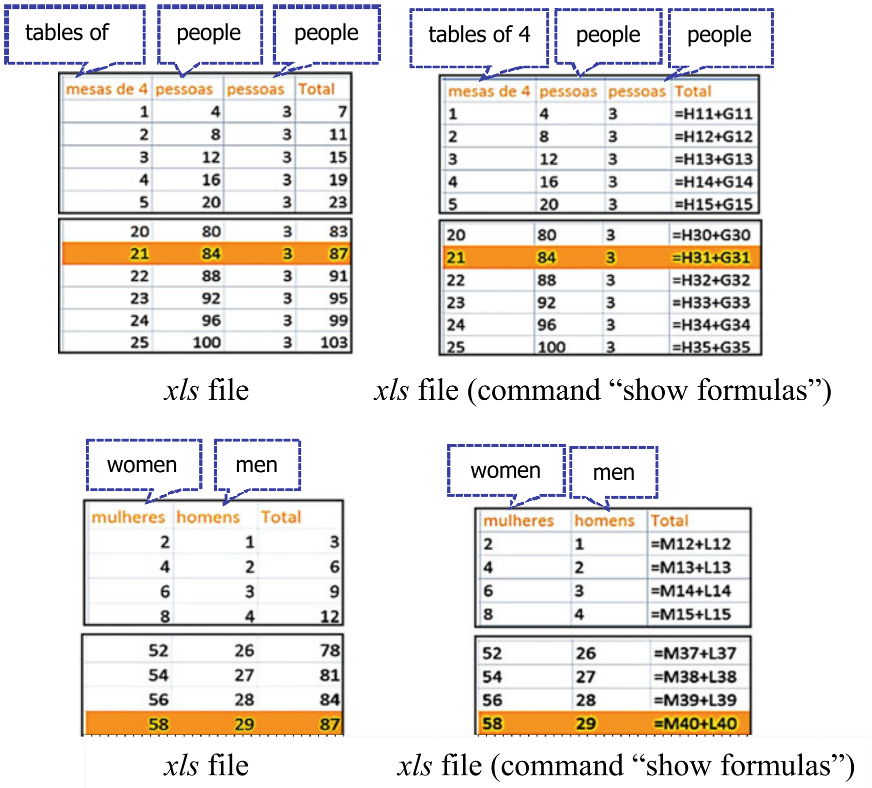
5.4 Results

As mentioned before, the students were not yet aware of the formal methods of solving systems of simultaneous equations. Besides, in middle school, students do not work with more than two equations and two unknowns. As such, from a symbolic algebra perspective, the given problem was beyond the students' knowledge of formal algebraic methods in terms of solving a system of three equations with three unknowns, as the following (Fig. 5.2).

Without the formal algebra method, students began to use the spreadsheet as a means for structuring the problem conditions in successive interconnected columns. They went through the translation, in numbers or formulas, of the relations between variables, thus obtaining numerical tables. Finally, the students controlled the data produced by searching the solution that satisfied the conditions imposed as those were displayed in the spreadsheet. To a certain extent, the representations provided by the spreadsheet were also a means of verifying the solution to the problem.

5.4.1 Solution 1

A pair of students, Maria and Jessica, started by addressing the condition of the problem on the number of people seated at tables of 4 (Fig. 5.3). In their model, the number of tables of 4 people is treated as a variable that changes within the set of whole numbers (represented as variable-column). Moreover, in the students' model of the situation it plays the role of an independent variable, in the sense that other variables are dependent on this one, such as the number of people sitting at the tables (therefore the multiples of four). That second set of values was generated in a second column using the Autofill to produce a linear sequence increasing by 4. Then Jessica entered a constant-valued column filled with the constant 3, which referred to the three people that were sat at a particular table of the restaurant. This third column mostly plays the role of a numerical parameter in the sense that it is a fixed value regardless of the number of tables in the restaurant or the number of people sitting in groups of four. Finally, the students created a fourth column to compute the total



mesas de 4	pessoas	pessoas	Total
1	4	3	7
2	8	3	11
3	12	3	15
4	16	3	19
5	20	3	23
20	80	3	83
21	84	3	87
22	88	3	91
23	92	3	95
24	96	3	99
25	100	3	103

mesas de 4	pessoas	pessoas	Total
1	4	3	=H11+G11
2	8	3	=H12+G12
3	12	3	=H13+G13
4	16	3	=H14+G14
5	20	3	=H15+G15
20	80	3	=H30+G30
21	84	3	=H31+G31
22	88	3	=H32+G32
23	92	3	=H33+G33
24	96	3	=H34+G34
25	100	3	=H35+G35

mulheres	homens	Total
2	1	3
4	2	6
6	3	9
8	4	12
52	26	78
54	27	81
56	28	84
58	29	87

mulheres	homens	Total
2	1	=M12+L12
4	2	=M13+L13
6	3	=M14+L14
8	4	=M15+L15
52	26	=M37+L37
54	27	=M38+L38
56	28	=M39+L39
58	29	=M40+L40

Fig. 5.3 Maria and Jessica's spreadsheet model

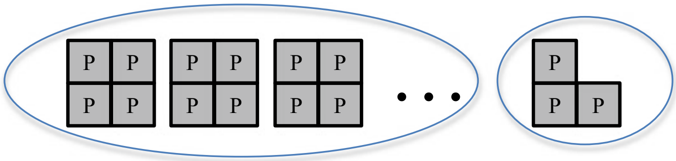


Fig. 5.4 First part of the model—People organized by tables

number of people sitting at the restaurant by adding the variable "people in groups of 4" and the parameter "3 more persons".

The following diagram (Fig. 5.4) illustrates the reasoning undertaken by the students in constructing the first table on the spreadsheet (the total of people is represented by the multiples of 4 plus 3).

Then, the students represented, in a separate spreadsheet table, the condition for the ratio of men and women in the total of clients, by creating new variable-columns (number of men, women, and total of individuals). They realized that the number

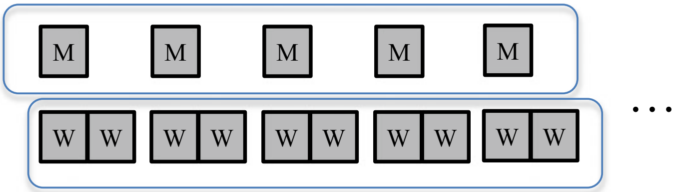


Fig. 5.5 Second part of the model—People grouped by gender

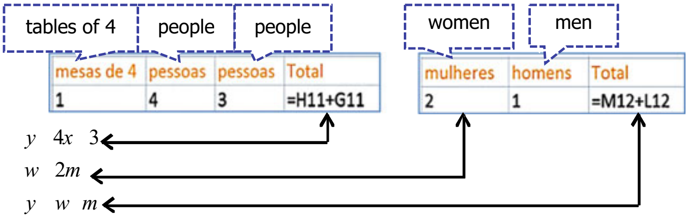
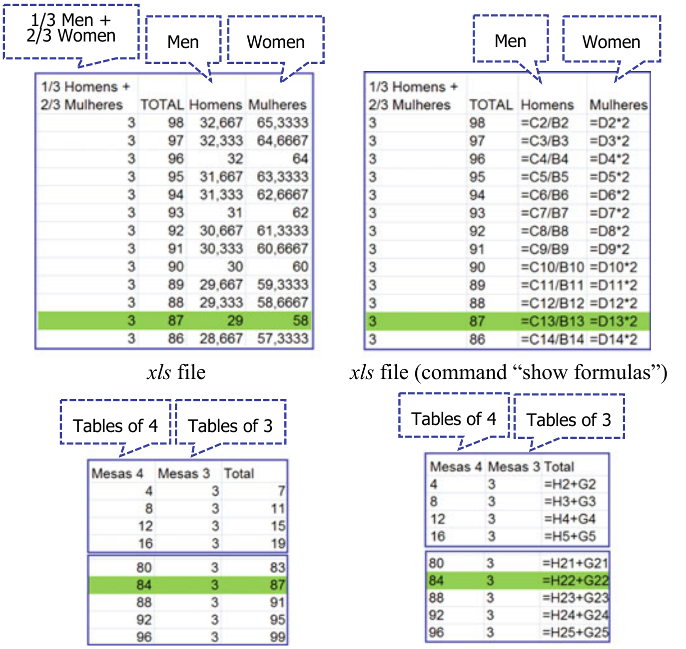


Fig. 5.6 The corresponding algebraic model (Maria and Jessica's model)

of women had to be an even number, as it was twice the number of men in the restaurant. This led them to the idea of creating a variable-column filled with the consecutive even numbers. A column for computing the number of men was then obtained by dividing the previous one by 2, which yielded the consecutive whole numbers. Finally, by adding the values of the two columns, a new column generated the total number of clients in the restaurant. The resulting total is obviously given by the multiples of 3. This line of reasoning is illustrated by the diagram of Fig. 5.5.

With this approach, the students developed a model based on the separation of two sets of conditions, each of them generating an output of values that would have to match at some point. The shaded rows (Fig. 5.5) show that the students carried out the inspection of the solution by comparing the columns of totals in the two separate tables (their answer is 87 people in the restaurant, not counting the remaining 4 people that booked a table). The solution of the problem, as their conceptual model highlights, is therefore the sum of a multiple of 12 with 3.

We may notice an interesting relationship between Maria and Jessica's model on the spreadsheet and the corresponding symbolic model (Fig. 5.6). The students initially separated the conditions in two unconnected tables (two separate equations) but later, by inspecting the results in the two columns of totals, they made the necessary connection (two simultaneous equations). As a result of the way they expressed the conditions on the spreadsheet, these students were also able to obtain additional information about the situation of the problem, namely the number of women and men already in the restaurant.



1/3 Homens + 2/3 Mulheres	TOTAL	Homens	Mulheres
3	98	32,667	65,3333
3	97	32,333	64,6667
3	96	32	64
3	95	31,667	63,3333
3	94	31,333	62,6667
3	93	31	62
3	92	30,667	61,3333
3	91	30,333	60,6667
3	90	30	60
3	89	29,667	59,3333
3	88	29,333	58,6667
3	87	29	58
3	86	28,667	57,3333

xls file

1/3 Homens + 2/3 Mulheres	TOTAL	Homens	Mulheres
3	98	=C2/B2	=D2*2
3	97	=C3/B3	=D3*2
3	96	=C4/B4	=D4*2
3	95	=C5/B5	=D5*2
3	94	=C6/B6	=D6*2
3	93	=C7/B7	=D7*2
3	92	=C8/B8	=D8*2
3	91	=C9/B9	=D9*2
3	90	=C10/B10	=D10*2
3	89	=C11/B11	=D11*2
3	88	=C12/B12	=D12*2
3	87	=C13/B13	=D13*2
3	86	=C14/B14	=D14*2

xls file (command "show formulas")

Mesas 4	Mesas 3	Total
4	3	7
8	3	11
12	3	15
16	3	19
80	3	83
84	3	87
88	3	91
92	3	95
96	3	99

xls file

Mesas 4	Mesas 3	Total
4	3	=H2+G2
8	3	=H3+G3
12	3	=H4+G4
16	3	=H5+G5
80	3	=H21+G21
84	3	=H22+G22
88	3	=H23+G23
92	3	=H24+G24
96	3	=H25+G25

xls file (command "show formulas")

Fig. 5.7 Excerpt of Carolina's solution in the spreadsheet

5.4.2 Solution 2

Carolina is another student who also organized the conditions by separating them in two distinct tables, as shown in Fig. 5.7.

In her resolution, the student used simultaneous increasing and descending sequences. She starts by separating the customers by gender. In a column, she generates a sequence of whole numbers accounting for the total number of people in the restaurant. By starting with 98, she takes into account some of the problem givens although she does not consider the fact that one table for four persons would be still available. In the next column, she calculates the division of the totals by 3, in order to get one-third of the totals (the number of men). In another column, she calculates twice the previous results (the number of women). The diagram of Fig. 5.8 can illustrate this first part of the model used by Carolina.

In the last three columns, she makes the distribution of customers by tables of 3 and of 4. She used increasing sequences where the successive multiples of 4 represent the varying number of people in tables of 4 and since there was only one table with 3 people, the number 3 is repeated along another column. Then she adds the values in the previous two columns, which yields a column for the total of people. This

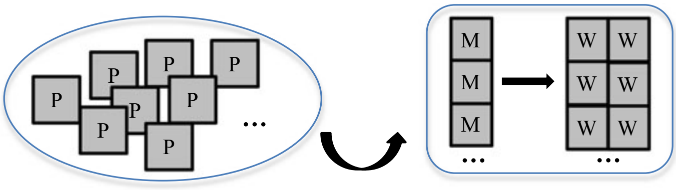


Fig. 5.8 First part of the Carolina's model—People organized by gender

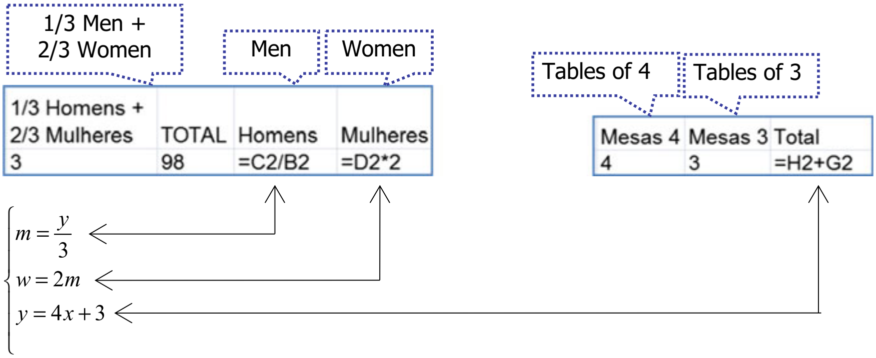


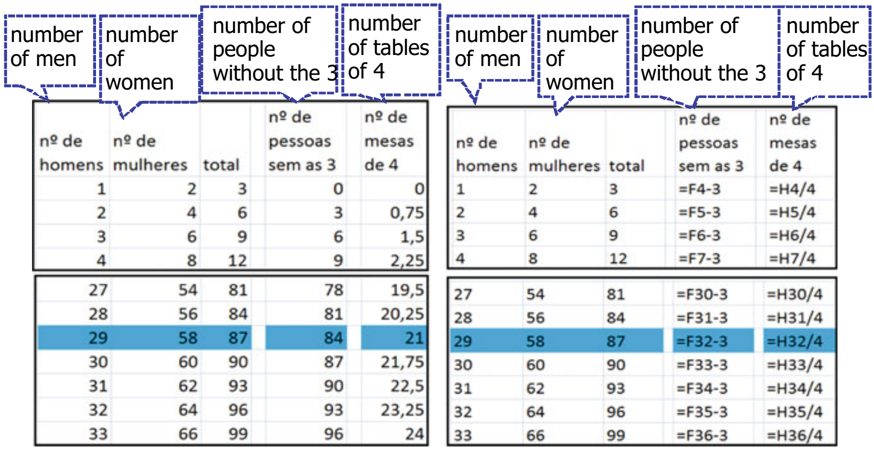
Fig. 5.9 Translation into algebraic language of Carolina's model on the spreadsheet

second part of Carolina's model is similar to the first one used by Maria and Jessica (Fig. 5.4).

To get the solution, Carolina compares the two columns with the totals and she finds the same number appearing in both columns (the shaded rows), which gives her the number of people in the restaurant.

In solving the problem, the student uses the idea of proportion to 'separate' the customers by gender, as mentioned in her answer: "Since the number of women is exactly twice the number of men, it can be said that the total of persons is represented as 3-thirds, being one third of men, and two thirds of women". She also uses the notion of multiples of four to define the number of people sitting at tables of four and a column with the number 3 to act as a constant standing for the three people seating in one table.

It is apparent that using the spreadsheet pushed her to identify all the relevant variables and constants and encouraged the search for dependency relations. In addition, it led to a strategy that allowed addressing the two conditions involved in the problem separately, and later making their connection by finding equal outputs in the two independent tables created. Her reasoning may be translated into algebraic symbolic language through the system of equations presented in Fig. 5.9.



nº de homens	nº de mulheres	total	nº de pessoas sem as 3	nº de mesas de 4
1	2	3	0	0
2	4	6	3	0,75
3	6	9	6	1,5
4	8	12	9	2,25
27	54	81	78	19,5
28	56	84	81	20,25
29	58	87	84	21
30	60	90	87	21,75
31	62	93	90	22,5
32	64	96	93	23,25
33	66	99	96	24

xls file

nº de homens	nº de mulheres	total	nº de pessoas sem as 3	nº de mesas de 4
1	2	3	=F4-3	=H4/4
2	4	6	=F5-3	=H5/4
3	6	9	=F6-3	=H6/4
4	8	12	=F7-3	=H7/4
27	54	81	=F30-3	=H30/4
28	56	84	=F31-3	=H31/4
29	58	87	=F32-3	=H32/4
30	60	90	=F33-3	=H33/4
31	62	93	=F34-3	=H34/4
32	64	96	=F35-3	=H35/4
33	66	99	=F36-3	=H36/4

xls file (command “show formulas”)

Fig. 5.10 Ana’s spreadsheet model

5.4.3 Solution 3

The approach of another student in the class reveals a similar start but it progressed in a way that prevented the separation of the problem conditions in two tables. In fact, Ana started by representing the relationship between the number of men and the number of women and then obtained a column for the total of people (Fig. 5.10). The student soon concluded that the totals in that column were multiples of 3. Then she subtracted 3 to the total of persons to account for the fact that only one table had 3 individuals, and then divided the result by 4. Her idea was to have the remaining people arranged in groups of 4 due to the fact that they were all at tables of 4. Therefore, in a new cell, Ana entered the title “Number of tables of 4” and bellow she created a formula that made the division by four of the total number of people and then dragged the fill handle. This way, she was aiming to find the number of tables of 4 that were taken in the restaurant. To obtain the solution, she just had to inspect the values in that column in search for the whole numbers and for the highest number lower than 100. In her answer to the problem, Ana wrote: “The maximum number must be 87 before the 4 friends came in; if I had considered the number 99 and added the 4 friends I would get 103 as the total, but the capacity of the restaurant is 100 people, which means that it is not the solution”.

We may also describe Ana’s conceptual model schematically to better see the similarities and differences from the previous cases (Fig. 5.11). Although hers is also a model that involved two steps, it did not require two separate tables and instead she managed to combine the various problem conditions in a single table. In fact, Ana expressed the conditions on the spreadsheet by chaining them in successive compositions.

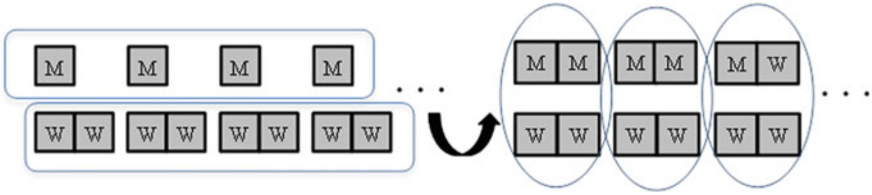


Fig. 5.11 People grouped by gender rearranged in groups of four

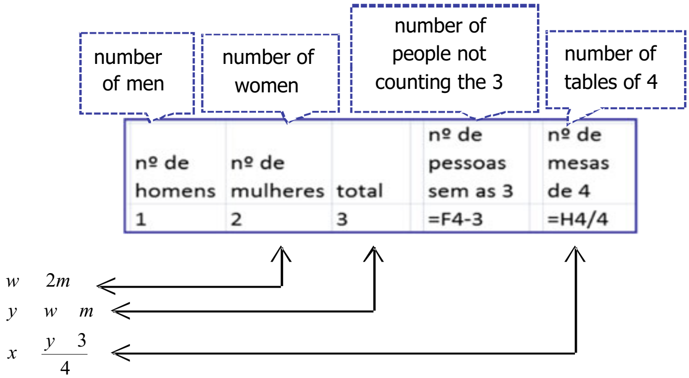


Fig. 5.12 Translation into algebraic language of Ana's model on the spreadsheet

Next, we propose a representation of Ana's spreadsheet-based model with the corresponding algebraic language (Fig. 5.12). The symbolic model shows that in this case the number of persons at the restaurant (minus 3) has to be a multiple of 3; moreover, the test to reach the solution means finding out the multiples of 3 that are also divisible by 4. Therefore, the solution belongs to the set of multiples of 12.

5.5 Discussion and Conclusion

Although the students had not yet learned the algebraic representations, namely a system of equations with several unknowns, the spreadsheet allowed them to undertake and explore other approaches, which reflected a variety of representations involving formulas and tables and consequently a diversity of conceptual models of the given problem.

The three solutions presented above indicate that the spreadsheet proved to be useful to solve the problem and that the students recognized it as an appropriate tool for solving this problem. They also took different approaches; some considered all the conditions of the problem jointly represented, while others used a strategy of separating two sets of conditions and then comparing the numerical values to find a common value to obtain the solution.

In general, the spreadsheet helped students dealing with a complex word algebraic problem that was beyond their mathematical knowledge of algebraic methods, namely of solving systems of simultaneous equations. One important aspect emerging from the set of solutions was the fact that students showed a clear notion of the solution as a value that would need to satisfy simultaneously a set of equations and conditions. In this sense, we may consider that students gave meaning to the concept of a solution of simultaneous equations, which is often a difficult idea in initial stages of the learning of the algebraic method.

By means of sophisticated numerical approaches based on defining numerical sequences that described functional relations, they were able to make use of variable-columns, using the specific syntax of the spreadsheet, and create chained relations between variables. We claim that those affordances of the spreadsheet were fundamental in structuring the students' problem solving approaches and in providing them a representational system to express the conditions given in the problem. This conclusion resonates with other studies such as Ainley et al. (2004) and it provides a clear indicator of how students interpreted the problems in light of their mathematical knowledge and their knowledge of the tool. For example, decisions such as the ways of defining variable-columns are intrinsically connected to the students' model on how some variables depend on other variables and to their choice of independent and dependent variables. Therefore, the different spreadsheet organizations appearing in a particular problem are a consequence, among other things, of conceptual choices and constitute a powerful mirror into students' successful ways of conceiving the structure of the problem situation. Such decisions can be seen as originating from the ongoing interaction between the subject and the tool, in a way that makes it impossible to separate the two. Evidences of such interdependence are quite clear in students' problem-solving activity, particularly in their ways of reporting the reasoning developed while working in the digital medium.

The work with the spreadsheet transforms the nature of students' mathematical representations to the extent that those become encapsulated in a medium with very specific characteristics. The solution of the problem solved with the spreadsheet arises from the student's ongoing interaction with the tool; both the student and the spreadsheet act and react to each other throughout the activity (Moreno-Armella & Hegedus, 2009). This type of work has significant consequences for the expression of students' mathematical thinking, particularly of algebraic thinking, during their problem solving-and-expressing (Carreira et al., 2016).

The three solutions here analyzed, in terms of their symbolic algebraic counterpart (the systems of equations that they mirror) highlight that the individual's interactions with the spreadsheet can lead to solutions with distinct algebraic features. Algebraic problems of some complex level seem to be especially useful for bringing out different conceptual models and simultaneously different forms of expression of these models based on the spreadsheet.

We suggest, following the results by Calder (2010), that different solutions may take place depending on the level of experience of the students with the spreadsheet, on their understandings of the ideas involved, and on their perceptions about the ways to model the conditions of the problem with the digital tool. The participants in this study took advantage of the spreadsheet in different ways, using different mathematical concepts and simultaneously different affordances of the digital tool.

Hegedus (2013) has underlined the idea that technological affordances must become mathematical affordances and argued that meaningful integration of technology in the learning environment should be developed through mathematization of technological affordances. Thus, he pointed out a set of future design principles (executable representations, co-action, navigation, manipulation and interaction, variance/invariance, mathematically meaningful shapes and attributes, magnetism, pulse/vibration, construction and aggregation) that must deserve further attention in the upcoming research and development efforts. The co-action, one of the characteristics in the list, is one that we find clearly important to the study of solving algebraic word problems with the use of a spreadsheet. Examples of co-action, as those presented in the solutions above, illustrate ways in which the spreadsheet affordances offer routes to obtain the solution to a problem.

Based on the data collected, we may state that the idea of transforming unknowns into variables and creating tables that translate functional relations is a mathematization of specific affordances of the spreadsheet. One of them is the use of formulas to create dependency relations between variable-columns. Another is the possibility of decomposing the set of conditions of a given problem into several tables representing pieces of the same problem. The comparison between those tables is one of the ways to obtain the solution, by searching for the value that verifies all the subcomponents simultaneously (subsystems of equations).

Finally, the many different possibilities of translating mathematical conditions into relations between variable-columns suggest that the spreadsheet favors the production of diverse conceptual models that are interesting and mathematically powerful. Such diverse models, when represented by means of the spreadsheet syntax, provide a rich image not only of the many ways of having a word problem translated into a set of equations but also of the many ways in which the solution of the simultaneous equations may be algebraically obtained, depending on the transformations and operations you perform in solving it.

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Part III
Inquiry and Problem Posing in Mathematics Education

Chapter 6
Is an Inquiry-Based Approach Possible at the Elementary School?

Magali Hersant and Christine Choquet

The value of problem solving to promote mathematical understanding and learning is recognized equally by mathematicians, teachers' trainers and teachers. However, in spite of this agreement to associate mathematical activity with problem solving, problem solving has had multiple and often contradictory meanings through the years (Schoenfeld, 1992, p. 337). This subject is regularly approached and questioned on international colloquiums such as ICME-13.

Santos-Trigo recognizes that "research in problem solving has generated interesting ideas and useful results to frame and discuss paths for students to develop mathematical knowledge and problem solving proficiency" (Santos-Trigo, 2013, p. 500). But he also notices that "it is not clear how teachers implement and assess their students' development of problem solving competencies" (ibid., 2013, p. 500). In this area, questions about *problem posing* especially emerge. As generation of new problems or reformulations of given problems (Silver, 1994) problem posing is epistemologically relevant for teaching and learning mathematics. Indeed, mathematicians, physicists and epistemologists like Hadamard, Einstein, Popper and Bachelard seem to agree that posing an interesting problem is more important than solving it. Following Singer, Ellerton, and Cai (2013), we can link problem posing experiences to "development of abilities, attitudes and creativity, and its interrelation with problem solving and studies on when and how problem solving sessions should take place" (Malaspina, 2016, p. 34). Likewise, it provides information about the ways to pose new problems and about the need for teachers to develop abilities to handle complex situations in problem solving contexts. So, problem posing seems to be an interesting topic to both study teaching and learning mathematics.

Inquiry is at the heart of problem posing (Singer et al., 2013). In Europe, over the last decade, the institutional willingness to promote Inquiry Based Learning (IBL) in

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mathematics revive interest on problem solving and posing to teach and learn mathematics at every level of education. But the conditions and constraints which might favor, or on the contrary hinder, implementation of IBL in mathematics and learning mathematics with IBL must still be specified. Dorier and Garcia (2013) considered that teachers play a central role in an institutional system and that attention should be paid to teachers' training, especially for primary school. They also mentioned the importance of the resources that should contained didactical comments.

In this context, we address the question of the possibilities of learning mathematics based on an inquiry approach at French elementary school, as far as most of elementary school teachers have a weak scientific background and therefore a weak experience on posing problems. To this end, we propose to study two well-contrasted case studies using IBL for mathematics learning in ordinary teaching context and in didactic engineering context and, within, describe and characterize some conditions of possibilities of learning mathematics with inquiry at French elementary school. Taking into account previous studies about inquiry-based learning in mathematics and its implementation in the classroom, in the European context (Sect. 6.1), we hypothesize that these conditions are both determined by the problem and by the activity of the teacher as he supports students' activity. To characterize the conditions on the problem we introduce the notion of *potential of inquiry*. To identify conditions attached to the teachers' practices we rely on the analyze of the students' activity with the *leaning by problematization* framework (Fabre & Orange, 1997). This theoretical framework is strongly anchored in science epistemology (Popper, 1972; Bachelard, 1970) and in inquiry (Dewey, 1938; Fabre, 2005), as we will explain it (Sect. 6.2). Within this framework posing problem is as important as solving it and problem posing is considered as a way to explore what conditions and possibilities for problems or situations to engage students in problem posing activities. Indeed, we can identify within this framework whether teachers' in-class activity allowed problem posing and solving for the students.

6.1 Inquiry-Based Learning: An Inquiry Processes That Is Difficult to Implementation in Classroom

The promotion of a teaching of mathematics by IBL appears as a world movement even if the epistemological outlines and the didactic stakes are to be specified. Attempts to implement IBL in mathematics are made and numerous research projects concerning the inquiry in sciences and in mathematics have been carried out during the last 20 years (Erh-Tsung, & Fou-Lai, 2013; Engeln, Euler, & Maaß, 2013; O'Shea & Leavy, 2013). Our review of literature shows that the definition of IBL is not stabilized in the field of the international didactics. However, IBL appears strongly connected with inquiry as we will bring it out. Furthermore, this survey also highlights difficulties to implement IBL in day-to-day teaching.

6.1.1 IBL and Inquiry

IBL is anchored at the same time in the investigation and in the construction of knowledge about reasoning in a critical way, in reference to the philosopher Dewey (1859–1952) (Linn, Davis, & De Bell, 2004; Rocard et al., 2007). This inquiry progresses through the interactions between unknown elements, that raise questions, and known elements that allow to analyze these unknown elements and to form hypotheses or still to connect some elements in already lived experimentations. Thus, an essential function of IBL is to organize the field of experimentation of the students and the development of attitudes of learning based on the practice of reflexive activities like inquiry (Dewey, 2011; Hétier, 2008). The term "inquiry-based learning" refers to student-centered ways of teaching by posing questions, exploring situations and developing their own ways towards solutions. It thus reaffirms the link between inquiry and problem posing (Maaß & Artigue, 2013).

In Europe, an institutional context has intended to promote IBL for teaching and learning mathematics (Rocard et al., 2007) and several European research projects have been conducted to help in the development of new practices of mathematics teaching. We can especially mention the Fibonacci project (2010–2013; led by the Ecole Normale Supérieure, France and the University of Bayreuth, Germany), the LEMA project (2008, 2010; 6 European countries) and the PRIMAS project (2010–2013: 14 universities, 12 European countries) both rooted in the Pedagogical University of Freiburg.

We notice that the cited above projects put forward the wealth of a work of modelling and then most of the time propose IBL from a modelling context. But we make assumption that some other kinds of problems can also lead to mathematical inquiry. In this paper, we illustrate this possibility with a discrete optimization problem (problem 2).

Outcomes of these projects include propositions of IBL situations that support the development of mathematics teaching practices and identification of difficulties of implementation for the teachers. This latter especially draws our attention for our work.

6.1.2 Difficulties of Implementation

Within PRIMAS project, which objective was the developing of devices of preservice and in-service teachers' training, Dorier and Garcia (2013) identified the conditions and constraints that might favor, or on the contrary, hinder a large-scale implementation of inquiry-based mathematics and science education.

In most countries, it seems that teachers find it difficult to choose statements and to implement in class activities based on inquiry (Dorier & Garcia, 2013; Schoenfeld & Kilpatrick, 2013). It also seems difficult to define and to distribute the responsibilities between students and the teacher in front of proposed tasks. The teachers do not feel at ease with sharing students' results to compare their productions, with organizing them into a hierarchy and with implementing mathematical debates (Inoue & Buczynski, 2011). These difficulties that are related to the process of institutionalization (Choquet, 2014) do not seem specific to the IBL. They are well-known about problem solving.

Having identified and explained these difficulties, research proposed improvements in order to promote and develop IBL in rights conditions leading to students' learning. First, types of resources (textbooks, websites) are different among countries and it seems important to propose to teachers' resources promoting IBL "accompanied by didactical comments on how it can be efficiently implemented in class and embedded into a device to be used for professional development" (Dorier & Garcia, 2013, p. 849). Second, studies show the central role played by the use of digital technology in problem solving and in particular in the IBL:

> "There is a need to develop or adjust current problem-solving frameworks […] to characterize the ways of reasoning, including the use of new heuristics, for example, dragging in dynamic representations, with which students construct learning in a result of using digital tools in problem-solving approaches" (Santos-Trigo, 2013, p. 500). And it also seems necessary "to develop methodological tools to observe, analyze, and evaluate group's problem-solving behaviors that involve the use of digital technology" (Ibid., p. 500). This use should not be reduced to the exploration of the problem to establish hypothesis but it has to be a part of all the resolution's process of the problem. (Artigue, 2012)

Third, even if curricula in all countries support IBL (Dorier & Garcia, 2013), the elementary teachers' mathematical and science competencies include a weak didactical qualification to implement an IBL approach in their class. That's why pre-service and in-service teachers' training might be increased especially on IBL.

In the French context, institutional aims greatly emphasized the fact that mathematics teaching should contribute to the development of students' inquiry competences. Low scores of French students to PISA problem solving samples, and disaffection of scientific programs at University are the two main reasons mentioned to support these injunctions. The mathematics curriculum of primary school (2016) affirms again the importance of developing problem solving students' competences, especially through the resolution of real-word problems. French elementary school teachers have to teach mathematics and sciences even if they are not specialist in these matters (Artigue, 2011). There exist strong links between IBL in sciences and in mathematics education, especially the fact that inquiry is the core of mathematical and scientific activities (Hersant & Orange-Ravachol, 2015). But, there also exist differences that make it difficulty to implement and require a specific teachers' training.

In the following part, taking into account research results presented here, we present the theoretical framework of learning by problematization (Fabre & Orange, 1997) that allows to envisage mathematical problems solving in terms of inquiry, which "can spread and produce solutions to [these] problems but also establish necessities to which they are subjected" (Hersant & Orange-Ravachol, 2015, p. 100).

6.2 Theoretical References and Research Design

We propose to identify conditions of possibilities for learning mathematics with IBL at the elementary school. To this end, we use the learning by problematization framework (Fabre & Orange, 1997) and introduce the notion of inquiry potential of a problem.

6.2.1 Learning by Problematization: A Theoretical Framework to Analyze Students' Mathematical Activity

We use this framework to analyze the students' activity. So, it is important to precise that posing and constructing problem is here seen from the students' point of view. Indeed, even if the teacher posed a question it does not mean that the problem is posed for the students and least of all that they construct it. But when students make attempts, formulate sub-problems or conjectures we can say that they at least pose the problem.

Learning by problematization is a theoretical framework developed by Fabre and Orange (1997) for the didactic of biology. It is yet well known and used in many didactics, especially in didactic of mathematics (Hersant, 2010; Grau, 2017). In this framework, the position and the construction of the problem have a more important place than its solution. This is connected with the importance of inquiring (Dewey, 1938), posing and constructing problems (Popper, 1972) in sciences.

Taking also into account Bachelard's epistemology, this framework considers that knowing is not "knowing that" but "knowing that it cannot be otherwise". Indeed, this framework makes a distinction between facts that come under opinions, and necessities that come under scientific constructions built into a scientific paradigm (Kuhn, 1962). Therefore, problematization is defined as a multidimensional process involving posing, building and solving problem in a dialectic of facts and of ideas (Orange, 2000). From an epistemological point of view, this above feature of the problematization process witch deals with scientific activity is also relevant for mathematics. Let's refer to mathematicians to explain this specific point. Regarding the multidimensional process, we can first refer to Poincaré (1905, 1970) who noticed the strong link between intuition as "an instrument of creation" (p. 37) and logical as an "instrument of proof" (p. 37). So, from his point of view, intuition plays a key role in posing and building problem, whereas logical, and especially deduction, mainly intervenes in solving problem. Moreover, this multidimensional process deals with an experimental dimension. Perrin who is a mathematician asserts this experimental dimension of the mathematics (2007) when he explains that "mathematics is also an experimental science". For Pòlya (1954, 1965) this experimental dimension underlies a similar dialectic of facts and ideas in biology and in mathematics. Indeed, for him (1965, pp. 110–111), "specific examples" (facts) suggests "new significations" (ideas) that lead to hypothesis and then proof (Pòlya, 1965, p. 111).

These distinctions between facts and ideas lead Orange to consider three structures of thinking summoned up during the search of a problem (Orange, 2000, 2005). The first one is the *empirical register* that corresponds to relevant facts for the problem, established during the search of the problem. The second one deals with the *register of necessities* that are established into an "epistemic structure" that Orange calls the *explicative register* (Orange, 2000, 2005). We shall explain these registers with an example. If the problem is to know if 46 is the sum of three consecutive numbers, 45 = 14 + 15 + 16 and 48 = 15 + 16 + 17 are relevant facts. Then relating and confronting these facts, in an "induction" and "more general statement" process (Pólya, 1954), make it possible to establish necessities: as 14 + 15 + 16 = 45 and 15 + 16 + 17 = 48, there is no other possibility to sum three consecutive numbers and to obtain 46; indeed 46 will never be the sum of three consecutive numbers. These latter propositions are not facts, nor opinions. But they are not only conclusions: they are built necessities. And building them we ensure that it cannot be otherwise. So, these elements come under the *register of necessities*. These necessities are established into a model. Indeed, to put up these necessities we consider arithmetic domain. But we could also envision the problem in a functional way (with a discreet function). Then the necessities will have to do with surjection function. In a way, this model matches with Piaget's "epistemic structure". Orange (2000, 2005) call it the *explicative register*. For Scientifics or mathematicians, excepted during paradigm shifts, models are well shared and known. But, for students who are in process of learning what sciences or mathematics are, these explicative models are in construction. And we have to take into account this in-process-building in our analysis of students' search of problem. For the previous problem, for example, at the end of the primary school many students think that 46 cannot be the sum of three consecutive numbers because, even if they try a lot, they do not find any such sum. Their model corresponds to "naïve empiricism" (Balacheff, 1987). It is not an acceptable model in mathematics, regarding to proof criteria but it explains the way they envision the solution of the problem that is in an empirical model (Hersant, 2010).

The space of constraints and necessities (Orange, 2000, 2005) is a way to represent the construction of the problem. It accounts for tensions between empirical facts and necessities into an epistemic structure that are realized by one student or a group of students. The pertinent facts and the tensions established by the students can be indeed inferred from their productions and the verbal interactions observed in class. Then these tensions are represented by linking facts and necessities (for examples, see Fig. 6.1). In these diagrams connections between the elements of the three registers are not represented by arrows but only by segments. Indeed they indicate no direction, nor logical or chronological links but mean putting in tense relations.

6.2.2 *Inquiry Potential of a Problem*

We suppose that conditions of learning by inquiry both depend on the way the teacher posed the problem—especially the problem's writing as the setting of the

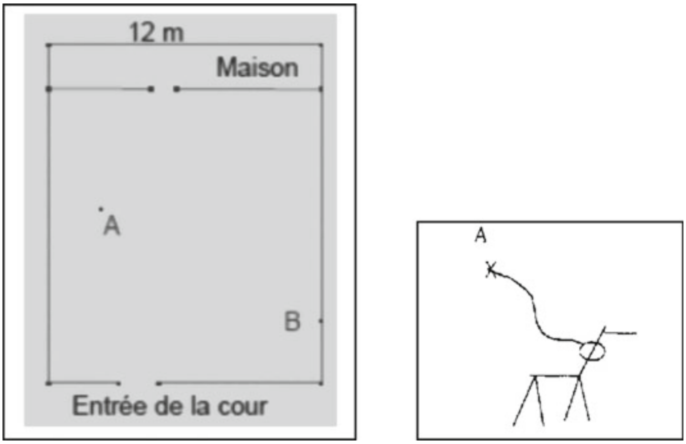


Fig. 6.1 Drawing at the scale given by the teacher for problem P0 (left) and for problem P1 (right)

search—and on the activity of the teacher as a help for the inquiry. Therefore, with respect to the first condition, in the first step of our study we look for the *inquiry potential* of the problem. This empowers us to estimate the possibilities, for the students, to pose and construct the problem from the question posed by the teacher. To define the inquiry potential of a problem we use the following questions:

(i) is the problem likely to engage students in a research activity for considerable time?
(ii) in particular, does the problem engage students in making attempts?
(iii) does the problem support the formulation of sub-problems? Of conjectures?

The first and second concern the possibility for the students to explore the problem and, therefore, to have great conditions to construct it. The third concerns the construction of the problem as the formulation of sub-problem helps it.

6.2.3 Research Design

We analyze the activity of students who are between 8 and 11 years old while they try to solve a mathematical problem. To this end, we use a corpus extracted from previous projects (Choquet, 2014; Hersant, 2010) and take at it a fresh look with new theoretical tools. The first case study deals with modelling. It focuses on the learning of the concept of circle—as the set of all points in a plane that are at a given distance from a fixed point—and of disc—as the set of points that are at a smaller than or equal distance to a fixed point. It corresponds to an ordinary classroom situation (Laborde, Perrin-Glorian, & Sierpinska 2005) as the researcher does not intervene in the choice of the problem nor in its management in the classroom.

From the teacher's point of view this situation corresponds to an investigation situation. The second case study is extracted from a didactic engineering. Its goal was to overcome a widespread misconception among the young students about the impossibility in mathematics: "it is impossible because I did not succeed in doing it". Hersant showed that discreet optimization problems are suitable to overcome this misconception and therefore proposed a set of didactical situations about discreet optimization. The design of these situations both relied on the theory of didactical situations in mathematics (Brousseau, 1997)—importance of a retroactive milieu and of the didactical contract—and on learning by problematization—importance of posing and constructing problem and of building necessities. These situations can also be considered as inquiry situations.

For these both cases, our analyze consists of two steps. The first one deals with doing a priori analyze and the second with a posteriori one. In the first step, we determine the inquiry potential of the problem and then, as a minimal investigation exists, we establish a priori space of constraints and necessities. That means an ideal space of constraints taking into account students' knowledge when they have to solve the problem. In the second step, we confront these results with the students' productions and the teacher's intervention. This brings elements on the role of the teacher in the inquiring process.

To study the possibilities to learn mathematics by inquiring at the elementary school, we will look into the way these students construct the problem, that means the pertinent facts they consider, the necessities they establish and the epistemic structure they summon up. We will represent this activity with a space of constraints and necessities. This space will help us to characterize their activity as a problematization or not. But it will also help us to understand the conditions that permit or not this problematization.

6.3 First Case Study: Modeling a Situation to Learn About Disc in an Ordinary Teaching Practice

Let's consider the two following problems. The first one is part of the French official instructions. The second one is part of a textbook, it's the problem that the teacher chose to submit to his students.

> Problem P0: Sophie has to fetch milk from the farm whose yard is shown below. In A and B are leashed two dogs. In A, Azor has a 6 m long leash; in B, Baltazar has a 5 m long leash. Can Sophie go to the door of the farm without being bitten?
>
> Problem P1: A pet is leashed to a post. The leash is 8 m long. Draw a picture of the area where the pet can move.

None of these problems has an immediate answer for a pupil who ignores the definition of a circle as the set of all points in a plane equidistant from a fixed point. For both of these problems, we can analyze as follows the inquiry potential.

6.3.1 Inquiry Potential of the Problem

Concerning the problem P0, the question allows students to easily propose an answer—that may no be the expected answer (i). Indeed, the wording of the question of P0 is non mathematical but the figure introduces the geometric framework as the posts of both dogs are represented by points. In an inquiry way, as Dewey describes it, they will make attempts (can Sophie go straight ahead? can she take this way?) (ii). Students can easily formulate a conjecture (it is possible or not) and then have to find a way to prove it (iii). Owing to the effect of the didactical contract (Brousseau, 1997; Hersant & Perrin-Glorian, 2005, Hersant, 2014), they will surely try to find one. As this task appears as a mathematical one, students know the teacher will not accept an unjustified answer and, therefore, they won't themselves accept it because in the mathematics classroom answers must be justified in accordance with the epistemological side of the didactical contract (Hersant, 2014). They can identify sub-problems to increase their understanding of the situation (when she is there, what happens?). Their tests will certainly lead them to conjecture that the border of the "unbitten" zone is made of circles (iii). So, this problem is likely to generate doubts and implication in the task to remove these doubts.

Concerning the problem P1, it is more difficult for students to have an idea of the expected answer (it is neither «yes because» nor «no, because») (i). Indeed, the answer matches with staked knowledge that students are supposed to ignore. This significant difference with the previous problem is due to the wording of the problem: the students have to «draw a picture of the area» and not to decide to the possibility of plowing a path. Moreover, counter to the previous problem, this one is not clearly a mathematical one. Indeed, even if the word "area" is used in the wording of the problem, the task and the draw accompanying this wording suggest the expected answer is not mathematical nor geometrical. We can suppose that the schema (Fig. 6.1, right) is given to help children to imagine the situation but it hinders the setup of a suitable didactical contract, especially a geometrical contract. Nevertheless, students can make attempts (the dog can be here, he cannot be there, etc.) (ii). They can identify sub-problems (can the dog go here?). But they probably will be satisfied with the identification of some places and will not seek further (iii).

6.3.2 A Priori *Space of Constraints and Necessities*

Figures 6.2 and 6.3 respectively picture the a priori spaces of constraints and necessities for P0 and P1.

For each problem, students can construct sub-problems and this leads them to do some tests. These tests allow them to constitute a corpus of possible and impossible ways either for Sophie or the pet. Indeed, for P0, students use drawing to scale and, so they can see on their drawing if the circles intersect. Therefore, the discussion about the number of intersections is moot. These new elements about the problem are

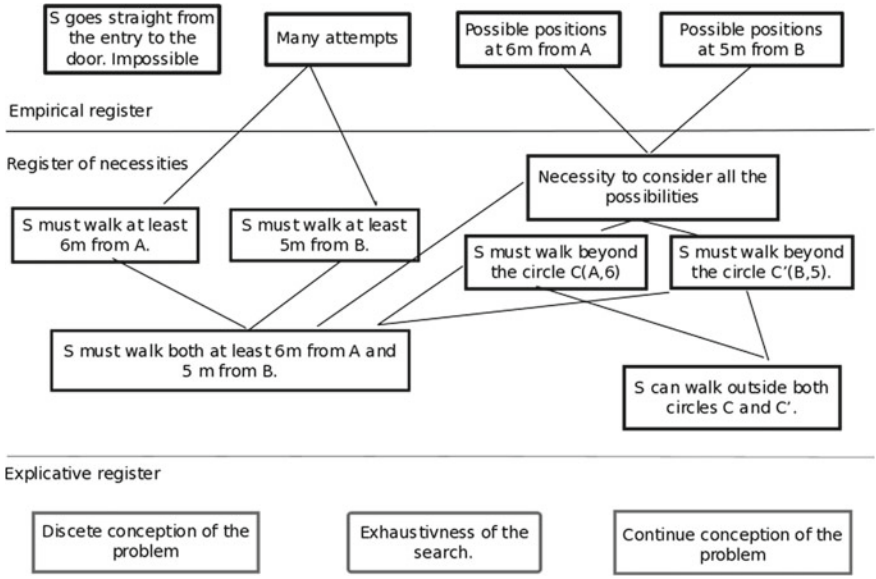


Fig. 6.2 A priori space of constraints and necessities for P0. "S" means Sophie, "C(A, 6)" means circle which center is A and which radius is 6; D(A,6) means the disc

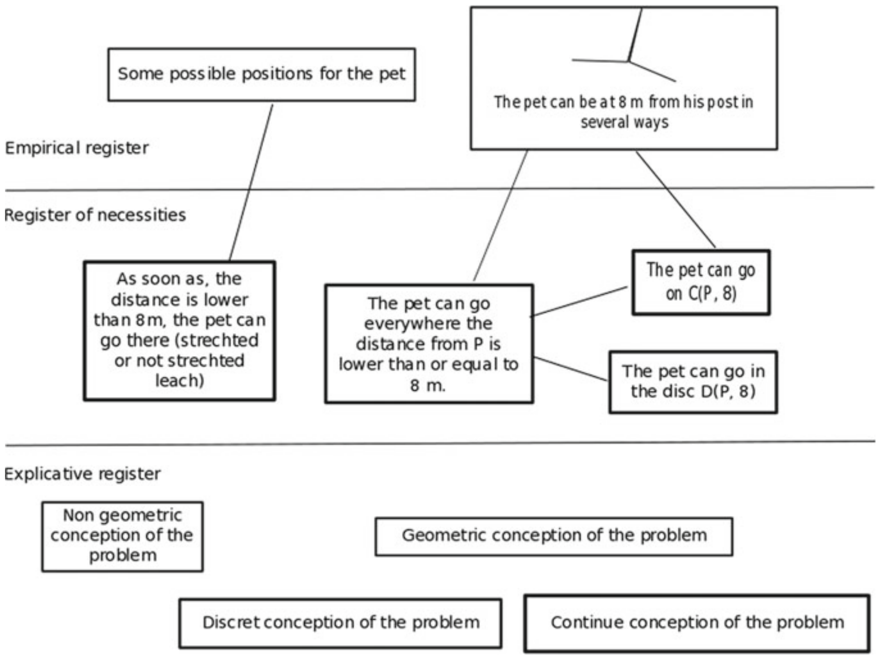


Fig. 6.3 A priori space of constraints and necessities for P1. "P" matches with the post

facts and match with the empirical register. Dually, these empirical elements allow students to give off necessities of the problem (for example: the pet must go beyond 8 m; he can go everywhere the distance to the post is less than 8 m; he always can go at 3 m of his post).

These two spaces highlight the necessity to move from a discrete representation (search for punctual solutions) of the problem to a continuous one (search for all solutions). This moving is also a crossing from the one-dimensional geometry to a two-dimensional geometry. For P0, the problem takes charge of this cognitive gap. Indeed, the two following necessities "Sophie has to walk at least at 6 m from A" and "Sophie has to walk at least at 5 m from B" only give positions where Sophie is not bitten. But they do not tell a possible way to the door. So, it does not close the problem and maintains the doubt to the possibility to reach the door. This doubt keeps the necessity to browse all the possibilities in an exhaustive manner. And thereby, it leads to encounter the move from the discrete to the continuous. The effect of an epistemological clause of the didactical contract also play a part (Hersant, 2014): students know that it is a mathematical task and especially a geometrical task (points are represented on the diagram) and that, therefore, they cannot be satisfied with a "yes" or "no" answer.

At the opposite, in problem 1, the expected answer is a drawing and the didactical contract is not clearly a mathematics one (the drawing indicates a drawing pet and a point) or a geometric representation. The epistemological clause of the contract cannot play for inducing an exhaustive research and moving from a discrete to a continuous envision of the problem. Moreover, the situation itself does no generate many doubts. In other words, the situation does not have the potential to lead the students to the research of every possibility. In this case the move from the discreet to the continuous envisions the problem that is not supported by the situation.

6.3.3 Problem P1: Implementation, Students' Productions and Sharing

One of the observed teacher turned the problem P0 into the P1. Its realization has been observed by Choquet (2014) in a 21 students' classroom at the end of the elementary school (10–11 years old students). They knew the signification of common vocabulary associated with a circle (ray, diameter, center, chord). They also knew how to draw a circle with a compass. But, they did not yet know the mathematical definition of a circle.

The teacher presents the activity and let students search individually for ten minutes. Then they work in small groups during fifteen minutes: each student has to search the problem and to give a written solution, but students are allowed to speak about the problem. After the students' research, the teacher selects three productions to be collectively discussed for ten minutes. Indeed, five students turn in a blank page and the productions of the sixteen other students can be split into three categories.

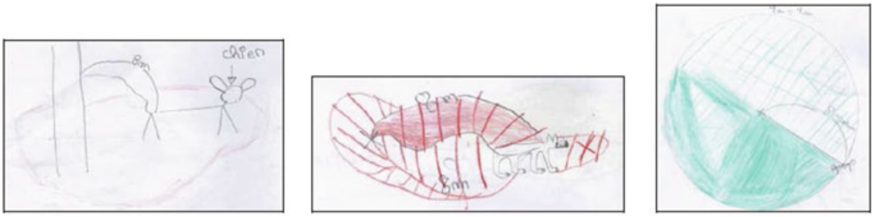


Fig. 6.4 Productions of students from first category (left), from second category (middle) and from third category (right)

The first category ties with students who draw the leash and a dog in a vertical plan (Fig. 6.4, left). They are seven in this case. Their production is closely linked to the drawing proposed by the teacher in the presentation of the problem (Fig. 6.1, right). Only one position of the dog is considered; they do not represent any area. These answers indicate that these students did not achieve to pose the mathematical problem, nor construct the problem, nor explored the field of possibilities. Indeed, they probably did not establish any necessity about this problem. Why? They did not consider the problem as a mathematical one because the statements of the problem implemented a didactical contract that is beside mathematics. In the second category, we gather two students who begin to schematize and envision several positions for the dog (Fig. 6.4, middle). We can suppose that these students lead a mathematical inquiry but they come up against the difficulty to move from discrete to continuous. These students most probably do not strike the problem of exhaustiveness of the answer. Their activity may correspond to our a priori space of constraints (Fig. 6.3). In the last category, there are seven students who draw the circle that bounds the zone without prior trials (Fig. 6.4, right). The observation suggests that these students already knew the definition of a circle as the set of all points in a plane that are at a given distance from a given point. So it is difficult to say that they have posed and constructed the problem.

Finally, it seems that the students have no approach of inquiry: those who have well conceptualized the notion of circle already reinvest it, probably without considering any sub-problems; the others stumbled on the exploration of a one-dimension problem.

6.3.4 *What Can We Learn from This Case?*

In this case, we think that the teacher changes the wording of the problem P0 without enough considering the effects of these modifications on the inquiry potentialities of the problem. That reveals a critical point for teaching by inquiry at the elementary school in France: most teachers have a literature Baccalaureate and it is not easy for them to fashion problems for their students. Moreover, this example mainly shows the limits of using real-world problems to impulse an inquiry process. Indeed, despite of an attempt of class discussion after group work, most of the students do not identify the task as a mathematical one. The didactical contract is not clear enough and the teacher does not intervene to make this contract explicit. So, we can imagine that the students will not learn about geometry with this problem. Indeed, during the research phase their activity do not allow them to build up a mathematical problem. Then, the solution given by the teacher will not be anchored in a problem research. In these conditions, can we still consider that these students learn by solving problem?

6.4 Second Case Study: Searching the Optimal Solution in a Discrete Optimization Problem in a Didactic Engineering

The situation has been designed within the framework of a didactic engineering involving a researcher (Hersant) and teachers (Hersant, 2010). It refers to the called “no three in line problem”. The wording of the problem is the following:

Problem 2: How many points can we put at most on this grid without forming any alignment of three points (see Fig. 6.5)?

6.4.1 Inquiry Potential of the Problem

This situation has been realized in several classes by teachers who all contributed to the design of the situation; the researcher did not intervene in the class management. The scenario was the following. First each student has to test possibilities, respecting constraints of the problem. This is the enumeration phase. As the task demands to make tests, all the students can do something and start to explore the problem (i and ii). So, the problem is likely to engage students in search. Moreover, this engagement in the problem is durable because students take to the game (i).

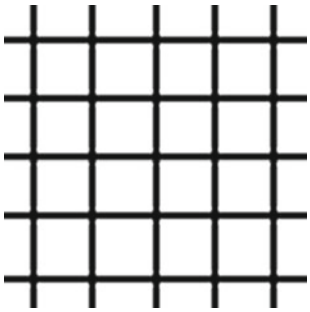


Fig. 6.5 The grid given to the students

If necessary, the teacher takes examples of putting points and asks students if the case correspond to an alignment. Then students can work by group and start to construct sub-problems (iii) like: is 7 the best solution? For example, the researcher often observed the following situation. A student comes against the possibility to put more than six points on the grid (ii), he is convinced that it is impossible to put seven points on it. Indeed, in a naïve empirical explicative register, he thinks mathematical impossibility matches with pragmatic impossibility: “it’s impossible because I search hard but I do not succeed”. But, when one of his classmate succeed in putting seven points, he wonders: is seven the solution? how can I be sure? All the students will not doubt the same, indeed teacher can help them to construct these sub-problems (iii).

When many students have a solution that they can’t improve, the teacher stops the research and ask each group to realize a poster with its best or one of its best solution(s). The solutions of each group are post up on the board and are collectively examined to check that they respect the no three in line constraint. The best solution(s) of the class is (are) identified. At this time, the question is to know if it is worth to keep searching, that means if we can improve the best solution of the class, or not. Then students are led to construct sub-problems: how can we know if this solution is the best? Is it possible to put *n* points on the grid?

So, with this problem, students are engaged in the search, they make attempts and construct sub-problems. The inquiry potential of the problem is real.

6.4.2 A Priori *Space of Constraints and Necessities*

The a priori space of constraints and necessities for this problem is the following (Fig. 6.6).

This figure highlights the empirical facts that can be built up by the students during the search of the problem, especially the enumeration phase. It also shows the possible conflict that can exist between an in progress-necessity based on an empirical conception of the impossible and a fact like “I can put 9 points on the grid”. These contradictions will lead students to search for necessities and to evolve their point of view on the problem: searching how to put points on the grid without any alignment of three points will never bring the solution, the proof of the problem also needs to mobilize short-cuts and proofs of impossibilities. Especially they will have to establish the following necessity: there are at most two points on a line (or on a column). Then, if a ten points solution has been found by some students, then they will conclude that the solution is ten.

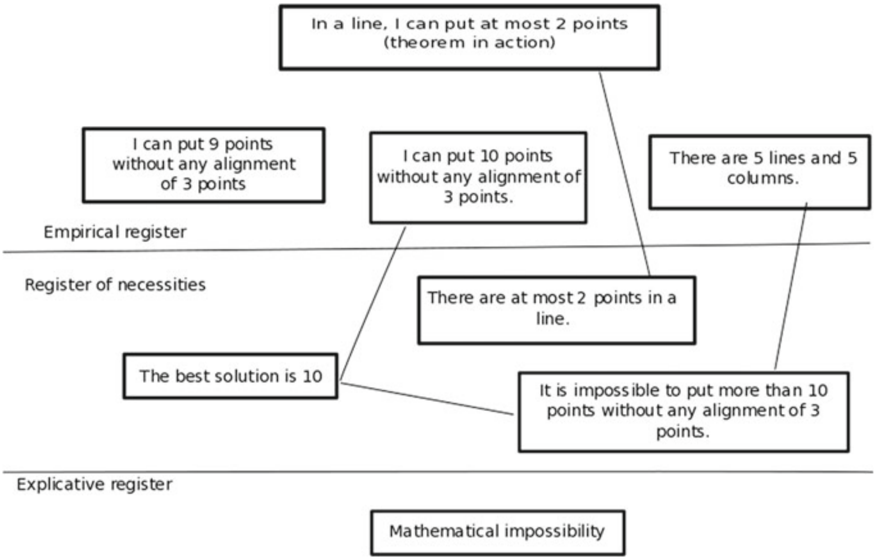


Fig. 6.6 A priori space of constraints and necessities for this problem

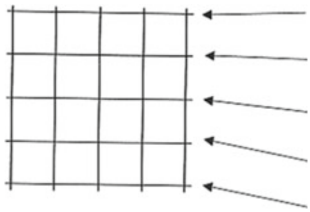
6.4.3 Students' Productions

We propose this problem in several classes of students of 8–11 years old (Hersant, 2010). We will here especially be interested in the first class where the situation has been tested (8 and 9 years old students). The first hour was dedicated to enumeration on an arrangement of 4 lines and 4 columns grid. The students found eight points solutions. The second session was dedicated to an enumeration of five lines and five columns grid. At the end of this session, students summarize the state of the search in the class. They formulate the conjectures:

> "Our record is 9, but I am sure than we can put 10 on the grid";
> "If we concentrate more, we can put more points on the grid";
> "There are two in each column".

The third session was devoted to the search of the solution of the problem. For this we introduce the following question: is it possible to put more than 10 points on the grid? By introducing this question, our aim was to oblige students to work on short-cuts necessities. We knew that if the teacher does not take this initiative, students will keep on enumerating the problem and have no chance to establish necessities. But we observe during this session that students have difficulties to envision the proposition «we can at most put two points on a line without forming any alignment of three points» as a mathematical necessity. Indeed, they envision it only as a theorem in action (Vergnaud, 1998) stemming from their experience of the enumeration. So only 2 pupils in 16 who expressed their views in an individual productions think that it is

Fig. 6.7 The box of points



impossible because we can only put 2 points on a line. The other produced arguments such as the following:

> "it's impossible because Jean said it" (1 in 16 students)
>
> "it's impossible because we have already well searched and we did not find more" (4 in 16 students)
>
> "because each time we try to add one more point on a 10 points grid there are three in a line" (3 in 16 students)
>
> "it's impossible, the grid is to small" or "there are not enough crosses" (6 in 16 view students)

So, we decide to introduce a box of points (see Fig. 6.7) and to ask them «You disagree about the reason. Now, using this table and without putting any points on the grid, tell us if it is possible to put more than ten points on the grid». This box of points involves switching from a geometrical setting (Douady, 1986) to an arithmetic one. Indeed, thus, implicitly, the question is: can we make *n* with a sum of 5 terms small or equal to 2? This helps students without killing the inquiry. It is also a great support to anchor the proof of the problem in the pupils' activity.

6.4.4 *What Do We Learn from This Case?*

This brief analysis of the students' activity shows that, in this case, students have posed and constructed the mathematical problem; they have carried out an investigation with the help of the teacher. So, this case study shows that inquiry is possible at the elementary school in France. It also empowers us to identify some conditions for its existence. First, it seems that the inquiry potential of the problem plays an important role. Here, he is mainly due to the doubt generated by the enumeration phase. Indeed, during this phase each student build certainties that could be destabilized by one of his classmate. Here, the milieu (Brousseau, 1997) of the situation plays an important role in problem posing. Moreover, the problem is easily identified as a mathematical problem by the students and it seems to facilitate the process of inquiry inside the mathematics field as the didactical contract is clear for the students. Finally, the engineering process provided the interventions of the teacher in case of students' difficulties and these interventions seem to effectively empowered students' inquiry.

6.5 Conclusion

We studied two cases to identify some conditions of possibilities of learning mathematics with inquiry in elementary French school. For this, we first characterized inquiry for learning mathematics as a problem posing and constructing issue that leads to the establishment of necessities of the problem, according to our framework. Taking this point of view allows to broaden our vision of inquiry-based learning in mathematics beyond modeling. Indeed, problem posing activity and establishing necessities is at the core of mathematics activity, whatever the statement of the problem is.

Our two cases are well contrasted considering their objectives of learning. Problem P1 is inspired from problem P0. Both of them deal with a real world problem and modelling. They both aim to learn about circle. But, P0 does not lead to investigation by the students. Problem P2 deals with the meaning of impossible in mathematics without any ambition about learning curricular knowledge. It nevertheless brings students to an investigation.

Furthermore, we also highlight that the three problems P0, P1 and P2 have a different inquiry potential. We can explain these differences in the following way. Comparing problem P0 and problem P1, we emphasize the importance of designing the statement of the problem and making available didactical comments to teachers as Dorier and Garcia (2013) proposed it. Indeed, we can suppose that from P0 to P1, we lost a part of inquiry potential because of the lack of didactical comments associated to P0. For the problem P2, instead, we can suppose that the didactic engineering insures the inquiry potential of the problem.

Moreover, important difference between the two cases concerns the possible link between the effective activity of the students during the search of the problem and the solution of the problem. In problem P1, we saw that for many students there may not exist such a link. Therefore, it is very difficult for the teacher to explain it. On the other hand, for problem P2, these links exist and it is easy for the teacher to highlight them. Thus, our study shows differences between the way each teacher manages the students' research and highlights the crucial role of supporting teachers in students' inquiry activity, through didactical comments on the management of the situation, either through their participation in didactic engineering that could contribute to their professional development.

Finally, which conditions of possibility of inquiry at elementary school can we identify? Two of the conditions presented by the second case appear especially important. First of all, the design of the situation conducts students to doubt and therefore enrolls them on inquiry. This property that is related with the potentiality of the problem seems to play a crucial role. Then, the teacher is able to support the students' inquiry activity taking into account their questions and introducing new sub-problem. This allows students to establish the necessities of the problem in direct link with their own exploration of the problem. Regarding this point, we can suppose that the teacher's participation in the modelling of the situation with the researcher have a significant influence that remains to be determined.

In conclusion, this study asks us new questions concerning knowledge at stake in inquiry based learning situations and also teachers' training to manage problem-posing and to use resources cautiously.

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Chapter 7
How to Stimulate In-Service Teachers' Didactic Analysis Competence by Means of Problem Posing

Uldarico Malaspina, Carlos Torres, and Norma Rubio

7.1 Introduction

Recently, researchers in mathematics education are becoming more interested in problem posing (Felmer, Pehkonen, & Kilpatrick, 2016; Malaspina, Mallart, & Font, 2015; Singer, Ellerton, & Cai, 2015; Torres & Malaspina, 2018). These scholars claim that it is very important for teachers to develop problem posing skill by both modifying given problems as well as posing them from concrete situations. Evidently, in a problem posing activity, the person's mathematical knowledge is brought into play, but if the problem posed is aimed at contributing to the student's knowledge—or more specifically, to understanding and solving other more complex problems—then the teacher's didactic-mathematical knowledge must also intervene. This aspect is closely related to the teachers' didactic analysis competence, which has been broadly studied within the onto-semiotic approach of cognition and mathematics instruction (OSA) (Breda, Pino-Fan, & Font, 2017; Rubio, 2012).

In our study, we focus on analyzing what representations, strategies, and resources teachers showed while dealing with tasks that involve problem posing activities (Font, Planas, & Godino, 2010; Torres & Malaspina, 2018).

Thus, we adopt the lines of research that use problem posing as a window of opportunities for students and teachers to understand mathematics, as well as the studies of specific strategies for problem posing (Kontorovich & Koichu, 2009; Malaspina et al., 2015; Milinkovié, 2015; Mallart, & Font, 2015). The work shown by 15 in-service high school mathematics teachers on problem posing is analyzed, and evidences of the close link between the didactic analysis competence and the problem posing skill that facilitate learning are discussed.

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In this study, we consider the following research question: *How can we use problem posing activities to stimulate the development of teachers' didactic analysis competence?* To answer this question, we use the problem posing strategy Episode, Pre-Problem, Post-Problem (EPP) used in Malaspina et al. (2015). To this end, the research team relies on tools from the OSA (epistemic and cognitive configurations) to analyze the solutions to the problems and to examine the teachers' didactic competence by analyzing *pre-problems*, whose fundamental aim is to facilitate the understanding and solution of the problem presented in the class episode.

7.2 Theoretical Framework

7.2.1 Problem Posing and Mathematics Teachers' Competences

In order to develop and assess students' mathematical knowledge and skills, instructional strategies mainly focus on problem solving. We consider that a teacher must not only have the ability to solve mathematical problems, but also choose, modify and pose problems with educational purposes, which means to facilitate or delve into his students' learning and stimulate their mathematical thinking (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016; Mallart, Font, & Malaspina, 2016; Tichá & Hošpesová, 2013). The tasks, in which a mathematics teacher analyzes a problem for the aforementioned educational purposes, imply the development of the didactic analysis competence, which entails reflecting on her/his mathematical practice of solving and posing problems, and analyzing to which extent it is contributing to facilitate the understanding and solving of other more complex problems.

It is worth mentioning that there are different positions in terms of what researchers understand by engaging in problem posing activities. In this research, we adopt the proposal from Malaspina (2015), according to which problem posing is a process through which a new problem is obtained. If the new problem is obtained by modifying a given problem, it is said that the new problem was obtained by *variation*. If the new problem is obtained from a given situation or from a specific requirement, whether mathematical or didactic, it is said that the new problem was obtained by *elaboration*. Malaspina (2015) also considers that problems have four fundamental elements: information, requirement, context (intra-mathematical or extra-mathematical) and mathematical environment; in that sense, problem posing by variation entails quantitative or qualitative modifications of one or more of these elements in a given problem; and problem posing by elaboration can be done specifying these four elements from the given situation. This approach for problem posing is complemented by the strategy proposed in Malaspina et al. (2015) to implement it in workshops with in-service teachers or teachers in training in order to stimulate their ability to pose problems by variation. In our study, we use this strategy to develop an empirical research. In such strategy, participants in the problem posing workshop are asked to modify the proposed problem in the context of a teacher's class episode. In the first phase, participants in the workshop are asked to propose a problem that helps students understand and solve the problem proposed in the episode; a problem with such characteristic is called *pre-problem*. In the second phase, participants in the workshop are asked to propose a more challenging problem than the one of the episode; a problem with such characteristic is called *post-problem*. In each one of the phases of this strategy, which we will call EPP since it stands for episode, pre-problem and post-problem; there is individual work, group work—preferably in pairs—and socialization with all participants in the workshop. The mathematical activity includes the solution to the posed problems, which contributes to the interaction between problem solving and problem posing.

In this study, we implemented the EPP strategy in a problem-posing workshop on affine functions with 15 in-service high school teachers, and we focused our attention on pre-problem posing, since it requires didactic criteria from the poser, because it should have the characteristic to facilitate the understanding and resolution of a previously given problem (*episode problem*). We analyze these didactic criteria taking into consideration the problems posed and solved by the participants, both during individual work and pair work. We use constructs of the onto-semiotic approach of cognition and mathematics instruction (OSA), which we explain in the following section.

7.2.2 Onto-Semiotic Approach of Cognition and Mathematics Instruction (OSA)

There are different views on the conception of competence in the context of teaching and learning. For the purposes of this research, we will adopt the point of view of Tardif (2006), which proposes to define competence as "a complex know how, supported by the mobilization and effective combination of a variety of internal and external resources within a family of situations" (p. 22). Knowledge, skills, abilities and attitudes of each individual are considered among these resources.

We adopt the OSA as theoretical framework because we are interested in documenting teachers' competences that they develop when they engage in problem posing activities. Likewise, we believe it is relevant to use an approach that provides us with categories to analyze both teachers' mathematical knowledge and didactic knowledge.

In the OSA, didactic-mathematical knowledge is understood as knowledge of mathematics and its teaching, which a mathematics teacher must have to design, implement and assess the complex processes of mathematics teaching.

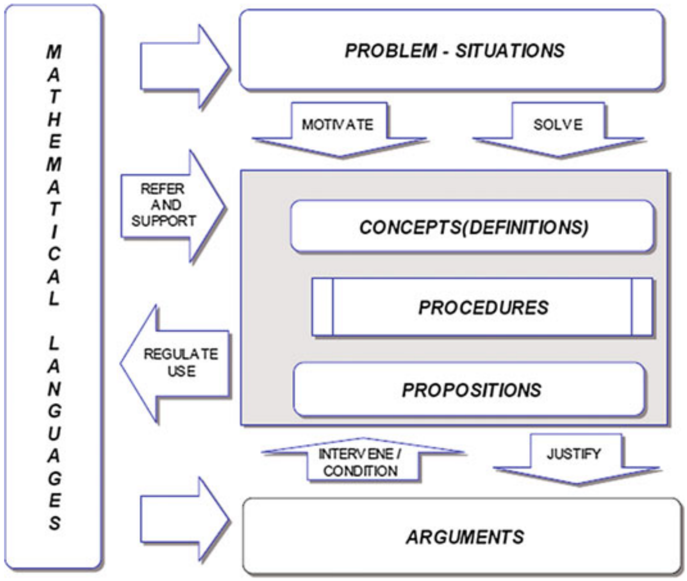


Fig. 7.1 Configuration of primary objects

An important OSA theoretical construct for the analysis of mathematical objects, such as concepts, procedures, propositions and arguments, are the epistemic and cognitive configurations, which we explain next.

According to Godino, Batanero and Font (2007), when a person carries out a mathematical practice and assesses it, he/she has to activate knowledge and resources of involved mathematical objects, that is to say: problem situations, languages, propositions, definitions, procedures and arguments. These elements will be interrelated, making configurations defined as webs of objects that intervene and emerge from the systems of practice (Fig. 7.1); such configurations are epistemic (EC) when they are webs of objects considered from an institutional perspective, and they are cognitive (CC) when they are webs of objects considered from a personal perspective. Analyzing these configurations allows us to obtain information about the *anatomy of a problem solution.*

The research was developed within this framework, drawing attention to the fact that these elements serve as reference to pose problems with emphasis on the didactic point of view. In terms of the framework, a teacher is competent in the analysis of mathematical practices, objects and processes when that teacher is able to answer questions such as: what are the meanings of the mathematical object being studied? what are the configurations of mathematical objects and processes involved in the solution of problems that are typical of the different meanings of the contents planned in teaching? (Epistemic configurations); what are the configurations of objects and processes brought into play by students in the solution of the proposed problems? (Cognitive configurations). In other words, it is about the competence in the analysis of mathematical practices involved in the resolution of tasks developed in the processes of mathematical instruction. In the OSA, mathematical practices are understood as actions made by a subject in order to solve or to pose a problem or to do a task, whether discursive-declaratory (indicating knowledge possession) or operative-procedural (indicating an ability or competence). Both types of practices are related, so carrying out operative practices efficiently involves putting declaratory knowledge into action.

In our study, we will consider aspects related to two levels of analysis of mathematical activities (Godino, Giacomone, Batanero, & Font, 2017): with mathematical and didactic practices, and with configurations of mathematical objects. In the former, we focus on problem posing and solving, and in the latter, we delve into the analysis, using the configurations.

7.3 Methodology

In this research, we used a multiple case study with 15 in-service public high school mathematics teachers who participated in the *problem-posing workshop on affine functions*. They did not have previous experiences in problem-posing tasks. Following Ponte (2006), our study is exploratory, descriptive and analytical. The units of analysis were the teachers' solutions to episode problem and to the problems posed by them.

In the *problem-posing workshop*, we used the EPP prompts to stimulate the development of the skill to pose pre-problems by varying a given problem. The workshop sessions included two days sessions working two hours per day. The dynamics were the following:

First session: the research team gave a very brief presentation on problem posing approach. Worksheets that included an episode class designed by the research team were handed out to the participants. The participants were asked to work individually to solve the episode problem and then to pose a pre-problem. Then, they shared what they had done individually in pairs. The participants gave back their worksheets including their individual and pair work, and a pre-problem posed by a couple was shared collectively.

Second session: Worksheets with the same episode class as the previous session were handed out so that they would pose a post-problem, first individually and then in pairs. After giving back their worksheets, one of the problems was made collective, and comparative comments were made of the pre- and post-problems posed.

The methodology of this research has the following phases:

(1) Workshop preparation. By taking into account teachers' didactic experiences in teaching the theme of functions in secondary school education, the research team selected the following episode from a class with students between 14 and 15 years old, in order to present it to the teachers participating in the workshop:

> Mr. Torres proposed the following problem to eighth-grade students in a mathematics class on functions:
>
> In the shop at the corner of my block, each kilogram of potatoes costs 3 PEN. In the wholesale market, which is far from home, each kilogram of potatoes costs 2 PEN, but I have to spend 5 PEN in bus tickets to get there and come back.
>
> Will it always be more convenient to buy potatoes in the market rather than in the shop? Why?
>
> After a few minutes, some students commented:
>
> **Juan**: Sure, it will always be more convenient to buy at the market because it is cheaper there.
>
> **María**: Not always… It depends…
>
> **Mateo**: It will be more convenient to shop at the market if you have to buy more than 8 kilos of potatoes.

In this workshop preparation phase, we also elaborated a configuration of primary objects based on an expert solution to the problem presented in the episode carried out by the research team. This expert solution brings out the primary mathematical objects which could intervene in the solution to episode problem. The configuration was considered as EC in order to have it as a reference to analyze it and compare it to the configurations of the participants' solutions, which were considered as CC. A pre-problem solution should not need more than the primary mathematical objects proposed by the expert solution.

(2) The episode is presented to the participants of the workshop, implementing the EPP strategy. In this phase two types of data are obtained: the solutions to the problem presented in the episode and the problems (and their solutions) posed by the participants.

(3) The problems solved and posed by the in-service teachers are analyzed. The research team examines overall the different solutions to the problem presented in the episode, examines the solutions to the posed pre-problems and elaborates their corresponding CC. Likewise, the research team elaborated the expert solutions to the posed pre-problems with their corresponding EC. Then, both configurations are compared in order to determine if the definitions, procedures, etc. appearing in the teachers' solutions imply a higher cognitive demand, or not, than the problem presented in the episode. We focused our attention on the pre-problems posed by the participants in the workshop, since they show aspects from the teachers' didactic analysis in a better way, in the sense that they should contribute to the understanding and solving of the problem presented in the episode. We examined the pre-problems posed individually and compared them with the ones they presented after working in pairs, taking in consideration the posed problem by each of one.

The EC and CC have been used within research methodologies based on OSA framework (e.g. Badillo, Font, & Edo, 2015; Malaspina, & Font, 2010), with the aim of examining the mathematical solutions of pupils. This technique allows systematically describing the mathematical activity carried out by the in-service teachers (mathematical practice of solution), the mathematical activity of posing (mathematical practice of posing), and the primary mathematical objects (language, problem situation, concepts, propositions, procedures and arguments). The epistemic and cognitive configurations elaborated by the research team were validated through the methodology of expert triangulation (OSA specialists). Thus, this analysis tool was built by involving researchers from the same theoretical field in order to have different reflections and points of view when analyzing the configurations; this is what Lincoln and Guba (1985) refer to as Member Checking.

7.3.1 Expert Solution and EC of the Episode Problem (ECPe)

The expert solution to the episode problem (Fig. 7.2) is mainly based on the definition of two functions: $f(x) = 3x$ and $g(x) = 2x + 5$, which express how much x kilograms of potatoes cost in a convenience store and in the wholesale market, respectively. The graphs for both functions are drawn in the same system of coordinates, and it is determined—visually and algebraically—that the amount spent in x kilos of potatoes is **not always** less in the wholesale market than in the convenience store.

While elaborating the EC of the solution, the languages used (verbal, symbolic, graphic and tabular representations), the information, requirement, context and mathematical environment of the problem are explicitly stated. Moreover, the concepts involved (linear function, affine function, expense function, slope, y-intercepts, graphs of functions, linear inequation) and the emerging proposition: If there are values of x for which $f(x) < g(x)$, then it is not always more convenient to buy in the wholesale market. In addition, the procedure is described and the arguments are explained to support and validate the given proposition, which derives in the conclusion.

7.4 Results

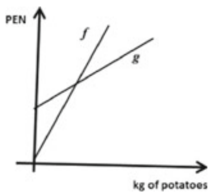
In relation to the teachers' solutions to the problem shown in the episode above, most of them do not reveal similar procedures to the ones of the expert solution. The use of tables prevails, as well as the calculations of the expenses in specific cases. Eight teachers define the functions of the case, but only two use their graphs. On the other hand, only two solve an inequation, and only four use the expression *not always* in their answer.

A thorough analysis of the CC of the participants' solutions reveals important aspects of their mathematical competence, which we will not explain in detail now. In relation to the problems posed by the teachers, in general terms, they show the ability to pose problems by varying a given problem; however, we perceive little consideration to the didactic characteristic such problems should have, like *pre-problem*, in the sense of helping students to clarify and solve the e problem presented in the episode. To make this perception more evident, as explained in the methodology, the research team has elaborated CCs from the solutions that the teachers presented, and we also elaborated expert solutions and their ECs to the proposed problems.

We did some comparisons between these configurations (see Fig. 7.3), which revealed important aspects of the teachers' didactic competences. Some of them gave us information that is strongly related to mathematical competences (M), and others gave us information that is strongly related to the didactic analysis (D).

To illustrate the type of data and the analysis done, we show the following: (1) an example of the pre-problem posing task; (2) the teacher's solution to his proposed problem; (3) the CC of such solution, elaborated by the research team.

Convenience store: 1 kg of potatoes 3 PEN, Wholesale market: 1 kg of potatoes 2 PEN, Bus ticket: 5 PEN. x: amount (in kg) of potatoes, $x \geq 0$. $f(x)=3x$: Function that defines expenses in x kg of potatoes in the convenience store. $g(x)=2x+5$: Function that defines expenses in x kg of potatoes in the wholesale market. We make sketches of these functions (f and g) using the slopes and intersections with the Y axis. We can clearly see the amount spent in x kilos of potatoes is **not always** less in the wholesale market than in the convenience store because the g graph is not below the f graph for all x values.



Using algebraic symbols:

$$g(x) < f(x) \Leftrightarrow 2x + 5 < 3x \Leftrightarrow 5 < x$$

So, it will be more convenient to shop in the wholesale market when we have to buy more than 5 kg of potatoes and not in other cases.

Tabular illustration:

Kg potatoes	x	0	1	2	3	4	5	6	7
Store	$f(x)$	0	3	6	9	12	15	18	21
Market	$g(x)$	5	7	9	11	13	15	17	19

We can clearly see that there are x values where $f(x) < g(x)$ and other values where $f(x) > g(x)$.

Fig. 7.2 Expert solution

In Fig. 7.4, we present a problem posed by an in-service teacher (we call him T3).

Translation of pre-problem posed by T3 (Fig. 7.4)

Lalo is a boy who *makes a living* by selling roses, earning 2 PEN each. To make sure he sells out, he decides to go to a concert of *romantic music*, where he has to pay 20 PEN to get in. How many roses will he have to sell to beat his regular sale, with which he earns 30 PEN daily on average?

In Figure 7.5, we show the teacher's solution to his posed problem:

According to the methodology presented, the research team elaborated the CC of this solution (CCPp)—Table 7.2—which was compared to the EC of the episode problem (ECPe) summarized in Table 7.1.

Analyzing the CC of the solution to this posed pre-problem (Table 7.2), we state the problem has some good characteristics such as a clear, interesting statement with an extra-mathematical context and related to the mathematical environment desired to work with eighth-grade students. Nevertheless, it is evident in the CC that in this problem posed by T3, and in the solution he proposes himself, are involved concepts, propositions, procedures and arguments—even though these last ones are not explicit

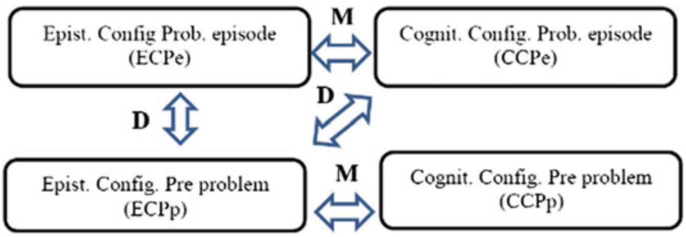


Fig. 7.3 Diagram to compare configurations

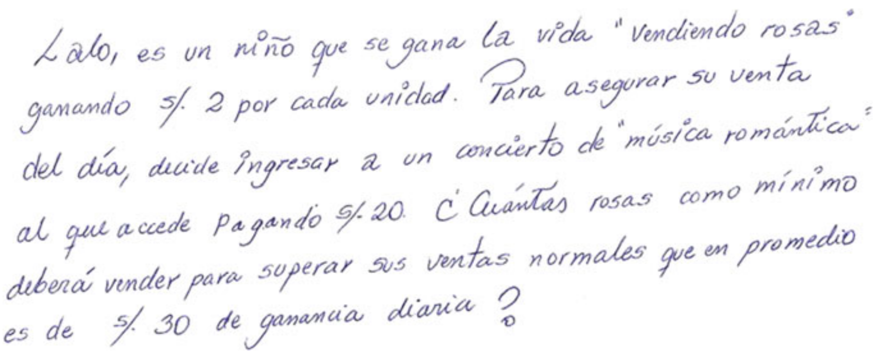
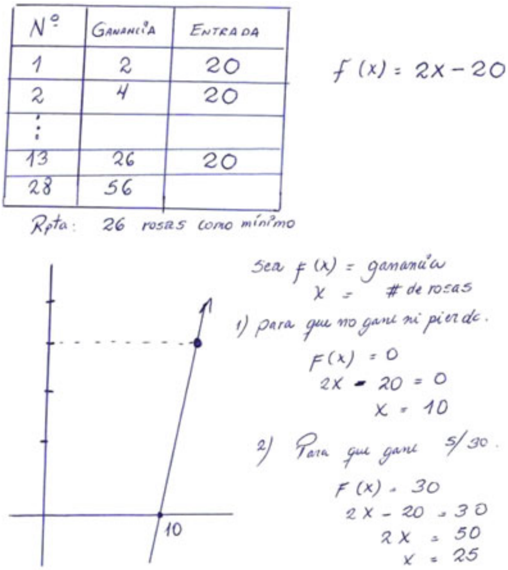

Lalo, es un niño que se gana la vida "Vendiendo rosas"
ganando S/. 2 por cada unidad. Para asegurar su venta
del día, decide ingresar a un concierto de "música romántica"
al que accede pagando S/. 20. ¿Cuántas rosas como mínimo
deberá vender para superar sus ventas normales que en promedio
es de S/. 30 de ganancia diaria?

Fig. 7.4 Pre-problem posed by T3



Translation

Profit Concert ticket

Answer: 26 roses minimum

Let $f(x)$ = profit

x = amount of roses

1) To break even

2) To earn 30 PEN

Fig. 7.5 Teacher T3's solution to his posed problem

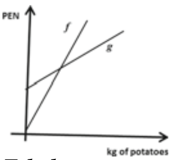
in his solution—which require a higher cognitive demand than the problem presented in the episode.

Another fact that also shows that the problem posing activities reveal little management of the criteria for didactic analysis is the decision made by teacher T3 and a colleague while working in pairs. They posed practically the same problem proposed by T3 as pre-problem of the group, even though his colleague (whom we will call T3A) posed a problem that would be preferable as pre-problem in relation to the episode problem. According to the CC of the solution shown, carried out by the research team, the concepts and procedures corresponding to T3A's problem do not require a higher cognitive demand in relation to the one required by the episode problem. However, both the problem from T3, which we just analyzed, and the pre-problem posed by the couple formed by T3 and T3A do require a higher cognitive demand.

Table 7.1 Epistemic configuration of the expert solution (ECPe)

Languages

Verbal representations:
Expenses, PEN, kilograms, purchase
x: amount (in kg) of potatoes which we buy
$f(x) = 3x$: Function that defines expenses in x kg of potatoes in the convenience store
$g(x) = 2x + 5$: Function that defines expenses in x kg of potatoes in the wholesale market
Symbolic representations:
S/3, S/2, $f(x) = 3x$, $g(x) = 2x + 5$, $g(x) < f(x)$, $\rightarrow$, …
Graphic representation:



Tabular representation:

Kg potatos	x	0	1	2	3	4	5	6	7
Store	$f(x)$	0	3	6	9	12	15	18	21
Market	$g(x)$	5	7	9	11	13	15	17	19

Problem situation

Information: Prices of a product in two places, fixed cost for bus tickets to buy in the wholesale market
Requirement: Determine the minimum amount of units that need to be sold to beat a given income amount.
Comparison of expenses for the same purchase in two places
Context: Extra-mathematical
Mathematical Environment: Related affine functions, linear equations.

Concepts

Linear function, affine function, expense function, slope, Y-intercepts, graphs of functions, linear inequation

Propositions

If there are values of x for which $f(x) < g(x)$, then it is not always more convenient to buy in the wholesale market.

Procedures

Define the x variable with $x \geq \mathbf{0}$. Algebraically write the functions for the expenses when buying x kilos of potatoes in the convenience store and in the market (f and g, respectively). Sketch the graphs of functions f and g, $f(x) \geq \mathbf{0}$ and $g(x) \geq 5$, assume f and g are continuous functions. Compare the images of x according to functions f and g. Solve the linear inequation $g(x) < f(x)$. Use the tabular representation for comparing correspondent $f(x)$ and $g(x)$ values.

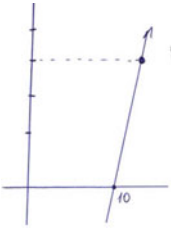
Arguments

Thesis 1: If there are x values where $f(x) < g(x)$, then it is not always more convenient to shop in the market.
Argument: It is more convenient to buy x kg of potatoes in the market if and only if $g(x) < f(x)$.
Thesis 2: When u and v are real functions of a real variable z. The u graph is below the v graph for every value of the z variable in a J interval if and only if $u(z) < v(z)$ and $z \in J$.
Argument: If $(p; q)$ and $(p; r)$ represent, respectively, points A and B of a vertical line in the Cartesian plane, A is below B if and only if $q < r$.
Thesis 3: It is not always more convenient to shop in the wholesale market.
Argument: There are points of the f graph that are below the g graph. So, according to Thesis 2, there are x values where $f(x) < g(x)$. The conclusion is drawn from Thesis 1.

Table 7.2 Cognitive configuration of the solution to the posed pre-problem (CCPp)

Languages

Verbal representations: Profit, ticket, minimum, amount, loss, average, sales, variable x, amount of roses sold, function f: profit
Symbolic representations: Amount of roses: 1, 2, 3,…, 13, 28. Profit: 2, 4,…, 26, 56. Ticket: 20. $f(x) = 2x - 20$, PEN (S/), =,
Graphic representation:



Tabular representation:

Nº	Ganancia	Entrada
1	2	20
2	4	20
⋮		
13	26	20
28	56	

Problem situation

Information: Income for selling a product, considering a fixed cost. Daily average income specified.
Requirement: Determine the minimum amount of units that need to be sold to beat a given income amount.
Context: Extra-mathematical
Mathematical environment: Related affine functions, linear equations

Concepts

Linear function, graphs of functions, X-intercept, strictly increasing function, income function, profit function, linear equation, variable, algebraic expression, minimum whole number of a lower enclosed set of real numbers.

Propositions

The solution to the equation $f(x) = k$ determines the number of units that need to be sold to have a profit k. If $f(x) = 0$, there are no earnings or losses.

Procedures

Define variable x. Write algebraically the profit function (f) for selling x roses, taking only in consideration a fixed cost (payment for the concert ticket). Elaborate a table to place the resulting amounts from substituting variable x for positive whole numbers. Compare the results in the table, taking into account the number obtained as income. Determine the number of units required to sell in order to get the given average profit, using the determined function. Graph the profit function and observe that it is strictly increasing. Conclude that the minimum number required will be the lowest whole number greater than the number gotten in the previous step.

Arguments

Thesis 1: If $f(x) = 0$, there are no earnings or losses.
Argument: A positive profit is a "real profit" and a negative profit is a loss.
Thesis 2: If f is a strictly increasing function, q is a given number and u solves the equation $f(x) = q$, then the lowest whole number v, greater than u, is the lowest whole number, as long as $f(x) > q$ for every $x \geq v$.
Argument: Since function f is strictly increasing, if $v > u$ then $f(v) > f(u) = q$. Since v is the lowest whole number greater than u, any x number, specially a whole number, as long as $x \geq v > u$ complies with $f(x) > f(u) = q$. That is, $f(x) > q$.

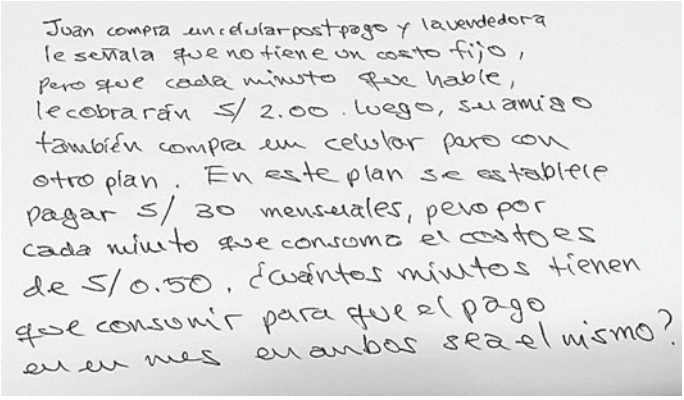

Juan compra un celular postpago y la vendedora
le señala que no tiene un costo fijo,
pero que cada minuto que hable,
le cobrarán S/ 2.00. Luego, su amigo
también compra un celular pero con
otro plan. En este plan se establece
pagar S/ 30 mensuales, pero por
cada minuto que consume el costo es
de S/ 0.50. ¿Cuántos minutos tienen
que consumir para que el pago
en un mes en ambos sea el mismo?

Fig. 7.6 Pre-problem posed by T3A

The following is the pre-problem posed by T3A:

Translation of the pre-problem posed by T3A (Fig. 7.6)

Juan bought a post-paid cellphone, and the salesperson pointed out that it does not have a fixed cost, but for every minute he talks, he will have to pay 2 PEN. Then, his friend Luis also buys a cellphone but with a different payment plan. The plan involves a monthly payment of 30 PEN, but the cost for every minute he talks is 0.50 PEN. How many minutes do they have to talk so both payments are the same?

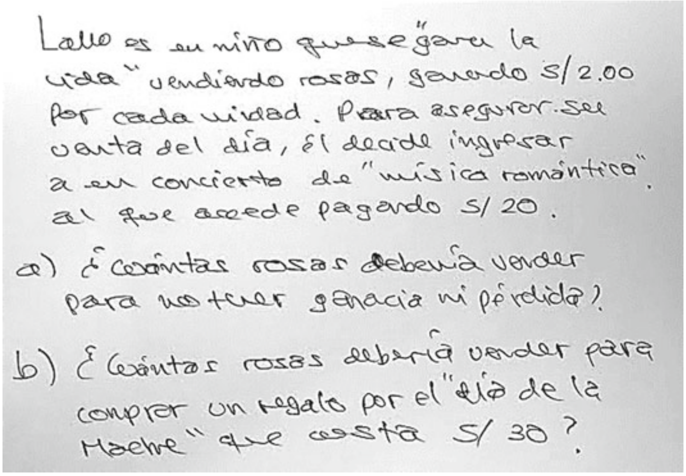
The following is the pre-problem posed by the T3 and T3A couple:

Translation of the pre-problem posed by T3 and T3A (Fig. 7.7)

Lalo is a boy who *makes a living* selling roses, earning 2 PEN per unit. To make sure he sells out that day, he decides to go to a romantic music concert, whose ticket costs 20 PEN.

(a) How many roses should he sell so there is no profit or loss?
(b) How many roses should he sell in order to buy a 30-PEN gift for Mother's Day?

We observe that the difference between this problem and the one posed by T3 (Fig. 7.4) is very little, even though it is worth mentioning that this one has a slight greater didactic consideration than the problem from T3, since its requirements are well separated, and it no longer refers to a minimum sales amount nor an average


Lalo es un niño que se gana la vida "vendiendo rosas, ganando S/ 2.00 por cada unidad. Para asegurar su venta del día, él decide ingresar a un concierto de "música romántica" al que accede pagando S/ 20.

a) ¿Cuántas rosas debería vender para no tener ganancia ni pérdida?

b) ¿Cuántas rosas debería vender para comprar un regalo por el "día de la Madre" que cuesta S/ 30?

Fig. 7.7 Pre-problem posed by T3 and T3A

daily profit. Either way, solving this problem requires a higher cognitive demand than solving the episode problem or the problem from T3A.

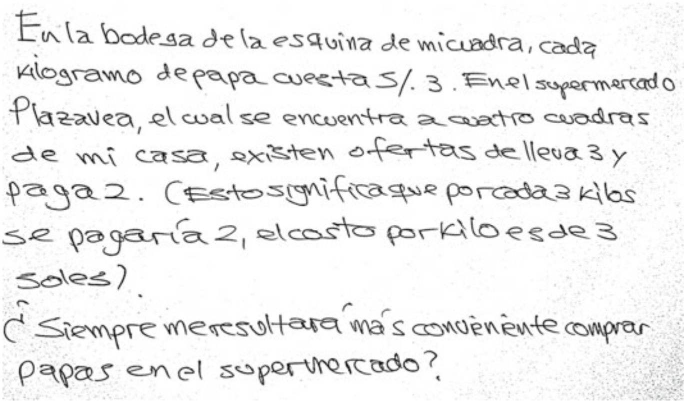
The EPP strategy considers pair work or small groups precisely to stimulate joint reflection on the problem posing activity particularly on a pre-problem. This pre-problem considers didactic aspects, but it is evident that it is necessary to guide that didactic reflection so it is more fruitful, since similar situations to the ones described occur in pre-problem posing by other teachers and other teacher couples. We could state that, by problem posing, the teachers from the sample are more careful with mathematical aspects than with didactic aspects.

Another example of a pre-problem that requires higher cognitive demand than the problem presented in the episode is shown in Fig. 7.8. Teacher T5 posed this problem.

Translation of the pre-problem posed by T5 (Fig. 7.8)

In the convenience store at the corner of my block, each kilogram of potatoes costs 3 PEN. In the supermarket, which is four blocks away from my house, there are sales offering to buy 3 kg for the price of 2 kg (this means that, for each 3 kg of potatoes, you would be paying for 2 kg; the price per kilo is 3 PEN). Will it always be convenient to buy potatoes at the supermarket?

This problem (Fig. 7.8) is perceived as requiring more cognitive demand than the problem presented in the episode since one has to consider buying a number of kilograms of potatoes that is multiple of 3. It might be concluded intuitively that it will always be more convenient to shop in the supermarket, but it is not easy to use



En la bodega de la esquina de mi cuadra, cada kilogramo de papa cuesta S/. 3. En el supermercado Plazavea, el cual se encuentra a cuatro cuadras de mi casa, existen ofertas de lleva 3 y paga 2. (Esto significa que por cada 3 kilos se pagaría 2, el costo por kilo es de 3 soles).
¿Siempre me resultará más conveniente comprar papas en el supermercado?

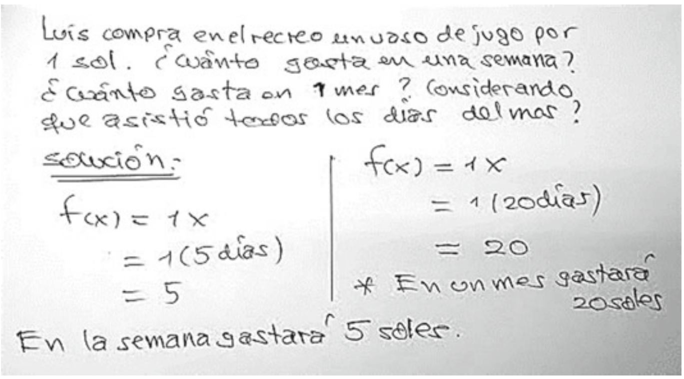
Fig. 7.8 Pre-problem posed by T5

functions for proper justification, being this the mathematical setting of the problem presented in the episode.

We conclude that the problem posed does not have the conditions of a *pre-problem*, whose main characteristic is to facilitate the understanding and resolution of the problem given in the episode.

Similarly to what was done in the case of the problem posed by T3, in Fig. 7.9 we present the pre-problem posed together by teachers T5 and T5A as a couple, as well as their solution:

Translation of the pre-problem posed by T5 (Fig. 7.9)



Luis compra en el recreo un vaso de jugo por 1 sol. ¿Cuánto gasta en una semana? ¿Cuánto gasta en 1 mes? Considerando que asistió todos los días del mes?

Solución:

$f(x) = 1x$
$= 1(5 \text{ días})$
$= 5$

En la semana gastará 5 soles.

$f(x) = 1x$
$= 1(20 \text{ días})$
$= 20$

* En un mes gastará 20 soles

Fig. 7.9 Problem posed by T5 and T5A

Luis buys a glass of juice for 1 PEN during break. How much does he spend in 1 week? How much does he spend in 1 month, considering he goes to school every day of the month?
Solution

$f(x) = 1x$ $= 1\,(5\text{days})$ $= 5$ He will spend 5 PEN in a week.	$f(x) = 1x$ $= 1\,(20\,\text{days})$ $= 20$ He will spend 20 PEN in a month.

In this case, the pre-problem presented by the teachers does not have enough didactic considerations because it is very simple, to the point that the use of functions seems forced and that does not contribute to a better understanding and resolution of the problem proposed in the episode. On the other hand, the language used in the problem is ambiguous, since it considers monthly and weekly expenses, and the solution considers 20 days. It is implicitly assumed that a month has four weeks and that *go to school every day of the month* means five days a week only. The solution uses the functional notation inadequately—both mathematically and didactically.

It is worth mentioning that the need to delve into didactic reflections became evident when the teachers participating in the workshop exchanged their opinions about the posed problems individually and in pairs during the extensive socialization. Even more so when the research team showed some mathematical objects of the corresponding epistemic and cognitive configurations and did the comparisons referred to the mathematical objects considered in each problem: language, concepts, propositions, procedures and arguments.

Generally speaking, we could say that the posed and solved problems reveal some teachers have mathematical competence, and most of them have serious limitations in terms of didactic analysis competence in relation to reflecting on their practices and the mathematical objects involved in such problems. This happens in every case of the posed *pre-problems*; for this reason, we consider that it is necessary to polish the EPP strategy, including a stage of reflection on the mathematical practice of problem solving and problem posing, which could serve as the basis for teachers to improve their didactic analysis in relation to problem posing.

7.5 Final Considerations

The analysis of the experience developed, made by the research team using OSA theoretical constructs, shows the teachers' difficulties to pose problems taking into account didactic considerations, which are revealed in the pre-problems posed by the teachers in the workshop, following the EPP strategy.

In virtue of the analyses performed and the importance of developing teachers' didactic analysis competence, particularly when posing mathematical problems that contribute to facilitate and carry on into their students' learning, we propose the EPP strategy for problem posing to be polished, considering a phase (R) of metacognitive and didactic reflection; therefore, the strategy name would be ERPP. In such phase, the teachers must elaborate a CC of their solutions to the problem presented in the episode and—based on it—reflect on their practices. Thus, the new strategy we propose has four phases: In the first one, a class episode is presented, proposing a problem and students' comments on its resolution, which reveal difficulties. In the second phase, each teacher solves the problem presented in the episode and reflects on the mathematical activity necessary to solve it by using the tool of configuration of mathematical objects (CC). In the third phase, each teacher poses a problem, called pre-problem, so that its solution facilitates students' understanding and solution of the problem presented in the episode.

In the fourth phase, each teacher poses a problem, called post-problem, whose characteristic is to have a higher cognitive demand than the problem presented in the episode.

We have already developed this ERPP strategy in Torres and Malaspina (2018), in which a phase of familiarization has been incorporated with the tool of configuration of primary objects, proposed in the OSA.

Thus, based on the study done, we consider that we answer the research question, proposing the ERPP strategy for mathematical problem posing, which in turn stimulates the development of teachers' didactic analysis competence, since a metacognitive and didactic reflection phase is considered, supported by the use of the OSA tools.

Stahnke, Schueler, and Roesken-Winter (2016) present a review of the empirical research done on mathematics teachers, and it concludes that these researches show teachers have difficulties to analyze mathematical tasks (and their educational potential) that their students propose. In order to overcome these difficulties, it is fundamental for teachers to have the ability to analyze their own mathematical tasks and, in that sense, we consider this proposal provides a specific means to do so, by means of a problem posing strategy with a phase of didactic reflection.

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Part IV
Assessment of and Through Problem Solving

Chapter 8
The Impact of Various Methods in Evaluating Metacognitive Strategies in Mathematical Problem Solving

Mei Yoke Loh and Ngan Hoe Lee

8.1 Introduction

Metacognition that is often described as 'thinking about thinking', is sometimes considered as elusive because it is concerned with the internal processing of information which is extremely difficult to capture through observation of students' behavior and analysis of written work. What is presented in written work hardly provides the insights into students' decision-making processes which are likely to be metacognitive in nature and such decisions and reasoning may lead to the success or failure in solving the problem. In addition, it is extremely difficult to differentiate between cognitive and metacognitive processes as a combination of the different elements of knowledge of both processes is at work at the same time (Garofalo & Lester, 1985; Goos & Galbraith, 1996).

Despite the difficulty in collecting data on metacognition, various approaches have been attempted. Self-reporting seemed to be a technique often associated with metacognitive studies (Wilson, 2001). While there were concerns that self-reporting measures may stimulate metacognition, students being their prime witnesses to their own thinking, there is no better way to find out about their thinking (Solas, 1992; Jacobse & Harskamp, 2012).

Some self-reporting techniques include survey, reflection log/journal, talk/think aloud protocol, concurrent probing, and retrospective probing. These self-reporting techniques could also be broadly classified by format (i.e. paper-pen format versus verbalization) and data type (i.e. qualitative versus quantitative data). Of these, three of the most commonly employed self-reporting techniques used in metacognitive

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studies, namely survey inventory (paper-pen and quantitative data), retrospective self-reporting (paper-pen and qualitative) and qualitative interview (verbalization and qualitative), were used in this study.

This paper presents the findings on the affordances of three self-reporting techniques, survey inventory, retrospective self-reporting and qualitative interview, used to collect data on metacognition in mathematical problem solving and how the findings using these approaches impact on data interpretation. The findings aim to establish greater insights into approaches taken towards instrumentation and data analysis in research relating to students' metacognitive practices.

8.2 Definition of Metacognition in Mathematical Problem Solving

Many researchers (Nietfeld, Cao, & Obsorne, 2005; Schraw & Dennison, 1994; Schraw & Moshman, 1995) define metacognition as having two main components: knowledge about cognition and regulation of cognition. Monitoring is considered as a sub-process of regulation of cognition as it is difficult to classify observed behaviors under monitoring and regulation separately. Some other researchers (Brown, 1987; Efklides, 2006; Jacobse & Harskamp, 2012; Pintrich, Wolfers, & Baxter, 2000) have surfaced monitoring as a distinct metacognitive process from regulation of cognition. The latter conceptualization of metacognition is more closely aligned to the Singapore Primary Mathematics Syllabus (Ministry of Education [MOE], 2012) which this study is based on.

Consequently, metacognition in mathematical problem solving for this current study is operationalized as consisting of three interdependent components: metacognitive awareness, metacognitive monitoring and metacognitive regulation. The three components are described as follows:

Metacognitive Awareness refers to an individual's awareness of his or her own cognitive and affective resources (Chang & Ang, 1999) in relation to the task. There are three kinds of metacognitive awareness, namely awareness of declarative knowledge (knowing what), procedural knowledge (knowing how), and conditional knowledge (knowing when and why) (Hacker, 1998; Schraw & Moshman, 1995). From this perspective, students' metacognitive awareness about the problem and the strategies or heuristics in problem solving would lead them to devise or select strategies to solve a mathematics problem.

Metacognitive Monitoring refers to periodic engagement in understanding the task performance while executing the cognitive actions (Schraw & Moshman, 1995). It includes actions that keep track of problem solving activities throughout the phases of problem solving such as asking one-self questions to stay on task as one work on the problem.

Metacognitive Regulation refers to decisions made after re-evaluation of cognitive and metacognitive activities throughout the problem solving process (Brown, 1987;

Efklides, 2006; Pintrich et al., 2000). Metacognitive regulation is a more observable aspect of metacognition when it results in decision made to existing plans, and/or monitoring actions, that lead to change in strategic actions based on existing knowledge. However, regulation may also result in non-actions e.g. student progresses with plan after checking on workings half-way through executing the plan.

8.3 Theoretical Background of Self-reporting Approaches

As mentioned in Sect. 8.1, this study will examine three of the most commonly employed self-reporting techniques used in metacognitive studies, namely survey inventory, retrospective self-reporting and qualitative interview.

8.3.1 Self-reporting Survey Inventory as a Research Method

In the 1990s, as there were few standardized or commercial inventory to measure metacognitive skills, a number of inventories were developed by different institutions to examine metacognition processes in mathematics. One of them, the National Centre for Research on Evaluation, Standards, and Student Testing (CRESST) developed a new measure on metacognitive and affective processes of children in the context of a large scale mathematical alternative assessment program, which was believed to result in higher level thinking or metacognitive skills (O'Neil & Abedi, 1996). This inventory was a set of self-regulation measures, with the concept of metacognition adapted from Pintrich and De Groot (1990). The inventory consists of planning, monitoring or self-checking, cognitive strategies and awareness. The construct 'awareness' was an added element that was not present in Pintrich and DeGroot's framework. It was added as the research team believed that there is no metacognition without the participant being conscious of it (see also Flavell, 1979). O'Neil and Brown (1998) used this inventory with 1032 8th grade students to find out if there was differential effect between open-ended problem and multiple-choice question on metacognition and affect.

Wilson (1997) conducted a pilot study to find out key aspects of students' metacognitive thinking as well as assessability of their metacognition. He used multiple methods for data collection, a questionnaire which included open-ended mathematics questions and self-reporting of the problem solving processes, an inventory of metacognitive behavior and an interview to clarify responses after solving a mathematics question. The metacognition inventory was developed as there was no available questionnaire that assessed metacognition of children at grade six. The inventory was based on metacognitive behavior on monitoring one's thinking about learning, behavior, abilities and progress; and, monitoring task such as thinking about choice of strategy, use of strategy and tools. The questionnaire was administered to 15 Grade Six students. Students were asked to indicate the frequency of metacognitive behavior practiced on a Likert scale for the inventory and record what they did as they solved two mathematics problems. The questionnaire was adapted from the mathematics questionnaires used by Stacey (1990) and Fortunato, Hecht, Kehr, Tittle and Alvarex (1991) who designed statements that focused on various stages of problem solving: planning, monitoring, evaluation and execution of the problem. Five students out of the 15 students who answered the questionnaire were interviewed immediately after the implementation of the questionnaire. They were asked to solve another problem and talk through their thinking strategies when interviewed. The questionnaire was reported as effective in data collection while the interview did not seem to provide additional information beyond what was revealed in the questionnaire. Students seemed to have difficulty reporting on their thinking when asked about the strategies they used when solving the mathematics problem.

Survey inventory may not be an optimal instrument to measure metacognition but in research involving large samples, survey inventory was used for pragmatic reasons as other more qualitative methods such as reflective journal and interview might be too time consuming (Sperling, Howard, & Murphy, 2002). Survey inventories are easily administered and scored and that made them useful large-scale assessment tools. For the same reason, the survey inventory is chosen as one of the data collection methods for this study as it involved a large sample.

8.3.2 Written Self-report of Problem Solving Processes as a Research Method

One of the self-reporting methods is to get students to report concurrently or retrospectively during a problem solving session. Students record in writing what they did in the problem solving process instead of verbalisation in an interview. Wilson (1997) found self-report in the form of recording what students did as they solved mathematics problems, a reliable method of data collection when there were other means to triangulate the data such as interviews and survey inventory. This finding was also supported by other researchers (Cohen & Manion, 1994; Schoenfeld, 1985).

Pugalee (2001) in his study used a different method of self-report. He worked with twenty 9th graders enrolled in an Algebra I course to find out whether students' written description of their problem solving processes showed evidence of metacognitive behavior. Each student solved 6 mathematics problems, one problem per day over 6 days. They recorded every thought that came to their mind while solving the problem. The writing demonstrated the existence of a metacognitive framework which is comparable to that of Garofalo and Lester's (1985) metacognitive framework: orientation, organization, execution and verification.

While writing appears to function as a vehicle in finding out the metacognitive behavior that are crucial to metacognition in mathematical problem solving, it is noteworthy that quality written response is highly dependent on students' linguistic skills (Thorpe & Satterly, 1990) and willingness to write in detail. Therefore, written response might not represent the actual thinking (Chi, Bassok, Lewis, Reimann, & Glaser, 1989). With all these limitations in mind, the written descriptions would be interpreted with caution in this study.

8.3.3 Qualitative Interview as a Research Method

Ericsson and Simon (1980) developed three dimensions to verbalization: talk/think aloud, concurrent probing and retrospective probing. Subsequently, other researchers followed with similar classifications (Genest & Turk, 1981; Ginsburg, Kossan, Schwartz, & Swanson, 1983). Ericsson and Simon (1980) defined the three dimensions as follows:

(a) talk/think aloud: students verbalize their thinking with undirected probing concurrent to working on a task. The information reported is that which engaged the student's attention during the problem solving process;
(b) concurrent probing: students report on specific aspects that researchers are interested in during problem solving;
(c) retrospect probing: students are prompted to recall after problem solving and then report on specific aspects that are of interest to the researchers.

While interviews may provide valuable in-depth information, there are limitations to the above methods. Clarke (1992) believed that verbalization would interfere with what it sought to monitor. It might have stimulated metacognition rather than provided data on cognition and metacognition (Wilson, 1997). Webb, Campbell, Schwartz, and Sechrest (1966) also claimed that probing intrudes into the students' thinking and may in the process of measuring, direct or change the thinking strategies and thus reconstruct atypical responses that would not otherwise have occurred.

Wilson (1998) attempted an unconventional interview method termed as 'multi-method interviews' which integrated oral Likert-type responses, self-assessment, observation and think aloud technique. It was used to find out what metacognitive and cognitive actions 30 Australian elementary school students employed during problem solving. Action cards with action statements describing the cognitive or metacognitive actions were used to encourage description of their thinking processes during problem solving. The problem solving process was video-taped. During the interview, the video of the problem solving attempt was replayed. Students could change the action cards to better describe their thinking during problem solving process when they reviewed the video. The interview seemed less disruptive to the problem solving process. However, such changes might not truly reflect the actual process as they were highly dependent on students' memory of the entire process. Besides, the use of action cards might have suggested actions for the students while performing a task so they might affect the metacognitive processes. Students appeared to regularly evaluate their progress during problem solving. The most frequently reported action statement was, "I checked my answer as I was working."

The validity of verbal methods of data collection in the area of thinking processes has been questioned. Articulation of elements of thinking may not represent actual thinking processes and information gathered may be incomplete. Also, it is believed that females excel in verbal ability in the general population (Moccoby & Jacklin, 1974) and this may point towards gender biasness in such a methodology that requires language skill to accurately or elaborately describe various thinking processes.

Schoenfeld (1985) addressed the questions on incompleteness and environmental influence, such as stress due to working under observation during interview or when using pair protocol method. He argued that students working in pairs were likely to produce more verbalization than working individually as both explain and defend their decisions. However, at the same time, such verbalization may also influence the thinking of the other and stimulate metacognitive activity. Furthermore, the verbalized thoughts do not represent the approach taken by each student when working on the task alone.

Having considered the above limitations from literature review, administration of qualitative interview has to be done with care. One main concern researchers raised is that in the process of the interview, students might be stimulated to be more metacognitive.

Besides the manner in which an interview is conducted, another perennial question to ask when interviews play a large part in the data collection process is 'how many interviews are sufficient?' To this question, different researchers have different answers but most agree on one point, that is, there is no formula for calculating the appropriate number of interviews for every study (Adler & Adler, 2012). Most researchers advise looking at the purpose of the interview (Back, 2012), and it is important that the data collected yield sufficient convincing evidence that lead to appropriate inferences and conclusion. For mixed qualitative methods, a small number of interviews suffice (Crouch & McKenzie, 2006). Flick (2012) suggested an alternative perspective on sampling. If a study involved comparison of a few variables, there should be at least two cases from each cell of the grid in the comparative structure. The number of interviews for this study was based on the recommendations in the above literature.

8.4 Problem Solving Metacognitive Framework (PSM)

For this study, a framework for metacognition involved during mathematical problem solving was developed in an attempt to better describe the possible metacognitive strategies occurred while solving mathematics problems. Many frameworks that described metacognition in mathematical problem solving tended to describe metacognitive strategies according to the phases or stages of problem solving (Garofalo & Lester, 1985; Pólya, 1957; Schoenfeld, 1982; Schraw & Dennison, 1994). Similarly, the framework developed for this study is called 'Problem Solving Metacognitive' framework (PSM) and it is two-dimensional. Types of metacognitive strategies are defined by phases and levels. For phases of metacognitive strategies, definitions

Fig. 8.1 PSM framework

were adapted from well-established work developed by Garofalo and Lester (1985), Pólya (1957) and Schraw and Dennison (1994).

In terms of describing the depth (level) of metacognitive strategies, it was lacking in literature. The learning approaches identified by Biggs (1987) were taken into account when developing the PSM framework. While metacognitive activities involving any of the three components of metacognition (i.e. metacognitive awareness, monitoring and regulation) could occur at any of the four phases of the problem solving process, such activities are linked more specifically to different levels of metacognitive strategies. Researchers (Brown, 1987; Flavell, 1979) have suggested that metacognitive awareness precedes monitoring and regulation skills and Schoenfeld (1985) has suggested that students who are better with self-regulating are better problem solvers. Therefore, metacognitive strategies that are related to metacognitive awareness are classified as Surface Strategies while those related to monitoring and regulation are considered Deep and Achieving Strategies respectively. Thus the three levels of strategies are assumed to be hierarchical in nature unlike phases. For example, 'reading over the text a number of times to understand and identify the important points' is a Surface strategy as it is related to metacognitive awareness while a strategy such as 'trying to use the information given in the problem to get more information so as to better understand the problem' is considered an Achieving strategy as it required elements of monitoring and regulating as well. Figure 8.1 below shows the PSM framework for the study. More details on the development of the PSM framework is found in Loh (2015).

Descriptors for each phase of the PSM framework is shown in Table 8.1 and the descriptors for each level of the PSM framework is shown in Table 8.2.

8.5 Research Questions

The research question for this study is:

Table 8.1 Phases of metacognitive strategies in problem solving

Phase 1: Understanding
Strategic behavior to assess and understand a problem: (a) Comprehension strategies (b) Analysis of information and conditions (c) Assessment of familiarity of the problem (d) Initial and subsequent representation (e) Assessment of level of difficulty
Phase 2: Planning
Choice of approach, heuristics and plan to solve the problem.
Phase 3: Execution
Strategic behavior to assess: (a) Execution of plan (e.g. computation, procedure) (b) Progress (c) Trade-off decisions (e.g. speed vs. accuracy, degree of elegance)
Phase 4: Reflection
Evaluation of decisions made on: (a) Processes involved during understanding, planning and execution phases (b) Product (e.g. answers and procedure, reasonableness of answers)

Table 8.2 Levels of Metacognitive Strategies in Problem Solving

Level 1: Surface Strategy
The strategies that reflect the basic essentials of the task requirement such as reading instructions, applying procedural and factual knowledge, and speed in completing task.
Level 2: Deep Strategy
The strategies that reflect sense making in terms of the relationship between different concepts and skills, prior relevant knowledge, and including checking for accuracy and reasonableness in working, procedures and answers.
Level 3: Achieving Strategy
The strategies that reflect on the efficiency in solving a mathematics problem; working beyond what is given and derive at new information to solve the mathematics problem.

> What are the affordances of self-reporting methods used to collect data on types of metacognitive strategies students employed during mathematical problem solving based on the PSM framework?

A mathematics problem in this study takes a broad-based definition of a mathematics question which a problem solver does not have a direct or immediate path to a solution. This has a similar definition in National Council of Teachers of Mathematics (2000) that states "problem solving means engaging in a task for which the solution method is not known in advance".

8.6 Methodology

The research design adopted is one of mixed method. Integration of quantitative and qualitative approaches took place at the research question, data collection and data analysis stages. The quantitative data from the survey inventory provided the frequency count for each type of metacognitive strategies in the PSM framework. The qualitative data from the retrospective self-reports was coded and transformed to frequency count for each type of metacognitive strategies in the PSM framework. These two sets of data were triangulated for analysis and reporting. Triangulation of the frequency data would address the research question quantitatively and supported by the illustration of the metacognitive strategies exhibited in the problem solving process from the qualitative descriptions in the retrospective self-report and qualitative interview which is task-based.

8.6.1 Participants

The current study adopted a convenience sample of 22 classes. There were a total of 783 Secondary One students (age 13 years old) from five Singapore secondary schools. Data was collected in the beginning of Secondary One school term before Secondary One mathematics content was taught. In this way, the results would be a reflection of the types of metacognitive strategies students, with Primary Mathematics education, employed during mathematical problem solving.

8.6.2 Research Methods

Cohen and Manion (1994) suggested that if two measures are used and they agree leading to 'convergent validity', validity is assumed. Based on this argument, the current study uses a mixed method triangulation approach that draws on both quantitative and qualitative data so that the findings in this study can be stated with greater confidence than studies based only on one data source as they draw upon several pieces of data (Lee, 2008; Wilson, 1998; Wong, 1989). The methodology involved in Wilson's study (1997) was taken into consideration as it has close resemblance with the current study that involved survey inventory, self-report on problem solving processes and student interview.

A survey inventory and a problem solving test consisting of four word problems with an element of retrospective self-report on the problem solving process were administered to 783 Secondary One students. Taking into consideration of recommendation by Flick (2012) that interviewees should be selected from various groups of students for comparison between variables, two students were selected from each

Table 8.3 Sample of statements from the survey inventory

Statements	Source	Phase	Level
I read over the text a number of times to understand and identify the important points.	Wong (1989)	Understanding	Surface
I try to determine what the problem required.	O'Neil and Abedi (1996)	Planning	Surface
I ask myself if I have considered all options when solving a problem.	Schraw and Dennison (1994)	Execution	Achieving
I check that my answers are reasonable.	MOE (2007)	Reflection	Deep

class for the task-based interviews. Therefore, a total of 44 students of equal number of male and female students were interviewed.

Survey Inventory. Items from the survey inventory were adapted and modified from validated survey inventories (O'Neil & Abedi, 1996; Schraw & Dennison, 1994; Wilson, 1997, 1998; Wong, 1989). For example, action statements on metacognitive behaviour from Wilson (1997, 1998) were taken into consideration when crafting the inventory items. Some of these action statements such as 'I thought about whether what I was doing was working' and 'I checked my answer as I was working' that described the metacognitive behavior during Execution phase were not found in other survey inventories and were adapted for the current study. Since this inventory was used for grade six students, the language used in the behavior statements was also taken as reference when crafting the survey inventory for current study, given the age of the students in consideration for this study. In addition to items that were modified from other validated inventories, some items were developed and added to the inventory based on the definition of metacognition in the Singapore mathematics syllabus (MOE, 2007). For example, 'I check that my answer is reasonable'. The survey inventory consists of statements that describe various metacognitive strategies involved during problem solving and students were to decide to what extent they used the strategies in each on the 42 statements on a 5-point Likert scale. The survey inventory was validated and it demonstrated an overall high reliability (42 items; $\alpha = 0.93$). More details on the validation of the survey inventory can be found in Loh (2015). The survey statements were also classified by phases and levels of metacognitive strategies. The mean value of each phase and level were tabulated for comparison and analysis. Some sample statements are shown in Table 8.3.

Problem Solving Test with Retrospective Self-report. Four mathematics problems were used as a basis for students to self-report retrospectively of the problem solving processes. The mathematics problems were based on content covered in Primary Mathematics (MOE, 2012). One of the mathematics problems is as follows:

Three students, Ali, Sam and Don were given the following problem:

2A4, 329 and 5B3 are 3-digit numbers.
When 2A4 is added to 329, it gives 5B3.
5B3 is divisible by 3.
What is the largest possible value of A?

Ali thought A could be 1.
Sam thought A was 5.
Don thought A was 4.
Who was correct?

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 2A4
+329
 5B3
```

The metacognitive strategies identified in the retrospective self-report of the problem solving test were coded with a set of codes developed based on the items from the survey inventory (such as those in Table 8.3) so as to facilitate comparison and triangulation. Each metacognitive activity identified in a self-report was coded for both phase and level. The frequency of codes for each phase and level in each self-report were tabulated for comparison and analysis. About 11.2% of the self-reports were coded twice by the researcher and a mathematics educator who has some knowledge of metacognition in mathematics. The inter-rater reliability index attained at 88.4%. The rest of the self-reports were coded by the researcher alone. These transcripts were coded twice by the researcher at intra-rater reliability of 92% as there was no available coder with the expert knowledge on metacognition and mathematical problem solving. More details on the coding of the retrospective self-reports is found in Loh (2015).

Task-based Interview. This task-based interview used one mathematics problem, similar to those in the problem solving test for retrospective self-report. Each student solved the mathematics problem and simultaneously use 'think aloud' strategy to verbalize his/her thoughts. It was deliberate to use concurrent reporting of the problem solving process in task-based interview so as to provide a different perspective from retrospective self-report in the problem solving test. With reference from literature review on studies by Clarke (1992) and Wilson (1997), it is important to ensure that the interviewer restrains from asking questions when the interviewees articulate their problem solving process based on a task. Minimizing prompts would minimize stimulation to their metacognition. However, if the student remained silent for more than 15 s, the researcher would prompt the student in a neutral manner e.g. 'keep talking', 'don't keep quiet/silent', 'tell me what's on your mind', 'what are you thinking now?'. These prompts were only used as incitements to sustain thinking aloud and would have little or no influence on the thought processes (Ericsson, 2006). The resulting interview protocols were concurrent rather than retrospective in nature which, hopefully, would be less vulnerable to memory distortion and a closer representation of their metacognitive activities at the time of the problem solving process (Veenman, 2005). This would also provide information on metacognition with a different perspective from retrospective self-reporting. It would provide a descriptive picture of metacognitive strategies employed by students while solving problem. This would serve to complement retrospective self-report of problem solving processes and the survey inventory. The interviews were tape-recorded and selected portions of the audio recordings were transcribed. The metacognitive strategies identified in the transcript were also coded with the same set of codes as the retrospective self-report.

While this set of data could be used for triangulation with the survey inventory and retrospective self-report, the small number of data as compared to survey inventory and retrospective self-report would have to be interpreted cautiously. Each metacognitive activity identified in the transcript of the interview was coded for both phase and level.

The transcripts were coded with the same list of metacognitive activities as those in the survey inventory and retrospective self-report. The transcripts were coded twice by the researcher for intra-rater reliability. Coding of the transcripts required careful analysis especially when description of these activities might be unclear or disjointed. The mean value of the frequency of student usage of metacognitive strategies by phases and levels in the transcripts were calculated to triangulate with those of survey inventory and retrospective self-report.

8.7 Results

Data from two data collection instruments, survey inventory and retrospective self-report, was used to analyze the frequency of student usage by phases and levels of metacognitive strategies first. After triangulation and comparison of results from these two instruments, data from the task-based interview would be analyze to support the findings qualitatively.

8.7.1 Frequency of Student Usage of Metacognitive Strategies by Phases in Problem Solving

There were two sources of data for analysis. The frequency of codes for each phase in the retrospective self-report was tabulated and then the phase mean would be calculated. For the survey inventory, the mean value of each phase would be calculated. Figure 8.2 shows the bar graphs of phase mean with data from survey inventory and retrospective self-report.

From the survey inventory, the highest frequency of student usage of metacognitive strategies is at Phase 4 (M = 3.71, SD = 0.71), followed by Phase 2 (M = 3.65, SD = 0.69). Comparatively, students scored lowest in Phase 1 (M = 3.46, SD = 0.59) indicating that students seemed least active in metacognition during Phase 1 of problem solving. However, it is important to note that the mean values of all the four phases are relatively closed, in the 3 to 4 range. From the retrospective self-report, the highest frequency of student usage of metacognitive strategies was at Phase 1 (M = 2.14), followed by Phase 3 (M = 1.23). Phase 4 (M = 0.08) had the least number of occurrences of metacognitive strategies.

The results from both data source ran contrary to each other. The highest frequency of student usage of metacognitive strategies occurred at Phase 1 in the retrospective

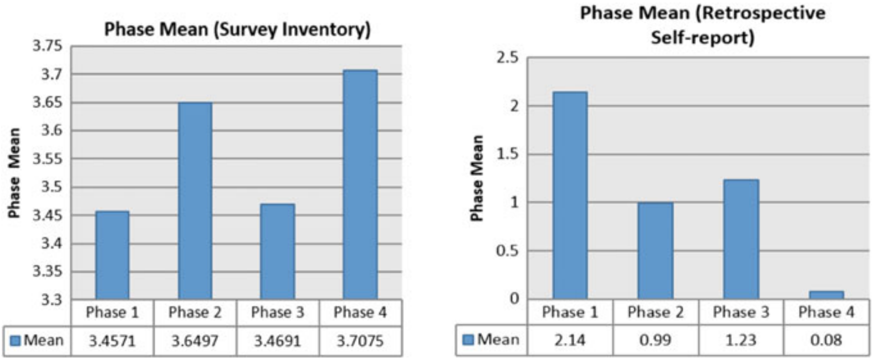

Fig. 8.2 Phase mean with data from survey and self-report

self-report but it was at Phase 4 in the survey inventory. On the other hand, the lowest frequency of student usage of metacognitive strategies occurred at Phase 4 in retrospective self-report but it was at Phase 1 in the survey inventory. The findings from the survey were similar to that from Wong (1989) who also used survey inventory as the data collection instrument.

In conclusion, there is a difference in the frequency of student usage for each phase of metacognitive strategies while solving mathematics problems. However, there is no clarity at which phase where the highest or lowest frequency of student usage of metacognitive strategies occurred since the results from the survey and the self-report do not agree.

8.7.2 Frequency of Student Usage of Metacognitive Strategies by Levels in Problem Solving

Similar to phases of metacognitive strategies, there were two sources of data for analysis. For the retrospective self-report, the frequency of codes for each level was tabulated and then the level mean would be calculated. For the survey inventory, the mean value of each level would be calculated. The data set used for analysis was the same as that for phases of metacognitive strategies, $N = 783$.

Coding of the retrospective self-report was based on matching the identified metacognitive strategies with the codes regardless of the hierarchical levels i.e. the statements would be coded literally without presumption of prior metacognitive activities. For example, the statement ‘I was thinking how to find out if … in an easier way’ was coded as an Achieving strategy only, even though the student was likely to also have checked the workings (a Deep strategy). Figure 8.3 shows the level mean with data from survey inventory and retrospective self-report.

For the retrospective self-report, there were more Deep strategies ($M = 2.68$) than Surface ($M = 1.25$) and Achieving ($M = 0.51$) strategies. Levels of metacognitive

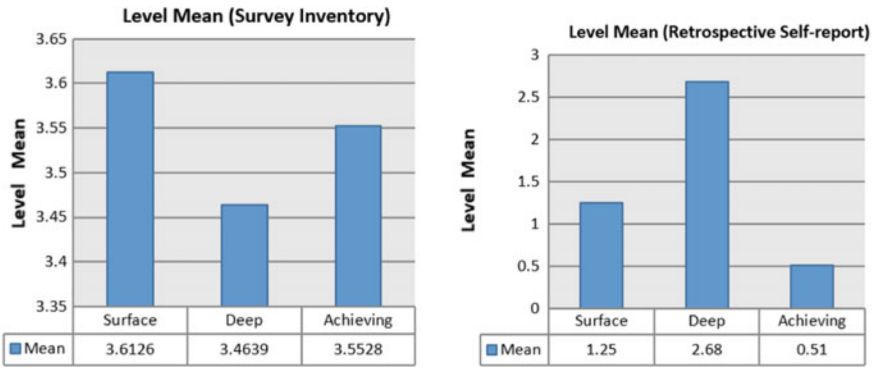

Fig. 8.3 Level mean with data from survey inventory and retrospective self-report

strategies being hierarchical in nature could mean that those who practiced deep strategies also practiced surface strategies. Similarly, those who practiced achieving strategies would have also practiced surface and deep strategies. However, the other way was not true.

For the survey inventory, the highest frequency of student usage of metacognitive strategies is at Surface Level (M = 3.61, SD = 0.49) followed by Achieving Level (M = 3.55, SD = 0.65), and lastly at Deep Level (M = 3.46, SD = 0.61). Again, the mean values of all the three levels are relatively closed, in the 3 to 4 range.

Similar to the results for frequency of student usage of metacognitive strategies by phases, the results by levels from retrospective self-report and survey inventory did not seem to match. Retrospective self-report showed that more deep strategies were exhibited in the problem solving process while the survey inventory reported that students employed more surface strategies.

In conclusion, there is a difference in the frequency of student usage for each level of metacognitive strategies while solving mathematics problems. However, there is no clarity at which level the highest or lowest frequency of student usage of metacognitive strategies occurred since the results from the survey inventory and the retrospective self-report do not agree.

8.7.3 Data from Task-Based Interview

As part of the methodology in this study, the task-based interview is a qualitative measure to triangulate and support the quantitative findings from the survey inventory and retrospective self-report. Figure 8.4 shows the bar graphs of phase and level means with data from the task-based interview.

The highest frequency of student usage of metacognitive strategies is at Phase 3 (M = 3.03), followed by Phase 1 (M = 2.10). Comparatively, students scored lowest

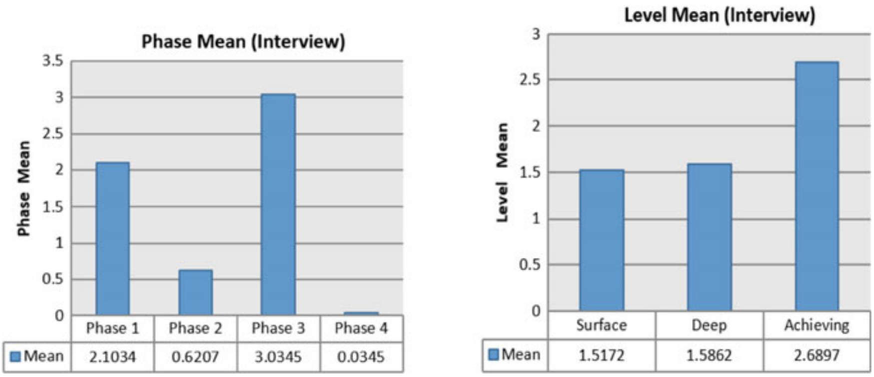

Fig. 8.4 Phase and level mean with data from task-based interview

in Phase 4 (M = 0.03) indicating that students seemed least active in metacognitive processes during Phase 4 of problem solving. The highest frequency of student usage of metacognitive strategies was at Achieving Level (M = 2.69), followed by Deep Level (M = 1.59). Surface Level (M = 1.52) had the least number of occurrences of metacognitive strategies.

As a third measure to triangulate with the results from survey inventory and retrospective self-report, the results from the interview also do not seem to match in terms of the frequency of student usage of metacognitive strategies by phases and levels.

8.8 Discussion

While the survey inventory, retrospective self-report and task-based interview are familiar instruments used in research on metacognition, they are often used singly and researchers have expressed concerns in their reliability (Thorpe & Satterly, 1990; Clarke, 1992; Sperling et al., 2002). This study explores the use of more than one approach in data collection with the intention to be able to state the findings with greater confidence as they draw upon several pieces of data. There were two considerations when choosing the data collection instruments: large sample size and enable analysis of individual student's metacognitive behaviors. Balancing theoretical and practical issues is a challenge. Sperling et al. (2002) supported the use of survey inventory for large sample and qualitative methods might be too time consuming (Schellings & Van Hout-Wolters, 2011). However, survey inventory only requires students to response to a set of pre-set statements on Likert scale and does not provide an avenue for students to describe their thinking individually. An open-ended approach that enables students to articulate in their own words their thought processes pertaining to specific problems would provide data for analyzing individual

Table 8.4 Frequency of student usage by phases of metacognitive strategies

Instruments	Most frequent	2nd most frequent	3rd most frequent	Least frequent
Survey inventory	Phase 4	Phase 2	Phase 3	Phase 1
Retrospective self-report	Phase 1	Phase 3	Phase 2	Phase 4
Task-based interview	Phase 3	Phase 1	Phase 2	Phase 4

Table 8.5 Frequency of student usage by levels of metacognitive strategies

Instruments	Most frequent	2nd most frequent	Least frequent
Survey inventory	Surface	Achieving	Deep
Retrospective self-report	Deep	Surface	Achieving
Task-based interview	Achieving	Deep	Surface

student's metacognitive behaviors, retrospective self-report of problem solving processes and task-based interview are such instruments. Self-reporting retrospectively and concurrently of problem solving processes provided different perspectives of metacognition during problem solving too. While the study by Wilson (1997) used such similar mix of approaches, that study had a small sample of fifteen students and the different approaches were meant to collect more data on students' metacognition rather than triangulation between data sets, therefore there was no report on conflicting results from that study. In fact, literature reviewed did not surface any study that have triangulated data collected through three different instruments.

8.8.1 Triangulation of Data from Three Data Collection Instrument

The data from the three data collection instruments seemed to offer different results in terms of most or least frequent student usage of metacognitive strategies by phases and levels. A summary of the results is showed in Table 8.4 for the frequency of student usage by phases of metacognitive strategies and Table 8.5 for the frequency of student usage by levels of metacognitive strategies.

While it seems like the instruments are not reliable tools for data collection in the current study, from another perspective, it could mean that the nature of each instrument may have drawn different aspects of metacognitive responses from students. The survey does not have specific mathematics tasks for students to relate to when responding to the survey items so the responses were based on a general problem solving situation. This differs from the retrospective self-report and task-based inter-

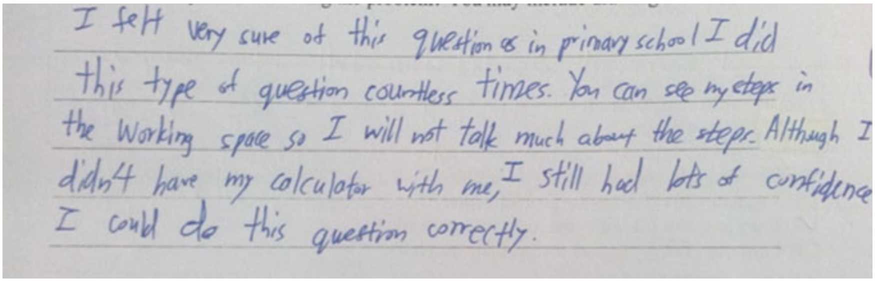
I felt very sure of this question as in primary school I did this type of question countless times. You can see my steps in the Working space so I will not talk much about the steps. Although I didn't have my calculator with me, I still had lots of confidence I could do this question correctly.

Fig. 8.5 An example of a student's self-report

view that are more closely associated with students' metacognitive behaviors when executing a mathematical problem solving task.

Students might have responded to survey items according to social desirability (Cromley & Azevedo, 2011) rather than what they actually do during problem solving. For example, mathematics teachers in Singapore might have often reminded students to 'check their answers' before handing in their work but in reality, students might not have checked the answers thoroughly and the manner of checking might not involve a metacognitive decision. This could be one of the reasons that the survey results showed Phase 4 (reflection) with the highest mean value in survey but not so for retrospective self-report and task-based interview where students worked on the tasks.

In addition, in terms of phases of metacognitive strategies, it is highly likely that students are not able to clearly differentiate metacognitive activities by phases in the survey inventory since they are not concurrently working on a mathematics problem. Students, for example, might not differentiate 'attending to instructions carefully' in Phase 1 (Understanding) from 'stop and reread the problem when I get confused' in Phase 3 (Execution). They might see both actions simply related to reading the problem and thus, might not accurately reflect the frequency of usage at different phases. Retrospective self-report and task-based interview, on the other hand, could better help students to relate to the phases in problem solving as they worked on an actual problem. This may explain some similarities in findings obtain from these two instruments. Both have the same least frequent phase while the first two highest frequent phases were Phase 1 and 3.

When describing the problem solving processes in the retrospective self-reports, students tended to describe their understanding of the problem (Phase 1) more than the other phases. The activities relating to Phase 3 and 4 are more evident in the 'working'. An example of a retrospective self-report that illustrated the above is shown in Fig. 8.5.

The example in Fig. 8.5 shows that the student deliberately omitted description of Phase 2, 3 and 4 processes which evidently were in the 'steps in the working space'. This is a typical example of a retrospective self-report. As such, metacognitive strategies exhibited in a retrospective self-report was limited to mainly Phase

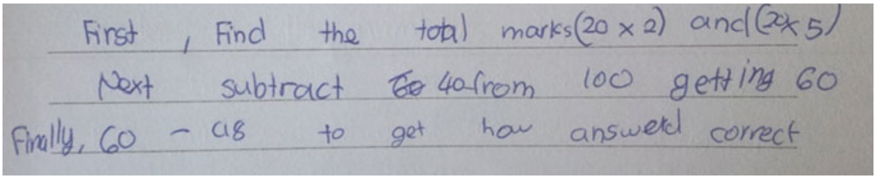

Fig. 8.6 An example of a self-report for Question 4 that described only the procedures in working

1 processes. For those who tried to describe the other phases in the retrospective self-report, many times students ended up describing the 'workings' i.e. stating the procedural steps in words, instead of describing the thinking behind the decisions made when solving a problem. An example that illustrates a typical retrospective self-report that described the procedural steps in solving a problem is shown in Fig. 8.6. Nonetheless, overall, retrospective self-report seems to present more accurate data on phases of metacognitive strategies than survey inventory.

Similarly, as students worked on the mathematics task during the interview, metacognitive activities during various phases are more explicit as students articulate their thoughts while solving the problem concurrently. Even when a student moved back-and-forth between different phases, their articulation would be easily identified by phases in problem solving. This method is probably even more accurate in capturing data by phases in problem solving than retrospective self-report which is still subjected to memory distortion, though it is also true that the 'think aloud' strategy during the task-based interview might have prompted students to be more metacognitive, as reflected by the literature.

In terms of levels of metacognitive strategies, it is still possible for the students to reflect the metacognitive strategies at each level in a survey inventory without actually working on a mathematics problem. Returning to the example cited earlier, similar metacognitive activities ('attending to instructions carefully' and 'stop and reread the problem when I get confused') were at the same level (i.e. Surface level) despite the differences in phases (i.e. Phase 1 and 3). Therefore, a survey would still be a reasonable instrument for use in the investigation of levels of metacognitive strategies. On the other hand, as observed in the retrospective self-reports, even a brief description of a metacognitive activity could be easily identified by phases of problem solving but it is not true about the depth of the strategy. Figure 8.7 shows an example of a student's self-report that explained the situation described.

In trying to understand the problem (Phase 1), the student wrote down notes followed by execution of the plan (Phase 3) as the student did the working, labelled and organized the working. However, the brief statement 'wrote down notes' described an action but the statement itself is insufficient to decipher whether it is a metacognitive strategy at Surface level such as 'attending to the instructions carefully' or Achieving level such as 'spending time to recall key points'. Sometimes with the help of the working, it could help to determine the level of metacognitive strategy while other times it remains vague.

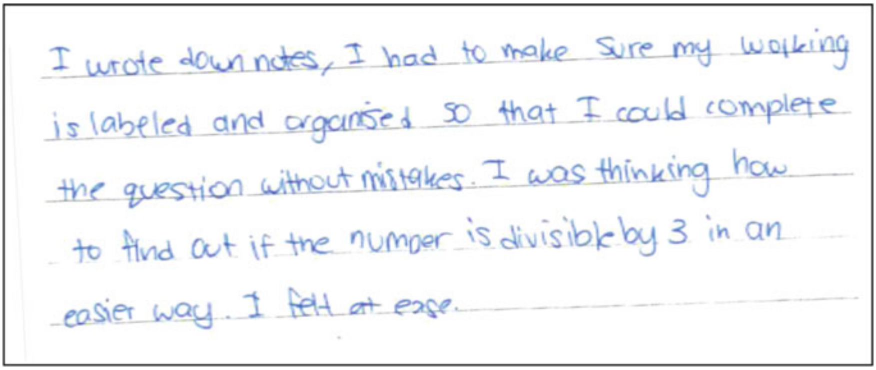

Fig. 8.7 Example of a student's self-report on Question 3 problem-solving working

8.9 Conclusion and Implications

The affordances of each data collection instrument would only provide one dimension of metacognition in mathematical problem solving. For large-scale study, the survey inventory is an instrument appropriate for use in the investigation of levels of metacognitive strategies while retrospective self-report and task-based interview is more appropriate for use in the investigation of phases of metacognitive strategies. Even though the data did not agree with each other, in this case, it would not imply that there is no 'convergent validity (Cohen & Manion, 1994) as different methods measure different dimension of metacognition. For a smaller sample size, perhaps task-based interview could be used to identify both the phases and levels of metacognitive strategies.

While it may be argued that if the objective of a study is only on one dimension of metacognition in mathematical problem solving, just an appropriate data collection instrument is sufficient. However, the findings may not provide a true picture of each student's overall metacognitive behavior.

In conclusion, a single measure would only provide a skew perspective about metacognition in mathematical problem solving in general. Thus, it would be difficult to compare various studies, internationally or locally, that used different data collection instruments. Findings from different research studies on participants, tasks, topics and methodologies varied substantially may not be generalized. There is, therefore, a need to use multiple approaches to collect data in order to have a good understanding of metacognition in mathematical problem solving from different perspectives before the findings could be generalized.

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Chapter 9
Assessing Inquiry-Based Mathematics Education with Both a Summative and Formative Purpose

Maud Chanudet

9.1 Introduction

Posing and solving problems have always been at the core of mathematics. Throughout history, mathematics has been constructed to answer questions and problems (Charnay, 1988). Halmos summarizes the place of problems in mathematics by considering it as a key component of the field: "What mathematics really consists of is problems and solutions" (1980, p. 519).

Problem solving, as a fundamental element in the construction of mathematics knowledge, has also been one of the major goals in mathematics education for a long time. Recently, the increasing interest in inquiry-based mathematics education (IBME), which broadly considers problem solving, encompasses common goals for mathematics and sciences education. In many countries, problem solving and IBME take a central part in the curriculum, as a means to develop specific mathematical contents and knowledge, but also as a goal in itself. "[Mathematical problem solving] has infused mathematics curricula around the world with calls for the teaching of problem solving as well as the teaching of mathematics through problem solving" (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016, p. 1).

This emphasis on problem solving and inquiry in mathematics education constitutes a change in mathematics education goals. As Artigue and Blomhøj (2013) remind us, there are tensions between the teaching and learning of specific concepts and techniques, organized according to a curricular progression and mathematics topics (space, numbers, operations, algebra, functions for instance) and the accent put on inquiry habits of mind and heuristics for problem solving. This approach raises a major question mark for teachers who struggle to assess such problem solving abilities and inquiry habits of mind.

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In that sense, new approaches to classroom assessment are needed in order to give teachers access to students' thinking and to improve teachers' assessment skills (Goos, 2014). Exploring new forms of classroom assessment seems all the more important as the assessment of students' learning has an impact on what is taught and how it is taught (Harlen, 2013). In the case of IBME, specific issues related to assessment arise: to understand what teachers want students to know and consequently what needs to be assessed, how to assess it, and how to improve students' problem solving competencies.

Thus, summative and formative functions of assessment pertaining to students' problem solving competencies remain a primary concern of teachers and the research community.

9.2 Research Questions and Outline of the Chapter

In this chapter, I partly address these issues in a specific context: the case of a problem solving centered course given in the canton of Geneva in French-speaking Switzerland and called *mathematics development course* (MDC). The global objective of my research is to find out if and how using an assessment tool such as a grid of criteria, can be useful for summative assessment, and can also encourage formative assessment processes and strengthen assessment for learning, in the case of a problem solving centered course. My research questions are: Is a common assessment tool useful for teachers to assess students in a summative way? Can a common assessment tool encourage formative assessment processes during specific problem solving lessons?

The remaining part of the chapter is structured as follows. First, I explain the context of this special course and present my theoretical framework based on theories from didactics of mathematics but also on theories from the field of the assessment of students' learning. I then expose the results of a survey focusing on the teachers' perception of MDC and their summative assessment reports on assessing students' problem solving competencies. After that, I explain how the survey leads me to work in a collaborative way with two teachers of MDC in order to construct a tool for mathematics teachers to assess students' problem solving performances with both summative and formative purpose. Finally, I present an exploratory study that analyzes a teacher's formative assessment practices within the MDC context in order to understand if and how she develops informal formative assessment and how she refers to this tool.

9.3 Theoretical Framework

9.3.1 Problem-Solving

Problem Solving and IBME as Learning Goals

Problem solving can be considered as "a response to a question for which one does not already know a method by which it can be answered" (Monaghan, Pool, Roper, & Threlfall, 2009, p. 24). In the French mathematics teaching tradition, problem solving has been seen for many years as a means to develop specific mathematical content and knowledge (Brousseau, 1998). It can be used "to attest the appropriation of mathematical knowledge but also to motivate the need for this knowledge and make its learning meaningful" (Artigue & Houdement, 2007, p. 368). However, for the past couple of years, many countries, and especially European countries, have been emphasizing problem solving in mathematics and inquiry in science education as a learning goal for its sake. Noticing that young people are less and less interested in mathematics and sciences, the European Rocard's report (Rocard, Csermely, Jorde, Lenzen, Walberg-Henriksson, & Hemmo, 2007) promotes a wider implementation of inquiry-based mathematics and science education (IBMSE) in classrooms as a tool to make sciences and mathematics more attractive to students. A lot of European projects have emerged recently to support the implementation and the development of IBME in mathematics teaching (Artigue & Blomhøj, 2013). However, this increasing interest in IBMSE has not been followed by a concise and commonly shared definition (Dorier & Garcia, 2013). To summarize, it

> refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions (like carrying out experiments, systematically controlling variables, drawing, diagrams, calculating, looking for patterns and relationships, making conjectures and generalizations), interpret and evaluate their solutions and communicate and discuss their solutions effectively. (Dorier & Maass, 2014, p. 300)

The generic goal of IBME is to make students work in ways similar to the one of mathematicians and scientists (Artigue & Blomhøj, 2013; Dorier & Garcia, 2013; Dorier & Maass, 2014) and to make students familiar with a scientific approach to solve problems. But as Dorier and Maass say "inquiry based mathematics education remains quite marginal in day-to-day mathematics teaching" (2014, p. 303) and implementing IBME in classrooms remains a crucial issue. This difficulty to make IBME appear in day-to-day teaching can be attributed to multiple reasons: lack of training for teachers, lack of time, weight of traditions (Dorier & Garcia, 2013).

The Place of IBMSE and Problem Solving in the Mathematics Curriculum in French-Speaking Switzerland

In French-speaking Switzerland, the shared curriculum for compulsory education promotes IBMSE in teaching mathematics and sciences. Moreover, in order to stress the strong link between mathematics and experimental sciences, these subjects are included in the same field and have general instructions in common. The shared goal is to promote students' scientific processes of thought. As a means to engage students in IBME, mathematics teachers are encouraged to focus on problem solving. In French speaking Switzerland, the implementation of IBME into teaching of mathematics is mainly approached through problem solving. It is though not unusual, as highlighted by Artigue and Blomhøj (2013) to make connections between problem solving and IBME because "problem solving competences and metacognitive skills can be interpreted in terms of inquiry habits of mind and related to the five essential ingredients attached to inquiry" (p. 802). These five ingredients are: valued outcomes, classroom culture, teacher guidance, type of questions and what students do.

Mathematics Development Course: A Special Course Aiming at Developing IBME and Student's Problem Solving Competencies

Therefore, in the canton of Geneva, the MDC has been created to support the integration of IBME in class. The main goal of this course is to improve the students' problem solving competencies, as described in the institutional guidelines (cf. Annexe 2, part I). This annual course is designed for a 45-min period per week and is delivered to 13–14 year[1] old students with a science profile. It is also important to mention that students do not necessarily have the same mathematics teacher as for the ordinary mathematics courses. Consequently, a teacher of the MDC does not necessarily know what students have already learnt in ordinary mathematics classes, and what problems they have already worked with.

Furthermore, according to the institutional guidelines of the MDC (Annexe 2, parts II and III), teachers have to provide students with specific activities related to mathematical strategies (analogical reasoning, study all cases, counterexample, introduction to proof, etc.) in order to establish debate rules and to develop a scientific approach following the pattern "make trials—conjecture—test—prove". These two goals aim to develop a systematic approach to solving problems. In that sense, teachers are invited to propose *open-ended problems* (Arsac, Germain, & Mante, 1991) to students, which is, in France and in French speaking Switzerland, a traditional way of introducing students to IBME (Annexe 2, part IV a). An open-ended problem is a problem that has a short statement, has no obvious solution and enables students to find an easy but not sufficient method to solve it. An example of a well-known open-ended problem is "How many diagonals does an n-sided polygon have?" Facing such a problem, students may use or learn some of the strategies mentioned above. These goals, establishing debate rules and developing a scientific approach, are described in a general way in the guidelines and can lead to wonder what these mathematical intended learning outcomes really are, and if it is possible to define it more precisely.

The Intended Learning Outcomes of MDC

For Hersant (2012), a scientific approach (as the quadruplet "make trials-conjecture-test-prove") cannot be considered as a relevant learning goal, especially because this goal is unclear, non-unique and too ambitious. When focusing on the intended

[1]Grade 8.

learning outcomes expressed in the curriculum, according to Hersant (2010) it is not possible to consider the scientific approach following the pattern of “make trials—conjecture—test—prove” as the only one, especially because the articulation between the step of tests and the step of proof is problem-specific. She also emphasizes that what gives this approach a scientific dimension is not the existence of trials, conjecture and proof but the articulation among these. She concludes her study with a last point concerning a possible intended learning outcome of problem solving for 10–11-year-old students. According to her, fundamental mathematics knowledge about problem solving is the articulation between the facts register and the reasoning register. Considering a specific problem, the intended learning outcome should be that students ponder whether solving this problem is possible or impossible, and why. Their reflection should be related to the link between the existence and the universality of the solution, in order to give them access to rationality and not only to empiricism. But these goals are not expressed in the curriculum and even if it was the case, it is easy to imagine that it would not be easy to implement in day-to-day practices.

The first goal, developing a scientific approach when solving problems, is not so obvious and unique, neither is the second. Debate rules can indeed refer to logical rules (for instance several examples do not prove a proposition, a counterexample is sufficient to disprove a conjecture, etc.) or to social rules (for instance listen to the others, etc.). It seems important to distinguish these two kinds of debate rules in order to make students aware of what are the specific logical rules that constitute the field of mathematics.

The Adaptation of Polya’s Model in of the Guidelines for MDC

A method of problem solving inspired by Polya (1945) is described within the institutional guidelines (Annexe 2, part IV b). It is presented both in the curriculum (for the teachers) and in the students’ theoretical book (for the students). This adaptation consists of three parts: part 1 (appropriation of the wording) corresponds with Polya’s model, part 2 (data processing) combines the steps 2–4 from Polya’s model (to devise a plan, to carry out the plan, to review and extend) and part 3 (communication of research procedures and results) related to communication has been added. This last part emphasises that in the curriculum of MDC, communication skills play a major role.

Even though, in his model, Polya insists on the importance of relying on past experiences to solve new problems (Liljedahl et al., 2016), and this not only during the step process related to devising a plan (Have you seen it before? Do you know a related problem?) but also in the last part linked to looking back (Can you use the result, or the method, for some other problem?), the curriculum does not seem to reflect this importance when adapting Polya’s model. Yet, this ability to recognize relevant elements in previously solved problems and to use them when solving a new problem is a key component of problem solving competencies. Julo (2002) claims that students are competent problem solvers when they succeed in recognizing such common and relevant elements among problems and that it is central to develop a memory of previously solved problems, so called *problems schemas*. But even if this skill (referring to previous experiences) is not emphasized in the model of problem solving developed in the curriculum, as mentioned above, one of the strategies students have to learn is analogical reasoning. Using analogical reasoning means that when facing a new problem we recognize similarities between situations or objects to transfer properties of solving methods from one to another (Weil-Barais, 1993). Thus analogical reasoning is close to the skill of referring to previous experiences.

According to research studies, "the identification of problem solving strategies and the process of modelling their use in instruction was not sufficient for students to foster their comprehension of mathematical knowledge and problem solving approaches" (Santos-Trigo, 2014, p. 498). The remaining key issue is to identify how teachers can help students recognise relevant common elements between problems, create appropriate links among these in order to transfer and apply their knowledge in new situations. Finally, the explanation of Polya's model has not been followed by the expected results (Julo, 2002). In that sense, one can wonder what is really expected according to the institutional guidelines by proposing this problem solving model to students. To ask students to solve problems following the steps described in the model? To give teachers some guidelines to elicit students' understanding, students' conceptions? There is no more information in the institutional guidelines for MDC. In practice, I observed that there are many classes in which teachers ask students to write down their research according to the three steps adapted from Polya's model (appropriation of the wording; data processing; and communication of research procedures and results). Students start their narration by rephrasing the problem, then they explain their research and finally they give their conclusion about the problem. In that sense, this method of problem solving inspired by Polya seems to be a guide for students to write down their research narrative more than a tool to help them solve the problem.

Consequently, the guidelines for MDC do not seem to clearly define what is institutionally expected about students' problem solving competencies and how teachers can foster such abilities. Identifying what students have to learn and to know about problem solving is still a problematic issue. It is also the case in England where Monaghan et al. claim that "expectations around problem solving are less well established and less secure and would need to be developed" (2009). The identification of the intended learning outcomes from IBME is by no means obvious, even for teachers, and the danger is that students might not be aware of what they are supposed to learn and to know. That is why IBME learning goals should be at the midst of specific discussions with students in class. Even though such discussions should also be encouraged during traditional mathematic classes, it is all the more important in the case of IBME.

Moreover, in MDC, teachers have to improve students' competencies and at the same time, assess them very frequently, about one time every four lessons. It raises a crucial question that is at the core of my research: how to assess and foster students' problem solving competencies? It leads me to move my attention to assessment of student's learning.

9.3.2 Assessment of Student's Learning

Formative and Summative Assessment of Student's Learning

According to Allal (2008b) assessment is summative as soon as a synthesis of the competencies and knowledge learnt is established by the student at the end of her/his course. The aim is to confirm that student's learning is aligned with system's intended outcomes. According to the distinction made by Scriven (1967) and then by Bloom (1968) between the summative and the formative functions of assessment, Black and Wiliam (1998) talk about formative assessment as soon as it is possible to get information, that they call feedback, about the gap between students' real level and the one they have to achieve. The notion of feedback is a key component of formative assessment. Formative assessment contains

> all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities. (Black & Wiliam, 1998, pp. 7–8)

However, feedback is considered formative only if the information gathered is used to improve performance. The French community, considering formative assessment in an expanded way, deals instead with the notion of regulation (Mottier Lopez, 2012) that takes into account feedback, but also the adaptation that can be provided to teaching and learning.

The definitions of summative and formative assessment emphasize that identifying assessment according to *when* it occurs (after a phase of teaching vs. within a teaching activity for instance) or *how* it occurs (paper-pencil test vs. worksheet for instance) seems less relevant than distinguishing assessment according to its function. But it does not mean that these two principal functions of assessment (summative and formative) cannot coexist. Thus some researchers (Allal, 2011; Harlen, 2012) argue that they can coexist in what Earl (2003) calls *assessment for learning*. That is, the same assessment activity can serve a summative and a formative purpose.

One the one hand it means that data collected by the teacher can be used to give students a mark, to acknowledge students' competencies but also to improve learning and teaching. On the other hand, for Shavelson et al., "formative assessment could serve summative needs" (2008, p. 298). Nevertheless, Harlen (2012) claims that it is necessary to be careful when using the same evidence for both, summative and formative purposes in order to protect the integrity of assessment and ensure that it has a positive impact on students' learning.

Criteria and Indicators

Whatever its function, assessment activity implies "the generation, interpretation, communication and use of data" (Harlen, 2013, p. 7). Data used by teachers to assess students within IBME are mostly judged "in relation to criteria, in which the standard of comparison is a description of aspects of performance" (Ibid., p. 7). Criteria are considered as ways to look at the students' production according to expected qualities (Gerard, 2010). If the criteria remain at a general level, teachers also need to define indicators related to each criterion. Indicators, more specific than criteria, make the evaluator aware of what she/he should look for in the students' production. Perhaps even more than for other mathematics' topics, defining criteria to assess students' problem solving competencies is difficult and deserves particular attention.

According to Allal (2008a), using an assessment tool such as a rubric of assessment criteria may help both: to control the action of the evaluator and to amplify her/his judgment skill. In that sense, teachers can use a tool to assess students' problem solving competencies, which can be a grid of criteria and indicators. Referring to this tool, teachers can assess students for summative purposes more efficiently. Moreover, teachers can get some information about students' difficulties and then adjust teaching in order to improve learning. Even though it is necessary for teachers to refer to criteria and standards to judge the quality of students' performance, the validity of assessment is enhanced when the teacher explicitly communicates these elements to students (Goos, 2014). Nicol and Macfarlane-Dick (2006) identify seven principles of good feedback practice. One of them is that a "good feedback practice helps clarify what good performance is (goals, criteria, expected standards)" (2006, p. 205).

Involvement of Students in Their Assessment

Many research studies demonstrate that the more students are involved in their assessment, the more competent they are. In that way, they can assess themselves (self-assessment), assess a peer and compare each other (peer-assessment) or assess themselves and compare it with the teacher's assessment (co-assessment). "Involving students in self-assessment can enhance metacognitive self-regulation and help students become familiar with criteria and standards that will be used to judge their performance" (Goos, 2014, p. 416). In that sense, a good feedback practice "facilitates the development of self-assessment (reflection) in learning and encourages teacher and peer dialogue around learning" (Nicol & Macfarlane-Dick, 2006, p. 205).

It is thus all the more crucial to ensure that students understand the target of their work and that they grasp what is expected as these elements are among the key components of formative assessment (Harlen, 2013). However, it means once again that "students need to have some understanding of the criteria to apply in assessing their work" (Harlen, 2013, p. 7). Therefore, specific discussions with the students about the assessment criteria and the expectations of their learning outcomes should be encouraged. Indeed, students' interpretation or understanding of the verbal description of criteria, such as that expressed in a grid of criteria, may differ widely from the teacher's interpretation. Having discussions with students about the meaning of the criteria and about what counts as good performances can prevent these misunderstandings (Goos, 2014).

Using a grid of criteria can ensure that the students get more involved in and thus more responsible for their assessment since they can compare their production with the expected qualities, and get feedback about what they do and what teachers expect them to do. This can result in self-assessment, peer-assessment and co-assessment and help students to play an active role in assessment and to regulate their learning.

From Formal to Informal Assessment

To classify classroom formative assessment, Shavelson et al. (2008) use a continuum that ranges from formal embedded assessment (teacher plans to obtain information about students' learning) to informal, on the fly, formative assessment (evidence of students' learning is obtained within usual activities). It means that formative assessment does not take a unique form but that it can be planned or not, it can refer to formal tools to collecting data or not, etc. Adopting this point of view, formative assessment can be considered as a practice integrated within the learning process (Lepareur, 2016). Referring to formative assessment in IBME is all the more relevant that

> the practice of formative assessment, through teachers and students collecting data about learning as it takes place and feeding back information to regulate the teaching and learning process, is clearly aligned with the goals and practice of inquiry-based learning. (Harlen, 2013, p. 20)

Ruiz-Primo and Furtak focus on informal formative assessment which "can take place in any student-teacher interaction" (2004, p. 3). Teachers, in order to acquire information about students' level, conceptions, skills, etc., may use daily and informal classroom talks. Such talks are called *assessment conversations*. In that sense, Black (2013) considers that any piece of instruction comprises an *interaction* step in which the learning plan is implemented and in which the collected evidence has to be used to develop a *learning dialogue.*

Duschl (2003) describes three domains that can promote the assessment of inquiry: the conceptual domain (reasoning and the use of science concepts), the epistemic domain (knowledge structures, rules and criteria used to determine what counts as key components of inquiry) and the social domain (representation and communication of concepts, knowledge, conceptual framework). Ruiz-Primo and Furtak (2004, 2007) encourage teachers to focus the assessment conversations on these three domains to enhance formative processes. They characterize teachers' activity during assessment conversations according to ESRU cycles, in which E means *eliciting* (use of strategies that allow students to share and make visible or explicit their understanding as completely as possible), S means *student responding*, R means *recognizing* (the teacher makes a judgment about the differences among students' responses, explanations, or mental models so that the critical dimensions relevant for their learning can be made explicit) and U means *using* (helping students to achieve a consensus based on scientific reasoning). In order to guide the identification of the different aspects of assessment conversations, Ruiz-Primo and Furtak (2007) develop some strategies, which are relevant to foster informal formative practices in the context of inquiry. I am using an adaptation of these strategies developed by Gandit (2016), who focuses more specifically on strategies related to inquiry in mathematics (Table 9.1). These strategies deal with only one of the three domains that is the epistemic frameworks. In this grid, T is used for teacher and S for student. Nevertheless, knowing how to design tasks and organize classrooms discussion in order to elicit students' mathematical concept and knowledge, and how to provide useful feedback that can let students move forward still remain a main subject of concern for teachers.

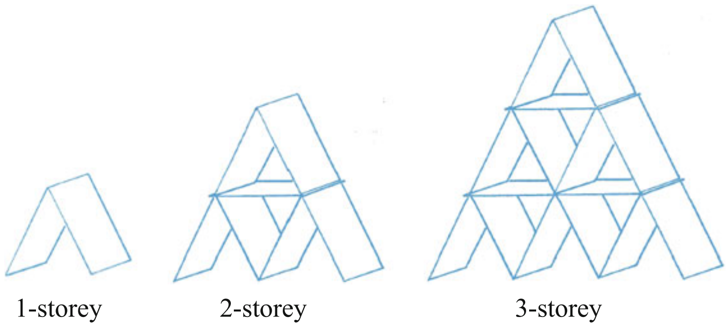
Table 9.1 Grid of analysis of ESRU cycles

Elicit	
T suggests a false answer, idea, concept to make S reflecting	E1
T asks S to give a proof, to argue	E2
T asks S to be more explicit, to clarify his ideas	E3
T asks for using well-known procedures	E4
T asks S to explain his strategy, what she/he has done	E5
Student	
S suggests a solution	S1
S justifies, explains his method, his reasoning	S2
S gives an example, a counter-example	S3
S explains how she/he had an idea, how she/he obtained a conjecture	S4
S questions another student about the topic under discussion	S5
S questions T	S6
S says that she/he does not agree, or that she/he agrees with T or with another student	S7
S agrees or disagrees with what is said	S8
S expresses ignorance or indecision	S9
S explains what she/he understood	S10
Recognize	
T repeats or rephrases a S's contribution	R1
T takes the S's idea as her/his own	R2
T clarifies or gives an answer based upon S's answers	R3
T encourages S to go on with her/his idea	R4
T readily offers right answer to a question	R5
T answers to a question saying yes/no or right/wrong	R6
T agrees, disagrees	R7
Use	
T suggests S a new activity that can help her/him	U1
T promotes argumentation	U2
T displays conclusions at the end of the discussion	U3
T asks S to go deeper	U4
T tries to focus S's attention on a point that can make her/him goes on	U5
T explains knowledge, strategies used by S	U6
T explains the goal, gives strategies, hints	U7
T comments on S's approach	U8
T leaves unanswered a question without consensus	U9

Adapted by Gandit (2016)

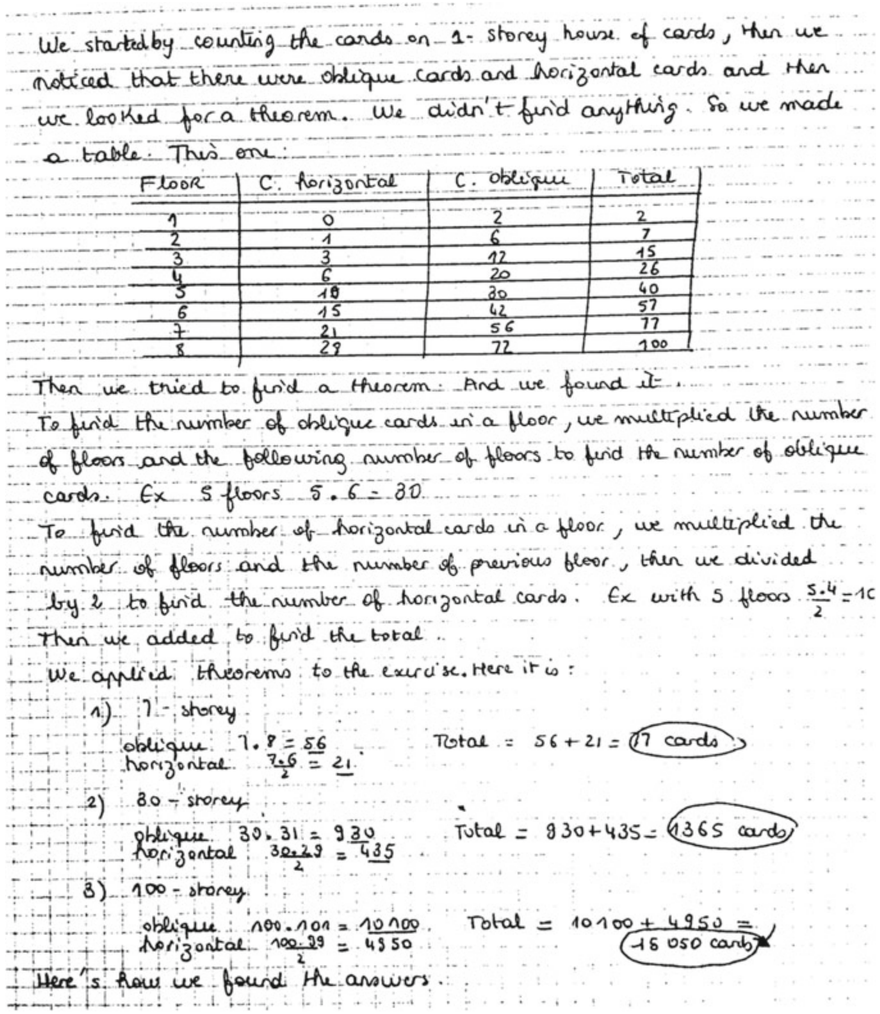
9.3.3 The Research Narrative: A Special Means to Assess and Develop Students' Problem Solving Competencies

Assessing problem solving remains difficult especially because some of "the challenges to assessment of problem solving in timed test arise from the fact that assessment of problem solving requires access to evidence of process" (Monaghan et al., 2009, p. 25). In order to give teachers access to students' research, thought processes and proposed solution, the MDC guidelines recommend *research narrative* (Bonafé, 1993; Bonafé, Sauter, Chevallier, Combes, Deville, Dray, & Robert, 2002; Chevallier, 1992; Sauter, 1998) as a particular means to assess students during MDC (see Annexe 2, part V, 1). Research narrative can be defined as a new contract between students and teachers in which students have to explain in writing, the best they can, how they solved (or tried to solve) the problem, including mistakes, wrong ways, dead-ends, help they received; and teachers have to assess students on these and only these points without taking into account whether the students found the right answer or not. There is below an example (Fig. 9.1) of such a production related to the following problem: To build a one-storey house of cards you need 2 cards. For a 2-storey house of cards, 7 cards are required. For a 3-storey house of cards, you need 15 cards. How many cards does it take to build a 7-storey house of cards? A 30-storey house of cards? A 100-storey house of cards?



With this activity, the fact that students have to explain all the strategies they tried and all the ideas they had to someone else, presupposes that they are first capable to do so for themselves. Students must reconstruct their reflection and make a synthesis of what strategies were effective, which ones were wrong ways or led to dead-ends, etc. This helps develop their reflection about what solving problems in mathematics means, about their own problem solving competencies and it can encourage the development of para-mathematical and proto-mathematical knowledge (Chevallard, 1994). Para-mathematical knowledge refers to auxiliary notions (equation, parameter, demonstration, etc.) which are not explicitly taught and usually not assessed.

Proto-mathematical knowledge is even more implicit than para-mathematical knowledge. It refers to notions that serve as tools to do mathematics and are deeply related to the didactical contract and to the classroom culture. For example, when



We started by counting the cards on 1- storey house of cards, then we noticed that there were oblique cards and horizontal cards and then we looked for a theorem. We didn't find anything. So we made a table. This one:

Floor	C. horizontal	C. oblique	Total
1	0	2	2
2	1	6	7
3	3	12	15
4	6	20	26
5	10	30	40
6	15	42	57
7	21	56	77
8	28	72	100

Then we tried to find a theorem. And we found it.
To find the number of oblique cards in a floor, we multiplied the number of floors and the following number of floors to find the number of oblique cards. Ex 5 floors 5 . 6 = 30
To find the number of horizontal cards in a floor, we multiplied the number of floors and the number of previous floor, then we divided by 2 to find the number of horizontal cards. Ex with 5 floors $\frac{5.4}{2}$ = 10
Then we added to find the total.
We applied theorems to the exercise. Here it is:

1) 7 - storey
oblique 7 . 8 = 56
horizontal $\frac{7.6}{2}$ = 21
Total = 56 + 21 = 77 cards

2) 30 - storey
oblique 30 . 31 = 930
horizontal $\frac{30.29}{2}$ = 435
Total = 930+435 = 1365 cards

3) 100 - storey
oblique 100 . 101 = 10100
horizontal $\frac{100.99}{2}$ = 4950
Total = 10100 + 4950 = 15 050 cards

Here's how we found the answers.

Fig. 9.1 A student's research narrative [We translated a student's research narrative. The original one (in French) is given at the end of this text (Annexe 1)]

a student has to factor the algebraic expression $4x^2 - 36x$, she/he is expected to answer $4x(x - 9)$. But if a student answers $4x^2 - 36x = 4x^2 - 2(2x \cdot 9) + 9^2 - 9^2 = (2x - 9)^2 - 9^2 = 2x(2x - 18)$ it should be considered as a wrong answer even if it is mathematically correct. Another example of proto-mathematical knowledge is the notion of pattern. This is not explicitly taught but students need to be able to recognize it to solve many problems.

On top of that, teachers can explore students' writing to provide them with "feedback giving [them] advice about the strength and weakness of the work and about how to improve" (Black, 2013, p. 170). Research narrative as a scheme used principally for summative assessment can also assume a formative function. Nevertheless, this choice to assess problem solving by research narrative can raise some issues. It is, indeed, necessary that students are able to produce clear evidence and communicate efficiently, without which other process capabilities cannot be reliably assessed (Monaghan et al., 2009).

9.4 First Part of the Research: Teachers' Point of View About the Assessment of Students' Problem Solving Competencies

9.4.1 Research Design

According to the results from the literature in the field, teaching and assessing problem solving remain a problematic issue for teachers. In the first part of my research, I aim to know the teachers' point of view about the assessment of students' problem solving competencies in the particular context of the MDC. In that sense, I submitted an online survey to teachers currently teaching or having taught this course in the past few years. It deals with three main topics: the type of problems the teachers give to the students, the assessment of problem solving competencies and the research narrative.

Teachers were asked to answer different types of questions: open-ended questions (for instance "give your two main criteria to choose a problem you will submit to students"), multiple choice questions (for instance "do you assess students: A. Individually B. Collectively") and propositions to order from the most significant to the least significant (for instance "classify these ten competencies from the most important for you to develop to the least important"). A group of 100 teachers were targeted by the survey but only 61 of them gave complete answers. I made a quantitative and qualitative analysis based on these 61 responses, depending on the type of the question.

9.4.2 Results and Discussion

Concerning the types of problems the teachers give to the students, they reported choosing them mostly on their perception of the degree of openness (50%[2]) which implies that the problem is a real problem for students, non-trivial, with no evident solution and no well-known method to solve it. Another important criterion is the

[2]For each question, the percentage expressed is the ratio between the number of teachers who choose this answer and the number of teachers who answered the question.

conformity between the difficulty of the problem and the students' supposed level (34%), and finally the notions and concepts at stake (34%). Most of the respondents say that in this course they want to develop students' competencies relative to the pattern try-conjecture-test-prove, to strategies useful to solve problems, to collaborative work, to communication and to critical thinking (between 70 and 85% for each one). One can notice that the competencies teachers would like to develop are really close and limited to the institutional learning outcomes described in the curriculum.

With regard to the assessment of students' problem solving competencies, the first interesting point is that most of the teachers assess their students more frequently than what is institutionally expected (85%). Almost all the teachers refer to using the research narrative and a grid of criteria to assess students (95%). Thus, referring to a grid of criteria to assess research narrative seems to be usual teaching practice. For more than half of the teachers (63%), assessing students in this particular context implies giving them a grade. Moreover, very few teachers (23%) claim encouraging the involvement of students in their assessment through self-assessment, peer-assessment or co-assessment, and within these 23%, only one teacher asks students to refer to the grid of criteria to assess themselves. Even though these results show that students are not directly involved in their assessment, some teachers declare that they would like the students to do so. This leads to wonder how teachers make students aware of what they are supposed to learn and of the criteria applied in assessing their work. Comparing what teachers say they want students to learn and what competencies they want to assess, one can see that it is deeply correlated. In most cases, teachers aim at developing and assessing the same competencies (78%). In other cases, the expected learning outcomes (what the teachers say they want the students to learn) and the competencies they want to assess are not aligned. One can distinguish two main configurations: competencies teachers say they want to develop but they do not say they want to assess, and competencies teachers say they want to assess but they do not want to develop in usual, non-assessed course. The first and the most representative case concerns the two following objectives: to respect debate rules and to work in a collaborative way, whereas the second scenario is identified with regards to the objectives such as to write a research narrative, to communicate the research or to solve a complex problem.

Concerning the research narrative, teachers think that the main difficulties for students are related to the narrative aspect of the activity (for instance to describe all steps, to justify) (40%), to their attitude towards the research (lack of perseverance for instance) (25%) and to the solving of the problem itself (for instance to use a strategy, to choose and follow another path) (25%). Almost all the teachers communicate their criteria of assessment to students either every time or sometimes (91%). The nature of the criteria that teachers principally transmit to students, refers to narrative competencies (related to clarity, consistency and comprehensiveness) (61%) and, though in a smaller proportion, to problem solving competencies (37%). It means that, for the majority of the students, the assessment of their research narrative deals more with narration than with problem solving. Nevertheless, it seems necessary to keep in mind that the main purpose of such an activity is to develop students' problem solving competencies and not only to develop narrative skills, even if it is an important transversal skill which is expected in mathematics. I do not have information about the way teachers communicate these criteria to students: do they give students the grid they use to assess themselves? Do they have specific discussion in class about the intended learning outcomes? To answer these questions, it could be useful to analyze the usual teachers' practices and not only the declared ones. On top of that, teachers say that assessing such open-ended activities is more complicated than assessing usual students' mathematical work (52%) and that they suffer from a lack of tools to assess students' research narrative (34%).

Finally, it is interesting to notice that teachers do not know their colleagues' practices. It can be surprising because MDC involves only about 800 students and 55 teachers a year. The small population involved and the specificity of this new course might lead to believe that teachers work in a collaborative fashion, especially when they work in the same school, and that they know their colleagues' practices well. According to the results, it is far from being the case. It can be problematic, for example when a student is moving in and out of schools during the school year, because what is expected from her/him in the course in her/his new class, is not necessarily the same as in her/his previous class. This lack of common expectations and common assessment practices is all the more alarming that this course counts as a major course. It means that the results obtained by a student in this course count as one of the four marks which are taken into consideration when deciding whether a student will pass grade 8 and thus be admitted to grade 9 or not.

These results give precious information about the teachers' point of view on the assessment of students' problem solving competencies in MDC. I would like here to highlight some significant results for the second part of my study. Almost all teachers use the research narrative to assess students, and referring to a grid of criteria to assess research narratives seems a usual teaching practice, even if grids of criteria are personal, and not shared among teachers. Competencies teachers would like to develop are really close to the institutional learning outcomes described in the curriculum. Teachers communicate to students some criteria, but these criteria are mainly relative to narration more than to problem solving. Finally, some teachers affirm that they lack assessment tools and that even though they are not promoting students involvement in their assessment, some of them would like to do so. Thus, it could be interesting to gather common expectations, common criteria about problem solving competencies in an assessment tool, such as a grid of criteria, which can promote the involvement in the assessment.

9.5 Second Part of the Research: A MDC Teacher's Informal Formative Assessment Practices

9.5.1 The Development of an Assessment Tool for Both Summative and Formative Purpose

As a consequence of the results of the first part of the research (lack of common expectations about MDC, teachers lacking tools to assess students, teachers wanting to involve students in their assessment), a working group that included two MDC teachers and myself was created in September 2015. The purpose of this working group was to give teachers of MDC a common tool to assess students' problem solving competencies with both summative and formative purpose and consequently to ensure common expectations about IBME (from teachers, and more globally from schools).

To develop this project, we used an existing tool developed by the Geneva team in the PRIMAS[3] project. The team collaborative work led to design a grid of criteria aiming at assessing research narrative in the context of the MDC. For that purpose, the team studied grids of criteria teachers used in their classes. They used these grids to evaluate numerous students' narrations in order compare criteria and to highlight the most important ones and to identify the ones that lacked. Thus, they noticed that the criteria linked to narration competencies were as numerous as those linked to problem solving competencies. Nevertheless, such an activity can easily be turned away from its original purpose. Even though it is important to develop communication skills, it is important to make sure that students do not only focus their attention on their narration, but also on the problem they have to solve. Asking students to write out a narration describing their research has to remain a means to access and develop their problem solving competencies and cannot become a goal in itself. But many students, especially when they are first confronted with research narrative, focus their narration on a complete description of what they do even though it does not deal with relevant mathematics content. Therefore, to prevent that risk, the Geneva team in the PRIMAS project emphasized criteria related to problem solving competencies more than those related to narration or presentation. They have also defined five dimensions: presentation, narration, research, technic and appropriation, which induce ways to look at students' production, and criteria related to each dimension, which describe expected qualities of the production. Keeping in mind that the grid should be an efficient tool to assess a large set of problems, they defined these criteria at a very general level. Then, teachers individually have to adapt the grid to the problem students are working on, to remove some criteria and to specify indicators related to criteria.

One year later, in order to ensure that these dimensions are relevant and that the criteria are efficient, exhaustive and well worded to assess students' problem solving competencies during MDC, our working group implemented this tool in classrooms.

[3] Available at http://www.primas-project.eu/fr/index.do.

The two teachers of the group used the grid for one year in their own MDC classes. At the beginning of the school year, they distributed the grid to students who could use it to compare their work with the expected performances during regular work but also during tests. It aimed at allowing discussions about criteria and dimensions, and at enhancing the involvement of students in their assessment. At the same time, teachers assessed each student's research narrative using the grid, for the whole year. The marks and comments students received were based on the grid.

Then, every three weeks on average, depending on their experience, we adjusted the grid, we clarified terms, we added and removed some criteria. The cycle occurred several times until we fixed the tool. To summarize the process and all the adaptations we made, we simplified the description of some criteria to make it easier to understand for the students, we took out some criteria that seemed to be too particular to apply to most problems and we added new criteria.

Then we worked with the members of another working group who focused on another course given to the same students, the following year. This course called *scientific approach* focuses on modeling, and involves mathematics and sciences. With this collaborative work, we aimed to ensure common expectations between these two courses. We modified a dimension, from appropriation to modeling, and once again we modified some criteria in order this time to emphasize, within a scientific approach, the steps of trials and conjecture and the articulation among these.

Finally, we asked some teachers who were not involved in the working group to use the grid in their classes, in order to ensure that the tool is also efficient in another school, with other teachers. That is why one teacher in each secondary school in Geneva (with the exception of one school, so 18 teachers in total) referred to the grid in order to assess their students. We wanted to collect their opinion after this experience to ensure that the use of the grid was close to their usual practices, that the grid was based on relevant dimensions and criteria, and that it could be an efficient tool to assess in a summative way students' problem solving competencies. We asked them to answer five questions, related to the appropriation of the grid, to the relevance of criteria and dimensions ("Do the five dimensions seem relevant to you?" for instance), to the way they use the grid with the students and to the adequacy with their general feeling about the students' research narratives ("Do the grades you obtained according to the grid seem aligned with your general feeling about the students' research narrative?").

The analysis of the answers confirms that the grid seems to be a useful tool for assessing students' research narratives with a summative purpose. Most teachers find that the dimensions and the criteria are relevant (83%), and that the grade they obtained was in alignment with those they would have given without the grid (94%). We did not modify the grid after this last step in the process and finally the grid of criteria we obtained is the one given in Table 9.2.

Table 9.2 Grid of criteria established by the working group

Dimensions	Criteria
Presentation	Care
Narration	Comprehensiveness Relevance of narration (centered on the research) and structure of the text (each step has a beginning, a development and a conclusion) The narration is complete and chronological (all steps are described and in the right order)
Modeling	Appropriation of the problem: rephrase the problem in French and/or express it with drawings, diagrams, tables Use pertinent mathematical tools, theories and strategies
Research	Follow a lead, have a strategy Make relevant trials, and try to eliminate randomness Explain and justify all the conjectures Expose a valid conjecture or a sufficient number of invalid conjectures Test (or prove) each conjecture Conclude each conjecture Express a global conclusion about the research, question the mathematical solution relative to the context of the problem
Technic	Use properly mathematical tools and theories (units, theorems, etc.) Introduce new codes, notations

9.5.2 Research Design

The analysis of this collaborative work occurs at two levels related to the two main functions of assessment: summative and formative.

The summative potential of the grid was studied according to discussions occurring during the meetings with the two other members of the working group. In summary, discussions regarding an assessment tool support negotiations between teacher and students about assessment criteria and more generally about the goal of IBME. According to the teachers, it helps them to better understand what they want the students to do and to learn. They base some of their feedback concerning students' written work on the grid. Thus, both students and teachers, refer to the same tool to discuss expectations about problem solving competencies.

It is all the more interesting that according to the results of the questionnaire, most teachers of MDC assess their students according to explicit criteria but only a small number have specific discussions with the students about their expected performances. Giving teachers and students a common tool to assess and compare their work with the expected qualities should help encourage such discussions. Another relevant point is that it is not obvious for students to understand how teachers consider the criteria and the dimensions to assess their narration. For instance, I observed during MDC some specific discussions between the teacher and students about how the teacher uses the criteria. Students wondered if each criterion had to appear explicitly in their narration. Moreover, some students thought that each criterion had to appear in their production and in the same order than in the grid. This illustrates that it is one thing to explain the criteria used to assess their work to the students, but it is another to explain to them how teachers use it specifically to give them a grade. Moreover, to emphasize problem solving competencies, teachers choose to adapt the grading scale during the year. They put more and more weight on research and technic instead of narration. The analysis of the answers of the 18 teachers who used the grid with a summative purpose gave us complementary information about its summative potential (see Sect. 9.5.1).

On top of that, the summative potential of the grid was at the core of a wider comparative study. The aim of this study was both, to establish real students' problem solving competencies, for all students involved in the MDC in 2016–2017, and to compare teachers' summative assessment practices. For the first part, all the students had to solve the same problem and were assessed with the same tool. It highlighted students' strengths and weaknesses in problem solving and in the narration of the process of solving problems. For the second part, 52 teachers giving this course in 2016–2017 assessed the narration of the same 3 students relative to the same criteria. It highlighted which dimensions and criteria of the grid led to different notations and consequently to different interpretations by teachers.

On top of that, I deal in my research with the formative potential of the grid in order to answer the question: is formative assessment fostered in the context of frequent summative assessments based on research narrative? To analyze this formative potential, I need to know how this tool is implemented in classroom, and how it could become a tool that fosters formative processes.

9.5.3 Methodology

To understand how the grid could be implemented in classroom, and how it could become a tool that fosters formative processes, I made an exploratory study focusing on a MDC teacher's practices. The results of this exploratory study should give me a first idea about how teachers refer to formative assessment in the particular context of MDC.

In that sense, I assisted and video-recorded two consecutive sessions of MDC facilitated by one of the teacher-member of the working group, at the end of the school year. The nine students of the class were working in four groups (3 groups of 2 students, 1 group of 3). They were working on two problems related to the introduction of algebra. At the end of the second period, students had to give a research narrative about the problems they were working on to the teacher. Consequently, half of the second session (about 25 min) was devoted to the narration and students were invited to write down their research process.

These students were using the grid of criteria since the beginning of the school year. The teacher gave each student a copy of the grid and allowed them to refer to the grid whenever they wanted. On top of that, every time she assessed students, the

Table 9.3 An assessment conversation and classification according to ESRU cycles

Assessment conversation	Strategy
T: Yes but how did you find this?	E5
S: I did a lot of stuff	S2
T: But try… what did you do? It's interesting to know how you were thinking	R4/E5
S: I did all of this but like everything in reverse	S2
T: Yes, so the first step. What did you do at the first step?	R7/E5
S: 24 minus 7	S2
T: Yes, you did reverse calculations. Yes. It's a good idea. Reverse calculation it's in fact a path	R2-U6

teacher gave them a copy of the grid with specific comments about their research narrative relative to each criterion.

According to my theoretical framework, and especially to ESRU cycles (Ruiz-Primo & Furtak, 2004, 2007), my research questions concerning formative assessment were:

– How did teachers encourage assessment conversations during IBME-centered courses?
– What kinds of feedback were provided?
– Was this feedback related to the criteria of the grid?

To interpret the data and make it relevant with my theoretical framework, I transcribed all interactions that occurred in class for the two consecutive lessons (about 67 min). I dissected the lesson in assessment conversations. I analyzed all the interactions according to the ESRU-cycles (Ruiz-Primo & Furtak, 2007) and more specially I reported every strategy used by the teacher related to each activity: eliciting, recognizing and using (Gandit, 2016) (Table 9.1). I specified for each assessment conversation the aim of the feedback: if it referred to the strategies used by students; to the goal of the problem; to mathematical knowledge, concepts and notions; to the material organization; to the solution or to the guidelines. I paid a special attention to reporting every reference to the criteria of the grid.

I give in Table 9.3 an example of a partial assessment conversation between a student and the teacher, and of how I classified each speaking turn according to the strategies related to ESRU cycles. T is used for teacher and S for student.

In this excerpt, one can see that the teacher tried to make the student explain his strategy, to make him conscious of what he has done, and gave him feedback about the validity of his strategy. She helped him to deconstruct, step by step, his reasoning. It illustrates that the teacher had several iterations with the same student and that the feedback was in this case about students' strategy. In addition to a ESRU complete cycle (E5-S2-R2-U6), there were also two partial cycles (E5-S2-R4 and E5-S2-R7).

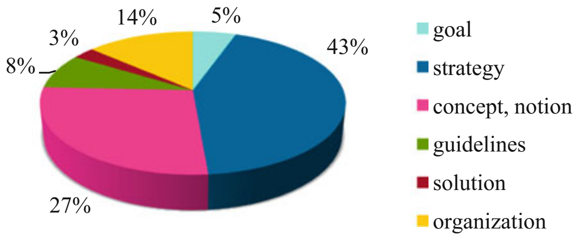


Fig. 9.2 Aims of feedback

9.5.4 Results and Discussion

There were no less than 21 assessment conversations during the 67 min of the lesson. Moreover, the teacher had between four and seven assessment conversations with each group of students (4 with the group 1, 7 with the group 2, 5 with the group 3 and 5 with the group 4). As she had specific assessment conversations with each student in each group, every student received numerous personal feedback during the course.

Concerning the aim of the feedback, it was mainly relative to the strategies used by the students, and in a smaller proportion, to mathematical knowledge, concepts and notions involved in the problem or used by the students. In most cases, assessment conversations were about how to solve the problem, about the validity of strategies used by the students and about some specific notions, concepts, and knowledge related to the topic of the problem. Figure 9.2 shows a global view of the proportion of each aim of the given feedback.

There were 48 cycles, among these, there were only 9 complete cycles (E-S-R-U) and 39 partial cycles (26 S-R-U cycles, 13 E-S-R cycles). As in the previous example (E-S-R/E-S-R/E-S-R-U), when the teacher ended a partial cycle with recognizing the student's response without using this response, she often elicited again which means that she used strategies that allow students to share and make visible or explicit their understanding as completely as possible. But these complete or partial cycles were not the most representative form of assessment conversations. Indeed, there were 426 steps in all the assessment conversations (all Eliciting, Student's response, Recognizing and Using activities taken into account) and only 146 steps constituting partial or complete cycles (E-S-R, S-R-U or E-S-R-U). The other steps were articulated in different ways, as S-R-S-U-S for instance. Moreover, even if the teacher recognized the student's response it did not necessarily imply that she used it. Indeed, the teacher recognized student's answer more often than she used her/his answer (she recognized student's response 127 times, and she used it 77 times). It illustrates that in the practices of this teacher, assessment conversations can have different structures. On top of that, one could have thought that the most represented form of assessment conversations would be a cycle, with four interactions E-S-R-U. But in most cases, assessment conversations articulated several partial cycles and were longer than only four interactions. On average, there were 13 interactions in an assessment conversation. Generally, the student did not explain her/his method, her/his strategy

Fig. 9.3 Distribution of strategies related to the activity of eliciting

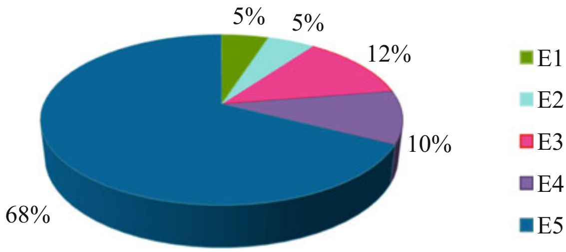
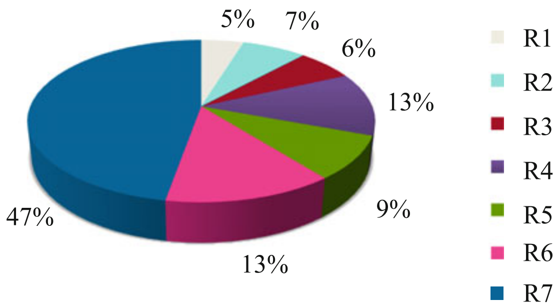


Fig. 9.4 Distribution of strategies related to the activity of recognizing



as explicitly as the teacher would like. Consequently, the teacher elicited again, she recognized the student's response and so on, until she knew exactly what the student has done or thought and that she could give her/him feedback about the validity of her/his strategy and about what the student had to do.

Concerning the activity of eliciting, the most represented strategy that allows students to share and make visible or explicit their understanding as completely as possible was E5: the teacher asked a student to explain her/his strategy, in other words what she/he has done. It means that the teacher mostly asked a student to explain her/his strategy, the way she/he solved the problem and what she/he has done in order to acquire information about student' level, conceptions, skills, difficulties. It occurred 27 times in 40 times. Other strategies were distributed following pretty much the same proportion (Fig. 9.3).

Focusing on students' responses, they mainly suggested a solution, a (partial) response to the problem the teacher asked them to answer (strategy S1, counted 39 times) and they justified, explained their method, their reasoning and their strategies (strategy S2, counted 51 times). Students used several different strategies to give the teacher access to their understanding, to their conceptions.

The teacher frequently recognized students' responses (127 times during the course). For that, she mostly agreed or disagreed with what the students explained, suggested, submitted, and gave them an implicit or an explicit validation or invalidation of their response or of their strategy (the strategy R7 occurs 60 times over 127). She referred to all strategies described in the grid (Fig. 9.4).

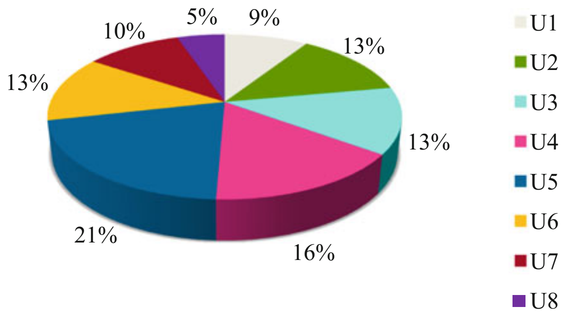


Fig. 9.5 Distribution of strategies related to the activity of using

After recognizing students' response, the teacher sometimes used their answer to give student feedback and to do so, she referred to numerous strategies in quite a balanced proportion (Fig. 9.5). It occurred 77 times.

She promoted argumentation, she displayed conclusions at the end of the discussion, she asked student to go deeper in her/his reflection, she tried to focus student's attention on a point that could have make her/him go further, and she explained knowledge and strategies used by students. Nevertheless, even though the teacher did not use students' response every time, she gave them feedback, which could be implicit, concerning their strategy, their mathematical concepts and notions.

Only six assessment conversations were initiated by the teacher and seventeen by a student. When a student started asking the teacher to help her/him, the teacher gave the student an answer and then tried to elicit her/his conceptions, strategy. The conversation initiated by a student was finally used by the teacher to take information about student's thinking.

Moreover, the teacher referred to some criteria of the grid, even if these references were not explicit. For example, when she said, "you will have to explain everything that happened during your research, so take as many notes as you can", she implicitly referred to the criterion "the narration is complete". She principally referred to the criteria relative to the narrative aspects of the research narrative and especially to the completeness aspect of the narration (10 times). She also referred to the criterion relative to the introduction of new codes and notations (3 times).

This criterion was particularly important in the problem students had to solve because the unknown was only represented by a question mark in the problem and students were referring to this unknown by using another symbol (a letter for most of them) to deal with it, without explaining what this new symbol referred to.

There were very few references to the grid of criteria with a formative assessment purpose. It did not seem to be an explicit means to communicate with students about what is expected from them.

9.6 Conclusion and Perspectives

The development of problem solving and inquiry in mathematics education calls for new approaches in classroom assessment. A course as MDC was an interesting context to observe such changes, because of new curriculum goals focusing on problem solving and IBME, and new assessment approaches based on the research narrative.

The first part of my research about teachers' point of view shows that problem solving assessment practices, and especially assessment criteria, are not commonly shared among teachers. This first part led me to develop, in collaboration with two teachers, a tool, a grid of criteria, which aimed to give teachers of MDC a common tool to assess students' problem solving competencies with both summative and formative purpose. This tool could be considered as an efficient tool for summative assessment in the context of MDC. The second part of my study shows that classroom assessment cannot be reduced to summative assessment, to tasks designed for assessment, or even to formal formative assessment. The teacher assessed students all the time and she used classroom conversations to support students' learning. In that sense she used strategies to elicit students' thinking, to recognize their response and to use it. The feedback she gave to students, aimed to help students identify the gap between their level and the standards, and situate themselves on the pathway to the solution. These assessment conversations were an opportunity to discuss the problem solving goals. Nevertheless, it is not possible to ensure that referring to a grid of criteria for summative assessment helps teachers develop formative assessment practices, because in that case, the teacher refers only to two criteria related to narrative aspect, and not to criteria related to more specific aspects of research. Obviously, as the research deals only with one teacher' assessment practices, it could be interesting to enlarge the study in order to confront the results to other teachers' practices.

I would like to raise some issues to be addressed. Even though IBME and problem solving carry more and more weight in mathematics curriculum, the goals are not so precisely defined and not so practical to implement in classes. The necessity of defining assessment criteria is nowadays commonly accepted but implies that teachers have specific discussions with students about these criteria in order to make them familiar with it and to encourage their involvement in assessment. To encourage formative assessment practices, and more precisely to help teachers develop assessment conversations that enhance informal formative assessment, it seems interesting to strengthen collaborative work with teachers to characterize strategies that promote eliciting, recognizing and using students' learning.

A main difficulty for the researcher remains to identify teachers' informal formative assessment practices. Contrary to formal assessment that is easier to pinpoint, for example a written summative test at the end of a teaching sequence, informal formative assessment can occur in many occasions during the class. Thus, the researcher has to be able to recognize when the teacher is gathering information even if it is usually not accessible to an external observer (Pilet & Horoks, 2015).

Annexe 1: A Student's Research Narrative (in French)

Nous avons commencé par compté les cartes sur des chateaux de cartes à 1 étage, 2 étages, 3 étages puis on à remarqué qu'il y avait des cartes oblique est des cartes horizontal, puis nous cherché un théorème on à rien trouvé alors on à fait un tableau.

celui ci :

étage	C. Horizontal	C. Oblique	total
1	0	2	2
2	1	6	7
3	3	12	15
4	6	20	26
5	10	30	40
6	15	42	57
7	21	56	77
8	28	72	100

nous avons essayé de trouve un théorème puis nous l'avons trouvé.

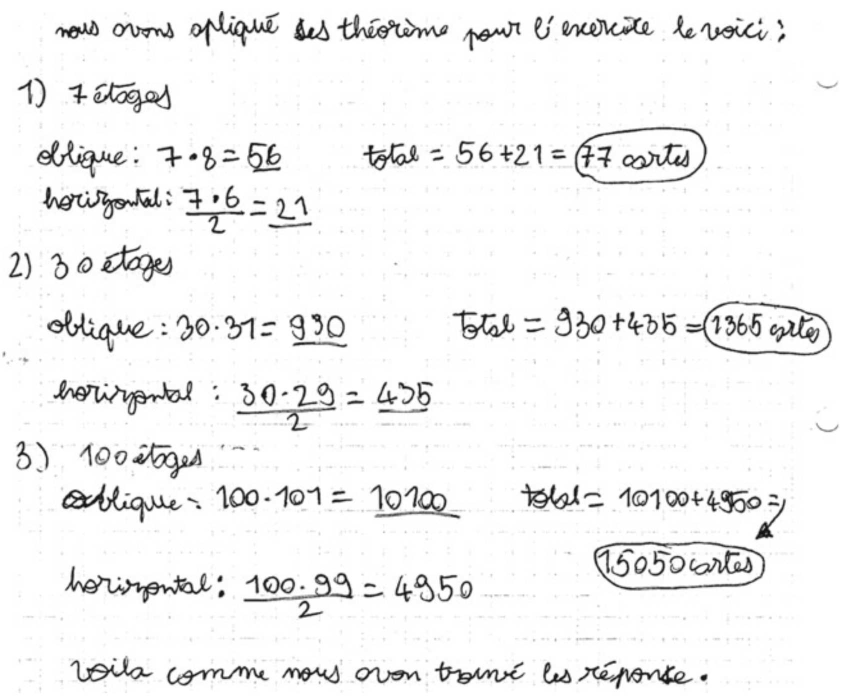
Pour trouvé le nombre de carte oblique dans un étage nous avons multiplié le nombre d'étage en question et le nombre d'étage suivant pour trouvé le nombre de carte oblique

ex 5 étage: $5 \cdot 6 = 30$

Pour trouvé le nombre de carte horizontal dans un étage nous avons multiplier le nombre d'étage enquestion et le nombre d'étage précédent puis on le divise par 2 pour trouvé le résulta du nombre de carte horizo

ex avec 5 étage: $\frac{5 \cdot 4}{2} = 10$

puis on les additions pour trouvé le total


nous avons apliqué ses théorème pour l'exercice le voici ;

1) 7 étages

oblique : $7 \cdot 8 = 56$ total = $56 + 21 = 77$ cartes

horizontal : $\frac{7 \cdot 6}{2} = 21$

2) 30 étages

oblique : $30 \cdot 31 = 930$ total = $930 + 435 = 1365$ cartes

horizontal : $\frac{30 \cdot 29}{2} = 435$

3) 100 étages

oblique : $100 \cdot 101 = 10100$ total = $10100 + 4950 =$ 15050 cartes

horizontal : $\frac{100 \cdot 99}{2} = 4950$

voila comme nous avon trouvé les réponse.

Annexe 2: Programme of Mathematics Development Course (MDC)

I. **Organization**
[…] This weekly period is intended to support a teaching that contributes to the strengthening and development of problem solving strategies and mathematical situations activities.

II. **Programme**

- The suggested activities are linked with three main topics:
  - Numbers and Operation
  - Space and measure
  - Function and algebra
- The problem solving strategies are:
  - Analogy
  - Trial and error—Example/counter-example
  - Inductive and deductive reasoning
  - Organized study of all cases and exhaustion of solutions
  - Introduction to proofs
- These strategies contribute to the development of:

  - Scientific procedures
  - The rules of scientific debate

III. **Mathematics development course: Introduction**
[…] The allocation of an additional period in the curriculum for grade 8 students with scientific profile, aims to enable these students to learn and become familiar with this important part of mathematical activity. The aim is not simply to solve problems "one by one", but also to discover and systematize problem-solving methods. In particular, the aim is to place the student in a learning situation where she/he will have to implement a "scientific approach", that leads her/him to the following scheme:

Try–Conjecture–Test–Prove

This part of mathematical activity is required when students are confronted with the so-called open-ended problems. This places the pupil in the most typical situation of mathematical activity, that of confronting a problem which enables her/him to work as a mathematician who is confronted with a problem to which she/he does not know the solution.

IV. **Open Ended Problem**

a. According to a definition proposed by a group of researchers at the IREM of Lyon, an "open-ended problem" has the following characteristics:
   - the wording is short
   - the wording does not introduce the method or the solution, the solution must not be reduced to the use or an immediate application of recent coursework
   - the problem must be situated in a conceptual field that students are familiar enough with, so that they can easily "take possession" of the situation and engage in trials, conjectures, draft resolutions, or counter-examples.

b. Solving a problem consists of a series of steps outlined in the official textbook:
   1. **Appropriation of the wording**: "understanding the problem to identify its purpose"
      At this stage, the teacher must ensure that all students are involved in the problem.
      That is to say that they are able to construct a correct representation of the data, understand the constraints and the goal to achieve. If necessary, the teacher answers questions, rephrases or makes the student rephrase the problem.
   2. **Data processing**: "design a plan", then "put the plan into action" and "get back to the solution".
      This stage corresponds to the research and resolution of the problem itself. A relatively short time slot can be allocated for individual research, followed by a second group work time.

During the individual research phase, the teacher can verify that each student has actually read the problem, has at least partially assimilated it, and that, during the group work, she/he will not only follow the ideas of the one who speaks first. Group work helps to avoid the discouragement of certain pupils, to stimulate the exchange of ideas among students, to learn how to collaborate, to listen to each other, to defend their point of view, to respect each other.

3. **Communication of research procedures and results**: "Write the results in a form that anyone can understand and follow the work done" At this stage, the student must account for all the resolution of the problem, individual phase and the group work included. Such a written report gives the teacher the first insight into the student's research work and provides an occasion for evaluation. The writing of this report is a basis for the evaluation and is therefore an important competence for the student. This is why the practice of "research narrative" has been chosen as a thread for this course. According to the textbook, a research narrative is "a comprehensive account of research, including trials and errors that didn't lead to a satisfactory result, or wrong conjectures, and the reasons which lead them to abandon them."

V. **Research narrative**

1. **Presentation of research narrative**
   […] The objectives of this pedagogical practice can evolve throughout the year. They may initially be:
   - **to develop students' curiosity and critical thinking**
   - **to provide a communication tool** that facilitates students' writing
   - **to put in place the rules of mathematical debate**, in particular the following ones: a counterexample is sufficient to invalidate a statement, examples that verify a statement are not sufficient to show its validity, an observation on a drawing is not sufficient to prove that a statement is true
   - **to allow the teacher to get a much better knowledge of the procedures of his pupils**.
2. **Correction and assessment**
   Criteria for a good research narrative
   The action of narration is not an easy activity, but one can retain some elements that are to be emphasized and encouraged by the corrector of the copies.
   - Writing style
   - The accuracy of the narrative: all ideas, all trials are described thoroughly
   - The sincerity of the narrative

   Criteria for Good Research
   To help students to better understand what is expected, it could be useful to refer to intermediate assessment means.

- An assessment of the analysis of a problem by the formulation and explanation of conjectures
- An assessment of the research phase: identifying and comparing strategies
- Another assessment of the "research" phase: using hints
- An assessment of the overall attitude
- An assessment of an oral presentation.

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Chapter 10
Beyond the Standardized Assessment of Mathematical Problem Solving Competencies: From Products to *Processes*

Pietro Di Martino and Giulia Signorini

10.1 Introduction

Many mathematicians recognize problem solving as the heart of mathematics (Halmos, 1980). At the beginning of the new millennium, addressing what the problems and challenges involve in mathematics education in Denmark, Niss (2003) reflected upon a series of educational issues, including the following two:

1. Which mathematical competencies do students need to develop at different stages of the education system?
2. How do we measure mathematical competence?

He answered the first question giving the following definition of mathematical competence:

> *Mathematical competence* then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. (Niss, 2003, pp. 121–122)

Then he proposed a subdivision of mathematical competence into eight sub-competencies. The second one was *posing and solving mathematical problems*, that was defined as follows for what concerns the *solving side*:

> Solving different kinds of mathematical problems (pure or applied, open-ended or closed), whether posed by others or by oneself, and, if appropriate, in different ways. (Niss, 2003, pp. 121–122)

The seminal work of Niss will affect the development of the current definition of mathematical competence, in particular the shared belief within the mathematics

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education community that solving mathematical problems is a crucial aspect in the development of mathematical competence. Mathematical problem solving has long been seen as an important aspect of the teaching and learning of mathematics (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016) and it has become one of the major goals of mathematics education at all school levels, and in different school systems.

As underlined in the famous document of the National Council of Teachers of Mathematics (NCTM, 2000), problem solving plays an important role in mathematics and should have a prominent role in mathematics education. However, for many teachers it is not obvious how to meaningfully include problem solving in their teaching practice and curriculum. Another non-trivial issue is how to assess[1] students' problem solving competencies.

10.2 The Standardised Assessment in Mathematics

Over the past twenty years the use of standardised tests for the assessment of students' mathematical competencies has been increasingly spreading in many countries, both at an international level—with the Programme for International Student Assessment (PISA) of the Organisation for Economic Co-operation and Development (OECD), and the Trends in International Mathematics and Science Study (TIMSS) of the International Association for the Evaluation of Educational Assessment (IEA) and at a national level (INVALSI in Italy), becoming increasingly significant in the educational context.

There are several reasons for this worldwide use of standardised assessments: they assess the outcomes of the teaching/learning process (Pellegrino, 2003) through numerical results, therefore they allow for an immediate (through questionable) comparison of students' results from different countries. The results of such kind of assessment are often used to certify the quality of the educational system, assessing whether students have reached the educational standards in a time in which the school autonomy asks for a greater accountability of outcomes (Kanes, Morgan, & Tsatsaroni, 2014).

Probably for these reasons, the results of the standardised assessments like TIMSS and PISA have a strong impact on the educational policies (Carvalho, 2012) and can influence political choices regarding school reforms (Mangez & Hilgers, 2012; Pons, 2012). In a way, the external and standardised assessments can affect what is considered relevant as educational outcome, also affecting teachers' educational choices and didactical practices (Nevo, 2001).

As a consequence, in the field of mathematics education there is a growing number of research papers related to the issue of the impact of external standardised assessments and their effects. The main lines of discussion concern their political impact (Kanes et al., 2014), their equity (Boaler, 2003), the instruments they use

[1]We prefer to use the verb *to assess* rather than the verb *to measure* in this context, because we strongly believe that competences are not measurable for their nature.

(Wiliam, 2008), the competences they assess (Ekmekci & Carmona, 2014), the reliability of the information that this kind of tests can (and cannot) have (Bodin, 2005). Researchers seems to focus mainly their interest on the students, attending to the possible causes of their successes and failures (Wijaya, Van den Heuvel-Panhuizen, Doorman & Robitzsch, 2014) and on the factors that influence their performance (Papanastasiou, 2000).

However, as Kanes et al. (2014) underlines, PISA has brought relatively few additional primary or secondary analyses of the data by national researchers, evaluators and experts.

10.3 Standardised Assessments and Problem-Solving Competence

The two most relevant international surveys—TIMSS and PISA—have different goals. TIMSS is curriculum-based, while PISA is focused on assessing the extent to which students at the end of compulsory education are able to apply their knowledge to solve or deal with real-life situations.

Despite their differences, both international surveys pay explicit attention to the assessment of students' problem solving competencies. Then again, problem solving now has a central role in many mathematical curriculums around the world (the focus of TIMSS), and it is clearly a key-competence for students to have in the modern society (the focus of PISA).

The TIMSS 2015 mathematics framework (Grønmo, Lindquist, Arora, & Mullis, 2015) recognizes three cognitive domains at grades 4 and 8: knowing, applying and reasoning. It is quite surprising (and questionable) that problem solving is considered central to the applying domain with an emphasis on dealing with more familiar and routine tasks.

A different approach is taken by the Programme for International Student Assessment (PISA) of the OECD. The specific PISA framework is strongly affected by the seminal work of Niss (2003). Stemming from Niss's definition, the PISA 2012 mathematics framework (the last PISA survey focused on mathematics) defines the "*mathematical literacy*" as:

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD, 2013b, p. 25)

This definition is a reformulation of the previous one (OECD, 2009), elaborated precisely to give greater emphasis to the role of students as active problem solvers. In particular, planning strategies for solving problems is one of the seven fundamental mathematical capabilities developed in the framework and—differently from the

TIMSS approach, but more in line with the idea of mathematical problem developed in mathematics education literature—problems are basically identified as non-routine exercises.

Therefore, comparing the two frameworks, it becomes central to define what "*problem*" means. We assume, as has been done within the PISA 2012 framework, the following definition of problem given by the Gestalt psychologist Karl Duncker, also mentioned in the PISA 2012 framework and therefore consistent with the kind of problems used in PISA survey:

> A problem arises when a living creature has a goal but does not know how this goal is to be reached. (Duncker, 1945, p. 1)

In this view, the emergence of a problem is characterized by two interrelated aspects: the presence of a goal and the non-immediacy of the way to reach it.

Related to this view, Mayer (1990) describes problem solving as a cognitive process directed at transforming a given situation into a goal situation when no obvious method of solution is available.

These ideas are included in the PISA framework where the "*problem solving competency*" is defined as[2]:

> Problem-solving competency is an individual's capacity to engage in cognitive processing to understand and resolve problem situations where a method of solution is not immediately obvious. It includes the willingness to engage with such situations in order to achieve one's potential as a constructive and reflective citizen. (OECD, 2013b, p. 122)

Once again, one key point of the definition is the need of a method of solution that is not immediately obvious: that is, a problem is characterized as a non-routine exercise. However, within the PISA framework, we can recognize a questionable assumption: the definition of problem solving competence refers only to cognitive aspects, neglecting a great amount of literature in mathematics education about the role of affective factors in problem solving activity started more than 35 years ago:

> The initial hypothesis of this project was that affect played an important role in problem solving, and that researchers who observed carefully would see the evidence of affect in both students and teachers. That hypothesis has been confirmed. (McLeod, 1989, p. 251)

The influence of affective factors in problem solving performance appears to be particularly relevant in standardised tests for at least two reasons.

On the one hand, a *strange* setting (i.e. different by the usual school setting) for students work on the test since it is not possible for them to get any clarifications or help from the teachers.

On the other hand, the time pressure is particularly strong during the tests. We note that there is a sort of contradiction between the use of problems (intended as non-routine exercises) in the tests, and the limited time students have to cope with them. This contradiction is also detected by the PISA framework itself:

[2]While in PISA 2012 the emphasis was on the assessment of individual problem-solving competency, in PISA 2015 a specific framework also for the assessment of collaborative problem-solving skills was added (OECD, 2017).

> The operational problem faced by OECD/PISA is how to assess whether 15-year-old students are mathematically literate in terms of their ability to *mathematise*. Unfortunately, in a timed assessment this is difficult because for most complex real situations the full process of proceeding from reality to mathematics and back often involves collaboration and finding appropriate resources and takes considerable time. (OECD, 2003, p. 28)

These two points, the impossibility of asking for clarification and the time pressure, appear to be particularly critical during the initial school grades.

In this chapter, we want to approach the issue of the assessment of students' problem solving competencies, discussing the limits, but also exploring the potential of standardised assessments in giving useful information to the education community about the levels of students' problem solving competencies.

We can speak of limits and potential of standardised assessment in giving useful information on two different fronts: the front of communication and the front of what information the assessments provide about students' problem solving competencies.

We have developed our analysis focusing on problems used in the National Students Standardised Assessment in Italy called INVALSI. INVALSI essentially shares the PISA framework: for this reason, here we discuss the recognized limits and potential of the PISA survey.

Concerning communication, the PISA regime (Kanes et al., 2014) includes texts produced for a variety of audiences, in particular reports on the outcomes of the tests, but, as Doig (2006) remarks, the information provided by the standard summative reporting methods has little effect on mathematics teachers' development and practice.

Concerning what PISA results can detect, surely PISA items often challenge students to solve interesting non-routine problems, but the structure of the tasks show several limitations in order to assess the processes involved in the problem solving. As a matter of fact, from the multiple choice or short answer items, we get practically no feedback about the students' methods of solution. This appears to be the main weakness of standardised assessments of students' problem solving competence: they assess the product (result) without considering the solution process. According to Duncker's definition of problem, the assessment of students' mathematical problem solving competence should consider both aspects: the product (i.e. assessing whether the student has reached the goal) and the process (i.e. assessing how the student has reached the goal).

Within this framework, during the academic year 2013/14, an Italian Project (called GRA-INVALSI) was designed to explore how to go beyond the students' results (the product) on the 2013 mathematical national test for grade 2, grade 5, grade 6 and grade 8, that is to discuss and interpret sources of students' answers (the processes) to the different items.

Thus, the focus of this chapter is on: to what extent does the consideration of the students' solution processes enrich our understanding of students' problem solving competencies and difficulties?

The answer to this question also leads to assessing the potential but, above all, to analyse the limits of standardised assessments of mathematical problem solving competence.

10.4 Research Design

10.4.1 The Context of the Study: The INVALSI Assessment

The INVALSI institute (National Institute for the Assessment of the Educational and Instructional System) was founded in the early 2000s, as a result of an intense cultural and political debate on the issue of external evaluation of the Italian educational system. The current configuration of the INVALSI as an independent part of the Ministry of Education is the result of a long evolution, lasting from 1999 to 2008, during which the functions of the institute were redefined and expanded, also because of the recognition of an increasing autonomy of the Italian schools and the consequent need for an evaluation system able to monitor and standardise the instruction offered throughout the national territory, in accordance with what happened in the international scene as well (OECD, 2013a).

At present, the mission of the INVALSI institute is to evaluate the efficiency and effectiveness of the Italian education system and to promote, through national and international evaluation initiatives, the improvement of education levels, and the culture of school accountability.

With this aim, since 2008, the INVALSI institute develops tests, implementing it every year in all Italian schools, assessing two subjects: Italian language and mathematics. These standardised tests are not administered on a sample basis, but they involve all Italian students attending grades 2, 5, 6, 8, 10. For each of these grades, the tests are the same for all schools, regardless of their specific features.[3]

For each grade, the mathematical problems within the tests are created by mathematics teachers of the same school level, according to the official curriculum promoted by the Italian Ministry of Education and essentially sharing the PISA framework (OECD, 2014). Therefore, the items are created trying to involve real problematic situations and following two principal dimensions: contents and processes.

The contents are divided into four categories, in line with the direction taken at the international level (Mullis, Martin, Ruddock, O'Sullivan, Arora, & Erberber, 2005; NCTM, 2000) and similarly to the four *overarching content categories* of PISA (OECD, 2006): Quantity, Space and Shape, Change and Relationships, Uncertainty and Data. For INVALSI, the four content categories are: Numbers, Space and Figures, Relations and Functions,[4] Data and Forecasts.

Concerning the cognitive processes involved in problem solving, INVALSI recognizes eight main processes that students should activate while solving the items, grouped into the same three macro-areas of processes indicated by the PISA 2012 mathematics framework (OECD, 2013b): Formulate, Employ, Interpret. Each item within the INVALSI tests is linked to the content and the process that it is supposed to assess, and also each item includes its designed assessment aim in the answer grid.

[3]This is particularly critical in high school (grade 10), since there are several types of school with different ministerial mathematical curricula.

[4]The content category "Relations and Functions" is not present in the grade 2 INVALSI test.

Once they are designed, the items on each test are preliminarily tested to check whether they are ambiguous or too difficult to answer (in this case they are deleted); then the tests are administered in each Italian school by the teachers themselves, with the exception certain classes—chosen on statistical bases—that constitutes the national sample for the statistical analyses. For the national sample the tests are administered and graded in these classes by external examiners.

In the other classes, the tests are corrected by school teachers using a correction grid provided by the institute. At the end of this correction phase, each school sends the results to the INVALSI. After a few months, the INVALSI returns to the schools the statistical data and results concerning the levels of students' learning. In particular, data received as feedback from the INVALSI offer the comparison of the results of the specific school with the national average and with other similar schools similar for some characteristic (such as numbers of students, social environment, percentage of non native students).

The INVALSI explicitly claims that tests are not designed to verify the individual student's achievement level, but to detect levels of learning in a global way. Therefore, the declared aim of the INVALSI is to evaluate the national educational system taken as a whole, according to the goals fixed by the Italian National Standards, in order to promote the pursuing of these goals and to improve the quality of the national school system. Taking this perspective, the INVALSI tests are, on the one hand, an interesting tool for informing features of learning and for evaluating the national educational system, withholding some interesting issues for educational research in the direction of skills assessment. On the other hand, they could provide a useful guide for both teachers and students, who have the opportunity to compare their teaching standards and their learning on a national scale. Engaging students in solving challenging problems and assessing their performance, the INVALSI might have the potential of promoting a problem-solving approach in the teaching of mathematics.

Nevertheless, the present situation in Italy is critical and needs to be analysed. INVALSI tests are often viewed with suspicion and hostility in the schools. As a result of this situation, on the one hand, the "INVALSI tests" have become a very popular topic in educational context: teachers, students, parents, experts of education, media and politicians show different opinions about the role and the action of INVALSI. Unfortunately, the debate often looks like a "religious war": radical opinions prevail and rarely these opinions concern students' difficulties. This situation gives rise to many unwanted consequences: strikes are not rare during the test days; there is a uncontrolled spread of students' *cheating* in the tests (Ferrer-Esteban, 2013); and the more evident effect on the didactical practices is a spreading of the *teaching to the tests* phenomenon.

10.4.2 The Methodology

The GRA-INVALSI Project, lasted one year; it involved 5 researchers in Mathematics Education and 26 teachers (14 primary school teachers and 12 lower secondary school teachers from different schools in Tuscany).

All school teachers participated voluntarily in the Project.

After a first initial meeting where researchers described the goals and the structure of the Project, the Project was developed according to a 3-phases schema for each of the four mathematical content categories: Numbers, Space and Figures, Relations and Functions, Data and Prediction.

The development of this kind of Team was strongly influenced by the idea of co-learning community introduced by Jaworski and Goodchild (2006). In particular, we wanted to conduct the analysis and discussions on the basis of different points of view, trying to fuse these perspectives in order to develop new knowledge and competences.

Phase 1: a 3-hours joint meeting.

In this phase, the items and the students' results in the 2013 INVALSI survey were analysed. After this analysis, at least three items for each school grade (2, 5, 6 and 8) were selected.

The selection of the items was based on various criteria developed on the basis of the teachers' opinions: the relevance of the specific mathematical content, the relevance of the hypothesised processes brought into play, unexpected students' results in the national survey.

For each selected item, the group of teachers of the corresponding school level and a researcher developed an a priori analysis, trying to foresee the main difficulties students would encounter. Based on this a priori analysis, the group decided how to use the selected items: either in their original version adding the request "*Explain how you reasoned*", or in a newly designed version, in order to test some a priori hypotheses about students' processes and difficulties (for example transforming the item into an open problem; or modifying the words used in the text of the item; changing or removing a figure of the original text).

Phase 2: two 1-hour classroom sections.

During the second phase, the different versions of the selected items were tested by the teachers of the corresponding grade level in their schools: 850 students from 39 classes were involved in this phase.

This second phase was divided into two class sections: in the first section, the teachers involved in the Project proposed the items to their students, without time constraints. For each item, students were asked to justify their answer. A posteriori, we noticed that usually students had spent not more than 30 min to elaborate their answer and provide a justification for a single item.

Students knew that they could ask the teacher for explanations regarding the wording of the item. The teachers took notes of all the students' requests for clarification. This point is strictly related to the consideration of the following consolidated result of the researches about problem solving: many of the difficulties met by students (in particular the youngest ones) during problem solving lie in the preliminary phase of the construction of an adequate representation of the problem situation (Verschaffel, Greer, & De Corte, 2000).

In the second classroom section of this phase, the teachers proposed and moderated a mathematical discussion—in the sense developed by Bartolini Bussi (1996)—in order to bring out the different processes used by the students to answer the item and to discuss the given justifications.

Teachers took notes of the discussion and, at the end of this phase, all the written productions of the students were collected.

Phase 3: a 3-hours joint meeting.

This second joint meeting occurred one month after the first joint meeting.

For each item, the corresponding group of teachers and researchers analysed the mathematical discussions, the students' solutions and the students' justifications in order to highlight the variety of processes shown beyond each possible answer and to identify the main sources of students' difficulties.

The differences between the a priori analysis, developed in the first phase, and this last analysis was assessed to highlight the main unexpected outcomes of the experimentation.

10.5 Results and Discussion

The analysis conducted during the Project allows us to highlight some general limitations of the standardised assessment in the evaluation of the problem solving competences.

We want to briefly discuss the results obtained regarding "the effects of the *time* variable in answering the questions". These effects were not only quantitative—the percentage of correct answers in our sample was always greater than the percentages of the national sample (however, we remind the reader that our sample was not selected on statistical criteria)—but qualitative, and related to the way of approaching the items. These qualitative effects were not only evident, but also particularly interesting.

In our experimentation, without time limit and pressure, students had the opportunity to read the text carefully, to use a trial and error approach, to consider all the answer options, to verify their first choice and, eventually, to change it. And, sometimes to the surprise of their teachers, they exploited this opportunity.

During the standardised test setting, students do not have this opportunity: they do not have much time to think, to evaluate all the options or to tinker with their ideas and eventually answer the question: during the test, with time constraint, if you believe you have somehow identified the right choice, you will immediately move to the next question.

During the official test, the time spent (the less the better) in answering is a quality. Therefore, it is not clear if standardised assessments intend to assess the students'

Table 10.1 Results of the national sample—Item D11 (grade 2)

Omissions	A	B	C
3.3%	43.2%	36.2%	17.3%

competence in solving problems or the students' competence in solving problems *quickly*.

A result of our Project in terms of teacher development, we observed that there was a progressive growth in teachers' awareness about the effects of the time constraint on students' performance. At first, many of the teachers on the Team believed that *poor* problem-solvers were poor solvers regardless of the time variable, but after analysing the answers given by their students without time pressure some of them changed their initial beliefs.

In the following section, we will focus on students' processes (how students arrive to an answer of a problem), discussing the potential of being aware of such processes in a meaningful assessment of students' problem solving competences (and difficulties).

We will discuss four different examples concerning the four test items—one for each school grade involved—about the content category Numbers.

10.5.1 Example 1: The Many Different Correct Answers—The Bus for a School Trip (Grade 2)

One of the items discussed in our group was the following, item D11 from the INVALSI 2013 survey, for grade 2:

A class made up of 9 males and 10 females, accompanied by Miss Gianna and Miss Luisa, takes the bus for a school trip. There are two empty seats left. How many seats are there in the bus for travellers in total?
(A) 19 (B) 21 (C) 23

This item seemed particularly significant to us, especially in consideration of the results of the national sample (Table 10.1): less than 1 out of 5 Italian grade-2 students chose the right option.

As we mentioned previously, the INVALSI items their designed assessment aim was explicit. In this case, the aim of the item was: "*to identify all relevant data needed to solve a problem*".

In order to understand why so many students chose options A and B, we proposed the item D11 in its original version (adding the request "*Explain how you reasoned*") in three grade 2 classes of three different schools (67 students in total).

As we said, our sample was not selected on a statistical basis, so the quantitative results are not very indicative. In any case, we believe that the increasing percentage of correct answers could be partially related to the fact that students knew that they had the time they needed to solve the problem, and in particular they knew they had time to carefully read the text of the problem.

Analysing the explanations for the different answers, there are no big *surprises* for the “right” choice C and for the choice A.

In particular, the latter case is almost always related to the well-known approach: find the digits (9 and 10 in this case), look for isolated key-words (“*in total*” in this case) and carry out the *corresponding* calculation (Sowder, 1989).

As Verschaffel, Greer and De Corte remark, this can be caused by the stereotyped idea of word problem that the pupils build up ever since the first school years:

> Although the numerical tasks are embedded in a context, the stereotyped nature of these contexts, the lack of lively and interesting information about the contexts, and the nature of the questions asked at the end of the word problems jointly contribute to children not being motivated and stimulated to pay attention to, and reflect upon, (the specific aspects of) that context. (Verschaffel et al., 2000, pp. 68–69)

Vice versa the analysis of the reasons for the choice B is surprising and very interesting. In most cases students identified and copied in their notes all the data (9 males, 10 females, the 2 teachers and the 2 free seats) but they did not use all the identified data to calculate the total number of seats on the bus. So on the one hand they showed to be able to identify all the data in the text, on the other hand they did not consider all the relevant data to answer the question. Therefore, the key question is: why didn’t they consider all the data to be relevant in their answer? Fundamentally, two main reasons emerged from the analysis of students’ justifications:

(i) Many students interpreted “travellers” not as “potential travellers” but as “current travellers”. There are two free seats, students in this group identified these two free seats, but they considered them as not being relevant for the answer because the question asks about seats for the current travellers, and the current travellers are 21 (9 male, 10 females and 2 teachers).

(ii) Some students identified all the data in the text but they did not use the data about the two teachers. Pupils in this group explained that the two free seats are the seats for teachers. They linked this answer to their real experiences: in their school trip experiences teachers do not sit in order to monitor students’ behaviour (see Fig. 10.1).

We wonder whether the answer 21 can actually be considered incorrect in these two cases. We believe that it could not. In both cases (i) and (ii) students showed to have the competence that the item was designed to assess—i.e. to identify all the *relevant* data in the text. Their argumentations highlight how they were assuming a definition of *relevant* different from that assumed the test designers, but nevertheless highly justified.

The answer 21, in the first case, is related to a different meaning attributed to the word “travellers”; in the second case to pupils’ construction of a representation of the problem situation based on their experiences.

As Zan (2011) underlines, narrative thinking should not be viewed as an obstacle to logical thinking, or anyway as a lack of rational thinking.

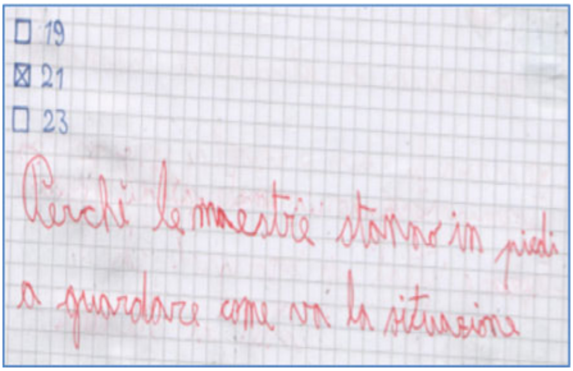


Fig. 10.1 "21, because teachers stand up to check the situation"

Table 10.2 Results of the national sample—Item D23 (grade 5)

Omissions	A	B	C	D
1.2%	3.8%	43.9%	6.5%	44.6%

10.5.2 Example 2: The Different Sources of Difficulties—The Number Closest to 100 (Grade 5)

Another item discussed in our group was the following D23 item from the INVALSI 2013 survey for grade 5:

Which of the following numbers is closest to 100?[5]
(A) 100,010 (B) 100,001 (C) 99,909 (D) 99,990

The explicit aim of this item was: *to compare decimal numbers*, but it appeared clear to us that the answer to this item requires more than simple comparison between decimal numbers, involving the crucial and difficult concept of distance (proximity) between numbers.

This item appeared to be interesting for many reasons: the significance of the concept involved, the fact that similar items are used also for grade 6 (therefore the involved argument is considered significant in the transition between primary and middle school), the result of the national sample (Table 10.2) that highlights that the most chosen option is not the correct one.

We proposed the item D23 in its original version, requiring also the justification for the given answer, to 63 grade 5 students of three different classes. After the individual resolution phase, a mathematical discussion orchestrated by the teacher was developed (Bartolini Bussi, 1996).

From a quantitative point of view, we obtained results similar to the national sample: the most chosen option was D. From a qualitative point of view, significant

[5]Underscore in the original text. Concerning closed questions of Italian national assessments, only the items for grade 2 have three answer options, while the items for all other grades have four options.

aspects to interpret students' answers (and difficulties) emerged from the discussions developed in the three classrooms. In particular, we identified three sources of difficulties related to the choice of incorrect options.

The first one is related to the explicit aim of the item: some students who chose A, C or D show difficulties in calculations with decimal numbers.

The second one is related to the *algorithm* to calculate the distance of X from 100. Without the absolute value, this algorithm is not *symmetric*: if X is greater than 100, I have to calculate X minus 100; if X is lesser than 100, I have to calculate 100 minus X. Some students *make the algorithm symmetric*: they calculate X minus 100 for each value of X, and—after that—they exclude the negative results (options A and B).

The third way of proceeding was the most common among the students and it is particularly interesting. The classroom discussions shed light onto a linguistic difficulty related to the term "*closest*". For many students, the expression "*closest to 100*" meant "*the number does not exceed 100*": "*we have not considered the numbers greater than 100 because 'the closest to 100' means that it does not reach 100*".

As Boero, Douek and Ferrari underline:

> Some difficulties generally arise from the differences in meanings and functions between the word component (i.e., the words and structures taken from ordinary language) of mathematical registers and the same words and structures as are used in everyday life. (Boero, Douek, & Ferrari, 2008, p. 265)

In this case, it seems that the students interpret the sentence "the number X is close to 100" as implying that X *precedes* 100. This is explicit in the following words of a student: "*we have to exclude 100,010 and 100,001* [the answer options A and B] *because they are over 100, therefore they exceed it and they move away from it*!". Moreover, in the discussion, the examples that students give to support their understanding and their use of the term "close to" are related to their everyday life experiences: "*when you are close to the finish line, you are not beyond the finish line*", "*some time ago, my dad said to friends of him that he earns close to 1300 euros. It doesn't mean that he earns 1310 euros, but 1290 euros!*", "*when you* [addressing to the teacher] *told me that my composition is close to being excellent, it means that my composition is not excellent, it is slightly worse: isn't it?*".

The discussion and experimentation of item D23 resulted particularly significant for teachers in our Project. On the one hand, they became aware of the role of mathematical discussions in highlighting a wide and detailed spectrum of sources for students' difficulties, and this awareness allowed teachers to plan and develop some appropriate and specific interventions. On the other hand, they realized the weight of linguistic aspects in choosing an answer for item D23.

Given the relevance of the concept "close to" in mathematics, we also discussed in the Team if there exist any "didactical reasons" for the meaning that many students gave to it. Some teachers noticed two typical features of the early activities (grade 1) with the number line drawn on the floor: usually teachers ask students to go forward on the number line for a fixed number of steps; the number line has an upper limit (usually the number 20) that pupils obviously never exceed. After this discussion, we proposed to a class involved in the experimentation of item D23 the task of reformulating the text of the problem in order to overcome the linguistic difficulty related to the meaning of "close to".

The result was particularly interesting and related to our discussion: after an intense debate, the class proposed the following version for item D23: "*Which of the following numbers is closest to 100, going back and forth on the number line?*".

Using this new version of item D23 in three different classrooms, we obtained a percentage of correct answers of about 90%!

10.5.3 Example 3: The Role of the Text of the Problem—The Chocolate Box (Grade 6)

An item discussed by the lower secondary school teachers in our group was item D16 for grade 6:

A box of chocolates contains 15 milk chocolates and 25 dark chocolates. With 100 milk chocolates and 180 dark chocolates, what is the maximum number of boxes with the same composition of the previous one that we can fill?
(A) 5 (B) 6 (C) 7 (D) 8

The explicit aim of this item was: *to find a solution that satisfies the constraints*. More precisely, it is explicitly linked to the competence "*to solve easy problems in all content areas, retaining control both on the solution process and on the results*", and to the ability "*to run the division with remainder between the natural numbers, to identify multiples and divisors of a number*". To approach and solve the problem correctly the student should realize (through products or divisions) that the number of milk chocolates allows to fill six boxes (each with 15 chocolates) with a surplus of 10 chocolates, while the number of dark chocolates allows to fill seven boxes (each with 25 dark chocolates) with a surplus of 5 chocolates, so the maximum number of boxes that satisfy the constraints is 6 (answer B). In this final phase the control over the meaning of the results comes into play, so the reference to the aforementioned competence "*to solve easy problems [...] retaining control both on the solution process and on the results*" becomes particularly evident; we note that is also one of the key elements in the process of problem solving according to the mathematization cycle by PISA (OECD, 2003).

We consider this item to be interesting because it refers to one of the most important mathematical competences, directly linked to the problem solving competence, and also because the results of the item revealed that it was particularly difficult for the Italian students (Table 10.3).

Concerning the *control issue* previously mentioned, the percentage of students who choose (c) appears to be particularly interesting.

In order to understand why so many students chose the incorrect options, we proposed item D16 in its original version (adding the request "*Explain how you*

Table 10.3 Results of the national sample—Item D16 (grade 6)

Omissions	A	B	C	D
5.5%	14.7%	26.6%	31.7%	21.5%

reasoned") in four grade 6 classes. Then the research group elaborated a modified version, which was tested in two other classes.

As before, also in this case, from a quantitative point of view, we obtained results similar to the national sample, with many students choosing answer C as a result of the division of the total number of chocolates (280) and the number of chocolates in each box (40). However, from the analysis of the explanations given by some students, it was clear that they do not ignore the constraints of the problem: in these cases the most important source of difficulties seems to be not the lack of problem solving competences or arithmetical abilities, but instead interpreting the text of the problem.

The text refers to a 'real' context, talking about chocolates and boxes to fill. This choice is very frequent in the first cycle, usually because recalling a concrete context is thought to be an element that can facilitate the understanding of the problem itself—referring to the students' experiences and thus evoking his/her knowledge of the things of the world—and also increase students' motivation in solving the problem. In fact, this choice does not always achieve these effects, and some research results indicate that the decision to set the problem in a concrete context sometimes introduces difficulties rather than eliminating them:

> Particularly meaningful from a narrative standpoint is therefore that piece of information that enables the child to grasp the problematic nature of the story and point out the link existing between the story itself and the posed question.
>
> It might also happen that the pieces of information needed to solve the problem are not necessarily consistent from a narrative viewpoint, and if they are inconsistent, they will probably be ignored by those who read in a narrative mode. (Zan, 2011, pp. 294)

In the case of item D16, despite the *concrete* context, the posed question asks to respect some constraints that appear artificial (15 milk chocolates and 25 dark chocolates in each box), especially in light of the consequences that they involve, i.e. the fact that this way the correct answer involves a surplus/waste of some chocolates. In this respect, one of the protocols collected during our experimentation is particularly insightful; the student performs the 15×7 and 25×7 multiplications and realizes (correctly) that the first exceeds 100 (the number of the total milk chocolates) by 5 chocolates, while the second is less than 180 (the number of dark chocolates) still by 5 chocolates, to conclude that it is possible to have another box of chocolates, yet different by the standard one (Fig. 10.2).

In this case the choice of the option C is conditioned by the artificiality of the text of the problem: the reasoning explained by the student shows that he is able to successfully execute the multiplications required and that he has perfectly understood the question and the problem constraints (so in this sense he fully achieved the learning objectives that the item was designed to verify), but since he perceives

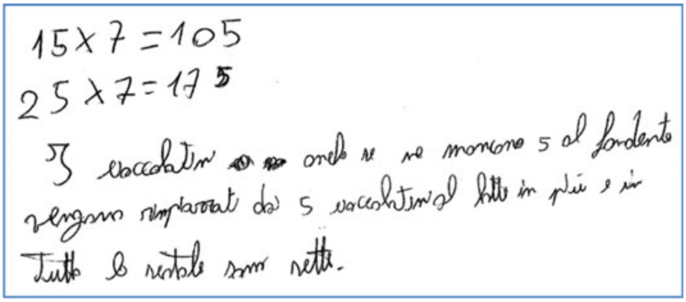


Fig. 10.2 "Though there are 5 dark chocolates missing, they are replaced by 5 extra milk chocolates and so the boxes are seven"

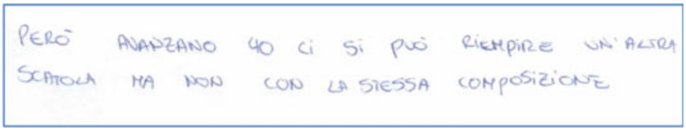


Fig. 10.3 "However, 40 [chocolates] exceed, with which you can fill another box, but not with the same composition"

them as artificial—because with the correct solution there would be many extra chocolates—he prefers to choose a more meaningful solution.

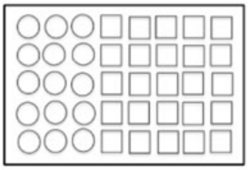
Another student who participated in this experimentation properly resolved the problem, but in the end he felt the need to specify that with the extra chocolates another box can be filled, even if without maintaining the same composition (Fig. 10.3).

The understanding of the text emerges as a crucial point in the resolution of problems. In this case such understanding seems to be hampered by the fact that the problem, despite the reality of the given context (chocolates, boxes to be filled), is not a realistic problem because of the artificial constraints it imposes. The understanding of problems related to the presence of non-realistic constraints can lead the student to ignore the constraint he does not understand, or to re-interpret it so that it acquires meaning for him.

The reformulation proposed by the research group tries to "solve" the artificiality of the constraints linking them to the shape of the chocolates, that is, by placing the same problem in a context where the constraint that must be respected is not imposed artificially and can acquire meaning for the student. The text of the reformulated item was the following: "*Marco works in a bakery, and the owner asked him to prepare boxes of chocolates for a sale on Sunday. In the bakery there are only boxes that hold 15 round-based chocolates, and 25 square-based chocolates. Marco counts the chocolates that are in the store: there are 100 chocolates with a round base and 180 chocolates with a square base. How many boxes can he fill?*". The item was accompanied by a drawing of the chocolate box (Fig. 10.4).

With this reformulation the number of correct answers increased to almost 50% of the students, although the mathematical complexity of the problems was equivalent.

Marco lavora in una pasticceria, e il proprietario gli ha chiesto di preparare delle scatole di cioccolatini da vendere per la domenica.
Nella pasticceria però ci sono solo scatole fatte per contenere 15 cioccolatini a base tonda, e 25 cioccolatini a base quadrata:



Marco conta i cioccolatini che ha nel negozio: ci sono 100 cioccolatini con la base tonda e 180 cioccolatini con la base quadrata.
Quante scatole potrà riempire?

Fig. 10.4 The reformulated item

We also found interesting the fact that some students, recognized the problem as realistic and after answering the problem, try to *complete* the story: “*We can sell the remaining chocolates separately*”).

We note, however, that the goal of this analytical work on the text is not to reduce the difficulties in order to increase the number of correct answers, but to separate out the difficulties due to the interpretation of the problem and its constraints (linked to the reality) from the difficulties due to low specific mathematical competences. A distinction that, in a standardised assessment performed through multiple choice items, is impossible to realize.

10.5.4 Example 4: The Role of Answer Options: Estimating the Root (Grade 8)

The lower secondary school teachers were interested in the analysis of item D19 for grade 8:

The number $\sqrt{6.4}$ *is approximately equal to:*
(A) 3.2 (B) 2.5 (C) 0.8 (D) 8.0

The item brings into play two significant aspects for mathematics education in middle school (grades 6–8): the definition of square root and the competence of estimating the value of a square root. The explicit aim of the item is: “*To estimate the value of the square root of a number*”. The result of the national sample highlights that the most chosen option was not the correct one (B), but also that less than 50% of the sample answers correctly (Table 10.4).

We experimented the item D19 in four grade 8 classes (87 students), requiring justification of the answer and orchestrating a classroom discussion after the individual resolution phase.

Table 10.4 Results of the national sample—Item D19 (grade 8)

Omissions	A	B	C	D
1.2%	3.8%	43.9%	6.5%	44.6%

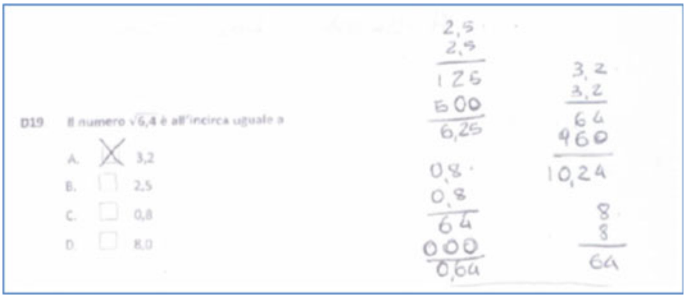


Fig. 10.5 Luca's protocol—Item D19

The first interesting observation is that very few students use words to explain their choices: in almost all protocols, justifications were simply calculations (see, for example, Fig. 10.5).

Luca's protocol is particularly interesting because it shows that Luca knows the definition of square root of a number and has no difficulties in carrying out calculations with decimal numbers. Nevertheless, Luca chose the wrong option. We elaborated on two a priori interpretations for Luca's choice: Luca may have difficulties related to the meaning of "*approximately equal*" (something similar to the discussion about the second example) or Luca could have made a careless mistake when selecting the option. In the discussion, we realized that the latter interpretation was the correct one.

This example clearly underlines the differences between the goal of a standardised assessment and the goal of the teacher in assessing students' mathematical competences. Standardised assessments have the goal of measuring a certain phenomenon, and in their framework Luca's protocol might be seen as a "psychometric noise"; whereas teachers should use assessments to understand students' difficulties and abilities case by case, therefore also being able to interpret *noises*.

The second observation concerns the *loud noise* provoked by the presence of answer options in the item. From the analysis of the students' protocols it became clear that the presence of answer options directs the students' thought processes: just like Luca, almost all the students develop their solutions starting from the answer options, calculating the square of the four given numbers and evaluating their distance from 6.4. If it is true that some of these answer options are related to the common misconceptions about square root, we wonder if item D19 is actually related to the competence of estimating a square root. It is not so obvious that students able to give the right answer are also those able to estimate the square root of 6.4 without seeing the options to choose from (and also the vice versa is not clear).

A particularly interesting aspect is that students were aware of this: "*The multiple choices produce confusion, the reasoning is driven by options, not by knowledge*"; "*We develop our solution process on the basis of the given options and this affects our reasoning*".

The discussion of this item confirms that the multiple choice items strongly affect the students' approach to the problem and their processes of thoughts, moving away from the original goals on the basis of which items have been developed.

10.6 Conclusions

The discussed examples highlight several significant aspects in the assessment of students' problem solving competences; in particular, the clear limitations of standardised assessments in assessing this kind of competences.

One first critical aspect is the time constraint: our experimentation confirms that we have to give students the time to reflect, act, verify and change their mind if we want to assess their problem solving competences. Trivially, if we want to assess students' reasoning, we have to give them the chance (and the time) to reason.

The second main critical point concerns with the attention and consideration of the students' thought processes. As a matter of fact, we need to consider the thought processes behind and beyond a given answer in order to really assess students' problem solving competences (Boero, 2011).

As shown by the discussed examples, the consideration of students' reasoning in problem solving activities, through the request of justifications and the classroom discussions, gives crucial elements to interpret students' answers and possible difficulties beyond an incorrect—but also a correct—answer.

On the other hand, the standardised assessments are usually not meant to *interpret* the students' mathematical behaviour, but rather to *measure*, the students' mathematical performance. A different aim should be that of researchers, but also of teachers, who should have the ambition of analysing, interpreting and understanding what is involved in students' answers.

As attested by the productive collaboration that took place within the community of practice developed in the GRA-INVALSI project, researchers and teachers can have similar aims in the approach to the results of standardised tests. Therefore, although the focus of this paper is not on the teachers, we believe that the results of the Project are interesting also (and especially) for teachers (Di Martino & Baccaglini-Frank, 2017). In particular, according to the discussed influence of standardised assessments on teachers' practices (Bodin, 2005; Nevo, 2001), it is crucial that teachers give the right weight to their students' results to the standardised assessment. On the one hand, it is important they are aware of what these results can and cannot illustrate. On the other hand, we argue the Project and its structure could describe a model for teachers' training aimed to develop teachers' *interpretative knowledge* (in the sense defined by Ribeiro, Mellone, & Jacobsen, 2016), and, more in general, a critical but constructive approach to the information given by standardised mathematics tests.

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Chapter 11
Toward Designing and Developing Likert Items to Assess Mathematical Problem Solving

James A. Mendoza Álvarez, Kathryn Rhoads, and R. Cavender Campbell

11.1 Introduction

Success in foundational mathematics courses provides critical access to science, technology, engineering, and mathematics (STEM) careers because foundational mathematics courses serve as prerequisites for courses in every STEM major (Bryk & Treisman, 2010; Fike & Fike, 2008; Jarrett, 2000). Whereas mathematicians and mathematics educators have identified prerequisite skills, procedural knowledge, and conceptual knowledge needed for courses such as calculus (e.g., Carlson, Oehrtman, & Engelke, 2010), little is understood about the mathematical problem solving (MPS) capacity students must develop to be successful in learning the mathematics central to their field of study (e.g., Selden, Selden, Hauk, & Mason, 2000). In fact, few efficient tools exist that can provide information on entry-level university students' MPS practices. The Mathematical Problem Solving Item Development (MPSI) Project aims to develop Likert MPS items that capture a student's use of MPS. Such a tool could be used to determine MPS profiles linked to success in mathematics courses such as college algebra or calculus.

In 1992, Schoenfeld noted that national calls for action in U.S. mathematics education indicated the "acceptance of the idea that the primary goal of mathematics instruction should be to have students become competent problem solvers" (p. 335). Since then, problem solving and mathematical practices continue to play central roles in standards established for school mathematics in the U.S. (e.g., National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), but college-level courses continue to have limited opportunities for MPS (Schoenfeld, 2013).

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As evidenced by U.S. high failure rates in college algebra and calculus, focusing solely on specific knowledge and skills may not be cultivating the mathematical reasoning or problem-solving capacity students need to progress through the STEM pipeline at the university level, or it may be that success in these courses does not distinguish between deep learning and surficial learning (Maciewjeski & Merchant, 2016). However, not only is little understood about the appropriate levels of facility in MPS needed to progress successfully through gateway mathematics courses, but also the complex nature of MPS makes it difficult to determine this capacity. Some researchers have used assessments involving problem posing tasks to determine students' thinking processes and abilities linked to MPS (e.g., Kwek, 2015; Silver & Cai, 1996; Van Harpen & Presmeg, 2015). Few would argue that the best measures of a student's MPS capacity involve task-based interviews and review of student work on problem posing or MPS tasks using detailed rubrics (cf. Dawkins & Epperson, 2014; Oregon Department of Education, 2000; Silver & Cai, 1996). However, these measures are time consuming and costly (e.g., Pugalee, 2004). With the goal of mitigating these costs and time constraints, the MPSI project will develop Likert items that can be machine scored. Our goal is that the MPSI Likert items can be used to create an efficient tool that will provide key information about MPS practices for entry-level university students.

In addition, we aim for the MPSI Likert items to measure MPS skills, independent of procedural knowledge and conceptual knowledge (beyond a secondary-school algebra threshold). We are particularly interested in using the items to explore the MPS skills needed for student success in courses such as college algebra or calculus. For example, students may possess prerequisite skills, procedural knowledge, and conceptual knowledge necessary for matriculation in an entry-level university course, yet not all of these students are successful in such courses. Efficiently capturing students' MPS practices may provide useful information about MPS thresholds necessary for success. In addition, once development of the MPSI Likert items is complete, the items could possibly be used in a variety of settings, such as to assess student readiness for university-level mathematics or to provide instructors with information regarding their students' MPS practices.

In this chapter, we describe the design and development of Likert MPS items to assess MPS and present current progress toward validating the items and their possible correlation to course outcomes. We also use the current data to provide initial interpretations on whether the MPS items being developed are separating MPS from mathematics content knowledge. Meaningful progress of the MPSI Project may open new avenues, which currently do not exist, for creating and designing assessments of MPS that present an efficient way to assess students' MPS capacity on a large scale.

11.2 Mathematical Problem Solving in the Research Literature

The development of Likert MPS items draws upon Lester's (2013) definition of a "mathematics problem" and Schoenfeld's (2014) generalized theoretical perspective. Campbell's (2014) systematic analysis of the mathematical problem solving research literature provides a framework, used in the development of the Likert MPS items, for identifying key aspects of mathematical problem solving we wish to capture in an MPS assessment tool.

11.2.1 What Is a Mathematics Problem?

Mathematicians and mathematics educators often use Polya's (1957) early work to describe the problem-solving process, but their definitions of mathematical problem solving or what constitutes a mathematics problem may differ widely (e.g. Schoenfeld, 1992; Wilson, Fernandez, & Hadaway, 1993). We ascribe to Lester's (2013) definition that "…a problem is a task for which an individual does not know (immediately) how to get an answer…" (p. 247). Thus, a problem is relative to the solver (Schoenfeld, 1985).

The development of the MPS items also attends to the developmental appropriateness of the items and incorporates the idea from Yee and Bostic's (2014) framework, in which they define a problem as "a developmentally appropriate challenge for which the participant has a goal but the means for achieving it are not immediately apparent." (p. 2).

11.2.2 Theoretical Perspective

In discussing MPS as a research and practice domain in mathematics education, Santos-Trigo (2014) asserts that problem solving as a research endeavor includes "analyzing cognitive, social, and affective components that influence and shape the learners' development of problem-solving proficiency" (p. 496). In addition, Lester and Cai (2014) assert that, "During the past 30 years, there have been significant advances in our understanding of the affective, cognitive, and metacognitive aspects of problem solving in mathematics" (p. 118).

As a practice domain, Santos-Trigo (2014) includes curriculum development and design that enhance MPS activities in the classroom and goes further to describe problem solving activities as those that involve sense-making, using different representations, exploration, natural extensions, and emphasis on appropriate language to discuss results. Lappan and Phillips' (1998) developed their middle school mathematics curriculum (*Connected Mathematics*) using a set of criteria they developed for good or "worthwhile problems." Cai, Wang, Moyer, and Nie (2011), for example, found this curriculum to be effective for fostering students' MPS.

Because our view of a mathematics problem is grounded in the problem solver's knowledge rather than an external hierarchical judgement of appropriate challenge, we view MPS as a process (cf. Silver, 1985). To explain someone's success or failure in a problem-solving attempt, we appeal to Schoenfeld's (2014) generalized theoretical perspective from his decades of research on essential components of mathematical problem solving to any knowledge-rich domain. This success or failure in a MPS attempt involves one or more of the problem solver's

a. domain-specific knowledge and resources,
b. access to productive "heuristic" strategies for making progress on challenging problems in that domain,
c. monitoring and self-regulation (aspects of metacognition), and
d. belief systems regarding that domain and one's sense of self as thinker in general and a doer of that domain in particular (…, one's domain-specific identity). (p. 405)

This is similar to Jonassen & Tessmer's description of problem solving as an activity that involves domain knowledge, structural knowledge, ampliative skills, and metacognitive skills as well as motivational/attitudinal components and knowledge about self (as cited in Jonassen, 1997, p. 66). Jonassen and Tessmer's structural knowledge involves information networking, conceptual networking, and mental models, which we believe rely on Schoenfeld's (2014) domain-specific knowledge and aspects of monitoring and access to productive heuristic strategies. Ampliative skills refer to constructing/applying arguments, analogizing, and inferencing, which in our view relate to Schoenfeld's (2014) resources, productive strategies, and monitoring.

11.2.3 Characterizing Mathematical Problem Solving

Campbell (2014) systematically analyzed research articles in MPS and cataloged explicitly-stated or implied definitions of MPS to determine characterizations of MPS in published research. His findings stabilized after analyzing 18 articles, and no new characterizations were required to classify definitions of MPS for an additional seven articles. He identified five key characterizations or domains: sense-making/orienting (e.g. Santos-Trigo, 1998; Schoenfeld, 1988); representing/connecting (e.g. Kieran, 2007; Wilson et al. 1993); challenge/difficulty (e.g. Chapman, 1999; Jonassen, 1997); reviewing/checking (e.g. Carlson & Bloom, 2005; Garofalo & Lester, 1985); and justification/defending (e.g. Jonassen, 1997; Szetela & Nicol, 2002). For example, Schoenfeld (1988) explicitly referred to sense-making as "an act of taking things apart and seeing what makes them tick" (p. 87), whereas Kieran's (2007) references to combining previously learned techniques imply connecting prior knowledge to problem situations. The domain of challenge/difficulty links to Jonassen's (1997)

Table 11.1 MPS domains and Schoenfeld's (2014) theoretical framework

Theoretical framework	MPS domains
Domain-specific knowledge and resources	Challenge, representing/connecting, sense-making
Access to productive "heuristic" strategies for making progress on challenging problems in that domain	Representing/connecting
Monitoring and self-regulation (aspects of metacognition)	Sense-making, reviewing, justifying
Belief systems regarding that domain and one's sense of self as thinker in general and a doer of that domain in particular	All five domains (indirect)

characterization that "problem solving, as an activity, is more complex than the sum of its component parts" (p. 65), and Garofalo and Lester (1985) included monitoring and checking as an important aspect of MPS. Finally, Jonassen (1997) indicated that justifying or defending generated solutions also engages learners in higher-order problem solving learning.

Campbell (2014) developed definitions for each domain using phrases and concepts from the relevant literature. We refined the definitions to focus on the significance of each domain and key aspects of MPS associated with it. Our MPS domain definitions are:

Sense-making: Identifying key ideas and concepts to understand the underlying nature of the problem. Attending to the meaning of the problem posed.
Representing/connecting: Reformulating the problem by using a representation not already used in the problem or connecting the problem to seemingly disjoint prior knowledge. Using multiple representations or connecting several areas of mathematics (e.g. geometric and algebraic concepts).
Reviewing: Self-monitoring or assessing progress as problem solving occurs, or assessing the problem solution (e.g. checking for reasonableness) once the problem-solving process has concluded.
Justifying: Communicating reasons for the methods and techniques used to arrive at a solution. Justifying solution method(s) or approach(es).
Challenge: The problem must be challenging enough from the perspective of the problem solver to engage them in deep thinking or processes toward a goal, "without an immediate means of reaching the goal" (Wilson et al. 1993, p. 57).

We associate these domains with components of Schoenfeld's (2014) theoretical framework (see Table 11.1), recognizing that although we aim to control for domain-specific knowledge in the MPS items, the degree of challenge, use of representing/connecting, and success in sense making could be linked to domain knowledge and resources. In addition, although belief systems and affect are important components in MPS, the domains only indirectly link to this component.

11.3 MPS Item Development and Design

To capture undergraduate students' use of the MPS domains, we have developed an MPS test which includes mathematics problems and associated MPS items. We describe problems, items, and the MPS test in Sects. 11.3.1 and 11.3.2; details regarding revisions and validation of the items are discussed in Sect. 11.3.3.

11.3.1 Problems and MPS Items

We aim to assess MPS through mathematics *problems* and associated *items*. Problems offer students an opportunity to engage in MPS, and the items are designed to provide information on each of the domains described in Sect. 11.2.3.

11.3.1.1 Problems

We currently use 10 mathematics problems. For example, one problem is *Ken's Garden*:

> Ken's existing garden is 17 feet long and 12 feet wide. He wants to reduce the length and increase the width by the same amount. He wants his new garden to be approximately half the size of the current garden, what dimensions are appropriate for Ken's new garden?

Our intent is to gather information regarding undergraduate students' MPS separate from conceptual knowledge and procedural knowledge associated with the problem solver's content knowledge and resources. As such, we developed problems at a secondary-school algebra level so that they are accessible to students in terms of prior mathematical knowledge and skills, but they require students to use problem solving methods to reach resolution. Students entering college algebra in the U.S. typically have had at least two years of secondary school algebra or have studied similar topics in a developmental mathematics program at a community college or university before enrolling in a college algebra course.

The problem statements are inspired by textbook problems, released items from high-stakes tests such as the State of Texas Assessments of Academic Readiness Algebra End-of-Course Exams (Texas Education Agency, 2011) as well as other sources such as the Partnership for Assessment for Readiness for College and Careers (PARCC) (Pearson Education, 2010), Algebra II Assessments (Charles A. Dana Center, 2007), and Smarter Balanced Assessments (2014). Although the conceptual knowledge and procedural skills should be familiar to the problem solver, the problems are designed to be consistent with our definition of problem (Lester, 2013). That is, the problem statements developed aim to create scenarios that require in-depth use of problem solving so that the associated items may explore a problem solver's facility in several MPS domains. To further minimize the effect of students' knowledge linked to specific algebra topics, the problems target a broad range of algebra

Table 11.2 Algebra concepts included in problems and format of problems

Problem name	Concepts	Format of problem statement
Extreme values	Quadratic functions	Abstract
Building functions	Graphs of basic functions; transformations of functions	Abstract
Intersecting graphs	Graphs of functions	Abstract
Fun Golf	Quadratic functions	Contextual
Cross-country race	Rate and linear functions	Contextual
Avoiding intersections	Graphs of linear and quadratic functions; transformations of functions	Abstract
Ken's garden	Quadratic relationship	Contextual
Air travel	Rate and linear functions	Contextual
Intersecting quadratics	Graphs of quadratic functions	Abstract
Robert's crew	Rate and proportional reasoning	Contextual

topics found in the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and the Texas Essential Knowledge and Skills (Texas Education Agency, 2012). In addition, some problems, such as Ken's Garden, are posed in a particular context, whereas other problems are posed abstractly. One example of a problem posed abstractly is *Intersecting Graphs*:

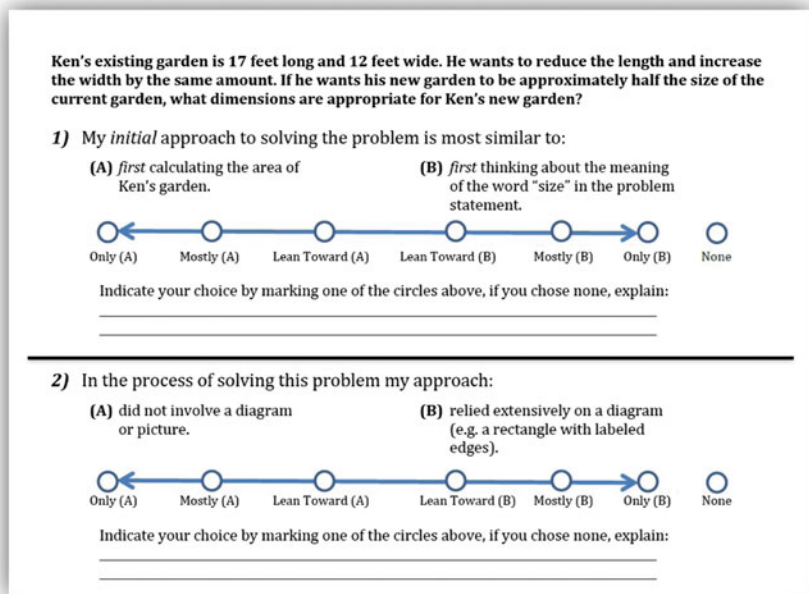
> Give two distinct functions f and g whose graphs intersect at the points $P(-1, -1)$ and $Q(3, 7)$. Explain how you are sure that your answer is correct.

Specific algebra concepts that can be used to solve the problems and the format of the problem are listed in Table 11.2.

11.3.1.2 Items

We currently use 54 items associated with the 10 problems. Each item is associated with one problem, and the items require that participants reflect on their work on the problems. By design, each item targets primarily one domain. We use a Likert scale to contrast approaches linked to the MPS domain that the item captures. For example, Fig. 11.1 shows two items associated with Ken's Garden. The first item is designed to capture the sense-making domain, and the second item is designed to capture the representing/connecting domain.

Answer choices indicate the degree to which a participant believes their approach is similar to the contrasting approaches given. One of the two approaches is identified by researchers as indicating a high use of the domain. For example, in Ken's Garden Item 1 (which captures sense making), the response "first calculating the area of



Ken's existing garden is 17 feet long and 12 feet wide. He wants to reduce the length and increase the width by the same amount. If he wants his new garden to be approximately half the size of the current garden, what dimensions are appropriate for Ken's new garden?

1) My *initial* approach to solving the problem is most similar to:

(A) *first* calculating the area of Ken's garden.

(B) *first* thinking about the meaning of the word "size" in the problem statement.

Only (A) | Mostly (A) | Lean Toward (A) | Lean Toward (B) | Mostly (B) | Only (B) | None

Indicate your choice by marking one of the circles above, if you chose none, explain:

2) In the process of solving this problem my approach:

(A) did not involve a diagram or picture.

(B) relied extensively on a diagram (e.g. a rectangle with labeled edges).

Only (A) | Mostly (A) | Lean Toward (A) | Lean Toward (B) | Mostly (B) | Only (B) | None

Indicate your choice by marking one of the circles above, if you chose none, explain:

Fig. 11.1 Selected items for "Ken's Garden" problem (fall 2016 version). Item 1 captures the sense-making domain. Item 2 captures the representing/connecting domain

Ken's Garden" represents a lower use of sense-making, whereas the response "first thinking about the meaning of the word "size" in the problem statement" represents a higher use of sense-making.

Items were developed by our research team, and we worked to reach agreement on the intended domain of the item whether the answer choices (posed on a Likert scale) allowed for an appropriate range of answers. The definitions of the MPS domains (see Sect. 11.2.3) guided the creation of the Likert items. For example, one part of the sense-making domain is "Attending to the meaning of the problem posed." For item 1 in Fig. 11.1, choice B indicates that the student attended to the meaning of the problem before making calculations. For the representing/connecting domain, items capture students' use of visual representations, such as diagrams and graphs, rather than symbolic representations. We made this choice because in the United States, visual representations in MPS are more non-traditional than symbolic representations in undergraduates' MPS, yet visual representations can be helpful in MPS (cf. Eisenberg & Dreyfus, 1991; Presmeg, 2006).

Across all items, high use of the linked domain varies as choice A or choice B to avoid the possibility of participants deriving a pattern that, say, choice A is the desired choice for all items. Participants are also given the option "none" and asked to explain if this option is used. When participants choose the latter, researchers

Table 11.3 Problems included on the three versions of MPST

Problem Name	MPST 1	MPST 2	MPST 3
Extreme values	✓		
Building functions	✓		✓
Intersecting graphs	✓		
Fun Golf	✓		✓
Cross-country race	✓		✓
Avoiding intersections		✓	✓
Ken’s garden		✓	
Air travel		✓	✓
Intersecting quadratics		✓	
Robert’s crew		✓	

use these responses to gauge whether the contrasting approaches accurately reflect participants’ MPS pathways on a given problem linked to a specific domain.

A single problem statement has approximately five to eight associated items that probe the level of a students’ use of four of the domains—all except challenge—identified in Sect. 11.2.3. Of the current 54 items, each of the four domains is captured by 11–17 items. The challenge domain is captured by a distinct type of item, explained in Sect. 11.3.2.

11.3.2 MPS Test

In a single administration, participants complete a two-part survey or test, which consists of five problems (Part I) and their associated items (Part II). For reference, we will call the collection of both parts an MPS test (MPST). By design, completion of an MPST takes no more than one hour. In fall 2016, we used three versions of the MPST, and each student took one version. Our 10 problems were distributed among the three versions as shown in Table 11.3.

In Part I, participants are asked to solve the five problems and record their work. After completing all problems in Part I, participants complete Part II of the MPST. Part II begins with having the participant rate the difficulty of each problem worked from “very easy” to “very difficult” (see Fig. 11.2). This information addresses the challenge domain and provides context for analyzing participant responses on Part II by attending to whether the participant experienced the problem as “a problem” as we define it as well as the role of content-specific knowledge and resources from Schoenfeld’s (2014) theoretical framework.

Participants are also given an example to guide their responses on the 25–35 items that follow (see Fig. 11.3).

Tests were scored by assigning the response for each item a score from 1 (low) to 6 (high), which correspond to the student’s choice on the Likert scale. Thus, depending

Name______________________________ UTA ID__100______________

Circle one number to complete the statements below.

1) On the scale, I thought the **Avoiding Intersections** problem was ________________.

1	2	3	4	5	6
Very Easy					Very Difficult

2) I thought the **Ken's Garden** problem was ________________.

1	2	3	4	5	6
Very Easy					Very Difficult

3) I thought the **Air Travel** problem was ________________.

1	2	3	4	5	6
Very Easy					Very Difficult

4) I thought the **Intersecting Quadratics** problem was ________________.

1	2	3	4	5	6
Very Easy					Very Difficult

5) I thought the **Robert's Crew** problem was ________________.

1	2	3	4	5	6
Very Easy					Very Difficult

Fig. 11.2 Rating problem difficulty at the beginning of Part II (MPST 2)

Answer the questions on the following pages as in the example.

Example:

Find the solution set for the equation $x^2 = 4$.

1) My *initial* approach to solving the problem is most similar to:

(A) using algebra. **(B)** using graphs.

Only (A) — Mostly (A) — Lean Toward (A) — Lean Toward (B) — Mostly (B) — Only (B) — None

Indicate your choice by marking one of the circles above, if you chose none, explain:

____

If you choose "none" explain here, otherwise do not use these lines.

What would each one of these choices mean?

Only A	I used algebra only and did not use graphs.
Mostly A	I used algebra far more than I used graphs.
Lean Toward A	I used algebra somewhat more than graphs.
Lean Toward B	I used graphs somewhat more than algebra.
Mostly B	I used graphs far more than I used algebra.
Only B	I used graphs only and did not use algebra

Fig. 11.3 Example provided as guide for answering items in Part II (fall 2016 version)

Table 11.4 Piloting and revision cycle

Time period	Action
Fall 2014—spring 2015	Piloted draft items with secondary mathematics teachers enrolled in a graduate mathematics program and College Algebra students, and conducted two think-aloud interviews
Summer 2015	Analysed data, revised existing items, and created new items
Fall 2015	Piloted items at the beginning of the semester with 108 College Algebra Students (nine course sections, five instructors), 405 Calculus I students (11 course sections, eight instructors), conducted 18 interviews (11 College Algebra & 7 Calculus), also administered post-test to 49 College Algebra and 165 Calculus students. Revision to items before administering post-test were made based upon student work on the pre-test and student interviews
Spring 2016	Piloted items in College Algebra, 133 pre-tests and 28 post-tests (four course sections, three instructors). Analysed student interviews. Items sent to MPS research experts for review and feedback. Data analysed by expert in psychometrics
Summer 2016	Revised problems and MPS Items based upon interview data and expert feedback and analysis
Fall 2016	Piloted revised items with 490 College Algebra students (11 course sections, six instructors) and 479 Calculus I students (eight course sections, eight instructors). Conducted ten think-aloud interviews
Spring 2017	Pilot items in College Algebra, expert review of revised items, psychometric analysis of fall 2016 data, analyse student interviews

on the arrangement of the contrasting choices for an item, the scores of 1 and 6 would correspond to choices “Only (A)” and “Only (B)” or vice versa. We used these scores to derive a participant’s average score in sense-making, representing/connecting, reviewing, and justifying based upon their answers to items linked to the respective domains on an MPST. Because we scored responses on a scale from 1 to 6, a middle score is 3.5. We also derived a maximum score from the highest average score across all domains. The maximum score is a participant’s peak use of at least one of the domains. That is, a participant who scores low in sense-making, but scores high in justifying will have a high maximum score and can be distinguished from a participant whose scores in all domains is low.

11.3.3 MPS Item Refinement and Validation

Prior to fall 2016, we created 15 mathematics problems and over 100 associated items. We have since used internal and external review, piloting, and validation processes to revise and refine our initial problems to develop the current 10 problems and 54 associated items. An overview of our processes is described in Table 11.4.

11.3.3.1 Past Validation Efforts

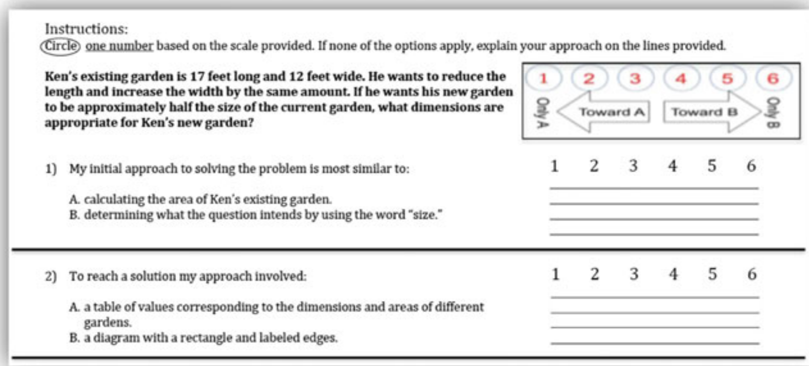
Drafts of problems and items began in 2014. Early piloting with practicing secondary mathematics teachers enrolled in a graduate mathematics education program as well as pilot testing and interviews with participants in college algebra courses informed revisions to the problem statements and associated items. These items were then piloted in a pre-/post-format in fall 2015 and spring 2016 with 256 college algebra students (108 in fall 2015, 148 in spring 2016) and 405 calculus students (fall 2015). We conducted task-based interviews with 26 students (19 from college algebra and 7 from calculus). Each interview was up to one hour in length and used a think-aloud protocol.

As part of the validation process, we analyzed data to compare participants' MPS levels as indicated by MPS items and their MPS behaviors during research interviews on three of the same problems worked during a pre- or post-MPST and one they had not seen previously. Each interview was assigned to two researchers to be coded. We coded verbal and written MPS behaviors in the interviews, using the five MPS domains as a coding framework (e.g., Miles & Huberman, 1994). The domain of representing/connecting was coded according to visual representations (written or mental)—as opposed to symbolic representations—to align with the item design, as described in Sect. 3.1.2. Each researcher independently coded three to five interviews, then coding was compared and contrasted, and the coding scheme was revised until we reached agreement.

The frequency of codes and the depth of the behaviors coded were compared to students' Likert scores on the MPS items. In cases where interview behaviors and Likert scores differed, we revisited interview transcripts and students' written work to seek reasons for the differences (Yin, 2009). This analysis revealed that the language used in some problems and items needed revision, and we revised accordingly. We also revised some of the Likert scale options to better align with the practices that we observed many students using in the interviews.

Through the process of interviewing and triangulating with participant MPS item responses and participant work, we were working toward validating the MPS items protecting against both external and internal threats to validity (American Educational Research Association, American Psychological Association, National Council on Measurement in Education, 1999; Fink, 2013). In addition, we consulted with an expert in psychometrics to investigate the validity and reliability of the items by statistical methods. Consultation in spring 2016 led to revisions in the format of the MPST—leading to the current three tests.

To ensure content validity, we sent revised problems and items to subject-matter experts in MPS for review in spring 2016. Reviewers were asked to comment on the problems as well as the items. For the problems, reviewers provided feedback on the clarity of the problem statement, the difficulty level of the problem for college algebra students, and the potential solution paths. For the items, reviewers provided feedback on the clarity of the item, the two options that were posed, and whether the options lent themselves to a Likert scale. Reviewers also commented on the domain that they believed each item captured. The results of this review helped our research



Instructions:
Circle one number based on the scale provided. If none of the options apply, explain your approach on the lines provided.

Ken's existing garden is 17 feet long and 12 feet wide. He wants to reduce the length and increase the width by the same amount. If he wants his new garden to be approximately half the size of the current garden, what dimensions are appropriate for Ken's new garden?

1 2 3 4 5 6
Only A — Toward A — Toward B — Only B

1) My initial approach to solving the problem is most similar to: 1 2 3 4 5 6

A. calculating the area of Ken's existing garden.
B. determining what the question intends by using the word "size."

2) To reach a solution my approach involved: 1 2 3 4 5 6

A. a table of values corresponding to the dimensions and areas of different gardens.
B. a diagram with a rectangle and labeled edges.

Fig. 11.4 Selected items for "Ken's Garden" problem (Original, fall 2015/spring 2016 version)

team to clarify our interpretations of the domains, revise the language in problems and items, and omit some problems and items.

For instance, based upon expert feedback, MPST data, and interview data, five problems and associated problematic MPS items were removed from the pool for piloting in fall 2016. For example, we found that the "Book Stacks" problem in the original pool (which could be solved using a system of linear equations) was too easy for participants (average participant-reported difficulty rating of 2.2 and 1.6 by College Algebra and Calculus participants, respectively), and the expert reviewers also remarked that the problem elicited limited MPS pathways. Anomalies in the data also indicated that some participants were misunderstanding how to use the Likert scale provided in the items, and a testing expert identified the placement of the scale and contrasting options as visually and cognitively problematic and possibly contributing to the anomalies in the data from fall 2015 and spring 2016. For the ten remaining problems and 54 associated MPS items, revisions in wording and presentation were made based upon expert feedback and data analysis. For example, Fig. 11.4 shows a snapshot of the fall 2015/spring 2016 version of the Ken's Garden problem and associated items. This can be compared to Fig. 11.1 which reflects the changes in the wording of the items and presentation of the rating scale. These revised problems and MPS items were used to create three versions of the MPSTs used in fall 2016.

11.3.3.2 Current Validation Efforts

We are currently continuing our validation efforts for the 10 current problems and three versions of the MPST. The current survey was piloted in fall 2016 with 492 college algebra students and 479 calculus students. We conducted 10 individual, task-based interviews with college algebra students and are analyzing them in ways similar to the fall 2015 interviews. Initial analysis on two of the interviews shows alignment between students' MPS practices and their scores on Likert items (Phan, 2017). The remaining interviews are currently being analyzed in a similar manner.

Graduate students analyzed written student work for four of the problems from the fall 2016 data, with the goal of characterizing solution paths and determining whether the MPS items aligned with typical solutions offered by students. In general, results showed that most student solutions paths—as indicated by written work—were captured in the items. Items are being revised in cases where items did not align with typical solution paths.

We are also considering the mean difficulty scores for each problem on the current MPSTs to ensure that problems are neither too difficult nor two easy for participants. These means are discussed further in Sect. 11.4.3.2.

We again consulted with an expert in psychometrics to analyze the three MPSTs administered in fall 2016. Cronbach's Alpha was used to assess the internal reliability of the items in each MPST. MPS domains that had low Alphas on an MPST were flagged, and we are exploring potential reasons for the low Alphas. For example, the reviewing items in MPST 1 had an Alpha of 0.396, whereas for MPST 3, the reviewing items had an Alpha of −0.056. These items are currently being revisited, and we are seeking potential reasons for the low Alphas as well as the large differences between the two exams. As part of this interpretation, we also investigated the correlation between an item and the corrected total of the items measuring the same domain on each MPST. For example, in MPST 1, the correlation between a student's average score on one of the reviewing items for the Intersecting Graphs problem and average score on all other reviewing items on MPST 1 was 0.035. This low correlation indicates that this item may be problematic. For each item, we also considered the Cronbach's Alpha for the domain, if the item were deleted from the MPST. For instance, for the same reviewing item from the Intersecting Graphs problem, the Alpha for the domain if the item was deleted was 0.418, also indicating a potential problem with the item.

For each MPST, factor analysis was done using principal component analysis to extract factors. A scree plot and our prediction of four factors (the four MPS domains) were used in a Varimax rotation with Kaiser Normalization. Some factors contained many items from the same MPS domain. For example, in MPST 1, three of the six justifying items loaded onto the same factor (at or above 0.32). However, the factors contain some inconsistencies. We are currently investigating the items within each factor to determine possible similarities and reasons for the grouping, and we plan to refine the items again to yield factors that align with our four MPS domains. The current 10 problems and 54 associated items are also under review by experts in MPS as well as master teachers of upper-secondary and lower-tertiary mathematics, and we will also use feedback from these experts in revision.

To ensure the construct validity (Kubiszyn & Borich, 2003) of the MPS items, the course materials used by the college algebra participants are being studied to determine the extent to which our MPS domains are emphasized or developed in these materials. In fall 2015, we collected written logs from Rhoads (researcher and author on this paper) who was the instructor for two sections of College Algebra

(120 students), indicating the extent to which she emphasized the MPS domains in lectures and lab. After each class period, Rhoads self-rated MPS domains that were emphasized and then was interviewed by Campbell (researcher and author on this paper) regarding how the MPS domains were emphasized during instruction. Classroom observations documenting instances of the MPS domains were also conducted, and the class meetings were video-recorded for further analysis to determine which aspects of a powerful learning environment were present (Engle, 2012; Engle & Conant, 2002). Because the course is in a relatively standard format and departmentalized, the course materials and instruction changed very little in subsequent terms. For this reason, video-recording of the lectures was not repeated.

Green (2016) and Peters (2017) examined the college algebra course materials for fall 2015 and spring 2017, respectively. The textbook for the course remained the same from fall 2016 to spring 2017, and the homework exercises and exam questions also changed very little. Green (2016) completed an analysis of all homework exercises used in fall 2015 in College Algebra and found that only 15% of the homework exercises elicited any of the MPS domains with most having only a minimal connection to the MPS domains. Moreover, none of the homework exercises or exam questions fit our definition of a problem (see Sect. 11.1). Peters (2017) results were similar; she claimed the MPS opportunities in the college algebra curriculum were limited, especially with respect to the domain of justification. Peters argued, "…students are not exposed to problems that intentionally and purposefully develop problem solving skills" (2017, p. 37). This corroborates Maciejewski and Merchant's (2016) assertion that first-year courses in mathematics primarily emphasize tasks with low-level cognitive demand.

By examining changes in participant MPS item choices from pre- to post-test, a modified grounded theory approach is being used to determine whether possible changes can be linked to features (or lack thereof) of the College Algebra course. Analysis focuses on construct validity—that is, whether the key areas of emphasis by the instructor or course materials correlate with participants' areas of change from pre- to post-assessment (Kubiszyn & Borich, 2003). If the items themselves provide reliable information regarding MPS domains, and the instruction lacked emphasis in a specific domain, but participants demonstrated strong improvement from pre- to post-scores in the same domain, then such anomalies in the data will also lead to the further refinement and development of the items. Ongoing analysis and results may also give rise to additional problems or questions that should be included as MPS items.

11.4 MPS Item Development Preliminary Results

In this section, we present demographic information on participants and describe data sources for fall 2015, spring 2016, and fall 2016. Preliminary results from participant interviews and MPST administrations are presented to explore whether

Table 11.5 Self-reported racial and ethnic identification of participants

Race or ethnicity	College Alg. (fall 15)	College Alg. (spring 16)	College Alg. (fall 16)	Calculus (fall 15)	Calculus (fall 16)
Hispanic	25 (23.1%)	45 (30.4%)	172 (35.0%)	95 (23.5%)	115 (24.0%)
White-Not Hispanic	32 (29.6%)	36 (24.3%)	140 (28.5%)	135 (33.3%)	139 (29.0%)
Black-Not Hispanic	22 (20.4%)	25 (16.9%)	58 (11.8%)	31 (7.7%)	39 (8.1%)
Asian	19 (17.6%)	16 (10.8%)	63 (12.8%)	107 (26.4%)	140 (29.2%)
Other	10 (9.3%)	26 (17.6%)	59 (12.0%)	37 (9.1%)	46 (9.6%)
Total	108	148	492	405	479

MPS practices during interviews aligns with MPST domain scores and how MPST domain scores for the current set of items may be linked to course performance.

11.4.1 Participants

The MPS items have been piloted in College Algebra and Calculus I at a large (>37,000 students, approximately 25% graduate students) urban university in the Southwestern U.S. The university student population is 25% Hispanic, 15% African American, 10% Asian, and 12% international students. Data reported here come from three semesters of piloting (fall 2015, spring 2016, and fall 2016). Females comprise 46% of the College Algebra participants (n = 748) and 27% of the Calculus participants (n = 884). Most students were in the first two years of study at the university with 18–19 year olds comprising 79 and 77% of the College Algebra and Calculus groups, respectively. The racial and ethnic composition of the groups deviated from the overall university composition, especially in Calculus with Asian students being overrepresented and Black students being underrepresented when compared to overall representation of these groups at the university (see Table 11.5). The "other" category in Table 11.5 includes the students who chose not to answer the question.

When asked to report the last mathematics course completed in high school, 73% of College Algebra participants and 83% of Calculus participants reported a course at the level of second-year algebra, pre-calculus, or calculus. If we include high school statistics courses in the latter, the percentages are 88 and 90% for College Algebra and Calculus participants, respectively. At the beginning of the semester, more than 80% of participants in each group reported feeling adequately to well-prepared in high school for their current mathematics courses.

Participants were enrolled in either College Algebra or Calculus I. The College Algebra course includes the study of linear, quadratic, polynomial, rational, radical absolute value, logarithmic, and exponential functions, relations and inequalities; graphs, basic characteristics, and operations on functions; real and complex zeros of functions; graphing techniques; and systems of equations and matrices. The course format follows an Emporium Model where students attend an 80-min lecture once per week and then are required to attend a computer-based (emporium) lab to work problems for 160 min per week. The labs are supervised by the instructor for the course and undergraduate or graduate student lab assistants. The Calculus I course includes standard topics from differential calculus and basic integral calculus topics through substitution. Students attend two 80-min lectures or three 50-min lectures per week, one 50-min recitation where a graduate teaching assistant answers homework questions, and one 50-min problem-solving lab supervised by their instructor and a graduate teaching assistant.

11.4.2 Data Sources

At the beginning of the fall 2015 term, College Algebra students in nine sections of the course (five instructors) were offered extra credit to complete a pre-MPST outside regular class meetings, whereas Calculus I students in 11 sections of the course (eight instructors) completed a pre-MPST during a regular class meeting. At the end of the fall term, students completed a post-MPST. Participants were not required to take both the pre- and post-tests. In fall 2015, 405 Calculus students and 108 College Algebra students completed an MPST at least once. Because an MPST contains only five of the problems from the pool, versions of the MPST varied in order to pilot all items. Because all course sections did not complete the same problems, the number of participants completing each problem ranged from 40 to 240. For example, the data on “Ken’s Garden” Item 2 includes 45 pre-test responses and 21 post-test responses from College Algebra and 177 pre-test responses and 75 post-test responses from Calculus I.

As in the fall 2015 term, the spring 2016 pilot of the MPSTs in four sections (three instructors) of College Algebra took place outside regular class meetings and participants were awarded extra credit for completing an MPST; 116 participants completed an MPST at least once. For the fall 2016 pilot, 492 College Algebra students in 11 sections of the course (six instructors) and 479 Calculus I students in eight sections of the course (eight instructors) completed at least one MPST. Three versions of the pre-MPST were administered during a regular class meeting for both College Algebra and Calculus. As in previous terms, the post-MPST for College Algebra was administered outside of regular class time, but during regular class time for the Calculus participants.

Table 11.6 gives an overview of the number of participants in each term. For the fall 2015 semester, 447 participants completed MPSTs (a pre-MPST, a post-MPST, or both) including responses to 15 problems and their corresponding items.

Table 11.6 Number of participants for each MPST

	Fall 2015	Spring 2016	Fall 2016
College algebra pre-MPST	69	113	492
College algebra post-MPST	44	28	129
Calculus pre-MPST	378	–	474
Calculus post-MPST	155	–	34

This included 69 pre-MPSTs and 44 post-MPSTs in College Algebra and 378 pre-MPSTs and 155 post-MPSTs in Calculus. In the spring of 2016, 116 College Algebra students completed 113 pre-MPSTs and 28 post-MPSTs. For fall 2016, College Algebra participants completed 492 pre-MPSTs and 129 post-MPSTs whereas Calculus participants completed 474 pre-MPSTs and 34 post-MPSTs.

In addition to collecting participants' work from the MPSTs, participants completed a demographic survey that included questions regarding the last course in mathematics studied in high school, how hard they expect to have to work to do well in their current mathematics course, planned field of study, and high school graduation date. Participants' in both the College Algebra and Calculus groups also agreed to give the project access to their final course grades.

Based upon work shown on the MPST, participants were invited to engage in hour-long think-aloud task-based interviews, as described in Sect. 11.4.3.1. Although we based initial invitations on cases where student work shown in Part 1 of the MPST seemed to conflict with their answers to Part 2 or where the work shown supported their answers to the items, the interview pool is mostly a convenient sample of participants willing to be interviewed.

11.4.3 Preliminary Results

The preliminary results described in this section derive from think-aloud interviews and from fall 2016 MPST data. The think-aloud interviews inform future revisions by comparing student performance on the MPST (i.e. their MPS domain scores) to the behaviors exhibited during interviews and looking for possible discrepancies in interpreting student responses to MPS items. Quantitative data analysis explores associations between MPS domain scores and course performance to better understand how content-domain knowledge may be influencing MPS domain scores or performance on particular items.

Table 11.7 Fall 2015 and spring 2016 interviews with coded instances of sense-making, representing/connecting, reviewing, and justifying

Domain	Number of interviews
Justifying	7 (27%)
Sense-making	25 (96%)
Representing/connecting	17 (65%)
Reviewing	24 (92%)

11.4.3.1 Results from Interviews

Presently, 26 of the think-aloud interviews have been completely transcribed and coded by two researchers. The number of interviews with any instance of sense-making, representing/connecting, reviewing, and justifying coded in the interviews are shown in Table 11.7. These numbers provide a broad overview of whether students were attending to a particular domain at all during an interview. A closer look at the quality and depth of students' use of the domains is further revealing.

In many cases, qualitative data shows that the participant responses to the MPS items align with their observed problem-solving approaches (see also Gonzales, 2017; Phan, 2017; Turner, 2017). This is particularly true for the sense-making domain. For example, for Ken's Garden Item 1 from Fig. 11.4, we anticipated that a score of 1 or 2 would indicate a low use of sense-making, a score of 3 or 4 would indicate a moderate use of sense-making, and a score of 5 or 6 would indicate an extensive use of sense-making. Data for this item from both fall 2015 and fall 2016 align with observed MPS approaches in interviews.

For instance, in fall 2015, for Ken's Garden Item 1, most participants, in both Calculus and College Algebra, chose 1 on the Likert scale, which we had linked to a low-level of sense-making. Although this result initially raised questions about the validity of the item, further insight for Ken's Garden Item 1 was provided from interviews and sense-making items for additional problems on the MPST. In fall 2015, we conducted nine individual interviews with participants in College Algebra who completed the Ken's Garden problem. Of these nine participants, six answered 1 or 2 to Ken's Garden Item 1 (indicating a possible low use of sense-making), and only one of these six had scored highly on other sense-making items on their MPST. Interviews supported the data on this item: The participant who had scored highly on other sense-making items on their MPST showed extensive use of underlining and frequent references to the original problem statement while working a problem conveyed a high use of the "taking apart" (Schoenfeld, 1988, p. 87) and putting back together process of sense-making. The other five participants did not noticeably use sense-making in their interviews. The three participants that did not choose 1 or 2 on Ken's Garden Item 1 revealed some instances of sense-making in their interviews (e.g. rereading the question to find important or noteworthy aspects of the problem). Hence, although there is one exception, the participant responses to this item seem to be strongly associated with a participant's use of sense-making in problem solving during the think-aloud interviews.

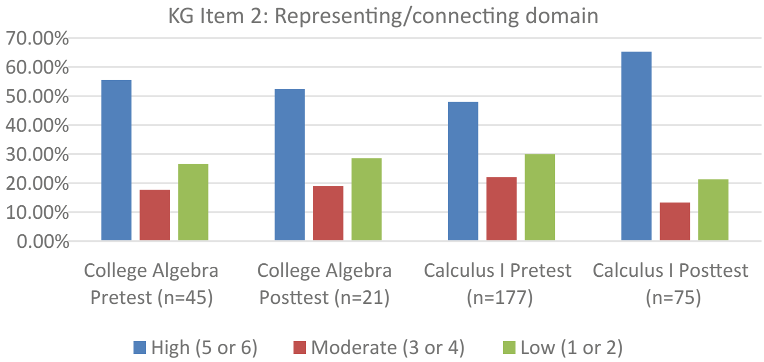


Fig. 11.5 Ken's Garden Problem Item 2 (KG Item 2) responses (fall 2015)

In the fall of 2016, each of the ten participants interviewed answered the Ken's Garden problem in either the Pre-MPST or the interview (see Fig. 11.1). Their interview responses on this item aligned with their scores on other sense-making items. Two participants' written responses to Ken's Garden Item 1 were not consistent with other responses on sense-making items. Each selected "mostly B" (coded 5 for this item) as a response but only averaged 2.6 and 1.0, respectively, for the other sense-making items. However, their responses did align with their work in the interviews with each indenting key parts of the problem statement using underlining and arrows and considering the length and width components of the problem separately. Four participants' scores coded as high use or low use on the MPST aligned strongly with behaviors observed during the interviews. Two scored an average greater than 4.5 in sense-making, and both selected "Mostly (B)" for the item. The other two participants scored an average less than 2.0 in sense-making and each indicated "Only (A)" or "Mostly (A)" as a response. The remaining four interview participants had averages between 2.0 and 4.5 on sense-making items which should align with a moderate use of sense-making. Only one of these chose a response other than 3 or 4 (moderate use) to Ken's Garden Item 1. In summary, the fall 2016 interview data indicates that Ken's Garden Item 1 also seems to be consistent with participants' sense-making scores on other problems.

In other cases, qualitative data has revealed instances in which participant responses to the MPS items differ from their observed problem-solving approaches, leading us to explorations of interview participants' trends in particular domains and identify items that may need revision. One of these is Ken's Garden Item 2 in Fig. 11.4, which corresponds to the representing/connecting domain. The results for Ken's Garden Item 2, shown in Fig. 11.5, show that for the College Algebra group, 25 of 45 participants answering Ken's Garden Item 2 on the pretest chose "Mostly (B)" or "Only (B)," with 21 of whom answered "Only (B)." For comparison, only 75 of 177 Calculus participants chose "Mostly (B)" or "Only (B)" on the pretest.

These results suggested that many students used diagrams, and College Algebra students may be more likely than Calculus students to use them. However, interviews revealed that although participants may draw a diagram, they do not always use the diagram in their solution approach. Six of the College Algebra participants, three of which answered "Only (B)" for this item, were interviewed regarding their work on this problem. In describing their problem-solving process, only one participant claimed to use their rectangular diagram in their approach. The other five participants explained that their processes involved trial and error or using a function or equation. Although four of these participants had drawn a rectangular diagram, only one participant described how they used the diagram. Further, in the written work from all College Algebra responses to the Ken's Garden problem, there is little evidence that diagrams were used in solving the problem, beyond the initial drawings. That is, many participants drew a rectangle and labeled the sides with the original lengths but made no apparent changes or additions to the drawing in the rest of their work. This result led us to revise Ken's Garden Item 2 to attempt to capture more extensive use of diagrams. (The fall 2016 version of the item is shown in Fig. 11.1.)

Interviews provided a further insight into the representing/connecting domain. An interview with "Annie" in fall 2015 illustrates that College Algebra participants may use non-algebraic approaches to problems only when their algebra skills do not support the solution pathway at hand (Campbell, 2017). Annie was a College Algebra student in her first year of the university. She had studied precalculus in high school and earned a B in College Algebra. In Fig. 11.6, we see Annie's work on her pre-MPST where she draws a diagram, labels it correctly, but scratches out the algebra and attempts a trial-and-error tabular-like approach. Her average score in representing/connecting was 3.33 on the pre-MPST. Later, when interviewed during the ninth week of the term, her problem-solving pathways relied mostly on computation and formulas. She then scored a 2.5 in representing/connecting on the post-MPST which shows a decrease in her average score by almost 1 point overall.

Further, three interview participants claimed that they only resorted to a visual or other non-algebraic approach when something seems to be going wrong with their calculations. For example, one College Algebra student described that he rarely used graphs in his solving process and adding, "But, usually that's forestalling the inevitable of settling on an answer I don't like anyway" (Rhoads, Epperson, & Campbell, 2017a, p. 134). The qualitative data related to the representing/connecting domain raises the question of whether students' experiences in College Algebra are linked to their propensity towards symbolic approaches.

Interviews also revealed an interesting trend with the domain of justification. As shown in Table 11.7, only 7 of 26 interviews contained any instance of justification, and in interviews with instances of justification, an average of only 1.5% of the transcript was coded as justifying. For an example of an instance coded as justifying, consider the "Extreme Values" problem in which participants were asked

> Given $f(x) = -\frac{3}{4}x^2 + 6$, $g(x) = -2x^2 - 5$, and $h(x) = \frac{1}{4}x^2 + 1$, is there a number, M, greater than the largest possible values of $f(x)$, $g(x)$, and $h(x)$? If there is, what is M? Explain your reasoning.

An instance coded as justifying occurred when a participant explains that M does not exist because the graph of h is a parabola and opens "up" (Rhoads, Epperson, &

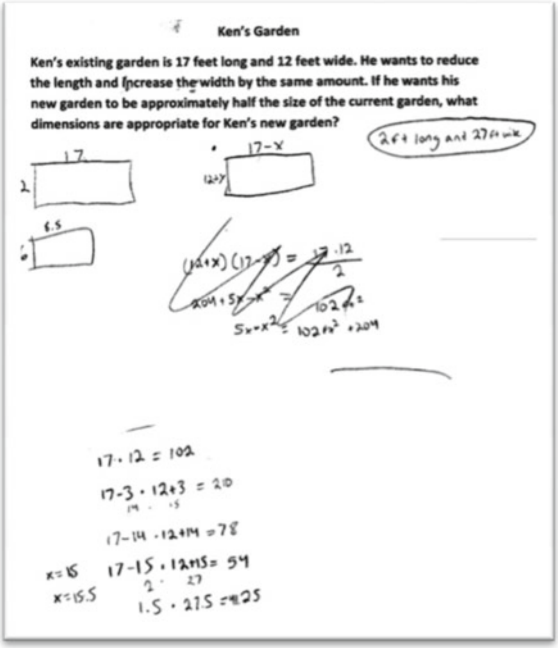


Fig. 11.6 Annie's written work on the Ken's Garden Problem

Campbell, 2017b). The extremely limited use of justifying in interviews leads us to wonder why this is occurring. Recall that Peters (2017) and Green (2016) also found few opportunities for justification in the College Algebra curriculum. We further discuss this potential link in the discussion.

11.4.3.2 Fall 2016 Quantitative Results

Several interesting trends and correlations arose in analysis of fall 2015 and spring 2016 data (cf. Rhoads et al. 2017a). However, because the problems and items were significantly revised and new data collection methods allowed for more participation in College Algebra during the fall 2016 data collection cycle, here we report only quantitative data from fall 2016 on average participant-reported difficulty level of the problems and to compare domain scores of Calculus participants and College Algebra participants.

The average participant-reported difficulty ratings for the problems indicate that problems may be at an appropriate difficulty level for the student population. For all ten problems used in fall 2016, the average participant-reported difficulty rating, where 1 indicates "Very Easy" and 6 corresponds to "Very Difficult," was determined. The pre-test self-reported difficulty rating average scores per problem for College Algebra participants ranged from 2.95 for the "Fun Golf" problem to 4.95 for the "Extreme Values" problem whereas pre-test average difficulty scores for Calculus

Table 11.8 Fall 2016 Pre- and post-MPST average scores by domain—sense-making (SM), representing/connecting (RC), reviewing (RV), and justifying (JU)—and maximum (MAX)

	SM	RC	RV	JU	MAX
College algebra pre (n = 474)	3.675	3.665	3.457	3.678	4.371
College algebra post (n = 122)	3.710	3.611	3.447	3.824	4.361
Calculus pre (n = 492	4.059	3.565	3.559	3.951	4.694
Calculus post (n = 29)	4.257	3.610	3.467	4.332	4.664

participants ranged from 2.65 for the "Fun Golf" problem to 4.00 for the "Intersecting graphs problem. The post-test self-reported difficulty rating average scores per problem for College Algebra participants ranged from 2.73 for the "Fun Golf" problem to 3.9 for the "Building Functions" problem whereas post-test average difficulty scores for Calculus participants ranged from 2.47 for the "Fun Golf" problem to 3.14 for the "Extreme Values" problem. The medians of the average self-reported difficulty ratings on the pre-tests were 3.98 and 3.33 for College Algebra and Calculus, respectively, whereas the medians of the average self-reported difficulty rating on the post-tests were 3.54 and 2.93 for College Algebra and Calculus, respectively. Because there are no extreme scores in these averages (i.e., scores at 2 or below or 5 or above), we hypothesize that the ten problems used in fall 2016 are posed at an appropriate level of difficulty for the students.

To try to gain a better understanding of how content knowledge and/or coursework may be affecting students MPS scores, we compared College Algebra participants' and Calculus participants' average scores in sense-making, representing/connecting, reviewing, justifying, and maximum. In Table 11.8, we see the average scores of each group in each of the MPS domains as well as a maximum score. The pre-MPST scores for College Algebra participants compared to scores for Calculus participants were statistically lower ($p < 0.05$) for all domains except representing/connecting. For sense-making, justifying, and maximum scores, the statistical significance held for $\alpha = 0.01$. The average post-MPST score for the College Algebra participants was lower than the average pre-MPST score for the Calculus participants in every domain except representing/connecting. This may suggest Calculus students have better use of the MPS domains, or it may suggest that Calculus students' stronger content knowledge is helping them to use the MPS domains.

We also compared participants' course grades to their domain scores (see Table 11.9). The comparison separates successful (ABC) participants—earned grade of A, B, or C on an A-F grading scale—from unsuccessful (DFW) participants—earned grade of D or F or W (withdrawn from course). A participant is considered successful if their earned grade is sufficient for progressing to a subsequent, higher-level mathematics course. We do not report the success rate comparison for the Calculus post-MPST due to the small number of posttests (n = 29) we collected; also, the focus of the MPS items is to eventually obtain MPS capacity information that can be used by College Algebra and other entry-level college course instructors and provide baseline information for students at the level of beginning Calculus.

Table 11.9 Fall 2016 pre- and post-MPST scores in sense-making (SM), representing/connecting (RC), reviewing (RV), justifying (JU), and maximum (MAX) by performance in course

	SM	RC	RV	JU	MAX
ABC college algebra pre (n = 230)	3.711	3.666	3.468	3.703	4.388
DFW college algebra pre (n = 242)	3.645	3.66	3.45	3.661	4.356
ABC college algebra post (n = 78)	3.697	3.592	3.391	3.864	4.342
DFW college algebra post (n = 44)	3.734	3.644	3.547	3.755	4.394
ABC calculus pre (n = 300)	4.117	3.581	3.603	3.954	4.697
DFW calculus pre (n = 192)	3.973	3.541	3.492	3.946	4.576

For College Algebra participants, 48% of those taking the pre-MPST earned an A, B, or C in the course and 63% of those taking the post-MPST earned an A, B, or C. For the Calculus participants, 60% of those taking the pre-MPST earned an A, B, or C in the course. When comparing successful participants to unsuccessful participants within the same course, we found that successful participants in Calculus scored statistically higher ($p < 0.01$) in sense-making. However, all other domain score differences within the same course were not statistically significant. This result is promising and—assuming that content knowledge is linked to course success—may suggest that the MPS items are somewhat separate from students' content knowledge. When comparing the successful participants in College Algebra to successful participants in Calculus, the scores on the pre-MPST were statistically higher ($p < 0.01$) for Calculus participants in all categories except representing/connecting. The sense-making and justifying scores on the pre-MPST were also statistically higher ($p < 0.01$) for participants who were unsuccessful in Calculus compared to participants who were successful in College Algebra. In other words, successful College Algebra students' MPS was neither as strong as successful Caclulus students' MPS (in three domains) nor as strong as unsuccessful Calculus students' MPS (in two domains). The reasons for this result require further exploration.

11.5 Discussion

As discussed in Sect. 11.3.3, analysis regarding the validity and reliability of the items in the most recent version is ongoing. However, the preliminary results presented in Sect. 11.4 give some insight toward the goal of developing MPS items that separate procedural and conceptual knowledge in algebra from aspects of MPS that can be used to leverage mathematical knowledge needed to succeed in mathematics.

One of our goals was to create problems at an appropriate difficulty level for entry-level undergraduates. Creating a problem that students do not readily know how to solve, evokes use of sense-making, representing/connecting, reviewing, or justifying, and draws from baseline knowledge in the content domain, presents several challenges as seen in our data. From data gathered on problem difficulty ratings in fall 2015 and spring 2016, we eliminated problems from the MPST pool that students rated too easy or too difficult. In fall 2016, the “Fun Golf” problem had the lowest average self-reported difficulty score for each group and for each testing period. However, the lowest score of 2.47 from the Calculus students indicates that, on average, “Fun Golf” is somewhat less challenging than other problems in the pool but not overly easy for students (otherwise we would expect a rating of 1–2). This less-challenging problem has the potential to affect student responses to the associated MPS items because routine problems do not require engaging in sense-making, representing/connecting, reviewing, or justifying. However, interviews have indicated that students may consider this problem easy because it is approachable and contains many pathways for entry, rather than because it is straightforward. Students have shown several MPS practices in interviews when working on this problem. In addition, Pineda (2016) focused on reviewing and representing/connecting when examining all student work collected in fall 2015 for Fun Golf. She found consistent patterns between student work and student responses for representing/connecting and reviewing MPS items, and she also found that most students did not use equations or symbolic methods to approach this problem but rather were more creative in their MPS approaches. This may suggest this least challenging problem in the set is above a (unknown) threshold that allows students to engage MPS. For the problems that participants rated most difficult on average, the concern is that lack of mathematics-specific content knowledge may be the reason for the rating. However, when students find a problem extremely difficult, it may not be affecting the item data: In many of these cases, participants do not attempt the problem and indicate “None” on the associated items giving the reason “I had no idea how to do this problem.” In participant interviews, we see some issues related to resources when problems are too difficult, but we may be seeing a reverse type of phenomenon on the items. That is, although many College Algebra students show evidence of using multiple representations to solve the problems, those with a high skill level favor symbolic-only approaches and their representing/connecting score is low (e.g. the case of Annie). This also indicates that our goal of minimizing the role of content-specific skills to bring forth use of the other MPS domains is still in progress, and data suggests it is critically important that problems are at the appropriate level of difficulty for the students. The expert reviewers for our project are also providing feedback on difficulty level of the problems.

The representing/connecting scores on the MPS items may be influenced by not only content-specific knowledge but also the course structures. Green (2016) found that the College Algebra homework elicited MPS domains only about 15% of the time. The course lectures, due to their compressed format, emphasized learning skills and procedures. This may be why we saw, for example in Ken’s Garden Item 2, in written work and interviews, that participants may draw diagrams, but do not use them in the process of solving the problem. In many cases, if they do draw a diagram, they will draw it at the beginning and then never refer to it. Again, the favoring of analytic-only approaches may be either explicitly or implicitly derived from the course structure. For the Calculus students, their representing/connecting scores increased from pre- to post-MPST, although this change was not statistically significant in fall 2016. We continue to investigate whether their experiences in the calculus course (e.g. many related rates problems require students to draw diagrams) bring the heuristic of drawing diagrams or other representations to the forefront and may influence the rise in representing/connecting scores.

Although we saw a low instance of justifying in the reported interview data, we see that there is a statistically significant difference between average justifying domain scores for College Algebra participants compared to Calculus participants. When comparing the average domain scores on the pre-MPST, which was typically administered on the first day of class, of successful (ABC) College Algebra students to unsuccessful (DFW) Calculus students, justifying is one of the two domains for which there is a statistically significant difference. This may indicate that important intervention must occur to improve students' capacity to justify their work in MPS in either College Algebra or the intermediate course (e.g. precalculus in the United States) so that successful College Algebra students reach justifying score levels that compare to those who eventually succeed in Calculus. However, differences in content-domain knowledge and resources may explain this difference rather than possible existence a certain justifying threshold score, as currently represented in the pool of items, needed to succeed in Calculus. Both Green (2016) and Peters (2017) found that there were few opportunities for justifying in the curriculum.

For the reviewing domain, the average scores on MPS items linked to reviewing were statistically lower for College Algebra participants compared to Calculus participants on the pre-MPST. In addition, when comparing successful College Algebra students to unsuccessful Calculus students, the Calculus students' average score is higher, but not statistically significant. In Table 11.7, we see that 92% (24 of 26) of the participants used reviewing at least once during the interviews and the relative use of reviewing is consistent with their average reviewing score on the pre-MPST. From this data, it may be that reviewing, as measured by the average scores on the reviewing items on the MPSTs, may not be affected as much by content-domain knowledge or an increase in procedural skills.

Some may argue that all MPS is sense-making and that in the process of sense-making to solve a problem—in the sense of Lester (2013)—all the aspects Schoenfeld's (2014) theoretical perspective are in play. Preliminary data analysis shows that sense-making captured by MPS items is consistent with participant work and interview data. For example, in Ken's Garden Item 1, the majority of participants did not claim to address the ambiguity of the Ken's Garden problem, and this raises the question of whether it is necessary for them to do so to successfully problem solve. It could be that addressing this ambiguity is automatic and requires little thought. However, with one exception, the interview data as well as the data from other sense-making items does support participants' responses to the Ken's Garden Item 1 sense-making question. That is, there is evidence that participants who claimed to address ambiguity in the Ken's Garden problem (i.e., chose 5 or 6) also showed evidence of sense-making elsewhere, whereas participants who chose 1 or 2 for this item showed limited evidence of sense-making elsewhere.

For the pool of fall 2016 items, sense-making is statistically significant when comparing pre-MPST scores for Calculus versus College Algebra participants, pre-

MPST scores for Calculus participants versus post-MPST scores for College Algebra participants, successful Calculus participants versus unsuccessful Calculus participants, and successful College Algebra participants versus unsuccessful Calculus participants. It may be that pre-MPST sense-making scores for successful Calculus students is statistically higher than pre-MPST sense-making scores for unsuccessful Calculus students because the sense-making items are capturing an aspect of problem solving that is not readily affected by learning or re-learning content-specific procedures and skills or that those who eventually pass Calculus begin the course with more content-domain knowledge and the sense-making items are capturing this. However, the significantly lower score of successful College Algebra participants in sense-making compared to unsuccessful Calculus students may suggest that College Algebra courses or courses that follow College Algebra before Calculus may need to increase opportunities for students to engage in sense-making.

Provided the MPS items reveal relatively accurate information on MPS, the fact that pre-MPST scores versus post-MPST scores showed no significant change indicates that perhaps the courses are having little impact on increasing MPS. This is consistent with findings by Dawkins and Epperson (2014) when investigating problem solving in calculus.

A major limitation in this work may be that the MPS Likert items cannot give information on social or affective components that impact problem solving proficiency (Santos-Trigo, 2014). At best, we have aspects of indirect measures of these components given by information on participant racial and ethnic diversity, past courses, preparedness, course grade outcomes, and course structures. As seen in Sect. 11.4.1, the participants come from a highly racially and ethnically diverse group and more than 73% of the College Algebra and 83% of the Calculus participants had completed high school courses at the level of second-year algebra and beyond. Also, more than 80% of the participants in each group reported feeling adequately well-prepared by past studies. Course structures (e.g. lecture-based format, computer-lab procedural work) possibly affected participants' growth in problem solving proficiency, but the MPS items do not capture that directly. The large number of unsuccessful (D, F, or withdrawn from course) students in each group is troubling and may have links to social factors.

The MPS items that we describe do not exist in textbooks, and their development and refinement involves contrasting approaches within appropriate domains, studying participant work on the items, and interviewing participants as well as expert review to validate links to the MPS domains and appropriate mathematical representations. The revised items have been piloted with over 900 participants, and ongoing analysis indicates consistency among participant work, follow-up interview data, and intended links to problem solving categories. We continue to triangulate student work, quantitative data, and qualitative data to refine problems and their associated items. The interaction between the low success rates in College Algebra (only 48% of fall 2016 participants earned an A, B, or C) and outcomes on the MPSTs continues to be explored as well as additional issues in item development that may also rest in the nuances of constructing items that distinguish sense-making from representing and connecting or reviewing from justifying.

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Part V
The Problem Solving Environment

Chapter 12
Creating and Sustaining Online Problem Solving Forums: Two Perspectives

Boris Koichu and Nelly Keller

12.1 Introduction

Mathematical problem solving in asynchronous or synchronous online discussion forums has gained increasing attention during the last decade. Many studies (e.g., Lachmy, Amir, Azmon, Elran, & Kesner 2012; Lazakidou & Retalis, 2010; Lin, 2011; Nason & Woodruff, 2003; Stahl, 2009; Stahl & Rosé, 2011; Tarja-Ritta & Järvelä, 2005) have shown that online discussions enable students to meaningfully use their mathematical knowledge, enhance self-regulation skills and support knowledge construction. It has been documented that sometimes students in such environments actively participated in solving complex problems for 2–3 weeks almost without teacher interventions (e.g., Moss & Beatty, 2006) and that some of those students who tended to be silent in classroom discussions could actively participate in online discussions (Schwarz & Asterhan, 2011).

Schwarz and Asterhan (2011) attribute the affordances of online discussions to their unique traits, such as fostering divergent rather than linear interactions, enabling flexible time schedules of participation in the discussions over relatively long periods, and encouraging explicit and accurate expression of the ideas in writing. Koichu (2018, 2015a, 2015b) argues that affordances of an online problem-solving forum stem from its fundamental characteristic of being a choice-affluent environment, that is, an environment, in which the students are empowered to make informed choices of a challenge to be dealt with, a way of dealing with the challenge, a mode of interaction, an extent of collaboration, and an agent to learn from. Furthermore, Koichu (2018, 2015a) presented an example suggesting that online discussions are particularly rich with the opportunities to enact three types of problem-solving resources: individual

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resources, shared resources having the potential to create an effect of group synergy (in the meaning specified in Clark, James, & Montelle, 2014) and resources stipulated by possible interactions of the members of the group with a source of knowledge about the solution, such as an internet resource, a textbook or a peer problem solver who acts as if she has solved the problem.

Stimulated by the above-mentioned findings and ideas, we designed a special learning environment comprising of a combination of problem-solving lessons in a classroom and out-of-classroom problem solving supported by online asynchronous discussion forums in social networks. We did so within a research project entitled "Heuristic and engagement aspects of learning through long-term collaborative mathematical problem solving."[1] Our experience in this project taught us that creating and sustaining online problem-solving forums is a truly challenging enterprise (Keller & Koichu, 2017; Koichu & Keller, 2017). Furthermore, we observed that past studies are informative about how the forums function at the mature stages of their development, that is, when the students are accustomed to participate in online discussions. Little is known about how to create the forums and bring them to maturation. The goal of this chapter is to fill in this lacuna, by presenting and theorizing our more successful and less successful attempts to create and sustain online forums as a platform for collaborative mathematical problem solving that can complement and extend classroom problem solving. The chapter is based on the evidence collected during one school year in two Israeli 10th grade mathematics classes (15–16 year old) and on interactions between the teachers of these classes with the research group of the project.

In this context, we explore the following research question: How do the classroom communities and the research group develop in interaction? In particular, what facilitates or hinders the implementation of online problem solving activities designed in a partnership between the research group and mathematics teachers of the participating classes?

12.2 Conceptual Framework

The conceptual apparatus of our study combines elements of two theoretical perspectives, the Theory of Diffusion of Innovation (Rogers, 2003) and the Community of Practice framework (Wenger, 2010). The former perspective puts forward individual decision-making in the struggle to adopt or reject an innovation (in our case, an online problem-solving forum) under multi-directional influences. The latter perspective puts forward boundary interactions among the participating communities as a driving force for their development.

[1] Selected findings of the project are reported in Keller and Koichu (2017), Koichu (2018, 2015a, 2015b), Koichu and Keller (2017), Lachmy and Koichu (2014).

12.2.1 Theory of Diffusion of Innovations[2]

Rogers (2003) defines innovation as "an idea, practice, or object that is perceived as new by an individual or other unit of adoption. It matters little, so far as human behavior is concerned, whether or not an idea is 'objectively' new as measured by the lapse of time since its first use or discovery" (p. 11). In our case, the idea of stretching the boundaries of a classroom by means of an online problem-solving forum was an innovation because it was new to the students and the teachers. Roger's (2003) Theory of Diffusion of Innovations (TDI) meticulously characterizes the innovation-decision process, in which individuals decide whether to accept an innovation or not. In particular, Rogers distinguishes five stages of the process: *knowledge, persuasion, decision, implementation* and *confirmation*.

At the *knowledge* stage, potential innovation adopters are exposed to the innovation's existence and obtain some information about how it functions. Sometimes individuals become aware of an innovation by accident, and sometimes they actively look for it in order to fulfil particular needs. It is also possible that the needs are formed as a result of one's exposure to an innovation. In the context of our study, the *knowledge stage* is described by focusing on how students were informed about the possibility to use social networks for collaborative problem solving, that is, to use them beyond their usual use for chatting on non-mathematical matters.

At the *persuasion* stage, an individual forms a favorable or unfavorable attitude towards an innovation. This stage presumes affective involvement with the innovation. In particular, the individuals may mentally apply the new idea to their present or anticipated future situation. They seek to answer such questions as "what are the innovation's advantages and disadvantages in my situation?", and seek the answer mostly from their near-peers, whose opinions based on their personal engagement with an innovation, are the most convincing. There is a discrepancy between forming a favorable attitude towards an innovation and an actual decision to adopt it. Adoption of an innovation can be influenced by a *cue-to-action*, an event that crystallizes an attitude into overt behavioral change. Accordingly, the description of the *persuasion stage* in our study is based on the data on the students' initial attitudes towards the use of social network for problem solving and their expectations. These became articulated during either specially designed or unforeseen events that occurred at the beginning of the project.

At the *decision* stage, an individual adopts (i.e., makes full use) or rejects an innovation. Any decision is not final however. The rejection can occur even after a prior decision to adopt; Rogers calls this phenomenon *discontinuance*. The theory distinguishes between *active rejection* and *passive rejection*. The former type of rejection consists of considering adoption of the innovation and then deciding not to adopt it. The latter one consists of never "really" considering the use of the innovation. The decision stage frequently includes a small-scope trial. The actual sequencing of the three stages presented so far can alter. Namely, both *knowledge–persuasion–decision* and *knowledge–decision–persuasion* sequences are possible. Evidence for inferring

[2] This sub-section is a slightly modified version of a section in Koichu and Keller (2017).

conclusions about the students' decisions came from their actual systematic appearance (or its lack) at the forums.

At the *implementation* stage, those individuals who decided to adopt an innovation put it into systematic use. Even though the decision has been made, the adopters may still feel a certain degree of uncertainty about the consequences of the innovation. In addition, problems of how exactly to use the innovation may emerge. Sometimes the adopters change or modify (in Rogers' terms, *re-invent*) the innovation at this stage. The implementation stage can be lengthy, but it ends when the idea that has once been innovative becomes institutionalized and regularized in the adopters' normal functioning. In our case, we characterize this stage by describing how the norms of participating in the forums has been established and gradually modified.

Finally, at the *confirmation* stage, an individual constantly seeks reinforcement for the decision to adopt or reject an innovation that has already been made. Because of positive or negative messages about the innovation, the decision can be reversed. Rogers points out that the *change agents* (i.e., those who influenced one's decision to adopt an innovation) have responsibility of providing supportive messages to the individuals who have previously adopted the innovation. Because of our focus on creating problem-solving forums and bringing them to maturation, this stage is indicated but not characterized in detail in this chapter.

12.2.2 Communities of Practice Perspective

Wenger (2010) asserts that a Community of Practice (CoP) can be viewed as a relatively simple social learning system, and a complex learning system can be viewed as a network of interrelated CoPs. He further notes that, in relation to the whole system, each CoP is engaged in its own practice production. Consequently, each CoP has boundaries defined by its practices, local engagement, local discourses and power relationships. Furthermore, characterizing a CoP includes attending to patterns of interaction and partnership among its members, who may have different roles, different types of expertise and contribute differently in performing a jointly approached task.

In the context of collaborative learning and problem solving in a group, Wit (2007) distinguishes between two main modes of partnership, *positive interdependence* and *negative interdependence*. *Positive interdependence* refers to the extent by which one group member's successful performance directly promotes the interests of fellow group members. By contrast, negative interdependence refers to the extent by which successful performance of one group member entails a loss by other group members. At the first glance, positive interdependence suits the CoP perspective whereas negative interdependence contradicts it. A deeper look reveals however that the mixture of positive and negative interdependences can serve as a driving force for productive group work (Wit, 2007). Namely, positive interdependence is a mechanism underlying collaborative effort when a group deals with a problem, which solution requires labor division and complementary contributions. Negative interdependence is a mechanism underlying the motivation of the group members to act in a self-interest way in order to get a fair share of costs and benefits out of the group work. That is, negative interdependence may hinder collaboration, but may as much be an impetus for maximizing individual contributions in service of the common goal and for negotiating the conflicting interests in order to improve collaboration.

As to across-communities communications, Wenger (2010) argues that these are "not necessarily peaceful or collaborative" (p. 183). The communities may merge, split, compete, complement each other, and even disappear because of *boundary interactions* defined as experiences of being exposed to foreign competencies, norms and practices. At the same time, the boundaries can also be a locus of productive meetings of different perspectives and reciprocal learning. According to Wenger (2010), one condition for boundary interactions to be productive is the existence of a shared history of learning. In his words:

> Without a shared history of learning, boundaries are places of potential misunderstanding arising from different enterprises, commitments, values, repertoires, and perspectives. In this sense, practices are like mini-cultures, and even common words and objects are not guaranteed to have continuity of meaning across a boundary. (Wenger, 2010, p. 183)

Solomon, Eriksen, Smestad, Rodal, & Bjerke (2017) present an additional condition for productive boundary interactions. They explored two intersecting but somewhat conflicting CoPs, a theory-oriented university education program and a practice-oriented program of early school placement, in which the same group of elementary school teachers simultaneously took place. Solomon et al. (2017) found that the boundary interactions between the CoPs were not particularly productive when teacher-mentors, who belonged to the school CoP, appeared to demand the alignment with the practices of their community without consideration of what the students learned in the university CoP. More productive boundary interactions were documented when the mentors provided advice but the alignment was not demanded so that the students had room for experimentation.

To summarize, the analyses in terms of the TDI and in terms of the CoP perspective complement each other in our study in the following way. The TDI enables us to characterize the process of incorporating online forums in the mathematics study as a dynamic innovation-decision process, in which individuals having different incentives decide whether and how to participate in the new for them learning activity. The CoP perspective enables us to look at characteristics of the classroom and the research communities as well as at the boundary interactions between them, with special attention to conflicting interactions that might facilitate or impede the development of collaborative problem solving in the project.

12.3 An Overview of the Project

As mentioned, a research project in the context of which we experimented with combining classroom and online problem solving was entitled "Heuristic and engagement aspects of learning through long-term collaborative mathematical problem solving". The main goal of this project was to produce a model of learning through mathematical problem solving, which would be attentive to cognitive, socio-affective and contextual aspects of this activity.

Presentation of the model is beyond the scope and focus of this chapter (see Koichu, 2018, for its exploratory version). For our current concerns, it is sufficient to mention that the explored in the project learning through problem solving was of participatory nature, in the meaning specified by Cobb (2000). To recall, Cobb argued that individual students' mathematics activity and their social learning practices complement each other. Namely, he argued that students contribute to evolving social practices of studying mathematics by reorganizing their individual mathematical activities and, conversely, these reorganizations are constrained by their participation in the evolving social practices.

12.3.1 The Project's Pedagogical Idea and Participants

The project's pedagogical idea was to extend the boundaries of a classroom by using online forums. We planned that each participating class would be engaged in the following activity at least three times during a school year. The students cope with a series of preparatory tasks during a 90-min lesson, and then they are offered an especially challenging geometry problem at the end of the lesson. Following the lesson, they engage, for 4–10 days, in solving the problem from home in a closed (that is, available only to the students of a participating class and the members of the research group) online forum.[3] The students use the online forum by means of uploading pictures of their hand-made drafts, openly sharing their problem-solving ideas and responding to ideas of the fellow participants.

Next, we planned that the teachers would refrain from structuring and orchestrating the students' performance, as it is frequently done in lessons when a given problem must be solved before the bell rings (e.g., Lampert, 1986, 1990; Stein, Engle, Smith, & Hughes, 2008). They instead would encourage the students' participation. When the forum becomes non-active because the students have eventually solved the problem and uploaded their solutions or, alternatively, because they give up or prefer presenting their solutions in a classroom, a concluding 90-min lesson is conducted in a classroom in order to get closure. The lesson consists of a series of whole-class and small-group discussions, during which the students share their experiences with the problem and present their solutions.

The described idea appeared to be appealing to several mathematics teachers, who appreciated it for several reasons, but mainly as an extra learning opportunity for their students. Two experienced mathematics teachers (hereafter, NK and AP) and two of their 10th grade classes took part in the first year of the project (2013–2014); five

[3]Our initial plan was to use Facebook and Moodle as technological platforms of the project. In practice, we quickly switched to Google+ and then added WhatsApp following the student choice.

more teachers (hereafter, AH, ES, RN, AG and LA) and their corresponding five classes joined the project during its second year (2014–2015). Mathematics in all participating classes was studied for five hours a week, in accordance with the Israeli high-level curriculum (see Movshovitz-Hadar, 2018, for details). For the concerns of this chapter, it is enough to mention that the geometry part of the curriculum (two hours a week) included the topics “triangle congruency”, “triangle similarity,” “quadrilaterals”, “circles” and “areas” and that the study of all the topics included dealing with challenging proving tasks.

In line with Cobb’s (2000) recommendations for conducting teaching experiments in collaboration with teachers, all aspects of the project were continuously discussed, either prospectively or reflectively, in weekly meetings of the research group. At different stages of the project, the group consisted from six to 10 members having M.Sc. or Ph.D. degree in mathematics education. It included the participating teachers and 3–5 additional members who assumed the roles of facilitators and researchers. In addition to the weekly meetings, each teacher was in contact (by means of email, telephone and one-on-one meetings) with an additional member of the group who was responsible for technological support and documentation of the activity. It is in place to mention here (this point is elaborated below) that special effort was made in order to establish equal partnership rather than mentor-trainee relationships in all teacher-researcher pairs as well as in the group as a whole.

12.3.2 Data Collection

Forty-two meetings of the research group were audiotaped (about 100 h) and, in addition, documented in the protocols of the meetings (more than 100 pages). The documents produced by the group and all relevant email exchange were stored. In addition, each member of the group, including the teachers, was required to keep a diary. The diaries were for writing anything their authors deemed important for the project, including their thoughts and feelings in relation to the project’s events. Among other uses, the diaries became an indispensable source of information about potentially important to the project events that otherwise could not be documented, such as: regular lessons preceding or following the lessons of the project, meetings with the students outside the classroom, occasional conversations between the members of the research group etc. The diaries were stored in shared Google Drive space of the group and were available for reading and commenting by the members.

As mentioned, we focus in this chapter on two 10th grade classes and on an online problem-solving forum that has gradually became active in one of the classes. The data on these classes and the forum were collected from videotaped lessons (14 lessons in two participating classes were videotaped during 2013–14), the aforementioned diaries, the content of the forum (more than 3000 posts) and the student interviews (10 out of 42 students were interviewed about their experiences in the project).

12.3.3 Data Analysis

We address the chosen research question by presenting three research narratives and an aggregated summary consisting of characteristics of the research group and two participating classes as CoP. In a way, the summary serves as a prelude to the narratives and the narratives serve as an empirical backup of some of the characteristics indicated (but not elaborated) in the summary.

The summary was produced by means of a *general inductive* approach (Thomas, 2006). Thomas (2006) argue that this approach enables researchers "(1) to condense extensive and varied raw text data into a brief, summary format, (2) to establish clear links between the research objectives and the summary findings derived from the raw data and to ensure that these links are both transparent (able to be demonstrated to others) and defensible (justifiable given the objectives of the research), and (3) to develop a model or theory about the underlying structure of experiences or processes that are evident in the text data" (p. 238).

Practically speaking, we, the authors of this chapter (hereafter, BK and NK; BK was a principle investigator, and NK was a teacher of the first participating class and a researcher), organized the data from different sources in portfolios for each CoP. We browsed each portfolio for excerpts representing events that were suggestive about the phases and characteristics indicated in *Diffusion of Innovations* and the *Community of Practice Perspective* sections (e.g., a particular lesson, meeting or conversation with the students). We then thoroughly studied, by means of reading, re-reading and interpreting, the chosen excerpts in order to distil brief characterizations of each community of interest. For example, aggregative summaries of *goals* and *practices* of the research group (see Table 12.1) were formulated based on agendas of 42 meetings of the group as reflected in the protocols of meetings. The summaries of *discourse* and *power relationships* were formulated as our (BK and NK) generalized reflections on the meetings, based on our memories, which were refreshed and backed up by careful listening of the audiotapes. Aggregated characteristics of the NK and AP classes were formulated in a similar manner based on appropriate data sources, including videotaped lessons and teaching diaries. Parts of the aggregated summary concerning the AP class were discussed with him and refined. Admittedly, the summary as a whole is rather silent about the across-communities interactions. Those are elaborated in the research narratives.

The research narratives concern the developmental aspects of the project. They were produced in accordance with the *narrative inquiry* tradition. We choose to adhere to this tradition because we, BK and NK, had been active members of the processes under exploration, and since the data included many field texts produced by us as well as by other project participants. As Clandinin and Caine (2008) explain, "Narrative inquiry is marked by its emphasis on relational engagement between researcher and research participants" (p. 542). Furthermore, they indicate that inquiring into the researchers' own stories of experience and living alongside participants are indispensable stages of narrative inquiry on the way of producing narratives/stories "representing lived and told experiences of participants and researchers" (p. 545).

Table 12.1 Characteristics of three CoP of interest

	Research group	NK's class	AP's class
Goals	– To answer a set of predefined research questions and explore the emerging ones – Sub-goals: to create and maintain an on-line knowledge base of the project; to develop/choose mathematics problems to be used in the project; to develop/refine data-collection tools; to serve as a scene for discussions	– To meaningfully study curriculum-prescribed mathematics, with particular attention to engaging all students in collaborative problem solving – To get prepared for a matriculation exam consisting of challenging mathematics problems	– To meaningfully study curriculum-prescribed mathematics, with particular attention to individual practicing of problem-solving methods – To get prepared for a matriculation exam consisting of challenging mathematics problems
Practices	– Planning the project events – Discussing events and issues presented by the individual members – Producing documents (e.g., lesson plans, "to-do" lists, guidelines, presentations)	– Individual and small-group problem solving – Participating in teacher-orchestrated problem-solving discussions – Asking the teacher and answering her questions – Doing homework – Participating in online problem-solving activities	– Mostly individual and occasionally small-group problem solving – Listening to the teacher's explanations – Asking the teacher and answering his questions – Occasional doing homework
Participants and roles	Teacher-researchers (NK and AP), facilitators and researchers responsible for parts of the project (IK, YB, and OM), project coordinator and researcher (RL), principle investigator (BK)	NK, a math teacher; 17 students who had different preparation levels and different aspirations in mathematics study. All students were involved in the same practices in the classroom (see the previous row), about half of the class were active participants of the forums	AP, a math teacher; 25 students who had different preparation levels and different aspirations in mathematics study. All students were involved in the same practices (see the previous row)

(continued)

Table 12.1 (continued)

	Research group	NK's class	AP's class
Discourse	Whole-group discussion of issues among the group members; discussions of particular issues in sub-groups	– Teacher-centered discussions of individual and group problem-solving contributions – Discussions among the students	Teacher-centered discussions of individual problem-solving contributions
Inter-dependence	Mostly positive. Occasional tensions between individual and collective interests (e.g., which issues to discuss and resolve first and how)	Mostly positive (i.e., problem-solving success of a peer is praised). Occurrences of negative interdependence in a struggle for the NK attention and for the leadership among the peers	Mostly neutral. Occurrences of negative interdependence in a struggle for the AP attention
Power relationships	BK sometimes acts as a leader, and sometimes as a fellow member. Each group member leads the development of a particular aspect of the project. The decisions are attempted to be made by consensus. When disagreements emerge in relation to classroom issues (e.g., which problem to choose), the teachers' voices are decisive	NK acts as a leader and facilitator. She orchestrates the lessons. Most of the students accept the NK's lead and participate At the forums, NK acts mostly as an observer of the student problem-solving attempts and occasionally encourages participation	AP acts as a leader and facilitator. He orchestrates the lessons. Most of the students accept the AP's lead and participate

Practically speaking, we focused on the most vivid in our memories stories within the project that in our view might have opened the window into "behind the scene" events related to establishing and sustaining an online problem-solving forum and gradual changes in classroom problem-solving norms. The narrative about AP's class was discussed with him and refined.

12.4 Aggregated Characterization of Three CoP

Table 12.1 presents aggregated characterizations of the research group and of the NK and AP's classes as CoPs. The data underlying the table (see above) were collected during the first year of the project; the table is organized in a way that highlights similarities and differences between the CoPs.

The first row of Table 12.1 implies that the goals of three CoPs were related and compatible. This is because they all could have been reached if and only if the core activity of the project—student mathematical problem solving—would be sufficiently rich. Indeed, the richer the problem-solving episodes in the classrooms and on the forum were, the richer data set and the better answer to the research question could have been produced. Conversely, the more relevant factors the research group identified and took into account when planning the lessons and choosing the problems for the forum, the better opportunities the students had for developing their problem-solving skills and learning through problem solving.

Of note is that NK and AP played multiple roles in the project, by being the leaders of their classroom communities and simultaneously active members of the research group. In this capacity, they essentially shaped the group working agenda and practices. NK and AP were committed to their students' success and to the project's success, but in somewhat different ways. As a rule, AP thoroughly followed the devised by the research group lesson plans, and NK used to reconsider the agreed plans (see Sect. 12.5.3).

In addition, Table 12.1 suggests that NK's and AP's classes had some features of an idealized mathematics community that Bielaczyc and Collins (1999) referred to as Lampert's mathematics classroom. To recall, Lampert's lessons (Lampert, 1986, 1990) usually started with a problem posed to the students, on which they worked alone or in small groups. Then the class discussed different ideas and solutions. The role of the teacher was to orchestrate the discussion by choosing certain student ideas and revoicing them. This lesson scheme was realized in both classes. However, student work in small groups was more featured in NK's than in AP's class.

12.5 Three Narratives

In contrast to the aggregated summaries presented in the previous section, the narratives presented in this section highlight the dynamic and dialectic aspects of the project. As explained in Sect. 12.3, the narratives are about the development of the participating classes and of the research group while acting in interaction.

12.5.1 The First Narrative: Evolution of the NK Class

The main events at the knowledge phase of the project in NK's class consisted of: (1) a conversation between NK and BK following BK's observation of one of NK's lessons; (2) a conversation between NK and her students. The first conversation resulted in NK's decision to take part in the project. NK decided so mainly because the idea to stretch the boundaries of a classroom by means of an online forum

Nine-Square Problem: There is a net of nine congruent squares (see the drawing). Prove that the two angles denoted in the drawing are equal.

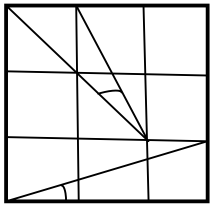

Fig. 12.1 The Nine-Square Problem

resonated well with her constant need to enrich her teaching repertoire and create new learning opportunities for her students.

In Rogers' terms, NK acted as a *venturesome innovator* who was able to cope with high degree of uncertainty about an innovation, and BK acted as a *change agent*. In her conversation with the students, NK acted as a *change agent*, and the students were potential *innovation-adopters* to be persuaded. NK argued that developing problem-solving skills was a strong benefit of participating in the project, and appealed to the students' curiosity to try something new and be a part of an interesting initiative. The students' reaction to the information about the project was favorable, though not exactly for the reasons that NK had presented. We know (from the interviews) that some students perceived the forum as an opportunity to improve their problem-solving skills, whereas some other students saw in the forum a way to overcome restrictions for the use of social networks imposed by their parents, and some other students – a way to get better prepared to the matriculation exam.

The first mathematical problem of the project, the Nine-Square Problem (see Fig. 12.1), was carefully chosen by the research group, including AP and NK.

Nine-Square Problem is representative of most of the problems of the project (see Appendix for additional examples). In particular, it had the following features: its solutions required theoretical knowledge studied in the NK and AP classrooms close to the day of opening the forums (namely, theorems on triangle similarity); it had several solutions, and it was of the type that Koichu, Berman and Moore (2006) tagged as ostensibly familiar problems. Namely, the problem looked similarly to problems the students were familiar with from classwork and homework. As such, the problem "invited" the students to approach it by means of mathematical ideas that worked well in the past. For example, the students might think of including the angles, which equality should be proved, into a pair of triangles and attempt proving their congruence by finding some equal elements. However, these ideas were insufficient in order to solve the problem; something else (e.g., some clever auxiliary constructions) should have been invented. Two auxiliary constructions representing two ways of solving the problem are presented in Fig. 12.2.

Three students worked on the Nine Square Problem when it was uploaded by IK to the Google+ forum. Their three-hour-long brainstorming session was unsuccessful. As a result, the forum was non-active during the next two days. This situation was discussed in a telephone conversation between NK and BK, and we decided that

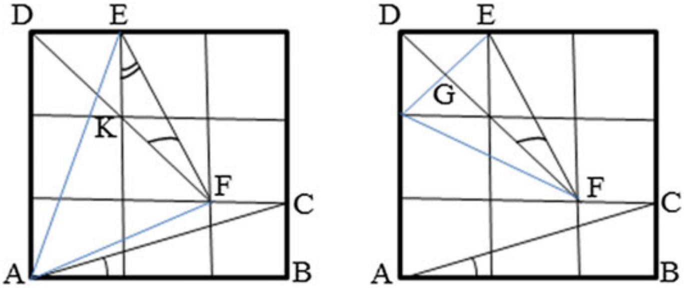

Fig. 12.2 Two auxiliary construction for solving the Nine-Square Problem

it would be appropriate to inquire the students about the reasons for the lack of participation in an informal setting. The day after that NK met the students in the schoolyard and asked: "why haven't you solved the problem? It is not that difficult". The students showed NK their hand-made drafts as evidence that they had tried. NK asked the students to upload their drafts to the forum and continue solving the problem together. In Rogers' terms, this conversation was one of the cues-to-action that eventually made the forum successful. That evening eight students entered the forum, cooperated and eventually solved the problem. From the forum:

Rachel: I am still stuck with computing angles. I made an auxiliary construction, an isosceles triangle…but from there again arrived at the angles equal 45 degrees.

Emuna: I have an idea. I think that it is possible to use the proportional segments that we studied at the last lesson. [I mean] the second theorem about triangle similarity.

Tehila: An isosceles triangle is good, and the second theorem is also good.

Rachel: But we should probably compute the angles.

Tehila: No, no, Rachel, focus on the sides and similarity!

Rachel: Wow! It seems me that I solved it. It is pity that we decided not to upload the solutions. Uff! I'll show tomorrow in the classroom. Or can I upload?

Rachel indeed solved the problem based on the ideas of Emuna and Tehila. It may be that Tehila has already solved the problem before Rachel's announcement, and that she attempted to help Rachel by providing a hint. By so doing, she acted in accordance with what had being agreed among the students before the forum began. Two solutions to the problem, by Tehila and an additional student, who was a silent observer of the forum, were presented at the mathematics lesson following the forum.

By the end of that lesson, NK collected the students' suggestions as to how to further run and improve the forum. As mentioned, less than half of NK's class took part in the forum at that stage, but the students who did not participate in the forum also took part in the conversation. Hence, the lesson became an event at which the early adopters of the innovation (eight students who had already tried the forum) shared their positive experience with additional potential adopters. Such events are characteristic of the persuasion stage of the innovation-decision process.

The students' suggestions included: to decide in advance at which times they could virtually meet at the forum in order to maximize simultaneous participation; to work at the forum in small group that would explore different ideas; to agree upon common mathematical notation from the beginning in order to make the communication easier. The first suggestion was later implemented, and the rest were not. Regardless of this, it is noteworthy that the students' suggestions reflected the existing in NK's mathematics lessons practices. That is, it seems us that the students suggested bringing to the forum what they had valued at their mathematics lessons.

Some of the NK's students were in position to appreciate the added value of the forum. From the interview with Emuna:

> For the students like me, it is difficult to express ourselves in a classroom, because there are students who understand the matter quicker [than us], and also because there are students who jump and answer every question immediately, regardless whether the answers are correct or not. When I am at the forum I feel in control. No one can interrupt my talk, and I have time to think before writing a post, and to post only when I am sure that I want to post. I can ask for help when I need it, and can just not look at the forum if I have my own idea to develop. In a classroom, when somebody talks – you hear it even if you don't want!

Three months later collaborative problem solving at the forum became a well-established practice in NK's class. The power relationship at the forum differed from those at the classroom. In the classroom, NK was an undisputed leader who orchestrated problem-solving discussions and served as a communication hub. In particular, she listened to the student suggestions, re-voiced some of them and re-addressed them to the class for further discussion. The above quote from the interview with Emuna confirms our impression from the videotaped lessons: as a rule, problem-solving discussions in the NK class were rather vociferous, and that the students frequently interrupted each other in the struggle for the NK attention.

The role of a leader was assumed by different students and sometimes by no one at the forum. Another difference was related to diversity of types of "expertise" among the students. The presented excerpt from the Nine-Square Problem forum is illustrative about these observations. In addition, it is illustrative of an additional phenomenon: whereas in the classroom all students were as a rule engaged in the same practices and were expected to master all material taught, the forum participants in time developed problem-solving specializations, such as a proposer of new ideas, a responder, a person who summarizes the proposed ideas, etc.

We also observed the movement of practices from the forum to the lesson at this stage. The following episode illustrates this phenomenon. NK planned to begin a regular (i.e., not planned in collaboration with the research group) geometry lesson by a brief discussion of a theorem about an angle between a chord and a tangent line (see Fig. 12.3).

She wrote on the whiteboard the theorem in "given – to be proved" form and asked the class if anyone had ideas as to how to prove it. She expected that the students would think on the question in silence for some time, and then she would orchestrate the discussion by revoicing and extending selected ideas asserted by the students in a whole-class discussion, as usual. Something unexpected to NK happened. Michal, Theorem: If a chord and a tangent line to a circle intersect in a point on the circle, the angle between the chord and the tangent line equals to the inscribed angle defined by the chord and a vertices lying on the other side of the circle.

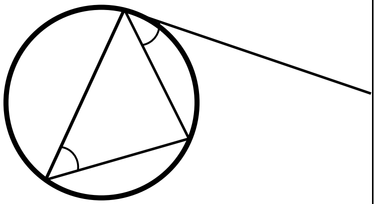

Fig. 12.3 A theorem about an angle between a chord and a tangent line

one of the most active forum participants, suggested: “Let’s solve this together, like we do at the forum”. NK moved aside and observed.

The class reorganized itself. Some of the students rearranged the tables in order to sit together in small groups, some others preferred to sit alone. The students divided the whiteboard into parts, and used it as if they wrote posts at the forum. Namely, every several minutes one of the students approached the board and wrote something. There were “posts” entitled “a new attempt” and “theorems that can be used in the proof”, there was also a “post” representing some proving plan, a post representing an unsuccessful attempt and more. There were also comments on the existing “posts”. For example, one student focused on reading the content of the whiteboard instead of inventing her own solution. She commented on part of one of the “posts”: “This point is not justified”. The activity continued autonomously (i.e., without any NK’s interventions) for about half an hour. It took much more time than NK planned, but as a result, the students independently found several different proofs of the theorem.

During the first year of participation in the project, the Google+ forum of NK’s class consisted of nine (instead of the planned three) problem-solving episodes of 100–300 posts each. Additional six problems were chosen by NK and discussed by the research group only post factum, as materials for the use in additional classes. More than 95% of the interactions were among the students; the role of NK was restricted to offering the students challenging problems and making encouraging remarks. The forum functioned as a platform for collaborative problem solving for three years until the class graduated in 2016. That is, at some point the forum stopped being an innovation and became a confirmed routine.

Today NK runs online problem-solving forums in her new classes and constantly refines the rules of the game. For instance, those students who are interested to early share their solutions and get feedback can now do so on the separate forum. The majority of the students learned to appreciate and enjoy the long-term collaborative work on challenging problems and do not hurry to enter that forum.

In summary, we deem that the forum in the NK class became successful because of a combination of factors and decisions, as follows:

- the idea of the forum addressed the existing in the classroom community learning needs, for example, to get prepared to a matriculation exam consisting of challenging problems;

- the teacher and some of her students were open to new experiences and can be characterized, in Rogers' terms, as venturesome;
- there were productive interactions between the early and the late adopters of the innovation;
- the existing in the classroom norms related to collaborative problem solving were supportive for initiating the forum, and the developed in the forum practices of sharing half-baked ideas and freely switching the modes of interactions were reinforced in regular mathematics lessons;
- cue-to-action events, some of which were planned and the others were unforeseen, occurred at the right time;
- the students were involved (along with NK and the research group) in establishing the rules of the game;
- the rules of the game were constantly adapted to the students' learning needs and choices.

12.5.2 The Second Narrative: Evolution of the AP Class

The beginning of the project in AP's class appeared to be different, even though the main events of the knowledge stage (i.e., a conversation between AP and BK following an observation of one of the AP lessons and a conversation between AP and his students in the classroom) occurred in the same manner as in the NK class. Moreover, the students' initial reaction to AP's invitation to take part in an online problem-solving forum was similar to that of the NK students—it was unequivocally favorable. In practice, however, the forum did not become active when the first and then the second problem had been offered. There were students who occasionally entered the forum, did not find there any activity, and after uploading one or two posts lost the interest.

Like NK, AP discussed the situation at the research group (see Sect. 12.5.3 for details) and talked with the students for several times. The students expressed their understanding of the importance of the forum and their intension to participate, but the forum did not revive. It seems that the AP students exercised what Rogers called passive rejection of an innovation. Interestingly, what worked well at the persuasion stage in the NK class did not work at all in the AP class. For example, only few of the AP's students persisted when solving the regular homework, so NK's pivotal request "upload your drafts to the forum and work together" (see Sect. 12.5.1) could not be implemented because the students just did not have drafts to share.

We discussed the situation in two meetings of the research group. During the first meeting, we talked about pros and cons of different technological platforms and considered if it was worthwhile to switch from one platform to another. We discussed whether it was worthwhile to give the students formal credit for participation in the forum (e.g., some bonus grades) or make the participation compulsory. An additional suggestion was to make the forum multi-functional and place on it materials for preparation for the tests and homework assignments so that the students would be accustomed to systematically enter the forum. One more suggestion was to revive the forum by simulating it at one of the mathematics lessons (see Sect. 12.5.1 for an example of such a lesson).

The turning point occurred at the next meeting. AP took the lead. He analyzed the situation acting in two roles, a teacher and a researcher. The discussion revealed that for different reasons some of the above suggestions could not be implemented, and some other could only give a fainted ray of hope for improvement. For AP and the rest of the group the conversation became a productive boundary interaction: circumstances and considerations of one CoP (i.e., AP's class) were confronted with circumstances and considerations of another CoP (i.e., the research group). AP made the conclusion that so far the discussion focused on the extraneous reasons for the lack of activity at the forum. From the protocol of the meeting:

AP: Today I think that the problem is in some classroom norms that hinder the forum.

RL: Indeed, the forum apparently reflects some processes in the classroom.

AP: Nothing happens in the forum, and it is a sort of reflection of what's going on in the class. This was unforeseen… I thought that my lessons provided a good foundation for the forum. Of course, there are extraneous reasons, but all together [extraneous reasons and some classroom norms] hinder the forum…

BK: I observed your lessons and saw that you offered the students quite difficult problems and that the students participated and presented ideas. And you as a teacher decided which ideas to use. As a result, the responsibility for the final solutions was yours. This is probably because it was not enough time to think about the problems. It also seemed me that the students were struggling for your attention and that doing homework for some of them was unimportant.

RL: But it is up to you [AP] to decide whether you want to change the existing norms.

BK: I know from personal experience that this [changing the norms] can be a very complicated process.

AP: I see the forum as an addition to our regular work. I am not ready to devote the precious lesson time in order to make the students work on the forum. It is not the end in itself for me.

Eventually, the forum did not worked as planned for AP's students. However, two months after the described meetings, AP shared with the research group good news about positive changes in his lessons' norms and dynamics. In particular, AP told the research group that more students wished to get help from the fellow students rather than from him, and that more students tried to assist their peers when solving problems.

BK: In your opinion, why did the change occur?

AP: I changed my behavior at the lesson. I forced myself not to respond immediately to the students' requests for help, but offered them to keep thinking by themselves or ask for the assistance from the other students. I had forced myself to talk less at the lessons, and the changes began after about two lessons.

The story had a continuation when an additional teacher from AP's school, ES, joined the project in 2014–2015. ES found himself in a situation similar to AP's situation at the beginning of the project. This time AP assumed the role of a facilitator and a researcher who worked in pair with ES. AP's advice took into account his own experience as well as NK's experience. However, attempts to enhance online collaborative problem solving as an addition to regular mathematics lessons deserved little response also in the ES class,[4] though some classroom norms eventually changed.

In summary, we attribute the lack of response to the idea of stretching the boundaries of a classroom by means of online forums in the AP and ES classes to the following factors:

- teacher-student interactions were more valued than student-student interactions at the AP and ES regular lessons;
- AP was ready to experiment with a new idea, but did not feel the need to deeply change his teaching practices;
- the norms of doing homework were underdeveloped in AP and ES's classes;
- we (the research group) did not succeed in designing effective cue-to-action events, despite of considerable effort made;
- teachers of the AP and ES's school delivered to their students a clear message that any learning activity must be graded. To recall, participation in the forum in the NK classes was voluntarily for the students and as such, it was detached from any formal assessment, and we decided (perhaps, mistakenly) that online forum would be a free-of-assessment zone also in the AP and ES classes.

We learn from the above story that boundary interactions between a classroom CoP and a research group CoP can result in productive learning opportunities for both communities. The story shows that something has changed in the AP classroom, namely, AP's ways of orchestrating problem-solving activities. Something has changed in the research group as well. Namely, we learned that the need to take into account specific circumstances of a classroom community could override the need to follow what initially seemed to be a feasible plan.

12.5.3 The Third Narrative: Evolution of the Research Group

Prior to running the first problem of the project in NK's and AP' forums, the research group conducted a series of preparatory meetings. The meetings were devoted to the following themes:

- developing a shared vocabulary (e.g., we discussed the research proposal and some research papers about mathematical problem solving in online learning environments);

[4]Let us mention here that another problem-solving-related idea worked very well in AP and ES classes. In brief, these teachers successfully engaged their students in long-term extracurricular mathematics research in the context of numerical sequences. This enterprise, which lasts for five consecutive years in their school, is presented elsewhere (Palatnik, 2016; Palatnik & Koichu, 2015, 2017).

- organizational aspects (e.g., creating an online knowledge base of the project, creating teacher-researcher pairs and defining the fields of individual responsibility within the project; collecting the informed consents from the parents of the students);
- research aspects (e.g., refining the data-collection procedures described in the proposal);
- technological aspects (e.g., choosing a technological platform for each class in accordance with the class characteristics);
- mathematical aspects (e.g., which geometry problems to choose and why);
- pedagogical aspects (e.g., how problem solving actually occurs in the participating classes, how to introduce the project to the students and how to run the forum).

These meetings were productive in many respects, and in particular they seemed to promote the atmosphere of partnership and positive interdependence among the members of the group. For instance, at the beginning NK was mostly interested in mathematical, technological and pedagogical aspects of the project, but in a short while she became interested also in its research aspects. BK was initially focusing on the research aspects and considered the rest of the issues as necessary tasks that needed to be done before "the real work begins". In a short while, he became intellectually and emotionally involved in all aspects of the project.

As it frequently happens, the crisis was unforeseen. This is what happened. The group invested much time and effort in developing a detailed plan of how to prepare the classes to approaching the first problem on the forum (the Nine-Square Problem, see Fig. 12.1). The lesson preceding the forum was thoroughly designed with a special attention to its mathematical and heuristic aspects. Specifically, a sequence of three challenging tasks was devised. Each task was carefully chosen so that its method of solution could serve as an indirect hint for the students when solving the target problem. Two days prior to the lesson and the forum, NK and AP had a long meeting at which they finalized and coordinated their actions in the classrooms.

AP fully realized the plan. NK drastically changed it at the last moment. From her diary written the night before the lesson:

> NK: I am very excited and concerned. For me, the lesson tomorrow has two goals: (1) the lesson must look as a regular one for the students. They should be busy enough and do not pay attention to a video camera and a man behind it. They should not feel that something unique happens; (2) still, the lesson should promote the emerging in the class wish to solve challenging problems, to solve them together, and to solve them even out of school.
>
> I've just arrived at the idea as to how to get the students to feel the difference [between different problem-solving modes] at the lesson. I plan to conduct a lesson consisting of three parts […] I think that I'll use this model for many times in the future.

Practically, NK prepared a new working sheet consisting of seven problems. The first two were relatively easy and preceded by the request "Think and answer". This part was for individual work. Problems 3–5 were more difficult and preceded in the worksheet by the title "Two is better than one! Think and answer in pairs". The last two problems were slightly more difficult. This part of the working sheet was entitled “The more – the better. Think and answer in small groups.” The Nine Square Problem was placed at the end of the working sheet. It was entitled “Together we’ll overcome any difficulty! This is a challenging problem to be solved at the forum”.

To summarize, the main difference between the agreed plan and the plan enacted by NK was as follows: the agreed plan allowed the teachers to discuss the mathematical ideas that later might be useful on the forum. NK’s plan was less related to the forum problem mathematically or heuristically, but allowed her to highlight at the lesson the advantages of collaborative problem solving when the problems were challenging to the students.

Two previous subsections has described how the story was developed in the participating classes. We now present the continuation of the story from the research group’s perspective. At the next meeting of the group, NK explained that the night before the lesson she suddenly realized that the agreed plan would not work in her class. She was not particularly explicit as to what exactly might have gone wrong in the agreed plan. Anyway, NK’s success was apparent, and the group turned to analyzing the NK’s forum and the subsequent tasks. An immediate organizational conclusion was “from now on, let’s leave devising the detailed lesson plans to the teachers. We should trust their intuitions even if they are unspoken in the group meetings”.

The reasons of the NK’s decision to change the agreed lesson plan became clear much later, in the third year of the project. In one of BK and NK’s conversations, NK deeply reflected on her personal experience of being a member of the research group at the beginning of the project. She explained that in spite of her active participation in all the discussions, she wished to change the agenda of the group to some practical issues. In particular, she repeatedly asked the question “Why should the students cooperate with us and solve problems together at the forum?” In her view, the question was not properly discussed by the group. NK felt that she was unable to change the group working agenda in the direction that was important to her. Specifically, convincing the students in benefits of collaborative problem solving was more important to NK than preparing the class mathematically-heuristically for solving a particular problem, which was the central idea of the lesson plan devised by the group. NK agreed with the plan because she did not have an alternative idea when the plan was suggested. As mentioned, an alternative idea came to NK at the last moment. Being truly committed to the project’s success as well as to her students’ success, she decided the night before the lesson that she should do what she believed was the best for her students, even on expanse of violating the agreed plan without noticing AP and the research group.

In summary, we deem that the presented story enables us to make several points. First, the story illustrates the Wenger’s (2010) note that boundary interactions between CoPs are “not necessarily peaceful or collaborative” (p. 183). We know now that this is true even when the interactions are smooth on the surface. It further substantiates the importance of the existence of a shared history of learning for making the boundary interactions productive (Wenger, 2010) and the importance of diversification of the modes of boundary interactions (Solomon et al., 2017). Second, the story illustrates that conflicting boundary interactions can reveal norms within the

CoPs that otherwise would remain unnoticed. Furthermore, the conflicting boundary interactions may serve as an impetus for gradual changing the norms. For instance, the story revealed to us that there has been the following norm in the research group: "all decisions should be made collaboratively based on explicit discussions of their pros and cons". This, seemingly productive norm, has been gradually complemented by the following one: "some decisions should be trusted to the specialists in the community, even if they (the decisions) are not discussed in detail by the group". In this way, the group members have more space for individual experimentation within the project. Speaking practically, we learned not to overestimate the detailed discussions of the emerging issues and do not think of our decisions as "final" or "optimal". We learned instead to make some decisions and move on in the hope to get smarter along with the further development of the shared history of mutual learning.

12.6 Concluding Remarks

The idea to use online discussion forums and social networks as means for enhancing collaborative mathematical problem solving has recently become widespread throughout the world. There are at least two reasons for this. First, it looks just right to capitalize upon the fact that online activity is an inalienable part of life of nowadays schoolchildren (Boyd, 2008). Second, very positive cases of collaborative online problem solving has been documented in several specially designed learning environments, such as Knowledge Forum (Moss & Beatty, 2006), Virtual Math Teams (Stahl, 2009) or threaded discussion asynchronous forums (Wentworth, 2009). Still, the promise of online communication technologies for enhancing mathematical problem solving in school is far from being fulfilled, and the gap between the scope of self-organized non-mathematical activity in social networks and the scope of mathematical activity in specially organized online learning environment is enormous.

Based on the premise that special effort should be made in order to incorporate technology-mediated communication into mathematics education, the present study enquired what could enhance or impede the development of online problem-solving forums in school practice when teachers and researchers collaborate in order to make the forums work. Of note is that our focus differs from the focus of many past studies, which have shed light on various affordances of online mathematical forums (e.g., Lachmy et al., 2012; Lazakidou & Retalis, 2010; Lin, 2011; Nason & Woodruff, 2003; Tarja-Ritta & Järvelä, 2005), but were rather silent about how to initiate and sustain them in conjunction with regular mathematics lessons.

The presented stories of two high-school classes and their long-term interactions with a research group have different happy ends: in one of the classroom communities, online problem solving has eventually become a routine practice and a valuable addition to classroom problem solving. In another classroom community, the forum did not become active despite considerable effort made, but enduring attempts to activate it led to enhancement of student-student interactions in the classroom.

In brief (see the detailed summaries in Sect. 12.5), some of the factors that enhanced or impeded the incorporation of the forums were related to school conditions, classroom norms, particular cue-to-action events, and our decisions and actions as a research group. We have also observed, more than once, the phenomena of discontinuance and of passive rejection for which we do not have convincing explanations, despite the extended data set in our possession.

One of the lessons learned from the present study is that the Community of Practice perspective (Wenger, 2010) and theories like the Theory of Diffusion of Innovation (Rogers, 2003) should be taken seriously. Either aggregative or detailed analysis of implementation of the project idea is helpful for us as a tool for refining the roadmap of the project, and we hope that it would be instrumental also in organizing and running similar projects elsewhere. In addition, we learned that creating conditions for implementation of an innovative pedagogical idea in a school reality should be given full attention prior to delving into a pursuit for "traditional" research questions, such as questions on cognition and affect in mathematical problem solving. Based on the accumulated experience, we call for reporting and analyzing not only those cases where problem solving was sufficiently rich, but also those cases where the designed activities did not work as planned. We conclude by suggesting that systematic attention to the latter cases may have not only practical, but also theoretical significance in mathematics education and enrich the existing models of mathematical problem solving in realistic educational contexts.

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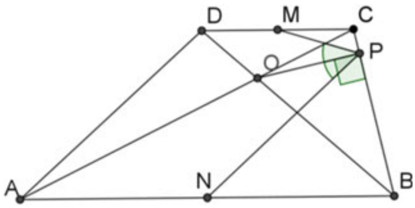
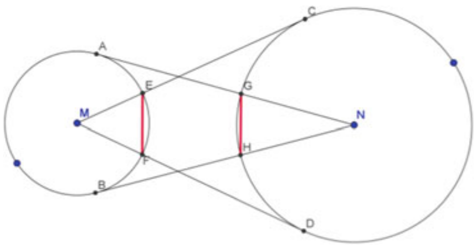
Appendix: Examples of Problems Used in the Project

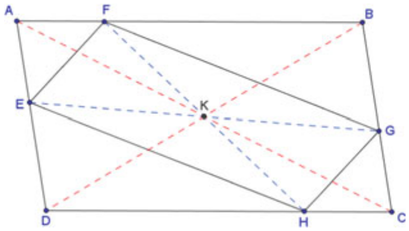
Trapezoid Problem (from Fraivert, 2016): Let ABCD be a trapezoid (see the drawing). M and N are the midpoints of AB and CD respectively, O is an intersection of the diagonals AC and BD, and OP is perpendicular to BC. Prove that OP is an angle bisector of the angle MPN.



Two Circles Problem (translated from Sharygin & Gordin, 2001, No. 3463): Two circles with centers M and N are given. Tangent lines are drawn from the center of each circle to another circle. The points of intersection of the tangent lines with the circles define two chords: EF and GH (see the drawing). Prove that EF = GH.



Nested Parallelograms Problem (translated from Sharygin & Gordin, 2001, No. 565,566): Given is a quadrilateral inscribed in a parallelogram. Prove that the inscribed quadrilateral is a parallelogram if and only if the intersection point of its diagonals coincides with the intersection point of the diagonals of the external parallelogram.



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Chapter 13
Conditions for Supporting Problem Solving: Vertical Non-permanent Surfaces

Peter Liljedahl

13.1 Motivation

My work on the research presented in this chapter began over 10 years ago when I was invited to help June, a grade 8 (12–13 year olds) teacher, implement problem solving in her classroom. At the time, problem solving was becoming more and more common in classrooms and had become an explicit part of the recently revised curriculum. June had never done problem solving with her students before, but with the shifting landscape around this idea she felt it was time. June was aware of my interest in problem solving as well as my willingness to help teachers to start implementing problem solving in their classrooms, so she reached out to me one day late into the school year.

June, as it turned out, was neither interested in co-planning nor co-teaching. What she wanted from me was simply a collection of problems she could try with her students. I was expecting to have a greater level of involvement in the lesson, but June was firm on her conditions. We eventually arrived at a compromise whereby I would supply the appropriate problems for June to use with her grade 8 students, and June would let me watch her implement them within her classroom.

The first problem I gave her to use was a problem that I had had much success with students of many different grades.

> If 6 cats can kill 6 rats in 6 minutes, how many will be needed to kill 100 rats in 50 minutes? (Lewis Carroll, cited in Wakeling, 1995, p. 34)

June accepted this problem in good faith and used it the next day. It did not go well. A mass of hands immediately shot up and June began quickly moving about the room to answer questions and provide help. Many students gave up almost as soon as a problem was presented, so June also spent much effort trying to motivate these students to try. In general, there was some work attempted when June was

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close by and encouraging the students, but as soon as she left the trying stopped. This continued for the whole 40 min period.

The following day I was back with a new problem. The results were as abysmal as they had been on the first day. The same was true of the third day. Over the course of three 40 min classes we had seen little improvement in the students' efforts to solve the problem, and no improvements in their abilities to do so. So, June decided it was time to give up. Her efforts to bring problem solving to her students had been met with resistance and challenge and resulted in few, if any, rewards.

I wanted to understand why the results had been so poor, so I asked June if I could stay and observe her and her students in their normal classroom routines. After three days of observing these routines I began to discern a pattern. That the students were lacking in effort was immediately obvious, but what took time to manifest was the realization that what was missing in this classroom was that the students were not thinking. More problematic was that June's teaching was predicated on an assumption that the students either could not, or would not, think. The classroom norms (Cobb, Wood, & Yackel, 1991; Yackel & Rasmussen, 2002) that had been established in June's class had resulted in, what I now refer to as, a non-thinking classroom. Once I realized this I proceeded to visit other mathematics classes—first in the same school and then in other schools. In each class I saw the same phenomenon—an assumption, implicit in the teaching, that the students either could not or would not think.

I wanted to better understand this phenomenon of non-thinking and then find ways to change it, to break the pattern of these non-thinking classrooms and build, in their place, thinking classrooms. In what follows I present three distinct research studies that are a small part of my journey in pursuit of these goals.

13.2 Study #1: Student Behavior

In order to better understand my experiences in June's class I conducted a series of studies, in collaboration with one of my doctoral students, in which we looked closely at student behaviors across a variety of more traditional mathematics classrooms and classroom activity settings—doing tasks in class, taking notes, homework, group work, review, lecture (Liljedahl & Allan, 2013a, b).

Research into these different activity settings began with the use of classroom videos, field notes, and post observation interviews with students. Using a constant comparative method (Charmaz, 2006; Glaeser & Strauss, 1967; Patton, 2002; Taylor & Borden, 1984) these data were continually analyzed between observations. From this analysis, over time, a number of interesting student behaviors began to emerge within each of these aforementioned classroom activity settings. As clarity was gained, coding for these, now known, student behaviors in subsequent observations becomes easier. Over time a form of saturation was reached as new observations of these activity settings no longer revealed new behaviors. When this occurred a taxonomy of student behavior within a certain activity setting had been achieved. This taxonomy then allowed us to analyze all of the student behaviors during an activity setting within one class. In what follows I look specifically at the taxonomy of student behavior around, what we came to call, the *now you try one* activity setting (Liljedahl & Allan, 2013a).

The *now you try one* activity setting is most often seen in classrooms wherein the dominant teaching method is direct instruction and is named for the oft use of *now you try one* tasks. These are the short tasks that teachers ask students to do immediately after s/he has done some direct instruction and presented some worked examples on a specific subtopic of the curriculum (e.g. how to factor a difference of squares, how to multiply two digit numbers, etc.). Although some teachers use these tasks for the purpose of students being able to practice the new skill, we found that the majority of teachers use *now you try one* tasks for the purpose of students being able to check their understanding of the new skill.

The cycle of direct instruction followed by a *now you try* one task is often repeated several times as the teacher moves through their planned lesson. As such, students work on them where they are sitting, and with whom they are sitting, during the direct instruction part of the lesson. Students are given 3–5 min to solve the task while the teacher moves around the room answering questions or stands at the front monitoring when student finish the question. When the time has expired the teacher goes over the solution, sometimes by calling for student input, before moving onto more direct instruction.

13.2.1 Methodology

Data for what I present here comes from a single lesson on completing the square as a way to graph quadratic functions being taught in a grade 11 classroom (n = 32). Because saturation had already been achieved and codes were already well established no video was used. Instead, we simply used the pre-established codes to annotate observed student behaviour on a supplied seating chart of the classroom during the *now you try one* phase of the lesson. Immediately after these observations, while students began to work on their assigned homework, as well as for a few minutes after class, we collected very brief interview data from a number of students selected based on their observed and different behaviours. The interviews were short (1–4 min) and were audio recorded using a portable digital recorder. For the most part these interviews consisted of a brief declaration of what we had observed them doing and one or two questions regarding their reasons for their behaviour. This was not foreign to the students as I had previously spent several lessons doing similar research in the same class; although not always in the context of *now you try one* problems. In all, data from 15 interviews was collected. Added to this were lengthier interviews with the teacher before and after the lesson in order to ascertain her goals for the lesson in general, and the *now you try one* problems in particular. In the post interview we shared with her some of the behaviours we had observed as well as some of the responses the students had given during our brief interviews and asked her to respond to these vis-à-vis her own goals.

These data were then analyzed through the lens of the aforementioned taxonomy of student behaviours that had emerged from the earlier work on the *now you try one* activity setting.

13.2.2 Results

From the analysis of these data our previously established taxonomy of five main student behaviors was confirmed. In what follows we present each of these student behaviors exemplified with excerpts from these data.

Amotivation
Of the 32 students observed for this study three (all boys) displayed a general lack of attention towards the lesson. They were generally disengaged and disinterested in the lesson. Visibly they paid little attention, took no notes, and when they were asked to try to solve an example on their own they made no attempt to do so, or to seek help. When asked about their lack of interest they each gave a different explanation.

Frank I don't get it. [shrugging his shoulders and looking back down at his cell phone]
Andrew My tutor will help me with this tonight.
Jason I'm just tired today.

When we shared these comments with the teacher after the class she replied that she was not surprised.

Teacher Frank and Andrew are never engaged. They're often absent or late and when they are here they don't do much. Andrew has a tutor and uses that as an excuse to not do anything in here … but he is still failing the course. Jason is always here but he isn't doing any better.

Ryan and Deci (2002) would refer to these students as amotivated. Amotivation is a deeper problem that goes well beyond the context that we were focused on. As such, we initially considered not including these cases in the taxonomy. However, we decided against this for two reasons—this behavior was seen in almost every class and its inclusion allows us to account for all of the behaviors seen during the *now you try one* context.

Stalling
Four students exhibited a behavior that we came to call stalling. Stalling behavior are actions that can be seen as legitimate—that are not out of place in a normal classroom or during the course of a lesson. What made these actions interesting to us was their timing. As soon as the students were asked to do a question on their own two students suddenly had to go to the bathroom, one needed to sharpen their pencil, and one couldn't find a calculator (even though the question didn't require one). When we asked the students about these coincidences they had a variety of superficial reasons justifying their actions:

Jessa I had to go. That's all.
Barry I was waiting until there was a break in the lesson.
Jenny My pencil broke.
Drew Calculators are allowed so I wanted to use one.

When pushed about these reasons, however, two things emerged that were common to each of these four students. First, all of them expressed that the *now you try one* was an unimportant part of the lesson; "like a break". The reason for this, they all revealed, was "because in a few minutes the teacher [was] going to provide the answer". Taken together, these students were seeing a redundancy between their efforts to solve the task (had they done so) and the teacher presented solutions. This redundancy exists only within a context where the purpose of the *now you try one* problem is the production of notes.

Faking
There is one final category of non-trying behavior—faking. Two students exhibited this behavior. These girls had two things in common—they had impeccable notes and from the front of the classroom they both appeared to be trying to solve the problem. It was only from our vantage point in the back (and side) of the classroom that we were able to detect what was really going on. Physically all of their actions were those of students who were working. Their heads were down and their pencils were moving. In reality, however, neither of them was actually writing anything on their paper, even though one of them even made the pretense of erasing a mistake. When asked about this they both gave the same general answer,

Keesha I don't want to mess up my notes.

When pushed on this point they both came back with the same answer that the stallers did—that the teacher will soon provide the solution. However, they added to this a nuance that the stallers did not mention, and perhaps did not care about.

Jennifer Not only will she give us the answer, she will give us the best answer. This is the one I want in my notes.

The importance of the best answer, as opposed to just a correct answer, is important when the goal is to produce perfect notes, a goal that both of these girls clearly shared.

Mimicking
The nine aforementioned students aside, the remaining 23 students all tried, at least in part, to solve the *now you try one* problem. Of these, 17 were mimicking. Visibly these students engaged in the task and tried to solve it. Some made mistakes, some gave up, but most succeeded in arriving at the correct answer. Successful or not, what these students all had in common was that they referred to their notes, or the notes on the board, often. Closer observation and our questioning revealed that the students in this category were not so much relying on understanding as much as simply following the solution pattern laid down by the teacher in the example that she had worked through immediately prior to the *now you try one* problem.

The constant referencing to the previously solved problem was symptomatic of the students' attempts to map characteristics of the example problem onto the current task. When asked about this mimicry behavior these students claimed that they were doing what the teacher wanted them to do.

John This is how we do things in this class. The teacher gives us an example and we write it down. Then she gives us one to try and we copy what we did in the example.
Charlotte Isn't this what we are supposed to be doing?

When we asked the students who had failed to get an answer about what happened their general response was that the *now you try one* question "must have been" different from the example question.

Samantha I got lost somehow. I'm not sure where. I thought I was following the rules.

For Samantha, like the rest of the students in this category, the "rules" is a solution pattern to be copied.

Reasoning
The remaining six students demonstrated a behavior of reasoning. These students not only attempted the problem but progressed through it in a reasoned and reasonable manner with minimal references to prior examples. This is not to say that the prior examples did not play a role in their solutions, but as a whole rather than the line by line copying that the mimics performed. Further observation of this group of students, as they tackled additional problems, confirmed that they had a good understanding of the mathematical relationships and skills at play. Given this, we asked these students if the teacher's examples had in any way contributed to their understanding of the *now you try one* problem. For the most part the students indicated that what the teacher's examples gave them was a new combination of things that they already knew.

Kenneth I don't know. Maybe. … I mean it all makes sense. If anything, maybe the examples just showed me what kinds of questions are possible.

That is, although they seemed to know all of the pieces they had never thought to combine their knowledge in this way.

The one exception to this was Ryan, who on several occasions (during the lesson that was observed for this study as well as others) anticipated the teacher's next example or next question. That is, unlike the others in this category, Ryan was able to combine his knowledge without being shown how to do this.

Table 13.1 Distribution of student behaviors

Behavior	n	%
Amotivation	3	9
Reasoning	6	19
Stalling	4	13
Faking	2	6
Mimicking	17	53

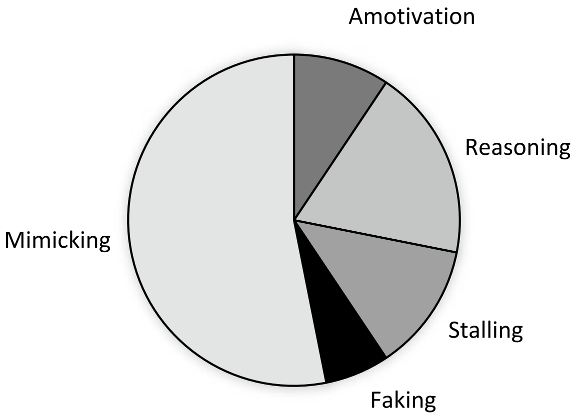

Fig. 13.1 Distribution of student behaviors

13.2.3 Discussion

Taken together, the distribution of the five different student behaviors during a *now you try one* context within the aforementioned class shows a disturbing trend (see Table 13.1 and Fig. 13.1).

Having spent time in this particular class before we knew that this teacher made extensive use of the *now you try one* problems. As such, prior to our observation we asked the teacher to explain to us what her intentions were with the tasks and what she expected the students to do with them.

Teacher Well, I use them to give the students a chance to check their understanding of what we had just learned. This way, if they don't understand something we can catch it right away.

Researcher And what do the students do with these problems?

Teacher For the most part they do the problems. You'll see when we are in there that there are a couple of boys in the back that don't do them but they don't really do anything. Everyone else, though, does them.

The teacher's expectation is that the students will do these problems as a way to test their understanding and she believes that, for the most part, this is what they do. In the post lesson interview she confirmed her expectation.

Teacher So, as predicted, those three boys in the back didn't do much. But everyone else was pretty much on task. I mean, they didn't all get the problems right, but they did them. And the ones that made mistakes had a chance to learn from their mistakes when we went over it.

The data does not agree with either her pre-lesson prediction or her post-lesson reflection. Of the 29 students in the class that the teacher thought were acting in alignment with her goals, only six actually were. The other 23 students were stalling, faking, or mimicking. Their actions were not actually what the teacher thought they were. That is, 23 out of 29 (79%) students were subverting the intentions of the teacher, and doing so in ways that the teacher was not aware of. It could be argued that those students who were mimicking understanding by mapping the solution process from one problem to another were exhibiting expected behavior, but keep in mind the words of John and Samantha. From the perspective of the students, they were not trying to test their understanding. They were *copying* and *following the rules*—neither of which is what the teacher intended.

These findings are consistent with our research in other contexts as well. Across the board students are finding ways to game the expectations of the teacher in ways that the teacher is not aware of. In many cases these behaviors are centered on proxies for learning and understanding, such as mimicking, that are not actually conducive to learning—but appear to be in alignment with the teacher's goals.

This behavior is consistent with what Fenstermacher (1986, 1994) has come to call *studenting*. Initially he used this term to describe the things that students do to help themselves learn; from paying attention to following instructions, from practicing to studying, from reviewing to seeking help, from trying to understand to ensuring they understand, etc. Later, however, he expanded this definition to also include the other things that students do while in learning situations—things that do not actually help them to learn.

> …things that students do such as 'psyching out' teachers, figuring out how to get certain grades, 'beating the system', dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments, threading the right line between curricular and extra-curricular activities, and determining what is likely to be on the test and what is not. (p. 1)

Taken together, the notion of studenting can be used to describe our results and helps us understand what students do while in a learning situation and expands our ability to talk about student behavior in classroom activity settings. More specifically, it gives us a name for the autonomous actions of students that may or may not be in alignment with the goals of the teacher. As such, studenting extends constructs such as the didactic contract (Brousseau, 1997) and classroom norms (Cobb et al., 1991; Yackel & Cobb, 1996) to encompass a broader spectrum of classroom behaviors—behaviors that are not predicated on an assumption of intended learning.

13.3 In Pursuit of Thinking Classrooms

June made extensive use of *now you try one* in her teaching and, not surprisingly, her students exhibited many of the studenting behaviors seen in the *now you try one* research. Under such norms it was unreasonable to expect that June's students were going to be able to spontaneously begin to engage in problem solving. What was missing for these students was much more than an exposure to problem solving activities. What was missing was a central focus in mathematics on thinking. The realization that thinking was absent motivated me to find a way to build, within these same classrooms, a culture of thinking, both for the student and the teachers. I wanted to build, what I now call, a *thinking classroom*—"a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion" (Liljedahl, 2016, p. 364).

My efforts to build and sustain such thinking classrooms has been an ongoing pursuit for over ten years. My initial effort in this regard was to do an inventory of classroom norms and practices. To do this I explored the practices of more than 40 classroom mathematics teachers. From this emerged an inventory of 11 discrete variables that permeate mathematics classroom practice everywhere, from primary to secondary, and can be used to inventory differences between mathematics classroom practices (Liljedahl, 2016). These variables are presented here as a set of questions.

1. What type of tasks are used, and when and how they are used?
2. How are these tasks given to students?
3. Do students work on tasks in groups and, if so, how are groups formed?
4. Where, and on what surfaces, do students work on tasks?
5. How the room is organized, both in general and when students work on tasks?
6. How questions are answered when students are working on tasks?
7. How are hints and extensions used while students are working on tasks?
8. How much autonomy do students have while working on tasks?
9. When and how does the teacher levels[1] their classroom during or after tasks?
10. When and how do students record notes?
11. When and how is assessment carried out, both in general and when students work on tasks?

In June's non-thinking class, for example:

1. Practice tasks were given after she had presented a number of worked examples (*now you try one*).
2. Students either copied these from the textbook or from a question written on the board.

[1]Levelling (Schoenfeld, 1985) is a term given to the act of closing of, or interrupting, students' work on tasks for the purposes of bringing the whole of the class up to a certain level of understanding of that task. It is most commonly seen when a teacher ends students work on a task by showing how to solve the task.

3. Students had the option to self-group to work on the homework assignment when the lesson portion of the class was done.
4. Students worked at their desks writing in their notebooks.
5. Students sat in rows with the students' desk facing the board at the front of the classroom.
6. Students who struggled were helped individually through the solution process, either part way or all the way.
7. There were no hints, only answers, and an extension was merely the next practice task on the list.
8. Students had little to no autonomy in how they engaged in tasks, usually having to complete worksheets or questions out of the textbook.
9. When "enough time" had passed June would demonstrate the solution on the board, sometimes calling on "the class" to tell her how to proceed.
10. Students wrote down what June wrote on the board at the front of the room.
11. Assessment was done through individual quizzes and test.

In my pursuit to foster and sustain thinking classrooms, I set out to explore each of these variables, on their own and in conjunction with of other variables. There were two mandates that guided this research. The first was to find practices around each variable that maximized students starting to thinking or, if already started, sustain their thinking. The second was that these practices had to be things that teachers were willing to adopt.

So, with the help of more than 400 teachers I embarked on a massive design-based research project (Anderson & Shattuck, 2012; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003; Norton & McCloskey, 2008). This approach allowed me to vary the practices around each of the variables, either independently or jointly, and to measure the effectiveness of that method for building and/or maintaining a thinking classroom. Results fed recursively back into teaching practice, each time leading either to refining or abandoning what was done in the previous iteration.

This method, although fruitful in the end, presented two challenges. The first had to do with the measurement of effectiveness. To do this I used what I came to call *proxies for engagement*—observable and measurable (either qualitatively or quantitatively) student behaviors. At first this included only behaviors that fit the *a priori* definition of a thinking classroom. As the research progressed, however, the list of these proxies grew and changed depending on the variable being studied and teaching method being used.

The second challenge had to do with the shift in practice needed when it was determined that a particular teaching method needed to be abandoned. Early results indicated that small shifts in practice. Did little to shift the behaviors of the class as a whole. Larger, more substantial shifts were needed. These were sometimes difficult to conceptualize. In the end, a contrarian approach was adopted. That is, when a practice around a specific variable needed to be abandoned, the new approach to be adopted was, as much as possible, the exact opposite to the practice that had shown to be ineffective for building or maintaining a thinking classroom. When students sitting showed to be ineffective, we tried making the students stand. When leveling to the top failed we tried levelling to the bottom. When answering questions proved to be ineffective we stopped answering questions. Each of these approaches needed further refinement through the iterative design-based research approach, but it gave good starting points for this process.

Over time results began to emerge and a set of practices for each of the aforementioned variables began to present themselves as most effective. In many cases, these practices were far from the norms that permeated mathematics teaching for so long. In what follows, I present the research into variable #4—*student work space*.[2]

13.4 Study #2: Student Behavior on a Variety of Work Spaces

Irrespective of the age students or the mathematics curriculum being presented, students all over the world sit at a desk or table and write in their notebooks. The teacher, on the other hand, stands and writes on some sort of vertical surface. These norms are so pervasive, so entrenched, that they are no longer negotiated. They have become non-negotiable norms. Even educational research, in its ever-present pursuit to improve classroom conditions and teaching practice have, for the most part, neglected to question these norms. Yes, desks and tables are a little nicer now they were 100 years ago. Blackboards have given way to whiteboards, which eventually gave way to "smart" boards. But the basic premise has remained unchanged. Students sit and teachers stands. Students write on horizontal surfaces and teachers write on vertical surfaces.

It became obvious early in this work that adherence to this norm was not conducive to the building of a thinking classroom. As such, almost immediately, a new space was explored. Following the contrarian approach established early on, the next space to test was to have students standing and working a vertical surface. The shift to having students work on whiteboards and blackboards was then an obvious extension.

In many classrooms where the research was being done, however, there were not enough whiteboards and blackboards available for all groups to work at. Some students would have to still be seated in their desks. This led to a phase of experimentation with alternative work surfaces, including poster board or flipchart paper attached to the walls, and smaller whiteboards laying on desks—with some classrooms using all three at the same time. Whenever this occurred there was a general sense shared between whatever teachers were in the room, as well as myself, that the vertical whiteboards were superior to any of the other options available to students. These observations led to the following comparative study focusing on this phenomenon.

[2]For results of the remaining variables see Liljedahl (2018, 2014, 2016).

13.4.1 Participants

The participants for this study were the students in five high school classrooms; two grade 12 (n = 31, 30), two grade 11 (n = 32, 31), and one grade 10 (n = 31).[3] In each of these classes students were put into groups of two to four and assigned to one of five work surfaces to work on while solving a given problem solving task. Participating in this phase of the research were also the five teachers whose classes the research took place in. Most high school mathematics teachers teach anywhere from three to seven different classes. As such, it would have been possible to have gathered all of these data from the classes of a single teacher. In order to diversify these data, however, it was decided that data would be gathered from classes belonging to five different teachers in five different schools.

These teachers were all participating in one of several learning teams which I was facilitating. Teachers participated in these teams voluntarily with the hope of improving their practice and their students' level of engagement. Each of these learning team consisted of between four and six, two hour meeting, spread over half a school year. Sessions took teachers through a series of activities, modeled on the most current results on the building and maintaining of thinking classrooms. Teachers were asked to implement the activities and teaching methods in their own classrooms between meetings and report back to the team how it went.

The teachers, whose classrooms these data was collected in, were all new to the ideas being presented and, other than having individual students occasionally demonstrate work on the whiteboard at the front of the room, had never used them for whole class activity.

13.4.2 Data

The students were put into groups of two to four by their classroom teachers and then each group was randomly assigned to one of five work surfaces: wall mounted whiteboard, whiteboard laying on top of their desks or table, flipchart paper taped to the wall, flipchart paper laying on top of their desk or table, and their own notebooks at their desk or table. Then all groups were assigned the same task to solve. As the objective of this research is to foster thinking in general, and during problem solving in particular, a problem solving task was used. For comparison sake, the task that chosen was the same one used in Jane's class years earlier.

> If 6 cats can kill 6 rats in 6 minutes, how many will be needed to kill 100 rats in 50 minutes? (Lewis Carroll, cited in Wakeling, 1995, p. 34)

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[3]In Canada grade 12 students are typically 16–18 years of age, grade 11 students 15–18, and grade 10 students 14–17. The age variance is due to a combination of some students fast-tracking to be a year ahead of their peers and some students repeating or delaying their grade 11 mathematics course.

To increase the likelihood that they would work as a group, each group was provided with only one felt or, in the case of working in a notebook, one pen. To measure the degree to which the work surface was affecting student thinking *proxies for engagement* were established. As mentioned, these proxies were used as a way to document observable student behaviors to gauge the degree to which the students are engaging with the assigned task. For the research presented here a variety of objective and subjective proxies were established.

1. **Time to task**
   This is an objective measure of how much time passed between the task being given and the first discernable discussion as a group about the task.
2. **Time to first mathematical notation**
   This is an objective measure of how much time passed between the task being given and the first mathematical notation was made on the work surface.
3. **Eagerness to start**
   This is a subjective measure of how eager a group was to start working on a task. A score of 0, 1, 2, or 3 was assigned with 0 being assigned for no enthusiasm to begin and a 3 being assigned if every member of the group were wanting to start.
4. **Discussion**
   This is a subjective measure of how much group discussion there was while working on a task. A score of 0, 1, 2, or 3 was assigned with 0 being assigned for no discussion and a 3 being assigned for lots of discussion involving all members of the group.
5. **Participation**
   This is a subjective measure of how much participation there was from the group members while working on a task. A score of 0, 1, 2, or 3 was assigned with 0 being assigned if no members of the group was active in working on the task and a 3 being assigned if all members of the group were participating in the work.
6. **Persistence**
   This is a subjective measure of how persistent a group was while working on a task. A score of 0, 1, 2, or 3 was assigned with 0 being assigned if the group gave up immediately when a challenge was encountered and a 3 being assigned if the group persisted through multiple challenges.
7. **Non-linearity of work**
   This is a subjective measure of how non-linear groups work was. A score of 0, 1, 2, or 3 was assigned with 0 being assigned if the work was orderly and linear and a 3 being assigned if the work was all over the place.
8. **Knowledge mobility**
   This is a subjective measure of how much interaction there was between groups. A score of 0, 1, 2, or 3 was assigned with 0 being assigned if there was no interaction with another group and a 3 being assigned if there was lots of interaction with another group or with many other groups.

These measures, like all measures, are value laden. Some (1, 2, 3, 6) were selected partially from what was observed informally when being in a setting where multiple work surfaces were being utilized. Others (4, 5, 8) were selected specifically because they embody some of what defines a thinking classroom—discussion, participation, and knowledge mobility. Non-linearity of work comes from the aforementioned research on non-thinking classrooms which showed that linearity of work is often correlated with mimicking (Liljedahl & Allan, 2013a).

As mentioned, these data were collected in the five aforementioned classes during a group problem solving activity. Across the five classes there were 10 groups that worked on wall mounted whiteboard, 10 that worked on whiteboard laying on top of their desks or table, 9 that worked on flipchart paper taped to the wall, 9 that worked on flipchart paper laying on top of their desk or table, and 8 that worked in their own notebooks at their desks or table. For each group the aforementioned measures were collected by a team of three to five people: the teacher whose class it was, the researcher (me), as well a number of observing teachers. These data were recorded on a visual representation of the classroom and where the groups were located with no group being measured by more than one person.

13.4.3 Results and Discussion

For the purposes of this chapter it is sufficient to show only the average scores of this analysis (see Table 13.2). These data confirmed the informal observations. Groups are more eager to start, there is more discussion, participation, persistence, and no-linearity when they work on the whiteboards. However, there are nuances that deserve further attention. First, although there is no significant difference in the time it takes for the groups to start discussing the problem, there are big differences between whiteboards and flipchart paper in the time it takes before groups make their first mathematical notation. This is equally true whether groups are standing or sitting. This can be attributed to the non-permanent nature of the whiteboards. With the ease of erasing available to them students risk more and risk sooner. The contrast to this is the very permanent nature of a felt pen on flipchart paper. For students working on these surfaces it took a very long time and lots of discussion before they were willing to risk writing anything down. The notebooks are a familiar surface to students so this can be discounted with respect to willingness to risk starting.

Although the measures for the whiteboards are far superior to that of the flipchart paper and notebook for the measures of eagerness to start, discussion, and participation, it is worth noting that in each of these cases the vertical surface scores higher than the horizontal one. Given that the maximum score for any of these measures is 3 it is also worth noting that eagerness scored a perfect 3 for those that were standing. That is, for all 10 cases of groups working at a vertical whiteboard, 10 independent evaluators gave each of these groups the maximum score. For discussion and participation 8 out of the 10 groups received the maximum score. On the same measures the horizontal whiteboard groups received 3, 3, and 2 maximum scores respectively. This can be attributed to the fact that sitting, even while working at a whiteboard, still gives students the opportunity to become anonymous, to hide, and not participate. Standing doesn't afford this.

Table 13.2 Average times and scores on the eight measures

	Vertical whiteboard	Horizontal whiteboard	Vertical paper	Horizontal paper	Note-book
N (groups)	10	10	9	9	8
1. Time to task (s)	12.8	13.2	12.1	14.1	13.0
2. Time to first notation (s)	20.3	23.5	144.1	126.3	18.2
3. Eagerness	3.0	2.3	1.2	1.0	0.9
4. Discussion	2.8	2.2	1.5	1.1	0.6
5. Participation	2.8	2.1	1.8	1.6	0.9
6. Persistence	2.6	2.6	1.8	1.9	1.9
7. Non-linearity	2.7	2.9	1.0	1.1	0.8
8. Mobility of knowledge	2.5	1.2	2.0	1.3	1.2

With respect to non-linearity it is clear that the whiteboards, either vertical or horizontal, allow a greater freedom to explore the problem across the entirety of the surface. Although the whiteboards provide an ease of erasing that is not afforded on the flipchart paper, and that this likely contributes to the shorter time to first notation, ironically, work is rarely erased by the students working on whiteboard surfaces. It seems that, rather than erasing to make room for more work, the work space migrates around the whiteboard surface representing the chronological nature of problem solving. In contrast, the groups working on flipchart paper tended to not write any work down until they were clear it would contribute to the logical development of a solution.

Finally, it is worth noting that groups that were standing also were more likely to engage with other groups that were standing close by. Although not measured, it was clear that this was more true for the vertical whiteboard groups. There are a number of reasons for this. Most obvious, vertical surfaces are more visible. However, there were very few observed instances of groups that were sitting down looking up to see what the groups that were standing were doing. Likewise, there were no instances of the students standing looking at the work of the groups that were sitting. Among those that were standing, there was a lot of interaction between those working on whiteboards, and almost none between those working on flipchart paper. Finally, there was very little interaction between those working on flipchart and those working on whiteboards. Part of this can be explained by proximity—the whiteboard groups were clustered on one or two whiteboards while the flipchart people were clustered elsewhere. But, it also is the case that the whiteboard groups had little reason to look to the flipchart groups. They worked slower and had little written on their work surface. This was also true between the flipchart groups—there was little to look at.

In short, and in answer to variable #4—*student work space*, groups that worked on vertical non-permanent surfaces demonstrated more thinking classroom behavior—persistence, discussion, participation, and knowledge mobility—than any of the

Table 13.3 Distribution of participants in VNPS study

	Elementary	Middle	Secondary	Totals
Learning team	21	43	41	105
Multi-session workshops	12	28	42	82
Single workshops	35	24	54	113
Totals	68	95	137	**300**

other type of work surface. Next most conducive was a horizontal whiteboard. The remaining three were not only not conducive to promoting thinking classroom behavior, they may actually have inhibited it. From this it is clear that non-permanence of surfaces is critical for decreasing time to task, as well as improving enthusiasm, discussion, participation, and persistence. It also increases the non-linearity of work which mirrors the actual work of thinking groups. Making these non-permanent surfaces vertical further enhances all of these qualities, as well as fostering inter-group collaboration, something that is needed to move the class from a collection of thinking groups to being a thinking classroom.

13.5 Study #3: Teacher Uptake of Vertical Non-permanent Surfaces

Having this evidence that vertical non-permanent surfaces (VNPS) are so instrumental in the fostering of thinking classroom behavior satisfied the first mandate. To satisfy the second mandate a follow-up study was done with teachers to gauge the willingness by teachers to take up this practice. In particular, the goal of this follow up study was to see the degree to which teachers, when presented with the idea of non-permanent vertical surfaces were keen to implement it within their teaching, actually tried it, and continued to use it in their teaching.

13.5.1 Participants

Participants for this portion of the study were 300 in-service teachers of mathematics—elementary, middle, and secondary school. They were drawn from three sources over a four year period: participants in variety of single workshops, participants in a number of multi-session workshops, and participants in learning teams. The breakdown of participants, according to grade levels, and form of contact is represented in Table 13.3.

There were a number of teachers who attended a combination of learning teams, multi-session workshops, and single workshops. In these cases their data was registered as belong to the group with the most contact. That is, if they attended a single workshop, as well as being a member of a learning team, their participation was registered as being a member of a learning team.

These participants are only a subset of all the teachers that participated in these learning teams, multi-session workshops, and single workshops. They were selected at random from each group I worked with by approaching them at the end of the first (and sometimes only) session and asking them if they would be willing to have me contact them and, potentially, visit their classrooms.

13.5.2 Data

Data consist primarily of interview data. Each participant was interviewed immediately after a session where they were first introduced to the idea of vertical non-permanent surfaces, one week later, and six weeks later. These interviews were brief and, depending on when the interview was, was originally designed to gauge the degree to which they were committed to trying, or continuing to use vertical non-permanent surfaces in their teaching and how they were using them. However, participants wanted to talk about much more than just this. They wanted to discuss innovations they had made, the ways in which this was changing their teaching practice as a whole, the reactions of the students and their colleagues, as well as a variety of other details pertaining to vertical non-permanent surfaces. With time, these impromptu conversations changed the initial interview questions to begin to also probe for these more nuanced details. For the purposes of this chapter, however, these data were analyzed only for the original aforementioned purpose.

In addition to the interview data, there was also field notes from 20 classroom visits. These visits were implemented for the purposes of checking the fidelity of the interview data—to see if what teachers are saying is actually what they are doing. In each case, this proved to be the case. It was clear from these data that teachers were true to their words with respect to their use of vertical non-permanent surfaces. However, these visits, like the interviews, offered much more than what was expected. I saw innovations in implementation, observed the enthusiasm of the students, and witnessed the transformational effect that this was having on the teaching practices of the participant.

13.5.3 Results and Discussion

In general, almost all of the teachers in this study, who were introduced to the notion of vertical non-permanent surfaces were determined to try it within their teaching and were committed to keep doing it, even after six weeks (see Fig. 13.2). This is a significant uptake rarely seen in the literature. This is likely due, in part, to the ease with which it is modeled in the various professional development settings. During

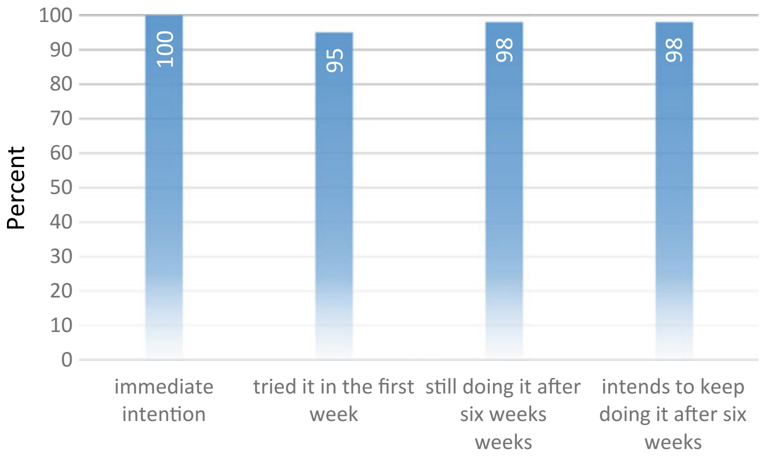

Fig. 13.2 Uptake of VNPS (n = 300)

these sessions not only is the methods involved easily demonstrated, but the teachers immediately feel the impact on themselves as learners when they are put into a group to work on a vertical non-permanent surface.

An interesting result from this aggregated view is that there were more teachers using non-permanent vertical surfaces after six weeks than there was after one week. This has to do with access to these vertical non-permanent surfaces. Many teachers struggled to find such surfaces. There were some amazing improvisations in this regard, from using windows, to bringing in a number of non-permanent surfaces, from shower curtains to glossy wall boards. One teacher even stood her classroom tables on end to achieve the effect. As time went on teachers were able to convince their administrators to provide them with enough whiteboards that these improvisations no longer became necessary. For some teachers, this took more time than others, and speaks to the delayed uptake for some. However, it also speaks to the persistence with which many teachers pursued this idea with.

A disaggregated look at these data shows that neither the grade levels being taught (see Fig. 13.3) or the type of professional development setting in which the idea was presented (see Fig. 13.4) had any significant impact on the uptake.

Literature on teacher change typically implies that sustained change can only be achieved through professional development opportunities with multiple sessions and extended contact. That is, single workshops are not effective mediums for promoting change (Jasper & Taube, 2004; Stigler & Hiebert, 1999; Little & Horn, 2007; McClain & Cobb, 2004; Middleton, Sawada, Judson, Bloom, & Turley, 2002; Wenger, 1998; Lord, 1994). The introduction of vertical non-permanent surfaces as a work space doesn't adhere to these claims. There are many possible reasons for this. The first is that the introduction of non-permanent vertical surfaces was achieved in a single workshop could be, as mentioned, due to the simple fact that it is a relatively easy idea for a workshop leader to model, and for workshop participants to experience.

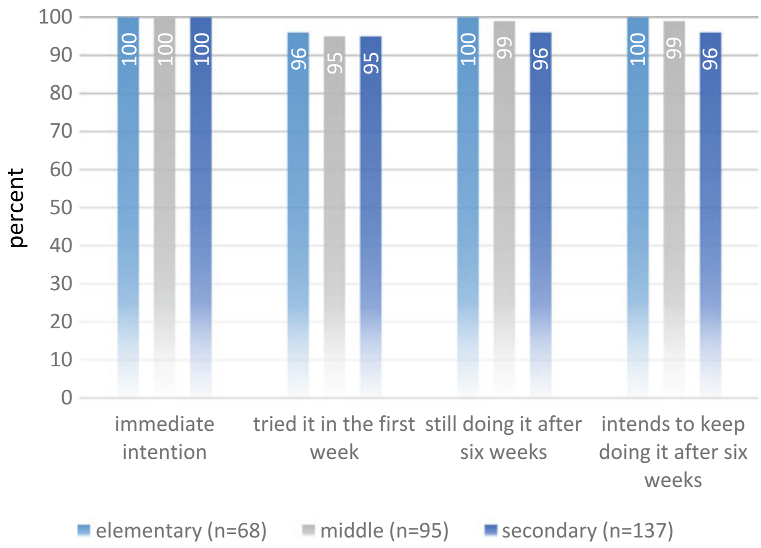

Fig. 13.3 Uptake of VNPS by grade levels (n = 300)

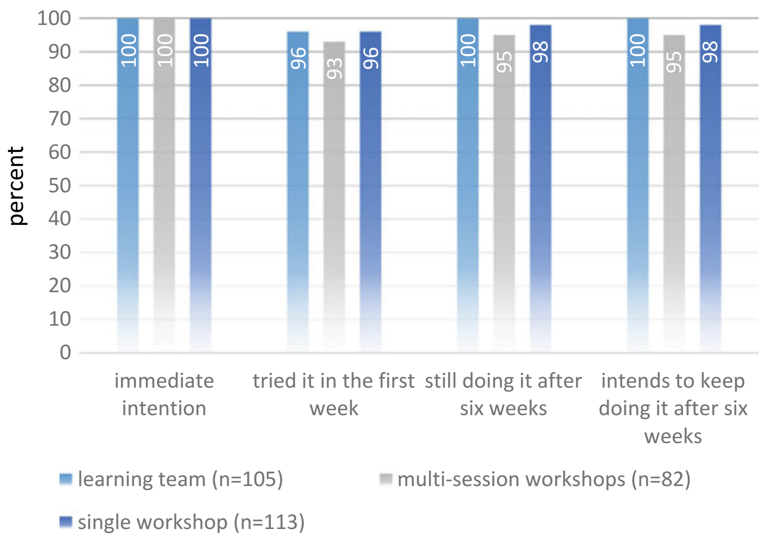

Fig. 13.4 Uptake of VNPS by professional development setting (n = 300)

Forty five minutes of solving problems in groups standing at a whiteboard, coupled with a whole group discussion on the affordances of recreating this within their own classrooms is enough to convince teachers to try it. And trying it leads to a successful implementation. Unlike many other changes that can be made in a teacher's practice, vertical non-permanent surfaces (as demonstrated in the first study) was well received by students, was easy to manage at a whole class level, and had an immediate positive effects on classroom thinking behavior. Together, the ease of modeling coupled with a successful implementation meant that vertical non-permanent surfaces did not need more than a single workshop to change teaching practice.

These possible reasons are supported by the comments of teachers from the interviews after week one and week six. The following comments were chosen from the many collected for their conciseness.

> I will never go back to just having students work in their desks.
>
> How do I get more whiteboards?
>
> The principal came into my class, now I'm doing a session for the whole staff on Monday.
>
> My grade-partner is even starting to do it.
>
> The kids love it. Especially the windows.
>
> I had one girl come up and ask when it will be her turn on the windows.

Not only is the implementation of vertical non-permanent surfaces immediately effective for these teachers, it is also infectious with other teachers quickly latching on to it and administrators quickly seeing the affordances it offers.

13.6 Conclusions

When June had asked me to help her implement problem solving in her classroom all she wanted from me to provide her with some problems. This, as it turns out, was not enough for her to have success in transforming a fundamentally non-thinking classroom into a thinking one. The norms that existed in her classroom, which she had established in her classroom, were working against her. Her students were not accustomed to thinking. They were used to direct instruction, multiple worked examples, and *now you try one* tasks. June needed something else, in addition to problem solving tasks, to help break her students out of this normative and non-thinking behavior. Had I given June the tool of VNPS along with the problems I provided I am confident that things would have gone quite differently. The aforementioned results show that June's students would have likely taken this on with great enthusiasm, and proceeded to work on the problems with greater participation and perseverance. Furthermore, the results show that with very little intervention June would have likely adopted this tool for use in her classroom.

When I began the research on students' work space the default was students sitting in desks—sometimes individually in rows, other times clustered in groups. The move from the desks to the vertical workspaces was made not because I saw something specifically wrong with students being in desks, but rather through adherence to the contrarian approach that was adopted early on in the more general research project. Looking back now at students working in desks, from the perspective of the affordances that having them stand around a non-permanent vertical surface offers, I see more clearly the problems that desks introduced into my efforts to build and maintain thinking classrooms. Primarily, this has to do with anonymity and how desks allow for, and even promote, this. When students stand around a whiteboards they are all visible. There is nowhere to hide. When students are in their desks it is easy for them to become anonymous, hidden, and safe—from participating and from contributing. It is not that all students want to be hidden, to not participate, but when the problems gets difficult, when the discussions require more thinking, it is easy for a student to pull back in their participation when they are sitting. Standing in a group makes this more difficult. Not only is it immediately visible to the teacher, but it is also clear to the students who is pulling back. To pull back means to step towards the center of the room, towards the teacher, towards nothing. There is no anonymity in this.

But these are the results for only one of the aforementioned 11 variables. Similar to the result of VNPS there are optimal practices for each of the other 10 variables that come together into a powerful framework that further transform the classroom, the role of the teacher, and the activities of students (Liljedahl, 2018, 2016). Taken together, the 11 practices that emerge out of this framework work together to form a cohesive and powerful practice for engaging, and sustaining, students thinking—for *build thinking classrooms.*

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Chapter 14
The ARPA Experience in Chile: Problem Solving for Teachers' Professional Development

Patricio Felmer, Josefa Perdomo-Díaz, and Cristián Reyes

14.1 Introduction

In the context of a national educational reform and a new curriculum that enhance the development of mathematical abilities, tremendous opportunities are opening for Chilean school mathematics development. Some of these opportunities are taken up by a team of mathematicians, mathematics educators, mathematics teachers, elementary teachers, engineers, educational researchers and other professionals who give life to the ARPA Initiative, where ARPA is the Spanish acronym for Activating Problem Solving in Classrooms (Activando la Resolución de Problemas en las Aulas). The ARPA Initiative's goal is the introduction of problem solving into regular teachers' practice through teacher professional development strategies, based on teachers experiencing problem solving in the way students will experience in classroom. The underlying central goal is having students to experience mathematics in its essence, giving sense to school mathematics and moving the class attention from teacher to students, opening the route for developing abilities intertwined with content.

We start this chapter describing the educational context in Chile in which the ARPA Initiative was born and where it is developing. Then we present the basic principles under which ARPA is devised and a brief general description of the different professional development strategies that are currently part of ARPA. The chapter continues with a more detailed description of the core strategy, PSClassroom, a year-long workshop where teachers have opportunities for doing and reflecting on problem solving and the way to introduce them into classroom. In the second part of the chapter we describe how the professional development program of ARPA has evolved during the last years and the research made in this context, both for informing practices and for communicating results to the mathematics education community.

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We conclude the chapter with some things we have learned from practice and describ the main challenges that the initiative faces for the future.

14.2 The Educational Context in Chile

Chile is undergoing an important educational reform that is transforming structural aspects of the educational system as a whole, at initial, primary, secondary, vocational and university level. This educational reform comes after years of economic growth, socio-economic segregation and deregulation in education, student unrest and unsatisfactory educational results. In particular, in the case of mathematics, the results of Chilean students in international tests have been practically stagnant at a low level for the last 10 years (Ministry of Education, 2016). These educational results do not correspond to results of a country with Chilean economic income, as it is shown in the publication by the Ministry of Education (2013). For example, the PISA test in 2015 shows that there is still a long way to go for the country to reach the average of countries with similar economic income, about 40 points, and even further to reach the average of countries in the Organization for Economic Cooperation and Development (OECD), more than 60 points. Moreover, even though Chilean results in mathematics are at the top of the Latin American region, they have remained practically constant since 2006, with almost 50% of students at level 1 or below[1] (Ministry of Education, 2016).

The reform is encouraged by international recommendations of OECD (2015) and the overall goal is to improve the quality of education for all students along the country. Before the reform started, in 2012 a new national curriculum was issued for primary and secondary school (1–10 grades). In the case of mathematics, this curriculum introduced an important change in the organization; it was based on three areas: Abilities, Content and Attitudes (Ministry of Education, 2012). Thus, a clear distinction between content and abilities in mathematics was drawn, in contrast with the earlier curricula where abilities were explicit, but blended with content, letting them be subordinated by the latter. The abilities declared in the new Chilean curriculum were: problem solving, reasoning and communicating, representing, and modeling. Among these, problem solving is central for the development of the other three abilities and is then crucial for the success in the mathematics education of students.

The unsatisfactory educational results of Chilean students together with the new curriculum pose enormous challenges to the whole educational system, including government and ministry of education, local county school organizations, educational foundations,[2] school leaders and teachers teaching mathematics. These challenges are especially critical for teachers, who undertake the most difficult aspects of these

[1]PISA test defines six level of achievement for students, being level 1 the lowest and level 6 the highest.

[2]In Chile, there are many private foundations administering a group of schools.

challenges and usually take the blame for the poor educational results. Furthermore, most do not currently have the adequate tools for change. In this context, professional development programs (PDP) should provide teachers with those tools, but even though there has been a great variety of PDP during the last years, regularly provided by the ministry of education, universities, educational foundations and some private organizations (Sotomayor & Walker, 2009), their effectiveness has not been well documented. There is currently no study that reports on the impacts of those PDP that are on offer, in terms of the actual changes in classroom, for teachers who attend them, and about the improvements of educational results of students. Without scientific studies, one may still conjecture from the commented stagnation on educational results of Chilean students in mathematics during the latest 10 years, that PDP has not been successful in changing classroom practices and improving student learning.

Consistent with the above, research allows to say that in Chilean mathematics school classrooms, problem solving has been virtually absent, even though it has been declared in national curriculum even before the 2012 reform. In a recent study by Felmer and Perdomo-Díaz (2016), novice high school teachers were observed while teaching and most of them did not use problem solving strategies. In other studies, similar conclusions were drawn by Alfaro and Gormaz (2009) when analyzing results of PISA 2006, by Preiss, Larraín, and Valenzuela (2011) from class observations, and Araya and Dartnell (2009) and Rodríguez, Carreño, Ochsenius, and Muñoz (2015) with data from various national teacher evaluations.

Since, after all, the educational reform is about educational quality for all students, the opportunities are there. The structural changes, the increase on the educational national expenditure, the teachers' career reform and the various sources of public funds for professional development open enormous opportunities for development in all areas of education, in particular in school mathematics. Thus, a great challenge is posed by the new curriculum, the poor students' results and the advantages of the new educational reality: how to make problem solving really happen in school classroom?

14.3 The ARPA Initiative

With the publication of Pólya's book *How to solve it* in 1945, problem solving entered into the arena of school mathematics teaching and learning. Nowadays, problem solving is internationally recognized as an essential component of school mathematics, and the reasons are rooted in the opportunities it offers students for mathematical development: establishing reasoned connections among mathematical elements, promoting skills of examining, representing and applying, and the use of mathematical thinking such as abstracting, analyzing, guessing, generalizing or synthesizing (Kilpatrick, Swafford, & Findell, 2009; NCTM, 2000; Niss, 2002). But, as Pólya also warned, teachers who has not '"experienced the tension and triumph of discovery"' would hardly offer their students problem solving opportunities in classroom (Pólya, 1966).

That Pólya's recommendations are not considered in the formation of Chilean teachers, that pre-service teachers are not experiencing problem solving in their education, could be drawn from Chilean students' results in mathematics and some few research results. Among these, we mention the studies by Felmer et al. (2015) and Varas et al. (2008), where evidence is provided of a scarce mathematics formation and almost no opportunities for problem solving strategies. In this context, the ARPA Initiative was born, with the main goal of giving the teachers opportunities to experience problem solving as Pólya proposed, looking for changing classroom practices and to improve student learning, designing, implementing and evaluating professional development strategies for teachers teaching mathematics. The starting point was to design and put in practice of mathematics teacher professional development strategies and the fundamental question at this moment was, what are the characteristics they should have to make them effective?

There is an important amount of research on what characteristics should a PDP have, in order to be effective in terms of change in the knowledge, skills and practice of teachers. To place teachers in the role of students, to create networks among the participants, to propose activities related to other programs or reforms and to carry out the program on a regular basis and during a long period of time (Borko, 2004; Desimone, Porter, Garet, Yoon, & Birman, 2002; Garet, Porter, Desimone, Birman, & Suk Yoon, 2001; Marrongelle, Sztajn, & Smith, 2013). In summary, these findings suggest that an effective PDP should:

- Privilege work activities in study groups, work with mentors, and create teacher networks and research projects.
- Contain long-term courses, with long sessions and extended over time.
- Involve collective participation of teachers, in groups of the same school, municipality or community of schools who all teach in the same course level.
- Include active learning, where teachers are involved in the analysis of teaching and learning.
- Be consistent with the objectives of the teachers, the school and its managers, the state and the curriculum.
- Focus on the content, that is, the activity focuses on content and math skills.

Having this knowledge in mind, Felmer and Perdomo-Díaz designed three professional development workshops, giving birth to the *ARPA Initiative*. These workshops are: PSAction, PSContent and PSClassroom (Felmer & Perdomo-Díaz, 2017; Perdomo-Díaz & Felmer, 2017), whose different characteristics will be explained in the next section. These workshops are based on two basic principles: *doing and reflecting* (Borko, 2004; Marrongelle et al., 2013) and this is the reason why they are called '"workshops"' instead of '"courses"', emphasizing their eminently practical spirit. These workshops have problem solving as articulating axis and they are aimed at teachers teaching mathematics, with the purpose of installing problem solving and effective school practices to improve teaching and learning of school mathematics.

As the ARPA Initiative progressed, and along the way as the needs were appearing and the capability of the team was ready, new workshops have been created, as a

Table 14.1 ARPA workshops

ARPA workshop	2013	2014	2015	2016
PSAction	x	x	x	x
PSAction II	–	–	x	x
PSContent	x	x	x	x
PSMonitor	–	x	x	x
PSClassroom	–	x	x	x
PSClassroom II	–	–	–	x

monitor formation workshop PSMonitor, and PSClassroom II and PSAction II for teachers willing to deepen on their knowledge and skills (Table 14.1).

All workshops in the ARPA initiative are characterized by two instances: problem solving blocks (*doing*) and plenary discussions (*reflecting*). During problem solving blocks, participants are given problems that they work with their peers in groups, with the support of a monitor who interacts with the groups through questions. The monitor provides each group with a problem and they work until solving it; it is considered that a group has solved a problem when all the members can explain the solution and the strategies used. When a group solves the problem, the monitor will provide the group with an extension problem, will ask a further question to keep them working or he/she will give them a new problem. In this way, each group works at its own pace and the difficulty of the problem is graded by the monitor, according to the participants' skills, so that the problem all time is an effectively challenge. This is the part of *doing* where teachers experience Pólya's recommendations.

ARPA's workshops also offer opportunities for teachers to reflect on their ability to solve problems, their mathematical knowledge and learning, the strategies used to solve the problems and emotions they have felt in this task, and on how the monitor interacts with them, how this may be a model for classroom implementation, and how actually to implement it.

On the other hand, each ARPA's workshop is changing in every new instance according to team learnings, either by scientific research or experience, and it may be adapted to specials audience to which is aiming in each case (elementary, high school teachers or post-secondary teacher, city or rural teachers, for example). Flexibility is a key concept in the ARPA Initiative in its construction, evolution and operation.

14.4 The ARPA's Workshops

We devote this section to briefly describe three of the main workshops that give shape to the ARPA Initiative: PSAction, PSContent and PSMonitor. PSClassroom and its second part PSClassroom II, are the most important in the ARPA Initiative and we devote the next section for their description. A more complete description of

PSAction, PSContent and PSClassroom may be found in Felmer and Perdomo-Díaz (2017) and Perdomo-Díaz, Felmer, Randolph, and González (2017).

PSAction is a workshop of 4–5 h, designed to disseminate the importance of problem solving in classrooms, to introduce teachers to this idea and to open their interest in going further. It starts with a short presentation, after which teachers participate in two problem solving blocks, organized in randomly chosen groups of three. During these blocks teachers solve problems involving contents from all axes of the math curriculum. During the last 45 min, a plenary discussion is held, where the monitor promotes discussion among teachers on the emotions experienced while solving problems, the strategies used to solve them, the role of the monitor, the way of interacting with teachers, and finally the possibility of introducing problem solving in their classrooms, using the work in the workshop as a model. The workshop concludes with a presentation.

This workshop has been offered many times, at different places along the country and abroad (Table 14.4). This workshop can be offered to any number of teachers, with the only restriction is the number of monitors (one for every 21 teachers). We have had experiences with more than 150 teachers. The interest of some teachers to participate in the workshop more than once, has motivated the design of PSAction II, a workshop with similar characteristics regarding problems solving activities, but with some time devoted to how to implement problem solving in classroom.

The second workshop was devised considering the need of Chilean teachers, specially, elementary teachers, for strengthening their disciplinary content for teaching. The PSContent workshop is a 25-hour strategy, with five hours a day during a week. It aims to create opportunities for teachers to deepen their knowledge of a specific content, considering elements of common knowledge, specialized knowledge of content and knowledge of content and teaching (Ball, Thames, & Phelps, 2008). The mathematical content that is treated in each workshop is chosen based on the needs of teachers, expressed through surveys, and on what literature and experience reveal as content with teaching and learning difficulties. PSContent workshops have been offered in statistics, probability fractions, geometry, numbers and operations, and arithmetic. Some of these topics have been considered for elementary, middle school and high school teachers, like geometry, statistics and probability. In the context of PSContent workshop another workshop was shaped up, this based only on problem solving, with problems from all axes of the curriculum, whose aim is for teachers to experience problem solving and to reflect with some deep on the possibility to introduce it in classroom. This version of problem solving workshop has been offered in two opportunities to kindergarten teachers.

PSContent workshop is intended for 21 teachers leaded by a monitor. It starts with a presentation where the contents and methodology are announced. Two blocks of problem solving and a plenary discussion are held in each session. The problems are articulated based on the disciplinary topic of the workshop, so as to provide instances in which teachers reflect on their knowledge of the content and its teaching. Only 3–5 key concepts of the topic are considered in each workshop, so to have time to deepen on them. After problem solving blocks, teachers work in a plenary discussion consisting in two parts, first a discussion lead by the monitor about the key concepts that have been worked out in the problems, and second on the strategies used and solutions obtained, ending with the identification of the key concepts involved. A 3-4-pages booklet with concepts treated is given three times in the week.

This workshop is offered usually in summer, but it also has a winter edition, and it has attracted an increasing number of teachers (Table 14.4). The first version of PSContent took place in January 2013, considering fraction as the focus and taking into account four key topics: (i) what is a fraction? (ii) common denominator (iii) subtraction and multiplication and (iv) division. In Perdomo-Díaz et al. (2017), an analysis of the contributions of the workshop to the development of teacher knowledge, based on one participant teacher case study, is presented.

Another ARPA's workshop is PSMonitor, devised for the formation of new monitors. Based on an experimental version in 2015, a PSMonitor workshops was devised during 2016. This workshop has 12 three-hours sessions distributed in 6 weeks. During the first 6 sessions, the emphasis is put on problem solving activities modeling problem solving in classroom, where participants play the role of students and monitor plays the role of teachers. Discussions on the different stages of these activities and the role of the monitor follows. The second half of the workshop is devoted to the various duties a monitor has when leading a PSClassroom workshop. This workshop is characterized by doing and reflecting, as all the other workshops of the initiative.

At this point it is convenient state what we understand by problem, since this notion is at the heart of the initiative:

> A problem is a mathematical activity for which the person that face it does not know a procedure leading to its solution. The person has interest in solving it, he/she considers it a challenge and he/she feels that he/she can solve it. The activity may be raised in a mathematical or non-mathematical context.

In the ARPA workshops, there are opportunities for teachers to solve problems and also to know and reflect on the notion of problem, as applied for them or their students as problem solvers. During workshops, teachers face problems that are prepared for them as problem solvers and, when they are solving them, they are asked to forget their role as teachers and play the role of a genuine problem solver. In PSClassroom workshop, during the first two sessions, they work on various problems. During the following six sessions, they first solve the problem that they will propose to their students as problem solvers, and only later they start to think on their students and the way they will present the problem to them. In some cases, they have to think in ways to simplify the problem to make it appropriate for their students.

It is also interesting to give some words on the way teachers are recruited for the various workshops. In the case of PSAction usually we recruit them by open invitations, in the contexts of various schools or a county or some other organization. These workshops are for free for the teachers and, in most cases, also free for institutions involved in the call. For PSContent workshops, which usually are offered in the form of summer courses, we recruit teachers by open call. Some teachers pay for the workshop by themselves, but for most of them payment will be provided by their school, county or other institution to which they belong. Some limited scholarships are also given. The case of PSClassroom is the most important, since it is the main

Table 14.2 Main characteristics of workshops

ARPA workshop	Duration (h)	Span	Main goal
PSAction	4–5	1 day	PS experience and diffusion
PSAction II	4–5	1 day	PS experience with classroom perspective
PSContent	25	1 week	Content through PS
PSMonitor	36	6 weeks	Formation of monitors
PSClassroom	50	1 year	PS introduction in school
PSClassroom II	50	1 year	Deepening PS introduction in school

workshop of the ARPA Initiative and the payment system has to do with the sustainability of the whole initiative. This workshop is organized usually by previous contacts with school director, educational foundations administrators or state officers at the level of counties, where interest of teachers and the feasibility of the workshop is analyzed. These educational authorities will provide the funds for the workshop and teachers will not pay for it. Moreover, the sessions of the workshop take place mostly during working hours.

It is important to say, that the Chilean educational system provides through the state for funds for professional development of teachers. These funds have been increased during the recent years due to the educational reforms. In the three cases, teachers receive a certificate of participation at the end of the workshop, but there is no bonus for them (Table 14.2).

14.5 PSClassroom

As it was mentioned above, PSClassroom workshop is the key workshop of the ARPA Initiative. Its declared goal is to introduce problem solving in classroom and changing teaching practices, towards an improvement of student achievements. PSClassroom is an annual workshop running usually from March to November[3] Teachers meet every month with a monitor, in sessions of 3:15 h, in a number of 21 at most 21 and 15 al least. These monthly sessions have emphasis on problem solving activities at the beginning of the year and the emphasis is gradually moving to preparing and

[3]School year in Chile runs from the beginning of March until the middle of December.

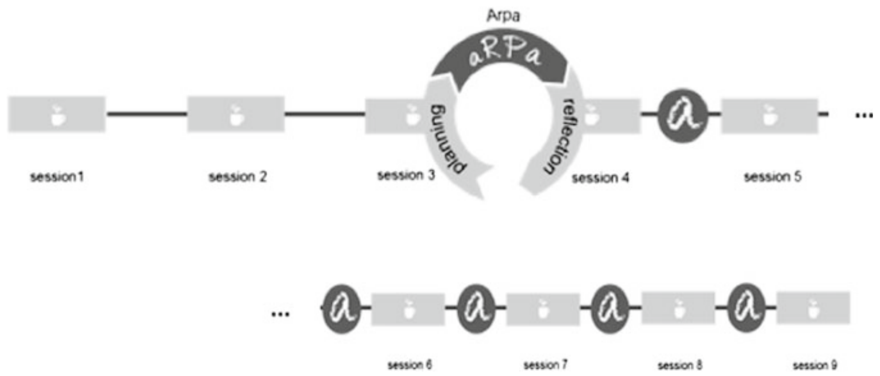

Fig. 14.1 PSClassroom workshop annual program

improving problem solving activities in classroom with students. These activities are characterized by random groups organization of students, autonomous work of students in solving a problem and a final plenary discussion, as we will describe later. We call this activity with a special name, Problem Solving Activity in Classroom (PSAC),[4] to distinguish it from other lectures.

The first two sessions are integrally devoted to problem solving activities with the aim that teachers experience by themselves the emotions taking place in this task, so they gain or regain ability to solve problems. Teachers work in various problem solving blocks in randomly chosen groups and they participate in plenary discussions whose topics are emotions, strategies and the way the monitor acts. During the next four sessions, teachers will continue having problems solving activities, but modeling PSAC, where the interaction between monitor and teachers model the way the interaction between teacher and student should be. These PSAC modeling activities are crucial learning opportunities for teachers to get the main characteristics of this activity that will be carried into classroom with their students (Fig. 14.1).

After the third session, teachers carry out PSAC into their classroom, repeating them six times along the year. Sessions combine analysis of the experiences occurring in the earlier PSAC and planning the next one. In sessions teachers analyze video episodes from co-participants and written reports of the activity. They learn how to plan questions to ask prior to PSAC, so they properly act in situations that may occur in classroom. Teachers are intended to learn, together with their students, how to conduct a productive problem solving lesson and to learn teaching techniques, like working in groups, questioning, giving students time to work by themselves, make conjectures, get things wrong and discuss with their peers. Once the basics of PSAC have been dominated, teachers also learn how to adapt mathematical activities into

[4]In Spanish, we use ARPA for these activities as the acronym of Problem Solving Activity in Classroom (Actividad de Resolución de Problemas en el Aula), which coincide with the initiative name.

challenging and reachable problems for their own students (Perdomo-Díaz & Felmer, 2017).

PSAC for students is organized in four stages: Delivery, Activation, Consolidation and Discussion. During delivery, students are organized in random groups, they receive the problem and start working. Activation starts when the group generates ideas of how to solve the problem. During this phase the teacher interacts with students, answering questions with another question, giving them the responsibility to go ahead. In some cases, a simplified problem is given and then the original problem may be returned to, after the group has experienced success with the simplified version. If a group does not understand the problem, the teacher asks a question; if a group gets stuck, the teacher asks a question; if a group makes a mistake, the teacher asks a question. When a group solves the problem, then the consolidation phase starts. The teacher asks questions to members of the group until either it is apparent that one of them does not know how to solve the problem, then the teacher leaves. If the monitor is convinced that all members know how to solve the problem, then he/she gives an extension, a more sophisticated and challenging version of the problem. Finally, the entire class engages in a plenary discussion which occurs about 10–15 min before the end of the PSAC. This phase is aligned with the last three steps described by Smith and Stein (2014).

At the end of PSClassroom workshop, there are still many features of PSAC to be improved by teachers—especially the plenary discussion and the interplay between content and student abilities. In order to face these needs and deepen the advances reached during the first workshop, we have devised a second part, we call PSClassroom II, and we have run a pilot workshop during 2016.

14.6 Three Years of Professional Development

The process of setting up the ARPA Initiative for teacher professional development started in 2012 or earlier, with some workshops where teachers solved problems, but it was in the triennium 2014–2016 where it really started to take form. Before 2014, PSAction and PSContent, even though related did not have the aspiration of becoming a unity with potential for development the dimension of a program.

14.6.1 Pre-pilot Year (2014)

During 2014, two one-year long professional development experiences were run, with 19 elementary teachers in one workshop and 6 high school teachers in the other. At the end of the year, teachers filled out an evaluation questionnaire with reflections on the workshop, where they narrated aspects of their experience, mostly positive for them and their students (Perdomo-Díaz & Felmer, 2017). This allowed to write a first version of a PSClassroom manual for the monitor and material for teachers and the main ideas for teacher formation were defined: problem solving for teachers, peer discussions, implementation of PSAC, video analysis, problem design, etc.

PSAction and PSContent workshops were taking a more definitive shape during 2014 and a manual for the monitor was written for PSAction. As a result of PSContent workshops in this year, first research results were obtained by Perdomo-Díaz et al. (2017), where a case study of a teacher that participated in a PSContent workshop on fractions is reported. The workshop was designed using problems selected using elements of common knowledge, specialized knowledge of content and knowledge of content and teaching (Ball et al., 2008). The study presents some insights on how this type of workshop may contribute to the development of teachers' knowledge for teaching fractions.

The activity taken place in 2014 and earlier, the gained experience and the interest showed by participant teachers allowed to conceive and shape, for the first time, a whole professional development program that could be developed during the years to come, with possibilities of having impact in classroom and improving student learning. Thus, a proposal for an implementation project, increasing the number of workshops was presented to Conicyt[5] national contest of R&D projects. The main goal was to set up the PDP, offering the three type of workshops during 2015 and 2016: 8 PSClassroom, 6 PSContent and 4 PSAction, per year. The project also proposed research on those workshops for studying their effectiveness, considering teachers' changes in beliefs, change in teaching quality and change in students' performance. The project's strategic goal was to experiment with PDP at a larger scale and to consolidate the workshops, in preparation for a wider development in the future, writing monitor and teacher manuals and developing related material, and learning to deal with team organization, data management, video recording and analysis, formation of monitors, and logistics aspects.

14.6.2 Pilot Year (2015)

This year the *ARPA Initiative* was officially born, ARPA for calling the conceived PDP with a catching name, well expressing the ideas behind, and Initiative instead of program or project, to allow flexibility in the up-coming scenarios, where more projects or programs could be part of ARPA.

Starting the year, funding of Fondef ID14I10338 project (2015) was granted, triggering a sudden and intense initiation process. The most urgent task during the first three months, that coincided with the beginning of the school year, was to set up at least the 8 promised PSClassroom workshops, recruiting teachers, setting up timetables for the 9 sessions, assigning monitors and providing materials. The main complexity of these actions was the short time and the distance of the four cities of implementation (Table 14.3). In particular, distances involved required the creation

[5]Conicyt is the national agency for funding of research and development in all disciplines.

Table 14.3 Participants in PSClassroom workshops during the three years

City	2014	2015	2016
Temuco	–	12	8
Concepción	–	39	45
Santiago	20	97	88
Valparaíso	–	7	19
San Pedro de Atacama	–	–	7
Copiapó	–	–	12
Total	20	155	167

of an effective communication system so teachers could communicate with monitors and monitors could communicate with the project central team.

Research had to be started simultaneously, collecting data in the different regions, putting an enormous challenge to the newly formed team, with few experience, both in research and teacher formation. The video-taping process for workshops and specially for three PSACs along the year for each participant teacher, was also an incredible logistical challenge. After the first semester, the situation was controlled and the year ended with a good amount of research data and with a team with extremely rich pedagogical and logistical experience.

14.6.3 Consolidating Year (2016)

As a result of 2015 experience, the next year was undertaken with greater skills, with more time for recruiting teachers, allowing the implementation of various organizational and, specially, pedagogical improvements. Monitors were much better trained in leading the workshops, and material for teachers and monitors were also improved, starting the year with PSClassroom manuals for all of them, properly distributed.

Regarding PSClassroom workshop participation, the 8 required by Fondef project were exceeded, reaching 17 workshops, four more than in 2015. This larger number was a consequence of the interest of teachers and school leaders for participating in the workshops. In Table 14.3 we present the number of participants disaggregated by cities and years.

The PSAction workshops were delivered in a number of cities throughout the country during 2015 and 2016, involving nearly 2500 teachers over the two years. This workshops helped in disseminating the ideas of problem solving, in particular in VI Region,[6] with more than 800 teacher participants there, helping to recruit teachers for PSClassroom workshops in the region, that took place during the second half of 2016. It is interesting to mention also that various PSAction workshops took place outside Chile, with around 600 teacher participants (Colombia, Dominican Repub-

[6]Chile is administratively divided in 15 regions. VI Region is located to the South of Santiago.

Table 14.4 Number of participants in the three workshops during 2013–2016

Workshop	2013	2014	2015	2016	2017	Total
PSAction	120	140	1339	1811	–	3410
PSAction II	–	–	30	25	–	55
PSContent	18	–	80	177	385	660
PSClassroom	–	19	125	163	–	307
PSClassroom*	–	–	–	288	–	288
PSClassroom II	–	–	–	18	–	18
PSMonitor	–	–	16	43	–	59
Total	138	159	1590	2525	385	4797

*Teachers participated only the second semester of 2016

lic and El Salvador). PSContent, was offered as summer courses in January 2016 and 2017 in Santiago. These workshops had an increasing number of participants along the years, with a leap passing from 150 in 2016 to almost 400 in 2017 (see Table 14.4). The topics of these summer courses were Numbers and Operations, Arithmetic, Fractions, Geometry, Algebra, various levels of Probabilities and Statistics and Problem Solving itself. In 2016, a winter version of the workshops was offered, with a participation of nearly 40 teachers.

In Table 14.4, special mention needs to be made on the fifth row, that corresponds to PSClassroom workshops taken place in VI Region only during the second half of the year. Regarding research data recollection, 2016 was intense again, but with a team with much more experience. During this year, data collected in 2015, questionnaires, video-tapes and written material, were prepared for research and the first research results were starting to emerge, as it will be presented in the next section.

The three years of professional development described, in the context of the state funded project, allowed the consolidation of a team with potential for further development of the ARPA Initiative. This team has added various professionals such as engineers, a graphic designer and a journalist, and it has consolidated its research capability with researchers dedicated to this aspect. These three years of experience allowed for the three strategies, PSAction, PSContent and PSClassroom, to consolidate, now having well established activities and time span, appropriate materials and manuals. A workshop for the formation of monitor and PSAction II and PSClassroom were created.

In the context of a collaboration between University of Chile and University of O'Higgins in the VI Region, a vast PDP was set up during 2016. A PSMonitor workshop was delivered in the first semester, with the purpose of preparing monitors from the region, that were able to run PSClassroom workshops during the second half of the year, involving almost 300 teachers. The PSClassroom workshop with a * in Table 14.4 are precisely the workshops that took place in this region.

Table 14.5 Research design

	Teachers	Classroom	Students
PSAction	Beliefs PS ability	–	–
PSContent	Beliefs PS ability	–	–
PSClassroom	Beliefs PS ability	Teaching quality Student opportunities	Beliefs Perceptions of teachers Math performance

14.7 Research on the PDP Carried Out by the ARPA Initiative

Whether the PDP carried out by the ARPA Initiative actually achieves its goal of improving student achievements and whether each workshop reaches its declared goal are the core questions of the research carried out during 2015 and 2016, in the context of Fondef project. In what follows we present an overview of the research design, some of the obtained results and we report on some ongoing research.

Research was designed in order to find changes as a result of the PDP development along a year. These changes are to be looked in three aspects: teachers, classroom and students, and in Table 14.5 we summarize what is looked for in each case. Data is collected through questionnaires, video recording and written production of students and teachers.

Regarding PSAction and PSContent workshops, we mention two ongoing research studies, on emotions and tensions when teachers solve problems in the first session of the workshops on one hand and on the relationship between the strategies that teachers develop in solving mathematical problems, the approaches to teaching and the corresponding level of metacognitive awareness after PSContent workshop on the other hand. Research on all three workshops is still ongoing, in particular collected data from 2016 has been just incorporated into the analysis. Research results about PSClassroom workshop is the most advanced and we present it in what follows.

14.7.1 Teachers' Beliefs Changes

It is well known that teachers' beliefs are very important when planning, managing and assessing student learning (Handal, 2003; Sullivan & Wood, 2008; Thompson, 1992). For example, teacher's conception of mathematics may affect the way how teachers encourage student to work, and what they expect their students have to learn. To measure the changes in teachers' beliefs, a specially designed questionnaire was applied to participants in PSClassroom, at the beginning of the year (March or April)

Table 14.6 Structure of questionnaire

Questionnaire parts	Dimensions
Nature of mathematics	Formalist view of mathematics
	Inquiry view of mathematics
Mathematics learning	Teacher-guided learning
	Active learning
Mathematics achievement	Mathematics achievement
Problem solving	Student-centered PS practices
	Teacher-centered PS practices
	Self-efficacy in PS
	Self-efficacy in teaching PS
	Value and importance of PS

and at the end of the year (November). The final sample was composed of 80 teachers participating in 2015 version of the workshop.

The questionnaire is composed of four parts. The first three parts measure teachers' beliefs on the nature of mathematics, mathematics learning, and mathematics achievement, using the TEDS-M questionnaire (Tatto et al., 2013). The fourth part has been developed by team members as an instrument highlighting mathematic teachers' motivational beliefs and practices regarding problem solving, both doing and teaching. This questionnaire has been independently validated and the documentation is in preparation to be published (Giaconi, Perdomo-Díaz, Cerda, & Saadati, n.d.). All items in the questionnaire use Likert scales. Table 14.6 presents in more details the questionnaire structure.

The results on pre/post application of the questionnaire show statistically significant changes in teachers' beliefs about the nature of mathematics in their formalist view and beliefs about learning of mathematics in teacher-guided learning (Cerda et al., 2017). Sample teachers reduced significantly their view that mathematics achievement is a fixed condition, associated with innate abilities, gender or with a particular ethnic group. These results are consistent with changes found in teacher-centered problem solving practices dimension, where teachers reported a lower frequency of such practices after the PSClassroom workshop. Regarding the other dimensions on problem solving, significant changes were found in student-centered problem solving practices and self-efficacy in problem solving and self-efficacy in teaching problem solving also changed after the workshop. Those unmentioned dimensions did not report significant changes.

In another study (Saadati, Cerda, Giaconi, Reyes, & Felmer, n.d.) with the same questionnaire data, a theoretical model of teachers' instructional beliefs and its impact on their practices on problem solving has been constructed. Results showed that teachers' formalist view of mathematics affects self-reported teacher-centered practices while, inquire view of mathematics has a large positive impact on teachers' self-efficacy beliefs and their beliefs about the value of problem solving, where both influence teachers' self-reported student-centered practices. This flow of influence improved after participating in the PDP, as the model remained valid. These findings inform about the importance of self-efficacy beliefs in student-centered practices, which provides evidence on the key role of teachers' problem solving abilities in their teaching. Based on these findings, improvements of workshops design are suggested. In ongoing research, the authors are also interested in measuring the effect of three ARPA workshops (PSAction, PSContent and PSClassroom) in teachers' instructional beliefs.

It may be important to point out the questionnaires about beliefs and practices only collect what teachers self-report on each of the different items. There is no way to know if what teacher report is a real change or an acquired knowledge during the workshop. However, it is interesting to mention that along the workshop there is no lectures or material teachers need to learn, but all knowledge is obtained through practice and conversations with other teachers and the monitor.

14.7.2 Teachers' Mathematics Teaching Changes

A study of teachers' mathematics teaching changes after the participation in the PSClassroom workshop was made through the video analysis of regular lessons and PSAC lessons recorded in 2015. One 'regular' lesson was video recorded in March-April and a second was recorded in November, at the end of the workshop. Additionally, among the six PSAC activities carried out by participant teachers along the year, the final one, that took place in November, was considered for video analysis also. The video analysis was made with the TRUmath rubric developed by Schoenfeld (2013), which was selected since it has a clearly defined notion of *good teaching* that seemed to align very well with the notion of good teaching promoted by PSClassroom workshop (Darragh, Espinoza, & Peri, n.d.).

The results, on a sample composed of 12 teachers, showed no real difference in teaching practices pre/post PDP, being consistent with the slow nature of this type of change, in contrast with the faster changes in beliefs reported above. These results also highlight the need for the PDP for emphasizing the possibilities of transferring pedagogical ideas and technics from the PSAC style lessons to regular mathematics lessons. Furthermore, analysis of the lessons showed in general low levels of cognitive demand, student agency, and teacher use of questioning for assessment purposes. All these findings support those of previous investigations within the context of Chile. In particular, the results are consistent with the study of Felmer et al. (2015). In contrast, evidence of difference in teaching practice was found between the regular lessons and those of PSAC lessons, both recorded in November. Higher scores were found, particularly, in cognitive demand, agency and uses of assessment, three areas of most need, demonstrating quantifiable improvement. These promising results are hoped to remain consistent when applied to a larger sample and incorporating the data from 2016. See Darragh et al. (n.d.) for more details.

Peri, Darragh, and Espinoza (2016) also conducted an analysis of questioning practices of teachers using the same data described above. Questions of teachers were categorized according to: Personal experience, Concept, Solution, Implementation, Explanation, Justification and Control (Radovic & Preiss, 2010). They found that the most frequently asked questions were Solution and Control, which is consistent with results obtained by Felmer et al. (2015) and this may be indicating a generalizable characteristic of Chilean mathematics teaching. An interesting finding was the decrease in the number of Control-type questions, that could be attributed to the PSClassroom workshop, because the more engaging nature of PSAC lessons.

14.7.3 Students' Performance Changes

Changes in students' performance were studied through a test consisting of a question (or exercise) given at the beginning of the year, in March-April, and at the end of the year, in November, to all students of those teachers in the sample used for studying teachers' mathematics teaching. The test was taken individually and the question is really an exercise, not a problem in the sense defined before, so this study is not meant to measure problem solving and other related abilities, but to measure mathematical knowledge in the traditional sense. The question, that involved equitable distribution, was adjusted depending on the level of students (1st to 8th grade) and with small changes in the context to distinguish pre/post problems. For example, for 3rd and 4th grade students, the problems were the following:

Pre-problem. *Three boys bought 24 chocolates and share them in equal parts.*

Eight girls bought 48 chocolates and share them in equal parts.

Who has more chocolates, one of the boys or one of the girls? Explain your answer.

Post-problem. *Three girls buy 27 cakes and share them in equal parts.*

Seven boys bought 42 cakes and share them in equal parts.

Who has more cakes, one of the boys or one of the girls? Explain your answer.

In a work in progress, the responses made by students to these questions were analyzed using a rubric created in the master thesis of Balboa (2015), which considered three aspects. The type of representation used by the student in solving the problem: Figurative, Numeric and Verbal; the stages through which the student go when solving the problem: Work, Answer and Explanation; and the mathematical concepts involved in solving the problem: Number, Comparison (larger-smaller) and Distribution (equal sharing). The analyzed data showed that the number of correct answers and the use of mathematics strategies improved along the year. This difference remains significant if we exclude from the sample the 43 first grade students, which may distort the results since they do not read at the beginning of the year. Another result is that the number of students who answered the question correctly and with the correct explanation, from only 30 (using mainly numerical representation) to 175 in the post-problem (using both numerical and figurative representations in similar proportions).

If we consider that students learn mathematics along the year, regardless of whether their teachers participate in PSClassroom workshop, the results may not seem remarkable. However, the available data allows us to compare the work by 3rd grade students in post-problem with 4th grade students in pre-problem, obtaining a significant difference in favor of 3rd grade students. In variables Number and Distribution, there are no significant differences, but Comparison is significantly different. Also in Explanation and Answer the difference is significant and in favor of 3rd grade students. Moreover, 36.2% of 3rd grade students answered the post-problem correctly and with correct explanation, whereas only 6.1% of 4th grade students answered the pre-problem correctly and with correct explanation. We hope to have a definitive publication of these results when a more complete analysis is performed and taking into account data from the 2016 students also.

In view of all results obtained so far, it is not possible to obtain conclusions regarding the effectiveness of the PDP. Still data involves only one year of systematic workshops, results are not complete and not all them show improvement. We have found that teachers' beliefs, based on their view, changed positively in terms of their conception of mathematics teaching and learning and on problem solving ideas. The introduced PSAC allows teachers to have lessons with higher cognitive demand, agency and uses of assessment, however regular lessons did not show changes. These changes are slower and the PSClassroom may help in this direction. Students improved their performance in solving problem, but it is not possible to attribute it to PSClassroom. However, a comparison of 3rd grade student at the end of the year with 4th grade students at the beginning of the year is suggestive. Inclusion of 2016 data and other future studies are on the way to obtain more solid evidences.

In terms of the ARPA Initiative, in three years of professional development we have learned to collect data for learning about the effectiveness of the workshops and for communicating results to researchers. We have a set of instruments and we have a team able to analyze and write about the collected. In this way, a research program to go along with development has been started.

14.8 How Teachers See ARPA Several Months After Workshops

In another effort to learn about the impact of the ARPA Initiative workshops regarding teacher changes and students learning, semi-structured interviews were conducted to participant teachers of some of the workshops taking place during 2015, about six months after completion of the PDP. The interviews are important by themselves but also they may be very useful in triangulating information obtained in classroom and with student written production. At the moment of preparation of this chapter, these interviews are in the phase of analysis and research results would appear in the future.

In this section our purpose is to communicate about experiences of change in the narratives of teachers. They tell about their experiences during the workshops as they had the opportunity to reconstruct the expectations, learnings and challenges they faced, and still face, in trying to implement problem solving as a new way of developing mathematical thinking in their students. The following text was composed by ARPA researcher Luz Valoyes.

Participation in PSAction and PSClassroom seeks for a transformation in teachers' own abilities to solve mathematical problems. It is well known the terror experienced towards mathematics by many people, and the participating teachers are not the exception. The workshops provide space for teachers to confront their own beliefs about mathematics and their fears about learning, giving them a learning experience that allows them to question those fears and reaffirm their self-confidence in their mathematics skills. This experience is told by Edmundo,[7] a teacher from the southern Coyhaique, who at the beginning of the workshop experienced a rebirth of old fears in relation to mathematics:

> Yes, first nervousness, because it was something new, right? eh, but then I realized that, I was able, then, although there were problems that I was not able to solve, I had, I had … I had a feeling that, giving them a little more time, I could solve them.

In this way, ARPA workshops become not only a space for didactic learning in which a new way of developing mathematical thinking is addressed, but also they constitute opportunities to strengthen positive mathematical identities on the part of teachers (Nasir, 2002), as subjects capable of advancing complex mathematical processes and developing them in their students. From the group work and the joint discussions, teachers find a space to strengthen their mathematical skills, as Claudia, a teacher from Puerto Montt, tells us:

> Yes, I remember … eh, we did some mathematical work, we had to come up with a solution. And I do not remember what, as well as very specifically what the problem was, but I remember how we all tried to arrive, and how I realized where I was wrong and how we tried to understand the others thoughts, so to be able to understand how to get the solution.

Teacher narratives reveal an important feature of ARPA workshops in relation to their beliefs and how their participation allows them to confront deep-rooted

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[7] All names of teachers are pseudonym.

beliefs in the educational systems. For example, children are generally thought to be better at learning mathematics than girls, or that students with Special Educational Needs (SEN) cannot develop complex mathematical thinking processes. From the implementation of ARPA workshops, teachers confront these ideas, as can be seen in the following narratives, where Adriana, a teacher from Rancagua, expresses her beliefs about women's problem-solving skills and how the workshop helped her to transform them:

> If we think about females' abilities to solve problems, my beliefs did change. I think that women and men alike [are good in math]. Because, look, I have always been trying that my fourth-grade girls get better grades in mathematics…And during my participation in ARPA I realized that we all have the same abilities [to solve mathematical problems]; therefore, we all can solve any problem, so girls can also do it.

Teachers also recognize that ARPA strategies enables and motivates more and better opportunities for participation in mathematics classes for those students who have difficulty engaging in mathematical practices because of some marginality condition. Such is the case of immigrant students, a population that has grown in recent years in Chile and which generally arrives looking for better living conditions. The following narrative is expressed by Carla, a teacher in Santiago, who has several immigrant students in her classroom. It describes the particular case of a Peruvian student, who has had difficulty integrating into the classroom and learning mathematics, but ARPA strategies has favored him, not only in terms of his participation, but also in his process of constructing positive mathematical identities (Martin, 2013):

> He has not got along with his classmates because he is a trouble-making. He always wants to be right. However, in mathematics and particularly during the ARPA classes he does participate, he raises his hand, he wants to draw. He wants to do everything. I think this is because he knows I will not say [his answer] is wrong or right. And like, in ARPA we do not say if [the answer] is right. I may ask them: there would not be another way [to solve the problem]? Did you check if you got the same number? I do it in that way and that has helped convince him.

As with all PDPs, teachers also experience challenges to implement the proposed changes. Such challenges are related to the dynamics of schools, as well as their own experiences and beliefs. In the narratives of the participant teachers it is possible to identify these challenges and how they have to face them to enable problem solving activity with their students as proposed by ARPA. One of the difficulties teachers point out is curricular constraints to advance the implementation of a strategy such as problem solving in the perspective proposed by ARPA. Although problem solving is proposed as one of the skills to be developed during schooling along the country, teachers struggle between covering content as proposed in the curriculum and textbooks and opening spaces for implementing a proposal that they find beneficial and whose aims is, precisely, to strengthen this ability. This dilemma is reflected in the following narrative of David, a teacher from Iquique:

> At the end of last year, I worked a little. To the extent that the program and contents allowed me. With few problems, it did not work … I cannot say that I worked 100% but I tried to include them in a part of the planning.

Finally, and related to the above, participant teachers find that time is a factor that limits the possibilities of implementing the ARPA strategy with their students. Paula, another teacher from Iquique states that:

> So, for example, as weakness could be that, many times it is not enough for everybody. Time is not enough, because those who are already finished start talking, you give them another exercise, but, but it is not enough then '"prof, you did not get to me"', '"but, let us see it in the next class"'. Then, time is the problem.

Thus, the successful implementation of ARPA strategies is faced with institutional constraints such as curricular organization, distribution of learning time, school modes or the mandatory use of certain textbooks that do not allow the development of other kinds of activities. Likewise, the narratives of participating teachers confirm the challenges related to their beliefs about their role in students' learning of mathematics. Although they recognize the benefits of ARPA strategies, they are constantly debating how to implement a strategy in which they must give up the leading role to their students. The above narratives are important elements for the strengthening of the proposal, as well as to measure the impact it has on the daily work of teachers.

14.9 What We Have Learned from Research and Practice

Three years of professional development practice and research results led members of the team naturally to propose changes in various ideas, concepts and procedures. To continuously introduce changes to modify what is not working and to deepen what is working is part of what the ARPA Initiative has as a characteristic: being flexible in the way to reach the goal.

14.9.1 Small, but Important Issues

When a new PSClassroom workshops starts, teachers and school leaders are informed about the workshop, duties, activities and outcome for them and students. However, experience tells that more information and reflection is badly needed. For this reason, we have devised an *Induction Event*. With participation of teachers and school leaders, in this event a presentation of the workshop, a problem solving block, and enough time for discussion and reflection is provided. Thus, they will have the opportunity to ask questions, express doubts and beliefs.

In another direction, practice tells that teachers do not adequately comply with planning reports before PSAC is implemented. Moreover, these planning reports are a source of displeasure, demotivation and sometimes annoyance on the part of teachers. On the other hand, an analysis of the quality of reports shows they are generally poor and repetitive. For these reasons, in future workshops we strengthen the planning instance that takes place during the session, so teachers complete it during the session, and they do not need to up-load it for review as before.

14.9.2 Tools Used in PSAC

One of the strategies used in PSClassroom to introduce problem solving as a regular mathematical practice in classrooms, a declared goal of the workshop is PSAC, a lesson where student experience problem solving. Research results by Darragh et al. (n.d.) and Espinoza, Darragh, and Peri (2016) show that in a way we have succeeded, but the same studies show that the good ideas of PSAC are not extended to regular classes. In order to stimulate these changes, we are introducing a subtle conceptual change so that PSAC is distinguished from the tools used in it. Thus, with suggestions during the sessions, we expect teachers start to use tools in other mathematics lessons, even in other subjects. Among the tools we may identify we have: random group work; teacher proposes challenging tasks; students start activities in an autonomous way; teacher interacts with students through questions; teacher asks questions for students to extend their reasoning; intentional selection of students presenting in plenary discussion; and teacher encourages dialogue among students. We expect that this modification in the conceptual plane, will suggest teachers to introduce changes to improve their regular classes which, as it is known, is a slow process anyway and depends on many factors.

14.9.3 The Case of Students with Special Educational Needs

During the last years, a national educational program called School Integration Program, stimulates the integration of students with special educational needs into regular schools. This program provides one special education teachers in each classroom where there are students with special educational needs. In this way, the regular teachers work together with the special education teacher (or should) to provide learning opportunities for all students. Those special education teachers, participating in this integration program, often want to participate in our workshops. Our policy is to accept them in PSAction workshops without restriction, but in PSClassroom we accept them only if the regular teacher to whom they accompany also participate in the workshop. During the sessions along the year, they do all activities as all teachers and during the execution of PSAC will share the class with the regular teacher, managing student work in equal conditions. Students will work in random groups, including students with special educational needs. During 2016 we have had numerous reports from teachers regarding how positive this modality has been for these students. One of these reports is published on the ARPA website (see http://arpamat.cl/?p=383) and other gave rise to an ongoing investigation, in which a student with special educational needs is being interviewed, their parents and her teachers, to build a case study. This situation should be the subject research in the future, because of the importance it has in the issues of inclusion and integration.

14.10 What Are the Main Challenges for the Future?

The main challenges that the ARPA Initiative is facing are related to scaling and sustainability, as any PDP that wants to impact in the educational system. Even though the initiative has not set a goal regarding number of teachers or schools to be involved in the workshops in a systematic change program, the needs of the system, the encouraging enthusiasm of some teachers and school leaders, and the suggestion from research lead the team to look for a consolidation and expansion in the future.

14.10.1 Education of Monitors

One of the main roles in our PDP is played by the monitor, which is the leader of the workshop and the model to be followed by teachers. The experience with the expansion of the ARPA Initiative to VI region has shown that the formation of local monitors, capable of leading the workshops and maintaining the integrity of ARPA principles, is a complex task that goes well beyond the PSMonitor workshop, needing a longer formation process. The challenge then is to devise and practice a monitor education model that adequately balance the need for maintaining integrity, the regional characteristics and the individualities of the monitors themselves. This model has to be flexible, it should consider research knowledge and it has to be developed along practice.

14.10.2 Sustainability of Changes in Schools

The work of the ARPA Initiative with an educational foundation, that run about a dozen of schools for more than three years, has shown that to give changes sustainability it is necessary to achieve the commitment of the school leaders and to institutionalize problem solving in the local school curriculum. In collaboration with this foundation, we are carrying out a project to incorporate in the local school curriculum, the implementation of PSAC on a regular basis. However, this work with a single foundation cannot be replied similarly with a higher number of schools. The challenge here is to develop the capacity to create networks with school leaders, for incorporating problem solving in their local school curriculum, based on the work with the foundation. The networks and the model have to be very flexible to deal with the special characteristic of each school and to keep the integrity of the program.

Both for monitor education and sustainability of school changes, research will be need for informing about the advances and suitability of the models and for communicating with researchers.

14.11 As a Way of Conclusion

This chapter has been devoted to describe the adventure of the ARPA Initiative as a research and development program. Our purpose has been to let the research community to know about this initiative and to call the attention on some important issues we have encountered on the way regarding professional development, problem solving and its installations in schools. Most of the ideas we are using in setting up our workshops are known, problem solving itself is the main activity of mathematicians and in education it has been more than 80 years, but perhaps originality have to be found in the way of putting them together to make up a professional development initiative, the way it is developed and the place where it takes place.

To conclude we would like to state some general guiding ideas we have taken into account or we have learned about in putting in practice the ARPA Initiative during the last five years:

- While centrality of problem solving in mathematics is obvious, its centrality in school mathematics has a long way to go.
- This centrality in school mathematics cannot be reached if teachers are not problem solvers.
- The introduction of problem solving in classrooms may provide teachers with valuable tools for improving their regular teaching.
- Problem solving may be the basis of a professional development program having teachers to solve problems and, in parallel, introducing problem solving in classroom.
- Any professional development program having the goal of changing school practices for improving students learning with success should have the aspiration of scaling up.
- Scaling up a program requires the creation of models for replication that have simultaneously consider different realities and a way to keep its integrity.
- Provoking changes in school practices with success creates the need of sustaining them along time.

But the most important guiding idea for the ARPA Initiative, above all the previous ones, is *flexibility*. There are no two equal teachers, classes, communities, schools, towns, cities, etc., so any program aspiring to scale up should be extremely flexible, in their methods, workshops, team, etc.

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Chapter 15
Understanding the Sustainability of a Teaching Innovation for Problem Solving: A Systems Approach

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15.1 Introduction

As early as the 1990s, the central theme of the primary and secondary mathematics curriculum in Singapore is Mathematics Problem Solving. In particular, the syllabi document published by the Ministry of Education identifies that Mathematics Problem Solving as the central theme because it presents an opportunity for "acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems" (MOE, 2007, p. 3). International comparative studies like PISA (Program for International Student Assessment) and TIMSS (Trends in International Mathematics and Science Study) have revealed that Singapore has achieved a high level of competence in mathematics in schools. Despite this relative success of Singapore mathematics instruction, studies have also noted a relatively weaker performance of students in solving unfamiliar problems (Kaur, 2009). Since the overarching aim of the Singapore mathematics curriculum at all levels of schooling is the development of Problem Solving (PS) ability, continued research in the development of PS in school mathematics is important to support classroom practice or inform curricular policy with research-based evidence better.

As to a practicable direction for PS research, Schoenfeld (2007) called for a concerted effort to translate decades of theory building about PS into workable classroom practices:

> That body of research—for details and summary, see Lester (1994) and Schoenfeld (1985, 1992)—was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved. … The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms. (Schoenfeld, 2007, p. 539)

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This was a reasonable direction in the Singapore context since numerous local studies (see for example Foong, 2009; Foong, Yap, & Koay, 1996) also attested to how PS was mostly theoretical talk but not common as classroom enactments.

We identified three major steps needed to realize the hard and unglamorous work of making PS instruction a staple in Singapore classrooms: (1) initialization of PS as an essential part of the mathematics curriculum in a school at a foundational year level; (2) infusion of PS as an embedded regular curricular and pedagogical practice across all year levels in the school; and (3) diffusion of this innovation from this school to the full range of schools in Singapore. Guided by the principles of design experiment (Middleton, Gorard, Taylor, & Bannan-Ritland, 2006), we embarked on the MProSE (Mathematical Problem Solving for Everyone) project. Focusing on the initialization step and the creation and trialing of a teaching innovation for PS, MProSE produced a redesign of the curriculum, assessment and teacher development structures for schools. Successful implementation was carried out in a high performing school called the initial school. The PS curriculum and the research outcomes of MProSE were reported extensively in Leong et al. (2011) and Leong, Tay, Toh, Quek, and Dindyal (2011).

A second project, MProSE: Infusion and Diffusion (MInD), launched in 2011 focused on the next two steps towards making PS instruction ubiquitous in Singapore classrooms. Built on MProSE's success in the initial school, MInD aimed to diffuse or scale up the innovation to schools catering for students across the ability spectrum. To that end, four mainstream schools (A, B, C and D) participated. These mainstream schools have some differences. 'A' is a high-performing mixed-gender school that runs an Integrated Programme, i.e., unlike in other secondary schools in Singapore, students in School A do not sit for the common 'O' level examination at the end of Year 10 that determines if students proceed to take Year 11 and 12. Instead, students in 'A' take a six-year program (Year 7 to Year 12) and will only sit for common Advanced Level examinations at the end of Year 12. 'B' is an all-girls school, while 'C' and 'D' are mixed-gender schools. Based on Ordinary Level examination results, 'B', 'C' and 'D' were consistently ranked in the top, lower and middle-tier (respectively). The MProSE design was adapted for use in these schools and implemented in the 2012–2015 period. After MInD ended, schools continued the PS teaching innovations that the project espoused. (From here onwards, we refer to this set of teaching innovations as "MInD" for convenience.) Upon follow-up, we found that MInD continued to thrive in all but one school. Adapted in ways relevant to 'A', 'B' and 'C', MInD became integral in teaching and learning mathematics. But this was not so for 'D', where MInD became confined to an enrichment program exhibited various "ailments".

Backed up by the positive feedback that all partner schools have expressed (see Leong et al., 2014), we believe that MInD is a promising innovation for realizing the ideals of positioning PS at the heart of school mathematics in every classroom. Further scaling-up MInD is something worth investing in. However, it would be prudent first to gain a better understanding of what sustained MInD in 'A', 'B' and 'C', but not 'D'.

This chapter aims to find out what are the crucial factors that affect sustainability of MInD as a teaching innovation across the various school systems. Existing research works carried out in diverse fields such as engineering and health-care, set out to identify those factors influence or inhibit sustainability of professional training, quality improvement or school-based health programs (Han & Weiss, 2005; Scheirer & Dearing, 2011; Shediac-Rizkallah & Bone, 1998; van Dyk & Pretorius, 2014; Vaughn, Klingner, & Hughes, 2000). One common theme threading through these works is the application of "Systems Thinking" as a way of understanding and studying program sustainability. Cognizant that systems approach is scientific method widely used in many different disciplines, we choose a more specific variant of the systems approach described in Berger and Brunswic (1981) that is best suited for analyzing educational contexts—Berger and Brunswic's systems approach has also been used in Gupta and Gupta (2013). In particular, we apply systems approach to compartmentalize and visualize the various components and interactions involved in studying the sustainment of MInD. We build a model for understanding how sustainability of MInD in mainstream schools is facilitated or impeded. We then demonstrate the usefulness of the model as attested by experiences of 'A', 'B', 'C' and 'D'. Finally, we discuss how the model may be used as a diagnostic tool for initiatives that propagate teaching innovations in PS.

15.2 Sustainability and the Systems Approach

For the past decade, research-based instructional innovations implemented in schools represent concerted efforts of collaboration between education authorities, schools and education-researchers. Most of these are research projects financially supported by state or institutional research grants or other sources of funding external to the school system that are no longer available after the research project is completed. Hence, program sustainability in the sense expressed by Han and Weiss (2005) is a legitimate concern, i.e., will there be continued implementation of an instructional innovation that stays faithful to its core design principles even after the withdrawal of the resources used to support initial training and implementation? As it is, Hogan and Gopinathan (2008) stated that

> instructional innovation is technically difficult and emotionally demanding, institutionally challenging, and risky for both teachers and schools since innovations often fail, and hard to sustain, and hard to scale-up. (p. 375)

These difficulties and challenges are to be expected since the modus operandi of these research-based instructional innovations manifests at three levels (see Florian, 2000; Gersten, Chard, & Baker, 2000; Huberman & Miles, 1984; Klingner, Arguelles, Hughes, & Vaughan, 2001; Vaughn et al., 2000): (1) Policy level—the identification of an educational policy advocated at the level of the district or country. (2) School level—the school's implementation of certain teaching innovations in response to the identified policy. (3) Program level—the teaching innovation itself, i.e., its design

Table 15.1 Generic components of a system based on Berger and Brunswic (1981)

Component	General description of component
Product (P)	Outcomes that result from the activities that take place.
Input (I)	Entities that are fed into the system and operated upon; these may be inputs of the previous situation that might include the system itself.
Resources (R)	Entities other than the input that are utilized to ensure the operation of the system.
Constraints (C)	Internal and external conditions imposed on the system, general conditions governing the institutional, social, cultural, economic environments. Note that the terminology of "Constraints" in Systems Approach is used in a neutral sense, and hence does not carry a negative connotation.
Strategy (S)	Organization of various components under given constraints to achieve optimal output.
Feedback & evaluation (FE)	The return flow of information back into the system.

principles, the necessary professional training, the intended learning outcomes, etc., as proposed by researchers.

To formulate the desired conceptual framework for program sustainability of instructional innovations, we use the metaphor of a system to represent the complex network of relations that exist between various agents. Here we follow the systems approach a lá Berger and Brunswic (1981): a system is defined by its components, and by their interrelationships. Table 15.1 lists the generic components of a system together with their general descriptions.

15.3 A Systems Model of Program Sustainability for MInD

In this section we follow Berger and Brunswic (1981) closely in giving a systemic description of program sustainability of MInD in a mainstream school: first the Program Level, and then the School Level. Suffice to say at this point that Singapore's Ministry of Education already recognized PS as the central theme in the Singapore mathematics curriculum since the MInD project clearly aligns with this policy, we omit Policy Level articulation here. To build the desired system, evidence found in the teachers' reports in the project schools concerning MInD at the end of its first year of implementation is used. These reports are presented as chapters in Leong et al. (2014).

15.3.1 Program Level

The Program Product (PP) comprises of both students and teachers involved in the PS lessons. PS students become effective mathematics problem solvers by internalizing Pólya's PS paradigm, i.e., the four-stage Pólya's (1957) PS model: Understanding the Problem, Devise a Plan, Carry out the Plan, Check and Extend, and by using PS heuristics in their PS endeavours. Thus, with regards to the PS students, the product comprises episodes of visible success, i.e., a sense of 'success' that is visible (that is, observable achievements or benefits) to teachers (Tirosh, Tsamir, & Levenson, 2015), change in the students' PS behaviour, i.e., changes in PS competency, PS disposition, and beliefs. PS teachers experience changes in competency in PS and teaching PS, and make adaptations appropriate to the school setting while maintaining program fidelity of MInD. The body of Product, comprising the preceding items, is given in the box on the *right* side of the systems diagram in Fig. 15.1. The items on the *left* side of the systems diagram concern Program Input and are labelled (PI1) and (PI2); we shall come to Program Input later.

Within the body of Product, there are dependence relations between some of the outcomes (see PP1, 2 and 3 in Fig. 15.1). We picked up a particular episode of visible success in one of the four project schools:

> Most students gave feedback that they enjoyed the lessons as they stimulated their thinking. The skills they learnt helped them with other mathematical problems. (Leong et al., 2014, Chap. 5, p. 89)

In Fig. 15.1, we use (X) → (Y) to denote (Y) depends on (X). For example, (PP1) → (PP2) indicates that (PP2) depends on (PP1). More precisely, visible success, such as this one, resulted in students finding PS skills learnt in the PS module meaningful,

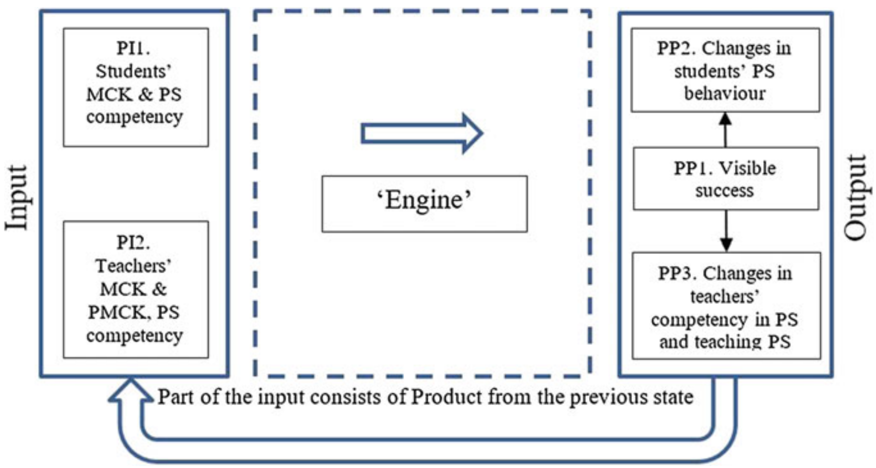

Fig. 15.1 Product and Input of a sustainable MInD (program level)

and this in turn brought about a positive change in the students' attitude towards PS. Repeated episodes of visible success improved the students' PS behavior.

Concerning the teacher-related outcomes, one of the schools reported having

> students who recognised that competence in mathematics did not only mean getting the correct solution, but rather, it also meant understanding and grasping the mathematical processes, and strategies involved as they attempted to solve these problems. … [I]t made us grow more confident about how MProSE can truly equip our students with essential skills for the 21st century. We [the PS teachers] thus took to preparing for the second implementation of MProSE… (Leong et al., 2014, Chap. 4, p. 74)

which is an instance of the dependency relation (PP1) → (PP3). Note that visible success, labelled as (PP1), amongst others plays a crucial role. Graph-theoretically, it is a source of the directed graph of Product—there are no incoming edges from vertices (PP2) and (PP3). Within the Product component, visible success reinforces both the changes in students' PS behavior and teachers' competency in PS and teaching PS. Because of its reinforcing role, one anticipates that a decline or absence of visible success will eventually lead to a decline in students' PS behaviour and teachers' competencies in PS and teaching PS, which in turn will affect the Input.

We now describe the Program Input (PI) component. The systems approach informs us that part of the input inevitably consists of the Product (P) of the system from the previous state: the PS competencies of the students and teachers are fed back as inputs as MInD progresses. In Fig. 15.1, we represent this Product-Input feedback with a thick arrow. The Input is determined partly by the level of the students' Mathematics Content Knowledge, and partly by the level of the teachers' Mathematics Content Knowledge and Mathematical Pedagogical Content Knowledge.[1]

From the set-up given in Fig. 15.1, we now turn to the 'engine' that operates on the given Input in order to yield the Product. One asks: Which entities are at play in the 'engine' part of this system? The systems approach answers this question by considering three components: the Resources, the Constraints, and the Strategies (see also Table 15.1 for their definitions).

Program Resources (PR) constitutes of three subcomponents. The first component is the MInD curriculum package which included 10 lessons, detailed lesson-plans, Mathematics Practical Worksheets and Assessment rubrics (see appendices in Leong et al., 2014; Toh, Quek, Leong, Dindyal, & Tay, 2011). The second component is all the time expended—for the students the length of the PS lesson, for teachers the actual classroom time, PS lesson preparation time and PS team meeting, the time spent with school leaders (Head of Department and principal) is part of the Resources. The third component is a Continual Professional Development package is available to ensure the continual training for new PS teachers. This training package equips PS teachers with PS terminologies, Pólya stages, Schoenfeld's (1985) framework, and the practical aspects of facilitating a PS lesson.

[1]Mathematics Pedagogical Content Knowledge refers to the "distinct body of instruction-related and student-related mathematical knowledge and skills—the knowledge that makes mathematics accessible to students" (Baumert et al., 2010, p. 142).

To understand the Program Constraints (PC), i.e., those conditions that are imposed on the system which are pertinent at the program level, it is helpful to understand this teaching innovation as conceptualized and designed by the project team to be implemented within a set of design parameters. These design parameters of MInD were set a priori. They were based on theoretical considerations developed as part of the process of a design experimentation guided by the intention of realizing the ideal of strengthening the place of PS at the heart of school mathematics. These design parameters are as follows (see Quek, Dindyal, Toh, Leong, & Tay, 2011 for details):

1. Place in the curriculum: The teaching of PS as espoused by the project must be part of the school's mainstream mathematics curriculum.
2. Model of mathematical problem solving: The model that will be used for PS will be Pólya's (1957) four-stage model consisting of Understanding the problem, Devising a plan, Carrying out a plan, and Checking and expanding. The use of the model will also be integrated with knowledge of Schoenfeld's (1985) PS framework which identify a solver's cognitive resources, beliefs, heuristics and metacognitive control as factors upon which a solver's success depended on.
3. Teacher autonomy: There must be investment in building teachers' capacity in PS and skill in teach PS. Teachers in school will ultimately teach the module themselves.
4. Infusion into regular mathematics content: Problem solving skills and habits learnt in the module must be infused into other mathematics modules to prevent atrophy.
5. Assessment of PS: PS should be a valued component in the school's assessment.

Given these design parameters, we identify the following key programmatic constraints. The first constraint was the design parameter (2) that bore upon the PS teachers to adhere to the Pólya's model of PS and Schoenfeld's framework concerning Heuristics and Control. The second constraint was that PS teachers were obliged to teach the PS module during the normal curriculum time as specified in design parameter (4). The third constraint is to grade the students' PS competencies as guided by design parameters (1) and (5) respectively. Lastly design parameter (3) is concerned with the teacher's capacity in PS, and thus a crucial constraint to take into consideration is the fourth constraint of teacher's minimum capacity of PS and teaching PS (Fig. 15.2).

Strategies are methods which are employed to work within the given constraints and resources of the system to achieve optimal product. Strategies executed at the program level is denoted by PST. The first strategy invokes *artefacts*. Artefacts refer to certain 'hardware' designed to enable processes to take place. (a) In the delivery of the 10 PS lessons, PS teachers used Mathematics Practical Worksheets to scaffold students' movement through the Pólya's PS stages (Understand the Problem-Devise a Plan-Carry out the Plan-Check and Extend). By filling out a Mathematics Practical Worksheet, students cultivated a habit of solving an unseen problem via these four stages. (b) *Assessment rubrics* are provided in the MInD package for teachers to grade students' PS work. (c) *Physical props and auxiliary materials* can be used

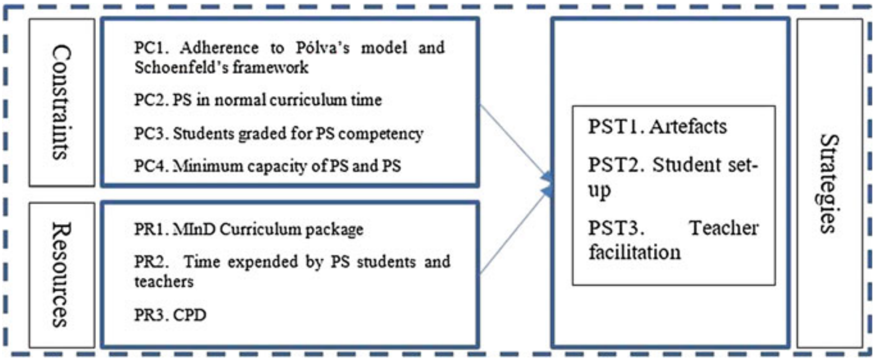

Fig. 15.2 Components of resources, constraints and strategies

to enhance learning experience. The second strategy makes use of *student set-up*. Student set-up refers to the physical set-up during the PS lesson. Each problem is solved in pairs in order to work on a larger resource of MCK with the additional awareness that such a PS team cannot have more than three members to reduce the chance of 'free riding'. The third strategy involves *teacher facilitation*. PS teachers facilitate the PS endeavors closely by monitoring the PS processes in each group, and intervening whenever necessary without giving away the solution directly.

We now turn to the component of Feedback and Evaluation (FE). From Product, we evaluate and feedback into the engine. As a result, strategies may change and resources may be re-deployed, given that little can be done to change the given constraints. During the PS lessons, students' PS behavior was exhibited through ongoing verbal and non-verbal responses between the students and the teacher, and among students themselves. Their PS behavior provided formative feedback to PS teachers. Such formative feedback then changed the style of teacher facilitation. Feedback and evaluation also included teachers' *modification*, *adaption* and *invention* (which remained faithful to the design principles) with the aim of improving the quality of the PS lessons. For example, teachers made physical props and manipulatives, re-designed parts of the Mathematics Practical Worksheet, made new problems that tie in better with the year level's mathematics syllabus, and changed the venue, time-duration and schedule of the PS lessons.

15.3.2 School Level

As Owston (2007) conveyed, "innovation benefits from leadership and a supportive organizational environment" (p. 62). Hence, it is important to study the sustainability of MInD at the School Level. In Fig. 15.3 we illustrate the relation between the School Level and the MInD Program Level. We elaborate on the School Level components as follows.

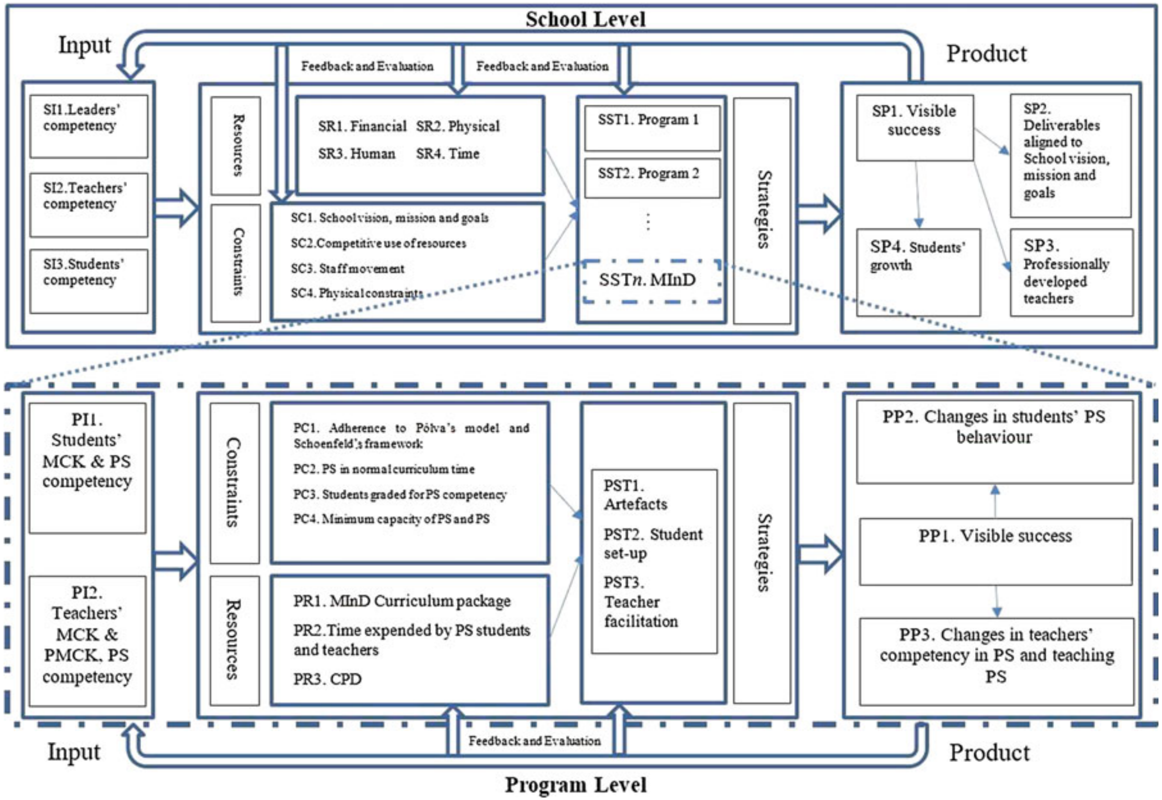

Fig. 15.3 Systems model of sustainable MInD: program and school levels

We begin with the Product component (SP) first. Similar to the Program Level, *visible success* is a very influential form of Product. The Focus Group Discussion carried out with the initial school in which MProSE was a success provided evidence about the importance of visible success:

> The initial school maintained the PS module as an examinable 9 lesson module in its modular curriculum. The module was also endorsed by their parent university as well as the Ministry of Education: "We have this curriculum review by [the university] as well as MOE … three or four years once. So I mean, recently, they went through our materials once, they saw this problem solving so they are also very encouraged by it. And they also hope to see it being infused in other modules as well. So I think, with this documented, yah, it should be staying for quite some time, you see, because this accreditation process is very important for us, especially we don't follow the mainstream syllabus." (Focus Group Discussion data from initial school)

In other words, the benefits of the innovative program must be seen by the school leaders and the policy makers so that resources can be subsequently channeled into keep the program alive.

Repeated episodes of achievement at the School Level would bring about *deliverables* that were aligned with the school vision and mission, short-term and long-term goals and their related Key Performance Indicators. One of the project schools, for instance, set the vision of developing students' character and capacity for life-long learning so that they are ready for the 21st century. This school recalled that it bought-in the ideals of MInD because what MInD proposes to produce aligns with the school goals:

> … we thought that it [MInD] could help us prepare out students for the 21st century by offering much more than what was promised in the theory. Its approach was strongly aligned with our school's holistic goals of character development and nurturing creative problem solvers and self-directed learners, preparing students to better face challenging problems in life and serving the community through problem solving. (Leong et al., 2014, Chap. 4, p. 62)

After the first round of implementation of MInD, positive changes in students' PS competency occurred, and these changes were seen to align with the school's vision and thus further reinforced the trust this school put into the teaching innovation:

> [MInD] can develop character and build lifelong skills and appreciation for mathematics. (Leong et al., 2014, Chap. 4, p. 73)

Visible success also resulted in an increase in *professionally developed teachers* who are prepared to teach students ready for the 21st century; specific to the MInD program, this translated to the outcome of PS teachers modelling as effective problem solvers and displaying competency in facilitating PS lessons. As for the students, visible success correspondingly yielded positive students' growth, e.g., enhanced mathematical abilities.

The Input at School Level (SI) comprises *leaders' competency*, *teachers' competency*, and *students' competency*. Here we refer to the general competencies (e.g., academic, administrative, etc.) associated to the respective roles of school leaders (Principal and Heads of Department), teachers and students.

School Level Resource (SR) refers to the *financial*, *physical*, *human* and *time* resources available to the school.

There are four types of Constraints at the School Level (SC), which we now describe in turn. *School's vision, mission and goals*: The top-management team set the school's vision, mission and laid down the short-term and long-term goals. *Competing use of resources*: Limited resources had to be shared among several programs and activities taking place simultaneously in the school. *Staff movement*: Internal and external movement of staff would take place constantly. *Physical constraints* include class size in a PS lesson (about 40 for each project school), physical setting, e.g., typical classrooms in Singapore schools are non-air-conditioned, hot and humid; in addition, long school hours often precede PS lessons, and so students feel tired and uncomfortable even before the start of the PS lesson.

The School deploy Strategies (SST) in the form of different programs in order to realize the goals set by the school and to produce the stated Product. In Fig. 15.3, the reader may wish to note that MInD may well be one of the many programs that are running simultaneously in the school.

Feedback and evaluation at the School Level (SFE) refer to information flowing back from the Product to the Input and the 'Engine'. Unlike the Program Level, the school leader when responding to the feedback and evaluation can make changes to alter certain constraints. Here, we focus on the internal information that arose from data collected from activities taking place in the school, meetings with teachers, lesson observations, and so on.

15.3.3 Interaction Between Program Level and School Level

Here we use the term 'transversality' to mean the 'criss-crossing' relations among different levels of the same system (see Berger & Brunswic, 1981, p. 24). Between the components operating at different levels, dependence relations act as channels for flow of information, activities, resources, etc.

Elsewhere in the literature of educational research (Kozma, 2003), the levels of program, school and policy are given different labels, namely, micro level (program), meso level (school) and macro level (policy: community, national, global). Following Kozma (2003), School Level subsumes Program Level in the sense that MInD, when implemented by a school, sits inside the School Level system as one of the many programs adopted by the school; it is to be viewed as a school strategy that aims to achieve the goals set by the school.

The Program Strategies employed by MInD generates visible success (see PP1 in Fig. 15.3) which is a critical vertex connecting the 'Engine' to Product, and also the sole source vertex of the Product component. Visible success of MInD yielded observable improvement in the students' PS behavior and the teachers' competency in PS and its facilitation. As a School Strategy, MInD's Program Product contributes towards the School Product. The overall positive changes in students and teachers become School visible success (see SP1 in Fig. 15.3), which is translated into concrete deliverables aligned with the school's goals. Elaborating on the point, consider one of the project schools. Prior to the implementation of MInD, this school already recognized that "[t]he 21st century knowledge-based society calls for mathematics education to train learners to be flexible, creative, confident and good team players who are able to solve new problems and deal with ambiguities" (Leong et al., 2014, Chap. 4, p. 61). After the students from this project school attended the PS module, they took an MInD Practical Test in which the average achievement of the students exceeded 80%. In the words of the Head of Department of mathematics of that school,

> We found these results encouraging since the [MInD] rubric placed premium on students' thinking processes and their efforts to extend their thinking through the Pólya's fourth stage. Thus, their performance in [MInD] reflected how students were beginning to develop more holistically by moving away from exam-oriented learning. (Leong et al., 2014, Chap. 4, p. 65)

The students' good performance, together with positive changes in students' PS behaviors, constitutes School Level visible success, and these affirmed the school leaders that the deliverables generated by MInD indeed met the school's goal within a short time—a form of School Level FE from Product to Constraints. School leaders' decisions directly influenced Competitive use of resources and Physical constraints favoring MInD. Initially, the PS module was run after school hours and thus PS was not run within the school curriculum (refer to PC2 in Fig. 15.3). But as a direct result of the School Level intervention, the PS module was allocated normal curriculum time the next year. Hence this change brought about the positive consequence of students and teachers not needing to stay back for extra hours after school. This time efficiency then produced more Program Level visible successes, and eventually more School Level visible successes. The flow becomes self-reinforcing and self-sustaining. In summary, our model (see Fig. 15.3) gives a systemic characterization of program sustainability for MInD: the system must operate in such a way to produce a significant level of visible success, at both the Program and School Levels, in order to secure continuous commitment and support from the School leadership, and thus achieving a self-sustaining and perpetual flow of processes and information between various components.

15.4 Method

We report on the qualitative feedback given by PS teachers and subject leaders of the Schools A, B, C and D at the end of the MInD project. All four mainstream schools submitted their detailed documentations of MInD's operation log after the project ended. To complement data from the written documents, the project team also conducted Focus Group Discussions. Focus Group Discussions collected at the early part of 2015 served as main field data. The Focus Group Discussions guiding questions covered teachers' personal beliefs about PS and teaching of PS, teachers' portrait of a PS lesson, ingredients for successful PS lessons and sustainment of MInD. Each

Focus Group Discussion, lasting 1.5–2 h, was facilitated by at least two researchers and videotaped with permission. The videos were all transcribed and analyzed by two researchers independently to surface relevant issues and implications towards answering the infusion and diffusion research questions of the MInD project, as well as finding out about the participants' perception regarding the factors contributing to the sustainability of MInD. The Focus Group Discussions conducted in the January 2015 were attended only by 'A', 'B' and 'C' but not 'D'. A separate Focus Group Discussion with 'D' was conducted in November 2016 and the data coded by two independent researchers. With these Focus Group Discussion data, we analyze program sustainability of MInD via the systems model previously derived. We report that 'A', 'B' and 'C' conformed to the program sustainability as described by the model; on the other hand 'D' displayed certain symptoms of faltering sustainability as diagnosed by the model.

15.5 Summary of Data and Findings

We begin by comparing, at the School level, the implementation of MInD across all the four schools, namely, A, B, C and D. Table 15.2 makes such a comparison in terms of the interactions taking place between the components:

We now compare the implementation of MInD, at Program level, for these same schools. Table 15.3 displays this comparison, again, in terms of the interactions between the various components.

With regards to the implementation at Program level, we provide additional details below.

- All four schools in the discussion kept the PS module that taught the students about PS and introduced Pólya's PS model.
- 'A' and 'B' reduced the number of lessons to 7 and 6 respectively, while 'C' modified the module to 8 lessons. The modifications were made to align with their tight schemes of work. In 'D', the school leaders moved the PS lessons out of the main curriculum time into an enrichment lesson after school hours.
- The language of PS (understand the problem, heuristics, being stuck, etc.) permeated into many other mathematics lessons in 'A', 'B' and 'C'. 'B': "And we also come out with an explicit way of maths language for our teachers so that for every problem that you do in class, you will do the same, 'how do you understand the problem', 'what is given', so we have a language for the teacher to help them to facilitate as a way for PS, as a whole curriculum be it in the syllabus or MProSE" (Focus Group Discussion data from 'B').
- 'A' suggested that teachers in pre-service training should learn how to teach PS: 'It will be definitely useful if fresh teachers start in this knowledge already." (Interview record taken from Focus Group Discussion with School A)
- 'A' and 'B' reported that their teachers work on solving new problems themselves. 'A': "Besides the teacher teaching this, we also included the PS sessions in the

Table 15.2 School-level comparison of implementation of MInD across four schools

Components	School A	School B	School C	School D
(SI1), (SC1, SC2), (SR) & (SP2), (SP3): School leaders' competency; School vision, mission & goals; Competing use of resources; School Resources & Deliverables aligned with school vision, mission and goals, Professionally developed teachers	Freedom granted by school leaders to plan and implement the mathematics curriculum from Years 7 to 12 Department leaders take a long term view of Continual Professional Development for PS. They recognize that building an PS culture takes a long time.	Need to convince each new school leadership the need for special provision to support curricular re-design for incorporating PS into lessons Head of Department takes a long term view of Continual Professional Development for PS	PS in line with school leaders' focus on quality of classroom instruction; PS also aligns to Ministry of Education's emphasis on "disciplinarity" as a 21st Century competency Head of Department is committed to PS as a mainstay	PS is not in line with school leaders' focus; new directions displaced PS from its mainstay. School leadership dictates timetabling, and PS is removed from main curriculum time, and placed as an enrichment lesson
(SI1), (SC3) & (SR3) School leaders' competency; Staff movement; Human resource	Stable and committed core of Heads of Department and Level Heads across Years 7 to 12; Communication between Head of Department and Principal concerning the importance of PS	Stable and committed core of Head of Department and Level Heads across Years 7 to 10. Kept the course after 2 changes of school leadership; Communication between Head of Department and Principal concerning the importance of PS	Stable core of departmental leadership that consists of the Head of Department and a Level Head; Communication between Head of Department and Principal concerning the importance of PS	Change of Principal and Head of Department, and Level Coordinator for PS. PS is not a KPI, and redeploy resources to other areas; Communication between Head of Department and Principal concerning the importance of PS was not so effective.
(SR4), (SC2) & (SP3) Time resources; Competing use of resources & Professionally developed teachers	Mathematics teachers support many programs offered by the school; hard to find a common Professional Development slot for mathematics teachers. They meet to solve mathematics problems	Weekly Professional Development slot allocated. Special provision for mathematics department.	Weekly Professional Development slot allocated. Mathematics teachers meet within their Year level teams to discuss improvements to teaching, including refinements to PS instruction	

Table 15.3 Program fidelity: comparison of implementation of MInD across four schools

Component	School A	School B	School C	School D
(PC1) & (PST3) Adherence to Pólya's model and Schoenfeld's framework & Teacher facilitation	Explicit coverage of Pólya's stages, heuristics, and Schoenfeld's "Control"; Introduced in Lesson 6 together with students' task. Continued in Lesson 7	Explicit coverage of Pólya's stages, heuristics, and Schoenfeld's "Control"; Introduced in Lesson 3 and incorporated in subsequent lessons	Tasks given to students included Stage 4 from Lesson 2	Exact copies of MProSE slides that explicate Pólya's Stages and Schoenfeld's framework; Tasks included Stage 4 from Lesson 5; Less stress on Stage 4
(PC2) & (PR2) PS in normal curriculum & Time expended by students and teachers	Placed in normal curriculum time; Terms 2 and 3 of Year 7; 7 this year, intends to stabilize at 8 PS lessons, each 45 min	Placed in normal curriculum time; Terms 1 and 2 of Year 7; Stable at 6 PS lessons, each 45 min, "minor tweaks" in future	Placed in normal curriculum time; First three terms of Year 8; Stable at 8 PS lessons, each 45 min	Not in normal curriculum time—Enrichment; Terms 1 and 2 of Year 7; Changed to 8 PS lessons, each 90–120 min
(PC1) & (PST1) Adherence to Pólya's model and Schoenfeld's framework & Artefacts			Emphasis on the use of Mathematics Practical Worksheets that indicate Pólya's Stages and heuristics	
(PC3) & (PST1) Students graded for PS & Artefacts	Within-module, end-of-module, in year-end exam	End-of-module, in year-end exam	End of Term 3	End of module; Assessed but did not count towards the final grade of the students
(PR1) & (FE) MInD Curriculum Package & Feedback and Evaluation in the form of Modifications, Adaptations and Inventions	7 from MProSE, 4 from the school; Teacher modelling of PS habits in regular teaching of mathematics; to do so, teachers must be habitual solvers of mathematics problems	7 from MProSE, 1 from the school; Redesign regular units of instruction that incorporates PS. Currently three units in Year 8. Plan to have one such unit every term for Years 7–9	6 from MProSE, 3 from the school; Teachers of other Year levels use the language of PS to reinforce the processes taught in the Year 8 PS module	8 or fewer from MProSE; remove "abstract problems"

protected time for the teachers to meet. It is to create and inculcate the culture of PS among the teachers. So we have it once every term, problems will be posted to teachers for them to try it out using the problems solving strategies" (Focus Group Discussion data from 'A').

- 'B' worked with the researchers to redesign 3 units in the mathematics curriculum to include PS as a pedagogical approach. The redesigned units are called Replacement Units or RU's. "And because we have done it in Secondary 1 as a whole module and in Secondary 2 we do the RU, the Replacement Unit. So when we teach the topic with the MProSE problem, either in the beginning, in the middle or towards the end also. So it's an infusion of MProSE. So they still keep in touch with it" (Focus Group Discussion data from 'B').
- Assessment of PS competency remained an important feature of MInD for 'A', 'B' and 'C'. For 'D', although the students were assessed using the Assessment Rubrics in the MInD curriculum package, the PS teachers decided not to count the PS marks towards the final grade for Mathematics since they managed to teach 8 out of the 10 lessons.
- Modifications were made to the types and difficulty of the problems.
- In 'D', PS teachers observed that students were quickly engaged in solving problems that were couched in real-life contexts, e.g., Jug Problem, Russian Roulette Problem. However, when it came to Problems that involved abstract mathematical definition, e.g. Nice Numbers, the students gave up after one or two attempts. In response to this situation, 'D' decided to remove these "abstract" problems from the list of problems in the MInD Curriculum Package.
- 'A', 'B' and 'C' used props and videos to support the PS module. 'B': "Because in terms of interest, they do not understand why they must go and pour jugs, why I must open lockers … So we came out with props, … we use videos. So we use props for the kids to understand the whole process. And to help them to be more interested and engaged." (Focus Group Discussion data from 'B'). 'A' concurred: "So props is a good idea. Because during the open house we did it like a props like that. 'Jumping Frog Problem'. They are very happy. They are able [to] see the generalization, some even can move up quite close to step 4 [Check and Extend]. They manage to see the whole thing is a quadratic equation rather than manually moving [the] frog." (Focus Group Discussion data from 'A'). In contrast, although the students in 'D' were very active in applying the heuristic of "Act It Out", no props or videos were used to support the teaching of PS lessons.

In summary, the data suggest that the PS module was stable and well-implemented in 'A', 'B' and 'C' based on the five design parameters. 'A', 'B' and 'C' have supported the module with teacher development materials and teachers' personal work on PS. The language of PS was infused into different levels and topics of the school mathematics curriculum. Notably, 'C' had used a PS approach to revamp some of its hard-to-teach topics. There was a time lapse of about one year between the submission of detailed documents and the conduct of Focus Group Discussion in 'D'. The Focus Group Discussion informed us that 'D' had started to deviate in their implementation of MInD from two of the design parameters: (i) PS in normal curriculum time: The new school leadership moved the PS lessons out of the normal curriculum time into an enrichment lesson slot outside school hours, and (ii) Assessment: The PS team decided not to include the PS scores of the students into their final mathematics grade. These two fundamental deviations alerted us that 'D' was starting to display symptoms of an "ailing" MInD.

15.6 Discussion

Zooming out from the details presented in the previous section, we now make sense of program sustainability of MInD at a higher vantage point by taking a systemic perspective offered by the Systems Model in Fig. 15.3. The ideal state of sustainability for MInD occurs when there is an unobstructed and self-sustaining flow of processes, information and activities that relate various components at both the Program and School Levels. The Systems Model we have constructed highlights visible success as the salient part of the Product both at the Program and the School Level. Visible success at the Program Level brings about positive changes to both the students' and teachers' competencies with regards to PS and the facilitation of PS lessons. Visible success at the School Level produces deliverables that are aligned with the school's vision, mission and goals, professionally developed teachers and students' growth. The flows which originate from the source of visible success to the respective parts of Products must maintain a healthy rate in order that a significant level of positive feedback and evaluation returns to the Input and the 'Engine' components. From the data collected from the project schools, we distil a set of the factors that would contribute towards a long-term sustainment of MInD as a teaching innovation.

School specific factors include the support by school leaders, and the autonomy enjoyed by the school teachers in implementing teaching innovations. Regarding support given by the school leaders, we note that schools that enjoy strong support by knowledgeable school leaders who believe that PS is the mainstay in the mathematics curriculum and are committed to running MInD, even when it requires a heavy investment of resources under real school constraints, are likely to operate a sustainable MInD.

> Knowledgeable and supportive school leadership can be instrumental in making a program a priority within the school, as reflected in the time, resources, incentives, and training allocated for the program as well as the expectation of accountability. (Kam, Greenberg, & Walls, 2003)

Importantly, leaders gain knowledge from external and internal sources. Externally, they are directed by current educational directives exerted by state authorities and by new educational trends taking place worldwide. Internally, they observe visible success contributed by the positive outcomes of teaching innovations currently taking place in the school. Principals, Head of Department and PS teachers in 'A', 'B' and 'C' kept on communicating about the program effectiveness of MInD in equipping students with the life-skill of creative problem solving because they saw the ongoing episodes of visible success in the PS classrooms as well as in the grade improvement in national examinations. Such successes must be made clearly visible to the school leaders to convince them that MInD is actively yielding concrete deliverables that are aligned with the school vision, mission and goals. In this respect, the Head of Department played the role of a mediator between the Principal and the PS teachers. Because the decisions made by the school's top management can facilitate or impede the operations of MInD, keeping the school leaders committed through constant positive feedback and evaluation is of paramount importance. Therefore, the core of the leadership in the Mathematics Department must be relatively stable to ensure a long-term commitment to PS as the way to teach and learn mathematics. Huberman and Miles (1984) and Sindelar, Shearer, Yendol-Hoppey, and Liebert, (2006) already warned us of the "second-wave" crisis, i.e., the first wave occurs when the innovation practice commences, and the second wave is the result of when teachers who demonstrated success with the innovation are promoted to higher positions, thus leaving a vacuum characterized by an absence of those 'enforcer' teachers for the teaching innovation. 'D' suffered from this second wave when its previous Mathematics Head of Department was promoted because of her contribution in running MProSE.

Concerning autonomy and flexibility, we can see that schools that enjoy a high degree of autonomy and flexibility in planning and implementing MInD are the ones that are likely to sustain. However, the trust given by the Principal can easily dissolve if there are no consistent visible successes emerging as a direct result of teaching innovation. The school culture developed in 'A', 'B' and 'C' attested to existing findings that those schools with shared vision and cultures of communications and shared decision-making, and schools that involved teachers in the design or modification of the innovation are more likely to sustain innovations (Florian, 2000; Huberman & Miles, 1984).

Program specific factors also come into the picture. Firstly, teachers' and students' attribution toward MInD. Klingner et al. (2001) said that innovations which were smaller in scope and those which placed fewer demands on the teachers were more likely to take root and be sustained. Innovations that required too many changes in the current functioning of the school were less successful than more proscribed innovations. MInD is a teaching innovation that is certainly complex, technically demanding and emotionally taxing. In order to ensure the sustainability of MInD, teachers' attribution towards MInD is critical: teachers must believe in the value of PS. This belief is affirmed in two ways: (i) teachers observe program effectiveness of MInD by witnessing episodes of visible success demonstrated in students' PS endeavors, and (ii) teachers' self-efficacy in PS strengthen their confidence in teaching PS.

Secondly, feasibility of implementation plays a big part in the sustainability of MInD as a teaching innovation. Schools 'A', 'B' and 'C' adopted suitable teaching strategies such as using Mathematics Practical Worksheet to scaffold students in learning Pólya's stages and Schoenfeld's framework as well as to help teachers manage the facilitation of large number of discussion pairs. Feasibility of implementation of PS lessons is increased by suitable modification, adaptations and inventions made by PS teachers themselves. However, such modifications must remain faithful to the design parameters of MInD. ‘D’ also made changes by removing the abstract problems and retaining only problems with real-life contexts. This modification restricted the variety of problems that students experienced, and hence deprived them of the chance of handling abstract mathematical definitions—the understanding of which forms a crucial experience in learning mathematics.

The third program specific factor is time. The Focus Group Discussions revealed that time resource (see SR4 and PR2 in Fig. 15.3) is the most crucial factor to be taken into consideration, given that there is a multitude of activities and programs in the school that are competing for resources (see SC2 in Fig. 15.3). Much as they believed in the idea of infusing PS into day-to-day mathematics lessons within main curriculum time, PS teachers in ‘D’ were hard-pressed for time to ‘cover’ the syllabus stipulated by the examination board. Successful and sustainable implementation of MInD in ‘C’ exploited the concept of ‘Replacement Units’, where difficult-to-teach topics were identified by teachers and with the help of the MInD team the usual units of lessons for those topics were replaced by units specially designed with the PS parameters in mind (see Leong et al., 2016). In this way, ‘C’ was able to optimize time their advantage in the sense that the difficult topic could be taught and learnt through PS within normal curriculum time.

Lastly, teacher training with regard to PS is a key program specific factor. Since the production of visible success is crucial as a feedback to the school leaders, high-quality teacher facilitation must occur in the PS classroom. During the implementation phase, the MInD project team was providing the expert advice and support, but during the sustaining phase, the PS teachers must continue to maintain their competency in PS and teaching of PS. Continual Professional Development at the Program Level must be in constant operation, where (i) mathematics teachers have regular meetings where PS is practiced, talked about and its pedagogy shared, and (ii) new teachers are professionally developed to be trained PS teachers. If needed, external consultancy provided by the MInD team could be engaged to provide timely training for new PS teachers. Additionally, project school have also suggested that the methods courses for pre-service teacher training conducted at the teacher training institute in Singapore should include PS facilitation (in the sense of MInD) as a cornerstone course.

15.7 Conclusion

The difficulties of implementing Mathematical Problem Solving are real, partly due to the technical difficulties involved in PS itself, and partly due to the lack of appropriate literature that can guide teachers effectively to “implement results of the problem-solving research into their everyday classroom praxis” (Zimmerman, 2016). In Singapore, we experienced the same difficulty of implementing PS in schools: although the Ministry of Education placed PS as the central theme of the framework for Mathematics Curriculum, there is a lack of widespread classroom practice schools in realizing this theme. MProSE, and subsequently MInD, can be seen as forerunners in terms of implementing a feasible, sustainable and scalable teaching innovation that promised to realize the lofty goal of teaching and learning PS in Singapore schools.

Although the design parameters built into the MInD were intended to optimize the sustainment of MInD in schools, adherence to the parameters is far from sufficient for guaranteeing its sustainability in actual implementation. Focus Group Discussions which we conducted revealed that while all project schools were on par with one another in terms of school demographics, resources and infrastructure, the operation of MInD could be very different. This is perhaps unsurprising—no schools are identical—each school has its organizational mission, goals, and culture. The task of characterizing program sustainability of MInD is thus challenging because of its multi-faceted nature: running a teaching innovation involves activities at the Program Level as well as the School Level. In this chapter, we made use of a systems approach to derive a model to describe how MInD can be implemented in a sustainable way. We viewed the situation at hand as a system which consists of several components (Input, Product, Constraints, Resources, Strategies and Feedback-Evaluation) operating at the Program and School Levels. Our theory proposes that MInD is able to sustain if and only if the system operates in such a way that it produces a significant level of visible success, at both Program and School Levels. In this way, it secures continuous commitment and support from the school leadership, and thus achieves a self-sustaining and perpetual flow of processes and information between various components. It might appear that a state of operation which is self-sustaining and perpetually smooth-flowing is too idealistic and can never be achieved realistically. However, the authors of this chapter think otherwise. The sustainability of MInD can be achieved once the project school adopts Problem Solving as its school culture of teaching and learning mathematics.

We validated the Systems Model by matching it against the data collected (in the form of detailed documentations and Focus Group Discussions) from the four project schools. 'A', 'B' and 'C' had institutionalized processes that promote smooth flow from component to component such that the two main channels of Feedback and Evaluation from Output to Input and 'Engine' are initiated by significant occurrences of School and Program Levels visible success. To ensure this smooth flow, many processes were in place, including effective communication between school leadership and the PS team, efficient deployment of resources, and establishment of continual professional development. On the flip side, we also learnt a great deal from the Focus Group Discussion conducted at 'D'. Because 'D' deviated from the fundamental design parameters that PS must have a place in normal curriculum time and that the students must be assessed of their PS competency, MInD was beginning to show signs of failing. The benefits of PS in teaching and learning mathematics did not become part of the knowledge, through visible success, that would otherwise inform the school leaders who could have deployed the limited school resources to support MInD. In terms of the Systems Model, the flow was slowing down which led to visible 'successes' to diminish. These were not observed by the school leader owing to lack of poor communication, and all these built up as a vicious cycle of 'malnourishment' that further worsened the health of MInD in 'D'.

If left unresolved, the problems experienced by 'D' in their implementation of MInD would eventually overcome the efforts of trying to run it as an enrichment program—the prospects are grim as school leadership support, teacher training and resources are withdrawn, and the attributions of teachers and students toward PS deteriorate. It would be helpful if the Systems Model we developed herein could not only provide a diagnosis of what went wrong but also a prescription of what actions could be taken to salvage the failing MInD. At this point of writing, we are inspired by the promise that Replacement Units seem to offer as our experience in 'C' suggests. We saw that Replacement Units could be employed as a strategy for teaching mathematics through PS within normal curriculum time, with special focus in teaching difficult-to-teach topics that teachers identified. At this point of writing, the project team has already re-established contact with 'D' to explore the possibility of discussing the prognosis offered by the Systems Model and of offering the Replacement Units as a strategy to revive the failing MInD.

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Conclusions

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Problem Solving in Mathematics Education has been a perennial topic study group (TSG) at the International Congress on Mathematics Education meeting. As such, it has become a place for researchers interested in the topic to gather, present, and discuss the most recent developments in the field. This book is a collection of some of the most current research on problem solving and, as with its predecessors, pushes the academic agenda further with new approaches, methods, and perspectives for not only understanding mathematical problem solving, but also fostering it.

In some cases, the contributions provide new directions for examining old themes, from the importance of heuristics to the currency of metacognition to the continued relevance in problem posing. Meanwhile, other contributions push into new territory to discuss different types of problem solving assessments as well as the environment necessary to occasion and nurture problem solving activity. Through all this a number of key outcomes have emerged that promise to continue to push the field of mathematical problem solving further in the years leading up to ICME-14.

1. The seminal work of Pólya and Schoenfeld continue to be an important referent in several chapters. In particular, Pólya's *looking back* stage is linked to the importance for students to always look for different ways to solve and discuss problems. However, emerging in this book is the antitheses of looking back in the idea of mathematical foresight as a construct to delve into the students' future-oriented thinking to solve problems.
2. The use of digital technologies continues to open new avenues to engage students in problem solving activities. Of particular interest is how new technologies can provide new problem solving strategies and ways of reasoning (such as dynamic modelling, foci tracing, quantifying parameters and attributes,

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using sliders, etc.) that emphasize both visual and empirical approaches. Digital technologies also affords us the possibility to move beyond the formal setting of the classroom through, for example, the use of a web-based mathematical competition.

3. Although problem posing activities are not new to the field of problem solving, the emergent focus on inquiry is bringing a new framing of problem solving approaches. Although relevant to both student and teacher, this new framing can afford teachers access to the problem formulation experience and may help them to recognize its value for themselves and their students.
4. Assessment continues to be an important area of research within mathematical problem solving and continues to be pushing towards recognizing the importance of assessing the problem solving process over the problem solving product. To this end, specific instruments that can be used to account for, and assess, students' problem solving behaviors continue to be developed. Emerging from this work is a reflexive critique of the wide use of standardized test and their limited ability to measure problem solving.
5. Finally, there is an emergence of research that explicitly looks at the engagement of students in problem solving—something that, until now, has only been implicitly addressed through the nature of the problem solving tasks as well as digital technologies. The emergent research on student engagement pushes well beyond this, challenging and experimenting with the normative classroom structures to shed light on what is important to take into account in order to construct learning environments conducive to occasioning and fostering student problem solving behaviors.

Taken together, the next few years promise to be an exciting time to be engaged in mathematical problem solving—as a researcher, as a teacher, and as a learner.