Chapter 7
The Height Datum Problem

7.1 Outline

Normal and orthometric heights are among the most widespread height coordinate systems in use for geodetic purposes. Yet in principle they can be determined only by ground gravimetric measurements combined with levelling so that $W(\mathrm{P})$ becomes available. Nevertheless, what the above measurements can really provide are at most potential differences, $W(\mathrm{P}_0) - W(\mathrm{P})$, for instance with respect to an origin point P_0 of which however the absolute value $W(\mathrm{P}_0)$ is unknown. When P_0 is a tide gauge, we know that we can assume $W(\mathrm{P}_0) \sim W_0$ with an error δW_0 such that $\left|\frac{\delta W_0}{\gamma}\right| < 2\,\mathrm{m}$ (cfr. Sect. 4.6); when P_0 is a point of known ellipsoidal height, e.g. a GNSS permanent station, we can always assume that $h^* \cong \widetilde{h}^* = h - \frac{T_b}{\gamma}$, where T_b is some global model that has been computed with biases and so it has an error which however is almost surely included in the above range.

In oceanic areas the information from radar altimetry and oceanography can be transformed into potential and gravity, yet biases seem to be pervasive and we can only say, after linearization and inversion, that we know $\Delta g + \frac{\gamma'}{\gamma}\delta W$, with the bias δW unknown for large portions of ocean where altimetric tracks can be readjusted at the crossovers (see Sansò and Sideris 2013, Chap. 9).

All in all we can say that instead of knowing $W(\mathrm{P})$, with known horizontal coordinates of P, $\sigma_\mathrm{P} = (\lambda_\mathrm{P}, \varphi_\mathrm{P})$, we rather have the information

$$\widetilde{C}_k(\mathrm{P}) = W(\mathrm{P}_{0k}) - W(\mathrm{P}) = W_{0k} - W(\mathrm{P}) \ , \tag{7.1}$$

which is valid for an area A_k where levelling on land, or track adjustment on ocean, are well connected to some origin P_{0k}.

Assuming for the sake of simplicity that P_{0k} is in any way close to the sea surface, we could say that in A_k we have the approximate potential

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$$\widetilde{W}(P) = W_0 - \widetilde{C}_k(P) = W_0 - W(P_{0k}) + W(P) \equiv \delta W_{0k} + W(P) \quad P \in A_k ; \quad (7.2)$$

so δW_{0k} has the meaning of the bias of the known $\widetilde{W}(P)$ in the area A_k. Putting together all the areas A_k, that we assume to cover the whole Earth sphere, we can represent our data as an approximate potential

$$\widetilde{W}(P) = W(P) + \delta W(P) \;, \quad (7.3)$$

where

$$\delta W(P) = \sum_{k=1}^{K} \delta W_{0k}\, \chi_k(P) \quad (7.4)$$

and

$$\chi_k(P) = \begin{cases} 1 & P \in A_k \\ 0 & P \notin A_k \end{cases} \;. \quad (7.5)$$

At this point we do not have anymore the telluroid S^*, i.e. we are not able to compute h^*_P by solving (4.80), but we can only put

$$\widetilde{W}(P) = W(\sigma, h_\sigma) + \delta W(\sigma) = U\left(\sigma, \widetilde{h}^*_\sigma\right) \;, \quad (7.6)$$

so deriving an approximate, or biased, telluroid $\widetilde{S} = \left\{h = \widetilde{h}^*_\sigma\right\}$, such that

$$DW = W - \widetilde{W} = -\delta W(\sigma) \neq 0 \;. \quad (7.7)$$

Accordingly, following the same linearization process as in Sect. 4.7 and recalling (4.78), we arrive at a BVP for the unknown anomalous potential T of the form

$$\begin{cases} \Delta T = 0 & \text{in } \widetilde{\Omega} \\ -T' + \left.\dfrac{\gamma'}{\gamma} T\right|_{\widetilde{S}} = Dg - \dfrac{\gamma'}{\gamma}\delta W & \text{on } \widetilde{S} \\ T = \mathcal{O}\left(\frac{1}{r^3}\right) \end{cases} \;. \quad (7.8)$$

Notice that $Dg = g(P) - \gamma\left(\widetilde{h}^*\right)$ is as a matter of fact what we can compute from gravimetry and the known approximate telluroid $\widetilde{S}$.

As we can see, (7.8) contains the K unknown parameters $\{\delta W_{0k}\}$, so that we can arrive to determine T and $\{\delta W_{0k}\}$ only by means of additional information; we will see in the chapter that this can be provided by points P where both $\widetilde{h}^*_P$ and h_P are known, to be precise at least one point per patch A_k, although knowing more can indeed improve the accuracy of the solution.

Let us note that, once $\{\delta W_{0k}\}$ are known, the potential $W(P)$ can be retrieved by

$$W(P) = \widetilde{W}(P) - \delta W(P)$$

and, since T is also known now without biases, we can return to compute all the transformations already studied in Chap. 5.

The solution of (7.8) is called the unification of the height datum problem, or more precisely, of the global height datum problem. In fact, if we consider as "height datum" the equipotential surface used as origin of orthometric heights, namely the geoid, we see that $\frac{\delta W(\mathrm{P})}{\gamma}$ can be interpreted as the separation between $\widetilde{S}$, which is composed by pieces of equipotential surfaces passing through P_{0k}, and the geoid, where $W(\mathrm{P})$ attains the value W_0. So knowing δW_{0k} means also to be able to transform local orthometric heights, referred to the equipotential through P_{0k}, into true orthometric heights, referred to the geoid.

An important point in the application of the above theory is that, when many points of known ellipsoidal height are present in the same patch A_k, one is led to use a least-squares adjustment to best estimate the $\{\delta W_{0k}\}$. However this requires that the covariance structure of the observations is known. This is particularly complicated for the oceanic areas where data have undergone a deep transformation process. On the other hand, we have already observed at the end of Sect. 4.7 that local models of T are available on continental areas with an overall error r.m.s. at centimetric level in geoid, in the area A of interest. This introduces the possibility of adjusting δW_{0k} for limited areas only, particularly continental areas, avoiding the problem of assigning a stochastic structure to the data in the ocean.

The whole subject of the unification of the height datum is still object of research and not completely assessed. So, in this chapter we aim at presenting the theory and evaluating the error budget with the purpose of demonstrating its feasibility. Some numerical examples, simulated or realistic, are also presented.

7.2 Formulation of the Global Unification of the Height Datum

As explained in the previous section, this problem is a combination of the solution of a GBVP with unknown additional parameters, $\{\delta W_{0k}\}$, and a set of additional data, corresponding to points P_i (at least one per patch A_k) where the ellipsoidal height $h_i = h(\mathrm{P}_i)$ has been observed.

As for the GBVP part, this has already been discussed in Sect. 7.1, leading to the formulation (7.8). Here we underline only that we know from the discussion of Sect. 4.7 that a linear solving operator exists, such that (7.8) can be written as

$$T = \widetilde{\mathcal{S}}\left(\mathrm{D}g - \frac{\gamma'}{\gamma}\,\delta W\right) = \widetilde{\mathcal{S}}\,(\mathrm{D}g) - \widetilde{\mathcal{S}}\left(\frac{\gamma'}{\gamma}\,\delta W\right)\;; \tag{7.9}$$

note that here the tilde stems from the fact that we solve with respect to the approximate surface $\widetilde{S}$. Let us observe that the operator $\widetilde{\mathcal{S}}$ is well defined when acting on functions in L^2_σ; this is the case in (7.9), also for the second term in the right hand side, because δW as a piecewise constant function is certainly in L^2_σ.

In this section we develop the theory as if the global model $\widetilde{\mathcal{S}}\,(\mathrm{D}g)$ would be really available, given that Dg is the only "observable" quantity at ground level available to us. Actually this is not the case with existing global models, in particular EGM2008. In fact space geodetic techniques, especially in the last two decades with the satellite gravimetry/gradiometry missions CHAMP, GRACE and GOCE, have provided an independent and direct information on the low degrees of the harmonic coefficients of $T\,(\mathrm{P})$; however this issue will be treated separately in the next section.

So we assume to know a biased anomalous potential

$$T_b\,(\mathrm{P}) = \widetilde{\mathcal{S}}\,(\mathrm{D}g)\ . \tag{7.10}$$

Subsequently, introducing (7.4) into (7.9), we arrive at the equation

$$T\,(\mathrm{P}) = T_b\,(\mathrm{P}) - \sum_{k=1}^{K} \delta W_{0k}\, \widetilde{\mathcal{S}} \left(\frac{\gamma'}{\gamma}\, \chi_k \right)\ , \tag{7.11}$$

with $T_b\,(\mathrm{P})$ known by hypothesis; for later use we can put $F_k\,(\mathrm{P}) = \widetilde{\mathcal{S}} \left(\frac{\gamma'}{\gamma}\, \chi_k \right)$, so that (7.11) is rewritten as

$$T\,(\mathrm{P}) = T_b\,(\mathrm{P}) - \sum_{k=1}^{K} \delta W_{0k}\, F_k\,(\mathrm{P})\ . \tag{7.12}$$

Let us consider now the observed $\{h\,(\mathrm{P}_i)\}$, $\mathrm{P}_i \in A_k$; recalling (7.2), we can write

$$\widetilde{W}\,(\mathrm{P}_i) = \delta W_{0k} + W\,(\mathrm{P}_i) = \delta W_{0k} + U\,(h_i) + T\,(\mathrm{P}_i) \qquad \mathrm{P}_i \in A_k\ . \tag{7.13}$$

On the other hand we have, according to (7.6),

$$\widetilde{W}\,(\mathrm{P}_i) = U\left(\widetilde{h}_i^*\right) \tag{7.14}$$

and indeed $\widetilde{h}_i^* = \widetilde{h}^*\,(\mathrm{P}_i)$ is known by hypothesis too. The practical situation is that, if P_i is a geodetic space station, this is connected to the local levelling line, so that $\widetilde{h}_i^*$ is directly known. Putting (7.13) and (7.14) together gives

$$U\,(h_i) - U\left(\widetilde{h}_i^*\right) = -T\,(\mathrm{P}_i) - \delta W_{0k}\ ,$$

which, linearized with respect to $h_i - \widetilde{h}_i^*$, yields

$$h_i - \widetilde{h}_i^* = \frac{T}{\gamma} + \frac{\delta W_{0k}}{\gamma}\ . \tag{7.15}$$

Notice that, as customary, in (7.15) we do not write explicitly where to compute T and γ, because choosing either h_i or $\widetilde{h}_i^*$, in the right hand side of this relation, produces only second order variations. Finally, introducing (7.12) into (7.15), we find

$$h_i - \widetilde{h}_i^* = \frac{T_b(\mathrm{P}_i)}{\gamma} - \frac{1}{\gamma}\sum_{j=1}^{K} \delta W_{0j} F_j(\mathrm{P}_i) + \frac{\delta W_{0k}}{\gamma} \qquad \mathrm{P}_i \in A_k\ . \tag{7.16}$$

As we see if we complement (7.16) with the proper error models for h_i, $\widetilde{h}_i^*$ and $T_b(\mathrm{P}_i)$, we have reduced the solution of our unification problem to that of a least-squares system.

Note should be taken that the functions $F_j(\mathrm{P})$ are generally small outside A_j, so (7.16) could become badly conditioned if one of the patches would be void of points P_i where h_i is known, as already stated before. It has to be stressed too that indeed the system (7.16) should be solved for all P_i in all patches together. This raises the question of how complicated could be the covariance matrix of (7.16). Even if one could reasonably assume (though not strictly) the errors of h_i and $\widetilde{h}_i^*$ to be independent, the same could not be true for the errors in the model $T_b(\mathrm{P})$; in fact, even if the gravity observations could be considered as being affected by independent measuring errors, the model is derived by solving the BVP, roughly by Stokes integration, and so it is expected to have a geographical correlation pattern. Not to be said, a correlation between the errors of $\widetilde{h}_i^*$ and T_b, as both are derived from Dg, should also be taken into account. Yet a simplification of the stochastic model, even a drastic one, would be acceptable in view of the large number of stations $\{\mathrm{P}_i\}$ that are generally available for each patch.

Nevertheless the weak point of the approach expressed by (7.16) is in the assumption that $T_b(\mathrm{P})$ is known. As a matter of fact, even the previous Earth models have always used the knowledge of low degrees coefficients of $T(\mathrm{P})$ from space geodetic observations (see for instance the paper by Rapp (1989) concerning the OSU86 model, complete up to degree and order 360). This creates models such that typically combine unbiased low degrees, derived from satellite observations, with biased gravity anomalies from ground data.

The problem will be more closely analyzed into the next sections at both local and global level.

7.3 On the Solution of the Unification Problem by a Suitable Global Model

The target of the section is to prove, by means of a careful but conservative error budget analysis, that already today we have global models that directly used in (7.15) provide us equations with errors below the 5 cm level. Since we can use several such equations for each δW_{0k}, we deem it reasonable to estimate such parameters with errors, in terms of geoid, i.e. of $\frac{\delta W_{0k}}{\gamma}$, of very few centimeters, at least in a global mean square sense. To go along this way we make beforehand two remarks.

The first is that we can free our problem from many mathematical complications if we can state a priori that all our harmonic functions can be expressed as a sum of spherical harmonics up to some finite maximum degree M; in our case M will be taken at the level of 2159, as the maximum degree of EGM2008.

This choice is justified by the following reasoning. Taking into consideration the discussion in Sect. 4.4, we start recalling the definition of full power degree variances C_n, namely

$$C_n = \sum_{m=-n}^{n} T_{nm}^2 \equiv \frac{1}{4\pi} \int \left[\sum_{m=-n}^{n} T_{nm} \, Y_{nm}(\sigma) \right]^2 d\sigma \, . \tag{7.17}$$

The plot of (7.17) for the EGM2008 model has been already displayed in Fig. 4.1. Because of their quite regular behaviour, C_n can be interpolated by some simple analytic expression. An exercise of this kind has been done by several authors with comparable results. The model that one can find in Sansò and Sideris (2013) has been computed by adapting to the empirical data the function

$$C_n = \frac{A\, q^n}{(n-1)(n-2)(n+4)(n+17)} \, . \tag{7.18}$$

A good matching, using only empirical values up to degree 1800, is obtained with

$$A = \left(\frac{\mu}{R} \right)^2 3.9 \cdot 10^{-5} \, , \quad q = 0.999443 \, .$$

Other authors (for example Hirt and Kuhn 2012) obtain slightly different values using all the empirical data; yet this does not change the order of magnitude of our guess. In fact adding our C_n given by (7.18) from 2160 up to 10000, we have an idea of the magnitude of the squared norm of the omitted part of T. More precisely we have the so called omission error, $\mathcal{OE}(T)$, for $M = 2159$ given by

$$\begin{aligned} \mathcal{OE}_{2160}(T) &= \left\{ \frac{1}{4\pi} \int \left[\sum_{n=2160}^{+\infty} \sum_{m=-n}^{n} T_{nm} \, Y_{nm}(\sigma) \right]^2 d\sigma \right\}^{\frac{1}{2}} = \\ &= \sum_{n=2160}^{+\infty} C_n \cong \sum_{n=2160}^{10000} C_n \cong 0.6 \text{ cm} \, \overline{\gamma} \, , \end{aligned} \tag{7.19}$$

i.e. this omission error in terms of geoid is globally well below the centimeter value. Indeed it is clear that this does not prevent us to have a value of some centimeters in some places on the Earth surface; however this seems compatible with the target of this section.

So we shall accept the above assumption. Then we claim that, neglecting second order terms, we have

$$\mathrm{D}g + \frac{\gamma'}{\gamma}\,\delta W\Big|_{\widetilde{S}} \cong \Delta g|_{S^*} \ , \tag{7.20}$$

with S^* the ordinary Marussi telluroid (see (4.80), (4.81)) and $\widetilde{S}$ the approximate telluroid defined by (7.14).

The relation (7.20) is proved by the following calculation

$$\begin{aligned}
&\mathrm{D}g\left(\widetilde{h}^*\right) + \frac{\gamma'}{\gamma}\,\delta W\left(\widetilde{h}^*\right) = g - \gamma\left(\widetilde{h}^*\right) + \frac{\gamma'}{\gamma}\left[W - U\left(\widetilde{h}^*\right)\right] = \\
&= g - \gamma\left(h^*\right) + \gamma\left(h^*\right) - \gamma\left(\widetilde{h}^*\right) + \frac{\gamma'}{\gamma}\left[U\left(h^*\right) - U\left(\widetilde{h}^*\right)\right] = \\
&= \Delta g\left(h^*\right) + \gamma'\left(h^* - \widetilde{h}^*\right) + \frac{\gamma'}{\gamma}\left[-\gamma\left(h^* - \widetilde{h}^*\right)\right] = \Delta g\left(h^*\right) \ .
\end{aligned}$$

We would like to acknowledge that this complies with a personal communication of T. Krarup to one of the authors.

A consequence of this remark is that, since the solution of the GBVP is unique, solving such a problem with known term Δg on S^* or with $\mathrm{D}g + \frac{\gamma'}{\gamma}\mathrm{D}W$ on $\widetilde{S}$ should give the same result in the linear approximation. Concisely, introducing the two solver operators $\mathcal{S}^*$ and $\widetilde{\mathcal{S}}$, the former referring to the GBVP with S^* as boundary, the latter to the same problem with $\widetilde{S}$ as boundary, we can claim that

$$T \cong \mathcal{S}^*\left(\Delta g\right) \cong \widetilde{\mathcal{S}}\left(\mathrm{D}g + \frac{\gamma'}{\gamma}\mathrm{D}W\right) \ . \tag{7.21}$$

Now we are ready to introduce our simple minded global model $\widetilde{T}$. We started by observing that we have available satellite-only models combining data from satellite geodesy of different missions, particularly the models derived by the three gravimetric/grodiometric missions CHAMP, GRACE and GOCE (Reigber et al. 2004; Tapley et al. 2004; Pail et al. 2011). Specifically we shall refer to the GOCO model T^{G} (Pail et al. 2010; Mayer-Gürr et al. 2015) up to degree and order 200, a level at which the cumulated error for the estimate of the coefficients becomes larger than the magnitude of the coefficients themselves, expressed by their degree variances C_n. So up to degree 200 we follow T^{G}, knowing that

$$T^{\mathrm{G}} = T_{200} + \varepsilon^{\mathrm{G}} \tag{7.22}$$

with

$$\frac{1}{\overline{\gamma}}\,\sigma\left(T^{\mathrm{G}}\right) = \frac{1}{\overline{\gamma}}\,\sigma\left(\varepsilon^{\mathrm{G}}\right) \cong 2\ \mathrm{cm}\,, \tag{7.23}$$

as it results from the estimates of the degree standard deviations provided with the model. The quantity $\sigma\left(\varepsilon^{\mathrm{G}}\right)$ in (7.23) is called the commission error $\mathcal{CE}\left(\varepsilon^{\mathrm{G}}\right)$ (see (4.40)).

We strengthen again that if we introduce the projection operator $\mathcal{P}_L$ that cuts every harmonic function at the degree L, i.e. for $M > L$

$$\mathcal{P}_L\left(\sum_{n=2}^{M} u_{nm}\left(\frac{R}{r}\right)^{n+1} Y_{nm}(\sigma)\right) = \sum_{n=2}^{L} u_{nm}\left(\frac{R}{r}\right)^{n+1} Y_{nm}(\sigma) \ , \tag{7.24}$$

then indeed, with $L = 200$, we have

$$\mathcal{P}_L T^{\mathrm{G}} \equiv T^{\mathrm{G}} \ , \tag{7.25}$$

so that (7.22) more precisely reads

$$T^{\mathrm{G}} = \mathcal{P}_L T + \varepsilon^{\mathrm{G}} \ . \tag{7.26}$$

Moreover the explicit interpretation of (7.23) is

$$\frac{1}{\gamma}\sigma\left(\varepsilon^{\mathrm{G}}\right) = \frac{1}{\gamma}\left[\mathrm{E}\left(\frac{1}{4\pi}\int\left(\varepsilon^{\mathrm{G}}\right)^2 \mathrm{d}\sigma\right)\right]^{\frac{1}{2}} \ ,$$

where the expectation E is taken on the stochastic structure of ε^{G}.

We assume that the information contained in T^{G} is better than the corresponding information on the low degrees contained in the EGM2008 model T^{E}. On the contrary, for degrees higher than 200 the only global information (in reality up to degree 2159) we have is contained in T^{E}, so we will take it as it is. Therefore we propose to create a kind of "Frankenstein model" according to

$$\widetilde{T} = T^{\mathrm{G}} + (I - \mathcal{P}_L)\, T^{\mathrm{E}} \ . \tag{7.27}$$

We note however that T^{E} has been computed from ground data, at least in the range of degrees higher than 200, and so it is affected by a bias because it could only be computed from the observations $\mathrm{D}g_0$ on the approximate telluroid $\widetilde{S}$. In other words

$$(I - \mathcal{P}_L)\, T^{\mathrm{E}} = (I - \mathcal{P}_L)\, \widetilde{\mathcal{S}}\left(\mathrm{D}g_0\right) \ . \tag{7.28}$$

Indeed $\mathrm{D}g_0$ is affected by some noise ε^g that propagates to the solution

$$\widetilde{\mathcal{S}}\left(\mathrm{D}g_0\right) = \widetilde{\mathcal{S}}\left(\mathrm{D}g + \varepsilon^g\right) = \widetilde{\mathcal{S}}\left(\mathrm{D}g\right) + \varepsilon^{\mathrm{E}} \ . \tag{7.29}$$

This ε^{E} is what in literature is called the commission error $\mathcal{CE}\left(\varepsilon^{\mathrm{E}}\right)$ of the model, and it is clear from our reasoning that ε^{E} has a maximum degree equal to 2159 too.

With such specifications, (7.27) reads

$$\widetilde{T} = \mathcal{P}_L T + \varepsilon^{\mathrm{G}} + (I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\,(\mathrm{D}g) + (I - \mathcal{P}_L)\,\varepsilon^{\mathrm{E}} \,. \tag{7.30}$$

Also for ε^{E} we have estimates that come together with the model T^{E}; the number cumulating all the errors between degree 200 and 2159 is

$$\mathcal{CE}\left(\varepsilon^{\mathrm{E}}\right) = \mathrm{E}\left[\frac{1}{4\pi}\int \left[(I - \mathcal{P}_L)\,\varepsilon^{\mathrm{E}}\right]^2 \mathrm{d}\sigma\right]^{\frac{1}{2}} = 3.6 \text{ cm} \,. \tag{7.31}$$

If we write the analogous of (7.30) for T, also taking into account (7.21), we see that

$$\begin{aligned} T &= \mathcal{P}_L T + (I - \mathcal{P}_L)\,\mathcal{S}^*\,(\Delta g) = \\ &= \mathcal{P}_L T + (I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\,(\mathrm{D}g) + (I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\left(\frac{\gamma'}{\gamma}\,\delta W\right) \,. \end{aligned} \tag{7.32}$$

Comparing (7.32) and (7.30), we find the total estimation error of $\widetilde{T}$, namely

$$\widetilde{T} - T = \varepsilon^{\mathrm{G}} + (I - \mathcal{P}_L)\,\varepsilon^{\mathrm{E}} - (I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\left(\frac{\gamma'}{\gamma}\,\delta W\right) \,. \tag{7.33}$$

If we can suppose that ε^{G} and ε^{E} have zero average, the same is not justified for $(I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\left(\frac{\gamma'}{\gamma}\,\delta W\right)$, which then assumes the meaning of the bias of $\widetilde{T} - T$, i.e.

$$b\,(\mathrm{P}) = \mathrm{E}\left\{\widetilde{T} - T\right\} = -\,(I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\left(\frac{\gamma'}{\gamma}\,\delta W\right) \,. \tag{7.34}$$

The construction of our error budget then continues with a majorization of the mean quadratic value of $b\,(\mathrm{P})$ over the unit sphere.

Now consider that $\widetilde{\mathcal{S}}$, the BVP solver, is as a matter of fact a combination of some kind of regularized downward continuation to the Earth ellipsoid and then a solution by quadrature with spherical harmonics (Sansò and Sideris 2013, Part II, Chap. 6). In any event, due to the smallness of the function

$$\frac{\gamma'}{\gamma}\,\delta W \cong -\frac{2}{r}\,\delta W \,, \tag{7.35}$$

(remember that $\mathcal{O}\left(\frac{\delta W}{\overline{\gamma}}\right) \cong 2$ m), we can approximate $\widetilde{\mathcal{S}}$ as applied to (7.35) by a simple spherical solver, namely the Stokes integral, which certainly constitutes the "large part" of $\widetilde{\mathcal{S}}$. So we can write (see (4.100))

$$b = -(I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\left(\frac{\gamma'}{\gamma}\,\delta W\right) \cong \frac{2}{R_0}\sum_{n=L+1}^{M}\sum_{m=-n}^{n}\frac{R_0}{n-1}\,\delta W_{nm}\,Y_{nm}(\sigma) \tag{7.36}$$

with R_0 the mean Earth radius. From (7.36) we then derive

$$\begin{aligned}\|b\|^2_{L^2_\sigma} &= \left\|(I - \mathcal{P}_L)\,\widetilde{\mathcal{S}}\left(\frac{\gamma'}{\gamma}\,\delta W\right)\right\|^2_{L^2_\sigma} = 4\sum_{n=L+1}^{M}\sum_{m=-n}^{n}\frac{\delta W^2_{nm}}{(n-1)^2} \le \\ &\le \frac{4}{L^2}\sum_{n=L+1}^{M}\sum_{m=-n}^{n}\delta W^2_{nm} = \frac{4}{L^2}\,\|(I-\mathcal{P}_L)\,\delta W\|^2_{L^2_\sigma} < \\ &< \frac{4}{L^2}\,\|\delta W\|^2_{L^2_\sigma}\ .\end{aligned} \tag{7.37}$$

Now we observe that, owing to its definition (7.4), δW^2 is given by

$$\delta W^2\,(\mathrm{P}) = \sum_{k=1}^{K}\delta W^2_{0k}\,\chi_k\,(\mathrm{P})\ ,$$

so that

$$\|\delta W\,(\mathrm{P})\|^2_{L^2_\sigma} = \frac{1}{4\pi}\int\sum_{k=1}^{K}\delta W^2_{0k}\,\chi_k\,(\mathrm{P})\,\mathrm{d}\sigma = \sum_{k=1}^{K}\delta W^2_{0k}\,\frac{|A_k|}{4\pi}\ , \tag{7.38}$$

where we have designated by $|A_k|$ the area of the patch A_k, projected on the unit sphere. As we see, (7.38) is a kind of weighted average of the δW^2_{0k} and, since $\max\left|\dfrac{\delta W_{0k}}{\overline{\gamma}}\right| \le 2$ m, we could reasonably hypothesize that

$$\frac{1}{\overline{\gamma}}\left\{\sum_{k=1}^{K}\delta W^2_{0k}\,\frac{|A_k|}{4\pi}\right\}^{\frac{1}{2}} \le 1\ \mathrm{m}\ . \tag{7.39}$$

Using (7.39) in (7.37), we receive

$$\frac{1}{\overline{\gamma}}\,\|b\|_{L^2_\sigma} < \frac{2}{200}\cdot 1\ \mathrm{m} = 1\ \mathrm{cm}\ . \tag{7.40}$$

Putting (7.23), (7.32) and (7.40) together, we formulate the following error budget

$$\begin{aligned}\frac{1}{\overline{\gamma}}\left\{\mathrm{E}\left[\|\widetilde{T} - T\|^2_{L^2_\sigma}\right]\right\}^{\frac{1}{2}} &= \frac{1}{\overline{\gamma}}\left\{\mathcal{C}\mathcal{E}^2\left(\varepsilon^{\mathrm{G}}\right) + \mathcal{C}\mathcal{E}^2\left(\varepsilon^{\mathrm{E}}\right) + \|b\|^2_{L^2_\sigma}\right\}^{\frac{1}{2}} \le \\ &\le \{4 + 12.96 + 1\}^{\frac{1}{2}}\ \ \mathrm{cm} = 4.24\ \mathrm{cm}\ .\end{aligned} \tag{7.41}$$

Let us remark that, if instead of (7.39) we had taken the upper limit of 2 m, then (7.41) would rise to 4.58 cm, which is not a very different number.

Let us further observe that certainly our analysis here is not very refined and in particular the model $\widetilde{T}$ on which the error budget has been constructed is not the optimal that one could calculate. Optimal solutions of the combination of satellite and existing global models can be found in literature (see for example Pavlis et al. 2012, 2013; Reguzzoni and Sansò 2012; Sansò and Sideris 2013, Part II, Chap. 6; Gilardoni et al. 2016).

On the other hand, we promised a conservative analysis that has generated the figure of 5 cm to majorize our global error; so we are confident that this is a reliable upper bound. Since the large part of the index (7.41) is due to $\mathcal{CE}\left(\varepsilon^{\mathrm{E}}\right)$, we know that this index has a great geographic variability, reaching the level of 30–40 cm in the Himalayas and in the Andes when ε^{E} includes also the first 200 degrees. However this is not the case in most areas of the globe and we can expect that a figure between 5 and 10 cm could be respected by the error in the stations chosen to construct the system (7.16). Therefore a first proposal is to use $\widetilde{T}$ (or a better model) in (7.16), so that we can write observation equations patch by patch and, hopefully, by averaging we can resort to an estimate of $\dfrac{\delta W_{0k}}{\gamma}$ with a few centimeters error.

A more refined proposal is to use the model $\widetilde{T}$ to arrive at a system of equations similar to (7.16); however we have now to pay attention to split the degrees below and above 200, as discussed in this section. In this case, from Eqs. (7.15), (7.33) and (7.36) we could write

$$\begin{aligned} h_i - \widetilde{h}_i^* &= \frac{\widetilde{T}}{\gamma} + \frac{T-\widetilde{T}}{\gamma} + \frac{\delta W_{0k}}{\gamma} = \\ &= \frac{\widetilde{T}}{\gamma} - \frac{1}{\gamma}\left(I - \mathcal{P}_L\right)\widetilde{\mathcal{S}}\left(\frac{2}{r}\,\delta W\right) + \frac{\delta W_{0k}}{\gamma} = \\ &= \frac{\widetilde{T}}{\gamma} - \frac{2}{\gamma}\sum_{n=L+1}^{M}\sum_{m=-n}^{n}\frac{\delta W_{nm}}{n-1}\,Y_{nm}\left(\sigma_{\mathrm{P}_i}\right) + \frac{\delta W_{0k}}{\gamma}\,; \end{aligned} \tag{7.42}$$

note that in (7.42) only the deterministic terms are reported, leaving the stochastic errors aside.

Now considering that

$$\delta W_{nm} = \sum_{j=1}^{K}\delta W_{0j}\,\langle\chi^j, Y_{nm}\rangle = \sum_{j=1}^{K}\delta W_{0j}\,\chi_{nm}^j\,,$$

Eq. (7.42) can be rewritten in the form

$$h_i - \widetilde{h}_i^* = \frac{\widetilde{T}\left(\mathrm{P}_i\right)}{\gamma} - \frac{2}{\gamma}\sum_{j=1}^{K}\delta W_{0j}\left[\sum_{n=L+1}^{M}\sum_{m=-n}^{n}\frac{\chi_{nm}^j}{n-1}\,Y_{nm}\left(\sigma_{\mathrm{P}_i}\right)\right] + \frac{\delta W_{0k}}{\gamma}\,, \tag{7.43}$$

where the unknown parameters $\{\delta W_{0k}\}$ appear explicitly and all the other terms are either observed or computed.

When all the quantities h_i, $\widetilde{h}_i^*$ and $\widetilde{T}(\mathrm{P}_i)$ are derived from observations, the Eq. (7.43) should be complemented with the proper error terms; if we assume that the errors in h_i and $\widetilde{h}_i^*$ are in the range of millimeters, and therefore negligible, and recalling (7.33), we can write

$$h_i - \widetilde{h}_i^* = \frac{\widetilde{T}(\mathrm{P}_i)}{\gamma} - \frac{2}{\gamma}\sum_{j=1}^{K}\delta W_{0j}\left[\sum_{n=L+1}^{M}\sum_{m=-n}^{n}\frac{\chi_{nm}^j}{n-1}Y_{nm}\left(\sigma_{\mathrm{P}_i}\right)\right] + \\ + \frac{\delta W_{0k}}{\gamma} + \varepsilon^{\mathrm{G}} + (I - \mathcal{P}_L)\,\varepsilon^{\mathrm{E}}\,. \tag{7.44}$$

7.4 On Local Solutions of the Height Datum Problem

We have already mentioned in the previous section that, when we have available a good model of the anomalous potential, like our $\widetilde{T}$ or better, we can safely substitute it in observation equations of the shape (7.15). This implies neglecting the bias term (7.36), which has been estimated to globally produce (cfr. (7.38)) a mean square error between 1 and 2 cm, and to accept a stochastic error, $\varepsilon^{\mathrm{G}} + (I - \mathcal{P}_L)\,\varepsilon^{\mathrm{E}}$, with an overall magnitude of the order of 4 cm. Including all the effects into the observation equation, we arrive at formula (7.44).

However two aspects limit this global approach to the determination of the height datum, i.e. of the biases $\{\delta W_{0k}\}$, namely that in oceanic areas we have observations for h_i ($\widetilde{h}_i^* = 0$ in this case) but this dataset is strongly correlated and the covariance structure of the error is not really known; moreover biases and stochastic errors can have a strong geographic signature which could deviate the estimates of $\dfrac{\delta W_{0k}}{\gamma}$, by one or more decimeters, at least for particular areas.

This is ultimately due to the fact that in such areas $\widetilde{T}$ is not a sufficient approximation to T; however we know that, apart from biases, we are able to compute a better estimate of T, for instance by using a local collocation solution

$$\widehat{T}_{\mathrm{loc}} = \widetilde{T} + T_{\mathrm{res}}\,, \tag{7.45}$$

for which a typical error-figure in terms of height anomaly could be 1–2 cm. We will call $\varepsilon_{\mathrm{res}}$ the error associated to the estimated residual potential T_{res}.

We want to examine whether and how we could take advantage of this improved knowledge to estimate one of the biases for a specific area. In this case we have to return to (7.44) and use $\widehat{T}_{\mathrm{loc}}$ instead of $\widetilde{T}$ and $\varepsilon_{\mathrm{res}}$ instead of $\varepsilon^{\mathrm{G}} + (I - \mathcal{P}_L)\,\varepsilon^{\mathrm{E}}$, thus arriving at an observation equation that we rewrite in the form

$$h_i - \widetilde{h}_i^* = \frac{\widehat{T}_{\text{loc}}\left(\text{P}_i\right)}{\gamma} - \frac{2}{\gamma} \sum_{j=1, j\neq k}^{K} \delta W_{0j} \left[\sum_{n=L+1}^{M} \sum_{m=-n}^{n} \frac{\chi_{nm}^{j}}{n-1} Y_{nm}\left(\sigma_{\text{P}_i}\right) \right] +$$
$$+ \frac{\delta W_{0k}}{\gamma} \left\{ 1 - 2 \left[\sum_{n=L+1}^{M} \sum_{m=-n}^{n} \frac{\chi_{nm}^{k}}{n-1} Y_{nm}\left(\sigma_{\text{P}_i}\right) \right] \right\} + \varepsilon_{\text{res},i} \quad \text{P}_i \in A_k \ . \quad (7.46)$$

Let us consider one element of the sum in the second term of the right hand side of (7.46), namely

$$\frac{2}{\gamma} \delta W_{0j} \left[\sum_{n=L+1}^{M} \sum_{m=-n}^{n} \frac{\chi_{nm}^{j}}{n-1} Y_{nm}\left(\sigma_{\text{P}_i}\right) \right] \cong$$
$$\cong \frac{1}{\gamma} \left(I - \mathcal{P}_L\right) \widetilde{\mathcal{S}} \left(\frac{2\, \delta W_{0j}}{R} \chi^j \right) \cong$$
$$\cong \frac{\delta W_{0j}}{\gamma} \frac{2}{4\pi} \int_{A_j} \sum_{n=L+1}^{M} \frac{2n+1}{n-1} P_n\left(\cos \psi_{\text{P}_i\text{Q}}\right) \mathrm{d}\sigma_{\text{Q}} \quad (j \neq k) \ . \quad (7.47)$$

As we see such a term represents the influence of the bias δW_{0j} of the zone A_j in the area A_k; when the two are well separated, it is known that the influence function

$$F_j\left(\text{P}\right) = \frac{1}{2\pi} \int_{A_j} \sum_{n=L+1}^{M} \frac{2n+1}{n-1} P_n\left(\cos \psi_{\text{P}_i\text{Q}}\right) \mathrm{d}\sigma_{\text{Q}} \ , \quad (7.48)$$

i.e. the integral on A_j of the truncated Stokes function, becomes quite small. However, if we could simply ignore $F_j\left(\text{P}\right)$, even when A_j is a neighbour of A_k, then we could delete the second term in (7.46), which at this point would become an observation equation for δW_{0k} only, i.e. we would have the possibility of a local determination of the bias.

Note that what we need now is a pointwise estimate for $\left|F_j\left(\text{P}\right)\right|$ and not the global mean square estimate that has already been found in the previous section. Unfortunately we do not have a strict proof, but only a guess based on the following example.

Example Assume A_j is just a spherical cap C_Δ of radius Δ, then we shall prove that the following approximate majorization holds

$$\left|F\left(\text{P}\right)\right| \lessapprox \frac{2}{\pi} \frac{1}{L+1} \quad (7.49)$$

when P is on the boundary of C_Δ, irrespectively of the value of Δ.

If we take the origin of the spherical coordinates at the centre of C_Δ, from (4.45) and using the summation theorem, we have

$$F(\mathrm{P}) = \frac{1}{2\pi} \sum_{n=L+1}^{M} \sum_{m=-n}^{n} \frac{Y_{nm}(\mathrm{P})}{n-1} \int_{C_\Delta} Y_{nm}(\mathrm{Q})\, d\sigma_\mathrm{Q} =$$
$$= \frac{1}{2\pi} \sum_{n=L+1}^{M} \frac{Y_{n0}(\mathrm{P})}{n-1} \int_{C_\Delta} Y_{n0}(\mathrm{Q})\, d\sigma_\mathrm{Q} =$$
$$= \sum_{n=L+1}^{M} \frac{(2n+1) P_n(\cos\theta_\mathrm{P})}{n-1} \int_0^\Delta P_n(\cos\theta)\sin\theta\, d\theta \,. \tag{7.50}$$

Since

$$(2n+1)\int_0^\Delta P_n(\cos\theta)\sin\theta\, d\theta = \int_{\cos\Delta}^{1} (2n+1) P_n(t)\, dt =$$
$$= \int_{\cos\Delta}^{1} \left[P'_{n+1}(t) - P'_{n-1}(t)\right] dt = P_{n-1}(\cos\Delta) - P_{n+1}(\cos\Delta) \,, \tag{7.51}$$

Eq. (7.50) becomes

$$F(\mathrm{P}) = \sum_{n=L+1}^{M} \frac{P_n(\cos\theta_\mathrm{P})}{n-1} \left[P_{n-1}(\cos\Delta) - P_{n+1}(\cos\Delta)\right] . \tag{7.52}$$

Now we apply a famous asymptotic expression for the Legendre polynomials (Abramowitz and Stegun 1964) claiming that

$$P_n(\cos\theta) = \sqrt{\frac{2}{\pi\sin\theta\cdot n}} \cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + \mathcal{O}\left(n^{-\frac{3}{2}}\right) . \tag{7.53}$$

In particular (7.53) holds for

$$\theta > \frac{3\pi}{4n+2} \,; \tag{7.54}$$

since we have in mind that $n > 200$ and Δ is at least $2°$ or (much) more, the condition (7.54) is met.

So we proceed noting that

$$P_{n-1}(\cos\Delta) - P_{n+1}(\cos\Delta) \cong$$
$$\cong \sqrt{\frac{2}{\pi\sin\Delta}} \left\{ \frac{\cos\left[\left(n+\frac{1}{2}-1\right)\Delta - \frac{\pi}{4}\right]}{\sqrt{n-1}} - \frac{\cos\left[\left(n+\frac{1}{2}+1\right)\Delta - \frac{\pi}{4}\right]}{\sqrt{n+1}} \right\} .$$

Since

$$\frac{1}{\sqrt{n \pm 1}} - \frac{1}{\sqrt{n}} = \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right) ,$$

the above asymptotic relation can be written as

$$\begin{aligned} & P_{n-1}(\cos \Delta) - P_{n+1}(\cos \Delta) \cong \\ & \cong \sqrt{\frac{2}{\pi \sin \Delta \cdot n}} \left\{ \cos\left[\left(n + \frac{1}{2} - 1\right)\Delta - \frac{\pi}{4}\right] - \cos\left[\left(n + \frac{1}{2} + 1\right)\Delta - \frac{\pi}{4}\right] \right\} = \\ & = \frac{2\sqrt{2}}{\pi} \sqrt{\frac{\sin \Delta}{n}} \sin\left[\left(n + \frac{1}{2}\right)\Delta - \frac{\pi}{4}\right] . \end{aligned}$$

So returning to (7.52) and applying (7.53) to $P_n(\cos \theta_P)$ too, we find

$$F(P) = \frac{4}{\pi} \sqrt{\frac{\sin \Delta}{\sin \theta_P}} \sum_{n=L+1}^{M} \frac{\cos\left[\left(n + \frac{1}{2}\right)\theta_P - \frac{\pi}{4}\right] \sin\left[\left(n + \frac{1}{2}\right)\Delta - \frac{\pi}{4}\right]}{n(n-1)} . \quad (7.55)$$

As soon as we put P on the boundary of C_Δ, i.e. we take $\theta_P = \Delta$, we get from (7.55)

$$\begin{aligned} |F(P)| \cong \frac{2}{\pi} \left| \sum_{n=L+1}^{M} \frac{\sin\left[(2n+1)\Delta - \frac{\pi}{2}\right]}{n(n-1)} \right| \lessapprox \\ \lessapprox \frac{2}{\pi} \sum_{n=L+1}^{+\infty} \frac{1}{n^2} \cong \frac{2}{\pi} \frac{1}{L+1} , \end{aligned}$$

and (7.49) is proved.

With this example we see that, at least when A_j is a spherical cap and $L = 200$, the influence of the bias δW_{0j} at its boundary is

$$\left| \frac{\delta W_{0j}}{\gamma} F(P) \right| \leq \left| \frac{\delta W_{0j}}{\gamma} \right| 3.2 \cdot 10^{-3} ,$$

namely well below the 1 cm level, even when $\frac{\delta W_{0j}}{\gamma} = 2$ m. Indeed when $\theta_P > \Delta$, we expect that $F(P)$ is even smaller, as shown in Fig. 7.1 when $\Delta = 5°$.

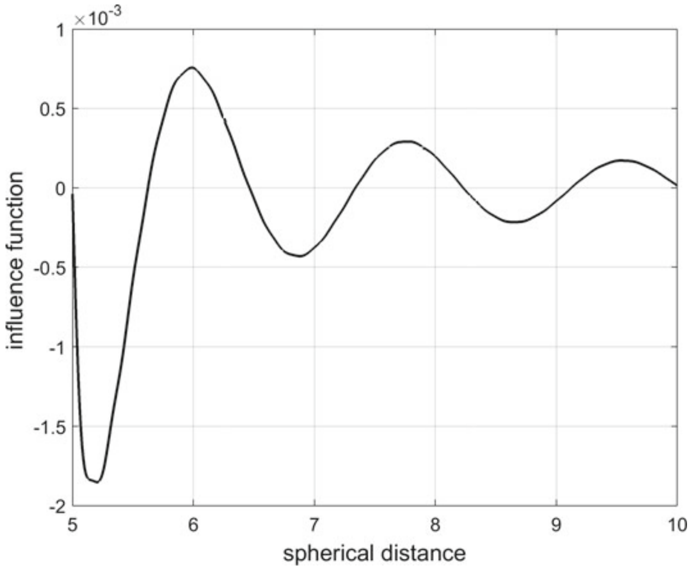

Fig. 7.1 Influence function $F(\theta_{\mathrm{P}})$ for $5° \leq \theta_{\mathrm{P}} \leq 10°$, in the case of a spherical cap C_Δ with $\Delta = 5°$, $L = 200$

Based on the guess supported by the above example, we propose that a local bias δW_{0k} is estimated from the set of observation equations

$$h_i - \widetilde{h}_i^* = \frac{\widehat{T}_{\mathrm{loc}}(\mathrm{P}_i)}{\gamma} + \frac{\delta W_{0k}}{\gamma}\left[1 - F_k(\mathrm{P}_i)\right] + \varepsilon_{\mathrm{res},i}\ , \tag{7.56}$$

where P_i are all the points in the area A_k where both h_i and $\widetilde{h}_i^*$ are available.

We close the section by observing that indeed we could have a situation where several $\{\delta W_{0k}\}$ can be estimated together, although they refer to some areas that do not cover the whole sphere, with an obvious modification of the above discussion. We underline however that in this case it is better that the local estimate of the potential $\widehat{T}_{\mathrm{loc}}$ is computed for the above areas together, because only in this case we shall have a consistent covariance matrix for $\varepsilon_{\mathrm{res}}$ (Reguzzoni and Venuti 2018).

7.5 An Example: The Italian Case

In this paragraph, the local solution of the height datum problem discussed in Sect. 7.4 is applied to the Italian case study. A similar computation has been applied as well to the determination of the geoid bias in Spain (Reguzzoni et al. 2018).

As a matter of facts, Italy has three different height systems based on three different reference tide gauges. The reference tide gauge for the mainland is in Genoa, while heights in Sicily are referred to the Catania tide gauge and those of Sardinia to Cagliari. Due to the different dynamic ocean topography in these three reference stations, inconsistencies at the decimetre level among heights in Italy mainland, Sicily and Sardinia are expected.

The equation to be used in estimating the local biases is (7.56) which can be further simplified for the present computation. In fact, in the Italian case presented here, it can be numerically proved that even considering the complete Eq. (7.16) accounting for the global unification, the term

$$\frac{1}{\gamma}\sum_{j=1}^{K}\delta W_{0j}\,F_j\left(\mathrm{P}_i\right)$$

is smaller than 1 mm. Thus, a fortiori, the corresponding local term in (7.56) can be disregarded.

So, the equation that will be used in the computation is

$$h_i - \widetilde{h}_i^* = \frac{\widehat{T}_{\mathrm{loc}}\left(\mathrm{P}_i\right)}{\gamma} + \frac{\delta W_{0k}}{\gamma} + \varepsilon_{\mathrm{res},i} \tag{7.57}$$

that can be rewritten as

$$\widetilde{\zeta}_k\left(\mathrm{P}_i\right) = \frac{\widehat{T}_{\mathrm{loc}}\left(\mathrm{P}_i\right)}{\gamma} + b_k + \varepsilon_{\mathrm{res},i} \tag{7.58}$$

where $\widetilde{\zeta}_k\left(\mathrm{P}_i\right)$ are the biased height anomalies in the k-th area and b_k the bias to be estimated on the same area.

It can be further assumed that $\widehat{T}_{\mathrm{loc}}$ is estimated as

$$\widehat{T}_{\mathrm{loc}}\left(\mathrm{P}_i\right) = T^L\left(\mathrm{P}_i\right) + T^H\left(\mathrm{P}_i\right) \tag{7.59}$$

where T^L is the prediction of the anomalous potential at point P_i coming from a satellite gravity model to degree L and T^H is the prediction derived from a high degree model, like e.g. EGM2008, from degree $L+1$ to degree H. Although by considering T^H we reintroduce biases through ground gravity data, it can be proved that the impact on the solution is of the order of some millimetres (Gatti et al. 2013). Thus, one can say that a feasible solution for the estimate of b_k can be obtained by the observation equation

$$\widetilde{\zeta}_k\left(\mathrm{P}_i\right) = \frac{T^L\left(\mathrm{P}_i\right) + T^H\left(\mathrm{P}_i\right)}{\gamma} + b_k + \varepsilon_{\mathrm{res},i} \tag{7.60}$$

By separating the observations and the unknowns to be estimated, one gets

$$\widetilde{\zeta}_k\left(\mathrm{P}_i\right)-\frac{T^L\left(\mathrm{P}_i\right)+T^H\left(\mathrm{P}_i\right)}{\gamma}+\varepsilon_{\mathrm{res},i}=b_k\ . \tag{7.61}$$

Now, if one considers N points in the K regions, with $N \geq K$, a linear system of N equations and K unknowns can be solved by least squares adjustment, once the observation error covariance matrix of ε is defined. This matrix has to account for the dispersion of the errors in the ellipsoidal heights derived from GNSS through the covariance matrix C_h, the errors in the normal heights derived from levelling and gravity measurements through $C_{\widetilde{h}^*}$, the commission errors of the satellite-only gravity model up to the degree L through C_{T^L}, and those in the high resolution model from degree $L+1$ up to degree H through C_{T^H}. Thus, assuming the above described errors independent from one another, the proper covariance structure to be used in the adjustment procedure is

$$C_\varepsilon=C_\zeta+C_{T^L}+C_{T^H}=C_h+C_{\widetilde{h}^*}+C_{T^L}+C_{T^H}\ . \tag{7.62}$$

In the Italian test case the least square problem is set by considering 1,068 points with known GNSS ellipsoidal heights and levelling derived heights. Among them, 43 points are in Sicily, 48 in Sardinia and the remaining 977 in the Italian mainland. The heights derived from levelling measurements were obtained by a least squares adjustment of the observations without any correction accounting for gravity effects (Betti et al. 2016). GNSS heights are referred to the ETRF2000 reference frame, epoch 2008.0.

Hence, it must be underlined that in Eq. (7.61) biased geoid undulations $\widetilde{N}_k\left(\mathrm{P}_i\right)$ are used (which, as said, are further biased since no gravity corrections have been applied).

The models components that have been considered in order to evaluate the T^L and the T^H terms are the GOCO-03S satellite gravity only model (Mayer-Gürr et al. 2012) and the EGM2008 global geopotential model. The GOC0-03S model basically combines the ITG-Bonn GRACE solution with the time-wise GOCE one (release R3, that is the third solution based on 1 year and a half GOCE data). The coefficients are available at the website of the International Center for Global Earth Models (ICGEM). Moreover, the GOC0-03S order-wise block diagonal error covariance matrix has been considered in the computation, which practically bears the same information as the full error covariance matrix (Gerlach and Fecher 2012). As for EGM2008 spherical harmonic coefficients, the error coefficient variances and a global grid of local geoid error variances are available. Consistently with GNSS data, the coefficients of the two global models are tide-free, while the levelling data are referred to the mean sea level at the three tide gauges of Genoa, Catania and Cagliari.

Before computing the left hand side of Eq. (7.61), reference frame transformations have to be considered.

The different coordinates have been referred to the most recent frame of the GOCE model. The Italian GNSS data are given in ETRF2000, epoch 2008.0, while GOCE data are in ITRF2008, with an unspecified epoch between 2010 and 2011. Transformations from ETRF2000-2008.0 to ITRF2008-2010/2011 can be performed in three steps. The EUREF transformation parameters have been applied from ETRF2000-2008 to ITRF2000-2008 and then the IERS transformation parameters have been used from ITRF2000-2008 to ITRF2008-2008. Finally, the ITRF2008 coordinates have been updated to epoch 2010/2011 using the mean velocity of a subset of Italian GNSS permanent stations (velocities published by IERS). To this aim, the GNSS permanent stations of Medicina, Genoa, Torino I, Cagliari, Matera, Padova and Perugia have been taken into account.

These transformations accounted for a displacement in the horizontal coordinates of about 50 cm and a 1 cm change in their heights. It can be proved that the impact of these shifts is negligible in terms of the bias estimation (Barzaghi et al. 2016). Similar transformations were not applied to EGM2008 since its reference time is not available.

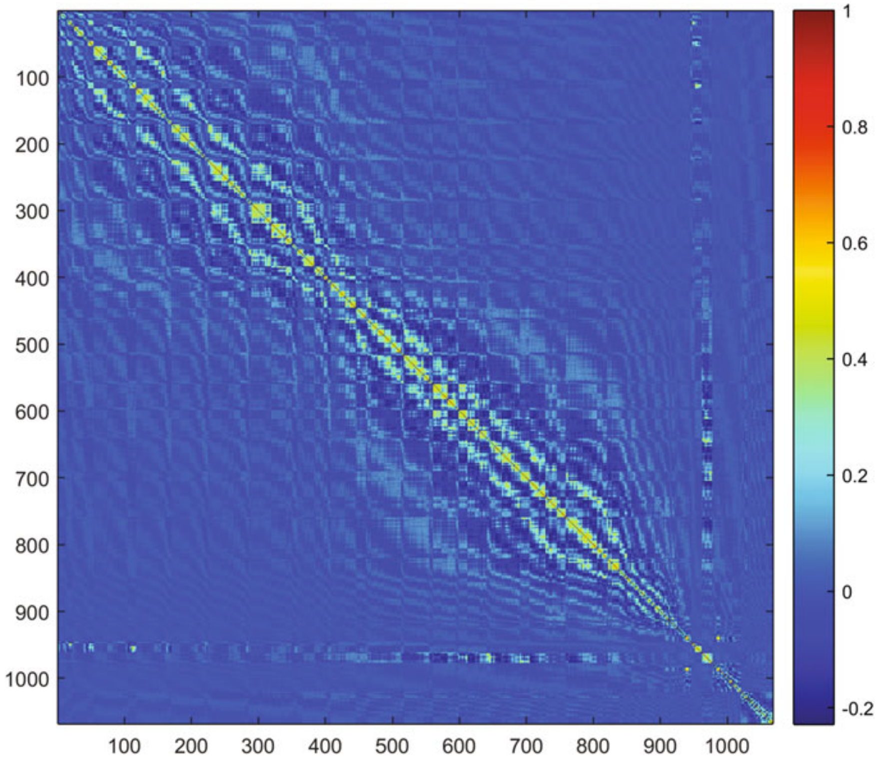

Fig. 7.2 The correlation matrix of ε with $L = 250$

Another key point in setting up the least squares problem in (7.61) is the definition of the stochastic model of the observations. This stochastic model, represented in (7.62), can be evaluated from the available error models. The set of differences between GNSS and levelling heights are assumed to be as an uncorrelated noise, so that it can be set

$$C_\zeta = \sigma_\zeta^2 I \tag{7.63}$$

where I is the identity matrix. The error covariance matrix C_{T^L} of the set of potential values T^L predicted in the GNSS-leveling points from GOC0-03S is obtained by propagation from the given order-wise block diagonal error covariance matrix. The covariance matrix C_{T^H} of the set of potential values T^H computed at the same points from EGM2008, is obtained by propagation from the coefficient error variances properly rescaled accordingly to the geographical map of local geoid errors (Gilardoni et al. 2013). The resulting error correlation matrix, with $\sigma_\zeta^2 = 1$ cm and $L = 250$, is plotted in Fig. 7.2.

Based on this covariance structure, the error in the estimated biases can be computed as a function of the degree L. In the Italian case study, it can be shown that the errors in the estimated biases of Italy mainland, Sicily and Sardinia are not strongly affected by the choice of L (Barzaghi et al. 2016) so that $L = 250$, the full GOC0-03S model resolution, has been selected in the computation.

Different biases estimates have been then computed using different values of σ_ζ, namely 1, 5, 10 and 12 cm. The least squares estimate satisfying the null hypothesis test

$$H_0 : \sigma_0^2 = 1$$

is the one based on $\sigma_\zeta = 12$ cm, which gives the values for the estimated biases that are listed in Table 7.1.

This first result is based on some quite strong simplifications and is hence affected by model errors. Particularly, the use of $\widetilde{N}_k$ (P$_i$), the biased geoid undulation, instead of the biased height anomaly $\widetilde{\zeta}_k$ (P$_i$), can induce distortions in the estimated biases. Nevertheless, the difference between the biases of Italy mainland and Sicily, that is 9.82 cm, is significantly close to the values reported by Istituto Geografico Militare, i.e. 14.1 cm. It is to be underlined that this value has been independently estimated using surveying techniques based on trigonometric levelling trough the Messina Strait coupled with spirit levelling in Sicily and Calabria to form a close loop across the two sides of the strait. So, despite the use of somehow improper data, acceptable results can be obtained by the devised least squares adjustment procedure. Thus, one

Table 7.1 The estimated biases with $\sigma_\zeta = 12$ cm

	Italy mainland (cm)	Sicily (cm)	Sardinia (cm)
b_k	77.22	67.40	97.90
σ_{b_k}	0.52	2.57	2.72

comes to a confirmation that the proposed method is effective in estimating the local biases and can be applied for solving the problem of the height systems unification at local level.

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