Chapter 6
The Relation Between Levelling, Geodetic and Other Unholonomic Heights

6.1 Outline

Spirit levelling is a complex geodetic measurement that combines several elementary steps, already defined in Sect. 4.6 (see also Fig. 4.6), each of them providing a step increment on a short baseline; such increments are then added along a levelling line, joining two stations P and Q. In this way, recalling (4.56), we have a measurement related to the extremes P and Q and to the path connecting them. If we call M a point running along the line, and with the idea to consider each step as infinitesimal, we can write

$$\delta L = \boldsymbol{n}_{\mathrm{M}} \cdot \mathrm{d}\boldsymbol{r}_{\mathrm{M}} \ , \tag{6.1}$$

$$\Delta_{\mathrm{PQ}} L = \int_{\mathrm{P}}^{\mathrm{Q}} \delta L = \int_{\mathrm{P}}^{\mathrm{Q}} \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{r} \ . \tag{6.2}$$

Since the approximate relation

$$\delta L \cong \mathrm{d}H \tag{6.3}$$

holds for a single step, for many practical applications the Eq. (6.2) has been considered as

$$\Delta_{\mathrm{PQ}} L = H_{\mathrm{Q}} - H_{\mathrm{P}} \ , \tag{6.4}$$

especially when levelling is restricted to an area of a few kilometers.

However, since $\mathrm{d}\boldsymbol{r}$ is in general pointing in an almost horizontal direction and certainly not along the vertical, the claim $\delta L = \mathrm{d}H$ is false, as proved and illustrated in Sect. 5.4, so (6.4) is false too; we could say that (6.4) holds with an acceptable approximation only when the path $\overline{\mathrm{PQ}}$ is at most a few kilometers long. So, as we see, there is an intimate connection between levelling observations and geodetic heights, though with some ambiguity that needs to be resolved.

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In Sect. 6.2 we shall first study the relation between $\Delta_{PQ}L$ and dynamic heights. Then in Sect. 6.3 we investigate the normal heights h^*_P, h^*_Q, showing that, contrary to the geodetic tradition, the observation equation in terms of h^* is more natural in that it requires only the knowledge of the anomalous potential T on the surface. On the contrary, in Sect. 6.4, studying the relation between $\Delta_{PQ}L$ and H_P, H_Q, we will show that such observation equation cannot avoid to introduce the knowledge of the topographic masses density; this is in fact intrinsic into the definition of orthometric height. Finally in Sect. 6.5 we shall discuss a different type of unholonomic height, namely the normal orthometric height, that is in fact used by some countries, so that its relation to $\Delta_{PQ}L$ and to other heights needs to be clarified. A final section, Sect. 6.6, of conclusions follows, with recommendations of practical nature.

6.2 The Observation Equation of ΔL in Terms of Dynamic Heights

The definition of dynamic height is (see (5.15))

$$H_P^D = \frac{C\,(P)}{\overline{\gamma}_0}$$

and, as commented in Sect. 5.3, since $\overline{\gamma}_0$ is just a constant, it bears the same information and geometry as the geopotential number $C\,(P) = W_0 - W\,(P)$, as well as the potential $W\,(P)$ itself.

Recalling (6.1) and (6.2), we start our reasoning from

$$\delta L = \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{r} = -\frac{\boldsymbol{g}}{g} \cdot \mathrm{d}\boldsymbol{r} = -\frac{\mathrm{d}W}{g} \tag{6.5}$$

and

$$\Delta_{PQ}L = -\int_P^Q \frac{\mathrm{d}W}{g}\,, \tag{6.6}$$

where the integral is meant to be computed along the levelling line, namely on the Earth surface.

Equation (6.6) can be further elaborated in the following way

$$\Delta_{PQ}L = -\frac{1}{\overline{\gamma}_0}\int_P^Q \frac{\overline{\gamma}_0}{g}\,\mathrm{d}W = -\frac{1}{\overline{\gamma}_0}\int_P^Q \left(\frac{\overline{\gamma}_0 - \gamma + \gamma - g}{g} + 1\right)\mathrm{d}W =$$

$$= \frac{W(\mathrm{P}) - W(\mathrm{Q})}{\overline{\gamma}_0} + \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\overline{\gamma}_0 - \gamma}{\overline{\gamma}_0}\, \delta L + \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\gamma - g}{\overline{\gamma}_0}\, \delta L \,. \tag{6.7}$$

Now, the difference $g - \gamma$ is called gravity disturbance δg and it is related to T by

$$\delta g = g(\mathrm{P}) - \gamma(\mathrm{P}) = \boldsymbol{e}_\gamma \cdot \nabla T \cong -\boldsymbol{\nu} \cdot \nabla T = -T' \,; \tag{6.8}$$

it is known that $\mathcal{O}(\delta g) \sim \mathcal{O}(\Delta g)$, i.e. $\mathcal{O}\left(\frac{\delta g}{\gamma_0}\right) \sim 10^{-4}$. Moreover it is

$$\frac{W(\mathrm{P}) - W(\mathrm{Q})}{\overline{\gamma}_0} = \frac{C(\mathrm{Q}) - C(\mathrm{P})}{\overline{\gamma}_0} = H_{\mathrm{Q}}^{\mathrm{D}} - H_{\mathrm{P}}^{\mathrm{D}} \,.$$

So from (6.7) we derive the observation equation

$$\Delta_{\mathrm{PQ}} L = H_{\mathrm{Q}}^{\mathrm{D}} - H_{\mathrm{P}}^{\mathrm{D}} - \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\gamma - \overline{\gamma}_0}{\overline{\gamma}_0}\, \delta L - \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\delta g}{\overline{\gamma}_0}\, \delta L \,. \tag{6.9}$$

A simple evaluation of the orders of magnitude shows that the first integral in the right hand side of (6.9) can amount up to meters per kilometer of height differences, while the second integral is at most one order of magnitude smaller.

6.3 The Observation Equation of ΔL in Terms of Normal Heights

In this case we return to Eq. (6.1) that we rewrite

$$\delta L = \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{r} = -\frac{\boldsymbol{g}}{g} \cdot \mathrm{d}\boldsymbol{r} = -\frac{\mathrm{d}W}{g} \,. \tag{6.10}$$

Now let us go back to (5.28) and write it in the form

$$\boldsymbol{n} = \boldsymbol{\nu} - \frac{1}{\gamma}(I - P_\nu)\nabla T + \widetilde{\boldsymbol{\delta}} \,, \tag{6.11}$$

where $\widetilde{\boldsymbol{\delta}}$ is given by (5.27) and $I - P_\nu$ is the projection on the horizontal plane, which is orthogonal to $\boldsymbol{\nu}$.

Using (6.11) in (6.10) yields

$$\delta L = \boldsymbol{\nu} \cdot \mathrm{d}\boldsymbol{r} - \frac{1}{\gamma}\nabla T \cdot \mathrm{d}\boldsymbol{r} + \frac{1}{\gamma}\boldsymbol{\nu} \cdot \nabla T\, \boldsymbol{\nu} \cdot \mathrm{d}\boldsymbol{r} + \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r} =$$
$$= \mathrm{d}h - \frac{\mathrm{d}T}{\gamma} + \frac{T'}{\gamma}\mathrm{d}h + \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r}\ . \tag{6.12}$$

We will elaborate the term

$$\frac{\mathrm{d}T}{\gamma} = \mathrm{d}\left(\frac{T}{\gamma}\right) - T\,\mathrm{d}\left(\frac{1}{\gamma}\right) = \mathrm{d}\zeta + T\,\frac{\nabla\gamma \cdot \mathrm{d}\boldsymbol{r}}{\gamma^2}\ . \tag{6.13}$$

To continue our reasoning on (6.13) we use a simplified version of $\dfrac{\nabla\gamma}{\gamma} \cong \dfrac{\nabla\gamma}{\gamma_0}$, namely (see Table 3.2 in Sect. 3.5)

$$\frac{\nabla\gamma}{\gamma_0} \cong -3 \cdot 10^{-4}\,\boldsymbol{\nu} + 0.8 \cdot 10^{-6}\sin 2\varphi\,\boldsymbol{e}_\varphi\ ,$$

and observe that

$$\mathcal{O}\left(\frac{T}{\gamma}\,\frac{\boldsymbol{\nu}\cdot\nabla\gamma}{\gamma}\right) \cong \mathcal{O}\left(\frac{T}{\gamma}\,\frac{\gamma'}{\gamma}\right) \cong 3 \cdot 10^{-5}\ ,$$

$$\mathcal{O}\left(\frac{T}{\gamma}\,\frac{|(I - P_\nu)\,\nabla\gamma|}{\gamma}\right) \cong \mathcal{O}\left(\frac{T}{\gamma}\,\frac{\left|\dfrac{1}{R}\,\dfrac{\partial\gamma}{\partial\varphi}\right|}{\gamma}\right) \sim 10^{-7}\ .$$

So the effect of the horizontal component of the term $\dfrac{T}{\gamma}\,\dfrac{\nabla\gamma}{\gamma} \cdot \mathrm{d}\boldsymbol{r}$, integrated over a 100 km line, is at most 1 cm, while the effect of the vertical component is 3 ΔL cm (ΔL in km), i.e. with a rise of 6 km along the line it can go up to 18 cm.

Therefore the vertical component of this term should be accounted for, especially in mountainous areas, while the horizontal one can be neglected. Therefore, returning to (6.13), we get

$$\frac{\mathrm{d}T}{\gamma} \cong \mathrm{d}\zeta + \frac{T}{\gamma}\,\frac{\gamma'}{\gamma}\,\mathrm{d}h\ ,$$

which, in (6.12), yields

$$\delta L = \mathrm{d}h - \mathrm{d}\zeta - \frac{T}{\gamma}\,\frac{\gamma'}{\gamma}\,\mathrm{d}h + \frac{T'}{\gamma}\mathrm{d}h + \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r} =$$
$$= \mathrm{d}\,(h - \zeta) - \frac{1}{\gamma}\left(-T' + \frac{\gamma'}{\gamma}\,T\right)\mathrm{d}h + \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r} =$$
$$= \mathrm{d}h^* - \frac{\Delta g}{\gamma}\,\mathrm{d}h + \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r}\ . \tag{6.14}$$

This is the sought observation equation of the levelling increment over one step; then $\Delta_{\mathrm{PQ}}L$ has the observation equation

$$\Delta_{\mathrm{PQ}}L = h^*_{\mathrm{Q}} - h^*_{\mathrm{P}} - \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\Delta g}{\gamma}\,\mathrm{d}h + \int_{\mathrm{P}}^{\mathrm{Q}} \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r} \ . \tag{6.15}$$

As promised, all the terms in (6.15) can be computed from surface anomalous quantities. In particular the term

$$\mathrm{DC} = \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\Delta g}{\gamma}\,\mathrm{d}h \ ,$$

also known in literature as dynamic correction, can amount up to 10^{-4} times the levelling increment, namely to several dozens of cm if P is by the sea and Q is on a high mountain.

On the contrary the last term, only recently reported in literature (see Betti et al. 2016),

$$\int_{\mathrm{P}}^{\mathrm{Q}} \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r} \cong 5.3 \cdot 10^{-3} \int_{\mathrm{P}}^{\mathrm{Q}} \sin 2\varphi \, \frac{h}{a} \, (\mathcal{M} + h)\,\mathrm{d}\varphi \ ,$$

can obviously give a sensible contribution only for a levelling line at altitude and developing in the north-south direction. For instance a levelling line on the Andes, 60 km long, around $\varphi = -45°$, at an altitude of 2 km, will have a correction term $\int_{\mathrm{P}}^{\mathrm{Q}} \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r}$ of about 10 cm.

All in all, we have shown that by calling NC, normal correction, the term

$$\mathrm{NC} = \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\Delta g}{\gamma}\,\mathrm{d}h - \int_{\mathrm{P}}^{\mathrm{Q}} \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r} \ , \tag{6.16}$$

the levelling increment has observation equation

$$\Delta_{\mathrm{PQ}}L = h^*_{\mathrm{Q}} - h^*_{\mathrm{P}} - \mathrm{NC} \ , \tag{6.17}$$

where the last term can be effectively computed by surface quantities. This means, for instance, that nowadays NC can be computed to a sufficient degree of accuracy from some global model of T, e.g. from EGM2008; note that, on the contrary, such a model could not be used to compute quantities inside the masses, where one should use the methods explained in Sect. 5.2.

Remark One possible objection to the computability of NC is that in principle the expression (6.16) should be reckoned along the "true" levelling line, the profile of which should therefore be known to compute the DC, while the second term is affected in any circumstance by a negligible error. Yet we can respond that assuming that the levelling line is known with some 10 m errors in height, what nowadays is easy to achieve e.g. by Real Time Kinematic GNSS observations, then Δg is known with at most 1 mGal error, implying that DC can be computed with an error of $10^{-6}\ \Delta_{PQ}L$. This is certainly negligible; a similar consideration holds for the term $\int_P^Q \widetilde{\boldsymbol{\delta}} \cdot \mathrm{d}\boldsymbol{r}$.

The conclusion of this section is that levelling networks should be compensated, after the application of normal corrections, directly in terms of normal heights and, to this aim, the use of global models to compute normal corrections can give accurate enough results, with particular caution in areas of rough topography.

6.4 The Observation Equation of ΔL in Terms of Orthometric Heights

The wanted observation equation is easily derived from (6.16) and (6.17), taking into account the following elementary relation, which takes advantage of (5.38) and (5.43),

$$\begin{aligned} h_Q^* - h_P^* &= h_Q - h_P - \zeta_Q + \zeta_P = \\ &= \left(h_Q - N_Q\right) - (h_P - N_P) + \left(N_Q - \zeta_Q\right) - (N_P - \zeta_P) = \\ &= H_Q - H_P + \int_{Q_0}^{Q} \frac{\Delta g}{\gamma}\,\mathrm{d}h - \int_{P_0}^{P} \frac{\Delta g}{\gamma}\,\mathrm{d}h\ , \end{aligned} \tag{6.18}$$

where Q_0 and P_0 are the projections of Q and P, respectively, on the geoid.

Defining the orthometric correction OC as

$$\mathrm{OC} = \mathrm{NC} - \int_{Q_0}^{Q} \frac{\Delta g}{\gamma}\,\mathrm{d}h + \int_{P_0}^{P} \frac{\Delta g}{\gamma}\,\mathrm{d}h \tag{6.19}$$

and substituting (6.18) and (6.19) into (6.17), we get

$$\Delta_{PQ}L = H_Q - H_P - \mathrm{OC}\ . \tag{6.20}$$

As we see, contrary to the case of the normal correction, in (6.19) we find that the orthometric correction cannot be computed without making hypotheses on the density of topographic masses. In fact, recalling (5.46), we can also write

$$\mathrm{OC} = \int_{\mathrm{P}}^{\mathrm{Q}} \frac{\Delta g}{\gamma}\,\mathrm{d}h - H_{\mathrm{Q}}\frac{\Delta g_{\mathrm{Q}}}{\gamma} + H_{\mathrm{P}}\frac{\Delta g_{\mathrm{P}}}{\gamma} + \int_{0}^{H_{\mathrm{Q}}} q\,\frac{h}{\gamma}\,\mathrm{d}h - \int_{0}^{H_{\mathrm{P}}} q\,\frac{h}{\gamma}\,\mathrm{d}h\ , \tag{6.21}$$

where $q = 4\pi G\rho$. Note that in (6.21) the first integral is along the levelling line, while the last two are inside the masses.

Since here we are reasoning apart from measurement errors, we could say that (6.21) can establish an orthometric coordinate system in a certain area if starting from a point P_0 on the geoid, we could reach every point Q in the area, connecting it to P_0 by a levelling line. In this case, noting that $\mathrm{P} = \mathrm{P}_0$, $H_{\mathrm{P}_0} = 0$, we have from (6.16) and (6.17)

$$H_{\mathrm{Q}} = \Delta_{\mathrm{P}_0\mathrm{Q}}L + \int_{\mathrm{P}_0}^{\mathrm{Q}} \frac{\Delta g}{\gamma}\,\mathrm{d}h - H_{\mathrm{Q}}\frac{\Delta g_{\mathrm{Q}}}{\gamma} + \int_{0}^{H_{\mathrm{Q}}} q\,\frac{h}{\gamma}\,\mathrm{d}h\ ; \tag{6.22}$$

this explicit formula is fundamental to understand the next remark.

Remark In geodetic literature it is often written that the relation (5.38)

$$h = H + N$$

can be used to assess the accuracy of a gravimetric geoid, by comparing it with the difference $h - H$, where h can be obtained by GNSS measurements, while H can be obtained by levelling.

We claim that the statement is wrong, at least at the level of accuracy of one centimeter. In fact we know from Sect. 4.7 that surface gravimetric data can provide only T, from the telluroid upward, and from this the height anomaly can be computed via Bruns' relation, $\zeta = \dfrac{T}{\gamma}$; so N can be derived only by making hypotheses on the topographic masses, e.g. by the (approximate) relation (5.47)

$$N_{\mathrm{Q}} = \frac{1}{\gamma}\left(T_{\mathrm{Q}} + \Delta g_{\mathrm{Q}} H_{\mathrm{Q}}\right) - \int_{0}^{H_{\mathrm{Q}}} q\,\frac{h}{\gamma}\,\mathrm{d}h\ .$$

On the other hand the above relation clearly shows that an error $\varepsilon\,(q)$, due to an error in ρ, would cause in the computation of N an error $\varepsilon\,(N)$ given by

$$\varepsilon\,(N) = -\int_{0}^{H_{\mathrm{Q}}} \varepsilon\,(q)\,\frac{h}{\gamma}\,\mathrm{d}h\ ,$$

while the same error in q would generate in H an error exactly equal in modulus but opposite in sign, so that the relation (5.38) can continue to hold, despite the fact that both H and N are affected by errors. As for the order of magnitude of such an error, one can use the rough appraisal

$$|\varepsilon(N)| = -\mathcal{O}\left(\int_0^{H_Q} \varepsilon(q)\, \frac{h}{\gamma}\, \mathrm{d}h\right) \sim \frac{2\pi G\, \varepsilon(\rho)\, H^2}{\gamma_0} \; ;$$

therefore, with an error of 10% in ρ, this would give

$$|\varepsilon(N)| \sim 10^{-5} H^2 \;, \quad (H \text{ in km}) \;,$$

which is 1 cm at $H = 1$ km, but 4 cm at $H = 2$ km, and so forth. So we expect that, in particular in mountainous areas, both H and N might be affected by centimetric errors without that (5.38) could reveal it.[1]

We think that the right approach would be to evaluate normal heights directly from levelling, as explained in Sect. 6.3, and then the height anomalies derived by some solution T of the GBVP, to be tested with the relation

$$h = h^* + \zeta \;,$$

where all terms can be observed and computed independently.

6.5 Levelling and Normal Orthometric Heights: An Unholonomic Coordinate

In a sense an unholonomic coordinate is a contradiction in terms, in that it is not a function of a point, as we defined it in Sect. 2.2, but rather a function of a point and a path, as it happens when we make line integrals of non-exact differential forms.

We shall deviate here from the approach of the previous sections and, instead of starting from the observation equation of δL or ΔL, we shall rather start from the other side, namely the definition of normal orthometric heights.

Borrowing for instance from the "Geodetic Glossary" of the National Geodetic Survey, we define the normal orthometric height, H^{no}, as

$$H_Q^{no} = \frac{1}{\overline{\gamma}^{no}} \int_{P_0}^{Q} \gamma\, \delta L \;, \tag{6.23}$$

[1]The authors are aware that while editing the book the same result has been independently published by (Sjoberg 2018); we are then happy to acknowledge this coincidence, confirming our findings.

where P_0 is an emanation point on the geoid and $\overline{\gamma}^{\text{no}}$ is the mean value of γ along the ellipsoidal normal, up to H^{no} itself, i.e.

$$\overline{\gamma}^{\text{no}} = \frac{1}{H_Q^{\text{no}}} \int_0^{H_Q^{\text{no}}} \gamma(z)\, dz \,. \tag{6.24}$$

Indeed $\overline{\gamma}^{\text{no}}$ is a function of point and in fact, using the approximate formulas of Table 3.2, we can even give its explicit form, namely

$$\overline{\gamma}^{\text{no}} = \gamma_e(\varphi) - \frac{1}{2}\gamma_1(\varphi)\, H^{\text{no}} + \frac{1}{3}\gamma_2(\varphi)\,(H^{\text{no}})^2 \,. \tag{6.25}$$

Indeed, as it already happened with Helmert's definition of orthometric height, (6.24) is an implicit equation for H^{no}.

On the other hand H^{no} is not a holonomic coordinate because

$$\gamma\,\delta L = -\frac{\gamma}{g}\, dW \tag{6.26}$$

is certainly not an exact differential: in fact $\frac{\gamma}{g}$ is not constant on equipotential surfaces. As a matter of fact, even going from P_0 to another point Q_0 on the geoid, we are not sure to find $H_{Q_0}^{\text{no}} = 0$.

Yet the rationale behind (6.23) as a substitute of H_Q, is that, as nicely stated by B. Heck (private communication), "at least the average variations of gravity due to latitude and height effect was considered, while the irregular variations of the gravity field had been neglected".

The integral in (6.23) is called spheropotential number, C', and for it one has

$$C' = \int_{P_0}^{Q} \gamma\,\delta L = \int_{P_0}^{Q} (\gamma - g)\,\delta L + \int_{P_0}^{Q} g\,\delta L \,. \tag{6.27}$$

On the other hand, as we have already seen in Sect. 6.2, it is

$$g - \gamma = \boldsymbol{e}_\gamma \cdot \nabla T \cong -\boldsymbol{\nu} \cdot \nabla T = -T' = \delta g \,.$$

Moreover

$$g\,\delta L = -\boldsymbol{g} \cdot d\boldsymbol{r} = -dW \,,$$

so that (6.27) becomes

$$C' = C + \int_{P_0}^{Q} \delta g\,\delta L$$

and (6.23) reads

$$H^{\text{no}} = \frac{C}{\overline{\gamma}^{\text{no}}} - \frac{1}{\overline{\gamma}^{\text{no}}} \int_{\text{P}_0}^{\text{Q}} \delta g \, \delta L \, . \tag{6.28}$$

Now consider that, by definition of normal height,

$$\begin{aligned} C &= W_0 - W(h) = U_0 - U\left(h^*\right) = -\int_0^{h^*} \boldsymbol{\gamma} \cdot \boldsymbol{\nu} \, \mathrm{d}h \cong \\ &\cong \int_0^{h^*} \gamma \, \mathrm{d}h = h^* \frac{1}{h^*} \int_0^{h^*} \gamma \, \mathrm{d}h \equiv h^* \overline{\gamma}^* \, ; \end{aligned} \tag{6.29}$$

here we have denoted by $\overline{\gamma}^*$ the mean of γ between 0 and h^*.

We anticipate that $\delta H^{\text{no}} = H^{\text{no}} - h^*$ is certainly smaller than 1 m, therefore we see from (6.25), keeping only the main term in γ_1 which is enough for the present calculation, that

$$\left|\overline{\gamma} - \overline{\gamma}^*\right| \lessapprox 0.15 \text{ Gal km}^{-1} \cdot 10^{-3} \text{ km} = 1.5 \cdot 10^{-3} \text{ Gal} \, .$$

Therefore

$$\frac{C}{\overline{\gamma}^{\text{no}}} = \frac{\overline{\gamma}^*}{\overline{\gamma}^{\text{no}}} h^* = h^* + \frac{\overline{\gamma}^* - \overline{\gamma}^{\text{no}}}{\overline{\gamma}^{\text{no}}} h^* \, ,$$

where the last term is of the order of $1.5 \; 10^{-7} \; h^*$, i.e. less than 1 mm even for $h^* = 6\,\text{km}$. So we can put

$$\frac{C}{\overline{\gamma}^{\text{no}}} \sim h^* \, ,$$

to find from (6.28)

$$H^{\text{no}} = h^*_{\text{Q}} - \frac{1}{\overline{\gamma}^{\text{no}}} \int_{\text{P}_0}^{\text{Q}} \delta g \, \delta L \, . \tag{6.30}$$

With (6.30) we can verify a posteriori that our guess that $H^{\text{no}} - h^*$ is less than 1 m is correct; in fact $\mathcal{O}\left(\int_{\text{P}_0}^{\text{Q}} \delta g \, \delta L\right) \sim 10^{-4} \, \Delta L$, i.e. 60 cm for $\Delta L = 6\,\text{km}$!

On a theoretical ground, (6.30) shows that there cannot be much advantage in using H^{no} instead of h^*. Yet, for the sake of completeness, let us further develop (6.30) to find the relation between H^{no} and the levelling observable $\Delta_{\text{P}_0\text{Q}} L$.

Going back to (6.17) and observing that when $\text{P}_0 \in \mathcal{G}$ then $h^*_{\text{P}_0} = 0$, we see that

$$h_Q^* = \Delta_{P_0Q}L + \int_{P_0}^{Q} \frac{\Delta g}{\gamma}\, dh - \int_{P_0}^{Q} \widetilde{\boldsymbol{\delta}} \cdot d\boldsymbol{r} \,.$$

Using this relation in (6.30), we get

$$H^{no} = \Delta_{P_0Q}L + \int_{P_0}^{Q} \frac{\Delta g}{\gamma}\, dh - \int_{P_0}^{Q} \frac{\delta g}{\overline{\gamma}^{no}}\, \delta L - \int_{P_0}^{Q} \widetilde{\boldsymbol{\delta}} \cdot d\boldsymbol{r} \,. \tag{6.31}$$

The integral of $\frac{\delta g}{\overline{\gamma}^{no}}\, \delta L$ can indeed be transformed into an integral of $\frac{\delta g}{\overline{\gamma}^{no}}\, dh$ because $\frac{\delta g}{\overline{\gamma}^{no}}$ is already of a maximum order of 10^{-4}. Moreover by writing

$$\frac{\Delta g}{\gamma} - \frac{\delta g}{\overline{\gamma}^{no}} = \frac{\Delta g - \delta g}{\gamma} + \left(\frac{1}{\gamma} - \frac{1}{\overline{\gamma}^{no}}\right)\delta g \cong \frac{\gamma'}{\gamma}\frac{T}{\gamma} - \frac{\gamma - \overline{\gamma}^{no}}{\gamma}\frac{\delta g}{\gamma} \,,$$

we easily verify that the integral in dh of the last term is irrelevant, so that (6.31) becomes

$$H^{no} = \Delta_{P_0Q}L + \int_{P_0}^{Q} \frac{\gamma'}{\gamma}\, \zeta\, dh - \int_{P_0}^{Q} \widetilde{\boldsymbol{\delta}} \cdot d\boldsymbol{r} \,. \tag{6.32}$$

A fast evaluation of the orders of magnitude of the correction terms in (6.32) shows that in general these are smaller than NC or OC; yet the price to pay in using (6.32) is that the so calculated value does depend on the path between P_0 and Q because H^{no} is unholonomic. A recent study with a precise numerical evaluation of the effects of using H^{no} in Australia, i.e. a nation that has officially adopted a normal orthometric height system (Featherstone and Kuhn 2006), can be found in Filmer et al. (2010).

6.6 Conclusions

Since the matter has a relevant practical impact on the adoption of national height systems for geodetic purposes, we like to summarize the relevant conclusions that one can draw from the discussions of the chapters:

1. levelling measurements should always be accompanied by corrections that depend on the height coordinate chosen,
2. to compute corrections we need an approximate position of levelling stations, say with 10 m accuracy; this is easily achievable by RTK observations that should always accompany levelling, especially in mountainous areas,

3. all corrections involving the knowledge of the anomalous potential on the Earth surface and outside can be computed to a sufficient accuracy by a good global model of T; obviously a good local model of T will do a better job,
4. the use of orthometric heights implies the application of the OC which depends on the knowledge of density of topographic masses; since such a detailed knowledge is usually not available, the OC can be computed only by making hypotheses on ρ, like $\rho = 2.67$ g cm^{-3}, which can imply a systematic error up to several centimeters, especially in mountainous areas,
5. normal heights, with their effectively computable NC, seem to be the most natural coordinates to compensate levelling networks; moreover they are consistent with the theory of the GBVP, which is one root of the foundations of Physical Geodesy,
6. the relation

$$h = H + N$$

should not be used to assess the accuracy of the estimated geoid at centimetric level, because H and N can hide errors equal in modulus and opposite in sign up to several centimeters; rather the relation

$$h = h^* + \zeta$$

can be used to assess the accuracy of the quasi-geoid, ζ, with systematic errors below the centimeter.

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