Chapter 5
Geodetic Coordinate Systems

5.1 Outline

In this chapter we return to the concept of coordinates, with a particular focus on various types of geodetic heights used in the geodetic environment.

We could say that there is a hierarchy of coordinate systems that goes from the most natural or physical ones, based on quantities related to the gravity field, to those more geometric, for which the gravity field plays almost no role.

The latter group includes the Cartesian triad, which constitutes the Terrestrial Reference System, co-rotating, in the mean, with the body of the Earth, with z axis along the rotation axis and the origin at the barycentre. Another coordinate system that shares the same characteristics is the terrestrial ellipsoidal coordinate system, which is centered at the same origin, namely the barycentre, has the polar axis along the rotation axis, is co-rotating with the Earth at the same mean angular velocity ω as (x, y, z) and has shape and dimension depending, as discussed in Sect. 3.5, on global gravimetric quantities like μ and J_2. The geometric properties of such systems have been discussed in Sects. 2.3.1 and 2.3.3, including the transformations of one into the other, so they will not be re-discussed here. We shall rather concentrate on the most natural coordinates, like the Hotine-Marussi system, the Helmert system, the Molodensky system. A particular care will be put in studying the transformations of such systems into ellipsoidal coordinates.

To achieve this, in particular for the so called orthometric heights, we will need to continue the potential and the gravity into the layer of the topographic masses. This can be done only by making some hypotheses on the mass density distribution and by applying suitable regularizing rules. Fortunately when this is needed only for the anomalous potential or gravity anomalies, as it is in our case, the result does not depend much on the error of the density model, so that the method can provide sensible answers.

It is for this reason that we shall open the chapter with a section on the subject of the continuation of the gravity field inside the masses.

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5.2 On the Continuation of Gravity into the Topographic Layer

We have already introduced in Sect. 3.3 the lines of the vertical $\{L_v\}$ and we have recalled the relation (3.28) that we repeat here

$$\frac{\mathrm{d}g}{\mathrm{d}\ell} = -2\,\mathcal{C}g + 4\pi G\rho - 2\,\omega^2 \; . \tag{5.1}$$

We observe that, if we assume to know the mean curvature $\mathcal{C}\,(\mathrm{P})$ and the mass density $\rho\,(\mathrm{P})$, the Eq. (5.1) can be taken as an ordinary differential equation for g that could be integrated along L_v from a point P on the surface S, where we assume to know the value $g\,(\mathrm{P}) = g_{\mathrm{P}}$, down to P_0 on the geoid $\mathcal{G}$, so to provide the value g_{Q} at any point on L_v (see Fig. 5.1).

We shall make two basic assumptions that will allow us to pursue the above program. Namely, calling $\ell = \ell_{\mathrm{Q}}$ the curvilinear coordinate along L_v, with origin in P_0 and positive upward, we shall assume that

$$\mathcal{C}\,(\mathrm{Q}) = \frac{1}{R+\ell} \; , \tag{5.2}$$

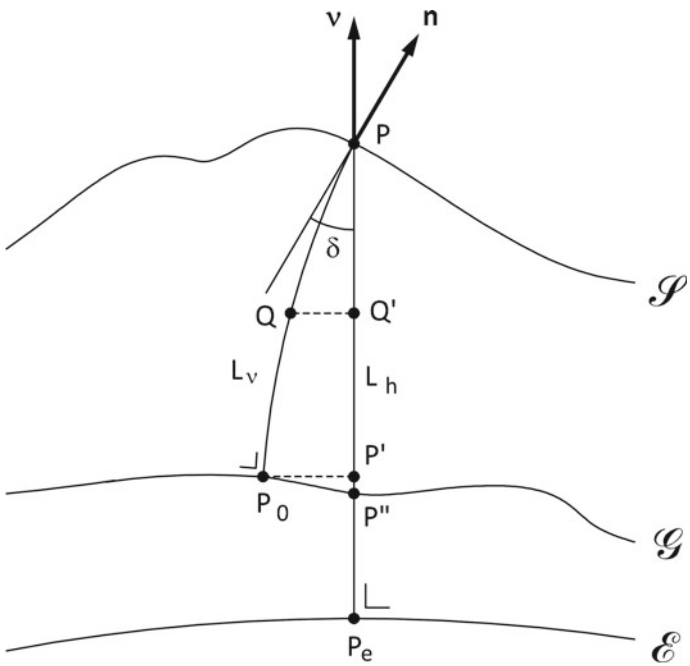

Fig. 5.1 The geometry of L_v and L_h (normal to $\mathcal{E}$) passing through the same point P, between the surface S and the ellipsoid $\mathcal{E}$

with R the mean radius of the Earth. A discussion of the error implied by such a drastic simplification can be found e.g. in Sansó and Sideris (2013, Chap. 2, page 92).

Moreover we shall assume that

$$\rho(\mathrm{Q}) = \rho_{\mathrm{C}} , \tag{5.3}$$

with $\rho_{\mathrm{C}} = 2.67 \text{ g cm}^{-3}$, the credited mean value of the density of the crust. We shall see later on how small (or large) can be the error induced by (5.3).

If we call

$$q = 4\pi G\rho_{\mathrm{C}} , \quad p = q - 2\omega^2 \cong 0.22274 \text{ mGal m}^{-1}$$

the Eq. (5.1) can be integrated, giving

$$g(\mathrm{Q}) \equiv g(\ell) = \frac{(R+\ell_{\mathrm{P}})^2}{(R+\ell)^2} g_{\mathrm{P}} - \frac{p}{3}\left[\frac{(R+\ell_{\mathrm{P}})^3}{(R+\ell)^2} - (R+\ell)\right] , \tag{5.4}$$

where indeed $g_{\mathrm{P}} = g(\ell_{\mathrm{P}})$, ℓ_{P} are constants that we assume to know.

We note here that

$$\left(\frac{R+\ell_{\mathrm{P}}}{R+\ell}\right)^2 \geq 1 ,$$

indicating that in principle the integration could become unstable if we go very deep. In reality, for $\ell_{\mathrm{P}} \leq 6$ km, one has

$$\left(\frac{R+\ell_{\mathrm{P}}}{R+\ell}\right)^2 \leq \left(1+\frac{\ell_{\mathrm{P}}}{R}\right)^2 \cong 1 + 2\cdot 10^{-3} ,$$

meaning that in the topographic layer the instability does not yet manifest itself sensibly. One has to remark as well that the hypothesis (5.2) has also a strong regularizing effect, because one can prove that $\mathcal{C}$ has a high frequency variability too, depending on the horizontal Laplacian of T. Yet the level of this effect can be considered as negligible in the present context, as commented in Sansó and Sideris (2013, Chap. 2, page 108). So we are left with the regularized downward continuation formula (5.4) for g.

Now, note that (5.4) can give us also the continuation of W to any point Q along L_v. In fact we have, by definition of L_v,

$$g = -\frac{\mathrm{d}W}{\mathrm{d}\ell}$$

so that

$$W(\mathrm{Q}) = W(\mathrm{P}) + \int_{\mathrm{Q}}^{\mathrm{P}} g \, \mathrm{d}\ell ,$$

the integral being computed along the vertical. With (5.4) and taking $Q \equiv P_0 \in \mathcal{G}$, so that $W(Q) = W_0$, we find

$$W_0 = W(P) + g_P\,\ell_P\left(1 + \frac{\ell_P}{R}\right) - \frac{p}{2}\,\ell_P^2 - \frac{p}{3}\,\frac{\ell_P^3}{R}\,. \tag{5.5}$$

A fast calculation of the order of magnitude of the last three terms in (5.5), divided by g_P to transform them in lengths and fixing $\ell_P \cong 6$ km, shows that the first term is of the order of ℓ_P, the second of the order of 3.6 m, the last of the order of 2.5 mm; this says that for all practical purposes, in the topographic layer, the last term can be safely neglected.

Finally let us assess the errors committed in continuing g down to the geoid $(\ell_Q = 0)$ due to a model error in ρ. We assume that, as a maximum value, $\delta\rho = 10^{-1}\rho$. Then

$$\delta g = \frac{\delta p}{3}\,R\left[\left(1 + \frac{\ell_P}{R}\right)^3 - 1\right] \cong -\delta p\,\ell_P\,,$$

with $\delta p \sim 2 \cdot 10^{-2}$ Gal km^{-1}. As we can see, in mountainous areas, where $\ell_P > 1$ km, the error in g can be very large, at least for such large errors in ρ. In any circumstance, in such areas we expect an error at least at the level of several mGal.

A further comment is that we assumed $g(P)$ and ℓ_P to be known; however, according to our discussion in Sect. 4.6, we can assume that $g(P)$ and $W(P)$ are known; so (5.5), where both $W(P)$ and W_0 are known, can be rather considered as a means to derive ℓ_P, neglecting as we said the last cubic term. In this case an error in ℓ_P is approximately given by

$$g_P\,\delta\ell - p\,\ell\,\delta\ell - \frac{\delta p}{2}\,\ell^2 = 0$$

or, with a justified simplification,

$$\delta\ell = \frac{\delta p\,\ell^2}{2g_P}\,.$$

With $\delta p = 2 \cdot 10^{-2}$ Gal km^{-1} and $g_P \cong 10^3$ Gal, this gives

ℓ_P	1 km	2 km	4 km
$\delta\ell$	1 cm	4 cm	16 cm

showing that the error is small, but not completely negligible, especially for high mountains.

A last point has to be raised before closing the section, namely the Eq. (5.4) is significantly plagued by a systematic error, because it is derived from (5.1), which is exact, under the hypothesis (5.2), which is very rough. In interpreting (5.1) an error in $\mathcal{C}$ multiplies g, i.e. about 10^6 mGal; if we were able to transform (5.1) into an equation for a variational quantity like Δg, then the error in $\mathcal{C}$ would multiply something of the order of 10^2 mGal, reducing significantly its impact. This is possible indeed, as shown in Sansó and Sideris (2013, Part I, Sect. 2.4), because, similarly to (5.1), one can write the equation for the normal gravity

$$\frac{d\gamma}{d\ell} = -2\,\mathcal{C}_0\,\gamma - 2\,\omega^2 \,, \tag{5.6}$$

where we have taken into account that the normal vertical lines have the same length as the ellipsoidal height (cf. (3.66)), so that $\ell \cong h$ in this case, and that $\rho = 0$ for the normal field outside $\mathcal{E}$. Then subtracting (5.6) from (5.1) and exploiting the appraisal

$$|\mathcal{C} - \mathcal{C}_0| \le \frac{10^{-3}}{R} \,, \tag{5.7}$$

one arrives at the equation $(q = 4\pi G\rho)$

$$\frac{\partial \Delta g}{\partial \ell} = -2\,\mathcal{C}_0\,\Delta g + q \,, \tag{5.8}$$

where the hypothesis $\mathcal{C}_0 \sim \dfrac{1}{R+\ell}$ produces an error of the order of 10^{-7} mGal $\text{m}^{-1} \ll q$.

5.3 The Hotine-Marussi Triad (Λ, Φ, W)

The two astrogeodetic coordinates $\Sigma = (\Lambda, \Phi)$, respectively longitude and latitude, are related to the direction of the vertical $\boldsymbol{n}$ and its Cartesian components in the geocentric (x, y, z), by the relation

$$\boldsymbol{n} = -\frac{\boldsymbol{g}}{g} = \begin{vmatrix} \cos\Phi\,\cos\Lambda \\ \cos\Phi\,\sin\Lambda \\ \sin\Phi \end{vmatrix} . \tag{5.9}$$

As $\boldsymbol{n}$, Σ can be determined by astrogeodetic observations, that first recover $\boldsymbol{n}$ in a celestial system and then rotate the vector to reckon its components in the terrestrial system (for details, see for instance Vanìcek and Krakiwsky 1986). W is just the gravity potential and it completes the triad.

As already observed in the introduction, the function $W(\mathrm{P})$ cannot be used as height throughout the whole exterior space. Indeed on the equatorial plane, at a distance of about seven times the radius of the Earth, W attains a minimum value and then it starts increasing for $r \to \infty$.

A different question is whether (Λ, Φ, W) can constitute a real, unambiguous coordinate system in our layer of interest. This is true if on the equipotential surfaces deployed in our relevant region, it never happens that $\boldsymbol{n}$ can become parallel at two different points; this in fact would mean that two different points have the same coordinates. This is the same to say that equipotential surfaces in the topographic layer are convex.

Although it is not impossible to find mass distributions that create non-convex equipotentials (Bocchio 1981), this seems not to be the case for the Earth. So we shall assume that in the topographic layer, (Λ, Φ, W) constitutes a coordinate system without singular points, at least in the correspondence $(\Lambda, \Phi, W) \to \mathrm{P}$; the inverse correspondence indeed displays the typical singularity of spherical coordinates already discussed in Sect. 2.3.

One important statement concerns the coordinate line L_w. This in fact is defined to be the line along which $\Lambda = \Lambda_0$, $\Phi = \Phi_0$, both being constant, i.e.

$$L_w \equiv \{\mathrm{P}\,;\ \ \boldsymbol{n}(\mathrm{P}) = \boldsymbol{n}_0\}\ , \tag{5.10}$$

with $\boldsymbol{n}_0 = \boldsymbol{n}(\mathrm{P}_0)$ and P_0 is any point, e.g. on the geoid, on which $W = W_0$. The point is that, if $\boldsymbol{n}_0$ is orthogonal to the geoid, $\{W = W_0\}$ at P_0, the same is not any more true for points P on which $W(\mathrm{P}) < W_0$, because the equipotential surfaces $\left\{W = \overline{W}\ ;\ \ \overline{W} < W_0\right\}$ are not parallel to the geoid, as discussed in Sect. 3.3. The lines L_w are called, according to T. Krarup, isozenithal lines and, as we see, they are not coinciding with the lines of the vertical L_v. The situation is illustrated in Fig. 5.2.

It is possible to write the differential equation of isozenithal lines by the reasoning that we sketch hereafter.

We start from (5.9) and we note that, by an elementary differential calculus, when we move $\mathrm{P} \equiv \{\boldsymbol{r}\}$ by an infinitesimal $\mathrm{d}\boldsymbol{r}$ so that $\boldsymbol{g}(\boldsymbol{r})$ goes into $\boldsymbol{g}(\boldsymbol{r} + \mathrm{d}\boldsymbol{r})$, we have

$$\mathrm{d}\boldsymbol{n} = -\frac{1}{g}\,(I - P_n)\,\mathrm{d}\boldsymbol{g}\ , \tag{5.11}$$

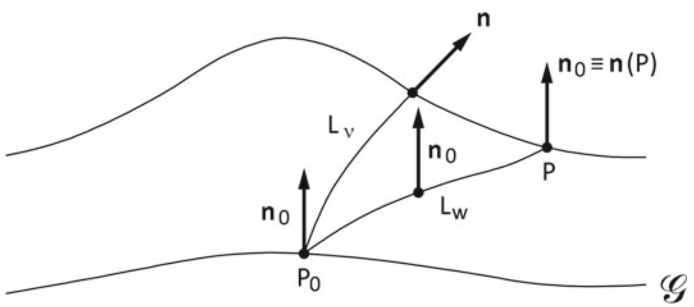

Fig. 5.2 The different paths of L_v and L_w through P_0

where $P_n \, \mathrm{d}\boldsymbol{g} = \boldsymbol{n} \, (\boldsymbol{n} \cdot \mathrm{d}\boldsymbol{g})$ is the projection of $\mathrm{d}\boldsymbol{g}$ on $\boldsymbol{n}$. Now, when $\mathrm{d}\boldsymbol{r}$ is along L_w, i.e. $\mathrm{d}\boldsymbol{r} = \boldsymbol{e}_w \, \mathrm{d}\ell$, we must have $\mathrm{d}\boldsymbol{n} = 0$. So along L_w

$$\mathrm{d}\boldsymbol{g} = P_n \, \mathrm{d}\boldsymbol{g} \tag{5.12}$$

has to hold. But (5.12) says that

$$\mathrm{d}\boldsymbol{g} \parallel \boldsymbol{n} \; . \tag{5.13}$$

On the other hand

$$\mathrm{d}\boldsymbol{g} = M \mathrm{d}\boldsymbol{r} = M \boldsymbol{e}_w \, \mathrm{d}\ell \; ,$$

with M the Marussi tensor (Marussi 1985), i.e. in Cartesian coordinates

$$M = \left[\frac{\partial^2 W}{\partial x_i \, \partial x_k} \right] \; .$$

So (5.13) says that

$$M \boldsymbol{e}_w \parallel \boldsymbol{n}$$

or, also recalling that $\boldsymbol{n} = \boldsymbol{n}_0$ along L_w,

$$\boldsymbol{e}_w \parallel M^{-1} \boldsymbol{n} \equiv M^{-1} \boldsymbol{n}_0 \; .$$

So finally the equation of L_w is determined by the tangent field

$$\boldsymbol{e}_w = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\ell} = \frac{M^{-1} (\boldsymbol{r}) \, \boldsymbol{n}_0}{|M^{-1} (\boldsymbol{r}) \, \boldsymbol{n}_0|} \; .$$

It might be a nice exercise for the reader to verify that, with a purely spherical potential, $W_S = \frac{\mu}{r}$, one has $\boldsymbol{n}_0 = \boldsymbol{e}_{0r}$ and $M = \frac{\mu}{r^3} \, (I - 3 P_r)$, so that $M^{-1} \div \left(I - \frac{3}{2} P_r \right)$ and we get then $\boldsymbol{e}_w = \boldsymbol{e}_r = \boldsymbol{e}_{0r}$; therefore, in this particular case, $L_v \equiv L_w$.

Remark 5.1 (Geopotential numbers and dynamic heights)
As it is obvious, W has the counterintuitive behaviour that it decreases when we move upward. This inconvenience can be eliminated by defining a *geopotential number* C as

$$C \, (\mathrm{P}) = W_0 - W \, (\mathrm{P}) \; , \tag{5.14}$$

because indeed $C \, (\mathrm{P})$ increases from lower to higher equipotential surfaces. We note that when $\mathrm{P} = \mathrm{P}_0$, a point on the geoid, then $W \, (\mathrm{P}_0) = W_0$ and $C \, (\mathrm{P}_0) = 0$.

We can observe as well that, if we had chosen the alternative definition of geoid as the equipotential surface passing through a given point P_0, e.g. a tide gauge station, then $C(P)$ would become observable by levelling and gravimetry (see Sect. 4.6), without knowing W_0. Since this is the practice adopted in many countries, we shall return in the last chapter to this point to explain how to unify the different datums.

Furthermore we can say that both $W(P)$ and $C(P)$ are dimensionally gravity potentials, namely the square of a velocity. To bring back a potential coordinate to the dimension of a length, as it seems intuitive for a height coordinate, sometimes a different coordinate is introduced, called *dynamic height* and defined by

$$H^D(P) = \frac{C(P)}{\overline{\gamma}_0} = \frac{W_0 - W(P)}{\overline{\gamma}_0} , \tag{5.15}$$

where $\overline{\gamma}_0$ is any *constant* value close to the actual gravity, i.e. to 10^3 Gal. As $\overline{\gamma}_0$, it could be convenient, for instance, to take the mean value of γ on the ellipsoid, namely

$$\overline{\gamma}_0 = 979.7614249 \text{ Gal} ,$$

although any other constant value, close by, would do. As we shall see later, H^D so defined results to be close to other types of heights, particularly to the orthometric heights.

It has to be remarked that in any way W, C, H^D, together with $\Sigma = (\Lambda, \Phi)$, share the same geometric behaviour, in particular in relation to isozenithal lines and coordinate surfaces.

Finally we have to understand how the Hotine-Marussi triad is related to geometrical coordinates. Since (x, y, z) and (λ, φ, h) can be just mathematically transformed one into the other, i.e. they are geometrically equivalent, we can study only the transformation between (Λ, Φ, W) and (λ, φ, h). The inverse transformation is obvious, in the sense that if we know $W(P) = W(\lambda, \varphi, h)$, then $P \to W$ is given and $\Sigma = (\Lambda, \Phi)$ is determined by inverting (5.9), namely

$$\begin{vmatrix} \cos\Phi \cos\Lambda \\ \cos\Phi \sin\Lambda \\ \sin\Phi \end{vmatrix} = \frac{\nabla W(P)}{|\nabla W(P)|} \tag{5.16}$$

or

$$\begin{aligned} \mathrm{tg}\Lambda &= -\frac{g_y}{g_x} \\ \mathrm{tg}\Phi &= -\frac{g_z}{g} . \end{aligned}$$

So turning to the direct transformation, we have to show how to pass from (Λ_P, Φ_P, W_P) to $(\lambda_P, \varphi_P, h_P)$. We continue to assume that $W(P)$ is a known function of the ellipsoidal coordinates of P and so $T(P)$ is known too.

We have already introduced in Sect. 4.7 the point P^*, which is characterized by the fact that it is on the same ellipsoidal normal as P, i.e.

$$\begin{cases} \lambda_{P^*} = \lambda_P \\ \varphi_{P^*} = \varphi_P \end{cases} \quad \text{or} \quad \boldsymbol{\nu}_{P^*} = \boldsymbol{\nu}_P$$

and it has an ellipsoidal height h^* such that (4.80) is satisfied, namely

$$U_{P^*} = W_P \; . \tag{5.17}$$

Now we introduce another point $P^{*\prime}$ (see Fig. 5.3), which is characterized by the conditions

$$\begin{cases} \lambda_{P^{*\prime}} = \Lambda_P \\ \varphi_{P^{*\prime}} = \Phi_P \end{cases} \quad \text{or} \quad \boldsymbol{\nu}_{P^{*\prime}} = \boldsymbol{n}_P \tag{5.18}$$

and

$$U_{P^{*\prime}} = W_P \; ; \tag{5.19}$$

as for (5.17), also (5.19) can be used to derive $h^{*\prime} = h(P^{*\prime})$. The couple of Eqs. (5.18) and (5.19) are known as Marussi mapping and by them the ellipsoidal coordinates of $P^{*\prime}$ are known.

The key point here is that the vector $\boldsymbol{r}_{P^{*\prime}P}$ is of the maximum order of

$$\mathcal{O}(|\boldsymbol{r}_{P^{*\prime}P}|) = 100 \text{ m} \tag{5.20}$$

and for such small vectors one can put

$$T(P) \cong T(P^{*\prime}) \; ; \tag{5.21}$$

in fact, if we take $\mathcal{O}(|\nabla T|) = 10^2$ mGal for a shift of 10^2 m, one has as a maximum $\frac{\delta T}{\gamma} \sim 1$ cm, i.e. $\delta T \sim 10^{-4}\, T$, which is acceptable.

First of all we notice that

$$h^{*\prime} \cong h^* \; ;$$

this is intuitive from Fig. 5.3 and it is confirmed by noting that P^* and $P^{*\prime}$ have the same value of U, so δh^* can be computed by differentiating the last formula in Table 3.2, truncated to the first order in h. Since U varies only with h and φ, the result is approximately

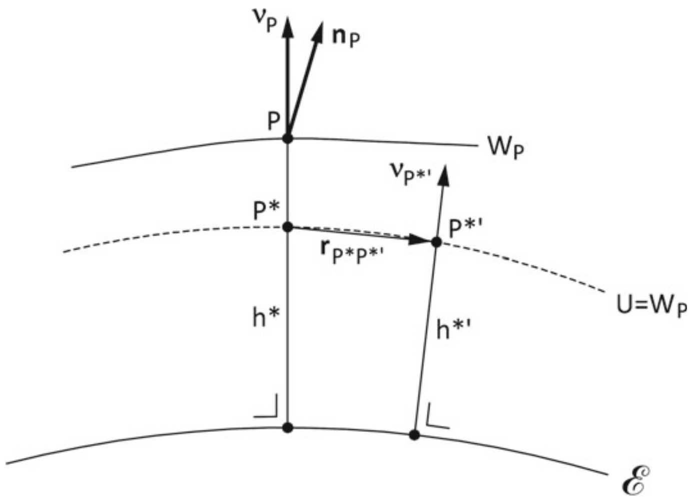

Fig. 5.3 The three points P, P^*, $P^{*\prime}$; $\mathcal{O}(\zeta) \sim \mathcal{O}(|\boldsymbol{r}_{P^*P^{*\prime}}|) \sim 10^2$ m; note that P^*, $P^{*\prime}$ lay on the same equipotential of $U(P)$, so $\boldsymbol{r}_{P^*P^{*\prime}} \cdot \boldsymbol{\nu}_P \cong 0$

$$\delta h^* \cong \frac{-\gamma_{e\varphi}\, h\, \delta\varphi}{\gamma_e} \; ;$$

for a shift of $\delta\varphi \sim 2 \cdot 10^{-5}$, corresponding to ~ 120 m, and an altitude of 6 km, this is below the μm and therefore we can forget it.

On the other hand we have as well

$$\zeta(P^{*\prime}) = \frac{T(P^{*\prime})}{\gamma(P^{*\prime})} \; ,$$

which is computable because we know the ellipsoidal coordinates of $P^{*\prime}$; but it is easy to verify that

$$\zeta(P^*) \cong \zeta(P^{*\prime}) \; ,$$

also taking (5.21) into account. Therefore we can put

$$h = h^* + \zeta(P^*) \cong h^{*\prime} + \zeta(P^{*\prime}) \; ,$$

so that h is now known.

Coming to the horizontal coordinates (λ_P, φ_P), we have first of all

$$\boldsymbol{n}_P = -\frac{\boldsymbol{\gamma}_P + \nabla T}{|\boldsymbol{\gamma}_P + \nabla T|} \; ; \qquad (5.22)$$

but

$$|\gamma_P + \nabla T| \cong \gamma_P + \boldsymbol{e}_\gamma \cdot \nabla T = \gamma_P - \widetilde{\boldsymbol{n}} \cdot \nabla T \ ,$$

because $\boldsymbol{e}_\gamma = \frac{\boldsymbol{\gamma}_P}{\gamma_P} = -\widetilde{\boldsymbol{n}}_P$.

On the other hand $\mathcal{O}(|\widetilde{\boldsymbol{n}} - \boldsymbol{\nu}|) \sim 5 \cdot 10^{-6}$, which multiplied by ∇T goes down to the μGal level, so we can say

$$|\boldsymbol{\gamma}_P + \nabla T| = \gamma_P - \boldsymbol{\nu} \cdot \nabla T$$

and therefore

$$|\boldsymbol{\gamma}_P + \nabla T|^{-1} = \frac{1}{\gamma_P}\left(1 + \frac{\boldsymbol{\nu} \cdot \nabla T}{\gamma_P}\right) . \tag{5.23}$$

Substituting (5.23) in (5.22) and keeping only first order terms, we get

$$\boldsymbol{n}_P = \widetilde{\boldsymbol{n}}_P - \frac{1}{\gamma_P}\left[\nabla T - \boldsymbol{\nu} \cdot (\boldsymbol{\nu} \cdot \nabla T)\right] . \tag{5.24}$$

The vector

$$\boldsymbol{\delta} = \boldsymbol{n}_P - \boldsymbol{\nu}_P \tag{5.25}$$

is called the vector deflection of the vertical and its modulus $\delta = |\boldsymbol{\delta}|$ just deflection of the vertical, a quantity that being generally small (of the order of $3 \cdot 10^{-4}$ at most) is approximately equal to the angle between $\boldsymbol{n}_P$ and $\boldsymbol{\nu}_P$.

The vector

$$\widetilde{\boldsymbol{\delta}} = \widetilde{\boldsymbol{n}}_P - \boldsymbol{\nu}_P \ , \tag{5.26}$$

that we already encountered in scalar terms in (3.63), is the normal vector deflection of the vertical and we know that in the topographic layer $\mathcal{O}\left(\widetilde{\delta}\right) = \mathcal{O}\left(\left|\widetilde{\boldsymbol{\delta}}\right|\right) \sim 5 \cdot 10^{-6}$; more precisely we know that $\widetilde{\boldsymbol{\delta}}$ is pointing northward, in the northern hemisphere, so that

$$\widetilde{\boldsymbol{\delta}} \cong \frac{\gamma_{e\varphi}}{\gamma_0}\frac{h}{a}\boldsymbol{e}_\varphi \cong 5.3 \cdot 10^{-3} \sin 2\varphi \frac{h}{a}\boldsymbol{e}_\varphi \ . \tag{5.27}$$

It is immediate to verify that computing $\widetilde{\boldsymbol{\delta}}$ with $\varphi = \Phi$ and $h = h^*$ does not change significantly its value, so we consider it a known vector.

So returning to (5.24) we can write, subtracting $\boldsymbol{\nu}_P$ to both members,

$$\boldsymbol{\delta} = \widetilde{\boldsymbol{\delta}} - \frac{1}{\gamma}(I - P_\nu)\nabla T \ , \tag{5.28}$$

with P_ν the projector in the $\boldsymbol{\nu}$ direction. Note that we do not specify any more where the terms multiplying ∇T are computed, because we know that it makes no difference whether this is in P, P* or P$^{*\prime}$; so we shall assume that they are computed in P$^{*\prime}$, which is known.

Finally, going back to the definition (5.25) and observing that

$$\boldsymbol{n}_{\mathrm{P}} = \boldsymbol{\nu}_{\mathrm{P}^{*\prime}} = \boldsymbol{\nu}\left(\Lambda, \Phi\right)$$

while

$$\boldsymbol{\nu}_{\mathrm{P}} = \boldsymbol{\nu}\left(\lambda, \varphi\right) \;,$$

if we put

$$\begin{cases} \Lambda = \lambda + \delta\lambda \\ \Phi = \varphi + \delta\varphi \end{cases} , \tag{5.29}$$

we see that

$$\boldsymbol{\delta} = \boldsymbol{\nu}\left(\Lambda, \Phi\right) - \boldsymbol{\nu}\left(\lambda, \varphi\right) = \boldsymbol{\nu}_\lambda \delta\lambda + \boldsymbol{\nu}_\varphi \delta\varphi \;.$$

On the other hand

$$\boldsymbol{\nu}_\lambda = \cos\varphi\, \boldsymbol{e}_\lambda \;, \quad \boldsymbol{\nu}_\varphi = \boldsymbol{e}_\varphi \;,$$

so that

$$\boldsymbol{\delta} = \cos\varphi\, \delta\lambda\, \boldsymbol{e}_\lambda + \delta\varphi\, \boldsymbol{e}_\varphi$$

and $(\cos\varphi\, \delta\lambda,\ \delta\varphi)$ are respectively the northward and the eastward component of $\boldsymbol{\delta}$, in geodetic literature also denoted as

$$\begin{cases} \eta = \cos\varphi\, \delta\lambda \\ \xi = \delta\varphi \end{cases} . \tag{5.30}$$

Therefore, returning to (5.28) and taking the scalar product with $\boldsymbol{e}_\lambda$ and $\boldsymbol{e}_\varphi$, we get respectively (see (2.85))

$$\begin{cases} \boldsymbol{e}_\lambda \cdot \boldsymbol{\delta} = \eta = -\boldsymbol{e}_\lambda \cdot \nabla T = -\dfrac{1}{(\mathcal{N} + h^*)\cos\varphi} \dfrac{\partial T}{\partial \lambda} \\ \boldsymbol{e}_\varphi \cdot \boldsymbol{\delta} = \xi = \widetilde{\delta} - \boldsymbol{e}_\varphi \cdot \nabla T = \widetilde{\delta} - \dfrac{1}{(\mathcal{M} + h^*)} \dfrac{\partial T}{\partial \varphi} \end{cases} . \tag{5.31}$$

The right hand side of (5.31) is known and therefore such a formula gives (η, ξ), i.e. $(\delta\lambda, \delta\varphi)$ and therefore also (λ, φ) by (5.29). The transformation between (Λ, Φ, W) and (λ, φ, h) is so accomplished, at least in a linear approximation, which amounts to a linear error in coordinates at most of 1 cm, as we have seen in various steps. As a remark, we see that the knowledge of the anomalous potential $T(\mathrm{P})$ is essential to perform our coordinate transformation and it is precisely for this reason that, even studying a geometric topic like heights, we need to know how to compute or at least to use $T(\mathrm{P})$. Finally, we observe that the term $\widetilde{\delta}$ in (5.31) has only recently been introduced (see Betti et al. (2016)) and, though small, it can produce sensible effects in long levelling lines in south-north direction at a relevant height h.

5.4 The Helmert Triad (Λ, Φ, H)

The couple $\Sigma = (\Lambda, \Phi)$ is defined by the relation (5.9), as for the Hotine-Marussi coordinates. The coordinate H, called orthometric height of the point P, is defined as follows: with reference to Fig. 5.1, we take the line of the vertical through P, L_v, and we consider the length of L_v between P and the geoid, i.e. the arc $\widehat{\mathrm{P}_0\mathrm{P}}$; then

$$H_{\mathrm{P}} = L_v\left(\widehat{\mathrm{P}_0\mathrm{P}}\right) , \tag{5.32}$$

taken positively outside the geoid and with the minus sign inside the geoid. A fast comparison shows that H_{P} is precisely equal to the curvilinear coordinate ℓ_{P} defined in Sect. 5.2.

In spite of its intuitive character, and the fact that for a long time it has been considered as a "natural" coordinate to describe the observation equations of spirit levelling (see Chap. 6), the orthometric height has some subtle properties that have made controversial its use in Geodesy.

The first surprising fact is that the lines of the vertical $\{L_v\}$ *are not* the coordinate lines of H. The family $\{L_v\}$ is used to define H, but its coordinate lines are defined by the condition on the other two coordinates,

$$\Phi = \Phi_0 \text{ (constant)}, \quad \Lambda = \Lambda_0 \text{ constant},$$

namely the lines $\{L_H\}$ coincide again with the family of isozenithals, already seen in Sect. 5.3. A little thought will show that, if we wanted a coordinate system where L_v was a coordinate line, we should have chosen a couple of coordinates (Λ_0, Φ_0) to accompany H or W, that are in fact the astrogeodetic coordinates of the projection of P on the geoid $\mathcal{G}$, along L_v itself.

Now it is obvious that, if we move along L_v, the arc length $\mathrm{d}\ell_v$ is

$$\mathrm{d}\ell_v = \mathrm{d}H \tag{5.33}$$

when we use H, while

$$d\ell_v = -\frac{dW}{g} \tag{5.34}$$

when we use W as a parameter. The relations (5.33) and (5.34), especially when they are written in the form

$$dW = -g\,dH\ , \tag{5.35}$$

have been already source of controversy because of its imprecise notation. In fact (5.33) and (5.34) are meant to be valid *only* along L_v; they are not equalities between total differentials. Borrowing from an old notation, the relation (5.35), which in general *is wrong*, should be written as

$$d_n W = -g\,dH\ , \tag{5.36}$$

meaning that the increment $d_n W$ is computed exclusively along L_v. In fact, if we move $\boldsymbol{r}$ by $d\boldsymbol{r} = \boldsymbol{n}\,dH$ (recall that $\boldsymbol{n} = -\frac{\boldsymbol{g}}{g}$ is always tangent to L_v), we have

$$d_n W = \boldsymbol{g}\cdot d\boldsymbol{r} = \boldsymbol{g}\cdot\left(\frac{\boldsymbol{g}}{g}\right)dH = -g\,dH\ ,$$

confirming the correctness of (5.36). On the contrary, if $d\boldsymbol{r}$ is pointing in any direction in space, the relation (5.35) cannot be maintained any more, because, if this would be true, we should have as well

$$dH = 0 \quad\Rightarrow\quad dW = 0\ ,$$

namely equipotential surfaces should have a constant orthometric height too. But in this case

$$g = \frac{dW}{dH}$$

should also be constant on an equipotential surface and this is known to be false on an empirical ground; on the other hand, even the normal gravity is not constant on the Earth ellipsoid, which is an equipotential of the normal potential. A deeper analysis (see Sansò and Vanìcek 2006) can show that the only field for which a relation like (5.36) is true is that with a purely spherical potential $\frac{\mu}{r}$.

Now we have to study the transformation of (Λ, Φ, H) into the other geometrical coordinates. As we shall see, to do that we will have to make in any way some hypothesis on the density of topographic masses, e.g. $\rho_c = 2.67\ \mathrm{g\ cm^{-3}}$. This introduces an unavoidable systematic error into the relation between H_P and the geodetic observables and therefore into the use of H_P itself. In principle we could say that writing (5.5) in the form

$$C(\mathrm{P}) = W_0 - W(\mathrm{P}) = g_{\mathrm{P}} H_{\mathrm{P}} \left(1 + \frac{H_{\mathrm{P}}}{R}\right) - \frac{p}{2} H_{\mathrm{P}}^2 \tag{5.37}$$

allows to compute $W(\mathrm{P})$ from H_{P} and therefore, once (Λ, Φ, W) are known, we can repeat the reasoning of Sect. 5.3 to derive (λ, φ, h). This is basically the solution developed by Helmert and we shall shortly report it in a remark at the end of the section. Yet, this is too intricate and, more important, subject to larger errors. We prefer here to go along a way that, exploiting relations between anomalous quantities only, implies smaller errors.

First, inspecting Fig. 5.1, we shall prove what we call the Operative Lemma of Orthometric Heights.

The Operative Lemma of Orthometric Heights: with an accuracy of about 1 cm, or better one can write everywhere on the surface S

$$h_{\mathrm{P}} = H_{\mathrm{P}} + N_{\mathrm{P_e}} \,. \tag{5.38}$$

Proof Looking at Fig. 5.1, (5.38) means

$$\overline{\mathrm{P_e P}} = \widehat{\mathrm{P_0 P}} + \overline{\mathrm{P_e P''}} \,; \tag{5.39}$$

we prove (5.39) by showing that

$$\widehat{\mathrm{P_0 P}} - \overline{\mathrm{P'' P}} \cong 0 \,, \tag{5.40}$$

at the approximation level of 1 cm.

Treating orders of magnitude, we can assume that δ is constant along L_v, because it is known (cf. (3.25)) that the variation of δ along L_v is one order of magnitude smaller than δ itself; moreover we know that $\delta \leq 3 \cdot 10^{-4}$.

Then we can write

$$\overline{\mathrm{P'P}} = \int_0^{H_{\mathrm{P}}} \cos\delta \, \mathrm{d}H \cong H_{\mathrm{P}} \left(1 - \frac{1}{2}\delta^2\right)$$
$$\overline{\mathrm{P''P'}} = \overline{\mathrm{P_0 P'}} \sin\delta \cong H_{\mathrm{P}} \cdot \delta \cdot \delta = H_{\mathrm{P}} \delta^2 \,.$$

Therefore

$$\overline{\mathrm{P''P}} \cong H_{\mathrm{P}} + H_{\mathrm{P}} \frac{1}{2}\delta^2$$

and then

$$\widehat{\mathrm{P_0 P}} - \overline{\mathrm{P''P}} = H_{\mathrm{P}} \frac{1}{2}\delta^2 \,;$$

The above, with $H_P = 6$ km and $\delta = 3 \cdot 10^{-4}$, attains the value of 0.27 mm, which is zero at our approximation level.

Now that (5.38) is proved, we have to show how to compute N_{P_e} by using anomalous quantities only. We first observe that the fundamental equation of Physical Geodesy, also recalling Bruns' relation (4.82), can be written as

$$-T' + \frac{\gamma'}{\gamma} T = -\gamma \frac{\partial}{\partial h} \frac{T}{\gamma} = -\gamma \frac{\partial}{\partial h} \zeta = \Delta g$$

or

$$\frac{\partial \zeta}{\partial h} = -\frac{\Delta g}{\gamma} . \tag{5.41}$$

Then, integrating (5.41) between P″ and P of Fig. 5.1, we get

$$N - \zeta_P = \int_{P''}^{P} \frac{\Delta g}{\gamma} \, dh \tag{5.42}$$

or, recalling that $\overline{P''P} \cong H_P$,

$$N_P = \frac{T(P)}{\gamma} + \int_0^{H_P} \frac{\Delta g}{\gamma} \, dh . \tag{5.43}$$

If we know $T(\lambda, \varphi, h)$, we can always compute $T(\Lambda, \Phi, H)$ committing an error of 1 cm at most, so the first term in (5.43) is known.

Now we use the identity

$$\int_0^x f(t)\, dt = x f(x) - \int_0^x t f'(t)\, dt ,$$

to compute the integral in (5.43). We obtain (remember that $\overline{P''P} \cong H_P$)

$$\int_0^{H_P} \frac{\Delta g}{\gamma} \, dh = H_P \frac{\Delta g_P}{\gamma} - \int_0^{H_P} h \left(\frac{\Delta g}{\gamma} \right)' dh . \tag{5.44}$$

On the other hand, recalling (5.6) and (5.8),

$$\begin{aligned} \left(\frac{\Delta g}{\gamma} \right)' &= \frac{\gamma \Delta g' - \gamma' \Delta g}{\gamma^2} = \frac{1}{\gamma} \left(-2\mathcal{C}_0 \Delta g + q \right) - \frac{\Delta g}{\gamma} \left(-2\mathcal{C}_0 - 2\frac{\omega_0^2}{\gamma} \right) = \\ &= \frac{q}{\gamma} + \frac{\Delta g}{\gamma} \frac{2\omega^2}{\gamma} . \end{aligned} \tag{5.45}$$

Now notice that $q \sim 0.2$ mGal m^{-1} and that $\left|\frac{\Delta g}{\gamma}\right| < 10^{-4}$ while $2\omega^2 \cong 10^{-2}$ mGal m^{-1}, so that the second term in (5.45) is five orders of magnitude smaller than the first.

So returning to (5.44), we find

$$\int_0^{H_P} \frac{\Delta g}{\gamma}\, dh \cong H_P \frac{\Delta g_P}{\gamma} - \int_0^{H_P} q\, \frac{h}{\gamma}\, dh\ . \tag{5.46}$$

An easy computation of orders of magnitude shows that (5.46) can amount up to a few meters for $H = 6$ km.

Summarizing we have the solution

$$N_P = \frac{1}{\gamma}\left(T_P + \Delta g_P H_P\right) - \int_0^{H_P} q\, \frac{h}{\gamma}\, dh\ , \tag{5.47}$$

which used in (5.38) provides the sought transformation. We note that such a formula, which is now standard in geodetic literature, gives the direct dependence of N_P on the profile of $\rho(Q)$ along the vertical of P, through the parameter $q = 4\pi G\rho$.

It is not difficult to see that, by taking $\rho = \text{const}$, a further approximation of (5.47) gives

$$N_P = \frac{1}{\gamma}\left(T_P + \Delta g_P H_P - \frac{1}{2} q\, H_P^2\right)\ . \tag{5.48}$$

We conclude this section by a remark on the so called Helmert heights.

Remark As commented before, we want to return to the relation between H_P and W_P, which was originally figured out by Helmert, following his definition of orthometric height.

This was derived by the following consideration: start with

$$W_0 - W(P) = -\int_{P_0}^{P} dW = \int_{P_0}^{P} g\, dH \equiv H_P \frac{1}{H_P}\int_0^{H_P} g\, dH = H_P\, \overline{g}\ , \tag{5.49}$$

where $\overline{g}$ is just the average of g along L_v, between P_0 and P. Then Helmert's reasoning continues with the computation of $\overline{g}$ under the hypothesis that $\rho = \text{const}$ and that g linearly depends on H. But we have already performed this calculation, leading to (5.5), when we disregard the third order term. So we can write

$$C(P) = W_0 - W(P) = g_P\, H_P + \left(\frac{g_P}{R} - \frac{p}{2}\right) H_P^2$$

and then, taking into account that $\frac{H_P}{R} \leq 10^{-3}$, we substitute into the second term of the above formula g_P with the mean value of the normal gravity, $\overline{\gamma}_0$, already computed in Sect. 5.3, Eq. (5.15).

All that gives

$$C(P) = W_0 - W(P) = g_P H_P + 0.0424 H_P^2 , \tag{5.50}$$

with g in Gal and H_P in km. Eq. (5.50) is exactly what one can find in literature (cf. Heiskanen and Moritz (1967, Eq. 4.4)).

5.5 The Molodensky Triad (λ, φ, h^*)

The coordinates (λ, φ) are taken as the ellipsoidal longitude and latitude and their knowledge implies that of $\boldsymbol{\nu}$, i.e. of the ellipsoidal normal passing through P. The h^* is the normal height, we have already defined in Sect. 4.7, and its defining equation is (see (4.80), (4.81))

$$U(\sigma, h^*) \equiv W(\sigma, h) , \tag{5.51}$$

namely the normal potential at height h^* along the ellipsoidal normal through P should be equal to the actual potential at P. The relation between h and h^* is provided by the Bruns relation (4.82), i.e.

$$h = h^* + \zeta = h^* + \frac{T}{\gamma} . \tag{5.52}$$

The relation (5.52) derives, as we know, from a linearization and as such it bears some approximation. In any way, as always, it assumes that we know T in ellipsoidal coordinates. To be precise, the term $\frac{T}{\gamma}$ in (5.52) should be computed with the ellipsoidal height fixed at h^*; so (5.52) gives the transformation from h^* to h. The inverse transformation can always be derived from (5.52), reversed in the form

$$h^* = h - \frac{T}{\gamma} , \tag{5.53}$$

where now $\zeta = \frac{T}{\gamma}$ can be computed at the ellipsoidal height $h = h_P$. In fact, since according to (5.41) we have

$$|\zeta(h) - \zeta(h^*)| \leq \int_{h^*}^{h} \frac{\Delta g}{\gamma} \, dh \cong \frac{|\Delta g|}{\gamma} |\zeta| ,$$

we see that, with $|\zeta| = 100$ m and $|\Delta g| = 100$ mGal,

$$|\zeta(h) - \zeta(h^*)| \leq 1\text{cm} .$$

Would such an accuracy be deemed insufficient, we can always resort to the defining equation (5.51). For that we can use the last equation in Table 3.2, neglecting the last term $\frac{1}{2}\frac{\tau_1^2}{\gamma_0}\frac{h^2}{3}$, which divided by γ_0 is of the order of magnitude of less than 1 μm even for $h = 6$ km; this can be put into the form

$$U_0 - U(h^*) = W_0 - W(\text{P}) = C(\text{P}) = \gamma_e(\varphi)h^* - \frac{1}{2}\gamma_1(\varphi)h^{*2} + \frac{1}{3}\gamma_2(\varphi)h^{*3} , \tag{5.54}$$

with γ in Gal and h^* in km.

Indeed, knowing h and so $W(\text{P})$ and $C(\text{P})$, one can solve (5.54) for h^*; yet, to avoid numerical instabilities, it is convenient to write (5.54) in the form

$$h^* = \frac{C(\text{P})}{\gamma_e(\varphi) - \frac{1}{2}\gamma_1(\varphi)h^* + \frac{1}{3}\gamma_2(\varphi)h^{*2}} \tag{5.55}$$

and solve it iteratively, starting with $h^* = 0$ at the right hand side.

In case we would like to transform h^* into H or vice versa, one can combine (5.38) and (5.52) to get

$$h^* = H + N - \zeta , \tag{5.56}$$

where $N - \zeta$ can be derived from (5.47) or even (5.48).

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