Chapter 1
Introduction

A height is a coordinate in $\mathcal{R}^3$, used in a certain subset of our space, particularly in the framework of physical sciences of the Earth, to discriminate higher from lower points, in some sense to be specifically stated by the type of the height chosen.

So first of all a height, as a coordinate for a subset of points in $\mathcal{R}^3$, is one coordinate in a triple and it makes little sense to define it without specifying the other two. Second, we need to restrict the physical purpose for which a height will be considered in this monograph.

We shall do that by a counterexample and then by stipulating a criterion to identify what kind of "heights" we are interested in, namely geodetic heights. For instance the height of a point on or above the Earth surface could be defined as the air pressure at that point, in a triple completed by two cartographic coordinates, in a specific area of the Earth. Such a concept of height is in fact used in atmospheric sciences to simplify the equations of the dynamics of the atmosphere, and even in common life in mountain excursions. However we rule out this concept, because we know that such a coordinate can significantly change from hour to hour at the same point, fixed with respect to the solid Earth. So we shall agree that we want to study heights that do not change in time, at least they do not change significantly over a time span in which the Earth can be considered as a stationary body.

One could object that the Earth undergoes not only slow geological movements, but also periodical deformations, for instance the body tides due to the attraction of the Sun and the Moon, that are in the range of 1 m and have a main semidiurnal period. Such effects will be considered as perturbations, globally known and subtracted to all the physical quantities considered in this work, so that we shall refer to an idealized static image of the Earth.

After these preliminary remarks, it is time to go the heart of the question, namely in which sense we intend to discriminate higher from lower points. This is primarily related to the gravity field and its direction. Locally, this is first of all related to our physiological sensations. A man standing on the ground defines the direction of the vertical and subsequently a small area under his feet, when they are kept orthogonal

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to the body axis, is horizontal. So a human body is itself a local gravity sensor; certainly not the only one! A short pendulum can be used to define the direction of the vertical at a point in space, also a small quantity of water, still in a bucket will define a horizontal plane at its centre and so forth. The vertical direction, that we shall identify by a unit vector $\boldsymbol{n}(\mathrm{P})$, is in the opposite direction of that of the gravity attraction, and once we know how to materialize it we can connect points that are close one to the other in the vertical direction. In this way starting from point, e.g. on the Earth surface, we can generate a line upward and in fact, assuming we are able to enter into the body of the Earth, also downward. This is a line of the vertical or plumb line; if we do the same at all points in the region of interest, we generate a family of lines, also referred to as the congruence of vertical lines. They have the property that at every point they are tangent to the direction of the vertical. It is a fundamental theorem, consequence of the famous theorem of existence and uniqueness of solutions of ordinary differential systems, that in the region of our interest, where the gravity attraction field never goes to zero and it is at least Hölder continuous, these lines can never intersect, nor even be tangent to one another at any point. In other words, in our region, through any point P there passes one and only one line of the vertical.

Once the congruence of vertical lines is established, one can also consider the family of surfaces that admit plumb lines as orthogonal trajectories. It turns out that these are equipotential surfaces of the gravity field, as we shall see later on, and as such they cannot intersect too. Moreover, always in the range of some kilometers up and down, it happens that the equipotentials are closed surfaces and therefore they are contained one into the other. Plumb lines and equipotentials are the two main ingredients of the geometry of the gravity field, which has been investigated in depth in the 50ies, 60ies and 70ies of the 20th century (Bomford 1952, Marussi 1985, Hotine 1969, Krarup 2006, Heiskanen and Moritz 1967, Grafarend 1975, just to mention a few). We shall use only some of these results, to be presented later on in the text.

We need now to better specify what is the region of our interest, where we want to establish and use geodetic height coordinates. Indeed this region has to include the surface of the Earth S; in particular we want coordinates which are good for all of S, a layer of points above and a layer below it. The reason why we want to cover the whole of S is because in the era of Global Navigation Satellite System (GNSS) measurements we are able to connect any point on the Earth surface by observations that need to be modelled by a unique coordinate system. The reason we limit the region by layers, say of $\pm 30\,\mathrm{km}$ width, is twofold. When we go far away from S in the upper direction, there is a quite substantial change of the geometry of the gravity field. For instance equipotentials at a distance of about 42,000 km from the barycenter become open surfaces (see Sansò and Sideris 2013, Sect. 1.9). This is basically the reason why the potential cannot be used as a height coordinate at least throughout the whole space. So 30 km above S is a good layer for both Geodesy

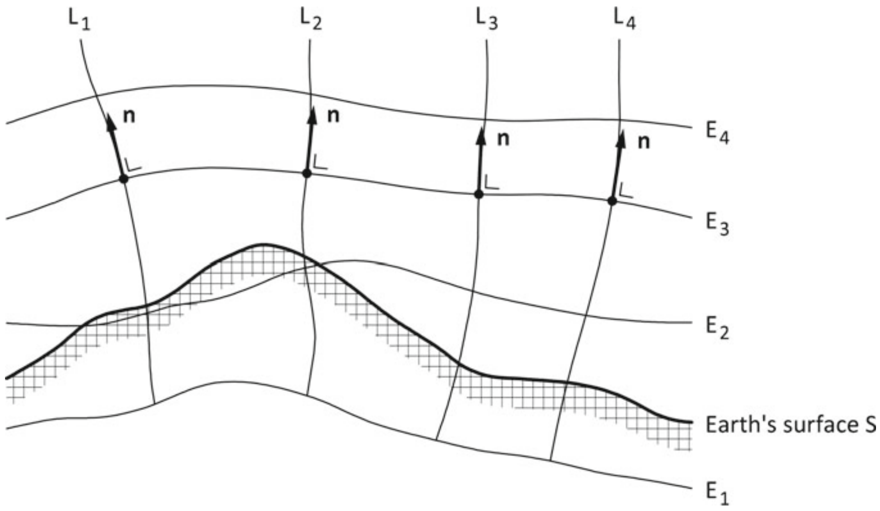

Fig. 1.1 A portion of the Earth surface S with part of the region of interest close to it; equipotential surfaces (E_1 E_2 E_3 E_4) and lines of the vertical (L_1 L_2 L_3 L_4)

and sciences of the lower atmosphere.[1] As for the layer below S one can observe that this depth is sufficient to include important isostatic compensation surfaces, like the Moho, which are traditionally discussed in the framework of Geodesy, as well as the region where most of crustal geophysical phenomena have a seat. When we go deeper we might incur into a more irregular behaviour of the gravity field and at the same time we have an increasingly poor knowledge of the distribution of the masses, which is the origin of the gravity field (Lambeck 1988; Anderson 2007). In the figure below we summarize in a pictorial form what we said above for a portion of our region of interest (Fig. 1.1).

It is worth underlining that given the properties of equipotential, i.e. horizontal, surfaces, they are naturally ordered from below to above, considering that also the plumb lines that they cross orthogonally have a natural positive verse inherited from the vertical unit vector $\boldsymbol{n}(\mathrm{P})$.

We are ready now to give a first definition of what we can consider a geodetic height, that we call here generally as q_3, the third coordinate in a triple (q_1, q_2, q_3). As all regular coordinates, q_3 will have a coordinate line ℓ_3 with a tangent unit vector $\boldsymbol{e}_3(\mathrm{P})$ attached at any point P on it; we say that q_3 is a geodetic height if the relation

$$\boldsymbol{n}(\mathrm{P}) \cdot \boldsymbol{e}_3(\mathrm{P}) \geq k > 0 \tag{1.1}$$

[1] *Note*: indeed Geodesy is also interested in satellite dynamics even for very high satellites, yet at that altitude not all the coordinates discussed in this book are of particular significance: whence the reason to limit our discussion to a bounded region.

is satisfied for all points in the region of interest and for some fixed, positive k. The meaning of (1.1) is that when we move along ℓ_3 in the positive sense, that we call upward, we expect to cross horizontal surfaces which are monotonously above one to the other.

To be more specific, since we do not like to use a coordinate whose lines cross the horizontal surface at a small angle, we could say that we expect that ℓ_3, i.e. $\boldsymbol{e}_3(\mathrm{P})$, is almost orthogonal to the equipotential passing by P. In practice we shall treat situations in which the constant k of (1.1) is

$$k = 1 - \varepsilon \qquad \varepsilon = \mathcal{O}(10^{-2}) \ ,$$

i.e. $\boldsymbol{n}$ and $\boldsymbol{e}_3$ form at most an angle of one or few degrees.

In this monograph we shall consider mainly four types of heights, that we name as orthometric, ellipsoidal, normal and orthonormal (or normal orthometric as they are called in literature), plus some variants. The focus of the book is on two issues: to find the relations between one system of coordinates and the other, which, as we shall see, implies a fine knowledge of the actual gravity field of the Earth; to find the relation between various heights and the quantities that are observable by geodetic techniques.

In this respect a last remark is in order; we often speak of a height system and by that we mean that not only we have a mathematical definition of the coordinate but we have also defined a reference system for it, which is essential to find the connection between this height and the observable quantities. This completely parallels what happens with all types of coordinates in Geodesy. In particular we fix a height system when, given a certain geometry of the coordinate lines and of the corresponding coordinate surfaces, one particular surface is chosen to which the value of $q_3 = 0$ is assigned. For instance in ellipsoidal coordinates the generating ellipsoid $\mathcal{E}$ is the height datum for ellipsoidal heights. Most of the other coordinate systems however try to use as reference for the height coordinate one particular equipotential surface of the gravity field. This is traditionally called the geoid and its choice, in the family of equipotentials, will necessarily occupy us in the book. In fact such a choice is not univocal and a lot of confusion has been generated by the practical custom of different nations to choose their own particular reference surface.

Nowadays with the important improvement in the knowledge of the global gravity field coming from space missions like GRACE (Tapley and Reigber 2001) and GOCE (Drinkwater et al. 2003), the time has come to make a precise choice defining one common height datum for the whole planet (Ihde et al. 2017).

While preparing the final issue of the book the authors have become aware of the existence of the work by Eremeev and Yurkina (1974). We would like to underline the notable closeness of that work to the actual spirit of this book, although the tools employed today take advantage of 50 years of geodetic research.

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