Chapter 7
Rayleigh–Bénard Convection

7.1 Heating a Fluid Layer from Below

The investigation of the fluid dynamics thermally induced by a vertical temperature gradient imposed on a fluid layer initiated with the experiments carried out by Bénard [1] at the beginning of the twentieth century. Such investigation was the subject of Henri Claude Bénard's doctoral thesis defended at Sorbonne University, in Paris. These experiments documented the formation of flow cells in a shallow fluid layer where the temperature on the lower wall is higher than on the upper free surface, provided that the prescribed temperature difference is higher than a threshold value. Bénard's experiments were carried out with the prescribed higher temperature within a range between 50 °C and 100 °C, by employing liquids such as wax and whale oil (spermaceti), which melt in this temperature range and do not display significant surface evaporation. For the readers interested in the scientific biography of Bénard, we recommend the review written by Wesfreid and published in Chapter 1 of the book edited by Mutabazi et al. [8].

Pearson [11] gave theoretical support to the idea that the thermal buoyancy force was not responsible of the phenomenon observed in Bénard's experiments. In Pearson's paper, his conclusion is: "we see that the buoyancy mechanism has no chance of causing convection cells, while the surface tension mechanism is almost certain to do so, and that observations support this". On the other hand, the theoretical scheme adopted for many years to explain Bénard's observations is that the thermal expansion of fluid elements close to the lower hot wall determines a vertical buoyancy force compensated by the viscous resistance. When these competing forces reach an equilibrium and, eventually, the buoyancy force prevails over the viscous resistance, the convection cellular flow is established [10]. The dimensionless parameter comparing the extent of the buoyancy force to that of the viscous resistance is nowadays well known as the *Rayleigh number*,

$$Ra = \frac{g\,\beta\,(T_1 - T_2)\,L^3}{\nu\,\alpha}\,. \tag{7.1}$$

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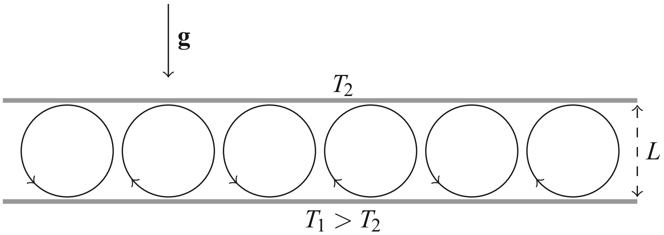

Fig. 7.1 A train of counter-rotating cells in a fluid layer bounded by two isothermal planes

As shown in Fig. 7.1, $T_1 > T_2$ is the temperature of the lower heated wall, while T_2 is the temperature of the upper free boundary, g is the modulus of the gravitational acceleration $\mathbf{g}$, and L is the thickness of the fluid layer. Actually, it is the Rayleigh number that displays a threshold, called the *critical value*, Ra_c, which defines the condition for the onset of the buoyancy-induced cells, namely $Ra > Ra_c$. Despite the correctness of Pearson's conclusions [11] about Bénard's experiments, there are several other experimental circumstances where the onset of the flow cells is in fact caused by the thermal buoyancy force and, hence, by the condition $Ra > Ra_c$. This happens, for instance, in the classical experiment reported by Schmidt and Milverton [15], as well as in many natural situations quite common in oceanography, meteorology, or geophysics [10]. Figure 7.1 shows that the flow pattern is a train of counter-rotating cells.

In the following, we will not investigate the role played by the surface tension, highlighted by Pearson [11], and focus our attention on the thermal buoyancy force as the cause of cells. This approach stems from the pioneering paper by Lord Rayleigh [13], and it has been developed by several authors, over an entire century, in a really huge literature. Extensive surveys on this topic can be found in many books. Just a few examples are Chandrasekhar [2], Koschmieder [6], Getling [4], Drazin and Reid [3].

7.2 The Rayleigh–Bénard Problem

The onset of buoyancy-induced cells is a classical problem of free convection in a horizontal fluid layer heated from below, viz. the well-known *Rayleigh–Bénard problem*. More precisely, in its simplified formulation, one assumes an infinitely wide horizontal fluid layer bounded by two isothermal planes. The lower boundary plane is kept isothermal at temperature T_1, while the upper boundary plane is kept isothermal at temperature $T_2 < T_1$. As is well known, buoyancy-induced cells appear when the Rayleigh number exceeds the critical value Ra_c. The critical value depends on the boundary conditions assumed at the isothermal boundaries. There are three main cases, classically devised in the literature:

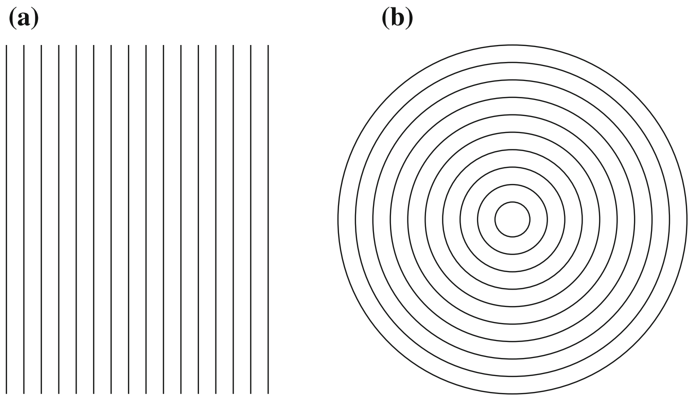

Fig. 7.2 Simple planforms of the convection cells: **a** straight rolls; **b** toroidal rolls

- Both boundaries are rigid and impermeable walls, so that impermeability and no-slip boundary conditions are prescribed on the velocity field. In this case, $Ra_c \approx 1707.76$.
- The lower boundary is a rigid and impermeable wall, while the upper boundary is a stress-free surface. With stress-free surface, we mean that the boundary conditions for the velocity are impermeability and vanishing tangential components of the viscous stress tensor, τ_{ij}. In this case, $Ra_c \approx 1100.65$.
- Both boundaries are stress-free surfaces. In this case, $Ra_c = 27\,\pi^4/4 \approx 657.511$.

The third case is the only one admitting a fully analytical solution, and it was originally regarded in the paper by Lord Rayleigh [13]. We mention that the stress-free boundary conditions embody a simplified physical model of the interface between a viscous liquid and a low-viscosity gas.

We have established that the boundary conditions prescribed at the horizontal boundary planes of the fluid layer influence the critical value of the Rayleigh number for the onset of the instability. The vertical sidewalls bounding laterally the shallow layer play an important role in shaping the planform of the buoyancy-induced cells. The planform is in fact the shape of the cells as detected on a plane cutting horizontally the fluid layer. The planform of the buoyancy-induced cells depends on several features of the system including the shape of the lateral confining walls, even when the fluid layer is extremely shallow. Two sample cases are illustrated in Fig. 7.2, namely that of the straight rolls, and that of the toroidal rolls. The latter planform is favoured when the sidewall is a vertical cylindrical surface with circular cross section.

The onset of buoyancy-induced cells in a fluid initially at rest may be viewed as a manifestation of the convective instability of the rest state, where the fluid velocity **u** is zero everywhere. In this sense, in the study of the Rayleigh–Bénard problem, we employ a linear stability analysis, so that the critical condition $Ra = Ra_c$ represents the threshold for the rest state to become convectively unstable.

At this stage, the reader may have noticed the twofold meaning of the terms "convection", "convective" and "convectively" in the present discussion. These terms naturally address a special type of heat transfer occurring in the fluid, i.e. the convection, and the specific type of instability arising in the fluid, i.e. the convective instability. As it should be clear from our definition given in Chap. 4, the convective instability may well emerge in flow systems where no convection heat transfer is present or may take place. There is no reasonable way to overcome this terminological conflict without introducing artificial terms different from those commonly employed in the literature. The author is confident that the context where the term is used makes its meaning unambiguous in every case.

If we consider a fluid layer, initially at rest, subject to an externally imposed temperature difference (heating from below), the rest state becomes unstable giving rise to buoyancy-induced cells when the Rayleigh number becomes sufficiently high.

In order to regard the Rayleigh–Bénard problem as a stability analysis, we need to develop the governing equations for the perturbations superposed onto a basic stationary state of the fluid.

7.3 Stability and Instability of Fluid Systems

As extensively discussed in Chap. 4, the basic idea behind Lyapunov's concept of instability is that we must consider an initial state of a system and a trajectory originating from this initial state. Then, we slightly perturb the initial state and examine the perturbed trajectory. If the small perturbation results, for a sufficiently large time, in a definitely different trajectory, then we have an unstable behaviour. Otherwise, we have stability. Instability is a consequence of an extremely strong dependence of the time evolution on the initial conditions.

If we apply Lyapunov's idea to the governing equations of a fluid, we must think of a trajectory as the time evolution of a given flow and we must think of an equilibrium state as a stationary flow. On checking the stability of a stationary flow, we must slightly perturb the velocity, pressure and temperature fields and see if the perturbation drives the system far away from its original stationary flow. If this happens, then we have an unstable flow. Otherwise, we have a stable flow.

A fluid flow can be unstable even in the absence of a thermal coupling, i.e. if the flow is isothermal or if the buoyancy force is negligible. In this case, the origin of the instability is in the governing mass and momentum balance equations and, in particular, in the nonlinear inertial term,

$$u_j \, \frac{\partial u_i}{\partial x_j} \, ,$$

of the local momentum balance equation (5.84),

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x_i} + \nu \nabla^2 u_i \ .$$

In the absence of this nonlinear term, every stationary flow without a thermal coupling would be stable. The instability of an isothermal, or forced convection, flow is called *hydrodynamic instability*. Since, in this case, the temperature field does not appear either in the local mass balance equation or in the local momentum balance equation, the analysis of the hydrodynamic instability does not involve the solution of the local energy balance equation.

Another kind of instability is that driven by the thermal coupling of the velocity field through the buoyancy force. This kind of instability is called *thermal instability*. The thermal instability depends not only on the nonlinearity of the local momentum balance, but it is also driven by the nonlinear convective term,

$$u_j \frac{\partial T}{\partial x_j} \ ,$$

as well as by the nonlinear viscous dissipation term, $2\,\nu\,\mathscr{D}_{ij}\,\mathscr{D}_{ij}/c$, of the local energy balance equation (5.85),

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \alpha \nabla^2 T + \frac{q_{\mathrm{g}}}{\rho_0\, c} + \frac{2\,\nu}{c}\,\mathscr{D}_{ij}\,\mathscr{D}_{ij} \ .$$

In order to illustrate the method for testing the stability or instability of a basic fluid flow, we refer to a Newtonian fluid and we consider the governing local balance equations (5.83)–(5.85), within the Oberbeck–Boussinesq approximation,

$$\frac{\partial u_j}{\partial x_j} = 0 \ , \tag{7.2}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\beta\,(T - T_0)\, g_i - \frac{1}{\rho_0} \frac{\partial P}{\partial x_i} + \nu \nabla^2 u_i \ , \tag{7.3}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \alpha \nabla^2 T + \frac{q_{\mathrm{g}}}{\rho_0\, c} + \frac{2\,\nu}{c}\,\mathscr{D}_{ij}\,\mathscr{D}_{ij} \ , \tag{7.4}$$

where the thermal power generated per unit volume, $q_{\mathrm{g}}(\mathbf{x}, t)$, is considered as a known function, independent of the fields $(\mathbf{u}, P, T)$. If we want to test the stability of a *basic solution*, $(\mathbf{u}_{\mathrm{b}}, P_{\mathrm{b}}, T_{\mathrm{b}})$, of Eqs. (7.2)–(7.4), we proceed as follows. We perturb the basic solution, i.e., we express the fields $(\mathbf{u}, P, T)$ as

$$u_i = u_{\mathrm{b}i} + \varepsilon\, U_i \ , \qquad P = P_{\mathrm{b}} + \varepsilon\, \Pi \ , \qquad T = T_{\mathrm{b}} + \varepsilon\, \Theta \ , \tag{7.5}$$

where ε is the perturbation parameter. The terms $\varepsilon\, U_i$, $\varepsilon\, \Pi$ and $\varepsilon\, \Theta$ express the perturbation of the basic solution. We remember that the basic solution $(\mathbf{u}_{\mathrm{b}}, P_{\mathrm{b}}, T_{\mathrm{b}})$ satisfies

Eqs. (7.2)–(7.4), and, on substituting Eq. (7.5) into Eqs. (7.2)–(7.4), we obtain

$$\varepsilon \frac{\partial U_j}{\partial x_j} = 0 , \tag{7.6}$$

$$\varepsilon \frac{\partial U_i}{\partial t} + \varepsilon U_j \frac{\partial u_{bi}}{\partial x_j} + \varepsilon u_{bj} \frac{\partial U_i}{\partial x_j} + \varepsilon^2 U_j \frac{\partial U_i}{\partial x_j}$$
$$= -\varepsilon \beta \Theta g_i - \frac{\varepsilon}{\rho_0} \frac{\partial \Pi}{\partial x_i} + \varepsilon \nu \nabla^2 U_i , \tag{7.7}$$

$$\varepsilon \frac{\partial \Theta}{\partial t} + \varepsilon u_{bj} \frac{\partial \Theta}{\partial x_j} + \varepsilon U_j \frac{\partial T_b}{\partial x_j} + \varepsilon^2 U_j \frac{\partial \Theta}{\partial x_j}$$
$$= \varepsilon \alpha \nabla^2 \Theta + \frac{4 \varepsilon \nu}{c} \mathscr{D}_{bij} \mathfrak{D}_{ij} + \frac{2 \varepsilon^2 \nu}{c} \mathfrak{D}_{ij} \mathfrak{D}_{ij} , \tag{7.8}$$

where

$$\mathscr{D}_{bij} = \frac{1}{2} \left(\frac{\partial u_{bi}}{\partial x_j} + \frac{\partial u_{bj}}{\partial x_i} \right) , \qquad \mathfrak{D}_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) . \tag{7.9}$$

We mention that the non-homogeneous term, q_g, in Eqs. (7.2)–(7.4) does not appear any more in the perturbation Eqs. (7.6)–(7.8), since (u_{bi}, P_b, T_b) is a solution of Eqs. (7.2)–(7.4).

Equations (7.6)–(7.8) express the governing equations for the perturbation fields (U_i, Π, Θ). We note that these equations contain a coupling to the basic solution (u_{bi}, P_b, T_b) only as a consequence of the nonlinear terms

$$u_j \frac{\partial u_i}{\partial x_j} , \qquad u_j \frac{\partial T}{\partial x_j} , \qquad \frac{2 \nu}{c} \mathscr{D}_{ij} \mathscr{D}_{ij} ,$$

that appear in Eqs. (7.3) and (7.4). Without these nonlinear terms, the perturbations would be uncoupled to the basic solution, so that the perturbation of the basic solution would be independent of the basic solution. This circumstance would result in a stability of the basic solution whatever it may be. Thus, we have established a link between the instability and the nonlinearity of the governing equations.

At this point, we have two alternatives: we may assume that the perturbations are small, or we may investigate perturbations of arbitrarily large amplitude. In the first case, we perform a linear stability analysis. In the second case, we investigate the nonlinear stability of the flow. The first option is the simplest one, and we will restrict all the forthcoming discussion to this case. Assuming small perturbations means assuming $\varepsilon \ll 1$, so that we can neglect the terms $O(\varepsilon^2)$ with respect to the terms $O(\varepsilon)$ in Eqs. (7.6)–(7.8). Therefore, we can simplify ε from Eqs. (7.6)–(7.8) and rewrite them as

$$\frac{\partial U_j}{\partial x_j} = 0 \,, \tag{7.10}$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial u_{\mathrm{b}i}}{\partial x_j} + u_{\mathrm{b}j} \frac{\partial U_i}{\partial x_j} = -\beta\, \Theta\, g_i - \frac{1}{\rho_0} \frac{\partial \Pi}{\partial x_i} + \nu\, \nabla^2 U_i \,, \tag{7.11}$$

$$\frac{\partial \Theta}{\partial t} + u_{\mathrm{b}j} \frac{\partial \Theta}{\partial x_j} + U_j \frac{\partial T_{\mathrm{b}}}{\partial x_j} = \alpha\, \nabla^2 \Theta + \frac{4\,\nu}{c}\, \mathscr{D}_{\mathrm{b}ij}\, \mathfrak{D}_{ij} \,. \tag{7.12}$$

One may solve Eqs. (7.10)–(7.12) and check what the time evolution of the perturbation is like: if it leads to an increasingly large departure from the basic solution, or if it leads to an asymptotic recovery of the basic solution. In the first case, we have a response of instability for the basic flow, while in the second case, we have an outcome of stability.

7.4 Formulation of the Rayleigh–Bénard Problem

In Sect. 7.2, we have seen that a crucial point in modelling the Rayleigh–Bénard system is the definition of the velocity boundary conditions. As illustrated in Fig. 7.3, the z-axis is taken as vertical, while the x and y axes are horizontal. For the sake of mathematical simplicity, we will initially model the boundaries $z = 0$ and $z = L$ as impermeable and stress-free. In doing this, we follow the approach chosen by Lord Rayleigh [13] in his pioneering paper. The determination of the onset conditions for the development of convection cells can be approached by a linear stability analysis, based on Eqs. (7.10)–(7.12).

7.4.1 Governing Equations

The critical condition for the onset of convection cells in the fluid layer is obtained starting from the basic state where the fluid is at rest,

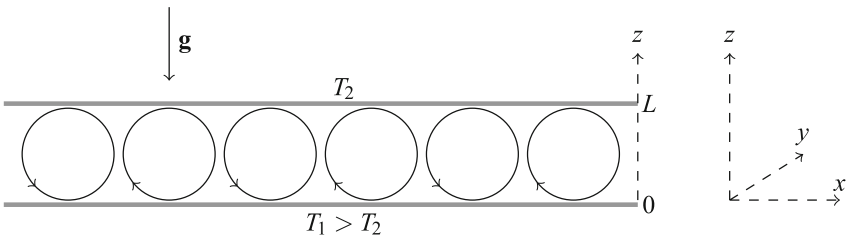

Fig. 7.3 Rayleigh–Bénard system: choice of the coordinate axes

$$u_{bi} = 0 \; , \qquad T_b = T_1 - (T_1 - T_2)\,\frac{z}{L} \; . \tag{7.13}$$

In fact, one may easily verify that Eq. (7.13) is a solution of the local mass, momentum and energy balance equations (7.2)–(7.4) under the assumption that no volumetric heat source is present in the fluid, namely $q_g = 0$. We also mention that Eq. (7.13) is compatible with the conditions of stress-free and impermeable boundaries, as the velocity is zero everywhere. Also the thermal boundary conditions of isothermal surfaces at $z = 0$ and $z = L$, with temperatures T_1 and T_2, are satisfied. Thus, Eqs. (7.10)–(7.12) yield

$$\frac{\partial U_j}{\partial x_j} = 0 \; ,$$

$$\frac{\partial U_i}{\partial t} = -\beta\,\Theta\,g_i - \frac{1}{\rho_0}\,\frac{\partial \Pi}{\partial x_i} + \nu\,\nabla^2 U_i \; ,$$

$$\frac{\partial \Theta}{\partial t} - W\,\frac{T_1 - T_2}{L} = \alpha\,\nabla^2 \Theta \; , \tag{7.14}$$

where we denoted as (U, V, W) the (x, y, z) components of the velocity perturbation U_i. The boundary conditions for the velocity and temperature fields model the constraints of uniform temperature, impermeability and vanishing tangential viscous stresses at $z = 0,\ L$. Here, we define the viscous stress tensor associated with the velocity perturbation,

$$\mathscr{T}_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \; . \tag{7.15}$$

Thus, the boundary conditions can be written either as

$$z = 0,\ L \;:\qquad W = 0 = \Theta \; , \quad \mathscr{T}_{zx} = 0 = \mathscr{T}_{zy} \; , \tag{7.16}$$

or, equivalently, as

$$z = 0,\ L \;:\qquad W = 0 = \Theta \; , \quad \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} = 0 = \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} \; . \tag{7.17}$$

Since $W = 0$ at $z = 0,\ L$, we can rewrite Eq. (7.17) as

$$z = 0,\ L \;:\qquad W = 0 = \Theta \; , \quad \frac{\partial U}{\partial z} = 0 = \frac{\partial V}{\partial z} \; . \tag{7.18}$$

The perturbation equations can be further simplified by allowing an appropriate scaling of the quantities, so that the study is carried out with a dimensionless formulation,

$$\frac{U_i}{\alpha/L} \to U_i \; , \qquad \frac{\Theta}{T_1 - T_2} \to \Theta \; , \qquad \frac{\Pi}{\rho_0 \nu \alpha / L^2} \to \Pi \; ,$$

$$\frac{x_i}{L} \to x_i \ , \qquad \frac{t}{L^2/\alpha} \to t \ . \tag{7.19}$$

Thus, Eqs. (7.14) and (7.18) can be rewritten in a dimensionless form as

$$\frac{\partial U_j}{\partial x_j} = 0 \ , \tag{7.20}$$

$$\frac{1}{Pr}\frac{\partial U_i}{\partial t} = Ra\,\Theta\,\delta_{i3} - \frac{\partial \Pi}{\partial x_i} + \nabla^2 U_i \ , \tag{7.21}$$

$$\frac{\partial \Theta}{\partial t} - W = \nabla^2 \Theta \ , \tag{7.22}$$

$$z = 0,\ 1\ : \qquad W = 0 = \Theta\ , \quad \frac{\partial U}{\partial z} = 0 = \frac{\partial V}{\partial z}\ , \tag{7.23}$$

where δ_{i3} is the $(i, 3)$ component of Kronecker's delta, namely the ith component of the unit vector $\mathbf{e}_z = (0, 0, 1)$, while the dimensionless parameters Pr and Ra are the *Prandtl number* and the Rayleigh number defined as

$$Pr = \frac{\nu}{\alpha}\ , \qquad Ra = \frac{g\beta(T_1 - T_2)L^3}{\nu\alpha}\ . \tag{7.24}$$

The term $-\,\partial\Pi/\partial x_i$ can be encompassed by taking the curl of the momentum balance equation so that one obtains:

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z}\right) = Ra\,\frac{\partial \Theta}{\partial x}\ , \tag{7.25}$$

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}\right) = Ra\,\frac{\partial \Theta}{\partial y}\ . \tag{7.26}$$

We derive Eq. (7.25) with respect to x, and Eq. (7.26) with respect to y. Then, we sum the two resulting equations, so that we obtain

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left[\nabla^2 W - \frac{\partial}{\partial z}\left(\frac{\partial U_j}{\partial x_j}\right)\right] = Ra\,\nabla_2^2\Theta\ , \tag{7.27}$$

where ∇_2^2 is the two-dimensional Laplace operator, defined as

$$\nabla_2^2\Theta = \frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\ . \tag{7.28}$$

By taking into account the local mass balance equation, $\partial U_j/\partial x_j = 0$, we can extract a set of two partial differential equations in the unknowns (W, Θ), which describe the linear stability problem,

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W = Ra\,\nabla_2^2\Theta\ ,$$

$$\frac{\partial\Theta}{\partial t} - W = \nabla^2\Theta\ ,$$

$$z = 0,\ 1\ :\qquad W = 0 = \Theta\ ,\quad \frac{\partial^2 W}{\partial z^2} = 0\ , \tag{7.29}$$

The boundary conditions $\partial^2 W/\partial z^2 = 0$, at $z = 0, 1$, are retrieved by deriving the stress-free conditions at $z = 0, 1$, given by Eq. (7.23),

$$\frac{\partial U}{\partial z} = 0\ ,\qquad \frac{\partial V}{\partial z} = 0\ ,$$

with respect to x and y, respectively, by summing them so that one obtains

$$\frac{\partial}{\partial z}\left(\frac{\partial U_j}{\partial x_j}\right) - \frac{\partial^2 W}{\partial z^2} = 0\ ,$$

and finally by employing the local mass balance equation, $\partial U_j/\partial x_j = 0$.

7.4.2 Normal Mode Analysis

Equations (7.29) can be solved by employing the Fourier transform method. We will follow a procedure similar to that described, for instance, in Sect. 4.2. The significant difference is that we now employ two-dimensional Fourier transforms, defined by Eqs. (2.91) and (2.92)

$$\tilde{W}(k_x, k_y, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x, y, z, t)\, \mathrm{e}^{-\mathrm{i}\left(k_x x + k_y y\right)}\, \mathrm{d}x\, \mathrm{d}y\ ,$$

$$W(x, y, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{W}(k_x, k_y, z, t)\, \mathrm{e}^{\mathrm{i}\left(k_x x + k_y y\right)}\, \mathrm{d}k_x\, \mathrm{d}k_y\ ,$$

$$\tilde{\Theta}(k_x, k_y, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta(x, y, z, t)\, e^{-i\left(k_x x + k_y y\right)}\, d x\, d y \ ,$$

$$\Theta(x, y, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\Theta}(k_x, k_y, z, t)\, e^{i\left(k_x x + k_y y\right)}\, d k_x\, d k_y \ . \tag{7.30}$$

In other words, we are seeking solutions expressed through a superposition of Fourier modes, or normal modes, propagating in the (x, y) plane along the direction of the wave vector (k_x, k_y). We are dealing with two-dimensional and, hence, double Fourier transforms. This means that the property of partial derivatives expressed by Eq. (2.18) applies to the derivatives both with respect to x and those with respect to y. This means that the Fourier transforms of $\nabla^2 W$, $\nabla^2 \Theta$ and $\nabla_2^2 \Theta$ are given, respectively, by

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right) \tilde{W} \ , \qquad \left(\frac{\partial^2}{\partial z^2} - k^2\right) \tilde{\Theta} \ , \qquad -k^2\, \tilde{\Theta} \ ,$$

where $k = (k_x^2 + k_y^2)^{1/2}$ is the wave number.

The use of the Fourier transform method, for the solution of Eq. (7.29), implies that $\tilde{W}$ and $\tilde{\Theta}$ are the new unknowns to be determined. This task can be accomplished by using the separation of variables, described in Appendix A, namely by separating the dependence on z and on t. Thus, we can express $\tilde{W}$ and $\tilde{\Theta}$ as linear combinations of separated solutions written as

$$\tilde{W} = f(z)\, e^{\lambda t} \ , \qquad \tilde{\Theta} = h(z)\, e^{\lambda t} \ , \tag{7.31}$$

where $\lambda = \eta - i\,\omega \in \mathbb{C}$ is a complex parameter, $\omega \in \mathbb{R}$ is the angular frequency, and $\eta \in \mathbb{R}$ is the growth rate. As usual, for a given k, $\eta > 0$ means convective instability, $\eta < 0$ means stability, while $\eta = 0$ indicates the threshold condition of neutral, or marginal, stability.

By evaluating the two-dimensional Fourier transform of Eq. (7.29), and by employing Eq. (7.31), the stability problem is formulated as

$$\left(\frac{1}{Pr}\,\lambda - \frac{d^2}{d z^2} + k^2\right)\left(\frac{d^2}{d z^2} - k^2\right) f + Ra\, k^2\, h = 0 \ , \tag{7.32}$$

$$\left(\lambda - \frac{d^2}{d z^2} + k^2\right) h - f = 0 \ , \tag{7.33}$$

$$z = 0, 1: \qquad f = 0 \ , \quad \frac{d^2 f}{d z^2} = 0 \ , \quad h = 0 \ . \tag{7.34}$$

We can combine the two Eqs. (7.32) and (7.33) into a single (sixth-order) ordinary differential equation in the sole unknown function h,

$$\left(\frac{1}{Pr}\,\lambda - \frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + k^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\,z^2} - k^2\right)\left(\lambda - \frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + k^2\right) h + Ra\,k^2\,h = 0\;, \quad (7.35)$$

with the boundary conditions

$$z = 0, 1: \qquad h = 0\;, \quad \frac{\mathrm{d}^2 h}{\mathrm{d}\,z^2} = 0\;, \quad \frac{\mathrm{d}^4 h}{\mathrm{d}\,z^4} = 0\;. \quad (7.36)$$

We mention that the boundary conditions $\mathrm{d}^2 h/\mathrm{d}z^2 = 0$ are obtained from Eq. (7.33) by taking the limits $z \to 0$ and $z \to 1$ and by using Eq. (7.34). Likewise, the boundary conditions $\mathrm{d}^4 h/\mathrm{d}z^4 = 0$ are obtained from Eq. (7.33) derived twice with respect to z.

A solution of the differential problem, expressed by Eqs. (7.35) and (7.36), is easily found, namely

$$h(z) = \sin(n\,\pi\,z)\;, \quad n = 1, 2, 3, \ldots\;, \quad (7.37)$$

provided that

$$\left(\frac{1}{Pr}\,\lambda + n^2\,\pi^2 + k^2\right)\left(n^2\,\pi^2 + k^2\right)\left(\lambda + n^2\,\pi^2 + k^2\right) - Ra\,k^2 = 0\;, \quad (7.38)$$

The additional algebraic equation (7.38) is the so-called *dispersion relation* of stability. Since $\lambda = \eta - \mathrm{i}\,\omega$, the imaginary part of the dispersion relation vanishes if and only if

$$\omega\left(n^2\pi^2 + k^2\right)\left[2\,\eta + (Pr + 1)\left(n^2\pi^2 + k^2\right)\right] = 0\;. \quad (7.39)$$

For convectively unstable or neutrally stable modes, i.e. for $\eta \geq 0$, this equation can be satisfied only if $\omega = 0$, meaning that only zero-frequency normal modes are allowed. This result is well known in the literature as the *principle of exchange of stabilities* [12]. As it has been pointed out by Pellew and Southwell [12], the physical meaning of this principle is that "while oscillatory motions are not excluded by this investigation, they are permitted only in circumstances making for stability, i.e. in which they decay". In fact, also stable modes cannot be oscillatory as it will be shown in Sect. 7.5.1.

7.4.3 Neutral Stability

Since $\omega = 0$ when $\eta \geq 0$, for convectively unstable or neutrally stable states, the real part of the dispersion relation (7.38) vanishes if

$$Ra = \frac{\left(n^2\pi^2 + k^2\right)\left(\eta + n^2\pi^2 + k^2\right)\left(Pr^{-1}\eta + n^2\pi^2 + k^2\right)}{k^2} \, . \tag{7.40}$$

Convective instability means that there exists a positive integer, $n = 1, 2, 3, \ldots$), such that the growth rate is positive, namely $\eta > 0$. On gradually increasing Ra starting from zero, one encounters instability first with $n = 1$, so that one has

$$Ra = \frac{\left(\pi^2 + k^2\right)^3}{k^2} \, , \quad \text{(neutral stability)},$$

$$Ra > \frac{\left(\pi^2 + k^2\right)^3}{k^2} \, , \quad \text{(convective instability)},$$

and thus, necessarily,

$$Ra < \frac{\left(\pi^2 + k^2\right)^3}{k^2} \, ,$$

implies stability. Figure 7.4 displays the neutral stability curve, namely the plot of function

$$Ra(k) = \frac{\left(\pi^2 + k^2\right)^3}{k^2} \, . \tag{7.41}$$

Its minimum defines the onset of convection cells,

$$k_c = \frac{\pi}{\sqrt{2}} \approx 2.22144 \, , \qquad Ra_c = \frac{27\pi^4}{4} \approx 657.511 \, . \tag{7.42}$$

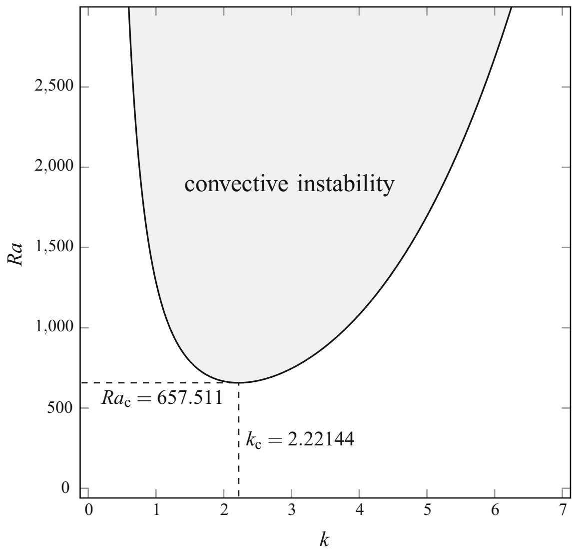

Fig. 7.4 Neutral stability curve for the Rayleigh–Bénard problem with stress-free and impermeable boundary conditions at $z = 0, 1$

7.5 Rayleigh–Bénard Problem with Other Types of Boundary Conditions

In cases where the boundary surfaces $z = 0, 1$ are not both stress-free, the convective stability analysis partly changes. For instance, when both surfaces $z = 0, 1$ are rigid impermeable walls, no-slip boundary conditions for the velocity have to be imposed at $z = 0, 1$. Thus, Eq. (7.29) changes to the form

$$\left(\frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 W = Ra \, \nabla_2^2 \Theta \; ,$$

$$\frac{\partial \Theta}{\partial t} - W = \nabla^2 \Theta \; ,$$

$$z = 0, \; 1 \; : \qquad W = 0 = \Theta \; , \quad \frac{\partial W}{\partial z} = 0 \; . \tag{7.43}$$

In the intermediate case, where $z = 0$ is subject to no-slip conditions and $z = 1$ is stress-free, we have

$$\left(\frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 W = Ra \, \nabla_2^2 \Theta \; ,$$

$$\frac{\partial \Theta}{\partial t} - W = \nabla^2 \Theta \; ,$$

$$z = 0 \; : \qquad W = 0 = \Theta \; , \quad \frac{\partial W}{\partial z} = 0 \; ,$$

$$z = 1 \; : \qquad W = 0 = \Theta \; , \quad \frac{\partial^2 W}{\partial z^2} = 0 \; . \tag{7.44}$$

In other terms, the partial differential equations for the perturbations are unaffected by changed boundary conditions, the only change being the boundary conditions for W and Θ. The reason is simple. The governing partial differential equations for the perturbations just depend on the basic solution that satisfies both stress-free boundary conditions and no-slip boundary conditions at $z = 0, 1$. In all these cases, the basic solution is given by Eq. (7.13). Equations (7.43) and (7.44) show that, when a boundary surface turns from stress-free to no-slip, one of the boundary conditions turns from $\partial^2 W/\partial z^2 = 0$ to $\partial W/\partial z = 0$. The reason is that the condition of vanishing second derivative $\partial^2 W/\partial z^2$ is a consequence of the vanishing tangential components of the viscous stress tensor. If a boundary, say $z = 1$, has impermeability and no-slip conditions, then one may write

$$z = 1 \; : \qquad U = 0 \; , \quad V = 0 \; , \quad W = 0 \; .$$

By employing such conditions, as well as the local mass balance equation (7.20) in the limit $z \to 1$, one readily reaches the conclusion

$$z = 1 \ : \qquad W = 0 \ , \quad \frac{\partial W}{\partial z} = 0 \ .$$

The method described in Sect. 7.4.2 can still be applied, together with the separation of variables expressed by Eq. (7.31). Hence, if no-slip boundary conditions for the velocity are imposed at both $z = 0$ and $z = 1$, Eqs. (7.32)–(7.34) are replaced by

$$\left(\frac{1}{Pr}\,\lambda - \frac{d^2}{d\,z^2} + k^2\right)\left(\frac{d^2}{d\,z^2} - k^2\right) f + Ra\, k^2\, h = 0 \ , \tag{7.45}$$

$$\left(\lambda - \frac{d^2}{d\,z^2} + k^2\right) h - f = 0 \ , \tag{7.46}$$

$$z = 0, 1 : \qquad f = 0 \ , \quad \frac{d\,f}{d\,z} = 0 \ , \quad h = 0 \ . \tag{7.47}$$

In the case of mixed no-slip and stress-free boundary conditions at $z = 0$ and $z = 1$, respectively, one has

$$z = 0 : \qquad f = 0 \ , \quad \frac{d\,f}{d\,z} = 0 \ , \quad h = 0 \ ,$$

$$z = 1 : \qquad f = 0 \ , \quad \frac{d^2 f}{d\,z^2} = 0 \ , \quad h = 0 \ , \tag{7.48}$$

instead of Eq. (7.47).

One may well say that, although possible, an analytical solution for either the differential problems, given by Eqs. (7.45)–(7.47) and by Eqs. (7.45), (7.46) and (7.48), is not the most convenient approach. An easier, reliable and accurate procedure to get the solution of either these differential problems is the use of a numerical solver for differential eigenvalue problems. We refer the reader to Chap. 10 for a discussion of the numerical method, and for the implementation of the code needed to develop this numerical solver. Figures 7.5 and 7.6 display, respectively, the neutral stability curves for the Rayleigh–Bénard problem with rigid and impermeable boundaries, i.e. for the conditions given by Eq. (7.47), and for the mixed case where the lower boundary is rigid while the upper boundary is stress-free, i.e. for the conditions given by Eq. (7.48). The shape of these neutral stability curves is not much different from that of the curve displayed in Fig. 7.4. We will see that this shape is surprisingly common for the diverse variants of the Rayleigh–Bénard problem. The most important difference between Figs. 7.4, 7.5 and 7.6 is in the position of the minimum, namely in the values of k_c and Ra_c. With the boundary conditions expressed by Eq. (7.47), we obtain

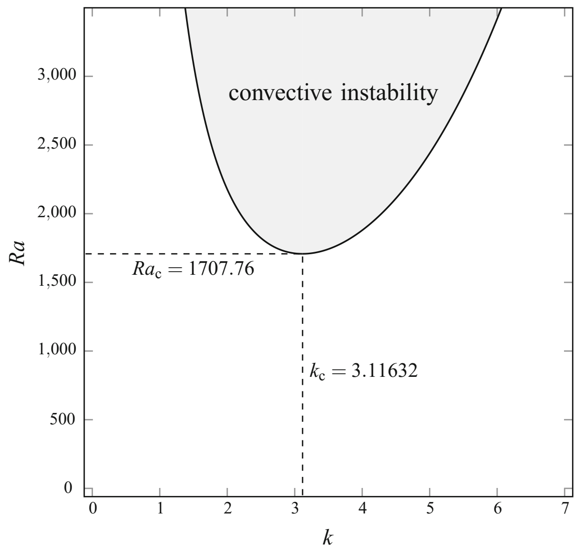

Fig. 7.5 Neutral stability curve for the Rayleigh–Bénard problem with rigid and impermeable boundary conditions at $z = 0, 1$

$$k_c = 3.11632 \,, \qquad Ra_c = 1707.76 \,, \tag{7.49}$$

while in the mixed case given by Eq. (7.48), we obtain

$$k_c = 2.68232 \,, \qquad Ra_c = 1100.65 \,. \tag{7.50}$$

These results allow one to conclude that the presence of stress-free boundaries tends to favour the onset of convective instability. In fact, the case where both the impermeable boundaries are rigid is the one where the instability requires the highest Rayleigh number. The mixed case is intermediate, while the case with two stress-free boundaries is that where the instability emerges at the lowest Rayleigh number. One can rephrase this conclusion by saying that the no-slip condition is a stabilising mechanism for the thermal instability.

7.5.1 The Principle of Exchange of Stabilities

By employing integration by parts over $z \in [0, 1]$, we can write

$$\int_0^1 \bar{f} \, \frac{d^4 f}{d z^4} \, dz = - \int_0^1 \frac{d \bar{f}}{d z} \, \frac{d^3 f}{d z^3} \, dz = \int_0^1 \left| \frac{d^2 f}{d z^2} \right|^2 dz \,, \tag{7.51}$$

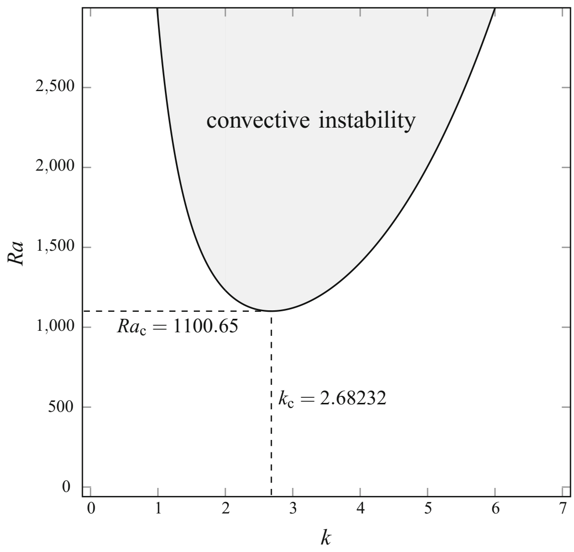

Fig. 7.6 Neutral stability curve for the Rayleigh–Bénard problem with rigid and impermeable boundary conditions at $z = 0$, and with stress-free and impermeable boundary conditions at $z = 1$

$$\int_0^1 \bar{f}\, \frac{d^2 f}{d z^2}\, dz = -\int_0^1 \left|\frac{d f}{d z}\right|^2 dz\,, \tag{7.52}$$

$$\int_0^1 h\, \frac{d^2 \bar{h}}{d z^2}\, dz = -\int_0^1 \left|\frac{d h}{d z}\right|^2 dz\,, \tag{7.53}$$

where the primes denote derivatives with respect to z, the overline denotes complex conjugation, and either the boundary conditions given by Eq. (7.47) or those expressed by Eq. (7.48) are employed.

We stress that the chain of integrations by parts in Eqs. (7.51)–(7.53) holds both with the set of boundary conditions (7.47) and with the set of boundary conditions (7.48). We finally mention that Eqs. (7.51)–(7.53) are valid also in the case where both boundaries are rigid and stress-free, described by Eq. (7.34).

Let us consider Eqs. (7.45) and (7.46). We multiply Eq. (7.45) by $\bar{f}$ and integrate over $z \in [0, 1]$. Then, by employing Eqs. (7.52) and (7.53), we obtain

$$\int_0^1 \left|\frac{d^2 f}{d z^2}\right|^2 dz + \left(2\,k^2 + \frac{\lambda}{Pr}\right) \int_0^1 \left|\frac{d f}{d z}\right|^2 dz + \left(k^2 + \frac{\lambda}{Pr}\right) k^2 \int_0^1 |f|^2\, dz$$
$$-\; Ra\, k^2 \int_0^1 \bar{f}\, h\, dz = 0\,. \tag{7.54}$$

Let us write the complex conjugate of Eq. (7.46),

$$\left(\bar{\lambda} - \frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + k^2\right)\bar{h} - \bar{f} = 0\,. \tag{7.55}$$

We multiply Eq. (7.55) by h and integrate over $z \in [0, 1]$. By using Eqs. (7.52) and (7.53), we obtain

$$\int_0^1 \left|\frac{\mathrm{d}\,h}{\mathrm{d}\,z}\right|^2 \mathrm{d}z + \left(\bar{\lambda} + k^2\right)\int_0^1 |h|^2\,\mathrm{d}z - \int_0^1 \bar{f}\,h\,\mathrm{d}z = 0\,. \tag{7.56}$$

We can combine Eqs. (7.54) and (7.56) to obtain

$$\int_0^1 \left|\frac{\mathrm{d}^2 f}{\mathrm{d}\,z^2}\right|^2 \mathrm{d}z + \left(2\,k^2 + \frac{\lambda}{Pr}\right)\int_0^1 \left|\frac{\mathrm{d}\,f}{\mathrm{d}\,z}\right|^2 \mathrm{d}z + \left(k^2 + \frac{\lambda}{Pr}\right)k^2\int_0^1 |f|^2\,\mathrm{d}z$$

$$- \; Ra\,k^2\left[\int_0^1 \left|\frac{\mathrm{d}\,h}{\mathrm{d}\,z}\right|^2 \mathrm{d}z + \left(\bar{\lambda} + k^2\right)\int_0^1 |h|^2\,\mathrm{d}z\right] = 0\,. \tag{7.57}$$

We recall that $\lambda = \eta - \mathrm{i}\,\omega$. Thus, the imaginary part of Eq. (7.57) is given by

$$\omega\left(\frac{1}{Pr}\int_0^1 \left|\frac{\mathrm{d}\,f}{\mathrm{d}\,z}\right|^2 \mathrm{d}z + \frac{k^2}{Pr}\int_0^1 |f|^2\,\mathrm{d}z + Ra\,k^2\int_0^1 |h|^2\,\mathrm{d}z\right) = 0\,. \tag{7.58}$$

The expression in round brackets on the left-hand side of Eq. (7.58) is positive, unless the perturbation is identically zero, i.e. $f = 0 = h$. Therefore, we can conclude that

$$\omega = 0\,. \tag{7.59}$$

so that the principle of exchange of stabilities holds. If we consider the real part of Eq. (7.57), we obtain

$$\int_0^1 \left|\frac{\mathrm{d}^2 f}{\mathrm{d}\,z^2}\right|^2 \mathrm{d}z + 2\,k^2\int_0^1 \left|\frac{\mathrm{d}\,f}{\mathrm{d}\,z}\right|^2 \mathrm{d}z + k^4\int_0^1 |f|^2\,\mathrm{d}z$$

$$- \; Ra\,k^2\left[\int_0^1 \left|\frac{\mathrm{d}\,h}{\mathrm{d}\,z}\right|^2 \mathrm{d}z + k^2\int_0^1 |h|^2\,\mathrm{d}z\right]$$

$$+ \ \eta \left(\frac{1}{Pr} \int_0^1 \left| \frac{\mathrm{d} f}{\mathrm{d} z} \right|^2 \mathrm{d}z + \frac{k^2}{Pr} \int_0^1 |f|^2 \, \mathrm{d}z - Ra \, k^2 \int_0^1 |h|^2 \, \mathrm{d}z \right) = 0 \, . \qquad (7.60)$$

Equation (7.60) leads to some interesting conclusions. At neutral stability, $\eta = 0$, we infer that Ra is positive for every k and Pr. Moreover, either the neutral stability curve $Ra(k)$ displays a singularity when $k \to 0$ or $\mathrm{d}^2 f/\mathrm{d}z^2$ is identically vanishing in this limit. Indeed, a singular behaviour of the neutral stability curve is implied by Eq. (7.41), for the case where both boundaries are rigid and stress-free.

Another feature which can be inferred from Eq. (7.60) is that, in the limit $Ra \to 0$, the growth rate η cannot be positive. An obvious feature on physical grounds as $Ra \to 0$ is achieved when the temperature difference between the bounding surfaces tends to zero. Under such conditions, the buoyancy force cannot activate and sustain any natural convection flow.

An important aspect of the principle of exchange of stability formulated by the integral method just described is that Eq. (7.59) holds independently of η being negative, zero or positive. This is a slight, but interesting, feature with respect to what we were able to infer from Eq. (7.39) for the case where both boundaries are rigid and stress-free.

7.6 The Horton–Rogers–Lapwood Problem

A stability analysis of the rest state not referring to a clear fluid layer, but to a fluid-saturated porous medium was performed by Horton and Rogers Jr [5], and by Lapwood [7]. The Horton–Rogers–Lapwood (HRL) problem is the porous medium analogue of the Rayleigh–Bénard problem for a clear fluid. The analysis of the HRL problem was originally performed by assuming the validity of Darcy's law and by employing linearised governing equations. During the years, several extensions of the HRL problem have been studied including treatment of Darcy–Forchheimer's model, of Brinkman's model and adopting a weakly nonlinear stability analysis. For a review of these results, one can refer to Rees [14] and Tyvand [16].

7.6.1 Formulation of the Problem

By analogy with the Rayleigh–Bénard problem, let us consider a horizontal fluid-saturated porous layer having thickness L, bounded by two impermeable planes. The lower boundary plane is maintained at temperature T_1, while the upper boundary plane has a uniform temperature $T_2 < T_1$.

For the mathematical formulation of the problem, we rely on the framework discussed in Sect. 6.4. By assuming the validity of Darcy's law, of the Oberbeck–Boussinesq approximation, the governing equations of the saturated porous medium, without any volumetric heat source, $q_g = 0$, can be written as

$$\frac{\partial u_j}{\partial x_j} = 0 \, ,$$

$$\frac{\nu}{K} u_i = -\beta \, (T - T_0) \, g_i - \frac{1}{\rho_0} \frac{\partial P}{\partial x_i} \, ,$$

$$\sigma \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \alpha \, \nabla^2 T + \frac{\nu}{K \, c} u_j u_j \, , \tag{7.61}$$

where $\alpha = \varkappa_{\text{eff}}/(\rho_0 \, c)$ and $\nu = \mu/\rho_0$. If we want to test the stability of a basic solution, (u_{bi}, P_b, T_b), we proceed as follows. We perturb the basic solution, i.e., we express the fields (u_i, P, T) as

$$u_i = u_{bi} + \varepsilon \, U_i, \qquad P = P_b + \varepsilon \, \Pi, \qquad T = T_b + \varepsilon \, \Theta,$$

where ε is a positive dimensionless quantity, the perturbation parameter. The terms $\varepsilon \, U_i$, $\varepsilon \, \Pi$ and $\varepsilon \, \Theta$ express the perturbation of the basic solution.

Thus, we obtain the perturbation equations,

$$\varepsilon \, \frac{\partial U_j}{\partial x_j} = 0 \, ,$$

$$\frac{\varepsilon \, \nu}{K} U_i = -\varepsilon \, \beta \, \Theta \, g_i - \frac{\varepsilon}{\rho_0} \frac{\partial \Pi}{\partial x_i} \, ,$$

$$\varepsilon \, \sigma \frac{\partial \Theta}{\partial t} + \varepsilon \, u_{bj} \frac{\partial \Theta}{\partial x_j} + \varepsilon \, U_j \frac{\partial T_b}{\partial x_j} + \varepsilon^2 \, U_j \frac{\partial \Theta}{\partial x_j} = \varepsilon \, \alpha \, \nabla^2 \Theta$$

$$+ \frac{2 \, \varepsilon \, \nu}{K \, c} u_{bj} \, U_j + \frac{\varepsilon^2 \nu}{K \, c} U_j \, U_j \, . \tag{7.62}$$

The perturbation equations (7.62) express the dynamics of the perturbation fields (U_i, Π, Θ). We note that these equations contain a coupling with the basic solution (u_{bi}, P_b, T_b) only as an effect of the nonlinear terms

$$u_j \frac{\partial T}{\partial x_j} \, , \qquad \frac{\nu}{K \, c} u_j u_j \, .$$

As in the case of a clear fluid, we have two alternatives: we may assume that the perturbations are small, or we may investigate perturbations of arbitrarily large amplitude. In the first case, we perform a linear stability analysis. In the second case, we carry out a nonlinear stability analysis.

The first option is the simplest one, and we will restrict all the forthcoming discussion to this case. Assuming small perturbations means requiring $\varepsilon \ll 1$, so that we can neglect the terms $O\left(\varepsilon^2\right)$ with respect to the terms $O(\varepsilon)$ in the perturbation equations.

Therefore, we can simplify ε from the perturbation equations (7.62) and rewrite them as

$$\frac{\partial U_j}{\partial x_j} = 0 \; ,$$

$$\frac{\nu}{K}\, U_i = -\beta\, \Theta\, g_i - \frac{1}{\rho_0}\, \frac{\partial \Pi}{\partial x_i} \; ,$$

$$\sigma\, \frac{\partial \Theta}{\partial t} + u_{\mathrm{b}j}\, \frac{\partial \Theta}{\partial x_j} + U_j\, \frac{\partial T_{\mathrm{b}}}{\partial x_j} = \alpha\, \nabla^2 \Theta + \frac{2\,\nu}{K\, c}\, u_{\mathrm{b}j}\, U_j \; . \tag{7.63}$$

One may solve these equations and check what the time evolution of the perturbation is like: if it leads to an increasingly large departure from the basic solution, or if it leads to an asymptotic recovery of the basic solution. In the first case, we have a response of instability for the basic flow, while in the second case, we have an outcome of stability. The critical condition for the onset of convection in the layer is obtained by a linear stability analysis carried out starting from the basic state,

$$u_{\mathrm{b}i} = 0 \; , \qquad T_{\mathrm{b}} = T_1 - (T_1 - T_2)\, \frac{z}{L} \; . \tag{7.64}$$

The nature of the basic state leads to a dramatic simplification of the linearised perturbation equations. In fact, Eq. (7.63) simplify to

$$\frac{\partial U_j}{\partial x_j} = 0 \; ,$$

$$\frac{\nu}{K}\, U_i = -\beta\, \Theta\, g_i - \frac{1}{\rho_0}\, \frac{\partial \Pi}{\partial x_i} \; ,$$

$$\sigma\, \frac{\partial \Theta}{\partial t} - W\, \frac{T_1 - T_2}{L} = \alpha\, \nabla^2 \Theta \; . \tag{7.65}$$

The perturbation equations can be further simplified by allowing an appropriate scaling of the physical quantities, in order to carry out the study through a dimensionless formulation. Hence, we define the dimensionless quantities by means of the scalings,

$$\frac{U_i}{\alpha/L} \to U_i \; , \quad \frac{\Theta}{T_1 - T_2} \to \Theta \; ,$$

$$\frac{\Pi}{\rho_0 \nu \alpha / K} \to \Pi \; , \quad \frac{x_i}{L} \to x_i \; , \quad \frac{t}{\sigma\, L^2/\alpha} \to t \; , \tag{7.66}$$

so that the dimensionless perturbation equations can be written as

$$\frac{\partial U_j}{\partial x_j} = 0 \ ,$$

$$U_i = R\,\Theta\,\delta_{i3} - \frac{\partial \Pi}{\partial x_i} \ ,$$

$$\frac{\partial \Theta}{\partial t} - W = \nabla^2 \Theta \ , \tag{7.67}$$

with the boundary conditions,

$$z = 0, 1 : \qquad W = 0 = \Theta \ . \tag{7.68}$$

In particular, the conditions $W = 0$ express the impermeability of the boundaries. The parameter R defines the *Darcy–Rayleigh number*,

$$R = \frac{g\beta(T_1 - T_2)KL}{\nu\alpha} \ . \tag{7.69}$$

A comparison with Eq. (7.24) reveals that the Darcy–Rayleigh number differs from the Rayleigh number of a clear fluid mainly due to the factor $K\,L$ instead of L^3.

The term $-\,\partial\Pi/\partial x_i$ can be encompassed by taking the curl of the momentum balance equation, which yields

$$\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z} = R\,\frac{\partial \Theta}{\partial x} \ , \tag{7.70}$$

$$\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} = R\,\frac{\partial \Theta}{\partial y} \ . \tag{7.71}$$

We derive Eq. (7.70) with respect to x and Eq. (7.71) with respect to y. Then, we sum the two resulting equations, so that we obtain

$$\nabla^2 W = R\,\nabla_2^2 \Theta + \frac{\partial}{\partial z}\left(\frac{\partial U_j}{\partial x_j}\right) \ . \tag{7.72}$$

By taking into account the local mass balance equation, $\partial U_j/\partial x_j = 0$, we can extract a set of two partial differential equations in the unknowns (W, Θ), describing the stability problem,

$$\nabla^2 W = R\,\nabla_2^2 \Theta \ ,$$

$$\frac{\partial \Theta}{\partial t} - W = \nabla^2 \Theta \ , \tag{7.73}$$

with the boundary conditions

$$z = 0, 1: \qquad W = 0 = \Theta \; . \tag{7.74}$$

7.6.2 Normal Modes

As in the Rayleigh–Bénard problem, the governing equations for the perturbations can be solved by employing the Fourier transform method. With the definitions formulated in Eqs. (7.30) and (7.31), we can express the Fourier transformed stability problem in the form

$$\left(\frac{d^2}{d\,z^2} - k^2\right) f + R\,k^2\,h = 0 \; ,$$

$$\left(\lambda - \frac{d^2}{d\,z^2} + k^2\right) h - f = 0 \; , \tag{7.75}$$

with the boundary conditions

$$z = 0, 1: \qquad f = 0 = h \; . \tag{7.76}$$

We can combine the two equations into a single (fourth-order) ordinary differential equation,

$$\left(\frac{d^2}{d\,z^2} - k^2\right)\left(\lambda - \frac{d^2}{d\,z^2} + k^2\right) h + R\,k^2\,h = 0 \; ,$$

$$z = 0, 1: \qquad h = 0 \; , \quad \frac{d^2 h}{d\,z^2} = 0 \; . \tag{7.77}$$

A solution of this differential problem is easily found, namely

$$h(z) = \sin(n\,\pi\,z), \quad n = 1, 2, 3, \; \ldots \; , \tag{7.78}$$

provided that

$$(n^2\pi^2 + k^2)\left(\lambda + n^2\pi^2 + k^2\right) - R\,k^2 = 0 \; . \tag{7.79}$$

This additional algebraic equation is the dispersion relation of stability. We recall that the complex parameter λ can be expressed in terms of its real part η and its imaginary part $-\,\omega$, that is $\lambda = \eta - i\,\omega$. Thus, the imaginary part of the right-hand side of Eq. (7.79) vanishes if and only if

$$\omega\left(n^2\pi^2 + k^2\right) = 0 \; . \tag{7.80}$$

This equation can be satisfied if and only if $\omega = 0$, meaning that only normal modes with zero frequency are allowed (principle of exchange of stabilities).

7.6.3 Neutral Stability

Since $\omega = 0$, the real part of the dispersion relation given by Eq. (7.79) yields

$$R = \frac{(n^2\pi^2 + k^2)\left(\eta + n^2\pi^2 + k^2\right)}{k^2} \ . \tag{7.81}$$

Stability means that, for all $n = 1, 2, 3, \ldots$), one has a negative growth rate, $\eta < 0$. In other words, one may conclude that

$$R < \frac{\left(\pi^2 + k^2\right)^2}{k^2} \ ,$$

implies stability, while

$$R = \frac{\left(\pi^2 + k^2\right)^2}{k^2} \ ,$$

yields neutral stability, and

$$R > \frac{\left(\pi^2 + k^2\right)^2}{k^2} \ ,$$

defines convective instability. The neutral stability curve, namely the plot of function

$$R(k) = \frac{\left(\pi^2 + k^2\right)^2}{k^2} \tag{7.82}$$

is displayed in Fig. 7.7. The minimum of this curve yields the conditions for the onset of convection cells in the porous layer,

$$k_c = \pi \approx 3.14159 \ , \qquad R_c = 4\,\pi^2 \approx 39.4784 \ . \tag{7.83}$$

It has been shown that the critical value of the Rayleigh number for the onset of convective cells in Rayleigh–Bénard convection is given by either Eq. (7.42), or Eq. (7.49), or Eq. (7.50), depending on the prescribed velocity boundary conditions. If one compares these results with Eq. (7.83), the first-glance conclusion is that it is easier to have convective instabilities in a Darcy porous medium than in a clear fluid. However, this is false as the Rayleigh number Ra is proportional to L^3, while the Darcy–Rayleigh number R is proportional to $K L$. Since the permeability K is usually very small [9], it is much more common having a clear fluid layer with $Ra \sim 10^3$ than a fluid-saturated porous layer with $R \sim 10$.

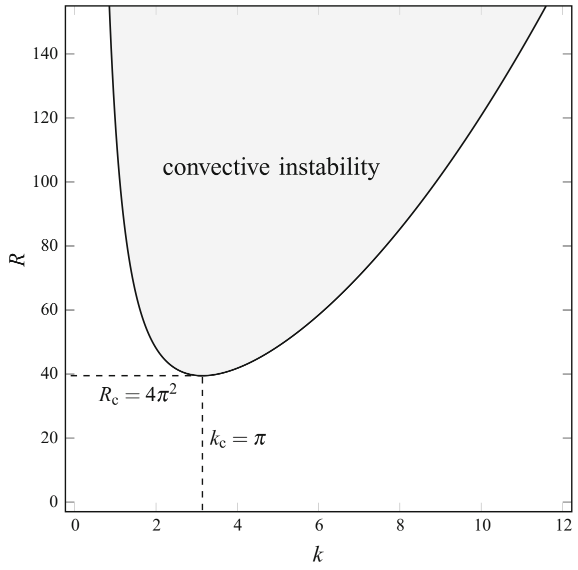

Fig. 7.7 Neutral stability curve for the Horton–Rogers–Lapwood problem with impermeable and isothermal boundary conditions at $z = 0, 1$

7.6.4 Form-Drag Effect

If one assumes Darcy–Forchheimer's model for local momentum balance, instead of Darcy's model, the set of governing equations is changed to

$$\frac{\partial u_j}{\partial x_j} = 0 \,,$$

$$\frac{\nu}{K}\left(1 + \frac{F\sqrt{K}}{\nu}\sqrt{u_\ell u_\ell}\right) u_i = -\beta\,(T - T_0)\,g_i - \frac{1}{\rho_0}\,\frac{\partial P}{\partial x_i} \,,$$

$$\sigma\,\frac{\partial T}{\partial t} + u_j\,\frac{\partial T}{\partial x_j} = \alpha\,\nabla^2 T + \frac{\nu}{K\,c}\left(1 + \frac{F\sqrt{K}}{\nu}\sqrt{u_\ell u_\ell}\right) u_j u_j \,. \tag{7.84}$$

The terms proportional to the form-drag coefficient F do not affect the linearised perturbation equations when the basic state is a rest state, namely when $u_{bi} = 0$. In fact, the terms proportional to F yield contributions of order ε^2 or ε^3 when the rest state is perturbed. In this case, both the basic solution

$$u_{bi} = 0 \,, \qquad T_b = T_1 - (T_1 - T_2)\,\frac{z}{L} \,. \tag{7.85}$$

and the linearised perturbation equations

$$\frac{\partial U_j}{\partial x_j} = 0 \; ,$$

$$\frac{\nu}{K} \, U_i = -\beta \, \Theta \, g_i - \frac{1}{\rho_0} \, \frac{\partial \Pi}{\partial x_i} \; ,$$

$$\sigma \, \frac{\partial \Theta}{\partial t} - W \, \frac{T_1 - T_2}{L} = \alpha \, \nabla^2 \Theta \; . \tag{7.86}$$

are exactly the same as those obtained by employing Darcy's law. Thus, the condition for the onset of convection cells is not affected by the form-drag effect.

7.6.5 Brinkman's Model

Changing the local momentum balance equation from Darcy's model to Brinkman's model and, thus, allowing for a Laplacian term contribution, as well as for no-slip conditions at the boundaries, sensibly affects the linear stability analysis.

A new parameter appears in the dimensionless perturbation equations, the *Darcy number*, namely

$$Da = \frac{\mu_{\text{eff}} \, K}{\mu \, L^2} \; , \tag{7.87}$$

where μ_{eff} is the effective viscosity. When $Da \to 0$, the results obtained by employing Darcy's law are recovered. On the other hand, we recover the results obtained for a Navier–Stokes fluid when $Da \to \infty$. Darcy's limit is $Da \to 0$ since the Darcy's law behaviour happens when the permeability is much smaller than the macroscopic scale of the porous medium, namely $K \ll L^2$. By the same reasoning, we can state that the clear fluid limit is approached when the porous medium has an extremely large permeability, so that $K \gg L^2$. In describing the transition from Darcy's flow to clear fluid flow, the Darcy number Da plays a key role. In general, the critical values (k_c, R_c) depend on Da.

In the case of Brinkman's model, the local mass, momentum and energy balance equations admit the same basic solution as with the other models, namely

$$u_{bi} = 0 \; , \qquad T_b = T_1 - (T_1 - T_2) \, \frac{z}{L} \; . \tag{7.88}$$

Therefore, the linearised local balance equations for the perturbation fields can be written as

$$\frac{\partial U_j}{\partial x_j} = 0 \; ,$$

$$\frac{\nu}{K} \, U_i - \nu_{\text{eff}} \, \nabla^2 U_i = -\beta \, \Theta \, g_i - \frac{1}{\rho_0} \, \frac{\partial \Pi}{\partial x_i} \; ,$$

$$\sigma \frac{\partial \Theta}{\partial t} - W \frac{T_1 - T_2}{L} = \alpha \nabla^2 \Theta , \tag{7.89}$$

where $\nu_{\text{eff}} = \mu_{\text{eff}}/\rho_0$. It is worth noting that the local energy balance equation for the perturbations given by the third Eq. (7.89), as expected, does not contain any contribution from the viscous dissipation effect, as such term is of higher order in the perturbation parameter ε and, hence, it is neglected in the linear approximation. This feature arises despite the uncertain form of the viscous dissipation function, either if it is given by Eq. (6.29) or by Eq. (6.30).

We introduce the same scaling defined by Eq. (7.66) in order to rewrite Eqs. (7.89) in a dimensionless form,

$$\frac{\partial U_j}{\partial x_j} = 0 ,$$

$$U_i - Da\, \nabla^2 U_i = R\, \Theta\, \delta_{i3} - \frac{\partial \Pi}{\partial x_i} ,$$

$$\frac{\partial \Theta}{\partial t} - W = \nabla^2 \Theta . \tag{7.90}$$

By evaluating the curl of the momentum balance equation, we obtain

$$\left(1 - Da\, \nabla^2\right) \left(\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z}\right) = R\, \frac{\partial \Theta}{\partial x} , \tag{7.91}$$

$$\left(1 - Da\, \nabla^2\right) \left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}\right) = R\, \frac{\partial \Theta}{\partial y} . \tag{7.92}$$

We now derive Eq. (7.91) with respect to x, and Eq. (7.92) with respect to y. Then, we sum the two resulting equations, so that we obtain

$$\left(1 - Da\, \nabla^2\right) \left[\nabla^2 W - \frac{\partial}{\partial z} \left(\frac{\partial U_j}{\partial x_j}\right)\right] = R\, \nabla_2^2 \Theta , \tag{7.93}$$

and, by employing the local mass balance equation, $\partial U_j / \partial x_j = 0$, we can write

$$\left(1 - Da\, \nabla^2\right) \nabla^2 W = R\, \nabla_2^2 \Theta . \tag{7.94}$$

As for the Rayleigh–Bénard problem, the boundary conditions can be expressed so that both boundary walls are isothermal, impermeable and stress-free, namely

$$z = 0, 1 : \qquad W = 0 = \Theta , \quad \frac{\partial^2 W}{\partial z^2} = 0 . \tag{7.95}$$

Thus, the stability problem is formulated in terms of the scalar fields W and Θ.

$$\left(1 - Da\,\nabla^2\right)\nabla^2 W = R\,\nabla_2^2 \Theta \;,$$

$$\frac{\partial \Theta}{\partial t} - W = \nabla^2 \Theta \;,$$

$$z = 0, 1: \qquad W = 0 = \Theta \;, \quad \frac{\partial^2 W}{\partial z^2} = 0 \;. \tag{7.96}$$

The Fourier transform is employed to determine the solution of Eqs. (7.96). Accordingly, we use the definitions given by Eqs. (7.30) and (7.31), so that Eqs. (7.96) yield

$$\left(1 - Da\,\frac{d^2}{d\,z^2} + Da\,k^2\right)\left(\frac{d^2}{d\,z^2} - k^2\right) f + R\,k^2\,h = 0 \;,$$

$$\left(\lambda - \frac{d^2}{d\,z^2} + k^2\right) h - f = 0 \;,$$

$$z = 0, 1: \qquad f = 0 \;, \quad \frac{d^2 f}{d\,z^2} = 0 \;, \quad h = 0 \;. \tag{7.97}$$

The solution of Eqs. (7.97) can be sought in the form

$$h(z) = \sin(n\,\pi\,z) \;, \quad n = 1, 2, 3, \ldots \;, \tag{7.98}$$

provided that the dispersion relation,

$$\left(1 + Da\,n^2\,\pi^2 + Da\,k^2\right)\left(n^2\,\pi^2 + k^2\right)\left(\lambda + n^2\,\pi^2 + k^2\right) - R\,k^2 = 0 \;, \tag{7.99}$$

holds. Since $\lambda = \eta - \mathrm{i}\,\omega$, the imaginary part of Eq. (7.99) yields

$$\omega\left(1 + Da\,n^2\,\pi^2 + Da\,k^2\right)\left(n^2\,\pi^2 + k^2\right) = 0 \;. \tag{7.100}$$

This means that the principle of exchange of stabilities is valid or, equivalently, that only non-travelling normal modes are allowed, i.e. those with $\omega = 0$. The real part of Eq. (7.99) yields

$$R = \frac{\left(1 + Da\,n^2\,\pi^2 + Da\,k^2\right)\left(n^2\,\pi^2 + k^2\right)\left(\eta + n^2\,\pi^2 + k^2\right)}{k^2} \;. \tag{7.101}$$

Instability is activated first by the $n = 1$ normal modes. Then, neutral stability happens with

$$R = \frac{\left(1 + Da\,\pi^2 + Da\,k^2\right)\left(\pi^2 + k^2\right)^2}{k^2} \;, \tag{7.102}$$

convective instability ($\eta > 0$) occurs with

$$R > \frac{\left(1 + Da\,\pi^2 + Da\,k^2\right)\left(\pi^2 + k^2\right)^2}{k^2}\,, \tag{7.103}$$

and stability is confined in the parametric region where

$$R < \frac{\left(1 + Da\,\pi^2 + Da\,k^2\right)\left(\pi^2 + k^2\right)^2}{k^2}\,. \tag{7.104}$$

The neutral stability condition, Eq. (7.102), suggests that the neutrally stable value of R for a given k increases with Da. The limit $Da \to 0$ yields a perfect agreement between Eqs. (7.102) and (7.82). In fact, in the limit $Da \to 0$, Brinkman's law reduces to Darcy's law. In order to recover the case of a clear fluid, whose neutral stability condition is expressed through Eq. (7.41), we must take the limit $Da \to \infty$. This limit can be taken consistently by employing the Rayleigh number,

$$Ra = \frac{R}{Da} = \frac{g\beta(T_1 - T_2)L^3}{\nu_{\mathrm{eff}}\,\alpha}\,, \tag{7.105}$$

instead of the Darcy–Rayleigh number. Here, Eqs. (7.69) and (7.87) have been employed. We note that there is a slight difference between the definitions of Ra given by Eqs. (7.1) and (7.105). The difference is in the denominator of Eq. (7.105) where ν_{eff} appears instead of the fluid kinematic viscosity ν. Such a discrepancy has no effect when the limit of a clear fluid is approached, i.e. the limit where the porosity tends to one, $\varphi \to 1$. In this limit, ν_{eff} and ν tend to coincide. This circumstance is evident by employing the definition $\nu_{\mathrm{eff}} = \mu_{\mathrm{eff}}/\rho_0$ and Eq. (6.10). The neutral stability condition given by Eq. (7.102) can be reformulated in terms of Ra as

$$Ra = \left(\frac{1}{Da} + \pi^2 + k^2\right)\frac{\left(\pi^2 + k^2\right)^2}{k^2}\,. \tag{7.106}$$

Evidently, Eq. (7.106) agrees with Eq. (7.41) when $Da \to \infty$. Plots of the neutral stability curves are displayed in Figs. 7.8 and 7.9, in the (k, R) plane or in the (k, Ra) plane, for different values of Da.

We have already mentioned that the critical values of k, R and Ra depend on the Darcy number. The evaluation of the minimum for the neutral stability functions $R(k)$ and $Ra(k)$ yields

$$k_{\mathrm{c}} = \frac{1}{2}\sqrt{\frac{\sqrt{\left(Da\,\pi^2 + 1\right)\left(9\,Da\,\pi^2 + 1\right)} - Da\,\pi^2 - 1}{Da}}\,,$$

$$R_{\mathrm{c}} = \frac{27\,Da^2\,\pi^4 + 18\,Da\,\pi^2 - 1}{8\,Da} + \frac{\left(9\,Da\,\pi^2 + 1\right)^{3/2}}{8\,Da}\,\sqrt{Da\,\pi^2 + 1}\,,$$

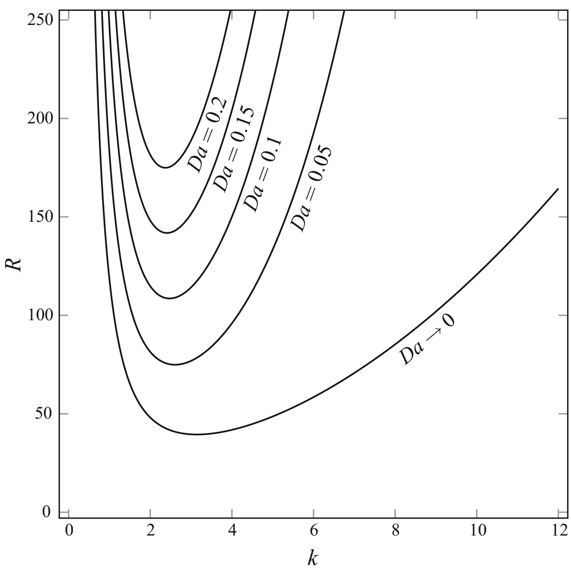

Fig. 7.8 Neutral stability curves $R(k)$ for the Rayleigh–Bénard problem in a porous layer, according to Brinkman's model, with impermeable and stress-free boundary conditions at $z = 0, 1$

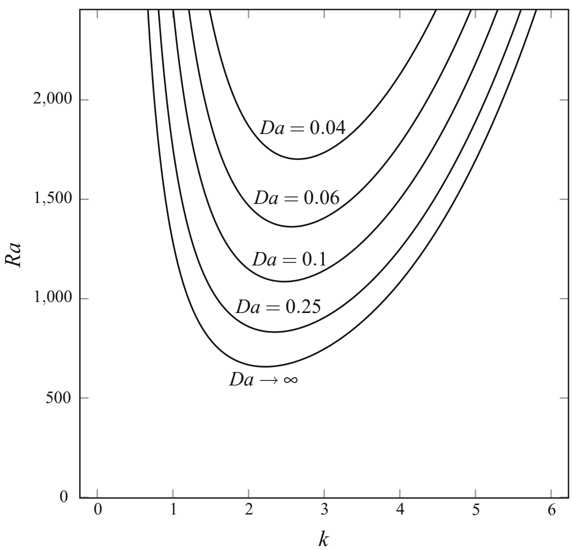

Fig. 7.9 Neutral stability curves $Ra(k)$ for the Rayleigh–Bénard problem in a porous layer, according to Brinkman's model, with impermeable and stress-free boundary conditions at $z = 0, 1$

$$Ra_c = \frac{R_c}{Da} \,. \tag{7.107}$$

By taking the limit $Da \to 0$ of k_c and R_c, one obtains the results given by Eq. (7.83). On the other hand, the limit $Da \to \infty$ of k_c and Ra_c yields the results given by Eq. (7.42). The plots reported in Figs. 7.8 and 7.9 suggest that the use of R is suitable to describe cases close to Darcy's regime, where Da is very small. The Rayleigh

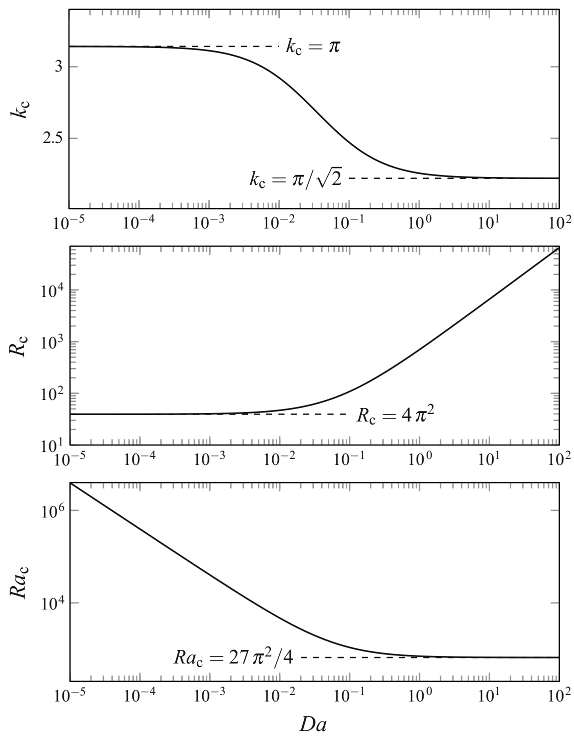

Fig. 7.10 Plots of k_c, R_c and Ra_c for the Rayleigh–Bénard problem in a porous layer, according to Brinkman's model, with impermeable and stress-free boundary conditions at $z = 0, 1$

number, Ra, is the suitable parameter when the flow takes place in a highly permeable medium, that is in a regime of large Darcy number, under conditions fairly close to those of a clear fluid. The behaviour of the critical values k_c, R_c and Ra_c versus Da is displayed in Fig. 7.10. These plots suggest that the critical values of k and Ra for a clear fluid are in fact almost attained when $Da \sim 1$. The Darcy's law regime, on the other hand, requires values of Da smaller than 10^{-3}.

7.7 A Porous Layer with Uniform Heat Flux Boundaries

Interesting variants of the Horton–Rogers–Lapwood problem come out when the thermal boundary conditions switch from isothermal to uniform heat flux. The mechanism of heating from below can be thermal contact, at the lower boundary, with an external thermal reservoir at a given temperature higher than that prescribed at the upper boundary. Alternatively, one can think to a given heat supply at the lower boundary provided through, say, an electric resistance. In this case, the boundary condition becomes one of uniform heat flux. Hence, we can devise a situation where, at $z = 0$, we have a uniform incoming heat flux q_0 and, at $z = L$, we have a uniform temperature T_0. In this case, we prescribe

$$z = 0 : \qquad -\varkappa_{\text{eff}} \, \frac{\partial T}{\partial z} = q_0 \; ,$$

$$z = L : \qquad T = T_2 \; . \tag{7.108}$$

For the sake of simplicity, we rely on Darcy's law. The basic state of the Horton–Rogers–Lapwood problem is slightly modified,

$$u_{\text{b}i} = 0 \; , \qquad T_{\text{b}} = T_2 + \frac{q_0 \, (L - z)}{\varkappa_{\text{eff}}} \; . \tag{7.109}$$

The differential equations for the perturbations of the basic state are still given by Eq. (7.65), provided that one defines

$$T_1 = T_2 + \frac{q_0 \, L}{\varkappa_{\text{eff}}} \; . \tag{7.110}$$

The dimensionless scaling of the governing equations can be carried out by employing Eq. (7.66), then Eqs. (7.67) are still valid, while Eq. (7.68) is replaced by

$$z = 0 : \qquad W = 0 = \frac{\partial \Theta}{\partial z} \; ,$$

$$z = 1 : \qquad W = 0 = \Theta \; . \tag{7.111}$$

We employ the Fourier transform method for the solution of Eqs. (7.67) and (7.111). Hence, we use Eqs. (7.30) and (7.31) to obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\, z^2} - k^2 \right) f + R \, k^2 \, h = 0 \; ,$$

$$\left(\lambda - \frac{\mathrm{d}^2}{\mathrm{d}\, z^2} + k^2 \right) h - f = 0 \; ,$$

$$z = 0 : \qquad f = 0 = \frac{\mathrm{d}\, h}{\mathrm{d}\, z} \; ,$$

$$z = 1 : \qquad f = 0 = h \; . \tag{7.112}$$

The difference with respect to the corresponding formulation of the Horton–Rogers–Lapwood problem, Eqs. (7.75) and (7.76), is just in the boundary condition at $z = 0$. This change makes a significant difference with respect to the complexity of the mathematical solution.

7.7.1 The Principle of Exchange of Stabilities

The boundary conditions in Eq. (7.112) allow one to write the following formulas of integration by parts:

$$\int_0^1 \bar{f}\,\frac{d^2 f}{d z^2}\,dz = -\int_0^1 \left|\frac{d f}{d z}\right|^2 dz\,, \qquad \int_0^1 \frac{d^2 \bar{h}}{d z^2}\,h\,dz = -\int_0^1 \left|\frac{d h}{d z}\right|^2 dz\,. \tag{7.113}$$

On multiplying by $\bar{f}$ the first Eq. (7.112), and multiplying by h the complex conjugate of the second Eq. (7.112), we obtain

$$\int_0^1 \left|\frac{d f}{d z}\right|^2 dz + k^2 \int_0^1 |f|^2\,dz$$
$$- R\,k^2 \left[\int_0^1 \left|\frac{d h}{d z}\right|^2 dz + \left(\bar{\lambda} + k^2\right)\int_0^1 |h|^2\,dz\right] = 0\,. \tag{7.114}$$

We recall that $\lambda = \eta - i\,\omega$, so that the imaginary part of Eq. (7.114) reads

$$\omega\,R\,k^2 \int_0^1 |h|^2\,dz = 0\,. \tag{7.115}$$

The integral on the left-hand side of Eq. (7.115) is positive, unless the perturbation is identically zero, i.e. $h = 0$. If h is identically zero, then Eq. (7.114) implies that also f is identically zero. Hence, we conclude that Eq. (7.115) can be satisfied by perturbations not identically zero, if $\omega = 0$. This means that the principle of exchange of stabilities holds, i.e., only Fourier modes with a zero phase velocity, $\omega/k = 0$, are allowed.

7.7.2 Solution of the Instability Eigenvalue Problem

The first and the second Eq. (7.112) can be rewritten as a single fourth-order equation in h. In the following, we will find the solution relative to the condition of neutral stability, so that we set $\lambda = \eta = 0$. Then, we can formulate a differential problem equivalent to Eq. (7.112), namely

$$\left(\frac{d^2}{d z^2} - k^2\right)^2 h - R\,k^2\,h = 0\,,$$

$$z = 0 : \qquad \frac{\mathrm{d}\,h}{\mathrm{d}\,z} = 0\,, \quad \frac{\mathrm{d}^2 h}{\mathrm{d}\,z^2} = k^2\,h\,,$$

$$z = 1 : \qquad h = 0\,, \quad \frac{\mathrm{d}^2 h}{\mathrm{d}\,z^2} = 0\,. \tag{7.116}$$

The characteristic equation associated with the differential equation (7.116) is given by

$$\left(s^2 - k^2\right)^2 - R\,k^2 = 0\,. \tag{7.117}$$

Its solutions are $s = \pm\,\chi_1$ and $s = \pm\,\mathrm{i}\,\chi_2$ where

$$\chi_1 = \sqrt{k\left(\sqrt{R} + k\right)}\,, \qquad \chi_2 = \sqrt{k\left(\sqrt{R} - k\right)}\,. \tag{7.118}$$

Hence, $h(z)$ can be written as

$$h(z) = C_1\,\mathrm{e}^{\chi_1 z} + C_2\,\mathrm{e}^{-\chi_1 z} + C_3\,\mathrm{e}^{\mathrm{i}\,\chi_2 z} + C_2\,\mathrm{e}^{-\mathrm{i}\,\chi_2 z}\,. \tag{7.119}$$

The coefficients C_1, C_2, C_3 and C_4 have to be chosen so that the four boundary conditions given by Eq. (7.116) are satisfied. This means that we can write the algebraic equation

$$\begin{pmatrix} \chi_1 & -\chi_1 & \mathrm{i}\,\chi_2 & -\mathrm{i}\,\chi_2 \\ 1 & 1 & -1 & -1 \\ \mathrm{e}^{\chi_1} & \mathrm{e}^{-\chi_1} & \mathrm{e}^{\mathrm{i}\,\chi_2} & \mathrm{e}^{-\mathrm{i}\,\chi_2} \\ \chi_1^2\,\mathrm{e}^{\chi_1} & \chi_1^2\,\mathrm{e}^{-\chi_1} & -\chi_2^2\,\mathrm{e}^{\mathrm{i}\,\chi_2} & -\chi_2^2\,\mathrm{e}^{-\mathrm{i}\,\chi_2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = 0\,. \tag{7.120}$$

Equation (7.120) admits only the trivial solution, where all coefficients C_1, C_2, C_3 and C_4 are zero, unless the determinant of the 4×4 matrix is zero. The coefficients cannot be identically zero, because this would imply a vanishing perturbation of the basic state. Then, the condition of zero determinant must hold, namely

$$\chi_2\,\sinh\chi_1\,\cos\chi_2 + \chi_1\,\cosh\chi_1\,\sin\chi_2 = 0\,. \tag{7.121}$$

Equation (7.121), together with Eq. (7.118), yields the dispersion relation at neutral stability in an implicit form where R cannot be explicitly expressed as a function of k. Although analytical, the expression of the dispersion relation given by Eq. (7.121) must be handled with care. In fact, Eq. (7.121) is just a condition of zero determinant and, as such, it may contain spurious solutions. An evident one is $\chi_2 = 0$ or, equivalently, $k = \sqrt{R}$. This solution must be excluded as writing Eq. (7.119) implies that the four solutions $\mathrm{e}^{\pm\chi_1 z}$, $\mathrm{e}^{\pm\mathrm{i}\,\chi_2 z}$ are assumed to be independent. This is untrue if $\chi_2 = 0$. In fact, one may easily check that Eq. (7.116) does not admit any nonzero solution h whenever $k = \sqrt{R}$.

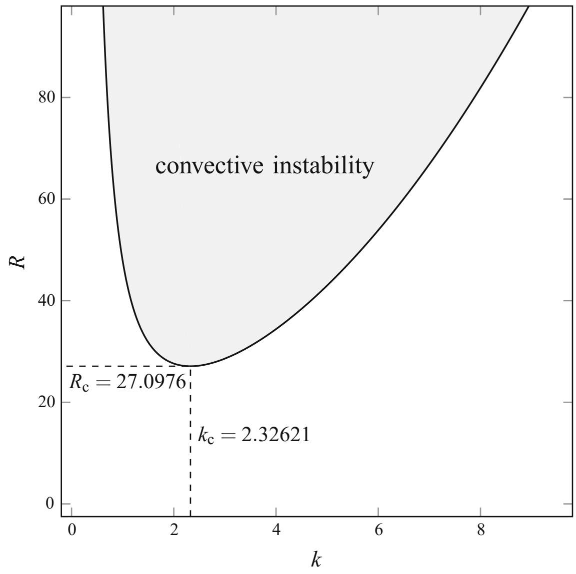

Fig. 7.11 Neutral stability curve for the Horton–Rogers–Lapwood problem with impermeable boundaries, uniform heat flux at $z = 0$ and uniform temperature at $z = 1$

An alternative to using the implicit dispersion relation (7.121) is adopting the numerical method described in Chap. 10 for the solution of Eq. (7.112) with $\lambda = 0$. Either way, one can gather the numerical data needed to draw the neutral stability curve in the (k, R) plane. A plot of the neutral stability curve and of the convective instability region is displayed in Fig. 7.11. This figure shows that the point of minimum R along the curve, namely the critical condition, is identified by

$$k_c = 2.32621\ , \qquad R_c = 27.0976\ . \tag{7.122}$$

This result was first pointed out in the paper by Lapwood [7].

7.7.3 Porous Layer with Uniform Heat Flux at Both Boundaries

The boundary condition of uniform heat flux can be prescribed both at the lower boundary and at the upper boundary. A situation can be imagined where all the heat supplied to the lower boundary is removed from the upper boundary, so that a steady condition can be allowed. Under such conditions, Eq. (7.108) is replaced by

$$z = 0,\ L: \qquad -\varkappa_{\text{eff}}\, \frac{\partial T}{\partial z} = q_0\ . \tag{7.123}$$

The basic state considered in Eq. (7.109) satisfies Eq. (7.123). The only important remark is that the constant temperature T_2 is now undefined or, stated differently, its value can be fixed arbitrarily. The reason is that, in a rest state, the temperature field is determined as a solution of the local energy balance equation which contains only derivatives of T. If the temperature boundary conditions are those given by Eq. (7.123), then one can conclude that T can be determined only up to an arbitrary additive constant.

The change needed in the eigenvalue problem expressed by Eq. (7.112) is just in the boundary conditions. Hence, we can write

$$\left(\frac{d^2}{d z^2} - k^2\right) f + R\, k^2\, h = 0\,,$$

$$\left(\lambda - \frac{d^2}{d z^2} + k^2\right) h - f = 0\,,$$

$$z = 0, 1: \qquad f = 0 = \frac{d h}{d z}\,. \tag{7.124}$$

One can easily check that the principle of exchange of stabilities holds. In fact, the integration by parts formulas reported in Eq. (7.113) are still valid, as a consequence of the boundary conditions specified in Eq. (7.124). Then, the same discussion and conclusions reached in Sect. 7.7.1 can be drawn.

The solution of Eq. (7.124) for the neutrally stable modes, with $\lambda = 0$, can be found analytically through the same procedure described in Sect. 7.7.2. Equation (7.116) now reads

$$\left(\frac{d^2}{d z^2} - k^2\right)^2 h - R\, k^2\, h = 0\,,$$

$$z = 0, 1: \qquad \frac{d h}{d z} = 0\,, \quad \frac{d^2 h}{d z^2} = k^2\, h\,. \tag{7.125}$$

No change is needed in Eqs. (7.118) and (7.119), as they rely only on the ordinary differential equation. On the other hand, Eq. (7.120) is replaced by

$$\begin{pmatrix} \chi_1 & -\chi_1 & i\,\chi_2 & -i\,\chi_2 \\ 1 & 1 & -1 & -1 \\ \chi_1\, e^{\chi_1} & -\chi_1\, e^{-\chi_1} & i\,\chi_2\, e^{i\,\chi_2} & -i\,\chi_2\, e^{-i\,\chi_2} \\ e^{\chi_1} & e^{-\chi_1} & -e^{i\,\chi_2} & -e^{-i\,\chi_2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = 0\,. \tag{7.126}$$

The condition of zero determinant for the 4×4 matrix yields the dispersion relation,

$$k^2\, \sinh\chi_1\, \sin\chi_2 + \chi_1\, \chi_2\, (\cosh\chi_1\, \cos\chi_2 - 1) = 0\,. \tag{7.127}$$

Through a numerical algorithm for root finding, Eqs. (7.118) and (7.127) can be employed to gather the numerical data needed to draw the neutral stability curve, viz. the lower bound to the convective instability region in the (k, R) plane. Again, the alternative is carrying out a fully numerical solution of the system of ordinary differential stability problem through the shooting method, along the lines discussed in Chap. 10.

The shape of the neutral stability curve is quite dissimilar to that illustrated in Fig. 7.11, relative to the hybrid case where the lower boundary is subject to a uniform heat flux and the upper boundary is kept isothermal. The dissimilarity can be easily revealed by looking for an asymptotic solution in the limit $k \to 0$. By relying on the inverse proportionality between wave number and wavelength, we can define this limit as one of large wavelengths. We express $h(z)$ and R as power series with respect to the small parameter k^2,

$$h(z) = \sum_{n=0}^{\infty} h_n(z)\, k^{2n} \;, \qquad R = \sum_{n=0}^{\infty} R_n\, k^{2n} \;. \tag{7.128}$$

This is perfectly legitimate as the wave number appears in Eq. (7.125) only through its square, k^2. By substituting Eq. (7.128) into (7.125) and collecting like powers of k^2, we obtain the zeroth-order boundary value problem, namely

$$\frac{d^4 h_0}{d z^4} = 0 \;,$$
$$z = 0, 1: \qquad \frac{d h_0}{d z} = 0 \;, \quad \frac{d^2 h_0}{d z^2} = 0 \;. \tag{7.129}$$

The solution of Eq. (7.129) is

$$h_0(z) = A \;, \tag{7.130}$$

where A is an arbitrary constant. To first order in k^2, we obtain the boundary value problem

$$\frac{d^4 h_1}{d z^4} - R_0\, A = 0 \;,$$
$$z = 0, 1: \qquad \frac{d h_1}{d z} = 0 \;, \quad \frac{d^2 h_1}{d z^2} = A \;. \tag{7.131}$$

Provided that

$$A\,(R_0 - 12) = 0 \;, \tag{7.132}$$

Equation (7.131) yields the solution

$$h_1(z) = B + \frac{A}{2}\, z^2 - A\, z^3 + \frac{A}{2}\, z^4 \;, \tag{7.133}$$

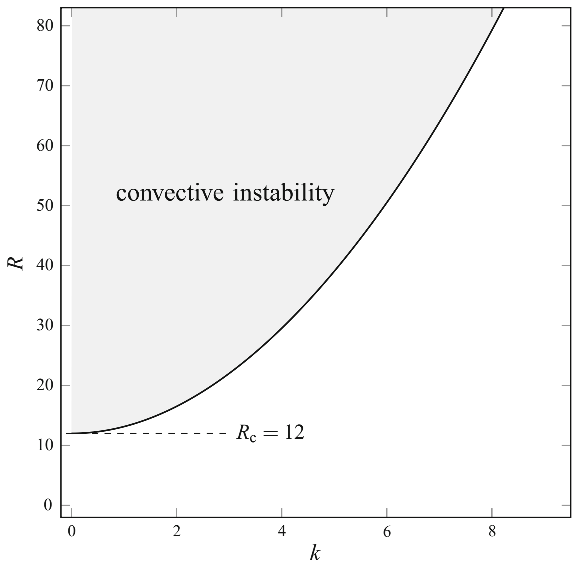

Fig. 7.12 Neutral stability curve for the Horton–Rogers–Lapwood problem with impermeable boundaries having uniform heat flux at $z = 0, 1$

where B is another arbitrary constant. If we assume $A \neq 0$, whatever is its value, we obtain

$$R_0 = 12 \ . \tag{7.134}$$

Equation (7.134), together with Eq. (7.128), leads to the conclusion that the neutral stability function $R(k)$ is not singular when $k \to 0$, as it happens in the cases illustrated in Figs. 7.7 and 7.11. On the other hand, it approaches the constant value 12 when $k \to 0$. Starting from this limiting value, the neutral stability function $R(k)$ is monotonic increasing, as shown in Fig. 7.12. This means that the critical values of k and R for the onset of convective instability are

$$k_c = 0 \ , \qquad R_c = 12 \ . \tag{7.135}$$

We have found that, on replacing the isothermal condition at the lower boundary with a uniform heat flux condition, the critical value of R decreases from $4\,\pi^2 \approx 39.4784$ to 27.0976. If also the upper boundary is subject to a uniform heat flux, then R_c further decreases to 12. Hence, we conclude that boundaries at uniform heat flux yield a destabilisation of the basic state with respect to isothermal boundaries.

7.8 A Note on the Shape of Convection Cells

A visualisation of the convection cells can be easily obtained when a single normal mode with a given wave vector (k_x, k_y) is considered,

$$W = \frac{1}{2\pi}\, \tilde{W}\, e^{i(k_x\, x + k_y\, y)} \;, \qquad \Theta = \frac{1}{2\pi}\, \tilde{\Theta}\, e^{i(k_x\, x + k_y\, y)} \;, \tag{7.136}$$

where Eq. (7.30) is taken into account. We have already pointed out, in Sect. 7.2, that manifold shapes of convection cells may arise at the onset of convective instability. The simplest geometry of the convection cells is straight rolls, whose planforms are illustrated in frame (a) of Fig. 7.2. Without any loss of generality, we can choose such straight rolls as having axes perpendicular to the (x, z) plane. The mathematical representation of this case is a wave vector directed along the x-axis, i.e. a situation where $k_x = k$ and $k_y = 0$. In this case, we have no dependence on y, so that the local mass balance equation, that is the condition of zero divergence for the velocity field U_i, can be written as

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0 \,. \tag{7.137}$$

This equation is identically satisfied by defining a *streamfunction*, $\Psi(x, z, t)$, such that

$$U = \frac{\partial \Psi}{\partial z} \;, \qquad W = -\frac{\partial \Psi}{\partial x} \,. \tag{7.138}$$

Obviously, Ψ is defined only up to an arbitrary additive function of t. This function of time can be fixed in a convenient way. For instance, on the basis of Eq. (7.136) and of the assumptions $k_x = k$ and $k_y = 0$, one can define it so that

$$\Psi = \frac{1}{2\pi}\, \tilde{\Psi}\, e^{i\,k\,x} \,. \tag{7.139}$$

The isolines of Ψ are called the *streamlines*. The streamlines provide a natural description of the two-dimensional velocity field $(U, 0, W)$, as the tangent to the streamlines is the field $(U, 0, W)$ itself. This result is an immediate consequence of the definition given by Eq. (7.138). On account of Eqs. (7.136) and (7.139), we can find a simple equation linking the fields W and Ψ, or $\tilde{W}$ and $\tilde{\Psi}$, namely

$$W = -i\,k\,\Psi \;, \qquad \tilde{W} = -i\,k\,\tilde{\Psi} \,. \tag{7.140}$$

As a consequence of Eq. (7.140), we infer that the streamlines are, in fact, coincident with the isolines of W. In order to get a graphical representation of the streamlines, we must remember that the physically significant field is not W, which is complex-valued, but its real part.

In the Rayleigh–Bénard problem, and in all its variants considered in this chapter included the Horton–Rogers–Lapwood problem with either isothermal or isoflux boundary conditions, the principle of exchange of stabilities holds. In particular, this means that the fields $\tilde{W}$ and $\tilde{\Theta}$ are real-valued. Then, we can write

$$\Re(W) = \frac{1}{2\pi}\, \tilde{W}\, \cos(k\,x) \;, \qquad \Re(\Psi) = -\frac{1}{2\pi\, k}\, \tilde{W}\, \sin(k\,x) \,. \tag{7.141}$$

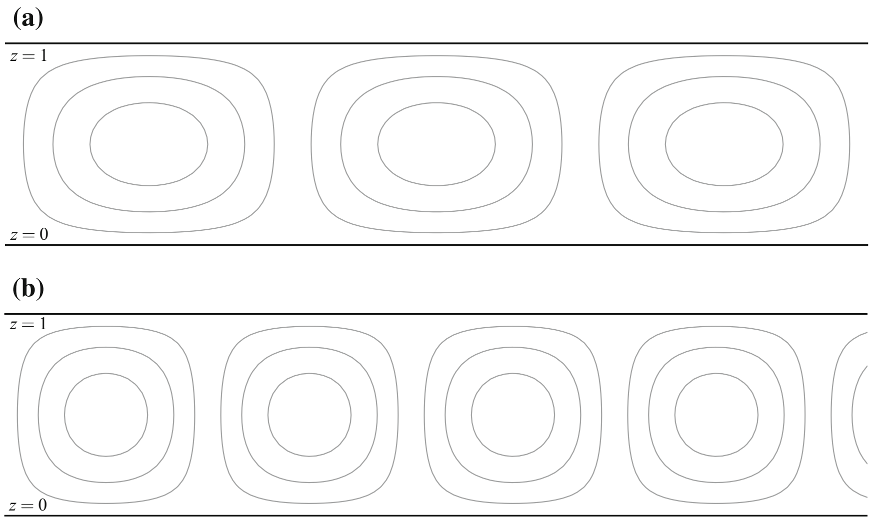

Fig. 7.13 Streamlines of the normal mode perturbation at the onset of convective instability for: **a** the Rayleigh–Bénard problem with stress-free and isothermal boundaries; **b** the Horton–Rogers–Lapwood problem with impermeable and isothermal boundaries

If we consider the Rayleigh–Bénard problem for a fluid layer bounded by stress-free and isothermal planes, or the Horton–Rogers–Lapwood problem for a saturated porous medium bounded by impermeable and isothermal planes, the expression of $\tilde{W}$ is such that

$$\tilde{W} = C \ \sin(n\,\pi\, z)\, \mathrm{e}^{\eta t} \ , \tag{7.142}$$

where C is a constant. Equation (7.142) can be deduced from Eqs. (7.31), (7.33) and (7.37).

A sensible case where one may wish to draw the streamlines of the perturbation normal mode is at the critical conditions for the onset of instability, namely $\eta = 0$, $n = 1$ and $k = k_c$. With these conditions, Eqs. (7.141) and (7.142) yield

$$\Re(\Psi) = -\frac{C}{2\,\pi\, k_c} \ \sin(\pi\, z) \ \sin(k_c\, x) \ . \tag{7.143}$$

We found that $k_c = \pi/\sqrt{2}$ for the Rayleigh–Bénard problem and $k_c = \pi$ for the Horton–Rogers–Lapwood problem, as reported in Eqs. (7.42) and (7.83), respectively. Thus, the streamlines can be easily represented in the (x, z) plane by employing equation (7.143) as the isolines $\Re(\Psi) = constant$.

Figure 7.13 shows the streamlines at the onset of convective instability, relative to a perturbation normal mode, with $k = k_c$ for either the Rayleigh–Bénard problem or the Horton–Rogers–Lapwood problem. On comparing frames (a) and (b) of Fig. 7.13, one may note the more stretched horizontal width of the Rayleigh–Bénard cells. The

Horton–Rogers–Lapwood cells have a characteristic square shape, due to the critical value $k_c = \pi$ implying the same periodicity of $\Re(\Psi)$ along the x and z directions.

If we move from those cases amenable to a fully analytical solution, then the function $f(z)$ adopted to express $\tilde{W}$ is not given by a simple sine function, but it is determined numerically. This does not change much in what we have said about plotting the streamlines of the normal mode perturbation at critical conditions, except that Eq. (7.143) is in fact replaced by

$$\Re(\Psi) = -\frac{1}{2\pi\, k_c}\, f(z)\, \sin(k_c\, x)\ . \tag{7.144}$$

Ultimately, the aspect ratio of the cells is determined uniquely by the value of k_c in each single case.

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