Chapter 6
Fluid Flow in Porous Media

6.1 The Basic Features of Flow in Porous Media

There are several excellent books regarding the dynamics of flow and convection in a fluid-saturated porous medium. We mention Bear [3], Kaviany [6], Nield and Bejan [17], Straughan [11] as examples of reference texts on these topics. This chapter is just an outline of the main mathematical models employed in the stability analysis of convection in porous media, with no ambition of being a complete or exhaustive presentation of the state of the art.

The oldest, the simplest and the most widely employed model of fluid flow in porous media is named after Darcy,[1] a French scientist with a strong professional interest in hydraulics. During his life, he was a civil engineer in the city of Dijon in France. He projected and built a pressurised water distribution system in Dijon. A few years before his death, he conducted the experiments that allowed him to formulate what today is well known as *Darcy's law*. His publication *The Public Fountains of the City of Dijon* contains an appendix written in 1856 entitled *Determination of the Laws of Water Flow Through Sand* where his law is formulated.

Fluid flow in porous media is of paramount importance both for geophysical applications such as filtration of water, hydrocarbons and gases in the soil and for engineering. For instance, one may point out the interest of porous media with reference to hydrology of aquifers, underground repositories used for sequestering nuclear waste, heat pipes, underground spreading of chemical waste, drainage and irrigation in agriculture, thermal insulation engineering, enhanced recovery of petroleum reservoirs, grain storage, water flow in geothermal reservoirs.

Before formulating Darcy's law, let us review the main features of the macroscopic description of fluid flow in porous media.

A porous medium is a solid material with void inner structures saturated by a fluid, liquid or gas. One can think to sand, pebbles, bread or to a metallic foam. One

[1] Henry Philibert Gaspard Darcy (1803–1858).

A. Barletta, *Routes to Absolute Instability in Porous Media*,
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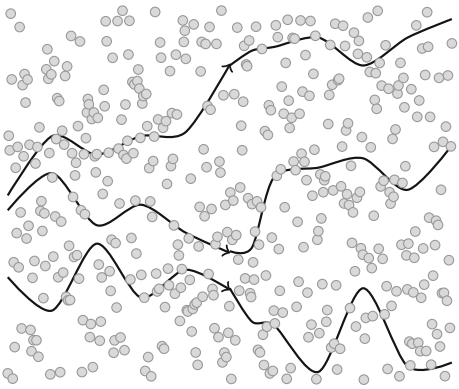

Fig. 6.1 Irregular trajectories of fluid elements moving inside a random cloud of solid particles

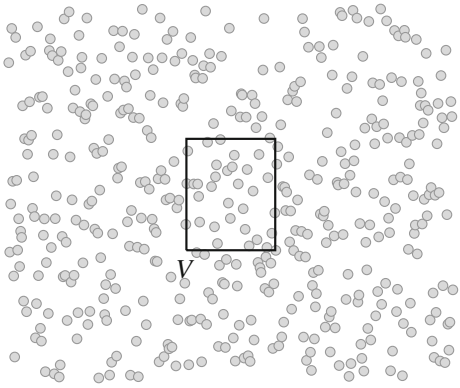

Fig. 6.2 Representative elementary volume V

can imagine that the void spaces within the solid are entirely filled by the moving fluid (see Fig. 6.1).

A basic quantity for the description of a porous medium is the ratio between the volume occupied by the fluid (voids) and the total volume including voids and solid. Referring to Fig. 6.2, one can consider a representative elementary volume V, often abbreviated as REV, small on a macroscopic scale even if large on the scale of the single grain, pebble or microchannel that may be present inside the porous medium. If V_f is the void part of V, we call *porosity*, φ, the ratio

$$\varphi = \frac{V_\mathrm{f}}{V} \,. \tag{6.1}$$

The porosity is a dimensionless quantity strictly smaller than unity.

The study of convection in porous media is based on the assumption that a fluid-saturated porous medium can be described as a continuum. This means that, in the representative volume V of the system, the number of pores is very high. Therefore, one can define a local fluid velocity field as an average value of the local fluid

velocity $\mathbf{u}^*$. There are two possible average values of $\mathbf{u}^*$ usually introduced: the *intrinsic velocity*, namely

$$\mathbf{U} = \frac{1}{V_f} \iiint_{V_f} \mathbf{u}^* \, \mathrm{d}V \ , \tag{6.2}$$

and the *seepage velocity* (also known as *Darcy's velocity*), namely

$$\mathbf{u} = \frac{1}{V} \iiint_{V} \mathbf{u}^* \, \mathrm{d}V \ . \tag{6.3}$$

The intrinsic velocity is defined as an average performed in the void part V_f of the representative volume V. Since $\mathbf{u}^* = 0$ in the part of V not included in V_f, the two integrals on the right-hand sides of Eqs. (6.2) and (6.3) are equal. Then, one can establish a very simple relationship between $\mathbf{U}$ and $\mathbf{u}$, namely

$$\mathbf{u} = \varphi \, \mathbf{U} \ . \tag{6.4}$$

This equation is well known as *Dupuit–Forchheimer relationship*.

The local value of the seepage velocity $\mathbf{u}$ depends on the shape and the size of the pores as well as on the causes that determine the fluid motion. The relationship between the local value of $\mathbf{u}$ and the forces acting on the fluid can be deduced by an appropriate local average over the representative elementary volume of the Navier–Stokes momentum balance, Eq. (5.59). However, due to the complexity of the system, in most cases this relationship is postulated through a constitutive equation validated experimentally.

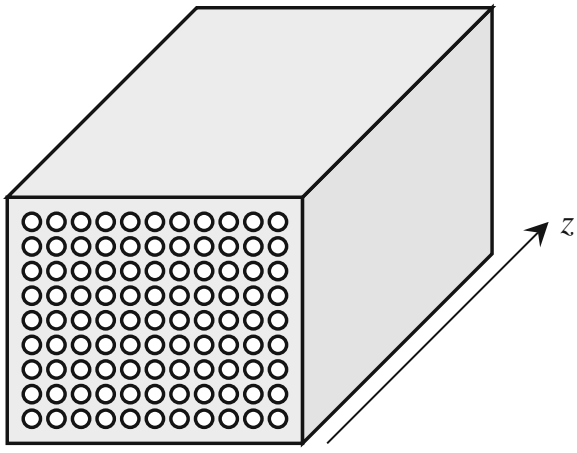

Fig. 6.3 Engineered porous medium described in Example 6.1

6.2 Local Momentum Balance in a Porous Medium

The simplest constitutive equation expressing the local seepage velocity of the fluid is *Darcy's law*, namely

$$\frac{\mu}{K}\,\mathbf{u} = -\,\nabla p + \mathbf{b}\,, \tag{6.5}$$

where K is a property of the system called *permeability*, μ is the dynamic viscosity of the fluid, p is the fluid pressure, and $\mathbf{b}$ is the external body force per unit volume applied to the fluid (in the simplest case, the gravitational body force $\rho\,\mathbf{g}$).

The rationale behind the assumption given by Eq. (6.5) relies on the observation that a porous medium can be thought of as a network of microscopic ducts where the fluid flows. In the absence of external body forces, the pressure gradient along a duct is proportional to the average fluid velocity in the duct itself, if the flow is laminar. On the other hand, if the flow is highly turbulent (hydraulic regime), the pressure gradient along a duct is proportional to the square of the average fluid velocity in the duct itself. Darcy's law refers to the case of laminar flow in the porous medium, so that the permeability K is considered as a property of the medium depending on the number of pores per unit area present in a cross-section transverse to the fluid flow, on the shape of the pores and on their size.

Example 6.1 Let us consider an engineered porous medium such that the pores form an ordered array of parallel infinitely long circular ducts each with a diameter d (Fig. 6.3). Let z be the axis parallel to the ducts, and let the number of ducts per unit area in a transverse section of the medium be N. Then, by comparison with the relationship between the average velocity u_m^* and pressure drop Δp in this kind of ducts, it is easily verified that

$$K = \frac{N\pi\, d^4}{128}\,.$$

In fact, it is well known that the average velocity u_{m}^* of fully developed laminar flow in a circular duct, namely *Hagen–Poiseuille flow*, is given by

$$u_{\mathrm{m}}^* = -\frac{d^2}{32\,\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z}\,.$$

One may notice that $u_{\mathrm{m}}^* = U$; i.e. u_{m}^* coincides with the intrinsic velocity U. Moreover, the porosity is given by $\varphi = N\,\pi\,d^2/4$. Then, on account of Dupuit–Forchheimer relationship, the Darcy velocity u is given by

$$u = \frac{N\pi\, d^2}{4}\,u_{\mathrm{m}}^* = -\frac{N\pi\, d^4}{128\,\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z}\,.$$

As a consequence of Eq. (6.5), one obtains the expression for K.

A variant version of Darcy's law is formulated by including an extra inertial term proportional to $\partial \mathbf{u}/\partial t$,

$$\rho\, C_{\mathrm{a}}\, \frac{\partial \mathbf{u}}{\partial t} + \frac{\mu}{K}\, \mathbf{u} = -\nabla p + \mathbf{b}\,, \tag{6.6}$$

where C_{a} is called *acceleration coefficient*. With a further generalisation, the dimensionless scalar coefficient C_{a} is replaced by a tensor coefficient, by taking into account the non-isotropic character of the inertial effect [11].

If the hypothesis of laminar fully developed flow in each pore cannot be applied, then proportionality between acting forces and the resulting fluid velocity must be relaxed in favour of a gradual transition towards a hydraulic regime where acting forces are proportional to the square of the fluid velocity in each pore. An extended form of Eq. (6.5) has been proposed which accounts for this effect, i.e. *Darcy–Forchheimer's model*,

$$\frac{\mu}{K}\left(1 + \frac{F\,\rho\,\sqrt{K}}{\mu}\,|\mathbf{u}|\right)\mathbf{u} = -\nabla p + \mathbf{b}\,. \tag{6.7}$$

In Eq. (6.7), $|\mathbf{u}|$ is the modulus of $\mathbf{u}$, ρ is the fluid mass density, and F is a property of the porous medium called the *form-drag coefficient*. It is easily verified that F is a dimensionless property of the porous medium.

Obviously, Darcy–Forchheimer's model includes Darcy's law as a special case, i.e. in the limit $F \to 0$. On the other hand, whenever $F\,\rho\,|\mathbf{u}|\sqrt{K}/\mu \gg 1$, transition to an hydraulic regime for the fluid flow inside the pores occurs. A widely accepted criterion to establish when Darcy's law must be abandoned in favour of Darcy–Forchheimer's model is formulated with the permeability-based Reynolds number,

$$Re_{\mathrm{K}} = \frac{\rho\,|\mathbf{u}|\,\sqrt{K}}{\mu}\,. \tag{6.8}$$

Darcy's law gradually loses its validity when $Re_K \sim 1/F$ or greater. From Eq. (6.8), a clever way to apply this criterion is taking $|\mathbf{u}|$ as the maximum value over the flow domain.

A common feature of Darcy's law and of Forchheimer's extension of this law is that they refer to a tightly packed solid with a fluid flowing in very small pores. Indeed, this is a circumstance very far from a free-flowing fluid. One can complete either Eq. (6.5) or (6.7) with just one velocity boundary condition on each boundary surface. This boundary condition can be, for instance, *impermeability* ($\mathbf{u}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit vector normal to the surface). However, one cannot allow also a no-slip condition on the same surface, as the problem would be over-specified. This is due to the low differential order of Darcy's and Darcy–Forchheimer's models of local momentum balance in a fluid-saturated porous medium.

The impossibility to prescribe no-slip conditions at the boundary walls creates a sharp distinction between the Navier–Stokes fluid model and the models of fluid-saturated porous media based either on Darcy's law or on Forchheimer's extension of this law. In some cases, a continuous transition from the momentum balance equation of a clear fluid (Navier–Stokes equation) to Darcy's law is considered as realistic. Along this direction, it has been proposed the so-called *Brinkman's model* for fluid flow in a porous medium. This model allows one to prescribe no-slip wall conditions as for a Navier–Stokes clear fluid. According to Brinkman's model, Eq. (6.5) must be replaced by

$$\frac{\mu}{K}\,\mathbf{u} - \mu_{\text{eff}}\,\nabla^2\mathbf{u} = -\,\boldsymbol{\nabla} p + \mathbf{b}\;, \tag{6.9}$$

where the quantity μ_{eff} is called *effective viscosity*: it depends on the fluid viscosity μ and on the porosity of the medium where the fluid flows. A commonly employed correlation for the effective viscosity is *Einstein's formula* for dilute suspensions, namely

$$\mu_{\text{eff}} = \mu\,[1 + 2.5\,(1-\varphi)]\;. \tag{6.10}$$

If the porosity is equal to 1, one has a clear fluid and Eq. (6.10) implies that $\mu_{\text{eff}} = \mu$. If $\varphi = 1$, Eq. (6.10) reduces to the Navier–Stokes equation without the inertial contribution (negligible acceleration), provided that the limit of infinite permeability is also taken ($K \to \infty$). On the other hand, in the limit of a very small permeability ($K \to 0$), the first term on the left-hand side of Eq. (6.10), $\mu\mathbf{u}/K$, becomes much larger than the second term, $\mu_{\text{eff}}\,\nabla^2\mathbf{u}$. Therefore, in the limit $K \to 0$, Brinkman's model reduces to Darcy's law, Eq. (6.5). It must be pointed out that the limit $K \to 0$ yields a singular behaviour next to the impermeable boundaries where the no-slip conditions cannot be applied anymore.

6.3 Local Mass and Energy Balance Equations

By employing a local volume averaging procedure, the mass and energy balance equations of a fluid-saturated porous medium can be expressed as

$$\varphi\,\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{u}) = 0\;, \tag{6.11}$$

and

$$\rho\,c\left(\sigma\,\frac{\partial T}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} T\right) = \varkappa_{\text{eff}}\,\nabla^2 T + q_{\text{g}} + \mu\,\Phi\;. \tag{6.12}$$

In Eq. (6.12), the thermal conductivities of the solid and of the fluid are considered as constant, σ is the *heat capacity ratio* defined as

$$\sigma = \frac{\varphi\,\rho\,c + (1-\varphi)\,\rho_{\mathrm{s}}\,c_{v_{\mathrm{s}}}}{\rho\,c}\ , \tag{6.13}$$

while $\varkappa_{\mathrm{eff}}$ is the *effective thermal conductivity* of the fluid-saturated porous medium defined as

$$\varkappa_{\mathrm{eff}} = \varphi\,\varkappa + (1-\varphi)\,\varkappa_{\mathrm{s}}\ . \tag{6.14}$$

In Eqs. (6.13) and (6.14), the properties ρ, c and $\varkappa$ refer to the fluid, while ρ_{s}, $c_{v_{\mathrm{s}}}$ and $\varkappa_{\mathrm{s}}$ refer to the solid matrix.

The term q_{g} in Eq. (6.12) is the power generated per unit volume in the fluid-saturated porous medium by, for instance, Joule heating or chemical reactions. The last term on the right-hand side of Eq. (6.12), $\mu\,\Phi$, is the power per unit volume generated by viscous dissipation.

Consistently with Eq. (6.11), if the seepage flow involves mass diffusion in the fluid, the concentration C of the diffusing species satisfies the partial differential equation

$$\varphi\,\frac{\partial C}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} C = \alpha_{\mathrm{m,eff}}\,\nabla^2 C + \varphi\,\dot{m}\ , \tag{6.15}$$

where $\alpha_{\mathrm{m,eff}} = \varphi\,\alpha_{\mathrm{m}}$ is the effective mass diffusivity of the fluid saturating the porous medium. Equation (6.15) is the porous medium version of Eq. (5.109), and it is relative to a binary mixture.

6.3.1 Local Thermal Non-equilibrium

In the *local thermal non-equilibrium model* (LTNE), the heat transfer across the fluid-saturated porous medium is described with two distinct temperature fields. Thus, one defines two different local temperatures, one for the solid and one for the fluid. There is an inter-phase heat transfer rate, modelled through a constant coefficient h multiplying the local temperature difference between the phases. Thus, two energy balance equations have to be written for the two phases [8, 11, 15],

$$(1-\varphi)\ \rho_{\mathrm{s}}\,c_{v_{\mathrm{s}}}\,\frac{\partial T_{\mathrm{s}}}{\partial t} = (1-\varphi)\,\varkappa_{\mathrm{s}}\,\nabla^2 T_{\mathrm{s}} + (1-\varphi)\,q_{g_{\mathrm{s}}} + h\,(T_{\mathrm{f}} - T_{\mathrm{s}})\ , \tag{6.16}$$

$$\varphi\,\rho\,c\,\frac{\partial T_{\mathrm{f}}}{\partial t} + \rho\,c\,\mathbf{u}\cdot\boldsymbol{\nabla} T_{\mathrm{f}} = \varphi\,\varkappa\,\nabla^2 T_{\mathrm{f}} + \mu\,\Phi + \varphi\,q_{g_{\mathrm{f}}} + h\,(T_{\mathrm{s}} - T_{\mathrm{f}})\ . \tag{6.17}$$

The inter-phase heat transfer coefficient h in Eqs. (6.16) and (6.17) describes the thermal energy exchange between the solid and the fluid phase. In Eqs. (6.16) and (6.17), two independent heat sources are envisaged in the solid and in the fluid, $q_{g_{\mathrm{s}}}$ and $q_{g_{\mathrm{f}}}$.

The lack of local thermal equilibrium between the solid and fluid phases is exploited by locally and instantaneously distinct values of the temperature. Such

situation actually arises when very fast transient processes occur. It is definitely favoured also under stationary conditions when there are markedly different values of the fluid and solid thermal conductivities, as well as for sufficiently low values of the inter-phase heat transfer coefficient h. The latter condition tends to favour an independent behaviour of the two temperature fields.

The two-temperature model based on Eqs. (6.16) and (6.17) actually leads to the local energy balance with local inter-phase equilibrium, Eq. (6.12), when the coefficient h tends to infinity. This feature is easily verified if one writes the sum of Eqs. (6.16) and (6.17), together with Eq. (6.16),

$$\begin{aligned}(1-\varphi)\ \rho_{\mathrm{s}}\, c_{\mathrm{v_s}} \frac{\partial T_{\mathrm{s}}}{\partial t} + \varphi\, \rho\, c\ \frac{\partial T_{\mathrm{f}}}{\partial t} + \rho\, c\, \mathbf{u}\cdot\boldsymbol{\nabla} T_{\mathrm{f}} \\ = (1-\varphi)\, \varkappa_{\mathrm{s}}\, \nabla^2 T_{\mathrm{s}} + \varphi\, \varkappa\, \nabla^2 T_{\mathrm{f}} + \mu\, \Phi + (1-\varphi)\, q_{\mathrm{g_s}} + \varphi\, q_{\mathrm{g_f}}\ ,\end{aligned} \tag{6.18}$$

$$(1-\varphi)\ \rho_{\mathrm{s}}\, c_{\mathrm{v_s}} \frac{\partial T_{\mathrm{s}}}{\partial t} = (1-\varphi)\, \varkappa_{\mathrm{s}}\, \nabla^2 T_{\mathrm{s}} + (1-\varphi)\, q_{\mathrm{g_s}} + h\,(T_{\mathrm{f}} - T_{\mathrm{s}})\ . \tag{6.19}$$

If one takes the limit $h \to \infty$ in Eq. (6.19), one just retains the leading contribution, namely the last term on the right-hand side. Thus, Eq. (6.19) yields in this limit the local thermal equilibrium condition, $T_{\mathrm{f}} = T_{\mathrm{s}}$. By substituting this limiting condition in Eq. (6.18), one is led to Eq. (6.12), completed by the definitions given by Eqs. (6.13) and (6.14). The power generated per unit volume is to be intended as

$$q_{\mathrm{g}} = (1-\varphi)\, q_{\mathrm{g_s}} + \varphi\, q_{\mathrm{g_f}}\ . \tag{6.20}$$

The use of the two-temperature model for the local energy balance poses a problem regarding the correct formulation of the boundary conditions to be prescribed for T_{f} and T_{s}. The condition of a prescribed boundary temperature has in fact a natural formulation where a local thermal equilibrium is assumed at the boundary,

$$T_{\mathrm{f}} = T_{\mathrm{s}} = T_{\mathrm{w}}(\mathbf{x}, t)\ , \tag{6.21}$$

where $T_{\mathrm{w}}(\mathbf{x}, t)$ is a known function expressing the wall temperature distribution which, in general, depends both on the position $\mathbf{x}$ and on time t.

Things are definitely more complicated when a condition of prescribed wall heat flux is to be allowed at the boundary. In this case, the solid and fluid phases are potentially independent absorbers of the incoming heat flux at the boundary. There are several studies of this problem, and a survey can be found, for instance, in Alazmi and Vafai [2]. The simplest model to be adopted when a wall heat flux distribution $q_{\mathrm{w}}(\mathbf{x}, t)$ is prescribed can be formulated as

$$T_{\mathrm{f}} = T_{\mathrm{s}}\ , \qquad -(1-\varphi)\, \varkappa_{\mathrm{s}}\, \frac{\partial T_{\mathrm{s}}}{\partial n} - \varphi\, \varkappa\, \frac{\partial T_{\mathrm{f}}}{\partial n} = q_{\mathrm{w}}(\mathbf{x}, t)\ , \tag{6.22}$$

where $\partial/\partial n$ denotes the normal derivative to the boundary. The model expressed by Eq. (6.22) makes sense when the bounding wall has a very high thermal conductivity so that a local equilibration of the otherwise different temperatures of the two phases, T_f and T_s, is a sensible assumption.

6.3.2 *Viscous Dissipation*

We pointed out that, either in Eqs. (6.12) or (6.17), the term $\mu\,\Phi$ yields the power per unit volume generated by viscous dissipation. The expression of $\mu\,\Phi$ depends on the model for momentum balance employed. As pointed out in Nield [10], the term $\mu\,\Phi$ can be evaluated according to the general rule,

$$\mu\,\Phi = \mathbf{f}_\mathrm{d}\cdot\mathbf{u}\,, \tag{6.23}$$

where

$$\mathbf{f}_d = -\nabla p + \mathbf{b}\,, \tag{6.24}$$

is the drag force. The drag force has an expression which depends on the model adopted for the momentum transfer,

$$\text{Darcy's law} \longrightarrow \qquad \mathbf{f}_d = \frac{\mu}{K}\,\mathbf{u}\,; \tag{6.25}$$

$$\text{Darcy–Forchheimer's model} \longrightarrow \quad \mathbf{f}_d = \frac{\mu}{K}\left(1 + \frac{F\,\rho\,\sqrt{K}}{\mu}\,|\mathbf{u}|\right)\mathbf{u}\,; \tag{6.26}$$

$$\text{Brinkman's model} \longrightarrow \qquad \mathbf{f}_d = \frac{\mu}{K}\,\mathbf{u} - \mu_\mathrm{eff}\,\nabla^2\mathbf{u}\,. \tag{6.27}$$

Nield's rule expressed by Eq. (6.23) is still the object of some controversies especially with reference to its application in the case of Brinkman's model. Let us refer for simplicity to an incompressible flow, $\nabla\cdot\mathbf{u} = 0$. One would expect that, in the limiting case of an infinite permeability $K \to \infty$, the expression of $\mu\,\Phi$ implied by Eqs. (6.23) and (6.27) is consistent with the expression of the dissipation function for a Navier–Stokes fluid, Eq. (5.56), namely

$$\Phi = 2\,\mathscr{D}_{ij}\,\mathscr{D}_{ij}, \qquad \mathscr{D}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\,. \tag{6.28}$$

On the contrary, in the limit $K \to \infty$ and $\varphi \to 1$, Eqs. (6.23) and (6.27) yield

$$\Phi = -\,\mathbf{u}\cdot\nabla^2\mathbf{u}\,, \tag{6.29}$$

since $\mu_{\text{eff}} = \mu$ in the limiting case of a fluid clear of solid material, as implied by Eq. (6.10). The difference between the expressions of Φ given in Eqs. (6.28) and (6.29) is apparent as Eq. (6.28) yields an expression containing only first-order derivatives of the velocity components, while the right-hand side of Eq. (6.29) contains second-order derivatives of the velocity components. Moreover, while Φ given by Eq. (6.28) can only be positive or zero, there could be flows such that the right-hand side of Eq. (6.29) is negative.

Al-Hadhrami et al. [1] proposed a different expression of Φ in the case of Brinkman's model, namely

$$\Phi = \frac{1}{K}\,\mathbf{u}\cdot\mathbf{u} + 2\,\frac{\mu_{\text{eff}}}{\mu}\,\mathscr{D}_{ij}\,\mathscr{D}_{ij}\ . \tag{6.30}$$

The advantage of the expression of Φ as given by Eq. (6.30) is that Φ cannot be negative, and the two limiting cases of Darcy's law ($K \to 0$) and Navier–Stokes fluid ($K \to \infty$, $\varphi \to 1$) are correctly recovered. However, the scientific debate on this subject is still open. An interesting contribution on this topic comes from the paper by Breugem and Rees [4].

6.4 The Buoyancy Force

The Oberbeck–Boussinesq approximation can be invoked for the seepage flow in porous media, along the same lines defined in Sect. 5.9. Again, the basic idea is assuming a variable density only in the gravitational body force $\mathbf{b} = \rho\,\mathbf{g}$. Then, a Taylor series expansion of the fluid density as a function of T, around the reference temperature T_0, truncated to first order is written, as in Eq. (5.66), and this leads to the definition of a piezometric head, as in Eq. (5.67), and a buoyancy force proportional to $T - T_0$.

If one invokes the validity of Darcy's law, the set of local balance equations according to the Oberbeck–Boussinesq approximation is given by

$$\frac{\partial u_j}{\partial x_j} = 0\ , \tag{6.31}$$

$$\frac{\mu}{K}\,u_i = -\rho_0\,\beta\,(T - T_0)\,g_i - \frac{\partial P}{\partial x_i}\ , \tag{6.32}$$

$$\rho_0\,c\left(\sigma\,\frac{\partial T}{\partial t} + u_j\,\frac{\partial T}{\partial x_j}\right) = \varkappa_{\text{eff}}\,\nabla^2 T + q_{\text{g}} + \frac{\mu}{K}\,u_j\,u_j\ . \tag{6.33}$$

In the domain of validity of Darcy–Forchheimer's model, Eqs. (6.31)–(6.33) are to be replaced by

$$\frac{\partial u_j}{\partial x_j} = 0 \,, \tag{6.34}$$

$$\frac{\mu}{K}\left(1 + \frac{F\,\rho\,\sqrt{K}}{\mu}\,|\mathbf{u}|\right) u_i = -\rho_0\,\beta\,(T - T_0)\,g_i - \frac{\partial P}{\partial x_i} \,, \tag{6.35}$$

$$\rho_0\,c\left(\sigma\,\frac{\partial T}{\partial t} + u_j\,\frac{\partial T}{\partial x_j}\right) = \varkappa_{\text{eff}}\,\nabla^2 T + q_{\text{g}} + \frac{\mu}{K}\left(1 + \frac{F\,\rho\,\sqrt{K}}{\mu}\,|\mathbf{u}|\right) u_j\,u_j \,. \tag{6.36}$$

Another formulation of the Oberbeck–Boussinesq approximation based on Brinkman's model can be written in a similar manner. In this case, the only conjectural element is the form of the viscous dissipation contribution in the local energy balance equation, as discussed in Sect. 6.3.2.

6.5 Non-Newtonian Flow in Porous Media

The seepage flow of non-Newtonian fluids in porous media has been the object of a widespread interest over the past decades. Many situations in the real world involve departure from the Newtonian fluid behaviour. This may happen with the flow of polymeric fluids, and of heavy oils or liquid hydrocarbons. The non-Newtonian character of the fluid yields modified formulations of the local momentum balance equation with respect to Darcy's law. Different formulations occur for different classes of non-Newtonian fluids [13, 16].

The simplest case is that of power-law fluids, where the relationship between the viscous stress tensor and the strain tensor is nonlinear. In the simple case where the velocity field is solenoidal, Eq. (5.48) is replaced by

$$\tau_{ij} = 2\,\hat{\mu}\;(\mathscr{D}_{k\ell}\,\mathscr{D}_{k\ell})^{(n-1)/2}\;\mathscr{D}_{ij} \,, \tag{6.37}$$

where $\hat{\mu}$ is the *consistency factor*, while n is the *power-law index*. For the seepage flow of a power-law fluid in a porous medium, Darcy's law is to be extended according to a formulation originally proposed by Christopher and Middleman [5],

$$\frac{\hat{\mu}^*}{K}\,|\mathbf{u}|^{n-1}\mathbf{u} = -\nabla p + \mathbf{b} \,, \tag{6.38}$$

where $\hat{\mu}^*$ is the *effective consistency factor*. Obviously, both Eqs. (6.37) and (6.38) yield their Newtonian counterparts when $n = 1$. In particular, Eq. (6.38) coincides with Eq. (6.5) when $n = 1$, provided that the effective consistency factor is identified with the dynamic viscosity of the Newtonian fluid.

A more complicated type of non-Newtonian fluids is that described by the *Bingham model*. Fluids within this class display a behaviour which is intermediate between a solid and a Newtonian fluid. In other words, a tangential stress imposed on the boundary of the fluid determines a corresponding nonzero strain only if the tangential stress exceeds a threshold value. When this threshold is exceeded, the excess tangential stress applied to the boundary is proportional to the strain as for Newtonian fluids. Several biological fluids behave in this manner. A thorough analysis of Bingham flow in porous media has been presented by Rees [14] and by Nash and Rees [9]. In particular, the simplest mathematical formulation of the extended Darcy's law for Bingham fluids was expressed by Pascal [12],

$$\begin{cases} \mathbf{u} = 0 \ , & |\mathbf{f}_\mathrm{d}| < \xi \ , \\ \dfrac{\mu}{K}\left(1 + \dfrac{K\,\xi}{\mu\,|\mathbf{u}|}\right)\mathbf{u} = \mathbf{f}_\mathrm{d} \ , & |\mathbf{f}_\mathrm{d}| > \xi \ . \end{cases} \tag{6.39}$$

Here, $\mathbf{f}_\mathrm{d}$ is the drag force, defined by Eq. (6.24), namely $\mathbf{f}_\mathrm{d} = -\boldsymbol{\nabla} p + \mathbf{b}$. The positive parameter ξ is the *yield pressure gradient* distinguishing the stagnant regime, $\mathbf{u} = 0$, from flow, $\mathbf{u} \neq 0$. Darcy's law for a Newtonian fluid is easily recovered from Eq. (6.39) in the limit $\xi \to 0$.

Several studies on convection in porous media involve *viscoelastic fluids*. These fluids display a nature that includes both the viscous damping, typical of Newtonian fluids, and the elastic response typical of solids. When the flow of these non-Newtonian fluids happens in porous media, the extension of Darcy's law is based on the so-called *Oldroyd-B model*, as mentioned by Khuzhayorov et al. [7],

$$\frac{\mu}{K}\left(1 + \tau_2 \frac{\partial}{\partial t}\right)\mathbf{u} = \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\mathbf{f}_\mathrm{d} \ . \tag{6.40}$$

Again, the drag force $\mathbf{f}_\mathrm{d}$ is given by Eq. (6.24), while τ_1 and τ_2 are the *relaxation time* and the *retardation time*, respectively. The formulation based on Eq. (6.40) is consistent only if $\tau_1 \geq \tau_2$. The special case $\tau_1 = \tau_2$ yields the Newtonian behaviour, while the limit $\tau_2 \to 0$ corresponds to the so-called *Maxwell model* for linear viscoelasticity. Equation (6.40) shows that, in the case of stationary fluid flow in porous media, there is no difference between the behaviour of a Newtonian fluid and that of a linear viscoelastic fluid, as the time derivatives in Eq. (6.40) are ineffective.

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