Chapter 5
The Equations of Fluid Flow

5.1 The Description of Fluid Flow

The basic idea behind the classical description of the fluid flow is that a fluid is a *continuous medium*. This means that, although a fluid (liquid or gas) has elementary constituents (atoms, ions and molecules), these constituents are so small that the length scale of every phenomenon involved in the macroscopic fluid flow will be much larger than the molecular scale. Hence, it is perfectly legitimate to consider *infinitesimal fluid elements* (see Fig. 5.1). In fact, the mathematical concept of infinitesimal scale is in any case relative to very small spatial domains where the number of elementary constituents is still extremely large (of the order of Avogadro's constant, 6.022×10^{23}).

The fluid is then described as partitioned in an infinite number of infinitesimal fluid elements each one evolving in time along its own trajectory. This description of the fluid flow presumes that we are able to know which is the spatial position $\mathbf{x} = (x, y, z)$ of every infinitesimal fluid element at every instant of time t. In doing this, we are implicitly assuming that the same spatial position $\mathbf{x} = (x, y, z)$ cannot be occupied by two different infinitesimal fluid elements at the same instant of time t (*localisation hypothesis*).

On tracing the trajectory of an infinitesimal fluid element, we can define its instantaneous velocity $\mathbf{v}$ at every instant of time t. If $\mathbf{x}(t) = (x(t), y(t), z(t))$ is the instantaneous position of an infinitesimal fluid element, its *instantaneous velocity* is defined as

$$\mathbf{v}(t) = \frac{\mathrm{d}\,\mathbf{x}(t)}{\mathrm{d}\,t} \, . \tag{5.1}$$

Therefore, we can define the *velocity field*, $\mathbf{u}(x, y, z, t)$, as the instantaneous velocity $\mathbf{v}(t)$ of the infinitesimal fluid element occupying the position $(x, y, z) = (x(t), y(t), z(t))$ at time t (see Fig. 5.2). The concept of velocity field is well-defined inasmuch as the localisation hypothesis holds.

We assume that the number of atoms, ions or molecules contained in an infinitesimal fluid element is large enough to consider it as a closed thermodynamic system.

A. Barletta, *Routes to Absolute Instability in Porous Media*,
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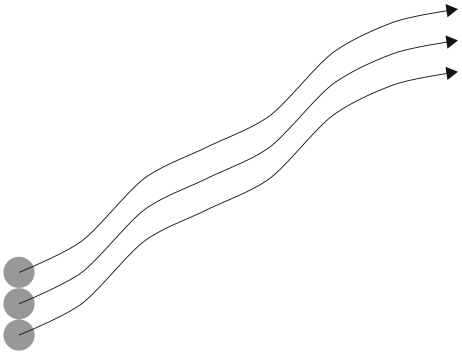

Fig. 5.1 A group of infinitesimal fluid elements evolving along their own trajectories

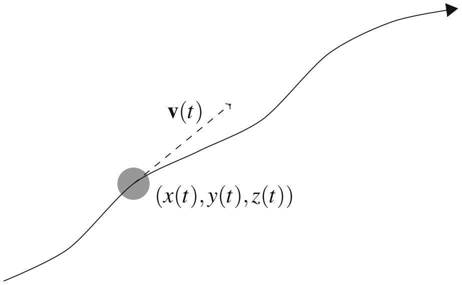

Fig. 5.2 Instantaneous velocity of an infinitesimal fluid element occupying the position $\mathbf{x}(t) = (x(t), y(t), z(t))$ at time t

Then, at every time, we can reasonably think of an "internal" thermodynamic state of the infinitesimal fluid element. The thermodynamic state of the infinitesimal fluid element can be described in a fairly simple way provided that it is a stable equilibrium state. The latter assumption, called *local equilibrium hypothesis*, is a reasonable one if the evolution of this very small thermodynamic system is sufficiently slow for having an instantaneous thermal equilibration of the fluid element. That an infinitesimal time is needed for reaching the thermodynamic equilibrium in a system of infinitesimal size appears as quite conceivable, even if exceptions may arise when the time evolution is so quick that the molecular processes lose coherence. An important characteristic property of the fluid, that will be defined in the last section of this chapter, is the thermal diffusivity α. For a given time scale Δt of the thermodynamic evolution, one can construct a corresponding length scale with $(\alpha\,\Delta t)^{1/2}$. In a gas, the breakdown of the local equilibrium hypothesis can be envisaged when this length scale is of the order of the mean free path of the elementary constituents (atoms, ions and molecules). The mean free path is the average distance travelled by a particle between two collisions and is of variable order of magnitude, ranging from 10^{-7} m for a gas at ambient pressure to several meters for a rarefied gas. In a liquid, the breakdown of the local equilibrium hypothesis is expected when $(\alpha\,\Delta t)^{1/2}$ is of the order of the molecular size (10^{-10} m at its smallest). A consequence of the local equilibrium hypothesis is that the "internal" thermodynamic state of the infinitesimal fluid element is determined by its temperature, T, and density, ρ. All the thermodynamic

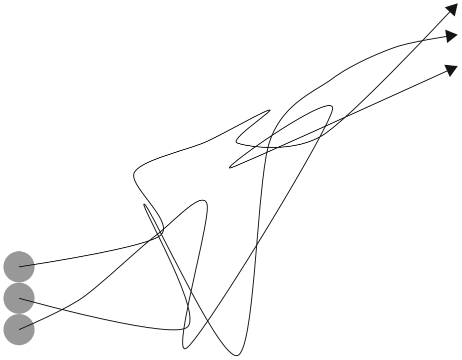

Fig. 5.3 Extremely tangled trajectories of three infinitesimal fluid elements in a turbulent flow

properties of the infinitesimal fluid element, such as the pressure, p, and the internal energy per unit mass, φ, are functions of (ρ, T).

The local equilibrium hypothesis implies that, at a given instant of time t, the infinitesimal fluid element is characterised by a temperature T, a density ρ and a pressure p. Thus, with a procedure perfectly similar to that invoked on defining the velocity field, one can also define the *temperature field*, $T(x, y, z, t)$, as the temperature T of the infinitesimal fluid element occupying the position (x, y, z) at time t. One can define the *density field*, $\rho(x, y, z, t)$, as the density ρ of the infinitesimal fluid element occupying the position (x, y, z) at time t. Finally, one can define the *pressure field*, $p(x, y, z, t)$, as the pressure p of the infinitesimal fluid element occupying the position (x, y, z) at time t. We stress that this procedure makes sense only if there is one and only one infinitesimal fluid element occupying the position (x, y, z) at time t. The localisation hypothesis does not hold for every flow regime, so that there exist fluid flows where the velocity field, the temperature field, the density field, the pressure field cannot be consistently defined. These flows generally take place when the flow rates are very high. When this happens, the flow experiences a transition from laminar to turbulent. For turbulent flows, the localisation hypothesis does not hold any more.

One can imagine the lack of validity of the localisation hypothesis as being a consequence of the extremely tangled geometry of the trajectories in a turbulent flow (see Fig. 5.3). The point is in fact even more complicated. The concept of infinitesimal fluid element and of trajectory of an infinitesimal fluid element loses any conceivability when the flow is turbulent. The physical reason is that molecular diffusion mechanisms arise, leading to an internal mixing at the molecular scale. Therefore, a given amount of fluid occupying an infinitesimal volume at the initial instant of time $t = 0$ may be spread over a finite region, even of very large size, at later times $t > 0$ (see Fig. 5.4).

We can conclude that we have a good paradigm for the description of fluid flows of laminar nature, but we have no consistent paradigm for the description of turbulent fluid flows. This problem prevents the possibility of a theory of turbulence, that is in fact still lacking. What physicists and engineers can say about turbulence has been

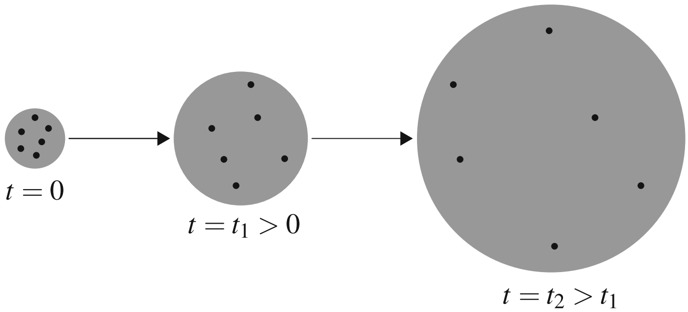

Fig. 5.4 Spreading over larger regions of an initially infinitesimal fluid element

conveyed during the last century into phenomenological models of turbulent flows, that cannot in any case be considered as theories of turbulence.

5.2 Reynolds' Transport Theorem

In order to describe fluid flow in a given region of space $\mathscr{R}$, it is convenient to investigate the displacements undergone, in a given time interval, by a *fluid body*, i.e. a fluid part bounded by an ideal surface impermeable with respect to mass flux. Obviously, this definition implies that a fluid body is a closed thermodynamic system. The concept of fluid body is the basis of the *Lagrangian description* of fluid flow.[1]

Let us consider a fluid body having mass M e let us denote by $\mathscr{R}_t \subset \mathscr{R}$ the region of space occupied by the fluid body at time t. Under fluid flow, the region of space $\mathscr{R}_t$ is continuously displaced driven by the moving fluid. The boundary of $\mathscr{R}_t$ is supposed to be a regular closed surface denoted by $\partial\mathscr{R}_t$. Each point in $\mathscr{R}_t$ is identified by the changing coordinates $(x(t), y(t), z(t))$. The outward unit normal to $\partial\mathscr{R}_t$ is denoted by $\mathbf{n}$ (see Fig. 5.5).

Let us consider any *extensive property*, Ψ, of the fluid body. In thermodynamics, one calls extensive any property of a system whose instantaneous value can be evaluated by summing up the values of the property assumed by all the subsystems of the given system. Mass, volume, energy and entropy are just a few examples of extensive properties. The mass of a system partitioned into two halves is the sum of the masses of the two halves.

To every extensive property Ψ of the fluid body, one can always associate a corresponding *specific property*, ψ, obtained by locally dividing the value of Ψ of any subsystem by its mass. If the subsystem employed to define ψ is an infinitesimal fluid element of mass $\mathrm{d}M = \rho\,\mathrm{d}V$, then one can intend ψ as a local field depending

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[1] As opposed to the Lagrangian description, the Eulerian description is based on a fixed volume through which the fluid flows. Thus, the Eulerian description is focussed on an open thermodynamic system.

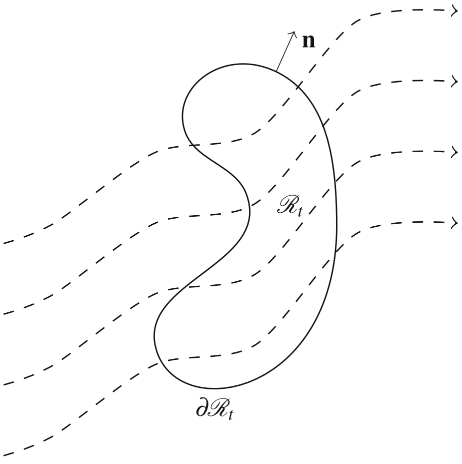

Fig. 5.5 Displacement of the fluid body occupying the moving region $\mathscr{R}_t$

on the local coordinates, as well as on time. Hence, the extensive character of the property Ψ allows one to write

$$\Psi(t) = \iiint\limits_{\mathscr{R}_t} \rho\, \psi \; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \;, \tag{5.2}$$

where the integration measure $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ describes the infinitesimal moving volume $\mathrm{d}V$. By tracing the trajectory of each infinitesimal fluid element, one has a one–to–one correspondence between the position at time t, $(x(t), y(t), z(t))$, and the corresponding position at initial time, $t = 0$, namely $(x(0), y(0), z(0)) = (X, Y, Z)$. Obviously, if $(x(t), y(t), z(t)) \in \mathscr{R}_t$, one has $(X, Y, Z) \in \mathscr{R}_0$. With this in mind, one can change the measure of integration on the right-hand side of equation (5.2), by introducing a suitable Jacobian, J. In particular, one obtains

$$\Psi(t) = \iiint\limits_{\mathscr{R}_0} \rho\, \psi\; J \; \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}Z \;. \tag{5.3}$$

The Jacobian J is the determinant of the matrix expressing the change of coordinates [1], from $(x(t), y(t), z(t))$ to (X, Y, Z). Hence, we can express J by employing the Levi-Civita symbol, ε_{ijk}, and Einstein's notation for repeated indices, namely

$$J = \det\begin{pmatrix} \dfrac{\partial x}{\partial X} & \dfrac{\partial x}{\partial Y} & \dfrac{\partial x}{\partial Z} \\ \dfrac{\partial y}{\partial X} & \dfrac{\partial y}{\partial Y} & \dfrac{\partial y}{\partial Z} \\ \dfrac{\partial z}{\partial X} & \dfrac{\partial z}{\partial Y} & \dfrac{\partial z}{\partial Z} \end{pmatrix} = \varepsilon_{ijk}\, \frac{\partial x_1}{\partial X_i}\, \frac{\partial x_2}{\partial X_j}\, \frac{\partial x_3}{\partial X_k} \;. \tag{5.4}$$

In Eq. (5.4), symbols (x_1, x_2, x_3) and (X_1, X_2, X_3) have been used instead of (x, y, z) and (X, Y, Z), respectively, as they are more convenient to implement Einstein's notation for sums over repeated indices. For readers unfamiliar with Levi-Civita symbol, or Kronecker's delta, or Einstein's notation for tensor operations, we refer to Appendix B. In particular, the expression of J in Eq. (5.4) is a consequence of equation (B.22) of Appendix B. We want to evaluate the time derivative of $\Psi(t)$. From Eq. (5.3), we obtain

$$\begin{aligned} \frac{\mathrm{d}\,\Psi}{\mathrm{d}\,t} &= \iiint\limits_{\mathscr{R}_0} \left[\frac{\partial(\rho\,\psi)}{\partial t}\;J + \frac{\partial(\rho\,\psi)}{\partial x_i}\;\frac{\mathrm{d}\,x_i}{\mathrm{d}\,t}\;J + \rho\,\psi\;\frac{\partial J}{\partial t} \right] \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \\ &= \iiint\limits_{\mathscr{R}_0} \left[\frac{\partial(\rho\,\psi)}{\partial t}\;J + \frac{\partial(\rho\,\psi)}{\partial x_i}\;u_i\;J + \rho\,\psi\;\frac{\partial J}{\partial t} \right] \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\;, \end{aligned} \tag{5.5}$$

where the definitions of instantaneous velocity over a trajectory, and of velocity field, $\mathbf{u}$ with components u_i, have been employed. We now express the derivative $\partial J/\partial t$, by using again the definition of velocity field,

$$\begin{aligned} \frac{\partial J}{\partial t} &= \varepsilon_{ijk}\,\frac{\partial u_1}{\partial X_i}\,\frac{\partial x_2}{\partial X_j}\,\frac{\partial x_3}{\partial X_k} + \varepsilon_{ijk}\,\frac{\partial x_1}{\partial X_i}\,\frac{\partial u_2}{\partial X_j}\,\frac{\partial x_3}{\partial X_k} + \varepsilon_{ijk}\,\frac{\partial x_1}{\partial X_i}\,\frac{\partial x_2}{\partial X_j}\,\frac{\partial u_3}{\partial X_k} \\ &= \frac{\partial u_1}{\partial x_\ell}\,\varepsilon_{ijk}\,\frac{\partial x_\ell}{\partial X_i}\,\frac{\partial x_2}{\partial X_j}\,\frac{\partial x_3}{\partial X_k} + \frac{\partial u_2}{\partial x_m}\,\varepsilon_{ijk}\,\frac{\partial x_1}{\partial X_i}\,\frac{\partial x_m}{\partial X_j}\,\frac{\partial x_3}{\partial X_k} \\ &\qquad + \frac{\partial u_3}{\partial x_n}\,\varepsilon_{ijk}\,\frac{\partial x_1}{\partial X_i}\,\frac{\partial x_2}{\partial X_j}\,\frac{\partial x_n}{\partial X_k} \\ &= \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \varepsilon_{ijk}\,\frac{\partial x_1}{\partial X_i}\,\frac{\partial x_2}{\partial X_j}\,\frac{\partial x_3}{\partial X_k} = (\boldsymbol{\nabla}\cdot\mathbf{u})\,J\;, \end{aligned} \tag{5.6}$$

where we recognised that ℓ can only be equal to 1, m can only be equal to 2, and n can only be equal to 3. In fact, expressions such as

$$\varepsilon_{ijk}\,\frac{\partial x_\ell}{\partial X_i}\,\frac{\partial x_2}{\partial X_j}\,\frac{\partial x_3}{\partial X_k}\;,$$

yield the determinant of a matrix with two equal rows, which is zero, unless $\ell = 1$. On account of Eq. (5.6), Eq. (5.5) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}\,\Psi}{\mathrm{d}\,t} &= \iiint\limits_{\mathscr{R}_0} \left[\frac{\partial(\rho\,\psi)}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}(\rho\,\psi) + \rho\,\psi\;\boldsymbol{\nabla}\cdot\mathbf{u} \right] J\;\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \\ &= \iiint\limits_{\mathscr{R}_0} \left[\frac{\partial(\rho\,\psi)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\psi\;\mathbf{u}) \right] J\;\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\;, \end{aligned} \tag{5.7}$$

so that one finally has

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \iiint\limits_{\mathscr{R}_t} \left[\frac{\partial(\rho\,\psi)}{\partial t} + \nabla \cdot (\rho\,\psi\,\mathbf{u}) \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{5.8}$$

Equation (5.8) represents the statement of Reynolds' transport theorem.

5.3 Local Mass Balance Equation

In this section, the first and simplest application of Reynolds' transport theorem is presented: the deduction of the local mass balance equation. Let us assume that the extensive property Ψ is the mass M of the fluid element. Since the fluid element is a closed thermodynamic system, M does not change with time. Hence, the left-hand side of equation (5.8) is zero. The specific property ψ is obtained as the ratio between Ψ and the mass, so that one has $\psi = 1$ in this case. As a consequence, Eq. (5.8) allows one to infer that

$$\iiint\limits_{\mathscr{R}_t} \left[\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\,\mathbf{u}) \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0\ . \tag{5.9}$$

Since Eq. (5.9) must hold for every possible choice of the fluid body and, thus, for every region of space $\mathscr{R}_t$ included in the domain $\mathscr{R}$ occupied by the fluid, the integrand must be identically zero. In other words, one has

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\,\mathbf{u}) = 0\ . \tag{5.10}$$

The partial differential equation (5.10) is the *local mass balance equation*, well known also as the *equation of continuity*. One can easily conclude that, if the density field ρ can be considered as time independent and uniform, i.e. for an *incompressible flow*, the equation of continuity is reduced to $\nabla \cdot \mathbf{u} = 0$. Hence, in this case, the velocity field is *solenoidal*.[2]

Equation (5.10) allows one to reformulate Reynolds' transport theorem through the following equation:

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= \iiint\limits_{\mathscr{R}_t} \left[\rho\,\frac{\partial\psi}{\partial t} + \psi\,\frac{\partial\rho}{\partial t} + \rho\,\mathbf{u}\cdot\nabla\psi + \psi\,\nabla\cdot(\rho\,\mathbf{u}) \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iiint\limits_{\mathscr{R}_t} \rho \left(\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi \right) \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = \iiint\limits_{\mathscr{R}_t} \rho\,\frac{\mathrm{D}\psi}{\mathrm{D}t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \end{aligned} \tag{5.11}$$

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[2] The term solenoidal comes from electromagnetism as the magnetic induction field in a solenoid is a vector field with zero divergence.

In obtaining Eq. (5.11), use has been made of the mathematical identity

$$\nabla \cdot (\rho\, \psi\, \mathbf{u}) = \rho\, \mathbf{u} \cdot \nabla \psi + \psi\, \nabla \cdot (\rho\, \mathbf{u}) \ , \tag{5.12}$$

and of the definition of *substantial derivative*,

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi \ . \tag{5.13}$$

Hereafter, for the sake of brevity, the measure $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ will be simply denoted by $\mathrm{d}V$ so that Eq. (5.11) reads

$$\frac{\mathrm{d}\,\Psi}{\mathrm{d}\,t} = \iiint\limits_{\mathscr{R}_t} \rho\, \frac{\mathrm{D}\psi}{\mathrm{D}\,t}\, \mathrm{d}V \ . \tag{5.14}$$

The alternative formulation of equation (5.10), based on Einstein's notation, is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\, u_j)}{\partial x_j} = 0 \ . \tag{5.15}$$

5.4 Forces Acting on a Fluid Body

The forces acting on the fluid body occupying the moving region $\mathscr{R}_t$ can be classified as *body forces* acting on the infinitesimal fluid elements in the interior of $\mathscr{R}_t$, and as *surface forces* acting on the boundary $\partial\mathscr{R}_t$. When dealing with surface forces, one defines the *traction*, meaning the force per unit area acting on the boundary of the fluid body.

Let b_i denote the ith component of the body force, i.e. of the force per unit volume acting on the fluid element and due to external fields of either gravitational, electric or magnetic origin. Thus, the ith component of the resultant body force acting on the fluid element is

$$F_i^{(\mathrm{b})} = \iiint\limits_{\mathscr{R}_t} b_i\, \mathrm{d}V \ . \tag{5.16}$$

In the following, it will be tacitly assumed that the body force is of gravitational origin, so that $\mathbf{b} = \rho\, \mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration. However, there can be applications where also contributions due to external electric and magnetic fields are important, as in the study of magnetohydrodynamics (MHD) [5].

In the evaluation of the total force F_i applied to the fluid element, one must consider, in addition to the gravitational body force contribution, also the resultant $F_i^{(\mathrm{s})}$ of the traction acting on the boundary surface of the fluid element, namely

$$F_i^{(\mathrm{s})} = \iint\limits_{\partial \mathscr{R}_t} f_i \, \mathrm{d}S \; , \tag{5.17}$$

where f_i is the ith component of the traction, i.e. the force per unit area, acting on the boundary surface $\partial \mathscr{R}_t$ of the fluid element, and $\mathrm{d}S$ is the measure of the surface integral.

The traction f_i can be expressed through the *mechanical stress tensor* σ as

$$f_i = \sigma_{ij} \, n_j \; , \tag{5.18}$$

where n_j is the jth component of the unit outward normal $\mathbf{n}$ to $\partial \mathscr{R}_t$ (see Fig. 5.5). By employing Gauss' theorem (see Appendix B), Eqs. (5.17) and (5.18) yield an expression of $F_i^{(\mathrm{s})}$ in terms of a volume integral, namely

$$F_i^{(\mathrm{s})} = \iiint\limits_{\mathscr{R}_t} \frac{\partial \sigma_{ij}}{\partial x_j} \, \mathrm{d}V \; , \tag{5.19}$$

so that the resultant force acting on the fluid body is given by

$$F_i = F_i^{(\mathrm{b})} + F_i^{(\mathrm{s})} = \iiint\limits_{\mathscr{R}_t} \left(\rho \, g_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right) \mathrm{d}V \; . \tag{5.20}$$

5.5 Local Momentum Balance Equation

Let the extensive property Ψ be the ith component of momentum. On account of the definition of momentum of a point-like object, one can easily conclude that the corresponding specific property ψ is the ith component of the velocity field, u_i. From elementary mechanics, we know that the time derivative of the ith component of momentum is equal to the ith component of the resultant force, F_i, acting on the mechanical system. Then, Reynolds' transport theorem (5.14) yields

$$\iiint\limits_{\mathscr{R}_t} \rho \, \frac{\mathrm{D} u_i}{\mathrm{D} t} \, \mathrm{d}V = F_i \; , \tag{5.21}$$

so that, by employing equation (5.20), one obtains

$$\iiint\limits_{\mathscr{R}_t} \left(\rho \, \frac{\mathrm{D} u_i}{\mathrm{D} t} - \rho \, g_i - \frac{\partial \sigma_{ij}}{\partial x_j} \right) \mathrm{d}V = 0 \; . \tag{5.22}$$

We are in a situation identical to Eq. (5.9). The integral on the left-hand side of equation (5.22) must be zero for every possible choice of the region of space $\mathscr{R}_t$, so that the integrand must be identically zero,

$$\rho \, \frac{\mathrm{D} u_i}{\mathrm{D} t} = \rho \, g_i + \frac{\partial \sigma_{ij}}{\partial x_j} \, . \tag{5.23}$$

Equation (5.23) is the *local momentum balance equation*.

5.6 Local Angular Momentum Balance Equation

One of the main properties of the mechanical stress tensor σ is its symmetry, $\sigma_{ij} = \sigma_{ji}$. The symmetry of the mechanical stress tensor is a consequence of the local balance of the angular momentum.

By remembering that the moment of a force is obtained through the vector product between the position vector $\mathbf{x}$ and the force, the resultant moment of the forces acting on the fluid body contained in $\mathscr{R}_t$ can be expressed, on account of equations (5.16) and (5.17), as

$$\mathbf{K} = \iiint\limits_{\mathscr{R}_t} \mathbf{x} \times (\rho \, \mathbf{g}) \, \mathrm{d}V + \iint\limits_{\partial \mathscr{R}_t} \mathbf{x} \times \mathbf{f} \, \mathrm{d}S \, . \tag{5.24}$$

Once more, we refer the reader to Appendix B for the properties of the Levi-Civita symbol. By using Eqs. (5.18) and (B.20), we can write the ith component of $\mathbf{K}$ as

$$\begin{aligned} K_i &= \iiint\limits_{\mathscr{R}_t} \rho \, \varepsilon_{ijk} \, x_j \, g_k \, \mathrm{d}V + \iint\limits_{\partial \mathscr{R}_t} \varepsilon_{ijk} \, x_j \, f_k \, \mathrm{d}S \\ &= \iiint\limits_{\mathscr{R}_t} \rho \, \varepsilon_{ijk} \, x_j \, g_k \, \mathrm{d}V + \iint\limits_{\partial \mathscr{R}_t} \varepsilon_{ijk} \, x_j \, \sigma_{k\ell} \, n_\ell \, \mathrm{d}S \, . \end{aligned} \tag{5.25}$$

By employing Gauss' theorem, the surface integral over $\partial \mathscr{R}_t$ can be rewritten as a volume integral, so that Eq. (5.25) reads

$$K_i = \iiint\limits_{\mathscr{R}_t} \varepsilon_{ijk} \left[\rho \, x_j \, g_k + \frac{\partial (x_j \, \sigma_{k\ell})}{\partial x_\ell} \right] \mathrm{d}V \, . \tag{5.26}$$

Let the extensive property Ψ be the ith component of the angular momentum, L_i. Then, from elementary mechanics, the corresponding specific quantity ψ is $\varepsilon_{ijk} \, x_j \, u_k$. As a consequence of Reynolds' transport theorem, Eq. (5.14), one can write

$$\begin{aligned}\frac{\mathrm{d}\,L_i}{\mathrm{d}\,t} &= \iiint\limits_{\mathscr{R}_t} \rho\;\varepsilon_{ijk}\,\frac{\mathrm{D}(x_j\,u_k)}{\mathrm{D}\,t}\,\mathrm{d}V = \iiint\limits_{\mathscr{R}_t} \rho\;\varepsilon_{ijk}\left[x_j\,\frac{\partial u_k}{\partial t} + u_\ell\,\frac{\partial(x_j\,u_k)}{\partial x_\ell}\right]\mathrm{d}V\\
&= \iiint\limits_{\mathscr{R}_t} \rho\left[\varepsilon_{ijk}\,x_j\,\frac{\partial u_k}{\partial t} + \varepsilon_{ijk}\,u_\ell\,\delta_{j\ell}\,u_k + \varepsilon_{ijk}\,x_j\,u_\ell\,\frac{\partial u_k}{\partial x_\ell}\right]\mathrm{d}V\\
&= \iiint\limits_{\mathscr{R}_t} \rho\left[\varepsilon_{ijk}\,x_j\,\frac{\partial u_k}{\partial t} + \varepsilon_{ijk}\,u_j\,u_k + \varepsilon_{ijk}\,x_j\,u_\ell\,\frac{\partial u_k}{\partial x_\ell}\right]\mathrm{d}V\\
&= \iiint\limits_{\mathscr{R}_t} \rho\;\varepsilon_{ijk}\,x_j\,\frac{\mathrm{D}u_k}{\mathrm{D}\,t}\,\mathrm{d}V\;, \end{aligned} \tag{5.27}$$

where we recognised that $\partial x_j/\partial x_\ell = \delta_{j\ell}$, and that $\varepsilon_{ijk}\,u_j\,u_k = 0$ as it represents the ith component of the vector product of $\mathbf{u}$ with itself. The mechanical balance of angular momentum prescribes that

$$\frac{\mathrm{d}\,L_i}{\mathrm{d}\,t} = K_i\;. \tag{5.28}$$

Then, we obtain

$$\iiint\limits_{\mathscr{R}_t} \varepsilon_{ijk}\left[\rho\,x_j\,\frac{\mathrm{D}u_k}{\mathrm{D}\,t} - \rho\,x_j\,g_k - \frac{\partial(x_j\,\sigma_{k\ell})}{\partial x_\ell}\right]\mathrm{d}V = 0\;, \tag{5.29}$$

which can be rewritten as

$$\iiint\limits_{\mathscr{R}_t} \varepsilon_{ijk}\left[\rho\,x_j\,\frac{\mathrm{D}u_k}{\mathrm{D}\,t} - \rho\,x_j\,g_k - x_j\,\frac{\partial\sigma_{k\ell}}{\partial x_\ell} - \sigma_{kj}\right]\mathrm{d}V = 0\;. \tag{5.30}$$

By employing the local momentum balance equation (5.23), one can rewrite Eq. (5.30) as

$$\iiint\limits_{\mathscr{R}_t} \varepsilon_{ijk}\,\sigma_{kj}\,\mathrm{d}V = 0\;. \tag{5.31}$$

Following the usual argument, since Eq. (5.31) must hold for every possible choice of the region $\mathscr{R}_t$, the integrand must be zero, so that we can write the local equation

$$\varepsilon_{ijk}\,\sigma_{kj} = 0\;. \tag{5.32}$$

Due to the properties of the Levi-Civita symbol (see Appendix B), Eq. (5.32) with $i = 1, 2, 3$ yields the equalities

$$\sigma_{32} = \sigma_{23}\ , \quad \sigma_{31} = \sigma_{13}\ , \quad \sigma_{12} = \sigma_{21}\ . \tag{5.33}$$

This just means that σ is a symmetric tensor, i.e. that $\sigma_{ij} = \sigma_{j\,i}$.

5.7 Local Energy Balance Equation

Let us now assume that the extensive property Ψ is the energy E of the fluid element. The energy per unit mass of the fluid, ψ, can be expressed as

$$\psi = \varphi + \frac{\mathbf{u}\cdot\mathbf{u}}{2}\ , \tag{5.34}$$

where φ is the internal energy per unit mass. In other words, we are assuming that ψ is the sum of the internal energy per unit mass and of the kinetic energy per unit mass. On account of equations (5.14) and (5.34), Reynolds' transport theorem yields

$$\frac{\mathrm{d}\,E}{\mathrm{d}\,t} = \iiint\limits_{\mathscr{R}_t} \rho \left(\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \right) \mathrm{d}V\ . \tag{5.35}$$

The thermodynamic energy balance can be written as

$$\frac{\mathrm{d}\,E}{\mathrm{d}\,t} = \dot{Q} + \dot{W}\ , \tag{5.36}$$

where $\dot{Q}$ and $\dot{W}$ are, respectively, the thermal power and the mechanical power received by the fluid body. At time t, the thermal power received by the fluid body that occupies the region of space $\mathscr{R}_t$ is the sum of two terms: the thermal power $\dot{Q}_\mathrm{s}$ that crosses the boundary surface $\partial\mathscr{R}_t$ of the body; the thermal power $\dot{Q}_\mathrm{g}$ generated within the fluid element (due to phenomena such as, for instance, the Joule effect in the case of a conducting fluid carrying an electric current). The quantity $\dot{Q}_\mathrm{g}$ can be expressed by introducing the thermal power generated per unit volume within the fluid, q_g, so that one has

$$\dot{Q}_\mathrm{g} = \int\limits_{\mathscr{R}_t} q_\mathrm{g}\ \mathrm{d}V\ . \tag{5.37}$$

The quantity $\dot{Q}_\mathrm{s}$ can be determined by means of the *heat flux density* $\mathbf{q}$,

$$\dot{Q}_\mathrm{s} = -\iint\limits_{\partial\mathscr{R}_t} \mathbf{q}\cdot\mathbf{n}\ \mathrm{d}S\ , \tag{5.38}$$

where $\mathbf{n}$ is the outward normal unit vector to $\partial\mathscr{R}_t$. By invoking Gauss' theorem, one can write

$$\dot{Q}_{\rm s} = -\iiint\limits_{\mathscr{R}_t} \boldsymbol{\nabla}\cdot\mathbf{q}\,{\rm d}V\ . \tag{5.39}$$

Then, one has

$$\dot{Q} = \dot{Q}_{\rm g} + \dot{Q}_{\rm s} = \iiint\limits_{\mathscr{R}_t} \left(q_{\rm g} - \boldsymbol{\nabla}\cdot\mathbf{q}\,\right)\,{\rm d}V\ . \tag{5.40}$$

The mechanical power $\dot{W}$ received from the fluid body coincides with the work per unit time produced by the gravitational body force and by the traction acting on the boundary $\partial\mathscr{R}_t$ of the fluid body. Inside $\mathscr{R}_t$, one has a work per unit time due to the gravitational field and acting on each infinitesimal volume element, given by $\rho\,\mathbf{g}\cdot\mathbf{u}\,{\rm d}V$. Moreover, at every position in $\partial\mathscr{R}_t$, one has an infinitesimal work per unit time $\mathbf{f}\cdot\mathbf{u}\,{\rm d}S$, so that

$$\dot{W} = \iiint\limits_{\mathscr{R}_t} \rho\,\mathbf{g}\cdot\mathbf{u}\,{\rm d}V + \iint\limits_{\partial\mathscr{R}_t} \mathbf{f}\cdot\mathbf{u}\,{\rm d}S = \iiint\limits_{\mathscr{R}_t} \rho\,g_i\,u_i\,{\rm d}V + \iint\limits_{\partial\mathscr{R}_t} f_i\,u_i\,{\rm d}S\ . \tag{5.41}$$

On account of equation (5.18), and by employing Gauss' theorem, Eq. (5.41) can be rewritten as

$$\begin{aligned}\dot{W} &= \iiint\limits_{\mathscr{R}_t} \rho\,g_i\,u_i\,{\rm d}V + \iint\limits_{\partial\mathscr{R}_t} \sigma_{ij}\,u_i\,n_j\,{\rm d}S = \iiint\limits_{\mathscr{R}_t} \left[\rho\,g_i\,u_i + \frac{\partial(\sigma_{ij}\,u_i)}{\partial x_j}\right]{\rm d}V \\ &= \iiint\limits_{\mathscr{R}_t} \left[\rho\,g_i\,u_i + u_i\,\frac{\partial\sigma_{ij}}{\partial x_j} + \sigma_{ij}\,\frac{\partial u_i}{\partial x_j}\right]{\rm d}V\ .\end{aligned} \tag{5.42}$$

On account of equations (5.35), (5.36), (5.40) and (5.42), one obtains

$$\begin{aligned}\iiint\limits_{\mathscr{R}_t} \Bigg[\rho\left(\frac{{\rm D}\varphi}{{\rm D}t} + u_i\,\frac{{\rm D}u_i}{{\rm D}t}\right) - q_{\rm g} + \frac{\partial q_j}{\partial x_j} - \rho\,g_i\,u_i \\ - u_i\,\frac{\partial\sigma_{ij}}{\partial x_j} - \sigma_{ij}\,\frac{\partial u_i}{\partial x_j}\Bigg]{\rm d}V = 0\ .\end{aligned} \tag{5.43}$$

Equation (5.43) can be rewritten as

$$\begin{aligned}\iiint\limits_{\mathscr{R}_t} \Bigg[\rho\,\frac{{\rm D}\varphi}{{\rm D}t} + \left(\rho\,\frac{{\rm D}u_i}{{\rm D}t} - \rho\,g_i - \frac{\partial\sigma_{ij}}{\partial x_j}\right)u_i \\ - q_{\rm g} + \frac{\partial q_j}{\partial x_j} - \sigma_{ij}\,\frac{\partial u_i}{\partial x_j}\Bigg]{\rm d}V = 0\ .\end{aligned} \tag{5.44}$$

Thus, by employing equation (5.23), Eq. (5.44) can be simplified to

$$\iiint\limits_{\mathscr{R}_t} \left[\rho \, \frac{\mathrm{D}\varphi}{\mathrm{D}t} - q_{\mathrm{g}} + \frac{\partial q_j}{\partial x_j} - \sigma_{ij} \, \frac{\partial u_i}{\partial x_j} \right] \mathrm{d}V = 0 \; . \tag{5.45}$$

Once again, we have reached a situation where an integral over $\mathscr{R}_t$ is zero. Due to the arbitrary choice of the integration domain, we an conclude that the integrand is zero. Thus, we obtain the *local energy balance equation*, namely

$$\rho \, \frac{\mathrm{D}\varphi}{\mathrm{D}t} = -\frac{\partial q_j}{\partial x_j} + q_{\mathrm{g}} + \sigma_{ij} \, \frac{\partial u_i}{\partial x_j} \; . \tag{5.46}$$

5.8 Viscous Stresses and Heat Flux

With fluids, the mechanical stress tensor is decomposed into an isotropic part and a traceless part,

$$\sigma_{ij} = -p \, \delta_{ij} + \tau_{ij} \; . \tag{5.47}$$

The term *isotropic* literally means independent of direction. In mathematics, a second-rank tensor is termed isotropic when it is the product of a scalar and Kronecker's delta. In fact, Kronecker's delta has the special feature that its components are the same in any Cartesian reference frame arbitrarily rotated around its origin. In Eq. (5.47), p is the pressure field and τ is a second-rank tensor with zero trace, that is $\tau_{ii} = 0$, called *viscous stress tensor*. It models the viscous behaviour of the fluid, which strongly depends on the specific fluid examined. There are several types of fluids that fall into two main categories: *Newtonian* and *non-Newtonian* fluids. Fluids falling in the former category are most of the fluids encountered in Nature, such as water, oil or gases. For Newtonian fluids, τ is usually represented as

$$\tau_{ij} = 2 \, \mu \, \mathscr{D}_{ij} - \frac{2}{3} \, \mu \, (\boldsymbol{\nabla} \cdot \mathbf{u}) \, \delta_{ij} \; , \tag{5.48}$$

where $\mathscr{D}_{ij}$ is the *strain tensor*, defined as

$$\mathscr{D}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \; , \tag{5.49}$$

and μ is the *dynamic viscosity*. The latter quantity is, in general, a thermodynamic property of the fluid depending on its local temperature and pressure. One may easily check from Eq. (5.49) that the trace of $\mathscr{D}$ coincides with $\boldsymbol{\nabla} \cdot \mathbf{u}$, so that Eq. (5.48) implies that the trace of τ is in fact zero. Equations (5.48) and (5.49) are consistent with the symmetry property of the mechanical stress tensor σ.

On account of equations (5.47)–(5.49), the term $\partial \sigma_{ij}/\partial x_j$ on the right-hand side of equation (5.23) can be rewritten as

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \\ -\frac{2}{3}\,\frac{\partial}{\partial x_i}\,(\mu\, \nabla \cdot \mathbf{u})\ . \tag{5.50}$$

A specially interesting case is one where μ undergoes negligible changes in space and time, so that it can be considered as a constant,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial^2 u_i}{\partial x_j\, \partial x_j} + \frac{\partial^2 u_j}{\partial x_i\, \partial x_j}\right) - \frac{2}{3}\,\mu\, \frac{\partial}{\partial x_i}\,(\nabla \cdot \mathbf{u}) \\ = -\frac{\partial p}{\partial x_i} + \mu\, \nabla^2 u_i + \frac{1}{3}\,\mu\, \frac{\partial}{\partial x_i}\,(\nabla \cdot \mathbf{u})\ . \tag{5.51}$$

Another quantity that involves the mechanical stress tensor is the term $\sigma_{ij}\, \partial u_i/\partial x_j$ appearing on the right-hand side of equation (5.46). Again, by invoking Eq. (5.47), one can write

$$\sigma_{ij}\, \frac{\partial u_i}{\partial x_j} = -p\ \nabla \cdot \mathbf{u} + \tau_{ij}\, \frac{\partial u_i}{\partial x_j} \\ = -p\ \nabla \cdot \mathbf{u} + \frac{1}{2}\, \tau_{ij} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -p\ \nabla \cdot \mathbf{u} + \tau_{ij}\, \mathscr{D}_{ij}\ , \tag{5.52}$$

where the symmetry of the mechanical stress tensor σ, and hence of the viscous stress tensor τ, has been employed. We mention that each term appearing in Eq. (5.52) has a specific name and implied physical meaning,

$$\begin{aligned} -p\ \nabla \cdot \mathbf{u} \quad &\Longrightarrow \quad \textit{pressure work}, \\ \tau_{ij}\, \mathscr{D}_{ij} \quad &\Longrightarrow \quad \textit{viscous dissipation}. \end{aligned} \tag{5.53}$$

Pressure work expresses, within the local energy balance, the contribution due to dilation or compression processes experienced by the fluid. On the other hand, viscous dissipation accounts for the frictional heat generation caused by the fluid viscosity. On account of equations (5.48) and (5.49), when the focus is on Newtonian fluids, the viscous dissipation term can be expressed as

$$\tau_{ij}\, \mathscr{D}_{ij} = 2\,\mu\, \mathscr{D}_{ij}\, \mathscr{D}_{ij} - \frac{2}{3}\,\mu\, (\nabla \cdot \mathbf{u})^2\ . \tag{5.54}$$

The double, implicit, sum over i and j means that $\mathscr{D}_{ij}\, \mathscr{D}_{ij}$ is expanded into nine terms. Three of them, namely the diagonal ones, yield $(\nabla \cdot \mathbf{u})^2$. The conclusion is that the right hand side of equation (5.54) cannot in any case be negative. Physically, this means that viscous dissipation is a heat source term contribution to the energy balance of the fluid, whose cause is the flow itself. We mention that the viscous dissipation term is often written in the form

$$\tau_{ij}\,\mathscr{D}_{ij} = \mu\,\Phi\ , \tag{5.55}$$

where Φ is called the *dissipation function*,

$$\Phi = 2\,\mathscr{D}_{ij}\,\mathscr{D}_{ij} - \frac{2}{3}\ (\nabla\cdot\mathbf{u})^2\ . \tag{5.56}$$

Equation (5.56) shows that the dissipation function depends only on the velocity field.

An important term of the local energy balance equation (5.46) is that expressing heat diffusion, namely $-\nabla\cdot\mathbf{q}$. This term can be rewritten on account of *Fourier's law*,

$$\mathbf{q} = -\varkappa\,\nabla T\ , \tag{5.57}$$

where $\varkappa$ is the *thermal conductivity* of the fluid. This quantity is a thermodynamic property of the fluid and, in general, it depends on both temperature and pressure.

By taking into account Eqs. (5.50), (5.52), (5.54) and (5.57), one obtains an expression of the local balance equations (5.15), (5.23) and (5.46) for a Newtonian fluid given by

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\,u_j)}{\partial x_j} = 0\ , \tag{5.58}$$

$$\begin{aligned}\rho\left(\frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j}\right) &= \rho\,g_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\\ &\quad -\frac{2}{3}\,\frac{\partial}{\partial x_i}\left(\mu\,\frac{\partial u_j}{\partial x_j}\right)\ ,\end{aligned} \tag{5.59}$$

$$\begin{aligned}\rho\left(\frac{\partial\varphi}{\partial t} + u_j\,\frac{\partial\varphi}{\partial x_j}\right) &= \frac{\partial}{\partial x_j}\left(\varkappa\,\frac{\partial T}{\partial x_j}\right) + q_{\mathrm{g}} - p\,\frac{\partial u_j}{\partial x_j} + 2\,\mu\,\mathscr{D}_{ij}\,\mathscr{D}_{ij}\\ &\quad -\frac{2}{3}\,\mu\left(\frac{\partial u_j}{\partial x_j}\right)^2\ .\end{aligned} \tag{5.60}$$

Equation (5.59) is well known as the *Navier–Stokes equation*.

The information conveyed by Eqs. (5.58)–(5.60) is not enough to determine theoretically the fluid flow. More details are needed such as the interplay between thermodynamic properties, say $\varkappa$ or μ, and the local fluid temperature or pressure. Convection studies often rely on an approximated scheme called the *Oberbeck–Boussinesq approximation*.

5.9 The Oberbeck–Boussinesq Approximation

When convection in fluids takes place, the mass density ρ can be approximated as a reference constant density, ρ_0, except for the gravitational body force term, $\rho\, g_i$ in Eq. (5.59), where the variability of ρ is taken into account. Moreover, both the dynamic viscosity and the thermal conductivity are assumed to be constant. Therefore, the local mass, momentum and energy balance equations can be simplified to

$$\frac{\partial u_j}{\partial x_j} = 0 \,, \tag{5.61}$$

$$\rho_0 \left(\frac{\partial u_i}{\partial t} + u_j\, \frac{\partial u_i}{\partial x_j} \right) = \rho\, g_i - \frac{\partial p}{\partial x_i} + \mu\, \nabla^2 u_i \,, \tag{5.62}$$

$$\rho_0 \left(\frac{\partial \varphi}{\partial t} + u_j\, \frac{\partial \varphi}{\partial x_j} \right) = \varkappa\, \nabla^2 T + q_{\mathrm{g}} + 2\, \mu\, \mathscr{D}_{ij}\, \mathscr{D}_{ij} \,. \tag{5.63}$$

The density ρ, in the term $\rho\, g_i$ is assumed to be a function of temperature only, $\rho(T)$, thus considering the dependence on the pressure as negligible. The linear equation of state

$$\rho(T) = \rho_0 \left[1 - \beta\, (T - T_0) \right] \,, \tag{5.64}$$

is assumed, where β is the *isobaric coefficient of thermal expansion*,

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p \,, \tag{5.65}$$

at the reference temperature T_0. The coefficient β is positive. In Eq. (5.64), the dependence on T is considered sufficiently weak as to be approximated linearly in a neighbourhood of the reference value T_0. This means that Eq. (5.64) can be intended as a Taylor series expansion of ρ around T_0,

$$\rho\,(T) = \rho_0 + \left.\frac{\partial \rho}{\partial T}\right|_{T_0} (T - T_0) + O\big((T - T_0)^2 \big) \,. \tag{5.66}$$

From Eq. (5.64), the expression

$$\rho\, g_i - \frac{\partial p}{\partial x_i}$$

in the local momentum balance equation (5.62) can be rewritten as

$$-\rho_0\, \beta\, (T - T_0)\, g_i - \frac{\partial P}{\partial x_i}$$

where P is called *piezometric head*, and it is defined as

$$P = p - \rho_0 \, g_i \, x_i \; . \tag{5.67}$$

The quantity P is the difference between the pressure p and the hydrostatic pressure $\rho_0 \, g_i \, x_i$. As a consequence, Eqs. (5.61)–(5.63) can be rewritten as

$$\frac{\partial u_j}{\partial x_j} = 0 \; , \tag{5.68}$$

$$\rho_0 \left(\frac{\partial u_i}{\partial t} + u_j \, \frac{\partial u_i}{\partial x_j} \right) = -\rho_0 \, \beta \, (T - T_0) \, g_i - \frac{\partial P}{\partial x_i} + \mu \, \nabla^2 u_i \; , \tag{5.69}$$

$$\rho_0 \left(\frac{\partial \varphi}{\partial t} + u_j \, \frac{\partial \varphi}{\partial x_j} \right) = \varkappa \, \nabla^2 T + q_{\mathrm{g}} + 2 \, \mu \, \mathscr{D}_{ij} \, \mathscr{D}_{ij} \; . \tag{5.70}$$

The term

$$-\rho_0 \, \beta \, (T - T_0) \, g_i$$

is directed vertically, either in the direction of $\mathbf{g}$ or in the opposite direction, depending on the sign of $T - T_0$. Due to this term in the momentum balance, fluid elements with a temperature higher than T_0 are pushed upwards, while fluid elements with a temperature lower than T_0 are pushed downwards. This term is usually called the *buoyancy force* and represents the common sense effect that the lighter (hotter) fluid floats on top of the heavier (cooler) one. When the buoyancy force is negligible with respect to the pressure force,

$$\rho_0 \, \beta \, |T - T_0| \, g \ll |\nabla P| \, ,$$

where g is the modulus of $\mathbf{g}$, the convection flow process is called *forced convection*. In this flow regime, the buoyancy force term can be neglected in Eq. (5.69), so that both Eqs. (5.68) and (5.69) do not contain any contribution of the temperature field. Therefore, in a forced convection problem, these equations can be solved independently of equation (5.70), i.e. the local energy balance equation.

When the buoyancy force cannot be neglected, the convection flow process is called either *free convection*, or *natural convection*, or *mixed convection*, or *combined forced and free convection*, or *buoyant flow*. In this flow regime, the buoyancy force term cannot be neglected in Eq. (5.69), so that this equation contains the temperature field. Therefore, in a buoyant flow problem, Eqs. (5.68)–(5.70) form a system of partial differential equations, so that they cannot be solved separately. We mention that the terms free convection or natural convection are used when the flow is driven only by the buoyancy force.

Thermodynamics ensures that $\varphi = \varphi(T, \rho)$ for every single-phase or two-phase stable equilibrium states [4, 6]. In the special case of a perfect gas, it is well known that $\varphi = \varphi(T)$, so that [6]

$$\mathrm{d}\varphi = c_{\mathrm{v}} \, \mathrm{d}T \; , \tag{5.71}$$

where c_v is the *specific heat* at constant volume. In the case of either a liquid or a real gas, one must rely on the Oberbeck–Boussinesq approximation by assuming that an approximate equation of state $\rho = \rho(T)$ can be applied. This implies that the pressure of the fluid does not change appreciably. Since $\rho = \rho(T)$ and since the pair (T, ρ) yields a unique stable equilibrium state, then one concludes that all the thermodynamic properties may be considered as functions of T. This conclusion holds for the internal energy per unit mass, so that a relationship,

$$\mathrm{d}\varphi = c\,\mathrm{d}T \ , \tag{5.72}$$

can be established. The thermodynamic coefficient c, in general, does not coincide either with c_v or with the specific heat at constant pressure, c_p. In fact, c is the total derivative of the function $\varphi = \varphi(T, \rho(T))$ with respect to T, and not the partial derivative of $\varphi = \varphi(T, \rho)$ with respect to T, when ρ is kept constant. As is well known, the latter is the correct thermodynamic definition of c_v. The equation of state $\rho = \rho(T)$ is one regarding a set of stable equilibrium states of the fluid with approximately the same pressure. Then, one has

$$c = \left(\frac{\partial\varphi}{\partial T}\right)_p \ . \tag{5.73}$$

Equation (5.73) is not the definition of the specific heat at constant pressure c_p. As is well known, c_p is in fact defined as

$$c_\mathrm{p} = \left(\frac{\partial h}{\partial T}\right)_p \ , \tag{5.74}$$

where $h = \varphi + p/\rho$ is the *enthalpy* per unit mass. Then, one can easily write the following relationship:

$$c = c_\mathrm{p} - \frac{p\,\beta}{\rho} \ . \tag{5.75}$$

Then, c is smaller than c_p and differs from c_v, except for the limiting case of a perfect gas. Indeed, in the latter case, one can easily show that the equation of state of the perfect gas and Eq. (5.75) ensures that $c = c_\mathrm{v}$, so that Eqs. (5.71) and (5.72) are perfectly consistent.

One can question about the extent to which c_p and c_v differ from c in the case of liquids. This topic has been examined by Barletta [3]. For water at atmospheric pressure, as well as for most liquids, the assumption

$$c \approx c_\mathrm{p} \tag{5.76}$$

is a definitely reliable one. By using Eq. (5.72), one has

$$\frac{\partial \varphi}{\partial t} = c\,\frac{\partial T}{\partial t}\,, \qquad \frac{\partial \varphi}{\partial x_j} = c\,\frac{\partial T}{\partial x_j}\,, \tag{5.77}$$

so that Eqs. (5.68)–(5.70) can be rewritten as

$$\frac{\partial u_j}{\partial x_j} = 0\,, \tag{5.78}$$

$$\rho_0 \left(\frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j} \right) = -\rho_0\,\beta\,(T - T_0)\,g_i - \frac{\partial P}{\partial x_i} + \mu\,\nabla^2 u_i\,, \tag{5.79}$$

$$\rho_0\,c \left(\frac{\partial T}{\partial t} + u_j\,\frac{\partial T}{\partial x_j} \right) = \varkappa\,\nabla^2 T + q_{\mathrm{g}} + 2\,\mu\,\mathscr{D}_{ij}\,\mathscr{D}_{ij}\,. \tag{5.80}$$

An alternative expression for Eqs. (5.78)–(5.80) is obtained on introducing the *kinematic viscosity*,

$$\nu = \frac{\mu}{\rho_0}\,, \tag{5.81}$$

and the *thermal diffusivity*,

$$\alpha = \frac{\varkappa}{\rho_0\,c}\,. \tag{5.82}$$

Thus, we can write

$$\frac{\partial u_j}{\partial x_j} = 0\,, \tag{5.83}$$

$$\frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j} = -\beta\,(T - T_0)\,g_i - \frac{1}{\rho_0}\,\frac{\partial P}{\partial x_i} + \nu\,\nabla^2 u_i\,, \tag{5.84}$$

$$\frac{\partial T}{\partial t} + u_j\,\frac{\partial T}{\partial x_j} = \alpha\,\nabla^2 T + \frac{q_{\mathrm{g}}}{\rho_0\,c} + 2\,\frac{\nu}{c}\,\mathscr{D}_{ij}\,\mathscr{D}_{ij}\,. \tag{5.85}$$

Typical boundary conditions prescribed for the velocity field are *impermeability* at the surface of a solid wall, meaning that the normal component of **u** is zero at this boundary. Also, the tangential components of velocity must vanish at the surface of a solid wall, and these conditions are termed *no-slip conditions*.

5.10 Governing Equations of Mass Diffusion

In general, the local mass balance equation of a fluid is given by Eq. (5.58). This equation holds either for a fluid with a single chemical constituent or for a multicomponent fluid. However, if one has a multicomponent fluid made up of N different chemical species, one can imagine the fluid as the superposition of N pure fluids coexisting in the same region of space. This means that, instead of a single velocity field **u**, one has to define N velocity fields $\mathbf{u}_n$, where $n = 1, 2, \ldots, N$, one for each constituent. Moreover, one must define a mass density field, C_n, for the nth constituent as

$$C_n = \frac{M_n}{V} , \tag{5.86}$$

where M_n is the mass of the nth constituent contained in an elementary volume V.

The overall fluid mass density is made up by the densities C_n of the constituents,

$$\rho = \sum_{n=1}^{N} C_n . \tag{5.87}$$

It is a common practice calling *concentrations* the mass densities of the N pure fluids.

The overall velocity field **u** of the multicomponent fluid can be defined as a weighted mean value of the velocity fields $\mathbf{u}_n$,

$$\mathbf{u} = \frac{1}{\rho} \sum_{n=1}^{N} C_n \, \mathbf{u}_n . \tag{5.88}$$

5.10.1 Transport Theorem for Mass Diffusion

We know that Eq. (5.58) arises from Reynolds' transport theorem relative to the multicomponent fluid. With reference to the nth pure fluid, the transport theorem can be formulated as

$$\frac{\mathrm{d}\,\Psi_n}{\mathrm{d}\,t} = \iiint\limits_{\mathscr{R}_t} \left[\frac{\partial (C_n \, \psi_n)}{\partial t} + \nabla \cdot (C_n \, \psi_n \, \mathbf{u}_n) \right] \mathrm{d}V , \quad n = 1, 2, \ldots, N , \tag{5.89}$$

where Ψ_n is any extensive property of the nth pure fluid, while ψ_n is the corresponding specific property, i.e. the property obtained from the division of Ψ_n by the mass of the nth constituent. The region $\mathscr{R}_t$ is that occupied by a given multicomponent fluid body at time t. Again, the region $\mathscr{R}_t$ defines a closed thermodynamic system. If $\Psi_n = M_n$ is the mass of the nth chemical species contained in the volume $\mathscr{R}_t$ at time t, then $\psi_n = 1$. Unlike the overall fluid mass M contained in $\mathscr{R}_t$ at time t, the mass M_n is not independent of time since chemical reactions may occur in the fluid system. Then, one may write

$$\frac{\mathrm{d}\,M_n}{\mathrm{d}\,t} = \iiint\limits_{\mathscr{R}_t} \dot{m}_n \, \mathrm{d}V , \tag{5.90}$$

where $\dot{m}_n$ is the local mass production rate per unit volume of the nth chemical species. On substituting $\Psi_n = M_n$ and $\psi_n = 1$ in Eq. (5.89), one obtains

$$\iiint\limits_{\mathscr{R}_t} \left[\frac{\partial C_n}{\partial t} + \boldsymbol{\nabla} \cdot (C_n \, \mathbf{u}_n) - \dot{m}_n \right] \mathrm{d}V = 0, \quad n = 1, 2, \ldots, N \; . \tag{5.91}$$

Since the region $\mathscr{R}_t$ has been chosen arbitrarily, the integral conditions can be satisfied only if the integrand vanishes locally at every position in the domain occupied by the fluid, namely if the local mass balance equations,

$$\frac{\partial C_n}{\partial t} + \boldsymbol{\nabla} \cdot (C_n \, \mathbf{u}_n) - \dot{m}_n = 0 \; , \quad n = 1, 2, \ldots, N \; , \tag{5.92}$$

hold. By summing all the N Eqs. (5.92), one obtains

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \, \mathbf{u}) - \sum_{n=1}^{N} \dot{m}_n = 0 \; , \tag{5.93}$$

where Eqs. (5.87) and (5.88) have been invoked. A comparison between Eqs. (5.93) and (5.58) allows us to infer that

$$\sum_{n=1}^{N} \dot{m}_n = 0 \; . \tag{5.94}$$

Equation (5.94) implies that the chemical reactions possibly occurring in the fluid system do not yield either a source or a sink for the overall fluid mass, i.e. the overall fluid mass is locally conserved.

5.10.2 Concentrations and Mass Fluxes

Mass diffusion occurs within the fluid when, locally, the overall fluid velocity **u** differs from the velocity $\mathbf{u}_n$ of the nth constituent. Then, if the mass diffusion takes place, one may define a *mass flux* for each constituent,

$$\mathbf{J}_n = C_n \, (\mathbf{u}_n - \mathbf{u}) \;\; , \quad n = 1, 2, \ldots, N \; . \tag{5.95}$$

As a consequence, Eq. (5.92) yields

$$\frac{\partial C_n}{\partial t} + \boldsymbol{\nabla} \cdot (C_n \, \mathbf{u}) + \boldsymbol{\nabla} \cdot \mathbf{J}_n - \dot{m}_n = 0 \; , \quad n = 1, 2, \ldots, N \; , \tag{5.96}$$

that can be rewritten as

$$\frac{\partial C_n}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} C_n + C_n \; \boldsymbol{\nabla} \cdot \mathbf{u} + \boldsymbol{\nabla} \cdot \mathbf{J}_n - \dot{m}_n = 0 \; , \quad n = 1, 2, \ldots, N \; . \tag{5.97}$$

5.10.3 The Oberbeck–Boussinesq Approximation

The Oberbeck–Boussinesq approximation is based on the assumption that the overall fluid density ρ may be treated as a constant in the local mass, momentum and energy balance equations of the fluid, except for the gravitational body force in the momentum balance. This exception allows one to model the density changes through the effects of the buoyancy force. When mass diffusion takes place, the overall fluid density ρ is considered not only as a function of the temperature T, but also of $N-1$ concentrations, $C_1, C_2, \ldots, C_{N-1}$, treated as independent variables. The latter conclusion may be inferred from Eq. (5.87). The fluid density ρ may be either considered as a function of $C_1, C_2, \ldots, C_N$ assumed as independent, or as a function of the independent variables $T, C_1, C_2, \ldots, C_{N-1}$. In fact, each concentration C_n is a function of T.

With small density changes occurring within the fluid, one can express ρ as a linear function of the independent variables $T, C_1, C_2, \ldots, C_{N-1}$,

$$\rho = \rho_0 \left[1 - \beta \left(T - T_0 \right) - \sum_{n=1}^{N-1} \beta_{C,n} \left(C_n - C_{n,0} \right) \right] , \tag{5.98}$$

where T_0 is the reference temperature, $C_{n,0}$ is the reference concentration of the nth component, ρ_0 is the overall fluid density evaluated in the reference state, while

$$\beta = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} , \qquad \beta_{C,n} = -\frac{1}{\rho} \frac{\partial \rho}{\partial C_n} , \qquad n = 1, 2, \ldots, N-1 , \tag{5.99}$$

are the isobaric coefficient of thermal expansion and the *concentration expansion coefficients*, respectively. While β is positive, the coefficients $\beta_{C,n}$ can be either positive or negative. In Eq. (5.99), these coefficients are evaluated at the reference conditions $T = T_0$ and $C_n = C_{n,0}$.

On the basis of equation (5.98), one can express the approximated momentum balance. Therefore, the set of local balance equations to be solved in the Oberbeck–Boussinesq approximation are

$$\nabla \cdot \mathbf{u} = 0 , \tag{5.100}$$

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \cdot \nabla \right) \mathbf{u} = & -\frac{1}{\rho_0} \nabla P - \beta \left(T - T_0 \right) \mathbf{g} \\ & - \sum_{n=1}^{N-1} \beta_{C,n} \left(C_n - C_{n,0} \right) \mathbf{g} + \nu \nabla^2 \mathbf{u} , \end{aligned} \tag{5.101}$$

$$\frac{\partial T}{\partial t} + \left(\mathbf{u} \cdot \nabla \right) T = \alpha \nabla^2 T + \frac{q_g}{\rho_0 c} + \frac{\nu}{c} \Phi , \tag{5.102}$$

$$\frac{\partial C_n}{\partial t} + \left(\mathbf{u} \cdot \nabla \right) C_n = -\nabla \cdot \mathbf{J}_n + \dot{m}_n, \qquad n = 1, 2, \ldots, N-1 . \tag{5.103}$$

We note that instead of the N equations (5.103), we have written just $N-1$ equations. The reason is that the overall fluid mass balance equation (5.100) and the N mass balance equations for the constituents of the multicomponent fluid are not independent, as a consequence of the constraint equation (5.87).

The local momentum balance equation (5.101) displays two kinds of buoyancy force: a *thermal buoyancy* induced by a possibly non-uniform temperature field,

$$-\beta\,(T-T_0)\,\mathbf{g}\;,$$

and a *mass diffusion buoyancy* induced by possibly non-uniform concentrations of the chemical components,

$$-\sum_{n=1}^{N-1}\beta_{C,n}\left(C_n-C_{n,0}\right)\mathbf{g}\;.$$

5.10.4 A Two-Component Mixture and Fick's Law

Here, we refer for simplicity to a two-component fluid ($N=2$). Since $N=2$, we have two concentrations C_1 and C_2, but just one of them is an independent variable as a consequence of equation (5.87). We denote this concentration as C and the corresponding mass flux as $\mathbf{J}$.

Thus, from Eqs. (5.100)–(5.103), we have four local balance equations that we can write, for convenience, according to Einstein's notation

$$\frac{\partial u_j}{\partial x_j}=0\;, \tag{5.104}$$

$$\begin{aligned}\frac{\partial u_i}{\partial t}+u_j\,\frac{\partial u_i}{\partial x_j}&=-\beta\,(T-T_0)\,g_i\\&\quad-\beta_C\,(C-C_0)\,g_i-\frac{1}{\rho_0}\,\frac{\partial P}{\partial x_i}+\nu\,\nabla^2u_i\;,\end{aligned} \tag{5.105}$$

$$\frac{\partial T}{\partial t}+u_j\,\frac{\partial T}{\partial x_j}=\alpha\,\nabla^2T+\frac{q_g}{\rho_0\,c}+2\,\frac{\nu}{c}\,\mathscr{D}_{ij}\,\mathscr{D}_{ij}\;, \tag{5.106}$$

$$\frac{\partial C}{\partial t}+u_j\,\frac{\partial C}{\partial x_j}=-\frac{\partial J_j}{\partial x_j}+\dot{m}\;. \tag{5.107}$$

We note that the set of four local balance equations (5.104)–(5.107) can be solved to determine the four unknown fields u_i, p, T and C. However, we have a further unknown: the mass flux J_i. Hence, the differential problem is under-determined (there are more unknowns than equations).

In 1855, a German physiologist and physicist, Adolf Eugen Fick, obtained a phenomenological law establishing a relationship between J_i and C. This law is now well known as *Fick's law* and can be formulated as [2]

$$J_i = -\alpha_\mathrm{m} \, \frac{\partial C}{\partial x_i} \, , \tag{5.108}$$

where the positive quantity α_m is the *mass diffusivity*. The mass diffusivity is a thermodynamic property of the two-component mixture and, coherently with the Oberbeck–Boussinesq approximation, it is considered as a constant. Then Eq. (5.107) can be rewritten as

$$\frac{\partial C}{\partial t} + u_j \, \frac{\partial C}{\partial x_j} = \alpha_\mathrm{m} \, \nabla^2 C + \dot{m} \, . \tag{5.109}$$

In this form, the local mass balance equation for the component with concentration C exploits the strong mathematical analogy between the heat diffusion, described by Eq. (5.106), and mass diffusion, described by Eq. (5.109).

Here, we have three fluid properties, α, α_m and ν having the same units, and thus defining two characteristic dimensionless ratios, the *Prandtl number* and the *Schmidt number*,

$$Pr = \frac{\nu}{\alpha} \, , \quad Sc = \frac{\nu}{\alpha_\mathrm{m}} \, . \tag{5.110}$$

5.11 Local Entropy Balance Equation

The entropy $\mathscr{S}$ of the fluid body contained in the region of space $\mathscr{R}_t$ is an extensive property. If we denote with s the specific entropy, then Reynolds' transport theorem, Eq. (5.14), yields

$$\frac{\mathrm{d}\,\mathscr{S}}{\mathrm{d}\,t} = \iiint\limits_{\mathscr{R}_t} \rho \, \frac{\mathrm{D}s}{\mathrm{D}\,t} \, \mathrm{d}V \, . \tag{5.111}$$

We rely on *Gibbs' equation* for an expression of ds [6],

$$\mathrm{d}s = \frac{1}{T} \, \mathrm{d}\varphi - \frac{p}{\rho^2 \, T} \, \mathrm{d}\rho \, . \tag{5.112}$$

From Eq. (5.112), one obtains an expression for the substantial derivative of s,

$$\frac{\mathrm{D}s}{\mathrm{D}\,t} = \frac{1}{T} \, \frac{\mathrm{D}\varphi}{\mathrm{D}\,t} - \frac{p}{\rho^2 \, T} \, \frac{\mathrm{D}\rho}{\mathrm{D}\,t} = \frac{1}{T} \, \frac{\mathrm{D}\varphi}{\mathrm{D}\,t} + \frac{p}{\rho \, T} \, \boldsymbol{\nabla} \cdot \mathbf{u} \, , \tag{5.113}$$

where Eqs. (5.10) and (5.13) have been employed. Substitution of Eq. (5.113) into Eq. (5.111) yields

$$\frac{\mathrm{d}\mathscr{S}}{\mathrm{d}t} = \iiint\limits_{\mathscr{R}_t} \left(\frac{\rho}{T}\,\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \frac{p}{T}\,\boldsymbol{\nabla}\cdot\mathbf{u} \right) \mathrm{d}V \ . \tag{5.114}$$

The second law of thermodynamics leads to an entropy balance equation where the infinitesimal change of entropy in a process is the sum of an entropy flux and an entropy production due to irreversibility [6]. Thus, we can write

$$\frac{\mathrm{d}\mathscr{S}}{\mathrm{d}t} = -\iint\limits_{\partial\mathscr{R}_t} \frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}S + \dot{\mathscr{S}}_{\mathrm{irr}} \ , \tag{5.115}$$

where $\dot{\mathscr{S}}_{\mathrm{irr}}$ is the *entropy production rate*. As the entropy is an extensive property, the entropy production rate can be expressed through a volume integral,

$$\dot{\mathscr{S}}_{\mathrm{irr}} = \iiint\limits_{\mathscr{R}_t} \sigma_{\mathrm{irr}}\,\mathrm{d}V \ . \tag{5.116}$$

Here, σ_{irr} represents the entropy production rate per unit volume. Its local value cannot be negative, due to the second law of thermodynamics,

$$\sigma_{\mathrm{irr}} \geqslant 0 \ , \tag{5.117}$$

where the equal sign implies reversibility. We also mention that the minus sign in front of the surface integral on the right-hand side of equation (5.115) is motivated by the need to express the incoming entropy flux across the boundary, whereas $\mathbf{n}$ is the outward normal to $\partial\mathscr{R}_t$.

By employing Gauss' theorem and Eq. (5.116), Eqs. (5.114) and (5.115) yield

$$\iiint\limits_{\mathscr{R}_t} \left(\frac{\rho}{T}\,\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \frac{p}{T}\,\boldsymbol{\nabla}\cdot\mathbf{u} - \frac{1}{T^2}\,\mathbf{q}\cdot\boldsymbol{\nabla}T + \frac{1}{T}\,\boldsymbol{\nabla}\cdot\mathbf{q} - \sigma_{\mathrm{irr}} \right) \mathrm{d}V = 0 \ . \tag{5.118}$$

We now invoke the local energy balance equation (5.46) and Eq. (5.52), so that we obtain

$$\iiint\limits_{\mathscr{R}_t} \left(\frac{q_{\mathrm{g}}}{T} + \frac{1}{T}\,\tau_{ij}\,\mathscr{D}_{ij} - \frac{1}{T^2}\,q_j\,\frac{\partial T}{\partial x_j} - \sigma_{\mathrm{irr}} \right) \mathrm{d}V = 0 \ . \tag{5.119}$$

As usual, Eq. (5.119) can be satisfied with an arbitrary domain of integration $\mathscr{R}_t$ if the integrand is zero, namely if

$$\sigma_{\mathrm{irr}} = \frac{q_{\mathrm{g}}}{T} + \frac{1}{T}\,\tau_{ij}\,\mathscr{D}_{ij} - \frac{1}{T^2}\,q_j\,\frac{\partial T}{\partial x_j} \ . \tag{5.120}$$

Equation (5.120) is the *local entropy balance equation*. Due to Eq. (5.117), the local entropy balance is effectively an inequality,

$$\frac{q_{\mathrm{g}}}{T} + \frac{1}{T}\,\tau_{ij}\,\mathscr{D}_{ij} - \frac{1}{T^2}\,q_j\,\frac{\partial T}{\partial x_j} \geqslant 0\,. \tag{5.121}$$

If one deals with a Newtonian fluid and if Fourier's law (5.57) is employed, the contributions of viscous dissipation, $\tau_{ij}\,\mathscr{D}_{ij}/T$, and of heat diffusion, $-(q_j\,\partial T/\partial x_j)/T^2$, are both non-negative. In this situation, the inequality (5.121) becomes an effective restriction only if one has to model heat sinks ($q_{\mathrm{g}} < 0$). This situation may occur, for instance, in the case of endothermic chemical reactions.

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