Chapter 3
Large-time Behaviour of Wave Packets

3.1 What is a Holomorphic Function?

The main elements of the theory of the functions of a complex variable can be found in many textbooks. Among these, the treatment presented in this section, as well as in Sects. 3.2 and 3.3, mainly follows the much more extended presentations of this topic available in Cartan [6], in Priestley [8], and in chapter 11 of Arfken et al. [4].

Let us recall the definition of set $\mathbb{C}$. The set $\mathbb{C}$ coincides with $\mathbb{R}^2$, in the sense that to every pair $(x, y) \in \mathbb{R}^2$ there corresponds one and only one complex number $z \in \mathbb{C}$ defined as

$$z = x + \mathrm{i}\, y \ . \tag{3.1}$$

The real number x is called the real part of z, while real number y is called the imaginary part of z,

$$x = \Re(z) \ , \quad y = \Im(z) \ . \tag{3.2}$$

The set $\mathbb{R}^2$ is also called the *complex plane*.

Unlike $\mathbb{R}^2$, the set $\mathbb{C}$ is structured as a field. This means that it has not only an inner operation of sum between any two elements, already present in $\mathbb{R}^2$, but also an operation of product, not present in $\mathbb{R}^2$. The product is defined as

$$\begin{cases} z_1 = x_1 + \mathrm{i}\, y_1 \ , \\ z_2 = x_2 + \mathrm{i}\, y_2 \ , \end{cases} \longmapsto z_1 z_2 = x_1 x_2 - y_1 y_2 + \mathrm{i}\,(x_1 y_2 + y_1 x_2) \ . \tag{3.3}$$

A special element of $\mathbb{C}$ is i, called the *imaginary unit*. On account of the definition of product between any two complex numbers, given by Eq. (3.3), the product of the imaginary unit and itself, i^2, is equal to -1.

To every complex number z, there corresponds one and only one complex number $\overline{z}$, called the *complex conjugate* of z and defined as

$$z = x + \mathrm{i}\, y \ , \quad \longmapsto \quad \overline{z} = x - \mathrm{i}\, y \ . \tag{3.4}$$

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The product of z and $\overline{z}$ is a real number called the square modulus of z,

$$z = x + \mathrm{i}\, y\ , \quad \longmapsto \quad z\,\overline{z} = x^2 + y^2 \equiv |z|^2\ . \tag{3.5}$$

The exponential of any $z \in \mathbb{C}$ is defined as the power series

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}\ . \tag{3.6}$$

The main property of the exponential function is

$$\mathrm{e}^z\, \mathrm{e}^w = \mathrm{e}^{z+w}\ , \quad \forall z, w \in \mathbb{C}\ . \tag{3.7}$$

The imaginary exponential function is an application $\mathbb{R} \to \mathbb{C}$ defined as $\theta \to e^{\mathrm{i}\,\theta}$. This function satisfies *Euler's formula*,

$$e^{\mathrm{i}\,\theta} = \cos\theta + \mathrm{i}\ \sin\theta\ , \quad \forall\theta \in \mathbb{R}\ . \tag{3.8}$$

Since $\sin^2\theta + \cos^2\theta = 1$, Eq. (3.8) allows one to express any complex number with modulus 1. Then,

$$\forall z \in \mathbb{C}\ , \quad \frac{z}{|z|} = \cos\theta + \mathrm{i}\ \sin\theta = e^{\mathrm{i}\,\theta}\ . \tag{3.9}$$

Equation (3.9) associates a real number θ to any complex number z. This real number is called the *argument* of z, i.e. $\theta = \arg(z)$. However, since the sine and cosine functions are periodic with period 2π, $\arg(z)$ is defined only up to integer multiples of 2π. Therefore, Eq. (3.9) gives rise to the so-called *polar representation* of a complex number,

$$\forall z \in \mathbb{C}\ , \quad z = |z|\ e^{\mathrm{i}\ \arg(z)} = |z| \Big\{ \cos[\arg(z)] + \mathrm{i}\ \sin[\arg(z)] \Big\}\ . \tag{3.10}$$

Since $\arg(z)$ is defined only up to integer multiples of $2\,\pi$, it is not strictly speaking an application $\mathbb{C} \to \mathbb{R}$, but the so-called multifunction, or *multivalued function*. In fact, for a given $z \in \mathbb{C}$, $\arg(z)$ can be a real number in the interval $[-\pi, \pi]$, and a real number in the interval $[\pi, 3\pi]$, and a real number in the interval $[-3\pi, -\pi]$, … . The terms of this infinite sequence of real numbers can be obtained by adding $2\pi k$, with $k \in \mathbb{Z}$, to the first real number (the value in the interval $[-\pi, \pi]$). The value in the interval $[-\pi, \pi]$ is called the *principal branch* of $\arg(z)$.

Another important multivalued function is the (natural) logarithm of z, defined as the inverse function of e^z. From Eq. (3.8), the main property of the logarithm is

$$\ln(z\, w) = \ln(z) + \ln(w)\ , \quad \forall z, w \in \mathbb{C}\ . \tag{3.11}$$

From Eqs. (3.10) and (3.11), one obtains

$$\ln(z) = \ln(|z|) + \mathrm{i}\ \arg(z)\ . \tag{3.12}$$

Equation (2.13) shows that the logarithm of z is a multivalued function $\mathbb{C} \to \mathbb{C}$. We have a principal branch of $\ln(z)$ defined by considering the principal branch of the argument of z, i.e. $\arg(z) \in [-\pi, \pi]$.

Example 3.1 In order to evaluate the logarithm of -1, we have just to recognise, from Eq. (3.8), that

$$-1 = \mathrm{e}^{\mathrm{i}\,(\pi+2\pi k)}\ , \quad \forall k \in \mathbb{Z}\ . \tag{3.13}$$

Then, we deduce that

$$\ln(-1) = \mathrm{i}\,\pi + 2\mathrm{i}\,\pi k\ , \quad \forall k \in \mathbb{Z}\ . \tag{3.14}$$

The principal branch value of $\ln(-1)$ is $\mathrm{i}\,\pi$.

3.1.1 Derivative of a Complex-Valued Function

The metric structure in $\mathbb{C}$ defined by the distance $|z - w|$ between any two complex numbers z and w allows us to extend the notions of limit and continuity defined in the elementary analysis of real functions. These notions are formally identical to those of the real analysis. The same holds for the notions of derivative and differentiability. A function $f : \mathscr{D} \to \mathbb{C}$, where $\mathscr{D}$ is an open connected subset of $\mathbb{C}$, is said to be differentiable at a point $z_0 \in \mathscr{D}$ if

$$\lim_{\substack{z\to z_0 \\ z\in\mathscr{D}}} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \in \mathbb{C}\ . \tag{3.15}$$

This means that, on considering $f(z) = f(x, y)$ where $x = \Re(z)$ and $y = \Im(z)$, there exists the double limit

$$\lim_{\substack{h_1\to 0, h_2\to 0 \\ h_1, h_2\in\mathbb{R}}} \frac{f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0)}{h_1 + i\,h_2} = f'(z_0) \in \mathbb{C}\ , \tag{3.16}$$

where $x_0 = \Re(z_0)$ and $y_0 = \Im(z_0)$. Obviously, the real numbers h_1, h_2 must be chosen as sufficiently small so that $(x_0 + h_1) + \mathrm{i}\,(y_0 + h_2) \in \mathscr{D}$. For the limit in the left-hand side of Eq. (3.16) to exist, its value $f'(z_0)$ must be independent of the special way it is evaluated. For instance, one may evaluate the limit by keeping $h_2 = 0$, so that

$$f'(z_0) = \lim_{\substack{h_1\to 0 \\ h_1\in\mathbb{R}}} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} = \left.\frac{\partial f(x, y)}{\partial x}\right|_{(x,y)=(x_0,y_0)}\ . \tag{3.17}$$

Alternatively, one may evaluate the limit by keeping $h_1 = 0$, so that

$$\begin{aligned} f'(z_0) &= \lim_{\substack{h_2 \to 0 \\ h_2 \in \mathbb{R}}} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{\mathrm{i}\, h_2} \\ &= \frac{1}{\mathrm{i}} \left. \frac{\partial f(x, y)}{\partial y} \right|_{(x,y)=(x_0,y_0)} = -\,\mathrm{i} \left. \frac{\partial f(x, y)}{\partial y} \right|_{(x,y)=(x_0,y_0)} . \end{aligned} \tag{3.18}$$

From Eqs. (3.17) and (3.18), one may easily infer that, if $f(z) = f(x, y)$ is differentiable at $z_0 = x_0 + \mathrm{i}\, y_0 \in \mathscr{D}$, then

$$\left. \frac{\partial f(x, y)}{\partial x} \right|_{(x,y)=(x_0,y_0)} + \mathrm{i} \left. \frac{\partial f(x, y)}{\partial y} \right|_{(x,y)=(x_0,y_0)} = 0 . \tag{3.19}$$

Definition 3.1 If $\mathscr{D}$ is an open connected subset of $\mathbb{C}$, a function $f : \mathscr{D} \to \mathbb{C}$ is *holomorphic* in $\mathscr{D}$ if it is differentiable at every point $z_0 \in \mathscr{D}$.

We note that a holomorphic function $f(z)$ has a very important feature. Let $f(z) = f(x, y)$, with $z = x + \mathrm{i}\, y$, and let $f(x, y) = u(x, y) + \mathrm{i}\, v(x, y)$, where u and v are real-valued functions. Then, Eq. (3.19) implies that

$$\frac{\partial f(x, y)}{\partial x} + \mathrm{i}\, \frac{\partial f(x, y)}{\partial y} = 0 , \tag{3.20}$$

namely

$$\begin{aligned} &\frac{\partial}{\partial x} [u(x, y) + \mathrm{i}\, v(x, y)] + \mathrm{i}\, \frac{\partial}{\partial y} [u(x, y) + \mathrm{i}\, v(x, y)] = 0 , \\ &\frac{\partial u(x, y)}{\partial x} - \frac{\partial v(x, y)}{\partial y} + \mathrm{i} \left[\frac{\partial v(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} \right] = 0 . \end{aligned} \tag{3.21}$$

From Eq. (3.21), one easily proves the following theorem.

Theorem 3.1 (Cauchy–Riemann equations) *Let $\mathscr{D}$ be an open connected subset of $\mathbb{C}$, and $f : \mathscr{D} \to \mathbb{C}$ be holomorphic in $\mathscr{D}$ with $f(z) = f(x, y) = u(x, y) + \mathrm{i}\, v(x, y)$, where u and v are real-valued. Then, the Cauchy–Riemann equations hold,*

$$\frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} , \quad \frac{\partial v(x, y)}{\partial x} = -\frac{\partial u(x, y)}{\partial y} . \tag{3.22}$$

Equation (3.22) reveals that a holomorphic function $f(z)$ is something more than a mere representation of a differentiable function $f(x, y)$ in an open subset of $\mathbb{R}^2$. On account of the definition of complex conjugation, we have

$$\begin{aligned} z = x + \mathrm{i}\, y , \quad \overline{z} = x - \mathrm{i}\, y , \\ x = \frac{z + \overline{z}}{2} , \quad y = \frac{z - \overline{z}}{2\,\mathrm{i}} , \end{aligned} \tag{3.23}$$

so that

$$\begin{aligned}\frac{\partial f(x,y)}{\partial \bar{z}} &= \frac{\partial x}{\partial \bar{z}}\frac{\partial f(x,y)}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial f(x,y)}{\partial y} \\ &= \frac{1}{2}\frac{\partial f(x,y)}{\partial x} - \frac{1}{2\,\mathrm{i}}\frac{\partial f(x,y)}{\partial y} = \frac{1}{2}\left[\frac{\partial f(x,y)}{\partial x} + \mathrm{i}\,\frac{\partial f(x,y)}{\partial y}\right].\end{aligned} \tag{3.24}$$

On account of Eqs. (3.20) and (3.24), we conclude that, when a differentiable function in an open subset of $\mathbb{R}^2$, $f(x, y)$, defines a holomorphic function in an open subset of $\mathbb{C}$, then $f(x, y)$ depends on z, but it cannot depend on the complex conjugate of z, namely

$$\frac{\partial f(x,y)}{\partial \bar{z}} = 0. \tag{3.25}$$

In a completely symmetric way, one can prove that if a differentiable function, $f(x, y)$, in an open subset of $\mathbb{R}^2$ defines a holomorphic function in an open subset of $\mathbb{C}$, then $f(x, y)$ can depend on $\bar{z}$, but it cannot depend on z.

Example 3.2 We can easily prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x, y) = (x^2 + y^2, 1 - x^2 - y^2)$ does not define a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. In fact, $f(x, y) = (x^2 + y^2, 1 - x^2 - y^2)$ is differentiable in $\mathbb{R}^2$. However, $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z\,\bar{z} + \mathrm{i}\,(1 - z\,\bar{z})$ cannot be a holomorphic function. In fact, f depends on both z and $\bar{z}$, so that, in particular, Eq. (3.25) is not satisfied.

Let us define a *harmonic function* as a twice differentiable function $f(x, y)$ with a vanishing Laplacian, namely

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0\,. \tag{3.26}$$

In other words, a harmonic function is any solution of Laplace's equation (3.26).

A general theorem can be proved.

Theorem 3.2 *A twice differentiable function $f(z, \bar{z})$ is harmonic if and only if it is the sum of a holomorphic function of z and a holomorphic function of $\bar{z}$.*

The proof of this theorem is as follows. Let us first assume that

$$f(z, \bar{z}) = F(z) + G(\bar{z})\,, \tag{3.27}$$

where $F(z)$ and $G(\bar{z})$ are differentiable. Then, from Eq. (3.23), we have

$$0 = \frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial F}{\partial x} - \frac{1}{2\,\mathrm{i}}\frac{\partial F}{\partial y}\,. \tag{3.28}$$

Thus, we have

$$\frac{\partial F}{\partial x} = -\,\mathrm{i}\,\frac{\partial F}{\partial y}\,. \tag{3.29}$$

As a consequence,

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -\,\mathrm{i}\,\frac{\partial^2 F}{\partial x\,\partial y} + \mathrm{i}\,\frac{\partial^2 F}{\partial y\,\partial x} = 0\ . \tag{3.30}$$

Moreover, we have

$$\frac{\partial G}{\partial z} = 0\ , \tag{3.31}$$

so that we obtain, by employing Eq. (3.23),

$$0 = \frac{\partial G}{\partial z} = \frac{\partial G}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial G}{\partial x} + \frac{1}{2\,\mathrm{i}}\frac{\partial G}{\partial y}\ . \tag{3.32}$$

Thus, we can write

$$\frac{\partial G}{\partial x} = \mathrm{i}\,\frac{\partial G}{\partial y}\ . \tag{3.33}$$

As a consequence,

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \mathrm{i}\,\frac{\partial^2 G}{\partial x\,\partial y} - \mathrm{i}\,\frac{\partial^2 G}{\partial y\,\partial x} = 0\ . \tag{3.34}$$

Therefore, we can conclude that $F(z) + G(\bar{z})$ is a harmonic function. Conversely, let us now assume that $f(z,\bar{z})$ is harmonic. Then, we can express

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right)\\ &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right) + \frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right)\\ &= \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial \bar{z}^2} + 2\,\frac{\partial^2 f}{\partial z\,\partial \bar{z}}\ ,\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y}\right) = \mathrm{i}\,\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)\\ &= -\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) + \frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)\\ &= -\frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial \bar{z}^2} + 2\,\frac{\partial^2 f}{\partial z\,\partial \bar{z}}\ .\end{aligned} \tag{3.36}$$

Therefore,

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\,\frac{\partial^2 f}{\partial z\,\partial \bar{z}}\ , \tag{3.37}$$

a condition which can be satisfied if and only if f is the sum of a function of z and a function of $\bar{z}$, namely

$$f(z, \bar{z}) = F(z) + G(\bar{z}) \ . \tag{3.38}$$

3.1.2 Path Integration in $\mathbb{C}$

A path or a contour in $\mathbb{C}$ is nothing but an oriented open or closed curve in the complex plane. Mathematically, a path in $\mathbb{C}$ is defined by a differentiable application $\gamma : [t_1, t_2] \to \mathbb{C}$, $\gamma = \gamma(t)$, where $[t_1, t_2] \subseteq \mathbb{R}$ is a real interval. Then, $\gamma(t)$ is the parametrisation of the path. For simplicity of notation, we will denote the path with the same symbol γ of its parametrisation. The *path integral* on γ of a function $f(z)$ is defined as

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{t_1}^{t_2} f[\gamma(t)]\,\gamma'(t)\,\mathrm{d}t \ . \tag{3.39}$$

It may be objected that the result of a path integration on a given oriented curve in the complex plane may be dependent on the chosen parametrisation of that curve. In fact, it may be proved that, under suitable conditions, two different parametrisations yield the same contour integral. In the case of closed contours γ, the mentioned suitable conditions mainly depend on the so-called winding number of the contour.

Example 3.3 To illustrate this point, let us evaluate

$$\int_\gamma \frac{\mathrm{d}z}{z} \ , \tag{3.40}$$

where γ is the unit circle centred in $z = 0$ and oriented counterclockwise. A parametrisation of γ can be given by

$$\gamma(\theta) = \cos\theta + \mathrm{i}\,\sin\theta \ , \quad \theta \in [0, 2\pi] \ . \tag{3.41}$$

On account of Eq. (3.8), one may equivalently write

$$\gamma(\theta) = \mathrm{e}^{\mathrm{i}\theta}, \quad \theta \in [0, 2\pi] \ . \tag{3.42}$$

Then, on account of Eq. (3.39), one has

$$\int_\gamma \frac{\mathrm{d}z}{z} = \mathrm{i}\int_0^{2\pi} \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi} \mathrm{d}\theta = 2\pi\,\mathrm{i} \ . \tag{3.43}$$

We note that one could have also employed other parametrisations of the unit circle, such as

$$\gamma(\theta) = \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\,\sin\theta, \quad \theta \in [0, 4\pi]\,, \tag{3.44}$$

or

$$\gamma(\theta) = \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\,\sin\theta, \quad \theta \in [0, 6\pi]\,. \tag{3.45}$$

The result of the integration would have been $4\pi i$ in the first case and $6\pi i$ in the second case. However, the *winding number* of the parametrisation defined on $[0, 4\pi]$ is 2, and the winding number of the parametrisation defined on $[0, 6\pi]$ is 3. This means that, in the first case, the point $z = \gamma(t)$ undergoes two complete turns around $z = 0$ and, in the second case, three complete counterclockwise turns around $z = 0$.

Incidentally, on relaxing the assumption of counterclockwise orientation of the path γ, one can devise both positive and negative winding numbers. The latter concept being relative to clockwise-oriented closed paths.

Here and in the following, if not differently specified, we will always assume that the winding number of a closed path is 1.

3.1.3 Homotopy

We consider an open connected subset $\mathscr{D} \subseteq \mathbb{C}$ and two closed paths γ_1 and γ_2 both oriented counterclockwise, or both oriented clockwise. If there exists a continuous map,

$$\Lambda : \mathscr{D} \times [0, 1] \to \mathscr{D}\,, \tag{3.46}$$

such that

$$\gamma_1(t) = \Lambda[\gamma_1(t), 0], \quad \gamma_2(t) = \Lambda[\gamma_1(t), 1]\,, \tag{3.47}$$

for every t, then γ_1 and γ_2 are homotopic. In other words, γ_1 and γ_2 are said to be homotopic in $\mathscr{D}$ if γ_1 can be continuously deformed into γ_2.

A special case is that of an oriented closed path γ which is homotopic in $\mathscr{D}$ to a point $z_0 \in \mathscr{D}$. In this case, γ can be continuously shrunk to a point z_0.

Theorem 3.3 *Let us consider an open connected subset $\mathscr{D} \subseteq \mathbb{C}$ and two closed paths $\gamma_1 \subseteq \mathscr{D}$ and $\gamma_2 \subseteq \mathscr{D}$ both oriented counterclockwise, or both oriented clockwise. If $f : \mathscr{D} \to \mathbb{C}$ is holomorphic, and if γ_1 and γ_2 are homotopic, then*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,. \tag{3.48}$$

Corollary 3.1 *Let us consider an open connected subset $\mathscr{D} \subseteq \mathbb{C}$ and a closed path $\gamma \subseteq \mathscr{D}$ homotopic to a point $z_0 \in \mathscr{D}$. If $f : \mathscr{D} \to \mathbb{C}$ is holomorphic, then*

$$\int_{\gamma} f(z)\,\mathrm{d}\,z = 0\,. \tag{3.49}$$

We note that the thesis of Corollary 3.1 is not incompatible with the result obtained working out the example regarding function $f(z) = 1/z$. In fact, in that exercise, the function $f(z) = 1/z$ is holomorphic in the punctured complex plane $\mathscr{D} = \mathbb{C} \setminus \{0\}$, due to the singularity in $z = 0$. Then, the unit circle centred in $z = 0$ is included in $\mathscr{D}$ while the point $z = 0$ is not. Thus, one cannot even question about the homotopy in $\mathscr{D}$ of the unit circle and the point $z = 0$.

If $\mathscr{D} \subseteq \mathbb{C}$ is open and connected, and if every closed path γ in $\mathscr{D}$ is homotopic to a point in $\mathscr{D}$, then $\mathscr{D}$ is called *simply connected*. Obviously, the punctured complex plane $\mathbb{C} \setminus \{0\}$ is not simply connected.

Corollary 3.2 *Let us consider a simply connected subset $\mathscr{D} \subseteq \mathbb{C}$ and a closed path $\gamma \subseteq \mathscr{D}$. If $f : \mathscr{D} \to \mathbb{C}$ is holomorphic, then*

$$\int_{\gamma} f(z)\,\mathrm{d}\,z = 0\,. \tag{3.50}$$

3.2 Laurent Expansions, Singular Points

Let us consider an annulus,

$$\mathscr{A} = \{z \in \mathbb{C} : R_1 < |z| < R_2\}\,. \tag{3.51}$$

A function $f : \mathscr{A} \to \mathbb{C}$ has a *Laurent expansion* in $\mathscr{A}$ if there exists a power series,

$$\sum_{n=-\infty}^{\infty} a_n\, z^n = \cdots + \frac{a_{-n}}{z^n} + \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1\, z + a_2\, z^2 + \cdots + a_n\, z^n + \cdots\,, \tag{3.52}$$

that converges in $\mathscr{A}$ and whose sum coincides with $f(z)$ for every $z \in \mathscr{A}$.

Theorem 3.4 *Any holomorphic function in an annulus $\mathscr{A}$, defined by Eq. (3.51), has one and only one Laurent expansion.*

An interesting special case is the limit $R_1 \to 0$, meaning a punctured disc

$$\mathscr{A}_0 = \{z \in \mathbb{C} : 0 < |z| < R\}\;. \tag{3.53}$$

Let us consider a holomorphic function $f(z)$ in the punctured disc $\mathscr{A}_0$. If $f(z)$ cannot be extended to a holomorphic function in the disc

$$\tilde{\mathscr{A}_0} = \{z \in \mathbb{C} : |z| < R\} \ , \tag{3.54}$$

the origin $z = 0$ is an *isolated singularity* of $f(z)$. In other words, $z = 0$ is an isolated singularity of $f(z)$ unless the Laurent expansion of $f(z)$ is such that

$$a_{-n} = 0 \ , \quad \forall n \in \mathbb{N} \ . \tag{3.55}$$

From the analysis of the Laurent expansion of $f(z)$, there are two possible kinds of isolated singularities.

- *A pole* — If only a finite number of coefficients a_{-n}, with $n \in \mathbb{N}$, is nonzero, the isolated singularity is a pole. If N is the largest $N \in \mathbb{N}$ such that $a_{-N} \neq 0$, we say that the pole is *multiple* with order N. If the largest $N \in \mathbb{N}$ such that $a_{-N} \neq 0$ is $N = 1$, we say that the pole is *simple*.
- *An essential singularity* — If there is an infinite number of nonzero coefficients a_{-n}, with $n \in \mathbb{N}$, the isolated singularity is an essential singularity.

We note that, if $f(z)$ is a holomorphic function in the punctured disc $\mathscr{A}_0$, Eq. (3.53), with a multiple pole of order N, then $z^N f(z)$ is holomorphic in the disc $\tilde{\mathscr{A}_0}$, Eq. (2.31).

We note that

$$f(z) = \frac{1}{z} \tag{3.56}$$

has a simple pole at $z = 0$, while

$$f(z) = \mathrm{e}^{1/z} \tag{3.57}$$

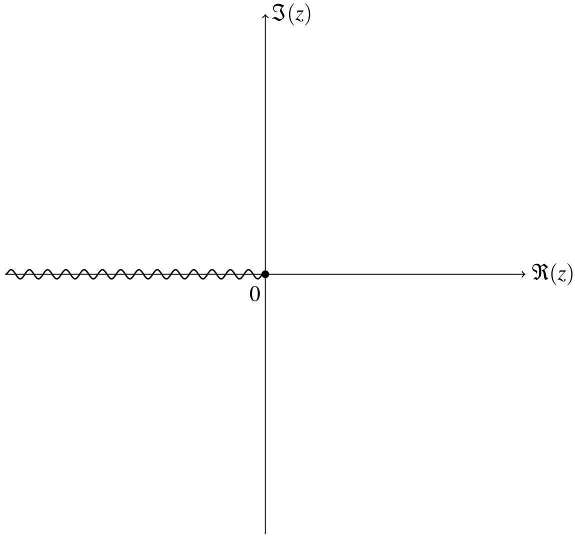
has an essential singularity at $z = 0$.

So far, we discussed the singularities of a function $f(z)$. We know that there also exist multivalued functions, an example being the logarithm $\ln(z)$, Eq. (3.12). We know that there exist infinite branches of $\ln(z)$, each one determining a different value associated with a given z. Other multivalued functions can be defined with the fractional powers of z. An example is

$$f(z) = \sqrt{z} = \sqrt{|z|}\ \mathrm{e}^{\mathrm{i}\ \arg(z)/2} \ . \tag{3.58}$$

If we consider the first branch $\arg(z) \in [-\pi, \pi]$, we obtain values of $\sqrt{z}$ with a positive or zero real part. If we consider another branch, say $[\pi, 3\pi]$, we obtain values of $\sqrt{z}$ with a negative or zero real part. Both in the case of $\ln(z)$ and in the case of $\sqrt{z}$, the multivaluedness can be represented by a *branch cut* in the complex plane (see Fig. 3.1). The branch cut is the wavy line on the half-axis $\mathfrak{Re}(z) \leqslant 0$. Every time we cross the branch cut and we enter a new branch of the multivalued function. The origin of the branch cut, $z = 0$, is to be considered as a singularity of the multivalued function, even if in a sense different from the isolated singularities

Fig. 3.1 Branch cut in the complex plane



of the functions discussed above. In fact, in this case we don't base our definition on the features of a Laurent series.

For the sake of simplicity, our definitions of Laurent series, isolated singularity, pole and essential singularity were relative to the origin. The same definitions may be relative to any other point $z = z_0 \in \mathbb{C}$ without any substantial difference. Indeed, we must consider an annulus,

$$\mathscr{A}_{z_0} = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}\,. \tag{3.59}$$

Then, a Laurent expansion of a function $f : \mathscr{A}_{z_0} \to \mathbb{C}$ exists if the power series,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n = \cdots + \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 \\ + a_1\,(z-z_0) + a_2\,(z-z_0)^2 + \cdots + a_n\,(z-z_0)^n + \cdots\,,\end{aligned} \tag{3.60}$$

converges to $f(z)$ for every $z \in \mathscr{A}_{z_0}$.

Let us consider an open connected subset $\mathscr{D} \subseteq \mathbb{C}$. If $f : \mathscr{D} \to \mathbb{C}$ is holomorphic in $\mathscr{D}$ except for a set of isolated singularities of $f(z)$ classified as poles, then $f(z)$ is said to be *meromorphic* in $\mathscr{D}$.

3.3 Residues

Let $f(z)$ be a holomorphic function in the punctured disc centred in $z = z_0$,

$$\mathscr{A}_{z_0} = \{z \in \mathbb{C} : 0 < |z - z_0| < R\} \ , \tag{3.61}$$

and let $z = z_0$ be a multiple pole of order N. Then, we may write the Laurent expansion

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n \ , \quad \forall z \in \mathscr{A}_{z_0} \ . \tag{3.62}$$

The coefficient a_{-1} is called the *residue* of $f(z)$ at $z = z_0$,

$$\text{Res}(f(z); z_0) = a_{-1} \ . \tag{3.63}$$

We can prove that, if $z = z_0$ is a simple pole of a holomorphic function $f(z)$ in the punctured disc $\mathscr{A}_{z_0}$, Eq. (3.61), then the residue of $f(z)$ at $z = z_0$ can be evaluated as

$$\text{Res}(f(z); z_0) = \lim_{z \to z_0} (z - z_0) f(z) \ . \tag{3.64}$$

The proof is as follows. We express $f(z)$ through its Laurent expansion

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \ . \tag{3.65}$$

Then,

$$(z - z_0) f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + a_2(z - z_0)^3 + \cdots \ . \tag{3.66}$$

By taking the limit $z \to z_0$ at both sides of this equation, we obtain

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1} = \text{Res}(f(z); z_0) \ . \tag{3.67}$$

Furthermore, we can prove that, if $z = z_0$ is a multiple pole of order $N > 1$ of a holomorphic function $f(z)$ in the punctured disc $\mathscr{A}_{z_0}$, Eq. (3.61), then the residue of $f(z)$ at $z = z_0$ can be evaluated as

$$\text{Res}(f(z); z_0) = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{N-1}}{\mathrm{d}\, z^{N-1}} \left[(z - z_0)^N f(z) \right] \ . \tag{3.68}$$

Equation (3.68) can be proved by expressing $f(z)$ through its Laurent expansion, so that we obtain

$$\begin{gathered} (z - z_0)^N f(z) = a_{-N} + a_{-(N-1)}(z - z_0) + \cdots \\ + a_{-1}(z - z_0)^{N-1} + a_0(z - z_0)^N + a_1(z - z_0)^{N+1} \ \ldots \ . \end{gathered} \tag{3.69}$$

One may easily verify that

$$\frac{\mathrm{d}^{N-1}}{\mathrm{d}\,z^{N-1}}\,(z-z_0)^n = 0\;, \quad 0 \leqslant n < N-1\;,$$

$$\frac{\mathrm{d}^{N-1}}{\mathrm{d}\,z^{N-1}}\,(z-z_0)^n = \frac{n!}{(n-N+1)!}\,(z-z_0)^{n-N+1}\;, \quad n \geqslant N-1\;. \tag{3.70}$$

Thus, by employing Eqs. (3.67)–(3.69), we obtain

$$\lim_{z\to z_0}\frac{\mathrm{d}^{N-1}}{\mathrm{d}\,z^{N-1}}\left[(z-z_0)^N f(z)\right] = a_{-1}\,(N-1)! = (N-1)!\,\mathrm{Res}(f(z);z_0)\;. \tag{3.71}$$

Theorem 3.5 (Cauchy's Residue Theorem) *Let us consider an open connected subset $\mathscr{D} \subseteq \mathbb{C}$ and a closed counterclockwise-oriented path $\gamma \subseteq \mathscr{D}$. Let $f(z)$ be a meromorphic function in $\mathscr{D}$ with a finite number of poles $z_1, z_2, \ldots, z_m$ inside the region bounded by γ, and such that γ does not pass through any singularities of $f(z)$. Then,*

$$\int_\gamma f(z)\,\mathrm{d}\,z = 2\pi\,\mathrm{i}\sum_{k=1}^{m}\mathrm{Res}(f(z);z_k)\;. \tag{3.72}$$

Cauchy's residue theorem is of paramount importance in the complex analysis, as it provides an extremely effective tool for the evaluation of integrals. For this purpose, Theorem 3.5 is completed by a useful lemma.

Lemma 3.1 *Let $f(z)$ be a meromorphic function in an open connected subset $\mathscr{D} \subseteq \mathbb{C}$ that includes the sector of the complex plane*

$$\mathscr{S} = \{z \in \mathbb{C} : \theta_1 < \arg(z) < \theta_2\}\;. \tag{3.73}$$

Let $\gamma_0(R;\theta_1,\theta_2)$ be the arc of the circle $|z| = R$ included in $\mathscr{S}$ and oriented counterclockwise. If

$$\forall z \in \mathscr{S}, \quad \lim_{|z|\to\infty} zf(z) = 0, \tag{3.74}$$

then

$$\lim_{R\to\infty}\int_{\gamma_0(R;\theta_1,\theta_2)} f(z)\,\mathrm{d}\,z = 0. \tag{3.75}$$

3.3.1 Evaluation of Integrals

Let us consider a few examples, in order to see how Cauchy's residue theorem can be a very useful method for the evaluation of integrals.

Example 3.4 We want to evaluate the integral

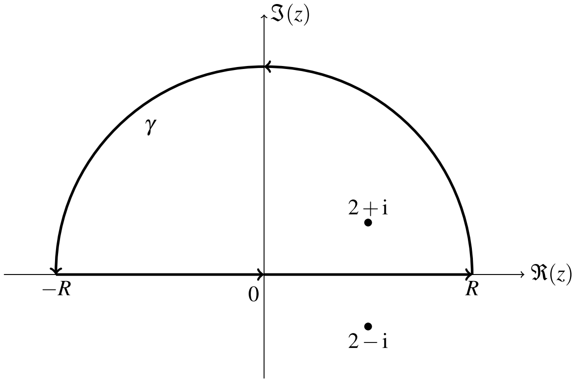


Fig. 3.2 Closed semicircular path used in Example 3.4

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}\,x}{x^2 - 4x + 5} \; . \tag{3.76}$$

Obviously, I is given by the limit $R \to \infty$ of the integral

$$I_R = \int_{-R}^{R} \frac{\mathrm{d}\,x}{x^2 - 4x + 5} \; . \tag{3.77}$$

Function

$$f(z) = \frac{1}{z^2 - 4z + 5} = \frac{1}{(z - 2 + \mathrm{i})(z - 2 - \mathrm{i})} \tag{3.78}$$

is meromorphic in $\mathbb{C}$ with two simple poles in $z = 2 - \mathrm{i}$ and in $z = 2 + \mathrm{i}$.

If we consider the closed semicircular path γ sketched in Fig. 3.2, the pole $z = 2 + \mathrm{i}$ is contained in the region bounded by γ, provided that R is sufficiently large. The following identity holds

$$\int_{\gamma} \frac{\mathrm{d}\,z}{z^2 - 4z + 5} = I_R + \int_{\gamma_0(R;0,\pi)} \frac{\mathrm{d}\,z}{z^2 - 4z + 5} \; , \tag{3.79}$$

where

$$\gamma_0(R; 0, \pi) = \{z \in \mathbb{C} : |z| = R, \; \mathfrak{Im}(z) > 0\} \; . \tag{3.80}$$

On account of Lemma 3.1, we can write

$$\lim_{R \to \infty} \int_{\gamma_0(R;0,\pi)} \frac{\mathrm{d}\,z}{z^2 - 4z + 5} = 0 \; . \tag{3.81}$$

Therefore, by invoking Cauchy's residue theorem, we may write

$$I = \lim_{R\to\infty} \int_{\gamma} \frac{\mathrm{d}\,z}{z^2 - 4z + 5}$$
$$= 2\pi\,\mathrm{i}\;\mathrm{Res}\left(\frac{1}{(z-2+\mathrm{i})(z-2-\mathrm{i})}; 2+\mathrm{i}\right) = 2\pi\,\mathrm{i}\,\frac{1}{2\,\mathrm{i}} = \pi \;, \tag{3.82}$$

where Eq. (3.64) has been used for the evaluation of the residue.

Example 3.5 Let us consider the integral

$$I = \int_0^{2\pi} \frac{\mathrm{d}\,x}{\sin x + \cos x + 5} \;. \tag{3.83}$$

We note that a parametrisation of the unit circle,

$$\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\} \;, \tag{3.84}$$

oriented counterclockwise, is

$$z = \mathrm{e}^{\mathrm{i}\,x} = \cos x + \mathrm{i}\,\sin x \;, \quad x \in [0, 2\pi] \;, \tag{3.85}$$

so that

$$\mathrm{d}\,z = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x}\mathrm{d}\,x = \mathrm{i}\,z\,\mathrm{d}\,x \;, \quad \mathrm{d}\,x = -\,\mathrm{i}\,\frac{\mathrm{d}\,z}{z} \;. \tag{3.86}$$

Then, on the unit circle $\mathscr{C}$, we have

$$\frac{1}{z} = \mathrm{e}^{-\mathrm{i}\,x} = \cos x - \mathrm{i}\,\sin x \;, \quad x \in [0, 2\pi] \;. \tag{3.87}$$

As a consequence, we may write

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin x = \frac{1}{2\,\mathrm{i}}\left(z - \frac{1}{z}\right),$$
$$\frac{\mathrm{d}\,x}{\sin x + \cos x + 5} = \frac{-\,2\mathrm{i}\,\mathrm{d}\,z}{(1-\mathrm{i})z^2 + 10z + (1+\mathrm{i})} \;. \tag{3.88}$$

Therefore, we have

$$I = -\,2\,\mathrm{i} \int_{\mathscr{C}} \frac{\mathrm{d}\,z}{(1-\mathrm{i})z^2 + 10z + (1+\mathrm{i})} \;. \tag{3.89}$$

The function

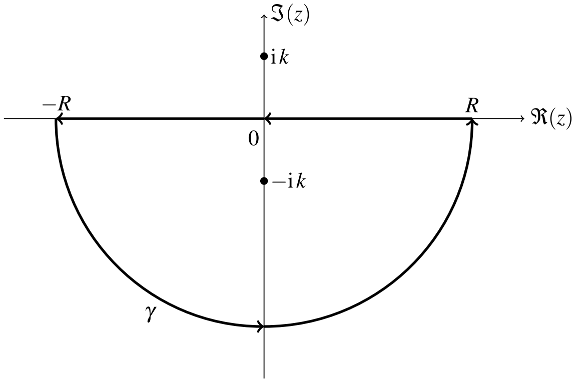


Fig. 3.3 Closed semicircular path used in Example 3.6

$$f(z) = \frac{1}{(1-\mathrm{i})z^2 + 10z + (1+\mathrm{i})} \tag{3.90}$$

has two simple poles at

$$z_1 = \frac{1+\mathrm{i}}{2}\left(\sqrt{23}-5\right), \quad z_2 = -\frac{1+\mathrm{i}}{2}\left(\sqrt{23}+5\right). \tag{3.91}$$

Only the pole z_1 is in the region bounded by $\mathscr{C}$, $|z_1| < 1$, while z_2 is outside this region, $|z_2| > 1$. Hence, on employing Cauchy's residue theorem, we obtain

$$I = 4\pi \ \mathrm{Res}(f(z); z_1) = \frac{2\pi}{\sqrt{23}} \ . \tag{3.92}$$

Example 3.6 We are now interested in evaluating the integrals

$$I_1 = \int\limits_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+1}\,\mathrm{d}x \ , \quad I_2 = \int\limits_{-\infty}^{\infty} \frac{\sin(kx)}{x^2+1}\,\mathrm{d}x \ , \quad k > 0 \ . \tag{3.93}$$

We note that, on account of Euler's formula (3.8), we may write

$$I_1 = \Re(I) \ , \quad I_2 = -\Im(I) \ , \tag{3.94}$$

where

$$I = \int\limits_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}kx}}{x^2+1}\,\mathrm{d}x \ . \tag{3.95}$$

A comparison with the definition given by Eq. (2.2) leads us to the conclusion that I is the Fourier transform of function $F(x) = \sqrt{2\pi}/(1+x^2)$ for the range $k > 0$. Then, we can focus on the evaluation of I. We change the integration variable,

$$y = kx \ , \quad x = \frac{y}{k} \ , \quad \mathrm{d}\, x = \frac{\mathrm{d}\, y}{k} \ , \tag{3.96}$$

so that

$$I = k \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\, y}}{y^2 + k^2} \, \mathrm{d}\, y \ . \tag{3.97}$$

On considering the closed semicircular path in Fig. 3.3, we write

$$\int_{\gamma} \frac{\mathrm{e}^{-\mathrm{i}\, z}}{z^2 + k^2} \, \mathrm{d}\, z = -\int_{-R}^{R} \frac{\mathrm{e}^{-\mathrm{i}\, y}}{y^2 + k^2} \, \mathrm{d}\, y + \int_{\gamma_0(R,\pi,2\pi)} \frac{\mathrm{e}^{-\mathrm{i}\, z}}{z^2 + k^2} \, \mathrm{d}\, z \ , \tag{3.98}$$

where

$$\gamma_0(R; \pi, 2\pi) = \{z \in \mathbb{C} : |z| = R, \ \Im(z) < 0\} \ . \tag{3.99}$$

Since $\Im(z) < 0$, we have

$$\lim_{|z| \to \infty} \frac{z\, \mathrm{e}^{-\mathrm{i}\, z}}{z^2 + k^2} = 0 \ . \tag{3.100}$$

Therefore, as consequence of Lemma 3.1, we obtain

$$\lim_{R \to \infty} \int_{\gamma_0(R,\pi,2\pi)} \frac{\mathrm{e}^{-\mathrm{i}\, z}}{z^2 + k^2} \, \mathrm{d}\, z = 0 \ , \tag{3.101}$$

so that

$$I = -k \lim_{R \to \infty} \int_{\gamma} \frac{\mathrm{e}^{-\mathrm{i}\, z}}{z^2 + k^2} \, \mathrm{d}\, z \ . \tag{3.102}$$

We employ Cauchy's residue theorem to evaluate

$$\int_{\gamma} \frac{\mathrm{e}^{-\mathrm{i}\, z}}{z^2 + k^2} \, \mathrm{d}\, z \ . \tag{3.103}$$

Function

$$f(z) = \frac{\mathrm{e}^{-\mathrm{i}\, z}}{z^2 + k^2} = \frac{\mathrm{e}^{-\mathrm{i}\, z}}{(z - \mathrm{i}\, k)(z + \mathrm{i}\, k)} \tag{3.104}$$

has two simple poles $z_1 = \mathrm{i}\, k$ and $z_2 = -\mathrm{i}\, k$. The closed path γ encircles the pole z_2, but not z_1, provided that R is sufficiently large ($R > k$). Then, we have

$$I = -k \lim_{R\to\infty} \int_\gamma \frac{\mathrm{e}^{-\mathrm{i}\,z}}{z^2+k^2}\,\mathrm{d}\,z = -2\pi k\,\mathrm{i}\,\mathrm{Res}(f(z); z_2)\ . \tag{3.105}$$

The residue is given by

$$\mathrm{Res}(f(z); z_2) = -\,\frac{\mathrm{e}^{-k}}{2\mathrm{i}\,k}\ . \tag{3.106}$$

Therefore, we conclude that

$$I = 2\pi k\,\mathrm{i}\,\frac{\mathrm{e}^{-k}}{2\mathrm{i}\,k} = \pi\,\mathrm{e}^{-k}\ . \tag{3.107}$$

This means that

$$I_1 = \pi\,\mathrm{e}^{-k}\ , \quad I_2 = 0\ . \tag{3.108}$$

3.4 The Laplace Transform

The Laplace transform of a function $f(t)$ is given by

$$\mathfrak{L}\{f(t)\}(s) = \tilde{f}(s) = \int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}\,t\ . \tag{3.109}$$

The transform $\mathfrak{L}\{f(t)\}$ is defined in the complex half-plane $\Re(s) > a$ where a is a real constant such that the following condition holds:

$$|\,f(t)| < C\,\mathrm{e}^{at}, \quad \forall t \geqslant 0, \tag{3.110}$$

with a proper choice of a positive real constant C. A sketch of the domain where $\tilde{f}(s)$ is defined is given in Fig. 3.4.

3.4.1 Inversion of the Laplace Transform

If the Laplace transform of a function $f(t)$ is known, one may determine $f(t)$. To achieve this task, there exists a procedure for the inversion of the Laplace transform. The inversion formula of the Laplace transform is as follows:

$$f(t) = \mathfrak{L}^{-1}\{\tilde{f}(s)\}(t) = \frac{1}{2\pi\,\mathrm{i}} \int_{p-\mathrm{i}\,\infty}^{p+\mathrm{i}\,\infty} \tilde{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}\,s\ , \tag{3.111}$$

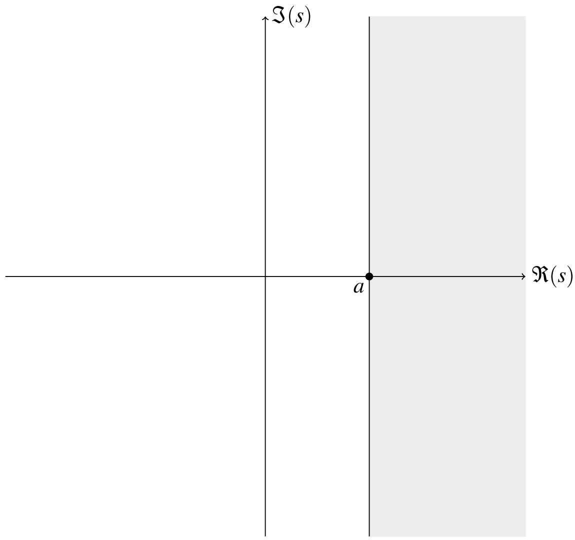


Fig. 3.4 Domain where the Laplace transform $\tilde{f}(s)$ is defined

for every real number $p > a$. The integration must be performed along a line path in the complex plane. This path is given by a vertical line that intersects the real axis, $\Im(s) = 0$, at $s = p$ (see Fig. 3.5).

Hence, the evaluation of the inverse Laplace transform implies the calculation of an integral in the complex plane. The theory of the integration in $\mathbb{C}$, and Cauchy's residue Theorem 3.5, is a strong basis for the inversion of the Laplace transform. In the simplest cases, one may utilise proper tables where the pairs $[f(t), \tilde{f}(s)]$ are reported (see, for instance, Debnath and Bhatta [7]).

3.4.2 Main Properties of the Laplace Transform

Let $f(t)$ and $g(t)$ be any two functions satisfying Eq. (3.110). Among the main properties of the Laplace transform, we mention the following:

- *Linearity*
  For every pair of real constants (C_1, C_2), we have

$$\begin{aligned}\mathfrak{L}\{C_1\, f(t) + C_2\, g(t)\}\,(s) &= C_1\, \mathfrak{L}\{f(t)\}(s) + C_2\, \mathfrak{L}\{g(t)\}(s) \\ &= C_1\, \tilde{f}(s) + C_2\, \tilde{g}(s)\ .\end{aligned} \tag{3.112}$$

- *Derivative*
  On considering the first derivative of $f(t)$ and evaluating its Laplace transform, we obtain

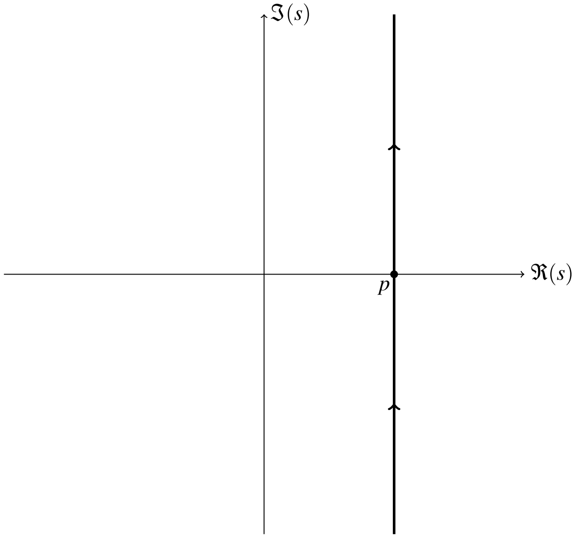


Fig. 3.5 Integration path for the inversion formula, Eq. (3.111)

$$\begin{aligned}\mathfrak{L}\left\{f'(t)\right\}(s) &= \int_0^\infty f'(t)\,\mathrm{e}^{-st}\,\mathrm{d}\,t = \left[f(t)\,\mathrm{e}^{-st}\right]_0^\infty + s\int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}\,t \\ &= s\,\mathfrak{L}\{f(t)\}(s) - f(0) = s\,\tilde{f}(s) - f(0)\;.\end{aligned} \tag{3.113}$$

In a similar way, one may evaluate the Laplace transforms of higher-order derivatives,

$$\begin{aligned}\mathfrak{L}\left\{f''(t)\right\}(s) &= s^2\,\tilde{f}(s) - s\,f(0) - f'(0)\;, \\ \mathfrak{L}\left\{f^{(n)}(t)\right\}(s) &= s^n\,\tilde{f}(s) - s^{n-1}\,f(0) \\ &\quad - s^{n-2}\,f'(0) - \cdots - f^{(n-1)}(0)\;,\end{aligned} \tag{3.114}$$

where $n \geqslant 2$.

- *Translation*
  The Laplace transform of $f(t)\,\mathrm{e}^{bt}$, where $b \in \mathbb{R}$ and $b \leqslant a$ so that the condition expressed by Eq. (3.110) is satisfied by $f(t)\,\mathrm{e}^{bt}$, is given by

$$\mathfrak{L}\left\{f(t)\,\mathrm{e}^{bt}\right\}(s) = \int_0^\infty f(t)\,\mathrm{e}^{-(s-b)t}\,\mathrm{d}\,t = \tilde{f}(s-b)\;. \tag{3.115}$$

In the special case where $f(t) = 1$, we get

$$\mathfrak{L}\left\{e^{bt}\right\}(s) = \int_0^\infty e^{-(s-b)t}\, \mathrm{d}\,t = \frac{1}{s-b}\ . \tag{3.116}$$

- *Scaling*
  Let us consider the Laplace transform of $f(bt)$, where $b > 0$. One has

$$\begin{aligned}\mathfrak{L}\left\{f(bt)\right\}(s) &= \int_0^\infty f(bt)\, e^{-st}\, \mathrm{d}\,t = \frac{1}{b}\int_0^\infty f(u)\, e^{-su/b}\, \mathrm{d}\,u \\ &= \frac{1}{b}\,\tilde{f}\left(\frac{s}{b}\right)\ . \end{aligned} \tag{3.117}$$

- *Convolution*
  The Laplace transform of the *convolution* between two functions $f(t)$ and $g(t)$, $f(t)\star g(t)$, defined as

$$f(t)\star g(t) = \int_0^t f(\hat{t})\, g(t-\hat{t})\, \mathrm{d}\,\hat{t}\ , \tag{3.118}$$

  is given by the product of the Laplace transforms of $f(t)$ e $g(t)$,

$$\mathfrak{L}\{f(t)\star g(t)\}(s) = \mathfrak{L}\{f(t)\}(s)\,\mathfrak{L}\{g(t)\}(s) = \tilde{f}(s)\,\tilde{g}(s)\ . \tag{3.119}$$

  Although the definition of convolution given by Eq. (3.118) differs from the definition of convolution stated for the Fourier transform, Eq. (2.19), it shares the same properties,

$$\textit{commutative} \quad \rightsquigarrow \quad f\star g = g\star f\ ; \tag{3.120}$$

$$\textit{associative} \quad \rightsquigarrow \quad f\star(g\star h) = (f\star g)\star h\ ; \tag{3.121}$$

$$\textit{distributive} \quad \rightsquigarrow \quad f\star(g+h) = f\star g + f\star h\ . \tag{3.122}$$

- *Ratio between two polynomials*
  Let us consider $\tilde{f}(s) = \tilde{G}(s)/\tilde{H}(s)$, where $\tilde{G}(s)$ and $\tilde{H}(s)$ are two polynomials such that the degree of $\tilde{H}(s)$ is greater than that of $\tilde{G}(s)$, and that $\tilde{H}(s)$ has only zeros with algebraic multiplicity 1. In that case, $\tilde{f}(s)$ can be expressed as the sum of partial fractions,

$$\tilde{f}(s) = \frac{c_1}{s-b_1} + \frac{c_2}{s-b_2} + \cdots + \frac{c_n}{s-b_n}\ , \tag{3.123}$$

  where $b_1, b_2, \ldots, b_n$ are the zeros of $\tilde{H}(s)$ and the coefficients c_i can be evaluated as

$$c_k = \lim_{s \to b_k} [(s - b_k)\, \tilde{f}(s)], \quad \forall k = 1, \ \ldots \ , n \ . \tag{3.124}$$

In other words, the coefficients c_k are the residues of $\tilde{f}(s)$ at the simple poles $s = b_k$. On account of the linearity and of the translation properties of the Laplace transform, we obtain

$$f(t) = c_1\, e^{b_1 t} + c_2\, e^{b_2 t} + \cdots + c_n\, e^{b_n t} = \sum_{k=1}^{n} c_k\, e^{b_k t} \ . \tag{3.125}$$

3.4.3 Meromorphic Functions

Let us assume that $\tilde{f}(s)$ has no essential singularities and that its poles $b_1, b_2, \ldots,$ $b_n, \ldots$ are in the complex half-plane $\Re(s) < p$. The integral expressing the inverse Laplace transform of $\tilde{f}(s)$, Eq. (3.111), can be evaluated through a limit of the integral of

$$\tilde{f}(s)\, e^{st} \tag{3.126}$$

evaluated on a closed path, γ, in the complex plane called the *Bromwich contour*,

$$f(t) = \lim_{R \to \infty} \left[\frac{1}{2\pi\, i} \int_{\gamma} \tilde{f}(s)\, e^{st}\, d\, s \right] , \tag{3.127}$$

where R is the radius of the curved part of the Bromwich contour. A sketch of this contour in the complex plane is given in Fig. 3.6.

In the limit $R \to \infty$, the integral on the semicircular part of the Bromwich contour tends to zero provided that, on this semicircle, the following condition holds:

$$|\tilde{f}(s)| < \frac{M}{R^{\kappa}} \ , \tag{3.128}$$

where M and κ are positive constants [8, 9]. Moreover, in the limit $R \to \infty$, the integral along the vertical line of the Bromwich contour tends to coincide with the integral that appears in the inversion formula of the Laplace transform, Eq. (3.111). In the limit $R \to \infty$, the Bromwich contour encloses all the poles of $\tilde{f}(s)$. Hence, the inverse Laplace transform of $\tilde{f}(s)$ can be evaluated by employing Cauchy's residue Theorem 3.5,

$$f(t) = \frac{1}{2\pi\,\mathrm{i}} \int_{p-\mathrm{i}\,\infty}^{p+\mathrm{i}\,\infty} \tilde{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}\,s = \lim_{R\to\infty} \left[\frac{1}{2\pi\,\mathrm{i}} \int_{\gamma} \tilde{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}\,s \right]$$
$$= \sum_n \mathrm{Res}\Big(\tilde{f}(s)\,\mathrm{e}^{st}; b_n \Big) \;, \tag{3.129}$$

where γ is the Bromwich contour.

Equation (3.129) is the basis for the evaluation of the inverse Laplace transform of $\tilde{f}(s)$ in all cases where $\tilde{f}(s)$ does not have either essential singularities or branch points.

3.5 Saddle Points

If $\mathscr{D}$ is an open connected subset of $\mathbb{C}$ and $\mathscr{D}^\circ$ its correspondent open connected subset of $\mathbb{R}^2$, a holomorphic function $f : \mathscr{D} \to \mathbb{C}$ can be rewritten as a function of two real variables, $f : \mathscr{D}^\circ \to \mathbb{R}^2$, by expressing $z = x + \mathrm{i}\,y$ and by evaluating the real and imaginary parts of $f(z)$. We have already pointed out that the resulting $f(x, y)$ is quite special on discussing the Cauchy–Riemann equations, Theorem 3.1. Other aspects of these special features are discussed in the following.

Let us denote by

$$u(x, y) = \Re(f(x + \mathrm{i}\,y)) \;, \quad v(x, y) = \Im(f(x + \mathrm{i}\,y)) \;, \tag{3.130}$$

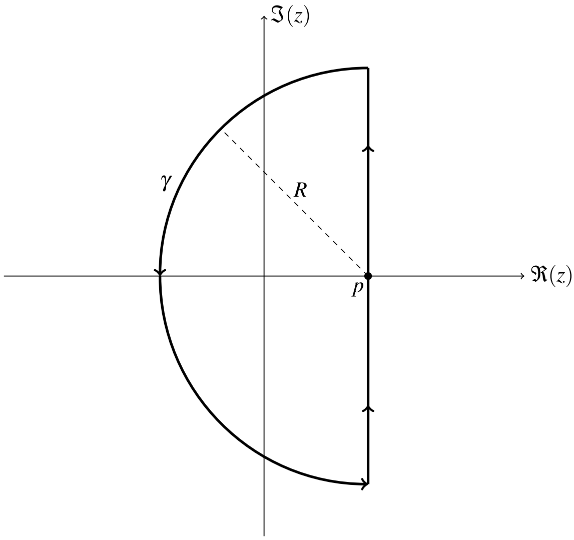


Fig. 3.6 Bromwich contour

the real and imaginary parts of a holomorphic function $f(z)$. Then, we can write

$$f(z) = u(x, y) + \mathrm{i}\, v(x, y) \; . \tag{3.131}$$

3.5.1 Stationary Point

Let us consider a point $z_0 = x_0 + \mathrm{i}\, y_0 \in \mathscr{D}$ such that $f'(z_0) = 0$. Since

$$x = \frac{z + \bar{z}}{2} \; , \quad y = \frac{z - \bar{z}}{2\mathrm{i}} \; , \tag{3.132}$$

we obtain

$$\begin{aligned} f'(z) &= \left[\frac{\partial u(x, y)}{\partial x} \, \frac{\partial x}{\partial z} + \frac{\partial u(x, y)}{\partial y} \, \frac{\partial y}{\partial z} \right] + \mathrm{i} \left[\frac{\partial v(x, y)}{\partial x} \, \frac{\partial x}{\partial z} + \frac{\partial v(x, y)}{\partial y} \, \frac{\partial y}{\partial z} \right] \\ &= \frac{1}{2} \left[\frac{\partial u(x, y)}{\partial x} - \mathrm{i} \, \frac{\partial u(x, y)}{\partial y} \right] + \frac{\mathrm{i}}{2} \left[\frac{\partial v(x, y)}{\partial x} - \mathrm{i} \, \frac{\partial v(x, y)}{\partial y} \right] \\ &= \frac{1}{2} \left[\frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial y} \right] + \frac{\mathrm{i}}{2} \left[\frac{\partial v(x, y)}{\partial x} - \frac{\partial u(x, y)}{\partial y} \right] . \end{aligned} \tag{3.133}$$

The condition $f'(z_0) = 0$ implies that, at $(x, y) = (x_0, y_0)$, the following equations hold

$$\frac{\partial u}{\partial x} = - \frac{\partial v}{\partial y} \; , \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \; . \tag{3.134}$$

By invoking the Cauchy–Riemann equations (3.22), one may conclude that Eqs. (3.22) and (3.134) can hold simultaneously at $(x, y) = (x_0, y_0)$ if and only if

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y} \; , \qquad \frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y} \; . \tag{3.135}$$

Equation (3.135) means that $(x, y) = (x_0, y_0)$, i.e. $z = z_0$, is a stationary point of both functions u and v.

The determinant of the Hessian matrix of either u or v may provide a characterisation of the stationary point [3]. We have to evaluate the second derivatives of $f(x, y)$,

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} = f'(z) \, \frac{\partial z}{\partial x} = f'(z) \; , \quad & \frac{\partial f(x, y)}{\partial y} = f'(z) \, \frac{\partial z}{\partial y} = \mathrm{i} \, f'(z) \; , \\ \frac{\partial^2 f(x, y)}{\partial x^2} = f''(z), \quad & \frac{\partial^2 f(x, y)}{\partial y^2} = - \, f''(z) \; , \end{aligned} \tag{3.136}$$

so that we may conclude that $f(x, y)$ is a *harmonic function*, i.e. a solution of Laplace's equation,

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0\,. \tag{3.137}$$

This is also a consequence of Theorem 3.2. Since $f = u + \mathrm{i}\,v$, Eq. (3.137) yields

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0 \quad \Longrightarrow \quad \begin{cases} \dfrac{\partial^2 u(x,y)}{\partial x^2} + \dfrac{\partial^2 u(x,y)}{\partial y^2} = 0\,, \\ \dfrac{\partial^2 v(x,y)}{\partial x^2} + \dfrac{\partial^2 v(x,y)}{\partial y^2} = 0\,, \end{cases} \tag{3.138}$$

meaning that both $u(x, y)$ and $v(x, y)$ are harmonic functions. Equation (3.138) allows one to conclude that the Hessian matrix of $u(x, y)$ has a non-positive determinant at the stationary point (x_0, y_0), and likewise for $v(x, y)$,

$$\begin{vmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 u}{\partial x\,\partial y} \\ \dfrac{\partial^2 u}{\partial y\,\partial x} & \dfrac{\partial^2 u}{\partial y^2} \end{vmatrix} = -\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 \leqslant 0\,. \tag{3.139}$$

As a consequence of Eq. (3.139), the eigenvalues of the Hessian matrix cannot be both positive or both negative, so that (x_0, y_0) can be neither a local maximum nor a local minimum. On the other hand, (x_0, y_0) can be a *saddle point* for $u(x, y)$ whenever the determinant of the Hessian matrix is strictly negative. With just the same argument, based on Eq. (3.138), this result can be achieved also for $v(x, y)$, namely

$$\begin{vmatrix} \dfrac{\partial^2 v}{\partial x^2} & \dfrac{\partial^2 v}{\partial x\,\partial y} \\ \dfrac{\partial^2 v}{\partial y\,\partial x} & \dfrac{\partial^2 v}{\partial y^2} \end{vmatrix} = -\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - \left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2 \leqslant 0\,. \tag{3.140}$$

We note that the determinant of the Hessian matrix of either $u(x, y)$ or $v(x, y)$ is strictly negative at the stationary point $(x, y) = (x_0, y_0)$ when $f''(z_0) \neq 0$. In fact, by employing the Cauchy–Riemann equations (3.22), one obtains

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{\partial^2 v(x,y)}{\partial x\,\partial y}\,, \qquad \frac{\partial^2 v(x,y)}{\partial x^2} = -\frac{\partial^2 u(x,y)}{\partial x\,\partial y}\,. \tag{3.141}$$

Thus, on account of Eqs. (3.136), (3.139), (3.140) and (3.141), one can infer that the determinant of the Hessian matrix of $u(x, y)$ at $(x, y) = (x_0, y_0)$ is given by

$$\begin{vmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 u}{\partial x\,\partial y} \\ \dfrac{\partial^2 u}{\partial y\,\partial x} & \dfrac{\partial^2 u}{\partial y^2} \end{vmatrix} = -\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \left(\frac{\partial^2 v}{\partial x^2}\right)^2 = -\left|\frac{\partial^2 f}{\partial x^2}\right|^2 = -|f''(z_0)|^2 \leqslant 0\,. \tag{3.142}$$

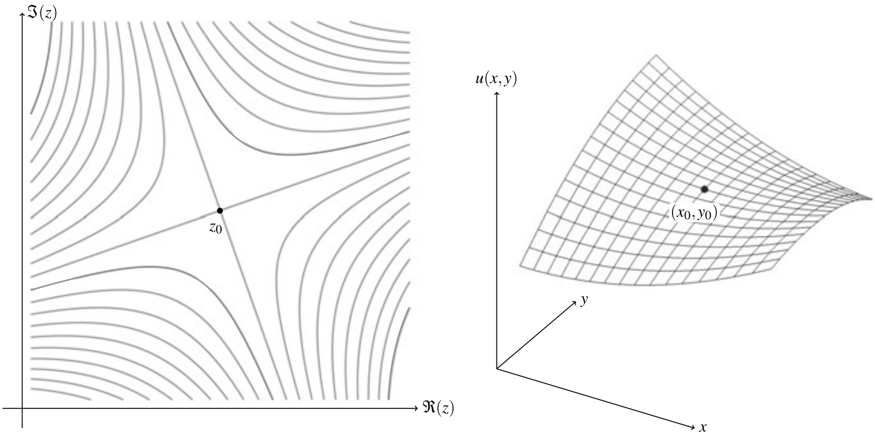


Fig. 3.7 Illustration of Theorem 3.6: contour lines of the real part of $f(z)$ around a saddle point $z_0 = x_0 + \mathrm{i}\, y_0$ and three-dimensional plot of $u(x, y) = \Re(f(x, y))$ at the saddle point

Likewise, for the determinant of the Hessian matrix of $v(x, y)$ at $(x, y) = (x_0, y_0)$, one obtains

$$\begin{vmatrix} \dfrac{\partial^2 v}{\partial x^2} & \dfrac{\partial^2 v}{\partial x\, \partial y} \\ \dfrac{\partial^2 v}{\partial y\, \partial x} & \dfrac{\partial^2 v}{\partial y^2} \end{vmatrix} = -\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \left(\frac{\partial^2 v}{\partial x^2}\right)^2 = -\left|\frac{\partial^2 f}{\partial x^2}\right|^2 = -|f''(z_0)|^2 \leqslant 0\,. \tag{3.143}$$

The conclusion of this reasoning can be stated in the form of a theorem.

Theorem 3.6 *Let $\mathscr{D}$ be an open connected subset of $\mathbb{C}$, and $f : \mathscr{D} \to \mathbb{C}$ be holomorphic in $\mathscr{D}$. If there exists $z_0 = x_0 + \mathrm{i}\, y_0 \in \mathscr{D}$ such that $f'(z_0) = 0$ and $f''(z_0) \neq 0$, then both the real and the imaginary parts of $f(x, y) = f(x + \mathrm{i}\, y)$ have a saddle point at $(x, y) = (x_0, y_0)$.*

The saddle-point concept as discussed in Theorem 3.6 is drawn qualitatively in Fig. 3.7. In the figure caption, it is mentioned the real part of $f(z)$, but there is no intrinsic difference in the graphical features if one deals with the imaginary part.

A comment on Theorem 3.6 can be useful. One may wonder what happens when $f''(z_0) = 0$. The answer is that, strictly speaking, one cannot employ the criterion based on the sign of the determinant of the Hessian matrix, as it becomes inconclusive when the determinant vanishes [3]. In fact, one may distinguish a case where all derivatives of $f(z)$ vanish at $z = z_0$, namely $f^{(n)}(z_0) = 0$ for all $n \geqslant 1$. In this case, a Taylor series expansion of $f(z)$ around $z = z_0$ is sufficient to prove that $f(z)$ is constant over the open connected subset of $\mathscr{D}$. A more interesting alternative is when there exists $n \geqslant 3$ such that $f^{(n)}(z_0) \neq 0$. In this case, strictly speaking, we do not have a saddle point at $z = z_0$. In fact, we are dealing with a saddle point in a generalised sense. A sketch of the geometrical features in a sample case with $n = 3$ is

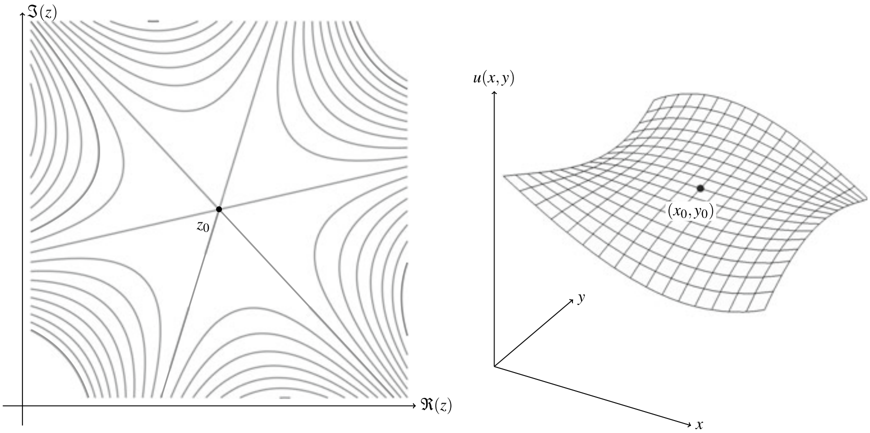


Fig. 3.8 Contour lines of the real part of $f(z)$ around a monkey saddle point $z_0 = x_0 + \mathrm{i}\, y_0$, where $f'(z_0) = f''(z_0) = 0$ with $f'''(z_0) \neq 0$, and three-dimensional plot of $u(x, y) = \Re(f(x, y))$ at the saddle point

presented in Fig. 3.8. This case is also called monkey saddle, as a saddle for monkeys should allow a place for the tail and not only for the legs. For the *generalised saddle points*, when the lowest n such that $f^{(n)}(z_0) \neq 0$ is greater than 2, we call n the *order* of the saddle point. A saddle point where $f''(z_0) \neq 0$ has order 2.

Whatever is the order of the saddle point, there exist ascending and descending paths that depart from z_0. This is clearly seen in the three-dimensional plots, reported in Figs. 3.7 and 3.8, displaying $\Re(f(z))$ versus (x, y). Among these ascending and descending paths, one may graphically detect those of *steepest ascent* and *steepest descent*. These paths are central in the formulation of the asymptotic approximation of wave packets at large times.

3.5.2 Paths from a Saddle Point

We consider an open connected subset of $\mathbb{C}$, namely $\mathscr{D}$, and an holomorphic function $f : \mathscr{D} \to \mathbb{C}$. Let $z_0 \in \mathscr{D}$ be a saddle point of order n.

In a small neighbourhood of z_0, one can express z as

$$z = z_0 + r\, \mathrm{e}^{\mathrm{i}\varphi} \,, \quad r \geqslant 0 \,, \quad \varphi \in [0, 2\pi] \,. \tag{3.144}$$

Moreover, one can write an approximate expression of $f(z)$ as a Taylor series centred in $z = z_0$ and truncated to the first two nonzero terms, namely

$$f(z) \approx f(z_0) + \frac{1}{n!}\, f^{(n)}(z_0)\, (z - z_0)^n \,. \tag{3.145}$$

Here, we are neglecting terms of order $|z - z_0|^{n+1}$, and we are assuming that all derivatives of $f(z)$ up to order $n-1$ are zero at $z = z_0$. We write $f^{(n)}(z_0)$ in its polar form as

$$f^{(n)}(z_0) = |f^{(n)}(z_0)|\, e^{\mathrm{i}\,\theta} \ , \tag{3.146}$$

where θ is the argument of $f^{(n)}(z_0)$. On substituting Eqs. (3.144) and (3.146) into (3.145), we obtain

$$\begin{aligned} f(z) &\approx f(z_0) + \frac{1}{n!}\,|f^{(n)}(z_0)|\, r^n\, e^{\mathrm{i}\,(\theta + n\,\varphi)} \\ &= f(z_0) + \frac{1}{n!}\,|f^{(n)}(z_0)|\, r^n \Big[\cos(\theta + n\,\varphi) + \mathrm{i}\, \sin(\theta + n\,\varphi) \Big] \ . \end{aligned} \tag{3.147}$$

On inspecting Eq. (3.147), one can conclude that the value of $\theta + n\,\varphi$ delineates if and how the real and imaginary parts of function $f(z)$ increase or decrease when z departs from the saddle point z_0 .

Let us consider $\Re(f(z))$. Equation (3.147) implies that $\Re(f(z))$ undergoes the steepest increase when z departs from z_0 if one chooses a path given by any line with $\cos(\theta + n\,\varphi) = 1$. Thus we define, for $\Re(f(z))$, the lines of *steepest ascent* from z_0 as those where

$$\begin{gathered} \theta + n\,\varphi = 2m\pi \quad \longmapsto \quad \varphi = \frac{2m}{n}\,\pi - \frac{\theta}{n} \ , \\ m = 0, 1, 2, \ \ldots\ , \ n-1 \ . \end{gathered} \tag{3.148}$$

Since there exist n different determinations of the angle φ, predicted by Eq. (3.148), there are n different paths of steepest ascent, for $\Re(f(z))$, departing from z_0. These paths can be easily detected in Figs. 3.7 and 3.8 and are explicitly displayed as thick dashed lines in Fig. 3.9 for a saddle point of order 2 and in Fig. 3.10 for a saddle point of order 3.

In an analogous way, we can easily detect those lines departing from the saddle point z_0 and such that $\Re(f(z))$ undergoes the steepest decrease. Those lines are termed of *steepest descent* and, on account of Eq. (3.147), they are defined by the condition $\cos(\theta + n\,\varphi) = -1$. Then, lines of steepest descent are such that

$$\begin{gathered} \theta + n\,\varphi = (2m+1)\pi \quad \longmapsto \quad \varphi = \frac{2m+1}{n}\,\pi - \frac{\theta}{n} \ , \\ m = 0, 1, 2, \ \ldots\ , \ n-1 \ . \end{gathered} \tag{3.149}$$

Again, there exist n different possible angles φ, predicted by Eq. (3.149) and, hence, there are n different paths of steepest descent, for $\Re(f(z))$, departing from z_0. These paths can be easily detected in Figs. 3.7 and 3.8 and are explicitly displayed as thick solid lines in Figs. 3.9 and 3.10 for saddle points of order 2 and 3, respectively.

We note that, along the lines of steepest ascent or steepest descent of $\Re(f(z))$ departing from a saddle point z_0, the imaginary part of $f(z)$ remains constant. In fact, Eqs. (3.148) and (3.149) imply that, along lines of steepest ascent or steepest

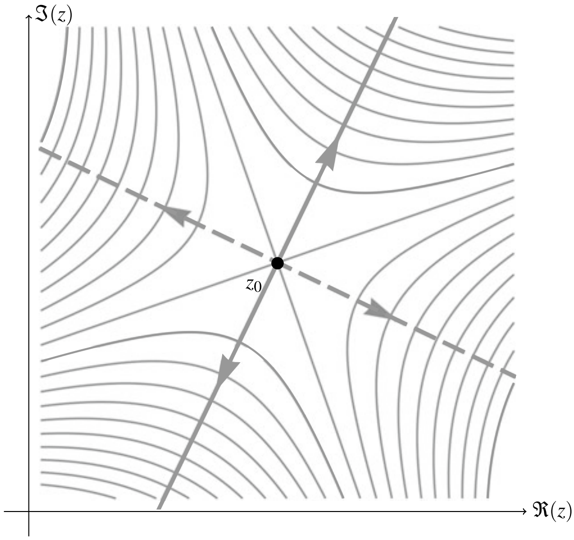


Fig. 3.9 Contour lines of the real part of $f(z)$ around a saddle point z_0 of order 2. The thick solid lines are the paths of steepest descent, while the thick dashed lines are the paths of steepest ascent

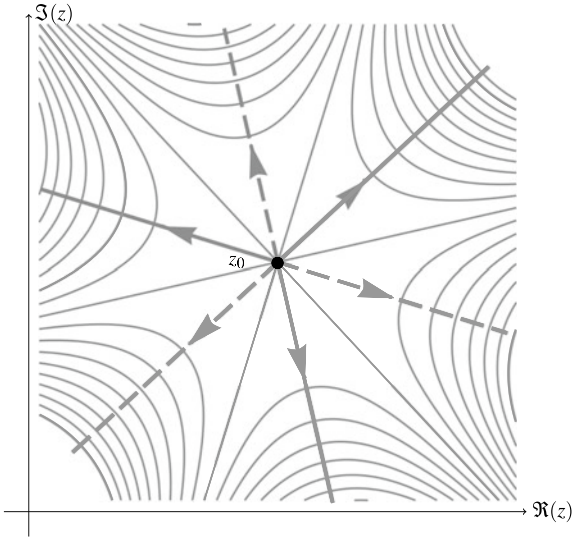


Fig. 3.10 Contour lines of the real part of $f(z)$ around a monkey saddle point, i.e. a saddle point z_0 of order 3. The thick solid lines are paths of steepest descent, while the thick dashed lines are paths of steepest ascent

descent of $\Re(f(z))$, $\theta + n\,\varphi$ is an integer multiple of π, so that $\sin(\theta + n\,\varphi)$ is zero. Thus, Eq. (3.147) implies that $\Im(f(z)) = \Im(f(z_0))$ along lines of steepest ascent or steepest descent of $\Re(f(z))$. In other words, the lines of steepest ascent or steepest descent of $\Re(f(z))$ are contour lines of $\Im(f(z))$.

3.5.3 Asymptotic Behaviour of Wave Packets at Large Times

Let us consider the three-dimensional wave packet given by Eq. (2.98),

$$\psi(\mathbf{x}, t) = \int_{-\infty}^{\infty} b(k, y, z, t)\, \mathrm{e}^{\mathrm{i}\,[kx-\omega(k)\,t]}\, \mathrm{d}\,k \;. \tag{3.150}$$

A particularly interesting case is one where the dependence on time of $b(k, y, z, t)$ is through an exponential function,

$$b(k, y, z, t) = \hat{b}(k, y, z)\, \mathrm{e}^{\eta(k)\,t} \;. \tag{3.151}$$

Then, on account of Eqs. (3.150) and (3.151), the expression of $\psi(\mathbf{x}, t)$, for a fixed position $\mathbf{x} = (x, y, z)$, is given by the time-dependent integral

$$I(t) = \int_{-\infty}^{\infty} \phi(k)\, \mathrm{e}^{\lambda(k)\,t}\, \mathrm{d}\,k \;. \tag{3.152}$$

Here, the complex function $\lambda(k)$ is defined as

$$\lambda(k) = \eta(k) - \mathrm{i}\,\omega(k) \;, \tag{3.153}$$

while

$$\phi(k) = \hat{b}(k, y, z)\, \mathrm{e}^{\mathrm{i}\,kx} \;. \tag{3.154}$$

The dependence of $\phi(k)$ on (x, y, z) is not explicitly declared with this notation as what really matters, in the forthcoming analysis, is just the dependence on time of the integral $I(t)$ or, equivalently, we can consider our reasoning as relative to a fixed position (x, y, z).

We aim to determine an approximate evaluation of $I(t)$ for large times t. This task can be managed by employing Theorem 3.3. In fact, integral $I(t)$ given by Eq. (3.152) can be considered as a path integral along a contour line γ coincident with the real axis in the complex plane and oriented along its positive direction,

$$I(t) = \int_{\gamma} \phi(k)\, \mathrm{e}^{\lambda(k)\,t}\, \mathrm{d}\,k \;. \tag{3.155}$$

Let us first imagine a situation where there exists a unique saddle point of function $\lambda(k)$, namely $k_0 \in \mathbb{C}$, and that $\phi(k)$ is not singular in k_0. We can imagine to deform path γ to γ^*, where γ^* crosses the saddle point k_0. A sketch of γ and γ^* is provided in Fig. 3.11. The question is whether $I(t)$ coincides with

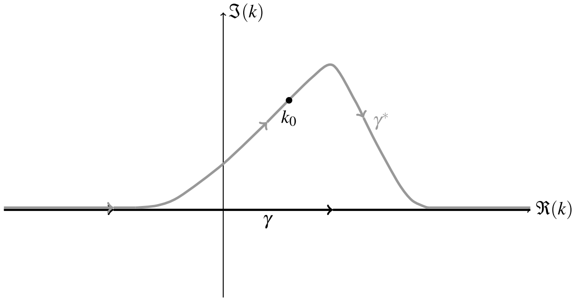


Fig. 3.11 Qualitative sketch of path γ, coincident with the real axis, and γ^* crossing the saddle point k_0 of $\lambda(k)$

$$I^*(t) = \int_{\gamma^*} \phi(k)\, \mathrm{e}^{\lambda(k)\,t}\, \mathrm{d}\,k\ . \tag{3.156}$$

The answer relies on Theorem 3.3. Integrals $I(t)$ and $I^*(t)$ coincide if path γ can be continuously deformed into γ^* within the domain where the integrand $\phi(k)\,\mathrm{e}^{\lambda(k)\,t}$ is holomorphic. In other words, one must check that no singularity of $\phi(k)\,\mathrm{e}^{\lambda(k)\,t}$ exists within the region bounded by $\gamma \cup \gamma^*$. This feature will be hereafter termed *holomorphy requirement*.

An interesting case is when γ^* locally coincides with a steepest descent path for $\Re(\lambda(k))$, crossing k_0. If k_0 is a second-order saddle point, in a small neighbourhood of k_0, we can approximate the integrand $\phi(k)\,\mathrm{e}^{\lambda(k)\,t}$, according to Eq. (3.144), as

$$\phi(k)\,\mathrm{e}^{\lambda(k)\,t} \approx \phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t}\,\mathrm{e}^{\lambda''(k_0)(k-k_0)^2\,t/2}\ . \tag{3.157}$$

Thus, following Eqs. (3.147) and (3.149) with $n = 2$, we get

$$\phi(k)\,\mathrm{e}^{\lambda(k)\,t} \approx \phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t}\,\mathrm{e}^{-\,|\lambda''(k_0)|r^2\,t/2}\ . \tag{3.158}$$

A change of r in the small interval $[0, \varepsilon]$, for a positive $\varepsilon \ll 1$, provides a local parametrization of γ^* in a small neighbourhood of k_0.

A key point in the formulation of the *steepest-descent approximation* is the following. The dominant contribution to $I^*(t)$ comes from a small neighbourhood of k_0, where the exponential $|\mathrm{e}^{\lambda(k)\,t}| = \mathrm{e}^{\Re(\lambda(k))\,t}$ is at its largest. In other words, an approximation of $I^*(t)$ is given by

$$I^*(t) = \int_{\gamma^*} \phi(k)\,\mathrm{e}^{\lambda(k)\,t}\,\mathrm{d}\,k \approx 2\,\mathrm{e}^{\mathrm{i}\,\varphi}\,\phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t} \int_0^{\varepsilon} \mathrm{e}^{-\,|\lambda''(k_0)|r^2\,t/2}\,\mathrm{d}\,r\ , \tag{3.159}$$

where the parametrization $k = k_0 + r\,\mathrm{e}^{\mathrm{i}\,\varphi}$, Eq. (3.144), has been used. We note that factor 2 comes from doubling the contribution of the integral over $r \in [0, \varepsilon]$ to include a piece of steepest ascent path to reach k_0 and one of steepest descent departing from

k_0. When t is very large, the integral of $\mathrm{e}^{-|\lambda''(k_0)|r^2 t/2}$ over $r \in [0, \varepsilon]$ does not differ much from the integral over $r \in [0, \infty]$, as the Gaussian function undergoes a rapid decay to 0 as r increases. Thus, we can write

$$I^*(t) \approx 2\,\mathrm{e}^{\mathrm{i}\varphi}\,\phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t} \int_0^\infty \mathrm{e}^{-|\lambda''(k_0)|r^2 t/2}\,\mathrm{d}r$$
$$= \mathrm{e}^{\mathrm{i}\varphi}\,\phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t} \sqrt{\frac{2\pi}{|\lambda''(k_0)|\,t}}\ . \tag{3.160}$$

From Eq. (3.149), we infer that φ is either $\pi/2 - \theta/2$ or $3\pi/2 - \theta/2$, where θ is the argument of $\lambda''(k_0)$. As a consequence, we obtain

$$I^*(t) \approx \pm\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\theta/2}\,\phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t} \sqrt{\frac{2\pi}{|\lambda''(k_0)|\,t}}\ . \tag{3.161}$$

Let us assume the validity of the holomorphy requirement, then Theorem 3.3 ensures that $I(t) = I^*(t)$ and we achieve the steepest-descent approximation of $I(t)$ at large times,

$$I(t) = \int_{-\infty}^{\infty} \phi(k)\,\mathrm{e}^{\lambda(k)\,t}\,\mathrm{d}k \approx \pm\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\theta/2}\,\phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t} \sqrt{\frac{2\pi}{|\lambda''(k_0)|\,t}}\ . \tag{3.162}$$

The ambiguity in the sign of the approximated integral is a consequence of the a-priori twofold choice in the definition of the steepest descent path that drives k away from k_0 along path γ^*, as suggested by Fig. 3.9. This is not a big problem when one is interested just in the large-time behaviour of $|I(t)|$, given by

$$|I(t)| \approx |\phi(k_0)|\,\mathrm{e}^{\Re(\lambda(k_0))\,t} \sqrt{\frac{2\pi}{|\lambda''(k_0)|\,t}}\ . \tag{3.163}$$

If we now relax the assumption that k_0 is a second-order saddle point of $\lambda(k)$ and assume a $n > 2$ order of the saddle point, Eq. (3.158) is now replaced by

$$\phi(k)\,\mathrm{e}^{\lambda(k)\,t} \approx \phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t}\,\mathrm{e}^{-|\lambda^{(n)}(k_0)|r^n t/n!}\ . \tag{3.164}$$

Equation (3.160) is modified into

$$\begin{aligned} I^*(t) &\approx 2\,\mathrm{e}^{\mathrm{i}\,\varphi}\,\phi(k_0)\,\mathrm{e}^{\lambda(k_0)\,t}\int_0^\infty \mathrm{e}^{-\,|\lambda^{(n)}(k_0)|r^n\,t/n!}\,\mathrm{d}\,r \\ &= \frac{2}{n}\,\mathrm{e}^{\mathrm{i}\,\varphi}\,\phi(k_0)\;\Gamma\!\left(\frac{1}{n}\right)\mathrm{e}^{\lambda(k_0)\,t}\left(\frac{n!}{|\lambda^{(n)}(k_0)|\,t}\right)^{1/n} . \end{aligned} \tag{3.165}$$

Here, $\Gamma(z)$ is *Euler's gamma function* [2],

$$\Gamma(z) = \int_0^\infty s^{\,z-1}\,\mathrm{e}^{-s}\,\mathrm{d}\,s \ . \tag{3.166}$$

Finally, Eqs. (3.162) and (3.163) are generalised to

$$\begin{aligned} I(t) &= \int_{-\infty}^{\infty} \phi(k)\,\mathrm{e}^{\lambda(k)\,t}\,\mathrm{d}\,k \\ &\approx \frac{2}{n}\,\mathrm{e}^{\mathrm{i}\,(2m+1)\pi/n}\,\mathrm{e}^{-\mathrm{i}\,\theta/n}\,\phi(k_0)\;\Gamma\!\left(\frac{1}{n}\right)\mathrm{e}^{\lambda(k_0)\,t}\left(\frac{n!}{|\lambda^{(n)}(k_0)|\,t}\right)^{1/n} , \end{aligned} \tag{3.167}$$

where $m = 0, 1, 2, \ \ldots\ , \ n-1$, and

$$|I(t)| \approx \frac{2}{n}\,|\phi(k_0)|\;\Gamma\!\left(\frac{1}{n}\right)\mathrm{e}^{\Re(\lambda(k_0))\,t}\left(\frac{n!}{|\lambda^{(n)}(k_0)|\,t}\right)^{1/n} . \tag{3.168}$$

Equation (3.167) shows the effects of the multiplicity of the possible steepest descent paths that depart from k_0, resulting in n possible values of the positive integer m. As m appears just in a phase factor, this multiplicity is ineffective when one deals with $|I(t)|$, as shown by Eq. (3.168).

We assumed the existence of just one saddle point of $\lambda(k)$. What if there are more? With several saddle points, the steepest-descent approximation just keeps that or those leading to the largest $\Re(\lambda(k_0))$, so that one filters the leading contribution to the integral $I(t)$. It is possible that two or more saddle points share the same value of $\Re(\lambda(k_0))$. In that case, their contributions have to be summed up in order to form the asymptotic approximation of the integral $I(t)$.

For a more detailed and exhaustive discussion of the steepest-descent approximation of time-dependent integrals, we refer the reader to textbooks on applied mathematics such as Ablowitz and Fokas [1], Bender and Orszag [5], or the more recent Arfken et al. [4]. All these books include a discussion of several examples where the steepest-descent approximation is employed.

Example 3.7 Let us consider a case where $I(t)$, given by Eq. (3.152), is defined with $\phi(k) = 1$ and

$$\lambda(k) = -4k^2 + 2k + 4\,\mathrm{i}\,k \ , \tag{3.169}$$

namely

$$I(t) = \int_{-\infty}^{\infty} e^{(-4k^2+2k+4\,i\,k)\,t}\, d\,k\ . \tag{3.170}$$

The integral on the right-hand side of Eq. (3.170) can be evaluated analytically, so that we obtain

$$I(t) = \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-3\,t/4}\, e^{i\,t}\ . \tag{3.171}$$

There is an interesting fact about Eqs. (3.170) and (3.171). The integrand in Eq. (3.170) tends to ∞ when $t \to \infty$, for every k such that $0 < k < 1/2$. On the other hand, Eq. (3.171) shows that $I(t)$ tends to 0 when $t \to \infty$. This situation is often reproduced with wave packets: although there are normal modes whose amplitude grows in time, the wave packet as a whole might tend to 0 in the limit $t \to \infty$.

One can apply to Eq. (3.170) the steepest-descent approximation. Since

$$\lambda'(k) = -8k + 2 + 4\,i\ , \qquad \lambda''(k) = -8\ , \tag{3.172}$$

there is a single saddle point,

$$k_0 = \frac{1+2\,i}{4}\ , \tag{3.173}$$

of order $n = 2$. We have

$$\lambda(k_0) = -\frac{3}{4} + i\ . \tag{3.174}$$

Function $\lambda(k)$ satisfies the holomorphy requirement over the whole complex plane. We can thus apply Eq. (3.163) to obtain

$$|I(t)| \approx \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-3\,t/4}\ . \tag{3.175}$$

In fact, in this case, the steepest-descent approximation yields the exact result for $|I(t)|$, as it can be easily checked by comparing Eqs. (3.171) and (3.175).

Example 3.8 A classical application of the steepest-descent method is given by *Stirling's approximation* of the factorial [2]. We base the evaluation on Euler's gamma function, defined by Eq. (3.166), and on its property that, if n is a natural number, then $n! = \Gamma(n+1)$ [2]. In fact, from Eq. (3.166), we can write

$$n! = \int_{0}^{\infty} s^n\, e^{-s}\, d\,s\ . \tag{3.176}$$

We change the variable of integration to $r = s/n$, so that we obtain

$$n! = n^{n+1} \int_0^\infty e^{[\ln(r)-r]\,n}\, d\,r \ . \tag{3.177}$$

We aim to achieve an approximate expression of the integral on the right-hand side of Eq. (3.177) when n is very large. Then, we invoke the steepest-descent approximation. We have

$$\lambda(r) = \ln(r) - r \ . \tag{3.178}$$

There is just one saddle point, $\lambda'(r_0) = 0$, namely $r_0 = 1$. We obtain

$$\lambda(r_0) = -1 \ , \qquad \lambda''(r_0) = -1 \ . \tag{3.179}$$

The saddle point is placed on the real axis and it is of order $n = 2$. The argument of $\lambda''(r_0)$ is $\theta = \pi$ and the steepest descent path just coincides with the real axis oriented along its positive direction. This is a simple case where $\gamma = \gamma^*$. From Eqs. (3.162) and (3.177), we can finally write

$$n! \approx \sqrt{2\pi}\; n^{n+1/2}\, e^{-n} \ , \tag{3.180}$$

which is the well-known Stirling's approximation for the factorial of a large natural number n.

References

1. Ablowitz MJ, Fokas AS (2003) Complex variables: introduction and applications. Cambridge University, Cambridge
2. Abramowitz M, Stegun I (1968) Handbook of mathematical functions. Dover, New York
3. Apostol TM (1967) Calculus, vol. 2: multi-variable calculus and linear algebra with applications to differential equations and probability. Wiley, New York
4. Arfken GB, Weber HJ, Harris FE (2012) Mathematical methods for physicists: a comprehensive guide. Elsevier, New York
5. Bender CM, Orszag SA (1999) Advanced mathematical methods for scientists and engineers I. Springer, New York
6. Cartan H (1995) Elementary theory of analytic functions of one or several complex variables. Dover, New York
7. Debnath L, Bhatta D (2014) Integral transforms and their applications. CRC, New York
8. Priestley HA (2003) Introduction to complex analysis. Oxford University, Oxford
9. Schiff JL (1999) The Laplace transform: theory and applications. Springer, New York