The Quaternion-Fourier Transform and Applications

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Abstract. It is well-known that the Fourier transforms plays a critical role in image processing and the corresponding applications, such as enhancement, restoration and compression. For filtering of gray scale images, the Fourier transform in $\mathbb{R}^2$ is an important tool which converts the image from spatial domain to frequency domain, then by applying filtering mask filtering is done. To filter color images, a new approach is implemented recently which uses hypercomplex numbers (called Quaternions) to represent color images and uses Quaternion-Fourier transform for filtering. The quaternion Fourier transform has been widely employed in the colour image processing. The use of quaternions allow the analysis of color images as vector fields, rather than as color separated components. In this paper we mainly focus on the theoretical part of the Quaternion Fourier transform: the real Paley-Wiener theorems for the Quaternion-Fourier transform on $\mathbb{R}^2$ for Quaternion-valued Schwartz functions and L^p-functions, which generalizes the recent results of real Paley-Wiener theorems for scalar- and quaternion-valued L^2-functions.

Keywords: Quaternion analysis · Paley-Wiener theorem
Quaternion-Fourier transform

1 Introduction

The original Paley-Wiener theorem [8] describes the Fourier transform of L^2-functions on the real line with support in a symmetric interval as entire functions of exponential type whose restriction to the real line are L^2-functions, which has proved to be a basic tool for transform in various set-ups. Recently, there has been a great interest in the real Paley-Wiener theorem due to Bang in [1] and Tuan in [11], in which the adjective "real" expresses that information about the support of the Fourier transform comes from growth rates associated

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to the function f on $\mathbb{R}$, rather than on $\mathbb{C}$ as in the classical "complex Paley-Wiener theorem". The Fourier transform of functions with polynomial domain supports, of functions vanishing on some ball, and even in the classical case the result obtained here are also new. The set-up is as follows. For any functions $f \in \mathcal{S}(\mathbb{R}^k)$, there holds

$$\lim_{n\to\infty} \|P^n(iD)f\|_p^{\frac{1}{n}} = \sup_{y\in \mathrm{supp}\hat{f}} |P(y)|$$

and

$$\lim_{n\to\infty} \| \sum_{m=0}^{\infty} \frac{n^m \Delta^m f}{m!} \|_p^{\frac{1}{n}} = exp\Big(- \inf_{y\in \mathrm{supp}\hat{f}} \Big) |y|^2,$$

here $P(y)$ is a non-constant polynomial and $P(iD)$ is the transmutation operator.

In this paper we will consider the real Paley-Wiener theorem for the quaternion Fourier transform (QFT) which is a nontrivial generalization of the real and complex Fourier transform (FT) to quaternion algebra. The four components of QFT separate four cases of symmetry in real signals instead of only two ones in the complex FT. The QFT plays an important role in the representation of signals and transforms a quaternion 2D signal into a quaternion-valued frequency domain signal. There are lots of efforts to devote to many important properties and applications of the QFT (see [2–4,6,7,9,10]).

Motivated by recent work [5] which derived a real Paley-Wiener theorem to characterize the quaternion-valued L^2-functions whose QFT has compact support, we systematically develop a real Paley-Wiener theorem for QFT on $\mathbb{R}^2$ for quaternion-valued Schwartz functions and L^p-functions, $1 \le p \le \infty$.

The paper is organized as follows. Section 2 is devoted to recalling some definitions and properties for quaternions and their analysis. In Sect. 3, we prove the real Paley-wiener theorems for the QFT.

2 Preliminaries

The quaternion algebra $\mathbb{H}$ and Clifford algebra are extensions of the algebra of complex numbers. The quaternion algebra is given by

$$\mathbb{H} = \{q | q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where the elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ obey Hamilton's multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i^2} = \mathbf{j^2} = \mathbf{k^2} = \mathbf{ijk} = -\mathbf{1}.$$

The conjugate of a quaternion $q \in \mathbb{H}$ is obtained by changing the sign of the pure quaternion part, i.e., $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. This leads to a norm of $q \in \mathbb{H}$, which is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{{q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2}.$$

A quaternion-valued function $f : \mathbb{R}^2 \to \mathbb{H}$ will be written as

$$f(\mathbf{x}) = f_0(\mathbf{x}) + f_1(\mathbf{x})\mathbf{i} + f_2(\mathbf{x})\mathbf{j} + f_3(\mathbf{x})\mathbf{k}, \qquad \mathbf{x} = (x_1, x_2),$$

with real-valued coefficient functions $f_0, f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}$. We introduce the space $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, as the left module of all quaternion-valued functions $f : \mathbb{R}^2 \to \mathbb{H}$ satisfying

$$\|f\|_p := \Big(\int_{\mathbb{R}^2} |f(\mathbf{x})|^p d\mathbf{x} \Big)^{1/p} < \infty, \qquad \text{if } 1 \le p < \infty,$$
$$\|f\|_\infty := ess \sup_{\mathbf{x} \in \mathbb{R}^2} |f(\mathbf{x})| < \infty, \qquad \text{if } p = \infty.$$

Definition 1. *The normalized right-sided QFT of a function $f \in L^1(\mathbb{R}^2)$ is defined by*

$$\mathcal{F}_q^r f(\lambda) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-\mathbf{i}x_1\lambda_1} e^{-\mathbf{j}x_2\lambda_2} d\mathbf{x}, \qquad \text{for all } \lambda \in \mathbb{R}^2. \tag{1}$$

So the corresponding inversion formula can be given as

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q^r f(\lambda) e^{\mathbf{j}x_2\lambda_2} e^{\mathbf{i}x_1\lambda_1} d\lambda, \qquad \text{for all } \mathbf{x} \in \mathbb{R}^2. \tag{2}$$

Similarly,

Definition 2. *The normalized left-sided QFT of a function $f \in L^1(\mathbb{R}^2)$ is defined through*

$$\mathcal{F}_q^l f(\lambda) = \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1\lambda_1} e^{-\mathbf{j}x_2\lambda_2} f(\mathbf{x}) d\mathbf{x}, \qquad \text{for all } \lambda \in \mathbb{R}^2, \tag{3}$$

and the corresponding inversion formula can be given as

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathbf{j}x_2\lambda_2} e^{\mathbf{i}x_1\lambda_1} \mathcal{F}_q^l f(\lambda) d\lambda, \qquad \text{for all } \mathbf{x} \in \mathbb{R}^2. \tag{4}$$

The QFT of a tempered distribution T is defined by

$$\langle \mathcal{F}_q^r T, \phi \rangle = \langle T, \mathcal{F}_q^l \phi \rangle, \qquad \phi \in \mathcal{S}(\mathbb{R}^2), \tag{5}$$

which is compatible with its definition on $L^1(\mathbb{R}^2)$.

In what follows, we recall the following important property of the QFT. For more properties and details, we refer to [5,6].

Proposition 1. (QFT partial derivatives). *If $\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(\mathbf{x}) \in L^1(\mathbb{R}^2)$, $m_1, m_2 \in \mathbb{N}_0$, then we have*

$$\mathcal{F}_q^r \{ \frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(\mathbf{x}) \mathbf{i}^{-m_1} \}(\lambda) = \lambda_1^{m_1} \mathcal{F}_q^r f(\lambda) \lambda_2^{m_2} \mathbf{j}^{m_2}, \tag{6}$$

and

$$\mathcal{F}_q^l\{\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1}\partial x_2^{m_2}}\mathbf{j}^{-m_2}f(\mathbf{x})\}(\lambda)=\mathbf{i}^{m_1}\lambda_1^{m_1}\mathcal{F}_q^l f(\lambda)\lambda_2^{m_2}. \tag{7}$$

Proposition 2. (QFT Plancherel). *If $f,g\in L^2(\mathbb{R}^2)$, then there holds*

$$(f,g)=\frac{1}{(2\pi)^2}(\mathcal{F}_q^r f,\mathcal{F}_q^r g). \tag{8}$$

In particular, if $f=g$, we have the following Parseval's Identity:

$$\|f\|_2=\frac{1}{2\pi}\|\mathcal{F}_q^r f\|_2. \tag{9}$$

3 Real Paley-Wiener Theorems for the Quaternion-Fourier Transform

First, we consider the functions vanishing outside a ball, which is the Paley-Wiener-Type Theorem.

Theorem 1. *Let $P(x)=x_1^{n_1}x_2^{n_2}$ for any fixed nonnegative integers n_1 and n_2. Suppose $P(\partial)^m\in L^p(R^2)$ for all $m\in N_0$ and $1\le p\le\infty$. Assume further that either $\mathcal{F}_q^r f$ has compact support or that the set $\lambda\in R^2:|P(\lambda)|\le R$ is compact for all $R\ge 0$. Then in the extended positive real numbers*

$$\lim_{m\to\infty}\|P^m(\partial)f\|_p^{\frac{1}{m}}=\sup_{\lambda\in supp\mathcal{F}_q^r(f)}|P(\lambda)|. \tag{10}$$

Proof. The case for $f\equiv 0$ is trivial, so we assume that $f\not\equiv 0$.

Step 1: If $2\le p\le\infty$, applying the Hausdorff-Young's inequality with $p^{-1}+q^{-1}=1$:

$$\begin{aligned}\|P^m(\partial)f\mathbf{i}^{-mn_1}\|_p&\le C\|P^m(\lambda)\mathcal{F}_q^r(f)\mathbf{j}^{mn_2}\|_q\\&=C\|P^m(\lambda)\mathcal{F}_q^r(f)\mathbf{j}^{mn_2}\|_{L^q(supp\mathcal{F}_q^r(f))}\\&=C\|P^m(\lambda)\mathcal{F}_q^r(f)\|_{L^q(supp\mathcal{F}_q^r(f))}\\&\le C\sup_{\lambda\in supp\mathcal{F}_q^r(f)}|P(\lambda)|^m\|\mathcal{F}_q^r(f)\|_{L^q(supp\mathcal{F}_q^r(f))},\end{aligned}$$

so we have

$$\begin{aligned}\lim_{m\to\infty}\sup\|P^m(\partial)f\mathbf{i}^{-mn_1}\|_p^{\frac{1}{m}}&\le\sup_{\lambda\in supp\mathcal{F}_q^r(f)}|P(\lambda)|\lim_{m\to\infty}\sup C^{\frac{1}{m}}\|\mathcal{F}_q^r(f)\|_q^{\frac{1}{m}}\\&=\sup_{\lambda\in supp\mathcal{F}_q^r(f)}|P(\lambda)|.\end{aligned} \tag{11}$$

For the case $1 \le p < 2$, using Hölder's inequality and Plancherel Theorem for the QFT, we get

$$\begin{aligned}\|f\|_p^p &= \int_{R^2} (1+|x|^2)^{-2p} |(1+|x|^2)^2 f(x)|^p dx \\ &\le \|(1+|x|^2)^{-2p}\|_{\frac{2}{2-p}} \|(1+|x|^2)^2 f(x)\|_2^p \\ &\le C\|(1+|x|^2)^2 f(x)\|_2^p \\ &= C\|(1-\Delta)^2 \mathcal{F}_q^r(f)\|_2^p, \end{aligned} \tag{12}$$

here $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ denotes the Laplacian.

Substituting f in the above inequality with $P^m(\partial)\mathbf{i}^{-mn_1}$, there holds

$$\|P^m(\partial)\mathbf{i}^{-mn_1}\|_p^p \le C\|(1-\Delta)^2 P^m(\lambda)\mathcal{F}_q^r(f)\mathbf{j}^{mn_2}\|_2^p.$$

By mathematical induction, we can show that

$$(1-\Delta)^2 (P^m(\lambda)\mathcal{F}_q^r(f)\mathbf{j}^{mn_2}) = P^{m-4}(\omega)\Phi_n(\omega)\mathbf{i}^{mn_2},\ m > 4,$$

where $\mathrm{supp}\Phi_n \subset \mathrm{supp}\mathcal{F}_q^r(f)$ and $\Phi_n(\omega) \le Cn^4$.

Hence,

$$\begin{aligned}\|P^m(\partial)f\mathbf{i}^{-mn_1}\|_p &\le C\|P^{m-4}\Phi_n(\omega)\mathbf{i}^{mn_2}\|_2 \\ &\le C \sup_{supp\mathcal{F}_q^r(f)} |P(\omega)|^{m-4} \|\Phi_n(\omega)\mathbf{j}^{mn_2}\|_2 \\ &\le Cn^4 \sup_{supp\mathcal{F}_q^r(f)} |P(\omega)|^{m-4}, \end{aligned}$$

which implies

$$\lim_{m\to\infty} \sup \|P^m(\partial)fi^{-mn_1}\|_p^{\frac{1}{m}} \le \sup_{supp\mathcal{F}_q^r(f)} |P(\omega)|. \tag{13}$$

In case $p = \infty$, we have

$$\begin{aligned}\|f\|_\infty &\le (2\pi)^{-1}\|f\|_1 \\ &= (2\pi)^{-1} \int_{R^2} (1+|x|^2)^{-2} |(1+|x|^2)^2 \mathcal{F}_q^r(f)| dx \\ &= (2\pi)^{-1} \|(1+|x|^2)^{-2}\|_2 \|(1+|x|^2)^2 \mathcal{F}_q^r(f)\|_2 \\ &\le C\|(1+|x|^2)^2 \mathcal{F}_q^r(f)\|_2. \end{aligned}$$

Therefore,

$$\begin{aligned}\|P^n(\partial)f\mathbf{i}^{-mn_1}\|_\infty &\le C\|(1+|x|^2)^2 P^n(\omega)\mathcal{F}_q^r(f)\mathbf{j}^{mn_2}\|_2 \\ &= C\|(1+|x|^2)^2 \mathcal{F}_q^r(f)P^n(\omega)\mathbf{j}^{mn_2}\|. \end{aligned}$$

Consequently,

$$\lim_{m\to\infty} \sup \|P^m(\partial)f\mathbf{i}^{-mn_1}\|_\infty^{\frac{1}{m}} \le \sup_{\omega\in supp(1+|x|^2)^2\mathcal{F}_q^r(f)} |P(\omega)|$$
$$= \sup_{\omega\in supp\mathcal{F}_q^r(f)} |P(\omega)|. \tag{14}$$

Step 2: Since $f \in S(R^2)$, the function f and its partial derivatives vanish at infinity, therefore, integration by parts gives

$$\int_{R^2} \overline{P^m(\partial)f}P^m(\partial)f(x)dx = \int_{R^2} P^m(\partial)\overline{f(x)}P^m(\partial)f(x)dx$$
$$= -\int_{R^2} \overline{f(x)}P^{2m}(\partial)f(x)dx.$$

Hence, by Hölder inequality, we have

$$\|P^m(\partial)f\|_2^2 \le \|f\|_q\|P^{2m}(\partial)f\|_p.$$

Replacing f by $P(\partial)f$ in above inequality, we have

$$\|P^{m+1}(\partial)f\|_2^2 \le \|P(\partial)f\|_q\|P^{2m+1}(\partial)f\|_p.$$

Since $f \in S(R^2)$, we have that $P(iD)f \neq 0$, and consequently,

$$\sup_{\omega\in supp\mathcal{F}_q^r(f)} |P(\omega)| = \lim_{m\to\infty} \|P^{m+1}(\partial)f\|_2^{\frac{1}{m+1}}$$
$$= \lim_{m\to\infty} \|P^{m+1}(\partial)f\|_2^{\frac{2}{2m+1}}$$
$$\le \lim_{m\to\infty} \|P(\partial)f\|^{\frac{1}{2m+1}} \lim_{m\to\infty} \inf \|P^{2n+1}(\partial)f\|_p^{\frac{1}{2n+1}}$$
$$= \lim_{m\to\infty} \inf \|P^{2m+1}(\partial)f\|_p^{\frac{1}{2m+1}}.$$

For another, applying formula for the proved case $p = 2$, there holds

$$\sup_{\omega\in supp\mathcal{F}_q^r(f)} |P(\omega)| = \lim_{m\to\infty} \|P^m(\partial)f\mathbf{i}^{-mn_1}\|_2^{\frac{1}{m}}$$
$$\le \lim_{m\to\infty} \|f\|_q^{\frac{1}{2m}} \lim_{m\to\infty} \inf \|P^{2m}(\partial)f\mathbf{i}^{-mn_1}\|_p^{\frac{1}{2m}}$$
$$= \lim_{m\to\infty} \inf \|P^{2m}(\partial)f\mathbf{i}^{-mn_1}\|_p^{\frac{1}{2m}}.$$

In summary, we get

$$\lim_{m\to\infty} \inf \|P^m(\partial)f\|_p^{\frac{1}{m}} \ge \sup_{\omega\in supp\mathcal{F}_q^r(f)} |P(\omega)|. \tag{15}$$

Inequality (15) together with inequalities (11), (13) and (14) give the formula (10). The theorem is proved. □

Remark 1. Due to the noncommutative property of quaternions, we only consider the special polynomials $P(x) = x_1^{n_1} x_2^{n_2}$. For the general polynomials in $\mathbb{R}^2$, we can only obtain the results in Step 1 in the above theorem.

Second, we consider the functions vanishing on a ball, which is the Boas-Type Theorem.

Theorem 2. *For any function $f \in S(R^2)$, the following equality holds:*

$$\lim_{n\to\infty} \| \sum_{m=0}^{\infty} \frac{n^m \Delta^m f}{m!} \|_p^{\frac{1}{n}} = \exp(- \inf_{y \in supp \mathcal{F}_q^r(f)} |y|^2), \ 1 \le p \le \infty. \tag{16}$$

Proof. From Proposition 1, we have for any function $f \in S(R^2)$:

$$\mathcal{F}_q^r \Big(\sum_{m=0}^{\infty} \frac{n^m \Delta^m f(x)}{m!} \Big) = \exp(-n|y|^2) \mathcal{F}_q^r(f)(y).$$

Follow the similar proof of the previous theorem, if $2 \le p < \infty$, applying the Hausdoff-Young's inequality with $p^{-1} + q^{-1} = 1$, there holds

$$\| \sum_{m=0}^{\infty} \frac{n^m \Delta^m f}{m!} \|_p \le C \| e^{-n|\lambda|^2} \mathcal{F}_q^r(f) \|_q \le C e^{-n \inf |y|^2} \| \mathcal{F}_q^r(f) \|_q.$$

Therefore,

$$\lim_{n\to\infty} \sup \| \sum_{m=0}^{\infty} \frac{n^m \Delta^m f}{m!} \|_p^{\frac{1}{n}} \le \exp(- \inf_{y \in supp \mathcal{F}_q^r(f)} |y|^2). \tag{17}$$

For the case $1 \le p < 2$, we first use the inequality (12) to get

$$\| \sum_{m=0}^{\infty} \frac{n^m \Delta^m f}{m!} \|_p \le \| (1 - \Delta)^2 e^{-n|y|^2} \mathcal{F}_q^r(f) \|_2.$$

Second, It's easy to show that

$$(1 - \Delta)^2 \exp(-n|y|^2 \mathcal{F}_q^r(f)) = \exp(-n|y|^2) \Phi_n(y),$$

with $\mathrm{supp}\Phi_n \subset \mathrm{supp}\mathcal{F}_q^r(f)$ and $\|\Phi_n\|_2 \le C n^4$.

Hence, we can obtain that

$$\lim_{n\to\infty} \sup \| \sum \frac{n^m \Delta^m f}{m!} \|_p^{\frac{1}{n}} \le \exp(- \inf_{y \in supp \mathcal{F}_q^r(f)} |y|^2). \tag{18}$$

In case $p = \infty$, using the inequality

$$\|f\|_\infty \le C \| (1 + |y|^2)^2 \mathcal{F}_q^r(f) \|_2$$

we get

$$\|\sum_{m=0}^{\infty}\frac{n^m\Delta^m f}{m!}\|_\infty \le C\|\exp(-n|y|^2)\mathcal{F}_q^r(f)(1+|y|^2)^2\|_2.$$

Therefore, we get inequality

$$\lim_{n\to\infty}\sup\|\sum_{m=0}^{\infty}\frac{n^m\Delta^m f}{m!}\|_\infty^{\frac{1}{n}} \le \exp(-\inf_{y\in supp\mathcal{F}_q^r(f)}|y|^2). \tag{19}$$

On the other hand, using the Plancherel theorem for the QFT and Hölder's inequality we have

$$\begin{aligned}\|\sum_{m=0}^{\infty}\frac{n^m\Delta^m f}{m!}\|_2^2 &= \int_{R^2}|\sum_{m=0}^{\infty}\frac{n^m\Delta^m f}{m!}|^2 dx\\ &= \int_{R^2} e^{-2n|y|^2}|\mathcal{F}_q^r(f)|^2 dy\\ &= \int_{R^2}\overline{\mathcal{F}_q^r(f)(y)}\exp(-2n|y|^2)\mathcal{F}_q^r(f)(y)dy\\ &= \int_{R^2}\overline{f(x)}\sum_{m=0}^{\infty}\frac{(2n)^m\Delta^m f(x)}{m!}dx\\ &\le \|f\|_q\|\sum_{m=0}^{\infty}\frac{(2n)^m\Delta^m f}{m!}\|_p.\end{aligned}$$

Similarly,

$$\|\sum_{m=0}^{\infty}\frac{n^m\Delta^m f}{m!}\|_2^2 \le \|\sum_{m=0}^{\infty}\frac{\Delta^m f}{m!}\|_q\|\sum_{m=0}^{\infty}\frac{(2n-1)^m\Delta^m f}{m!}\|_p.$$

In summary, we get

$$\lim_{n\to\infty}\inf\|\sum_{m=0}^{\infty}\frac{n^m\Delta^m f}{m!}\|_p^{\frac{1}{n}} \ge \lim_{n\to\infty}\|\sum_{m=0}^{\infty}\frac{n^m\Delta^m f}{m!}\|_2^{\frac{1}{n}} = \exp(-\inf|\omega|^2). \tag{20}$$

Combining inequalities (17), (18) and (20) we have the final result. □

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