

Generation of Spirals in Excitable Media



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Abstract The generation of dynamic spirals under conditions of excitability is presented. After a short description of some basic principles of nonlinear dynamics illustrated in the phase plane, we explain what excitable systems are, how excitation waves propagate, and why external forces influence rotating or moving spirals. Some images of such dynamic spirals are exhibited.

1 Introduction

There are two major categories of spirals we have learned about in chapter **Appearance in Nature**: those that have an immobile and static structure (horns of blackbucks or sheep, snails and seashells, pine cones or corals). Others rotate in time and space (galaxies, cyclones, air turbulence, the tail of a chameleon, a proboscis of butterfly, fish swarms or colonies of amoebae).

Artists will naturally create spiral pictures of a quiescent nature, as Gustav Klimt did in his frieze “Tree of life”. His colleague Johannes Itten, on the other hand, painted a spiral which starts to move by the dynamic choice of colors within the structure. Motion becomes a central theme for Paul Klee who considers spirals that open and close towards the environment. Stormy spirals are captured in the works of Leonardo da Vinci (with his drawings of water turbulence), Vincent van Gogh (in the “excited” sky of his masterpiece “Starry Night”) and Joseph Turner (living through a chaotic tempest in his painting “Snow storm”). See chapter **Arts and Beyond**.

In a preceding chapter **Spirals, Their Types and Peculiarities** mathematical presentations have provided some insight into the shape of various spirals that follow a spatially fixed path. In this chapter we will focus on rotating spirals. Dynamic spirals

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moving in space and forming vortices are frequently found in “excitable media” [1]. In fact, one finds that excitable media are dominating the scene: they occur in a large variety of systems of natural science, including chemistry, biology, neurobiophysics, biomedicine, and others.

Before dealing with excitable media coupled to physical transport processes, we will explain some of the general and basic principles about the phase plane and processes occurring in this plane. The brief introduction to the theory of dynamical systems may support readers to understand the following presentations, as well as articles in Part V.

In many experiments in excitable media rotating spirals can be generated by external forces. We will show schematically how an air blow breaks the excitable front, leading to curling of the open ends [2]. This way rotating spirals are created. The motion of their tips depend on chemical components and aging of the system. Further examples will illustrate how spiral motions are affected by electric fields [3] or laser light [4].

2 The Phase Plane

Let us write down a pair of ordinary differential equations (ODEs) of 1st degree for variables x and y , which are **linearly** coupled (by “+” or “-”) in the form

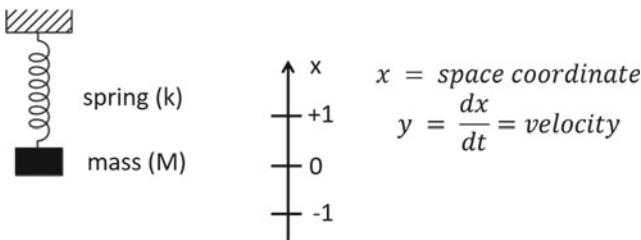
$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

with a, b, c, d real and constant coefficients.

These ODEs are soluble.

Example: Harmonic oscillator



Equation of motion

$$M \frac{d^2x}{dt^2} + kx = 0$$

(ODE of 2nd degree).

With $M = 1, k = 1$ (dimensionless quantities):

$$\frac{d^2x}{dt^2} + x = 0.$$

By writing

$$\frac{dx}{dt} = y, \quad \frac{d^2x}{dt^2} = -x \quad \rightarrow \quad \frac{dy}{dt} = -x$$

we obtain the 2-variable first order system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x.$$

Special solution:

$$x = \sin t$$

$$y = \cos t$$

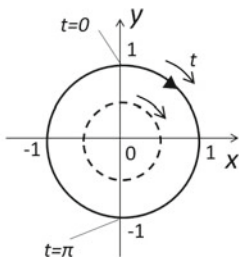
(t : dimensionless time),

with initial conditions (for $t = 0$):

$$x_0 = 0$$

$$y_0 = 1$$

Representation in the phase plane:



Cartesian coordinate system with variables x, y :
Evolution along trajectory (no explicit time)

Trajectory is a circle.
For different amplitudes of x and y → ellipse

Isoclines:

These are curves with the same gradient (slope) of the phase plane trajectory: e.g., for the equation

$$\frac{dy}{dx} = f(x, y)$$

Set $f(x, y)$ equal to a constant m :

$$f(x, y) = m.$$

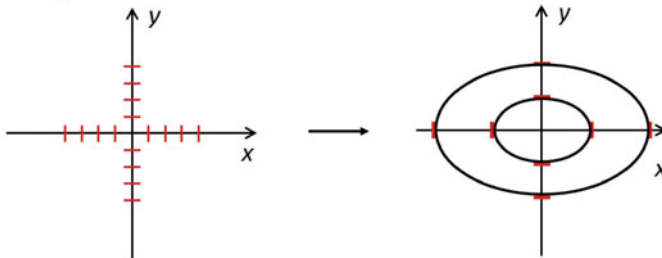
Especially useful are nullclines with $m = 0$.

Simple case: Harmonic oscillator

Nullclines:

$$\begin{array}{lcl} f(x, y) = y = 0 & & y = 0 \\ & \rightarrow & \\ g(x, y) = -x = 0 & & x = 0 \end{array}$$

Solution trajectories cross the nullclines on the axes according to the points marked in red.



The nullcline picture fits qualitatively for a circle or ellipse.

Nonlinear Case

A pair of nonlinear ODEs, (explicit and autonomous, i.e., no explicit dependence on t) can be written as

$$\begin{aligned} \frac{du}{dt} &= f(u, v, \alpha, \beta, \dots) \\ \frac{dv}{dt} &= g(u, v, \alpha, \beta, \dots) \end{aligned}$$

with f, g some nonlinear functions of the variables u, v and α, β, \dots parameters. (Certain mathematical conditions must be fulfilled to ensure existence and uniqueness.)

The symbols u, v are used here for concentrations of chemical variables, e.g.,

$u = \text{activator}$

$v = \text{inhibitor}$

In general, these ODEs cannot be solved, but there are approximative techniques used according to the specific problem at hand.

3 Excitable Systems

Excitable systems are an important special case. In Fig. 1 the main properties are illustrated in the u, v -plane. u is supposed to vary quickly (the activator), v slowly in time (the inhibitor). According to appropriate kinetic models the nullcline of function

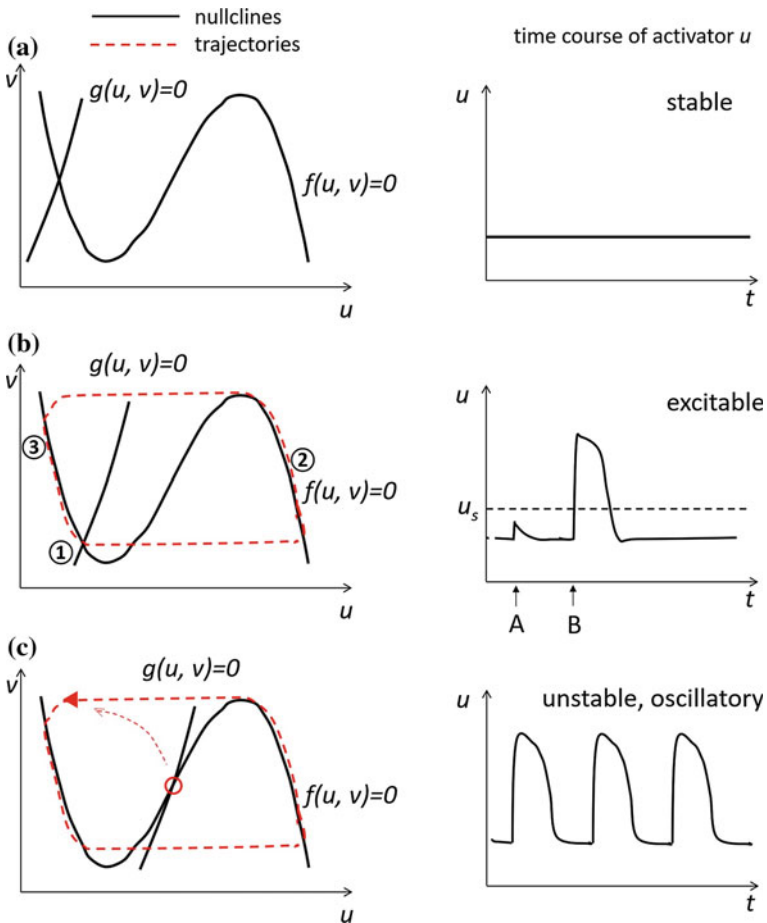


Fig. 1 Nullclines, see text

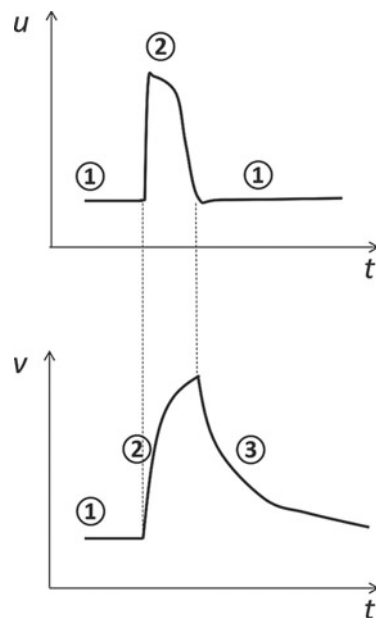
f [$f(u, v) = 0$], as a general supposition, has one minimum and one maximum. The parts with negative slope are stable, the rising part is unstable. The nullcline of function g [$g(u, v) = 0$] is monotonically increasing and stable. If these nullclines intersect, we obtain a fixed point, stable or unstable, without temporal evolution. There are the following scenarios (see Fig. 1):

- (a) Both stable nullclines intersect \rightarrow stable node, i.e., perturbations die out.
- (b) Again there is at ① a stable intersection. The system is resting and excitable. But only perturbations smaller than the threshold value u_s decay (see A). If $u > u_s$ there follows a large but short excursion of $u \rightarrow$ excitable state ② (see B). After returning to $u \leq u_s$, the system recovers (relatively slowly) and moves to the original excitable state \rightarrow refractory period ③. After reaching ① it waits for another superthreshold perturbation. Thus, in this case the system has mainly three states: excitable, excited (active), and refractory (resting).
- (c) A stable nullcline intersects with the unstable (rising) part of $f(u, v) = 0$. The system will move away from the crossing point towards a stable trajectory, which in this case develops towards a limit cycle, i.e., to autonomous oscillations (without any rest state).

Excitation Pulse

Following the trajectory (dashed line) in Fig. 1b we derive for an excitable system the characteristic temporal profiles of variables u and v (Fig. 2). Activator u is produced

Fig. 2 Excitation pulse as a function of time



as a pulse at short time scale, inhibitor v responds on slower time scale with long refractory tail: separation of variables possible.

4 Excitation Waves

Propagation of waves is frequently observed in spatially extended oscillatory or excitable systems. They are described by solutions of reaction-diffusion equations.

For one spatial coordinate x these write:

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u, v)$$

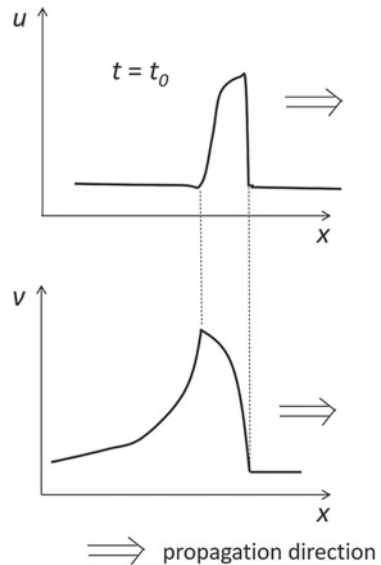
$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + g(u, v)$$

with partial derivatives written as $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \dots$ and D_u, D_v diffusion coefficients of species u and v , respectively. For three-dimensional space with coordinates $x, y,$ and $z,$ one uses the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The spatial profiles in a one-dimensional excitable system at a given time $t = t_0$ can be drawn as in Fig. 3.

Fig. 3 Spatial profiles of an excitation wave



Introducing diffusion as a transport process opens the possibility of spatial contact between species leading to local coupling. The excitation pulse of activator u moving in x -direction will transfer the excited state to the molecules in front. This causes the propagation of excitation in that direction and leaves behind an unexcited refractory tail which will recover (slowly) to a newly excitable state. This way a backward propagation becomes impossible. (Often the spreading of a forest fire front is taken as an analogon: a small enough distance between trees will support the motion of the front, whereas the fire is extinguished behind the front due to lack of “fuel”. The forest then recovers during a new growth phase.)

Such reaction-diffusion waves propagate with constant velocity, which depends on local concentrations and front curvature. They are frequently initiated in a thin excitable solution layer by a localized disturbance (immersion of a thin wire, gas bubble, dust particle, or laser spot). From there a circular wave would travel in outward direction. A periodically applied external disturbance would result in concentric rings - a so-called “target pattern”. Then, the wave velocity will be a function of the distance between the waves governed by a dispersion relation.

5 Spiral Formation

Remarkable is the shape of spirals evolving and rotating in thin solution layers. Spirals can be created by disrupting a small section of a propagating wave front, for instance, with a gentle blast of air ejected from a pipette [2] or with a laser spot. Either well spontaneous breaks may occur due to a bubble or a dust particle.

The two open wave ends develop in time as shown in Fig. 4.

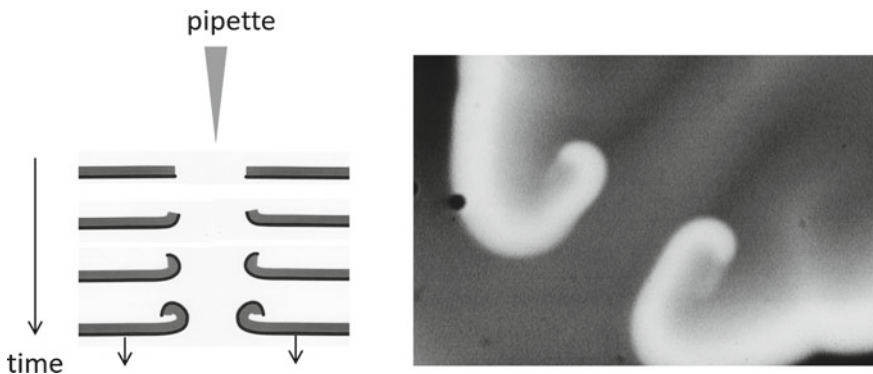


Fig. 4 Left: Evolution of open wave ends after disruption of the front. Black curve is excitable front, and grey part is refractory tail. The open ends curl around the refractory regions; right: image of the curling process

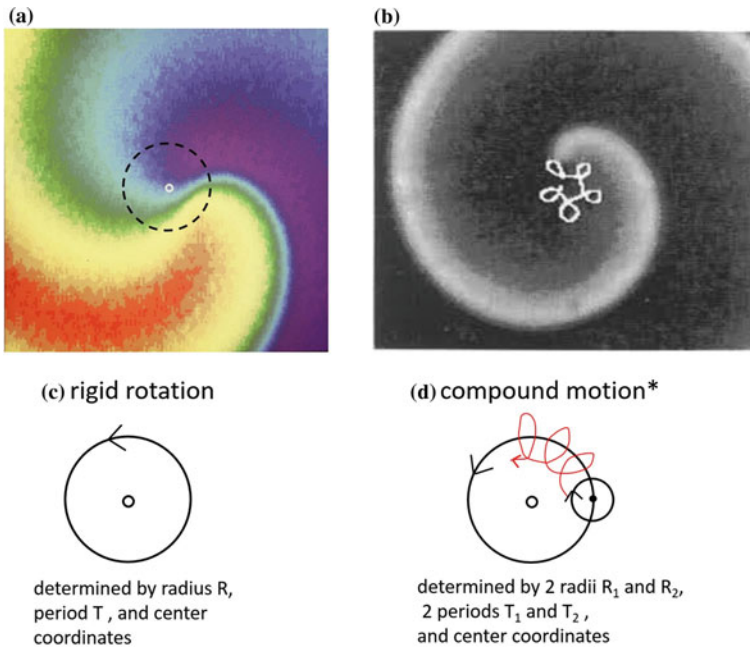


Fig. 5 **a** Enlargement of a spiral tip from Fig. 7 (see below). The black circle shows the rigid rotation of the tip around a central axis (white open circle). **b** When concentrations of the propagation medium are changed, the trajectory of the spiral tip may assume a compound (meandering) path [4]. Reprint permission from Nature. *This trace of a spiral tip is often erroneously called “hypocycloidal”. **c** and **d** Simple drawings show these dynamic differences

After an initiation stage and under “standard” conditions the tip of the spiral rotates uniformly on a circle around a spatially fixed center (Fig. 5a). The spiral arm has an almost Archimedean shape.

Due to the rotation a new curled wave emanates from the central region after each turn. With this property the spiral is a true self-organized structure in that it keeps rotating and producing new curling waves without any external influence. This is different from the case of target patterns, where every additional circular wave needs an extra excitation pulse at the center for initiation. The spiral manages its independent dynamics through the action of diffusion. In fact, since its tip traces a small circle around the spiral center, it enters, after each turn, an area where the excitability has been again restored. Thus a “reentrant” activity flares up and the next spiral turn commences.

When conditions in the excitable medium are changed, the trajectory of a spiral may assume quite different paths. A first step is normally a transition to a “compound motion”, when a second rotation period is superposed to the basic one (Fig. 5b).

A number of other, more complex traces of open wave ends have been observed: epicycles, loopy lines, irregular motion, straight or shrinking lines. We choose a

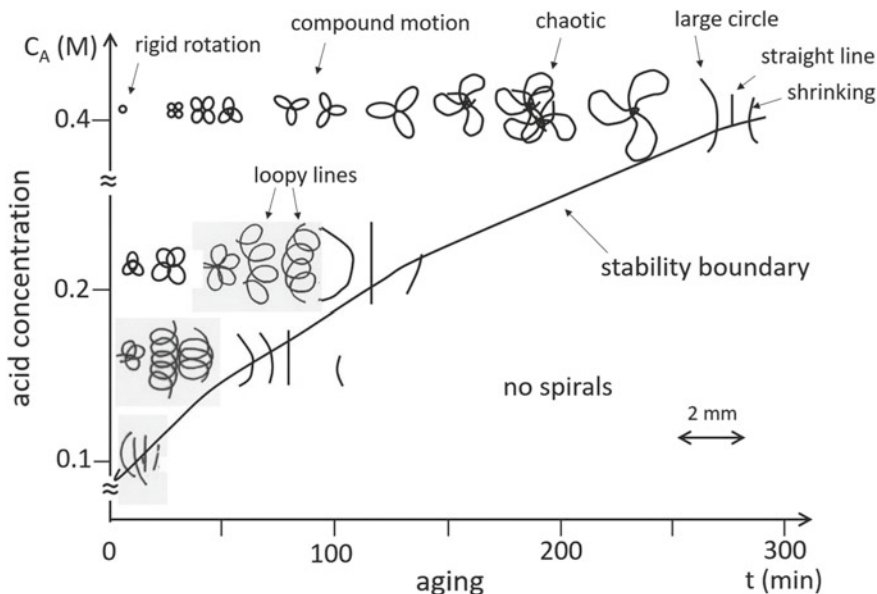


Fig. 6 Different trajectories of the spiral tip depending on acid concentration c_A and aging time of solution

graphic representation of trajectories measured during the aging of a chemical auto-catalytic reaction (the Belousov-Zabotinsky (BZ) reaction, see below), when the system passes through many of these trajectories during the course of time (Fig. 6).

6 Gallery of Dynamic Spirals

We proceed with a gallery of spiral images obtained from a special chemical system, the BZ-reaction which is, up to today, the most suitable medium to create spirals of various shapes and dynamics. This reaction will be introduced in the following chapter **Chemical Oscillations and Spiral Waves**.

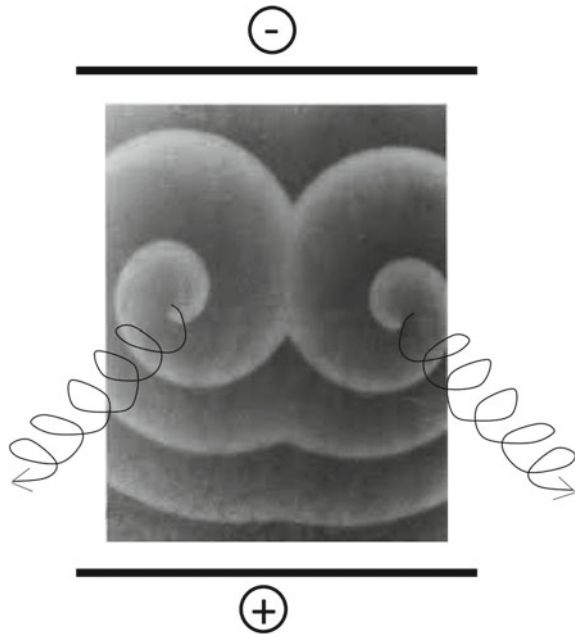
Images were taken with sensitive monochrome cameras and partially converted to color images by means of appropriate software and pseudo-color techniques [5] For an example, see Fig. 7.

Among the techniques to influence the dynamics of an excitation wave by external means the application of an electrical current proves to be a most efficient tool. For the BZ reaction an electric field was applied via two parallel electrodes to the short edges of a rectangular dish containing a thin layer of solution. Numerous chemical species in this reaction are of an ionic nature, especially important the small negatively charged bromide ion (the inhibitor in the reaction). The microscopic driving force is electromigration of ionic species. Ions are pulled towards one of the electrode along



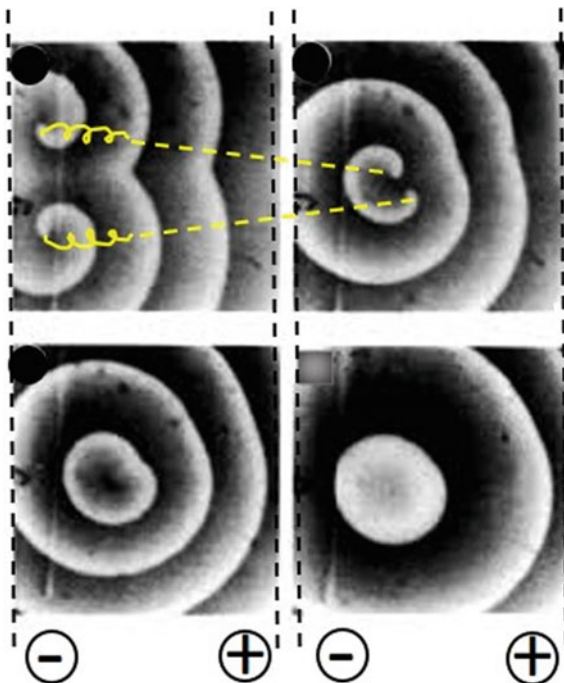
Fig. 7 Counter-rotating spirals in the BZ reaction, visualized in pseudo-color. Each of these spirals have an almost Archimedean shape and their tips rotate uniformly around a spatially fixed center. Note that wave fronts mutually annihilate each other upon collision. (Circular inclusions are CO₂ bubbles.) Reprint from [6], permission obtained from Springer Basel AG

Fig. 8 A pair of rigidly rotating spiral waves is perturbed by a constant electric field. Field lines are parallel and oriented vertically with the anode located at the bottom side of the figure. The electric field induces a spiral drift toward the anode and a strong deformation of the Archimedean spiral geometry. Loopy lines are directed along oblique angles determined by the chirality of spiral rotation



the electric field lines which can result in a local change of the concentration. In this case the major contribution is likely to stem from the small bromide ion migrating from the cathode towards the anode [7].

Fig. 9 By switching the electrical field polarity with respect to Fig. 8 the drift of the spiral pair is reversed and the loopy lines of the tips approach each other in time. Due to their symmetry and opposite chirality they finally collide, leading to mutual annihilation. This is a convenient method to get rid of unwanted vortices. Original picture from [3], Nature



The direction of this field-induced drift depends on the chirality of the spirals, as shown in Fig. 8. The left spiral, rotating clockwise, possesses an additional drift to the left side. The other spiral, rotating counterclockwise, drifts towards the right side. Switching off the current immediately stops the core drift. Changing the polarity of the field causes a core drift towards the initial position [3] (Fig. 9).

Another powerful way for forcing the dynamics of a rotating spiral is offered by laser light. When the excitability of the medium is modulated by a sinusoidal signal over the entire solution layer, by an expanded laser beam, the effect of the external frequency is reflected in the response of the trace of the spiral tip, which has been investigated already in Fig. 6 during the aging of the system. Even more complex behavior is now induced as a function of period and amplitude of the impinging light [8]. A quite simple example is plotted in Fig. 10. The compound motion of a spiral turns into a circular arrangement of multiple loops, due to phase-locking with the external rhythm. The figure shows the case of twofold periodicity.

A different method to exploit the photosensitivity of the excitable solution lies in the use of laser spots. If a laser spot of small diameter is directed into the self-sustained core region of a rotating spiral, it may pin the spiral tip to this spot, if the light has an inhibitory effect on the reaction (which can be produced by the choice of a photosensitive catalyst). The increased size of the “quiescent” core region then leads to a larger rotation period and consequently to a larger spiral pitch [9] (Fig. 11).

Fig. 10 Loopy line with twofold periodicity under the influence of an external periodic light field. Reprinted from [4], Nature

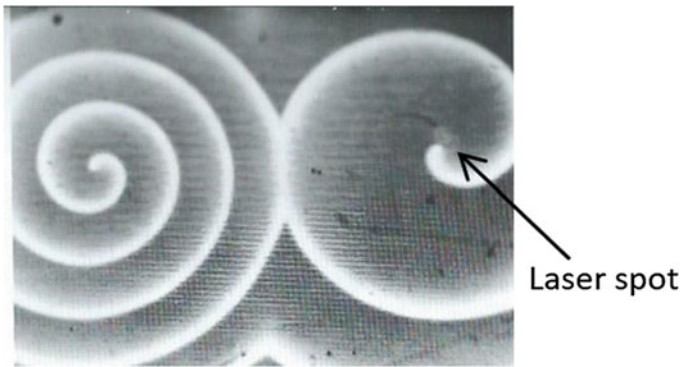


Fig. 11 In a photosensitive system, a laser spot can increase the the size of the autonomous spiral core (left) artificially (right) and thus cause the spiral to assume a much larger pitch. Reprint permission obtained from Elsevier

With help of such a spot open wave ends may be pinned and led along preselected paths. This way multiple spiral waves have been assembled [10]. In Fig. 12 (left) four open wave ends were created which, in the following, continue their dynamic paths and thus collide with each other. As a result a four-armed spiral is formed with an intricate mutual interaction.

A snapshot of this intricate interaction is shown in pseudo-color (Fig. 13).

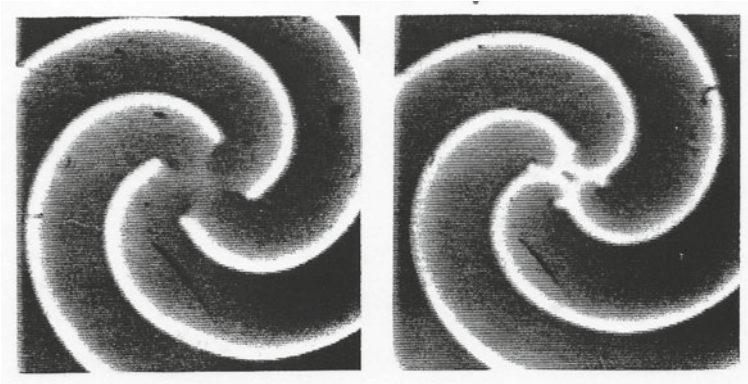
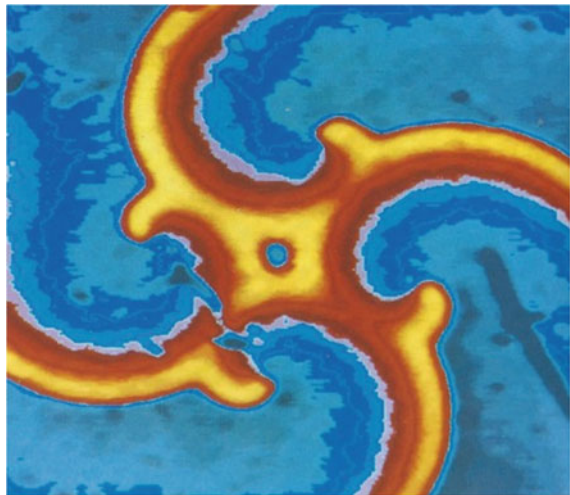


Fig. 12 Multiple spiral arms created by using a laser spot as a guiding obstacle. Left: four open wave ends assembled close to each other; right: their mutual collision

Fig. 13 Pseudo-color image of Fig. 12, right [11]. Reprint permission obtained from Springer Nature



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