Chapter 6
Weakly Nonnegative Quadratic Forms

In the previous chapters we met notions like positivity, nonnegativity and weak positivity, and applied to them various techniques like deflations, inflations, one-point extensions, reflections and edge reductions. Here we turn our attention to *weakly nonnegative forms*, that is, semi-unit quadratic forms $q : \mathbb{Z}^n \to \mathbb{Z}$ such that $q(v) \geq 0$ for all positive vector v in $\mathbb{Z}^n$. The above-mentioned methods are used to extend earlier results and algorithms to the weak nonnegative context, where now the existence of maximal sincere q-roots plays a key role, and hypercritical forms take the place of critical forms.

6.1 Hypercritical Forms

A quadratic semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is called *hypercritical* if it is not weakly nonnegative, but every proper restriction q^I is. For instance, the m-Kronecker form $q_m(x_1, x_2) = x_1^2 + x_2^2 - mx_1x_2$ is weakly nonnegative if and only if $m < 3$, and is hypercritical exactly when $m \geq 3$. Theorem 5.2 tells us that if the number of variables is at least three, then a critical (nonweakly positive) form is nonnegative with radical generated by a positive vector z, called a *critical vector*. In Proposition 6.2 below we give an analogous result for hypercritical forms.

Lemma 6.1. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a hypercritical semi-unit form.*

a) If q is also critical, then $n = 2$ and q is the Kronecker form $q_m = x_1^2+x_2^2-mx_1x_2$ for some $m \geq 3$. In particular, q has no critical vector.
b) If q is nonunitary then $n = 2$ and q is (up to order of variables) one of the forms q'_m or q''_m below, with $m > 0$,

$$q'_m(x_1, x_2) = x_1^2 - mx_1x_2 \qquad \text{and} \qquad q''_m(x_1, x_2) = -mx_1x_2.$$

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Proof. If q is also critical and $n \geq 3$, by Theorem 5.2 the form q is nonnegative, in particular weakly nonnegative. This is impossible since q is hypercritical.

Then $n = 2$, that is, $q(x_1, x_2) = x_1^2 + x_2^2 - mx_1x_2$ for some $m \in \mathbb{Z}$ (since q is unitary by Lemma 5.5(a)). Observe that if $m \leq 2$ then q is weakly nonnegative, and see Proposition 1.23 for the claim on critical vectors.

To verify (b) observe first that the forms q'_m and q''_m are hypercritical precisely when $m > 0$. Consider a vertex $c \in \{1, \ldots, n\}$ such that $q(e_c) = 0$, and take $x^{(c)}$ to be the vector in $\mathbb{Z}^{n-1}$ obtained by deleting the variable x_c. Then

$$q(x) = q^{(c)}(x^{(c)}) + x_c \left(\sum_{i \neq c} q_{ic} x_i \right).$$

Now, if q is hypercritical then there is a positive sincere vector x such that $q(x) < 0$. Moreover, $q^{(c)}(x^{(c)}) \geq 0$ implies that the second summand above is negative. Since x is a positive vector, there must be a $d \neq c$ such that $q_{cd} < 0$. Then the restriction $q' = q^{\{c,d\}}$ is one of the hypercritical forms q'_m or q''_m above. Since q is itself hypercritical, then $q = q'$ and the result follows. □

By Lemma 6.1(b) we may focus only on hypercritical unit forms, which can be characterized as follows.

Proposition 6.2. *For a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ with $n \geq 3$ the following are equivalent.*

a) The form q is hypercritical.
b) The form q is not weakly nonnegative, and for every critical restriction q^I of q there is an index i with $I = \{1, \ldots, n\} - \{i\}$, and a positive critical vector z of q^I such that $q(z|e_i) < 0$.

Proof. Assume q is hypercritical and consider a positive vector v with $q(v) < 0$. Since any proper restriction q^I is weakly nonnegative, the vector v is sincere. If q^I is critical, since $n \geq 3$ then q^I is a proper restriction of q by Lemma 6.1(a). Moreover, since $q_{ij} \geq -2$ for all $i \neq j$ (for q does not contain any Kronecker form q_m with $m > 2$) by Theorem 5.2 we may take a critical positive vector z for q^I, which we identify with its inclusion in $\mathbb{Z}^n$.

Take positive numbers m and k such that $kv - mz$ is a positive but nonsincere vector, say $(kv - mz)_j = 0$. (Such numbers exist: take an index $j \in \{1, \ldots, n\}$ such that $\frac{z_i}{v_i} \leq \frac{z_j}{v_j}$ for all $i \in \{1, \ldots, n\}$ and take $k := z_j$ and $m := v_j$.) Therefore

$$0 \leq q^{(j)}(kv - mz) = k^2 q(v) - kmq(z|v) + m^2 q(z) < -km \sum_{i \notin I} v_i q(z|e_i),$$

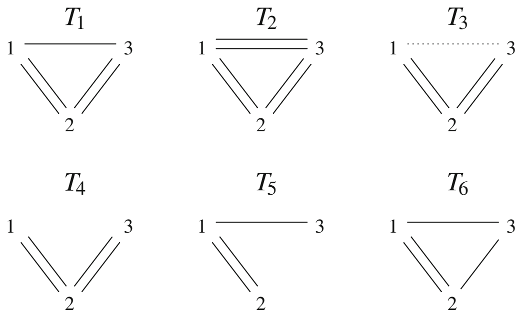
and since $v_i > 0$ for all $i \in \{1, \ldots, n\}$ there must exist an $i \notin I$ with $q(z|e_i) < 0$. Observe now that $2z + e_i$ is a sincere vector for q is hypercritical and

$$q(2z + e_i) = 4q(z) + 2q(z|e_i) + 1 = 2q(z|e_i) + 1 < 0.$$

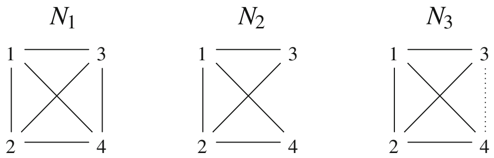
Hence $I = \{1, \ldots, n\} - \{i\}$. For the converse assume that $q^{(i)}$ is not weakly nonnegative for some $i \in \{1, \ldots, n\}$ and take $I \subset \{1, \ldots, n\} - \{i\}$ such that q^I is a critical restriction of $q^{(i)}$ (thus a critical restriction of q). By hypothesis (b) we have $q^I = q^{(i)}$. Therefore $q^{(i)}$ is hypercritical as well as critical, and by Lemma 6.1, it is the Kronecker form q_m for some $m \geq 3$, which contradicts the existence of a critical vector for $q^{(i)}$. Then $q^{(i)}$ is weakly nonnegative for all $i \in \{1, \ldots, n\}$, that is, q is a hypercritical form. □

Lemma 6.3. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a hypercritical unit form with at least three indices i, j, k.*

a) If $q_{ij} = -2$ then $n = 3$. In particular, the bigraph B_q associated to q is one of the following six bigraphs:



b) If $q_{ij} = q_{ik} = q_{jk} = -1$ and $q^{(k)}$ is critical, then $n = 4$ and the bigraph of q is one of:



Moreover, the quadratic form q_Δ represents numbers -1 and -3 for

$$\Delta \in \{T_3, T_4, T_5, T_6, N_2, N_3\}.$$

Proof. Since q has at least three vertices, B_q does not contain any Kronecker form q_m as a restriction for $m \geq 3$, that is, $q_{rs} \geq -2$ for all vertices $r < s$.

Assume first that $q_{ij} = -2$ for some vertices $i < j$. Then $q^{\{i,j\}}$ is a critical restriction of q. By Proposition 6.2 we have $n = 3$. Set $(i, j, k) = (1, 2, 3)$ and notice that $z = (1, 1)$ is a critical vector of $q^{(3)}$, thus again by Proposition 6.2 we have

$$0 < q(z|e_3) = q_{13} + q_{23}.$$

This implies that B_q is one of the bigraphs $T_1, \dots, T_6$. Moreover, the forms q_{T_i} for $i = 1, \dots, 6$ are all hypercritical (Exercise 1 below).

Assume now that $q_{ij} = q_{ik} = q_{jk} = -1$. Then $q^{\{i,j,k\}}$ is a critical restriction of q, hence $n = 4$ and we may take $(i, j, k, \ell) = (1, 2, 3, 4)$. Since $q^{(3)}$ is critical (with critical vector $e_1 + e_2 + e_4$), we have $q_{14} = q_{24} = -1$ and $0 > q(e_1 + e_2 + e_3|e_4) = -2 + q_{34}$. Therefore $q_{34} \in \{-1, 0, 1\}$, with corresponding cases N_1, N_2 and N_3.

For the last claim simply verify $q_{T_i}(v_i) = -1 = q_{N_j}(v'_j)$ and $q_{T_i}(w_i) = -3 = q_{N_j}(w'_j)$ for the vectors in the following list (for $i = 3, \dots, 6$ and $j = 2, 3$)

$$v_3 = (1, 2, 1) \text{ and } w_3 = (2, 5, 2),$$
$$v_4 = (1, 1, 1) \text{ and } w_4 = (2, 3, 2),$$
$$v_5 = (2, 2, 1) \text{ and } w_5 = (4, 4, 1),$$
$$v_6 = (1, 1, 1) \text{ and } w_6 = (2, 2, 1),$$
$$v'_2 = (1, 1, 1, 1) \text{ and } w'_2 = (2, 2, 2, 1),$$
$$v'_3 = (2, 2, 2, 1) \text{ and } w'_3 = (4, 4, 4, 1).$$

□

We now show that almost all hypercritical forms represent numbers -1 and -3. The importance of these two numbers will be clear in the proof of Theorem 6.16. In what follows, by a *slender* quadratic form we mean a unit form q with $q_{ij} \geq -1$ for all $i < j$. The bigraph associated to the Kronecker form q_m is denoted by $\mathbb{K}_m$ for $m \neq 0$.

Proposition 6.4. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a hypercritical unit form whose associated bigraph is not $\mathbb{K}_m$ ($m \geq 3$), T_1, T_2 or N_1 (see Lemma 6.3 for notation). Then there are positive (sincere) vectors v and w such that $q(v) = -1$ and $q(w) = -3$.*

Proof. Consider B_q, the bigraph associated to q. Since $\mathbb{K}_m$ is not contained in B_q for $m \geq 3$ we have $q_{ij} \geq -2$ for all $i < j$. If $q_{ij} = -2$ for some $i < j$, by Lemma 6.3(a) the bigraph B_q is one of $T_3, \dots, T_6$, which represent -1 and -3. Therefore we may assume that q is a slender form. We may also assume, using Proposition 6.2, that $q^{(n)}$ is a critical form with critical vector z, and $q(z|e_n) = -s < 0$. Moreover, by Proposition 5.4 we may take $z_1 = 1$.

First we show that $0 < s \leq 3$. By the above assumptions, the vector $x := z - e_1 + e_n$ is positive and not sincere. Thus, since $q(e_1|e_n) = q_{1n} \geq -1$,

$$0 \leq q(x) = q(z) + 2 - q(z|e_1) + q(z|e_n) - q(e_1|e_n) = 2 - q_{1n} - s < 4 - s.$$

Notice now that $s \neq 3$. Indeed, for $s = 3$ and $x = z - e_1 + e_n$ we have $0 \leq q(x) = 2 - q(e_1|e_n) - 3$, that is, $q_{1n} = -1$ and $q(x) = 0$. Then $q^{(1)}$ is not weakly positive, and again by 6.2, the form $q^{(1)}$ is critical. Since $x_n = 1$ the vector x is critical for $q^{(1)}$. We may assume that $q_{2n} = -1$, therefore

$$0 = q(x|e_2) = q(z|e_2) - q_{12} + q_{2n} = -q_{12} - 1,$$

that is, $q_{12} = -1$. Now use Lemma 6.3(b) to conclude that $B_q = N_1$, which is impossible.

The proof is completed by showing that in cases $s = 1, 2$ such vectors v and w may be given explicitly:

Case $s = 1$.

$$q(2z + e_n) = 1 + 2q(z|e_n) = -1 \text{ and } q(4z + e_n) = 1 + 4q(z|e_n) = -3.$$

Case $s = 2$.

$$q(z + e_n) = 1 + q(z|e_n) = -1 \text{ and } q(2z + e_n) = 1 + 2q(z|e_n) = -3.$$ □

As an immediate consequence we have:

Corollary 6.5. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a hypercritical unit form with $n \geq 5$. Then q represents numbers -1 and -3.*

For integers $a \leq b$ denote by $[a, b]$ the set of integers ℓ with $a \leq \ell \leq b$.

Lemma 6.6. *For any hypercritical unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ there is a (sincere) vector $v \in [0, 12]^n$ such that $q(v) < 0$.*

Proof. The statement is clear for Kronecker forms $q_m = q_{\mathbb{K}_m}$ with $m \geq 3$, and for the forms with associated bigraphs $T_3, T_4, T_5, T_6, N_2, N_3$ by simple inspection of the proof of the last claim in Lemma 6.3.

If q is not the form associated to graphs T_1, T_2, then by Proposition 6.4 we may assume that $q^{(n)}$ is a critical restriction with positive critical vector z, and -1 is either represented by $2z + e_n$ or by $z + e_n$. Since by Corollary 3.31 we have $z_i \leq 6$ for all i (cf. also Proposition 2.22), then both $2z + e_n$ and $z + e_n$ belong to $[0, 12]^n$.

To deal with cases T_1, T_2 (resp. N_1) evaluate at the vector $v = (1, 1, 1)$ (resp. $v = (1, 1, 1, 1)$) to get $q_{T_1}(v) = -2$, $q_{T_2}(v) = -3$ (resp. $q_{N_1}(v) = -2$). □

Exercises 6.7.

1. Show that all bigraphs in Lemma 6.3 correspond to hypercritical forms.
2. Prove that the form $q_m = q_{\mathbb{K}_m}$ does not represent the number -3 for any $m \geq 2$.
3. Show that the form q_Δ does not represent the number -1 for any
$$\Delta \in \{\mathbb{K}_m, T_1, T_2, N_1\}_{m \geq 4}.$$
4. Which of the forms associated to T_1, T_2 or N_1 represents the number -3?

6.2 Maximal and Locally Maximal Roots

For a weakly nonnegative semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ denote by $\mathbf{rad}^+(q)$ the set of positive vectors x with $q(x|e_i) = 0$ for $i = 1, \ldots, n$ (called the *positive radical* of q). Observe that if $x \in \mathbf{rad}^+(q)$ then q has no maximal positive root (with partial order $x \geq y$ if $x - y \in \mathbb{N}_0^n$), since $1 = q(v) = q(v + mx)$ for any positive q-root v and any $m \in \mathbb{N}$. As in the weakly positive case, we say that a weakly nonnegative semi-unit form q is *sincere* if it has a sincere positive root.

Proposition 6.8. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a sincere weakly nonnegative unit form. The following are equivalent.*

a) There are finitely many sincere positive roots of q (and we call q finitely sincere*).*
b) There is a maximal sincere positive root of q.
c) $\mathbf{rad}^+(q) = \emptyset$.

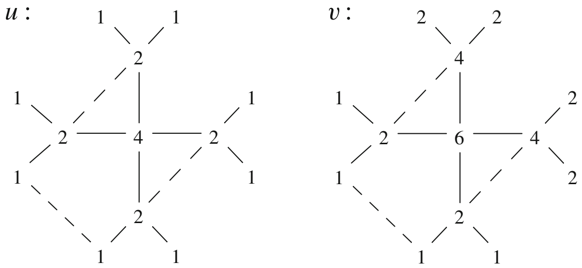
Proof. Clearly we have that (a) implies (b) and that (b) implies (c). To show that (c) implies (a) let us assume that q has infinitely many sincere positive roots. Then we may take a sequence of sincere positive q-roots $y^1, y^2, \ldots$ with $y^m < y^{m+1}$ for $m = 1, 2, \ldots$ (see Lemma 5.12). Notice that $|q(y^m|e_i)| \leq 2$ for all $i = 1, \ldots, n$ and $m \geq 1$. This follows from the sincerity of y^m and the inequality

$$0 \leq q(y^m \pm e_i) = 2 \pm q(y^m|e_i).$$

Hence there are $y^\ell < y^m$ with $q(y^\ell|e_i) = q(y^m|e_i)$ for all $i = 1, \ldots, n$ and $0 \neq y^m - y^\ell \in \mathbf{rad}^+(q)$. □

A positive root v of a semi-unit form q is said to be *locally maximal* if $q(v|e_i) \geq 0$ for all $i = 1, \ldots, n$. For v a maximal positive q-root, since $\sigma_i(v) = v - q(v|e_i)e_i$ is again a q-root where σ_i is the i-th reflection for q (Sect. 1.2), v is a locally maximal root. The converse is false in general, as the following example shows.

Example 6.9. Consider the quadratic form q given by the following bigraph, and selected vectors u and v.



Then q is weakly nonnegative and $u < v$ are positive q-roots with u a locally maximal root.

Proposition 6.10. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a finitely sincere weakly nonnegative unit form. Then a sincere positive root y of q is maximal if and only if y is locally maximal.*

Proof. We only need to show that local maximality implies maximality. Assume y is a locally maximal sincere positive root of q, and that x is a root with $y < x$. Then

$$q(y|x) = \sum_{i=1}^{n} x_i q(y|e_i) \geq \sum_{i=1}^{n} y_i q(y|e_i) = q(y|y) = 2.$$

Thus $0 \leq q(x-y) = 2 - q(y|x) \leq 0$, that is, $q(x-y) = 0$. Notice that $v := x - y$ satisfies $q(v|e_i) \geq 0$ for all $i = 1, \ldots, n$ (for if $q(v|e_i) < 0$ then $q(2v + e_i) = 4q(v) + 1 + 2q(v|e_i) < 0$). We also have $q(v|y) = q(x|y) - 2 = 0$, therefore

$$0 = q(v|y) = \sum_{i=1}^{n} y_i q(v|e_i),$$

which implies that $v \in \mathbf{rad}^+(q)$, in contradiction with Proposition 6.8. □

A vertex i such that $q(y|e_i) > 0$ for a locally maximal positive q-root y of a weakly nonnegative semi-unit form q is called an *exceptional index (or vertex)* for y (cf. Lemma 5.9).

Lemma 6.11. *For a locally maximal positive root y of a weakly nonnegative semi-unit form q, one of the following situations occur:*

a) There are exactly two exceptional indices $i \neq j$ and $q(y|e_i) = y_i = 1 = y_j = q(y|e_j)$.
b) There is only one exceptional index i, and $q(y|e_i) = 1$ and $y_i = 2$.
c) There is only one exceptional index i, and $q(y|e_i) = 2$ and $y_i = 1$.

Furthermore, if y is also maximal then situation (c) never occurs.

Proof. Let y be a sincere locally maximal positive q-root. Then we have

$$2 = q(y|y) = \sum_{i=1}^{n} y_i q(y|e_i),$$

thus clearly one of (a), (b) or (c) occurs. If (c) holds then $q(2y - e_i) = 5 - 2q(y|e_i) = 1$ and $2y - e_i > y$, therefore y is not a maximal root. □

The following lemma will be useful to determine the maximality of positive sincere roots. For instance, this criterion is used below in the proof of Lemma 6.13.

Lemma 6.12. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form with a maximal sincere positive root y. Then for any positive vector v with $q(v) = -1$ we have $q(y|v) = 0$.*

Proof. Since v is positive and y is locally maximal we have $q(y|v) \geq 0$. Then

$$\sigma_v(y) = y - \frac{q(y|v)}{q(v)} v = y + q(y|v)v,$$

which is a positive q-root with $y \leq \sigma_v(y)$. By maximality $q(y|v) = 0$. □

Lemma 6.13. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form such that there are indices $1 \leq i < j \leq n$ with $-5 \leq q_{ij} \leq -3$. Then q has no maximal sincere positive root.*

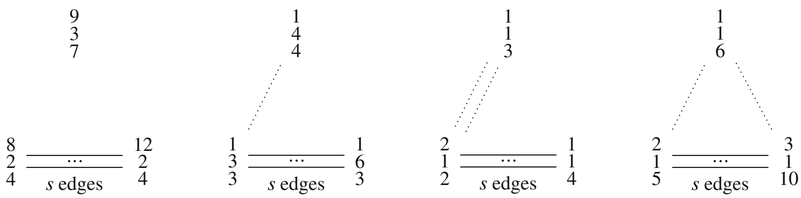
Proof. Let us assume that y is a sincere maximal positive root of q (hence locally maximal). Consider the triple $r = (e_i, e_j, y)$ and the root induced unit form q_r given by

$$\begin{aligned} q_r(x_1, x_2, x_3) &:= q(x_1 e_i + x_2 e_j + x_3 y) \\ &= x_1^2 + x_2^2 + x_3^2 + x_1 x_3 q(y|e_i) + x_2 x_3 q(y|e_j) - s x_1 x_2, \end{aligned}$$

where $q_{ij} = -s$ for some integer s. Let B be the bigraph associated to q_r. The shape of B depends on the values of $q(y|e_i)$ and $q(y|e_j)$. Since y is a root we have

$$2 = q(y|y) = \sum_{k=1}^{n} y_k q(y|e_k),$$

and since y is sincere, positive and locally maximal then $q(y|e_k) > 0$ for at most two vertices k (and $q(y|e_k) = 0$ for the rest), and in particular $m := q(y|e_i + e_j) \in \{0, 1, 2\}$. Thus we consider four cases: Case 1) $m = 0$; Case 2) $m = 1$; Case 3) $m = 2$ and $q(y|e_i)q(y|e_j) = 0$, and Case 4) $q(y|e_i) = 1 = q(y|e_j)$. These cases correspond to the four possibilities for B as depicted below (from left to right, observe that in all cases we have $(q_r)_{12} = q_{12} = -s$).



Each vertex of B contains a column with three natural numbers, corresponding to three vectors in $\mathbb{Z}^{B_0}$ which are, from top to bottom, sincere positive roots of q_r for $s = 3, 4, 5$ respectively. Then q_r has a sincere positive root $x = (x_1, x_2, x_3)$ and $y' := x_1 e_i + x_2 e_j + x_3 y$ is a root of q with $y' > y$, which is impossible. □

The following technical lemma imposes restrictions on sincere weakly nonnegative unit forms which fail to be unitary.

Lemma 6.14. *Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semi-unit form, and $x \in \mathbb{Z}^I$ a positive sincere root. For $r = 0, 1$ take $I^r = \{i \in I \mid q_{ii} = r\}$ and consider the restriction $x^r = x|_{I^r}$ in $\mathbb{Z}^{I^r}$. Then one and only one of the following assertions holds:*

a) $q(x^1) = 1$ *and* $q_{ij} = 0$ *for any* $i \in I^0$ *and* $j \in I$.
b) $q(x^1) = 0$ *and there exist* $i \neq j$ *in* I^0 *such that* $x_i = x_j = q_{ij} = 1$. *Moreover, if* $s \in I^0$ *and* $t \in I$ *is a different index satisfying* $q_{st} \neq 0$ *then* $\{s, t\} = \{i, j\}$.

c) $q(x^1) = 0$ *and there exist* i *in* I^0 *and* $j \in I^1$ *such that* $x_i = x_j = q_{ij} = 1$. *Moreover, if* $s \in I^0$ *and* $t \in I$ *is a different index satisfying* $q_{st} \neq 0$ *then* $\{s, t\} = \{i, j\}$.

Furthermore, if x *is maximal and* $I^0 \neq \emptyset \neq I^1$, *then* (c) *holds and* I^0 *contains exactly one element.*

Proof. Let us suppose that $I = \{1, \dots, n\}$ and $I^0 = \{1, \dots, m\}$ for $m \leq n$. Then

$$1 = q(x) = q(x^1) + q(x^0) + q(x^1|x^0), \quad \text{where} \quad q(x^1|x^0) = \sum_{\substack{i \in I^1 \\ j \in I^0}} q_{ij} x_i x_j.$$

Now, by Lemma 6.1 we have $q_{ij} \geq 0$ for $i \in I^0$ and $j \in I$ (for q'_m and q''_m are hypercritical if $m > 0$). Hence the three summands on the right of the equation are nonnegative, therefore exactly one of them is nonzero. This leads to the three assertions above, since x is sincere.

For the last claim we give a root $y > x$ for both cases (a) and (b). For (a) take $y = 2x^0 + x^1$, whereas for (b) take $y = x^0 + 2x^1$. Notice that $y > x$ since $I^r \neq \emptyset$ for $r = 0, 1$. Thus if x is a maximal root then (c) holds, that is, $1 = q(x) = q(x^1|x^0)$. Further, if $k \in I^0$ with $k \neq i$ then

$$q(x + e_k) = q(x^1|x^0) + q(x^1|e_k) + q(x^0|e_k) = q(x^1|x^0) = 1,$$

that is, $x + e_k$ is a root of q larger than x. □

Exercises 6.15.

1. Do hypercritical unit forms have to be connected?
2. Show that the quadratic form q in Example 6.9 is weakly nonnegative.
3. Show that if q contains a bigraph with shape T_1, T_2 or N_1 (as in Lemma 6.3), then q does not have a maximal sincere positive root.
4. In Example 6.9, verify that u and v are roots of q.
5. For a weakly nonnegative semi-unit form q, a positive q-root x and a positive isotropic vector z of q, show that the following assertions hold:
   i) $q(x|e_i) \geq -2$ and $q(z|e_i) \geq -1$ for $i = 1, \dots, n$.
   ii) If $x_i > 0$ then $q(x|e_i) \leq 2$, and if $z_i > 0$ then $q(z|e_i) \leq 1$.
   iii) $q_{ij} \leq 3$ if $x_i \neq 0 \neq x_j$, and $q_{ij} \leq 2$ if $z_i \neq 0 \neq z_j$.
6. Let q be a nonzero connected weakly nonnegative semi-unit form. Must q be unitary?
7. Consider the unit form in three variables $q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - s x_1 x_2$. Show that if $s \geq 3$ and $s - 2$ is not the square of an integer, then q has a sincere positive root.

6.3 Criteria for Weak Nonnegativity

Here we prove a **Weak Nonnegativity Criterion** due to Happel and de la Peña in [31]. Ovsienko showed in [44] that this result also holds without the condition $q_{ij} \geq -5$ for all $i < j$.

Theorem 6.16 (Happel–de la Peña). *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form with $q_{ij} \geq -5$ for all $1 \leq i < j \leq n$. If q has a maximal sincere positive root, then q is weakly nonnegative.*

Proof. Let y be a maximal sincere positive root of q. By Lemma 6.13 and Exercise 6.15.3, the form q does not contain bigraphs of type N_1, T_1, T_2, $\mathbb{K}_3$, $\mathbb{K}_4$, $\mathbb{K}_5$, nor, by assumption, bigraphs $\mathbb{K}_m$ for $m \geq 6$. If q is not weakly nonnegative, there is a hypercritical restriction q^I of q, and by Proposition 6.4 there exist positive vectors v and w with support I and with $q(v) = -1$ and $q(w) = -3$. It follows from Lemma 6.12 that $q(y|v) = 0$. Since v and w are positive vectors with same support and y is locally maximal, then $q(y|w) = 0$. Therefore

$$q(2y + w) = 4q(y) + q(w) = 1,$$

in contradiction with the maximality of y. □

Lemma 6.17. *Let q be a hypercritical unit form and i an index such that $q^{(i)}$ is not critical. Then $q^{(i)}$ is a positive form.*

Proof. Observe first that $q^{(i)}$ is weakly positive, since otherwise it would contain a critical restriction q^I, contradicting Proposition 6.2. Again by Proposition 6.2 there must exist a vertex c such that $q^{(c)}$ is a critical restriction of q (hence $c \neq i$), with critical positive vector z such that $q(z|e_c) < 0$. Then $q^{(c)(i)}$ is a positive unit form by Corollary 5.3.

If $q^{(i)}$ is not positive, there is a nonzero vector v such that $q^{(i)}(v) \leq 0$. In particular $v_c \neq 0$ since $q^{(c)(i)}$ is positive, so we may assume that $v = v' + v_c e_c$ with $v'_c = 0$ and $v_c > 0$. Notice that for $\alpha, \beta > 0$ we have

$$q(\alpha v + \beta z) = \alpha^2 q^{(i)}(v) + \alpha\beta q(z|v' + v_c e_c) \leq \alpha\beta v_c q(z|e_c) < 0.$$

Since $q^{(i)}$ is weakly positive the vector v' has a negative entry. But z is a critical positive vector of $q^{(c)}$, therefore we may find $\alpha, \beta > 0$ such that $\alpha v + \beta z$ is a positive nonsincere vector (take for instance $\alpha = z_a$ and $\beta = -v_a$ where a is an index such that $\frac{v_a}{z_a}$ is minimal among all fractions $\frac{v_j}{z_j}$ for $j \in \mathbf{supp}(z) = \{1, \ldots, n\} - \{c\}$). This is impossible since $q(\alpha v + \beta z) < 0$ and q is hypercritical, hence $q^{(i)}$ is a positive unit form. □

The following immediate consequence may be considered as a partial analogue of Theorem 5.2 (see also Corollary 5.3).

Corollary 6.18. *Any proper restriction of a hypercritical unit form is nonnegative.*

Proof. The result is clear for Kronecker forms q_m with $m \geq 3$. Therefore we may assume the hypercritical form q has at least three vertices (in particular $q_{ij} \geq -2$ for all $i < j$). By Lemma 6.17, if $q^{(i)}$ is not positive then $q^{(i)}$ is critical, thus nonnegative by Theorem 5.2. □

We now prove a generalization of the (Jacobi-like) Zeldych Criterion 5.26 given in [55]. Again we do not assume the quadratic form to be unitary. Let $\mathbf{ad}(B)$ denote the *adjugate* of a square matrix B.

Proposition 6.19. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form with associated symmetric matrix A (that is, $q(x) = x^t Ax$ for any $x \in \mathbb{Z}^n$). The following assertions are equivalent:*

a) The form q is weakly nonnegative.
b) For every principal submatrix B of A we have either $\det(B) \geq 0$, *or* $\mathbf{ad}(B)$ *has a negative entry.*

Proof. Let B be a principal submatrix of A and assume that $\mathbf{ad}(B)$ is nonnegative (that is, it has no negative entry). By Perron–Frobenius Theorem 1.36 there exists a positive eigenvector $v \in \mathbb{R}^n$ of $\mathbf{ad}(B)$ with eigenvalue $\rho > 0$. Assuming that q is weakly nonnegative and considering q as a real function $q_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}$ we have by continuity

$$0 \leq q_{\mathbb{R}}(v) = v^t Bv = \frac{1}{\rho} v^t B(\mathbf{ad}(B)v) = \frac{1}{\rho} \det(B)\|v\|^2,$$

and therefore $\det(B) \geq 0$.

Suppose now that q satisfies (b) but is not weakly nonnegative. Since property (b) is preserved by principal minors, by induction on n we may assume that q is hypercritical. By Corollary 6.18, every proper restriction of q is nonnegative, therefore by Proposition 1.33 we have $\det(B) \geq 0$ for each proper principal submatrix B of A.

Thus $\det(A) < 0$ since otherwise q would be nonnegative. Take $\mathbf{ad}(A) = (v_{ij})$. By hypothesis there must exist i, j with $v_{ij} < 0$. Let v be the j-th column of $\mathbf{ad}(A)$, so that $Av = \det(A)e_j$ and $q(v) = \det(A)v_{jj}$. Further, let $w > 0$ be a sincere positive vector with $q(w) < 0$. For $\lambda = -\frac{v_{ij}}{w_i} > 0$ we have $(v + \lambda w)_i = 0$ and (since the restriction $q^{(i)}$ is nonnegative)

$$\begin{aligned} 0 &\leq q(v + \lambda w) \\ &= q(v) + 2\lambda w^t Av + \lambda^2 q(w) \\ &< \det(A)[v_{jj} + 2\lambda w_j] \\ &= \frac{\det(A)}{w_i}[v_{jj} w_i - 2v_{ij} w_j]. \end{aligned}$$

As in the proof of Proposition 5.26, if $v_{jj} < 0$ we take $i = j$, thus

$$0 \leq q(v + \lambda w) < \det(A)(-v_{jj}) \leq 0,$$

and if $v_{jj} \geq 0$ then $v_{jj}w_i - 2v_{ij}w_j \geq 0$ and we have

$$0 \leq q(v + \lambda w) < \frac{\det(A)}{w_i}[v_{jj}w_i - 2v_{ij}w_j] \leq 0.$$

Both cases yield a contradiction. □

The following practical criterion is useful for the computational verification of weak nonnegativity.

Theorem 6.20. *A semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is weakly nonnegative if and only if $q(z) \geq 0$ for every $z \in [0, 12]^n$.*

Proof. If q is weakly nonnegative then $q(z) \geq 0$ for all $z \in [0, 12]^n$. If q is not weakly nonnegative, then there is a hypercritical restriction q' of q, and by Lemmas 6.1 and 6.6 there is a vector $z \in [0, 12]^n$ with $q(v) < 0$. □

We say that a weakly nonnegative semi-unit form q is 0-*sincere* if there exists a sincere vector $y \in \mathbf{rad}^+(q)$. We point out that in this case any isotropic vector $y \in \mathbb{N}_0^n$ belongs to the positive radical $\mathbf{rad}^+(q)$ of q. In fact, we have the following more general result.

Lemma 6.21. *Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semi-unit form and take $\mu \in q^{-1}(0)$.*

a) If $x \in \mathbf{rad}^+(q)$ and $\mathbf{supp}(\mu) \subset \mathbf{supp}(x)$, then $\mu \in \mathbf{rad}(q)$.
b) If μ is positive and $z \in \mathbb{Z}^I$ is such that $q(z|\mu) = 0$ and $z + n\mu$ is a positive sincere vector for some $n \geq 0$, then $\mu \in \mathbf{rad}^+(q)$.

Proof. Assume there is an index $i \in I$ such that $q(\mu|e_i) \neq 0$ and take $\epsilon = \pm 1$ such that $\epsilon q(\mu|e_i) > 0$. Taking $y = e_i - 2\epsilon\mu$, we observe that

$$q(y) = q(e_i) - 2\epsilon q(\mu|e_i) \leq -1.$$

By the requirement on the supports in (a), notice that there exists a $k \geq 0$ such that $y + kx$ is a positive vector, thus we arrive at the contradiction

$$0 \leq q(y + kx) = q(y) \leq -1.$$

This shows (a). For (b) assume that $\mu \notin \mathbf{rad}(q)$, thus there exists $i \in I$ with $q(\mu|e_i) > 0$ (for μ is a positive vector). In particular, there is $k \geq 0$ such that

$$q(z + k\mu|e_i) = q(z|e_i) + kq(\mu|e_i) \geq q(z) + 2.$$

Take $m := \max(k, n)$ and $y := z + m\mu$. Then $q(y|e_i) \geq q(z) + 2$ and $q(y) = q(z)$ since $q(z|\mu) = 0$ and $q(\mu) = 0$. Therefore

$$q(y - e_i) = q(y) + q(e_i) - q(y|e_i) = q(z) + 1 - q(y|e_i) \leq -1,$$

which is impossible since $y \geq z + n\mu$ is positive and sincere. □

For a semi-unit form q with a sincere positive radical vector z, we trivially observe that any vector x may be taken into a positive vector $x + kz$ with $k \in \mathbb{N}$, so that $q(x) = q(x + kz)$. This proves the following lemma.

Lemma 6.22. *Any* 0*-sincere form is nonnegative.*

6.4 Iterated Edge Reductions

Recall from Sect. 5.3 that for a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ and indices $i \neq j$ with $q_{ij} < 0$ we construct a quadratic form $q'(x) = q(\rho(x)) + x_i x_j$, with $\rho : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ given by

$$\rho(e_k) = \begin{cases} e_k, & \text{if } 1 \leq k \leq n; \\ e_i + e_j, & \text{if } k = n + 1, \end{cases}$$

called the *edge reduction of q with respect to i and j*. The same construction can be performed when q is a semi-unit form (or even a pre-unit form, that is, an integral quadratic form q with $q(e_i) \leq 1$ for all indices i) satisfying $q(e_i) = 1 = q(e_j)$ and $q_{ij} < 0$.

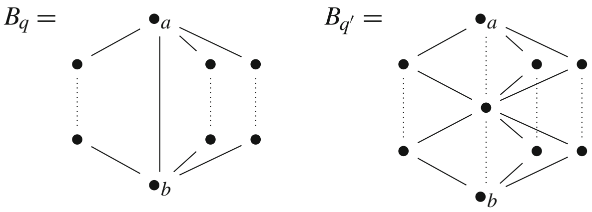
The quadratic form q can be recovered from q' using the nonlinear map $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ defined as follows,

$$\pi(x)_k = x_k, \quad \text{for } k \notin \{i, j, n + 1\} \text{ and}$$

$$(\pi(x)_i, \pi(x)_j, \pi(x)_{n+1}) = \begin{cases} (0, x_j - x_i, x_i), & \text{if } x_i \leq x_j, \\ (x_i - x_j, 0, x_j), & \text{if } x_i > x_j. \end{cases}$$

Since $\rho \circ \pi = \mathbf{Id}$ we have $q(x) = q'(\pi(x))$ for any vector $x \in \mathbb{Z}^n$.

Example 6.23. Consider the unit form q with associated bigrah B_q as shown below (left). Its edge reduction with respect to vertices a, b is the form q' with bigrah $B_{q'}$ (right).



For a quadratic form q denote by $\Sigma^+(q)$ the set of isotropic vectors of q with nonnegative entries.

Proposition 6.24. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form and $q' : \mathbb{Z}^{n+1} \to \mathbb{Z}$ be obtained from q by edge reduction with respect to indices i and j. Then q is weakly nonnegative if and only if q' is weakly nonnegative. In this case the maps ρ and π are bijections (inverse to each other) between the sets $\Sigma^+(q)$ and $\Sigma^+(q')$.*

Proof. Take a positive vector y in $\mathbb{Z}^{n+1}$. If q is weakly nonnegative, since $\rho(y) > 0$ we have

$$q'(y) = q(\rho(y)) + y_i y_j \geq 0.$$

Conversely, if $0 < x \in \mathbb{Z}^n$ and q' is weakly nonnegative, then $\pi(x) > 0$ and

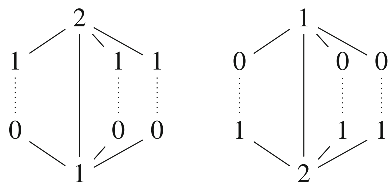
$$q(x) = q'(\pi(x)) \geq 0.$$

Assume that q and q' are weakly nonnegative. By the identity $q(x) = q'(\pi(x))$ the mapping π restricts to a function $\pi : \Sigma^+(q) \to \Sigma^+(q')$. If $y \in \Sigma^+(q')$ then

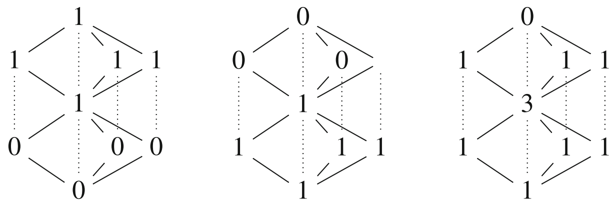
$$0 = q'(y) = q(\rho(y)) + y_i y_j.$$

Since both summands on the right are nonnegative, it follows that $y_i y_j = 0$ (thus $y \in \mathbf{Im}(\pi)$) and that $\rho(y) \in \Sigma^+(q)$. In particular, $\pi : \Sigma^+(q) \to \Sigma^+(q')$ is a surjective mapping, and the result follows since $\rho \circ \pi = \mathbf{Id}$. □

Even though there is a bijection between $\Sigma^+(q)$ and $\Sigma^+(q')$ when q is a weakly nonnegative semi-unit form and q' is an edge reduction of q, it is not always true that q and q' have the same number of critical vectors (a vector z is critical for q if the restriction of q to the support $\mathbf{supp}(z)$ of z is critical having z has positive generator of its radical). For instance, if q and q' are the forms shown in Exercise 6.23, then the following vectors v_1 and v_2 are critical vectors for q,

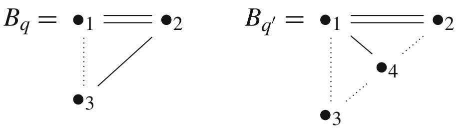


while q' has the following different critical vectors, $\pi(v_1)$, $\pi(v_2)$ and a third vector w with an entry 3.



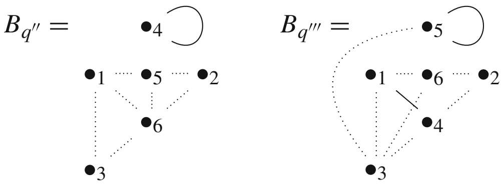
This behavior, along with notions like *positive corank* and *conformality* for edge reductions, are further explored in [54].

For a unit form q which is not weakly positive there might be an arbitrarily long iterated edge restriction for q, which is evident from the following example,



where B_q is a subbigraph of the bigraph $B_{q'}$ associated to the edge reduction q' of q with respect to the vertices 2 and 3. Notice that this example is actually weakly nonnegative.

An *iterated edge reduction* for a semi-unit form $q' : \mathbb{Z}^n \to \mathbb{Z}$ is a quadratic form $q : \mathbb{Z}^m \to \mathbb{Z}$ with $m \geq n$ that is obtained iteratively from q by a sequence of edge reductions. For instance, for the example in three variables q above, consider the iterated edge reductions q'' by edges $\{1,2\}$, $\{1,2\}$ and $\{2,3\}$, and the reduction q''' by edges $\{2,3\}$, $\{1,2\}$ and $\{1,2\}$ respectively, as shown below.



The following is a suitable generalization of Theorem 5.24 to the weakly nonnegative setting.

Theorem 6.25. *A semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is weakly nonnegative if and only if any iterated edge reduction q' of q is semi-unitary.*

Proof. The necessity follows from Proposition 6.24.

For the converse assume that q is a semi-unit form which is not weakly nonnegative. If there are vertices $a \neq b$ with $q_{ab} < -2$, then the edge reduction of q with respect to a and b is not semi-unitary. Therefore we may assume that $q_{ab} \geq -2$ for all vertices $a \neq b$. By Proposition 6.2 there is a critical vector z and $i \notin \mathbf{supp}(z)$ such that $q(z|e_i) < 0$. In particular,

$$q(2z + e_i) = q_{ii} + 2q(z|e_i) < 0.$$

Take vertices a and b with $q_{ab} < 0$ and consider the reduction q' of q with respect to a, b. First we notice that there exists a $j \notin \mathbf{supp}(z)$ such that

$$q'(2\pi(z) + e_j) < 0.$$

If $a = i$ and $b \in \mathbf{supp}(z)$ then take $j = n + 1$, so that

$$q'(2\pi(z) + e_{n+1}) = q(\rho[2\pi(z) + e_{n+1}]) = q(2z + e_i + e_b) \leq q(2z + e_i) < 0.$$

If $i \notin \{a, b\}$ or $\{a, b\} \cap \mathbf{supp}(z) = \emptyset$ then take $j = i$ and observe that

$$q'(2\pi(z) + e_i) = q(2z + e_i) < 0.$$

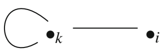
Now, if the weight $|z| = \sum_i |z_i|$ of z is greater than one, taking $a, b \in \mathbf{supp}(z)$ we have $|\pi(z)| < |z|$. By the above argument, replacing q for some iterated reduction of q, we may assume that $|z| = 1$, that is, $z = e_k$ for some $k \in \{1, \ldots, n\}$. Hence

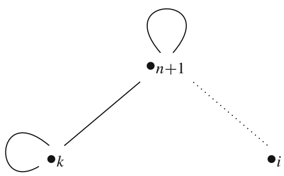
$$0 > q(2e_k + e_i) = 4q_{kk} + q_{ii} + 2q_{ki}.$$

Since $q_{ii}, q_{kk} \in \{0, 1\}$ we have $q_{kk} = 0 > q_{ki}$. Then the bigraph B associated to the restriction $q^{\{k,i\}}$ has one of the following forms,

corresponding to cases $q_{ii} = 0$ (left) and $q_{ii} = 1$ (these restrictions are the hypercritical semi-unit forms q'_m and q''_m from Lemma 6.1). For the reduction q' of q with respect to k and i we have

$$q'_{n+1,n+1} = q_{kk} + q_{ii} + q_{ki} = q_{ii} + q_{ki},$$

thus q' is not a semi-unit form unless B has the form . In this case the restriction $(q')^{\{k,i,n+1\}}$ has the following associated bigraph,



hence the reduction of q' with respect of k and $n+1$ is not a semi-unit form, which completes the proof. □

Following [54], by an *exhaustive reduction* for a semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ we mean an iterated edge reduction q' of q satisfying the following conditions:

i) Every edge reduction involved in the construction of q' is with respect to vertices i and j satisfying $1 \leq i < j \leq n$.
ii) For any $1 \leq i < j \leq n$ we have $q'_{ij} \geq 0$.

Notice that all exhaustive reductions involve the same number K of edge reductions, namely

$$K = (-1) \sum_{i<j \text{ and } q_{ij}<0} q_{ij}.$$

The forms q'' and q''' right before Theorem 6.25 are examples of exhaustive reductions of the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + x_1x_3 - x_2x_3.$$

Furthermore, we may consider a sequence $q^0, q^1, q^2, \ldots$ of semi-unit forms such that $q^0 = q$ and for $k > 0$ the form q^k is obtained from q^{k-1} by an exhaustive reduction. Then we say that q^k is obtained from q by an *iterated exhaustive reduction* (of length k). Notice that there is a sequence of integers

$$n = n_0 < n_1 < n_2 < \ldots < n_k$$

such that q^i is a semi-unit form in n_i variables for $i = 0, \ldots, k$. It is not known whether a semi-unit form q is weakly nonnegative if and only if any iterated exhaustive reduction of q stops, after finitely many steps, in a quadratic form having only nonnegative coefficients. However, the following criterion (which is an alternative version of Theorem 6.25) was proved in [54].

Remark 6.26. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form, and $q^k : \mathbb{Z}^{n_k} \to \mathbb{Z}$ be a sequence of iterated exhaustive reductions of q for $k = 0, 1, 2, \ldots$ Then q is weakly nonnegative if and only if q^k is semi-unitary for all $k \leq 31$.

6.5 Semi-Graphical Forms

The following result, known as the *reduction theorem by deflations* of weakly nonnegative forms, gives the main procedure to obtain graphical forms from weakly nonnegative semi-unit forms, which is one of the main tools in next section. We present a useful generalization.

Theorem 6.27. *Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semi-unit form with a maximal sincere positive root x. If $I = J \cup K$ is a nontrivial partition of the index set I, then there is an iterated deflation T for q concentrated in J such that the form $q' = qT$ satisfies the following.*

a) The form q' is a weakly nonnegative semi-unit form.
b) The form q' has a maximal positive root x' with $x = T(x')$.
*c) We have $q'_{ij} \geq 0$ for all $i, j \in J \cap$ **supp**(x').*
d) There are inclusions,

$$R^+(q') \xrightarrow{\quad T \quad} R^+(q) \quad \text{and} \quad \Sigma^+(q') \xrightarrow{\quad T \quad} \Sigma^+(q).$$

Proof. Take a deflation T_{ij}^- for q and the form $q^- = qT_{ij}^-$. Consider a positive vector $y \in \mathbb{Z}^n$ and take $y^- = T_{ij}^-(y) = y + y_i e_j$. Then y^- is a positive vector and

$$q^-(y) = q(T_{ij}^- y) = q(y^-) \geq 0,$$

which shows (a). For (b) we take i and j with $x_j \geq x_i$ so that $x^- := (T_{ij}^-)^{-1}x$ is a positive q^--root. If y^- is a positive q^--root with $y^- \geq x^-$, then $y := T_{ij}^-(y^-)$ is a positive q-root with $y \geq x$. Hence $y = x$, that is, the vector x is a maximal root. The claim (d) follows as in Lemma 2.19. Therefore points (a), (b) and (d) hold for iterated deflations.

For (c), as long as there are vertices i and j such that $q_{ij} < 0$ we may take a deflation T_{ij}^- or T_{ji}^- and continue with the reduction. The process must stop since in each step the weight $|x^-| = \sum_i x_i^-$ of x^- is smaller than the weight $|x|$ of x. □

Following Dräxler, Golovachtchuk, Ovsienko and de la Peña [22], we say that a semi-unit form $q : \mathbb{Z}^I \to \mathbb{Z}$ is *semi-graphical* if there exists a vertex $\omega \in I$ such that $q_{\omega i} < 0$ for all $i \neq \omega$, and $q_{ij} \geq 0$ for all $i, j \neq \omega$. As defined by Ringel [46], a *graphical form* is a semi-graphical unit form q such that $|q_{ij}| \leq 1$ for all $i \neq j$. According to Sect. 5.5, a centered form q is a semi-graphical unit form with $q_{\omega i} = -1$ for all $i \neq \omega$. Therefore graphical forms are centered.

Lemma 6.28. *Let q be a finitely sincere weakly nonnegative semi-unit form. Then B_q is a connected bigraph. Moreover, $q_{ii} = 0$ for a vertex i if and only if $q_{ij} \geq 0$ for all $j \neq i$.*

Proof. If B_q has a nontrivial partition supported by the sets of vertices I^1 and I^2, and x is a sincere positive q-root, then $x = x^1 + x^2$ with **supp**$(x^i) = I^i$ for $i = 1, 2$. Since $1 = q(x^1) + q(x^2)$ we may assume that $q(x^1) = 1$ and $q(x^2) = 0$, and thus conclude that all vectors $x^1 + mx^2$ are sincere positive q-roots for $m > 0$, in contradiction with q being finitely sincere.

For the second assertion notice that if $q_{ii} = 0$ and $q_{ij} < 0$, then $q(2e_i + e_j) = q_{jj} + 2q_{ij} < 0$. Conversely, assume that $q_{ij} \geq 0$ for all $j \neq j$ and that $q_{ii} = 1$. Since B_q is connected, there exists $j \neq i$ such that $q_{ij} > 0$. Then for any sincere positive root x we have

$$q(x|e_i) = 2x_i + \sum_{k \neq i} x_k q_{ki} \geq 3,$$

which is impossible since $0 \leq q(x - e_i) \leq 2 - q(x|e_i)$. □

The Kronecker form q_m for $m \geq 2$ is a semi-graphical form which is critical and hypercritical for $m \geq 3$. All other critical semi-graphical forms are actually graphical.

Lemma 6.29. *Any critical semi-graphical form q in $n \geq 3$ variables is a graphical form.*

Proof. Since $n \geq 3$ we have $q_{ij} \geq -1$ for all $i, j \neq \omega$. We show that $q_{ij} \leq 1$ for $i, j \neq \omega$. Since the vector $e_i - e_j$ is not sincere, and proper restrictions of critical forms are positive (cf. Corollary 5.3), we have

$$0 < q(e_i - e_j) = 2 - q_{ij},$$

thus the result. □

The list of critical semi-graphical forms with $n \geq 3$ is precisely that of Table 5.3. It will be useful to have a classification of centered hypercritical forms (equivalently, hypercritical semi-graphical forms with $n > 3$ variables). In Table 6.1 we exhibit such forms.

Recall that by a 0-sincere form we mean a weakly nonnegative semi-unit form q having a sincere positive radical vector. We say that a 0-sincere (weakly nonnegative) unit form is *reduced* provided $q_{ij} \leq 1$ for all vertices i, j (compare to slender forms). The following lemma justifies this definition. Recall from Sect. 5.5 that a unit form q is obtained from q' by doubling a vertex k if q is the one-point extension $q = q'[-e_k]$ (cf. also Exercise 3.32.4).

Lemma 6.30. *Let q be a 0-sincere (weakly nonnegative) unit form. Then q is not reduced if and only if there is a vertex i such that q can be recovered from the restriction $q^{(i)}$ by doubling a vertex.*

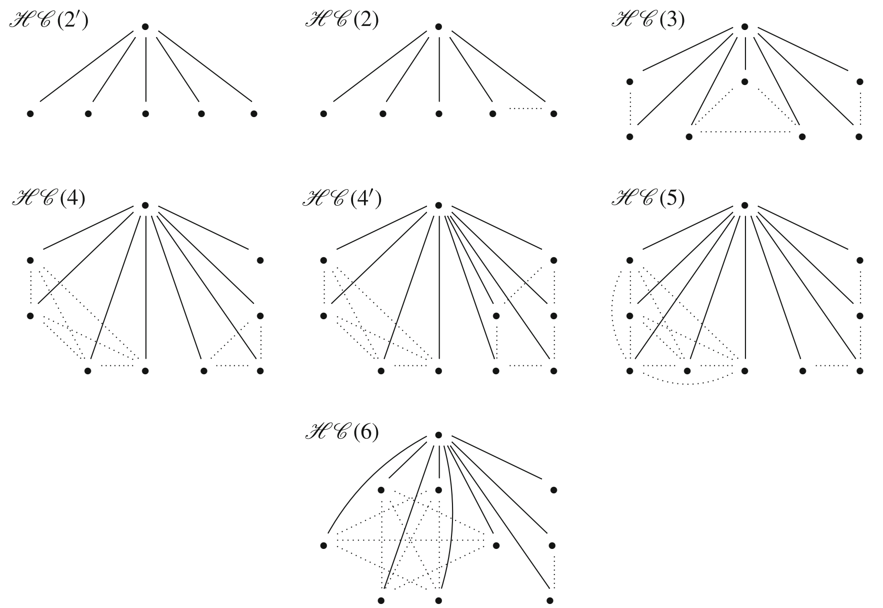
Proof. Assume $q : \mathbb{Z}^n \to \mathbb{Z}$ and take for simplicity $i = n$ and $q' = q^{(n)}$. Then clearly $q'[-e_k]_{kn} = 2$, thus $q = q'[-e_k]$ is not reduced.

For the converse assume that $q_{ij} > 1$ for some vertices $i \neq j$, and take z to be a sincere positive radical vector of q. Then we have

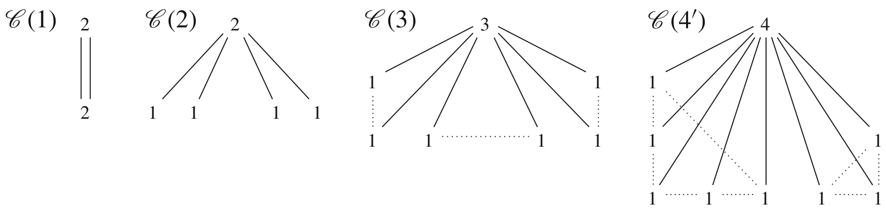
$$0 \leq q(z + e_i - e_j) = q(e_i - e_j) = 2 - q_{ij},$$

that is, $q_{ij} = 2$. In particular $q(e_i - e_j) = 0$, and since q is a nonnegative unit form (Lemma 6.22), by Lemma 3.2(a) the vector $e_i - e_j$ is radical, that is, q is obtained from q' by doubling vertex j. □

Table 6.1 Hypercritical graphical forms



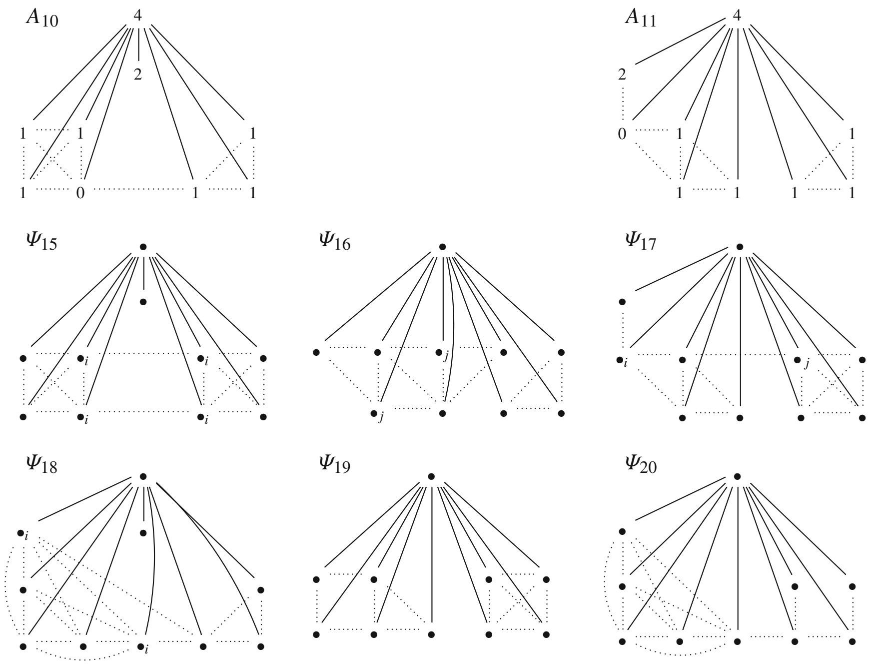
In the last part of this section we begin with technical preparations to end our discussion on integral quadratic forms with a generalization of Ovsienko's Theorem 5.25 to the weakly nonnegative context. In Table 6.2 we show some 0-sincere forms of small corank. The reason why we exclude those forms associated to bigraphs $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ and $\mathscr{C}(4')$ is the content of the following result (cf. Table 5.3 and the figure below).



The following classification result of graphical weakly nonnegative unit forms of small corank, due partially to Ringel [46] (cf. [23] for comments and proofs), will be used in the last steps in the proof of our last result Theorem 6.37.

We say that a 0-sincere graphical form q is *triangular* if there are precisely three critical restrictions q^{I_1}, q^{I_2} and q^{I_3} of q such that for any $i \neq j$ in $\{1, 2, 3\}$ the restriction $q^{I_i \cup I_j}$ is a 0-sincere form of corank 2.

Table 6.2 Reduced 0-sincere semi-graphical forms of corank one or two, without the forms associated to $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ and $\mathscr{C}(4')$ appearing as critical restrictions



In cases A_{10} and A_{11} the vector shown as integers at the vertices is the positive generator of the radical. A vertex marked as $\bullet_i$ or $\bullet_j$ represents a critical restriction $q^{(i)}$ or $q^{(j)}$ of shape A_{10} or A_{11}, respectively

Theorem 6.31. *Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a 0-sincere graphical form without critical restrictions having associated bigraph of the shape $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ or $\mathscr{C}(4')$.*

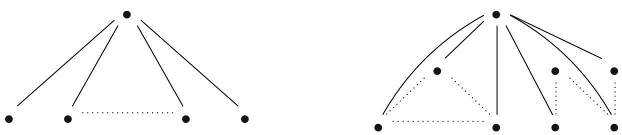
a) If $\mathbf{cork}(q) = 3$ then q is either triangular or one of the forms associated to Θ_1 or Θ_2 (see Table 6.3).
b) $\mathbf{cork}(q) = 2$ if and only if $q = q_{\Psi_\ell}$ for $\ell = 15, \dots, 20$ (see Table 6.2).

Remark 6.32. Let q be one of the forms q_{Φ_ℓ} for $\ell = 15, \dots, 20$. If $\mu^{(1)}$ and $\mu^{(2)}$ are critical vectors of q one can show by inspection that there are vertices i and j such that $\mu_i^{(1)} = 1$ and $\mu_j^{(1)} = 0$, and $\mu_i^{(2)} = 0$ and $\mu_j^{(2)} = 1$. In particular, for any positive radical vector μ of q, there are positive numbers m_1 and m_2 such that $\mu = m_1\mu^{(1)} + m_2\mu^{(2)}$.

Similarly, it can be shown that if q is a triangular 0-sincere form, then there are vertices $\{i, j, k\}$ such that the restriction of the critical vectors $\mu^{(1)}$, $\mu^{(2)}$ and $\mu^{(3)}$ are the canonical vectors with three entries. Therefore, for any sincere positive radical vector μ there are positive numbers m_1, m_2 and m_3 such that $\mu = m_1\mu^{(1)} + m_2\mu^{(2)} + m_3\mu^{(3)}$.

Exercises 6.33.

1. Show that the solid star $\mathbb{T}_{r_1,\dots,r_s}$ is equivalent to the centered form q where $q^{(\omega)}$ has associated bigraph $B = \sqcup I(r_i - 1)$ and where $I(m)$ is the complete dotted bigraph on m vertices.
2. Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative centered unit form with center ω, and for $i \in I$ consider the set $S_i = \{j \in I \mid q_{ij} > 0\}$. Show that if x is a positive sincere vector, $S \subset S_i$ with $i \neq \omega$ and
$$x_i - q(x|e_i) \geq x_\omega - \sum_{j \in S} x_j,$$
then $S = S_i$ and $q_{ij} = 1$ for all $j \in S_i$.
3. Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semi-graphical form with center ω. Suppose that x is a maximal sincere positive root with $x_\omega \geq 7$ and only one exceptional vertex.
   a) Show that q is a centered form, and that $q_{ij} \leq 1$ for all $j \neq \omega$.
   b) Set $S'_i = \{j \neq i \mid q_{ij} > 0\}$ and show that the restriction of B_q to S'_i is a complete graph with dotted edges. Moreover, $x_j = 1$ for all $j \in S'_i$ and if $j \in S_i$ and $k \in I$ satisfy $q_{jk} > 0$, then $k \in S_i$.
   c) Prove that S_i has exactly $x_\omega - 2$ elements.
   d) Notice that q is not weakly positive (why?) and show that if $J \subset I$ and the restriction q^J is critical, then $S_i \subset J$.
   e) Conclude that $x_\omega = 7$. [Hint: use (c) and (d) to verify that the restriction q^I may be identified with the critical form $q_{\mathscr{C}(6)}$, see Table 5.3].
4. Which of the hypercritical centered forms in Table 6.1 have as restriction the following bigraphs?



6.6 Generalizing Ovsienko's Theorem

Our objective in this section is to show that any maximal positive root x of a weakly nonnegative unit form q satisfies $x_i \leq 12$ for any index i, following arguments by Dräxler, Golovachtchuk, Ovsienko and de la Peña in [23]. We say that $x \in \mathbb{Z}^n$ is a 2-*layer root* of an integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ if x is a positive q-root and there exist positive isotropic vectors μ and μ' such that $x = \mu + \mu'$ (in particular $1 = q(x) = q(\mu|\mu')$).

Theorem 6.34. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly nonnegative semi-unit form with a maximal positive root x.*

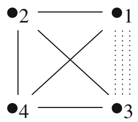
a) If there is a positive isotropic vector μ with $\mu < x$ then x is a 2-layer root.
b) If x is a 2-layer root then $x_i \leq 12$ for all $i = 1, \ldots, n$.

Proof. Without loss of generality we may assume that x is a sincere vector. To show (a), by maximality of x we have $\mu \notin \mathbf{rad}(q)$, and therefore $q(x|\mu) \neq 0$ by Lemma 6.21(b). That $q(x|\mu) = 1$ follows from the equations

$$0 \leq q(x - \mu) = q(x) - q(x|\mu) = 1 - q(x|\mu),$$
$$0 \leq q(x + m\mu) = q(x) + mq(x|\mu) = 1 + mq(x|\mu), \qquad \text{for all } m \geq 0.$$

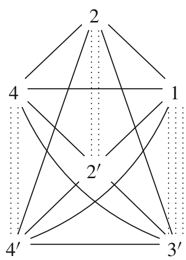
Hence $q(x - \mu) = q(x) - q(x|\mu) + q(\mu) = 0$, that is, $\mu' := x - \mu$ is an isotropic vector.

We now turn to the proof of (b), which we illustrate with an example. Take $x = \mu + \mu'$ with μ and μ' positive isotropic vectors of q.



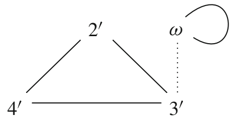
$x = \begin{smallmatrix}1&1\\1&0\end{smallmatrix} + \begin{smallmatrix}1&0\\1&1\end{smallmatrix}.$

Step 1. First we *double all vertices* $I = \{1, \ldots, n\}$ of the form $q : \mathbb{Z}^n \to \mathbb{Z}$ (cf. Exercises 3.32.4 and 5) to get a weakly nonnegative form $\overline{q} : \mathbb{Z}^{I \cup J} \to \mathbb{Z}$, where $J = \{n + 1, \ldots, 2n\}$. Consider μ as a vector in $\mathbb{Z}^{I \cup J}$ and define $\overline{\mu} = \sum_{i=1}^{n} \mu'_i e_{i+n}$. Then the projection $\pi : \mathbb{Z}^{I \cup J} \to \mathbb{Z}^I$ given by $\pi(e_{i+n}) = e_i = \pi(e_i)$ for $i \in I$ satisfies $\pi(\overline{x}) = x$ where $\overline{x} = \mu + \overline{\mu}$ is a maximal positive root of $\overline{q}$ (see Exercise 3.32.4(d)).
Take $I' = \mathbf{supp}(\mu)$ and $J' = \mathbf{supp}(\overline{\mu})$, and replace $\overline{q}$ by its restriction to $I' \cup J'$ (figure below for our example).



Step 2. Apply now the Reduction Theorem 6.27 to $\overline{q}$ with respect to I' to get an iterated deflation T (concentrated in I') and a weakly nonnegative quadratic form $\overline{q}' = \overline{q}T$ with a positive maximal root η such that $T(\eta) = \overline{x}$, and $\overline{q}'_{ij} \geq 0$ for all $i, j \in I' \cap \mathbf{supp}(\eta)$.

By Lemma 6.14, there is a vertex $\omega \in I'$ such that the support of η is $J' \cap \{\omega\}$, $\overline{q}'_{\omega\omega} = 0$ and the restriction q' of $\overline{q}'$ to J' is a unit form (in the example below the iterated flation is $T = T^{-}_{12}T^{-}_{14}$). Moreover, there exists a $j \in J'$ such that $\eta_\omega = \eta_j = \overline{q}'_{\omega j} = 1$ (in particular $\eta = \overline{\mu} + e_\omega$), and j is the unique element in J' satisfying $\overline{q}'_{\omega j} \neq 0$.



Step 3. By Lemma 6.21(b) we have $\overline{\mu} \in \mathbf{rad}^+(q')$, thus $\overline{\mu}$ belongs to the set

$$U = \{y \in \mathbf{rad}^+(q') \mid y_j = 1\}.$$

If U has infinitely many elements, there exist $y' < y \in U$, therefore $y - y' \in \mathbf{rad}^+(\overline{q}')$. This contradicts the maximality of η.

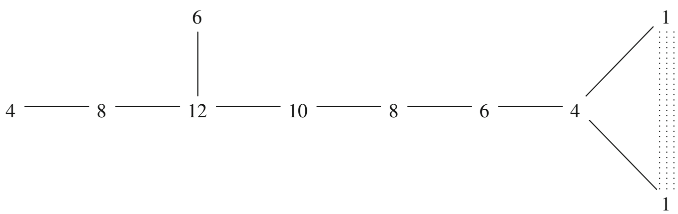
We conclude by pointing out that U is a finite set, thus by Lemma 6.35 below we have $\overline{\mu}_i \leq 6$ for $i \in I$. Since by symmetry we also have $\mu_i \leq 6$, then $x_i \leq 12$ for all $i \in I$. □

Lemma 6.35. *Suppose $q : \mathbb{Z}^I \to \mathbb{Z}$ is a (weakly nonnegative) 0-sincere semi-unit form such that there is an index $i \in I$ with $q^{(i)}$ unitary. If the set U of positive radical vectors y of q with $y_i = 1$ is finite, then $y_i \leq 6$ for any $y \in U$ and $i \in I$.*

Proof. We claim that the restriction $q^{(i)}$ is a weakly positive unit form. Otherwise there exists a positive isotropic vector μ with $i \notin \mathbf{supp}(\mu)$. By Lemma 6.21($a$) the vector μ is radical, contradicting the finiteness of U.

If $y \in U$ then $q(y - e_i) = q(e_i) = 1$, thus $y - e_i$ is a positive root of the weakly positive form $q^{(i)}$. The result follows from Ovsienko's Theorem 5.25. □

The following example shows that the bound 12 in Theorem 6.34 is optimal. The example is constructed by identifying all but the exceptional vertices of two copies of $q_{\widetilde{\mathbb{E}}_8}$, where the vector shown (a maximal positive root) is the sum of the corresponding positive generators of the radicals of $q_{\widetilde{\mathbb{E}}_8}$ (one for each copy).



The example above is not a 0-sincere form, which is a direct consequence of the following lemma.

Lemma 6.36. *Suppose $q : \mathbb{Z}^n \to \mathbb{Z}$ is a (weakly nonnegative) 0-sincere unit form. Then $q_{ij} > 1$ if and only if q is obtained from $q^{(i)}$ by doubling vertex j.*

Proof. Assume that $q_{ij} > 1$. By Exercise 6.15.5 we have $q_{ij} = 2$. Since $e_i - e_j$ is an isotropic vector ($0 = 2 - q_{ij} = q(e_i - e_j)$), by Lemma 6.21($a$) the vector $e_i - e_j$ is radical. Therefore by Exercise 3.32.6 the form q is equal to $q^{(i)}[j]$ (up to a reordering of vertices if necessary). The converse is evident. □

The following generalization of Ovsienko's Theorem is the main result in [23].

Theorem 6.37. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly nonnegative semi-unit form with a maximal positive root x. Then $x_i \leq 12$ for all $i = 1, \ldots, n$.*

Sketch of Proof. Suppose on the contrary that x is a maximal positive root of q with $x_\omega > 12$ for some $\omega \in \{1, \ldots, n\}$.

Step 1. *We may assume that $q : \mathbb{Z}^I \to \mathbb{Z}$ is a weakly nonnegative centered form without critical restriction of shape $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ or $\mathscr{C}(4')$. In this case, the maximal root x has two exceptional vertices. We may further assume that the cardinality $|I|$ is minimal among all such forms.*
Apply the Reduction Theorem 6.27 with respect to the set $I' = I - \{\omega\}$ and the maximal root x to get an iterated deflation T concentrated in I' and a maximal positive root x' of $q' = qT$ such that $x = T(x')$. Deleting some vertices if necessary, we may assume that x' is sincere, thus $q'_{ij} \geq 0$ for all $i, j \neq \omega$.
If there exists an $i \neq \omega$ such that $q'_{i\omega} \geq 0$ then by Lemma 6.28 we have $q'_{ii} = 0$. In particular e_i is a positive isotropic vector of q' with $e_i < x'$, therefore by Theorem 6.34, x' is a 2-layer root and $x_\omega = x'_\omega \leq 12$, a contradiction.
Moreover, if $q'_{i\omega} < -1$ then $q_{i\omega} = -2$ and the vector $e_\omega + e_i$ is isotropic for q' with $e_\omega + e_i < x'$, which is again impossible. Hence q' is a centered form.
Observe from Table 5.3 (see also the graphs after Lemma 6.30) that if q^J is a critical restriction of q' with associated bigraph $\mathscr{C}(2)$, $\mathscr{C}(3)$ or $\mathscr{C}(4')$, then there is a positive isotropic vector $\mu < x$, which once more by Theorem 6.34 yields a contradiction.
Finally, the statement about the exceptional vertices of x' is worked out in Exercise 6.33.3. Write q for q' and x for x'.

Step 2. *Let i and j be the exceptional vertices of x and consider the quadratic form $\overline{q}(y) = q(y) - y_i y_j$. Then $\overline{q}$ is a 0-sincere centered form with sincere positive radical vector x.*
By Lemma 6.11 we have $x_i = 1 = x_j$, therefore $i, j \neq \omega$.
First notice that the restriction $q^{(i)(j)}$ is weakly positive (otherwise there is a critical restriction with a critical positive vector μ, and $q(\mu + x) = q(x) = 1$ since $i, j \notin \mathbf{supp}(\mu)$, contradicting the maximality of x). This implies that $2 \leq q_{ij} \leq 3$. Indeed, by Exercise 6.15.5 the inequality $0 \leq q_{ij} \leq 3$ holds. If $q_{ij} \leq 1$ then $q(e_i + e_j + e_\omega) \leq 2$ and the claim below yields a contradiction with $z = e_i + e_j + e_\omega$.

Claim. If z is a positive vector with $q(z) \leq 2$ satisfying $z_k \leq 1$ for all $k \neq \omega$ and $z_i = 1 = z_j$, then $z_\omega > 6$.

Proof. If $z_\omega \leq 6$ then $x - z$ is a positive vector and since $q^{(i)(j)}$ is weakly positive we have

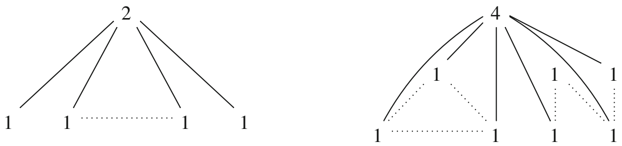
$$0 < q^{(i)(j)}(x - z) = q(x - z) = q(x) + q(z) - q(x|z)$$
$$= 1 + q(z) - (z_i q(x|e_i) + z_j q(x|e_j)) = q(z) - 1 \leq 1.$$

Then $x - z$ is a positive root of $q^{(i)(j)}$, and by Theorem 5.25 we have $x_\omega - z_\omega \leq 6$, in contradiction with $x_\omega > 12$. □

Observe that the bilinear form associated to $\overline{q}$ has the following shape,

$$\overline{q}(v|w) = q(v|w) - v_i w_j - v_j w_i,$$

hence $\overline{q}(x|e_k) = 0$ for all k since $q(x|e_k) = x_k = 1$ for $k = i, j$. Then x is a sincere positive radical vector for $\overline{q}$, and we only need to show that $\overline{q}$ is weakly nonnegative. Observe that $\overline{q}^{(i)} = q^{(i)}$ and $\overline{q}^{(j)} = q^{(j)}$. If $\overline{q}$ is not weakly nonnegative, then there is a hypercritical restriction $\overline{q}^J$ where $J \subset I$ contains both i and j. From Table 6.1 we see that $q_{ij} \neq 3$. Furthermore, if $q_{ij} = 2$ then $\overline{q}^J$ has a restriction including i and j with one of the following bigraphs (see Exercise 6.33.4)



Using the claim above with the vector z as indicated by the vertices in the figure, which satisfies $q(z) \leq 2$, we get a contradiction. Then $\overline{q}$ is a 0-sincere form with sincere positive radical vector x.

Step 3. *If for some vertices $s, t \in I$ we have $q_{st} > 1$, then $\{s, t\} = \{i, j\}$.*
Assume on the contrary that i does not belong to the set $\{s, t\}$ and consider the restriction $q' = q^{(i)}$, which has the vector $y = x - e_i$ as positive root. If q' is weakly positive, then $y_\omega = x_\omega \leq 6$, contradicting Ovsienko's Theorem. Then there is a critical restriction $(q')^J$ of q' with critical positive vector μ.
Since $\overline{q}_{st} = q_{st} > 1$, by Lemma 6.36 the 0-sincere form $\overline{q}$ is obtained from its restriction $\overline{q}^{(t)}$ by doubling vertex s. Consequently the vector $w = \mu - \mu_t e_t + \mu_t e_s$ is a positive isotropic vector for q' (thus also for q). Since $i, j \notin \mathbf{supp}(w)$ implies that $q(w|x) = 0$ we get the equation

$$q(x + w) = q(x) = 1,$$

which contradicts the maximality of x.

For a weakly nonnegative unit form $q : \mathbb{Z}^I \to \mathbb{Z}$ consider the union I^+ of the supports of all positive radical vectors of q. By Lemma 6.21, the restriction $q^+ := q^{I^+}$ is a 0-sincere form, called the 0-*sincere kernel* of q.

Step 4. *Let $\xi^+ : \mathbb{Z}^K \to \mathbb{Z}$ be the* 0*-sincere kernel of the restriction $q^{(i)}$. Then ξ^+ is nontrivial and satisfies* $\mathbf{cork}(\xi^+) \leq 2$.

Notice that $y = \sigma_i(x) = x - e_i$ is a sincere positive root of $q' := q^{(i)}$. Since $y_\omega > 12$, the form q' is not weakly positive, thus the 0-sincere kernel ξ^+ is nontrivial.

Now, since $q' = \overline{q}^{(i)}$, by Step 3 the form q' is graphical. Assume that $\mathbf{cork}(\xi^+) \geq 3$. Then we may take a 0-sincere restriction ξ of ξ^+ such that $\mathbf{cork}(\xi) = 3$ (cf. Lemma 6.22 and Remark 3.21).

Apply Theorem 6.31 to the form ξ, and notice first of all that Θ_1 is not the bigraph associated to ξ (by Theorem 6.34, since the vector z with $z_\omega = 5$ and $z_k = 1$ for all other vertices is isotropic with $z < x$). Thus if ξ is triangular or B_ξ is Θ_2, it can be seen that there exist critical vectors μ_1, μ_2 and μ_3 of ξ such that

$$|(\mu_s - \mu_t)_\omega| \leq 2, \quad \text{and} \quad |(\mu_s - \mu_t)_k| \leq 1 \quad \text{for } k \neq \omega,$$

for any $s \neq t$ in $\{1, 2, 3\}$ (see Exercise 2 below). Hence $x - (\mu_s - \mu_t) > 0$. Suppose that there are $s \neq t$ such that $q(x|\mu_s - \mu_t) \geq 2$. Then

$$q(x|\mu_s - \mu_t) = q(y + e_i|\mu_s - \mu_t) = q(e_i|\mu_s - \mu_t) \geq 2,$$

and since $x - (\mu_s - \mu_t) > 0$, we get the contradiction

$$0 \leq q(x - (\mu_s - \mu_t)) = q(x) + q(\mu_s - \mu_t) - q(x|\mu_s - \mu_t) < 0.$$

In particular, in the set $\{q(e_i|\mu_k)\}_{k=1,2,3}$ there are at least two equal elements, say $q(e_i|\mu_1) = q(e_i|\mu_2)$. We may also assume that $(\mu_1 - \mu_2)_\omega \geq 0$. Then $\mu_s - \mu_t$ is a radical vector of q, and taking $d = \min(x_k \mid (\mu_s - \mu_t)_k = -1)$ we get a nonsincere positive q-root $z = x + d(\mu_s - \mu_t)$ satisfying $z_\omega > 12$. Using Exercise 1 below, the vector z is a sincere maximal positive root of the restriction of q to the (proper) support of z, obtaining in this way a contradiction to the minimal choice of $|I|$ established in Step 1.

Step 5. *The form $\overline{q}$ admits no critical restriction with associated bigraph of shape $\mathscr{C}(1), \mathscr{C}(2), \mathscr{C}(3)$ or $\mathscr{C}(4')$.*

Since the form $\overline{q}$ is centered (Step 2), its bigraph does not contain the bigraph $\mathscr{C}(1)$ as a restriction. In all other cases notice that the support of the critical vector μ must contain both i and j (otherwise it would be a critical vector for q). Thus μ would be a positive root of q, and using the claim in Step 2 we get $\mu_\omega > 6$, a contradiction.

Step 6. *Final analysis of the case* $\mathbf{cork}(\xi^+) = 2$.

Consider that $\xi^+ : \mathbb{Z}^K \to \mathbb{Z}$ is a 0-sincere graphical form with $\mathbf{cork}(\xi^+) = 2$, which is by construction a restriction of the quadratic form $q^{(i)} = \overline{q}^{(i)}$ where

$q : \mathbb{Z}^I \to \mathbb{Z}$ is our original form. First we notice that $K = I - \{i\}$, that is, that $\xi^+ = q^{(i)}$. Indeed, if there is a $k \neq i$ in $I - K$ then the restriction of $q^{(i)}$ to the set $K \cup \{k\}$ has corank 3 by Exercise 5 below. This is impossible since ξ^+ has corank 2. Hence $K = I - \{i\}$. We will reach a contradiction by considering two cases.

Case $\overline{q}_{ij} = 2$. By Lemma 6.30 the form $\overline{q}$ is obtained from $\xi^+ = q^{(i)}$ by doubling vertex j. Define the vector $u := x - e_i + e_j$, which can be shown to be a sincere isotropic vector for ξ^+ with $u_j = 2$. Indeed, we have

$$\xi^+(u) = \overline{q}(x - e_i + e_j) = \overline{q}(x) = q(x) - x_i x_j = 0.$$

Taking $\mu^{(1)}$ and $\mu^{(2)}$ to be critical vectors of the two critical restrictions of ξ^+, there are positive integers m_1 and m_2 such that $u = m_1\mu^{(1)} + m_2\mu^{(2)}$ (see Remark 6.32). Since $u_j = 2$, up to exchanging the roles of $\mu^{(1)}$ and $\mu^{(2)}$ we may suppose that $\mu_j^{(1)} = 0$ or $\mu_j^{(1)} = 1$. But notice that in both cases we have $\mu^{(1)} < x$, therefore x is a 2-layer root by Theorem 6.34(a). This contradicts $x_\omega > 12$ by part (b) of that theorem.

Case $\overline{q}_{ij} = 1$. Again by Exercise 5 and Theorem 6.31, either $\overline{q}$ is a triangular form, or the form associated to one of the bigraphs Θ_1 or Θ_2. If $\overline{q}$ is triangular, then by Remark 6.32 there are positive integers m_1, m_2, m_3 such that

$$x = m_1\mu^{(1)} + m_2\mu^{(2)} + m_3\mu^{(3)},$$

where $m_1\mu^{(1)}$, $m_1\mu^{(2)}$ and $m_1\mu^{(3)}$ are critical vectors of $\overline{q}$. Since $x_i = 1$ we may assume that $\mu_i^{(1)} = 0$, therefore $\mu^{(1)} < x$. This is again impossible by Theorem 6.34. A similar argument can be formulated for case Θ_2 (see Exercise 3 below). Finally, if $\overline{q} = q_{\Theta_1}$, then the vector z given by $z_\omega = 5$ and $z_i = 1$ for $i \neq \omega$ is a positive q-root, contradicting the claim in Step 2 (see Table 6.3 and Exercise 4).

Step 7. *Final analysis of the case* **cork**$(\xi^+) = 1$.

We assume now that ξ^+ is itself a critical form, and let μ be its critical vector. Suppose first that ξ^+ is the form associated to one of the graphs $\mathscr{C}(5)$ or $\mathscr{C}(6)$. It can be shown then (see Exercise 6(b) and (c) below) that ξ^+ is the (one-point) restriction of a form of corank 2. Therefore we have again $\xi^+ = q^{(i)} = \overline{q}^{(i)}$. As before we consider separately the cases $\overline{q}_{ij} = 2$ and $\overline{q}_{ij} = 1$.

Case $\overline{q}_{ij} = 2$. By Lemma 6.30 the form $\overline{q}$ is obtained from ξ^+ by doubling vertex j. Hence $u := x - e_i + e_j$ is a sincere positive radical vector of ξ^+. Because $u_j = 2$ we have $u = m\mu$ for some $m \in \{1, 2\}$. However, recall from Proposition 5.4 that $\mu_\omega \leq 6$, therefore $x_\omega \leq 2\mu_\omega \leq 12$, a contradiction.

Case $\overline{q}_{ij} = 1$. A direct inspection of the bigraphs $\Psi_{17}, \ldots, \Psi_{20}$ given in Exercise 6 shows that, since $x_i = 1$, we may find a critical restriction of $\overline{q}$ avoiding vertex i, and such that its critical vector μ satisfies $\mu < x$. The contradiction is again derived from Theorem 6.34.

By Step 5 and the discussion above we may finally suppose that ξ^+ is the form associated to the graph $\mathscr{C}(4)$. Let us first assume that $K = I - \{i\}$ (that is, that $\xi^+ = q^{(i)}$). Then if $\overline{q}_{ij} = 2$ we can argue as above, while if $\overline{q}_{ij} = 1$ then by Exercise 6(a) the form $\overline{q}$ fails to be 0-sincere, in both cases a contradiction.
Therefore we may fix a vertex $k \neq i$ in the set $I - K$. Let us now assume that $I - K = \{i, k\}$. If $\overline{q}_{ij} = 2$ then one can check that $\overline{q}$ is not 0-sincere, and if $\overline{q}_{ij} = 2$ then by Exercise 6(d) the form $\overline{q}$ is associated to one of the bigraphs $\Psi_{15}, \dots, \Psi_{18}$, and one can proceed as above to find a critical vector μ with $\mu < x$, obtaining a contradiction using Theorem 6.34.
Assume now that we can find a second vertex $\ell \neq i$, different from k, in the set $I - K$. Consider the restriction $\widetilde{q} = q^{K \cup \{k,\ell\}}$, and take $q' = \widetilde{q}^{(\ell)}$. Hence $(q')^{(k)} = \xi^+$, which is the form associated to the graph $\mathscr{C}(4)$. By Exercise 6(a), the graph associated to q' has shape A_{10} or A_{11}. By Exercise 6(d), the form $\widetilde{q}$ is either not 0-sincere, or is associated to one of the bigraphs Ψ_{15}, Ψ_{16}, Ψ_{17} or Ψ_{18}. It is shown in [23, Sect. 9.9] that all these cases imply that $\overline{q}$ itself is not 0-sincere, a contradiction.

This completes the proof. □

Exercises 6.38.

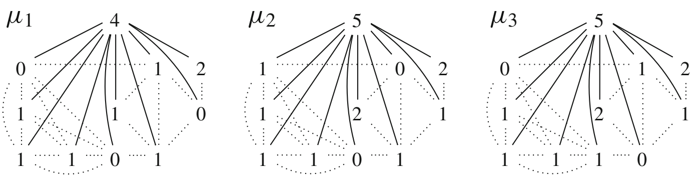
1. Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semi-unit form with a maximal sincere positive root x. If $\mu \in \mathbf{rad}(q)$ and $x + \mu$ is a positive vector, show that $x + \mu$ is a maximal sincere positive root of the restriction of q to the support of $x + \mu$.
2. Let q be a 0-sincere graphical form of corank 3 without having as restriction a form associated to the bigraphs $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ or $\mathscr{C}(4')$.

   a) If q is a triangular form, let μ_1, μ_2 and μ_3 be the positive critical vectors of q. Show that for $s \neq t$ in $\{1, 2, 3\}$ we have

   $$|(\mu_s - \mu_t)_\omega| \leq 2, \quad \text{and} \quad |(\mu_s - \mu_t)_k| \leq 1, \quad \text{for } k \neq \omega.$$

   [Hint: Use Theorem 6.31.]

   b) If $B_q = \Theta_2$ consider the vectors

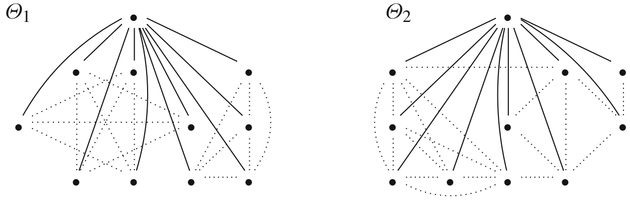


   Show that μ_1, μ_2 and μ_3 are critical vectors of q, and that for $s \neq t$ in $\{1, 2, 3\}$ we have

   $$|(\mu_s - \mu_t)_k| \leq 1, \quad \text{for all } k.$$

   Why is q not a triangular form?

Table 6.3 Some 0-sincere semi-graphical forms of corank 3



3. Show that if q is the quadratic form associated to Θ_2 (Table 6.3), and $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$ and $\mu^{(4)}$ are its critical vectors, then there are nonnegative integers $m_1, \ldots, m_4$ such that any sincere positive radical vector μ can be written as

$$\mu = m_1\mu^{(1)} + m_2\mu^{(2)} + m_3\mu^{(3)} + m_4\mu^{(4)}.$$

   Show also that we may assume, up to a reordering of variables, that m_1 and m_2 are positive integers.
4. Consider the quadratic form q with bigraph Θ_1 with center ω (Table 6.3), and let z be the vector with $z_\omega = 5$ and $z_i = 1$ for all other vertices. Show that z is an isotropic vector for q. Is it a radical vector?
5. Let $q : \mathbb{Z}^K \to \mathbb{Z}$ be a 0-sincere graphical form without critical restriction of shape $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ or $\mathscr{C}(4')$, and take $k \in K$.

   a) Show that if the restriction $q^{(k)}$ is a 0-sincere form of corank 2, then q is 0-sincere of corank 3.
   b) Show that in the situation of point (a), either q is a triangular form, or q is one of the forms Θ_1 or Θ_2 shown in Table 6.3.

6. Let $q : \mathbb{Z}^J \to \mathbb{Z}$ be a weakly nonnegative graphical form having no critical restriction of shape $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ or $\mathscr{C}(4')$. Consider a vertex $j \in J$.

   a) Show that if $q^{(j)} = q_{\mathscr{C}(4)}$, then q is the form associated to one of the bigraphs A_{10} or A_{11} below, and $\mathbf{cork}(q) = 1$.
   b) Show that if $q^{(j)} = q_{\mathscr{C}(5)}$, then q is the form associated to Ψ_{17}, Ψ_{19} or Ψ_{20}, and $\mathbf{cork}(q) = 2$ (see Table 6.2).
   c) Show that if $q^{(j)} = q_{\mathscr{C}(6)}$, then q is the form associated to Ψ_{18} or Ψ_{20}, and $\mathbf{cork}(q) = 2$ (see Table 6.2).

d) Show that if $q^{(j)} = q_{A_{10}}$ or $q^{(j)} = q_{A_{11}}$, then either q is the form associated to Ψ_{15}, Ψ_{16}, Ψ_{17} or Ψ_{18}, or q is not 0-sincere.

7. Let $q : \mathbb{Z}^J \to \mathbb{Z}$ be a graphical 0-sincere form having no critical restriction of shape $\mathscr{C}(1)$, $\mathscr{C}(2)$, $\mathscr{C}(3)$ or $\mathscr{C}(4')$. Show that if there is a vertex $j \in J$ such that $q^{(j)}$ has associated bigraph A_{10} or A_{11}, then B_q is Ψ_{15}, Ψ_{16}, Ψ_{17} or Ψ_{18}. In particular q has corank 2.