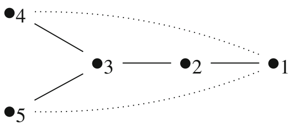
# Chapter 5
# Weakly Positive Quadratic Forms

Consider the quadratic form $q$ associated to the bigraph $G$ below (left). On one hand we observe that $q$ is not a positive form, since $T = T^-_{12}T^-_{13}T^+_{31}$ is an iterated flation for $q$ such that $qT$ is the form associated to extended Dynkin diagram $\widetilde{\mathbb{D}}_4$ (alternatively calculate $q(-1, 0, 1, 1, 1) = 0$). In particular $q$ has infinitely many roots (Theorem 2.16).




On the other hand, the positive roots $R^+(q)$ of $q$ are contained in the set of positive $q_\Delta$-roots $R^+(q_\Delta)$, where $\Delta$ is the Dynkin diagram $\mathbb{D}_5$. Indeed, for a vector $x \in \mathbb{Z}^5$ we have $q(x) = q_\Delta(x) + x_1(x_4 + x_5)$, and if $x$ is a positive root of $q$, then $x_1(x_4 + x_5) = 0$ and $x$ is a positive root of $q_\Delta$. Hence the set of positive roots $R^+(q)$ of $q$ is finite. The equality $q(x) = q_\Delta(x) + x_1(x_4 + x_5)$ also shows that if $x \in \mathbb{Z}^5$ is a positive vector, then $q(x) > 0$.

A semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is said to be *weakly positive* if $q(x) > 0$ for every positive vector $x \in \mathbb{Z}^n$ (recall that a vector $x \in \mathbb{Z}^n$ is positive given $x \neq 0$ and $x_i \geq 0$ for $i = 1, \ldots, n$).

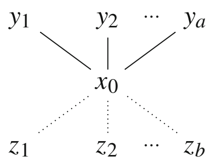
*Examples 5.1.* The following are examples of weakly positive unit forms.

a) A positive unit form is weakly positive.
b) Let $B$ be a bigraph with only dotted edges, and take $q_B$ to be its associated quadratic form (that is, $q_B$ is a unit form with $(q_B)_{ij} \geq 0$ for $i \neq j$). Then $q_B$ is weakly positive.



M. Barot et al., *Quadratic Forms*, Algebra and Applications 25,
https://doi.org/10.1007/978-3-030-05627-8_5

c) Consider the quadratic form $q^{a,b}$ associated to the following bigraph, with integers $a, b \geq 1$



Then $q^{a,b}$ is a unit form in $a + b + 1$ variables which is weakly positive exactly when $a \leq 3$. Indeed, we may write

$$q(x_0, y_1, \ldots, y_a, z_1, \ldots, z_b) = \sum_{i=1}^{a} \left( y_i - \frac{1}{2}x_0 \right)^2 + \frac{4-a}{4}x_0^2 + \sum_{j=1}^{b}(z_j^2 + x_0 z_j),$$

and verify the claim.

## 5.1 Critical Unit Forms

A unit form $q$ is called *critical nonweakly positive*, or for short just *critical*, if every proper restriction of $q$ is weakly positive, but the form $q$ itself is not weakly positive (compare with critical nonpositive forms defined in Sect. 2.3). The following characterization of critical forms was shown by Ovsienko in [43] (see also [52]). For the proof we follow Ringel in [46].

**Theorem 5.2.** *Let $q$ be a unit form. Then $q$ is critical if and only if $q$ is the Kronecker form $q_m(x_1, x_2) = x_1^2 - mx_1x_2 + x_2^2$ for some $m \geq 3$, or $q$ is nonnegative of corank one with radical generated by a sincere positive vector (referred to as a* critical vector *of $q$).*

*Proof.* Clearly the stated conditions are sufficient (see proof of Theorem 2.12). For the converse let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a critical form and $v > 0$ a positive vector with minimal weight $|v| := \sum_{i=1}^{n} |v_i|$ such that $q(v) \leq 0$.

Since any proper restriction of $q$ is weakly positive, the vector $v$ is sincere. Then for each vertex $i \in \{1, \ldots, n\}$ we have by minimality

$$0 < q(v - e^i) = q(v) + 1 - q(v|e^i),$$

and therefore $q(v|e^i) \leq q(v)$ for all $i$.

If $q(v) = 0$ then it follows from $q(v) = \frac{1}{2}\sum_{i=1}^{n} v_i q(v|e^i)$ that $q(v|e^i) = 0$ for all $i$ (since $v$ is sincere and positive), that is, $v$ is a radical vector. For any other positive $w$ with $q(w) \leq 0$ we choose an index $a$ such that $\frac{w_a}{v_a} \leq \frac{w_i}{v_i}$ for all $1 \leq i \leq n$.

Take $z := v_a w - w_a v$ and notice that $z$ is a positive vector in $\mathbb{Z}^n$ with $z_a = 0$. Then

$$0 \leq q^{(a)}(z) = q(v_a w - w_a v) = v_a^2 q(w) \leq 0,$$

and since $q^{(a)}$ is weakly positive, $q^{(a)}(z) = 0$ implies $v_a w = w_a v$. Again by minimality of $v$ all its entries are mutually coprime, therefore $v_a$ divides $w_a$, that is, $w$ is an integral multiple of $v$. This shows that if $q(v) = 0$ then $q$ is nonnegative with radical generated by $v$.

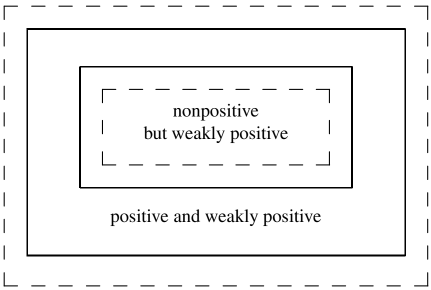
If $q(v) < 0$ then we proceed as in the proof of Theorem 2.12 to obtain $n = 2$ and $q(x_1, x_2) = x_1^2 + q_{12}x_1x_2 + x_2^2$ with $q_{12} \leq -3$. □

In particular notice that all critical forms $q$ in $n \geq 3$ variables are nonnegative with radical generated by a sincere positive vector. Using Theorem 3.5, if $q$ is connected there exists an iterated inflation $T$ for $q$ and an extended Dynkin graph $\widetilde{\Delta}$ such that $qT = q_{\widetilde{\Delta}}$.

**Corollary 5.3.** *A critical unit form is always critical nonpositive, that is, any proper restriction of a critical unit form is positive.*

*Proof.* The claim is clear for critical forms $q : \mathbb{Z}^n \to \mathbb{Z}$ with $n = 2$ (Kronecker forms $q_m$ with $m \geq 2$). If $n > 2$, it follows from Theorem 5.2 that a critical unit form is nonnegative with radical generated by a sincere vector. Therefore any proper restriction of $q$ is positive, that is, $q$ is critical nonpositive. □

Using Corollary 5.3 we are ready now to correct the picture drawn in Sect. 2.3.




Recall that the *one-point extension* $q[v] : \mathbb{Z}^{n+1} \to \mathbb{Z}$ of a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ with respect to a $q$-root $v$ is defined as

$$q[v](x_1, \ldots, x_n, x_{n+1}) = q\left(\sum_{i=1}^{n} x_i e_i - x_{n+1} v\right),$$

which is again unitary, see Lemma 3.26.

**Proposition 5.4.** *Let $q$ be a unit form in more than two variables.*

*a) The form $q$ is critical nonpositive if and only if $q = p[v]$, where $p$ is a positive unit form and $v$ is a sincere root of $p$.*
*b) The form $q$ is critical if and only if $q = p[v]$, where $p$ is a positive unit form and $v$ is a positive sincere root of $p$.*

*In both cases the radical* $\mathbf{rad}(q)$ *of $q$ is generated by a vector $z$ having a vertex $i$ with $z_i = 1$, while for all vertices $j$ we have $|z_j| \leq 6$.*

*Proof.* This is a direct consequence of Theorems 2.12 and 5.2, since $q[v]$ is a nonnegative unit form with $\mathbf{rad}(q[v])$ generated by the vector $v + e_n$ (cf. Lemma 3.26).

For the last statement, see Corollary 3.31. □

The following technical lemma will be widely used throughout this chapter. Recall that for $v \in \mathbb{Z}^n$, the support of $v$ is given by $\mathbf{supp}(v) = \{i \in \{1, \ldots, n\} \mid v_i \neq 0\}$.

**Lemma 5.5.** *For a weakly positive semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ the following statements hold:*

*a) The form $q$ is a unitary.*
*b) For every pair of indices $i \neq j$ with $q_{ij} < 0$ we have $q_{ij} = -1$.*
*c) If $v \in \mathbb{Z}^n$ is a positive $q$-root then $-1 \leq q(v|e_i)$. Moreover, if $i$ and $j$ are different indices in the support of $v$, then $q(v|e_i) \leq 1$ and $q_{ij} \leq 2$.*

*Proof.* Point ($a$) is clear. For ($b$) we evaluate $q$ at the vector $e_i + e_j$ to get

$$0 < q(e_i + e_j) = q(e_i) + q(e_j) + q(e_i|e_j) = 2 + q(e_i|e_j) = 2 + q_{ij}.$$

To show ($c$) notice that the inequality $-1 \leq q(v|e_i)$ holds in general (evaluate $q$ at $v + e_i$). Now, if $i, j \in \mathbf{supp}(v)$, the nonzero vector $v - e_i$ has no negative coordinates, therefore $0 < q(v - e_i) = 2 - q(v|e_i)$. For the second inequality assume that $q_{ij} \geq 3$, and notice that

$$q(e_i - e_j) = q(e_i) + q(e_j) - q(e_i|e_j) = 2 - q_{ij} < 0.$$

Since we may assume that $q(v|e_i - e_j) \leq 0$ (change the roles of $i$ and $j$ otherwise), for $y = v + e_i - e_j$ we have

$$q(y) = q(v) + q(e_i - e_j) + q(v|e_i - e_j) \leq 0,$$

a contradiction since $y$ is a positive vector. □

We say that a weakly positive unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is *sincere* if there exists a positive sincere root $v$ of $q$.

**Corollary 5.6.** *For $n \geq 1$ there are finitely many sincere weakly positive unit forms in $n$ variables.*

*Proof.* If $q : \mathbb{Z}^n \to \mathbb{Z}$ is a sincere weakly positive unit form, then by Lemma 5.5 we have $-1 \leq q_{ij} \leq 2$ for all $1 \leq i \neq j \leq n$. Thus the result follows. □

In Sect. 1.2 we have defined, for a quadratic unit form $q$, the $i$-th simple reflection $\sigma_i : \mathbb{Z}^n \to \mathbb{Z}^n$ given as $\sigma_i(x) = x - q(x|e_i)e_i$ for $x$ in $\mathbb{Z}^n$. In the following Proposition we resume some basic facts related to reflections when applied to weakly positive unit forms. We need some preliminary observations.

**Lemma 5.7.** *Let $q$ be a unit form.*

*a) If $v$ is a $q$-root, then $\sum_{i=1}^n v_i q(v|e_i) = 2q(v) = 2$.*

*If moreover $q$ is a weakly positive form and $v$ is a nonsimple positive root, then:*

*b) For all $i \in$ **supp**$(v)$ we have $|q(v|e_i)| \leq 1$.*
*c) There exists an $i \in$ **supp**$(v)$ with $q(v|e_i) = 1$.*

*Proof.* Part $(a)$ is a direct calculation. For $(b)$, by hypothesis we have $v \pm e_i > 0$. Therefore $0 < q(v \pm e_i) = 2 \pm q(v|e_i)$, which implies that $|q(v|e_i)| \leq 1$. Part $(c)$ follows directly from $(b)$ and $(c)$. □

Let $q$ be a unit form. Recall that a positive $q$-root $v$ is called *maximal* if for any $q$-root $w$ with $w \geq v$ (that is, such that $w - v$ is a nonnegative vector) we have $w = v$. Maximal roots play a key role in understanding weakly positive roots. Furthermore, since the restriction of a weakly positive unit form is again weakly positive, we may want to first understand those forms which are sincere.

**Proposition 5.8.** *The following are equivalent for a positive root $v$ of a weakly positive unit form $q : \mathbb{Z}^n \to \mathbb{Z}$.*

*a) The $q$-root $v$ is maximal.*
*b) We have $\sigma_i(v) \leq v$ for all $i = 1, \ldots, n$.*
*c) We have $q(v|e_i) \geq 0$ for all $i = 1, \ldots, n$.*

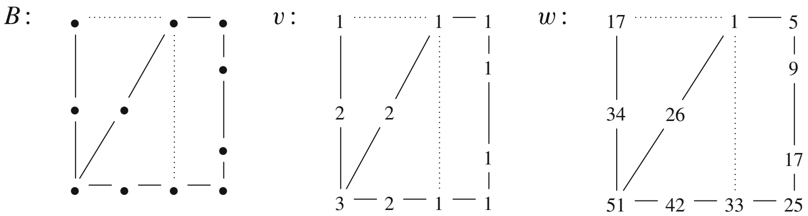
*Proof.* Assume $(a)$ holds. By definition $\sigma_i(v) = v - q(v|e_i)e_i$, thus we have either $\sigma_i(v) \leq v$ or $\sigma_i(v) > v$. Since $\sigma_i(v)$ is also a root of $q$ (Lemma 1.5$(c)$), by maximality of $v$ we have $\sigma_i(v) \leq v$, therefore $(b)$ holds.

That $(b)$ implies $(c)$ is obvious. We show that $(c)$ implies $(a)$. Let $w$ be a $q$-root with $w \geq v$. Then $w_i \geq v_i$ and $q(v|e_i) \geq 0$ for any index $i$, therefore

$$0 \leq q(w-v) = q(w)+q(v)-q(w|v) = 2-\sum_{i=1}^{n} w_i q(v|e_i) \leq 2-\sum_{i=1}^{n} v_i q(v|e_i) = 0,$$

showing that $q(w - v) = 0$, that is, $w = v$ since $q$ is weakly positive. □

The hypothesis that $q$ is weakly positive is essential to show that $(c)$ implies $(a)$ in Proposition 5.8, as the following example shows. Let $q = q_B$ be the form associated to the bigraph $B$ below and take $v$ and $w$ to be the vectors as indicated by the integers at the vertices.




Then it is easy to show that in fact $q(v) = q(w) = 1$ and $q(v|e_i) \geq 0$ for any vertex $i$, but clearly $v < w$. It also clear that $q$ is not weakly positive since $q(w-v) = -16$.

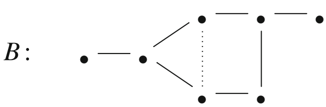
In view of the preceding result, for a maximal $q$-root $v$ it is natural to distinguish between vertices $i$ for which $q(v|e_i) > 0$ and those vertices $j$ where $q(v|e_j) = 0$. A vertex $i$ is called *exceptional* for the maximal $q$-root $v$ if $q(v|e_i) > 0$. The following result was observed by Ringel [46] in the context of sincere representation finite algebras.

**Lemma 5.9.** *Let $v$ be a sincere maximal positive root of a weakly positive unit form. If $v \neq e_i$ for $1 \leq i \leq n$ then either there exist exactly two exceptional vertices $i \neq j$ with $v_i = v_j = 1$, or there is exactly one exceptional vertex $i$ with $v_i = 2$.*

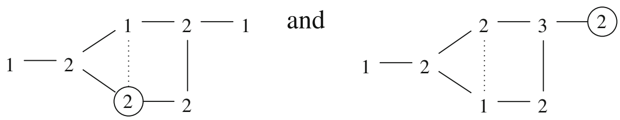
*Proof.* By Proposition 5.8($c$) and Lemma 5.7($b$) we have $q(v|e_i) = 0, 1$ for any vertex $i$. Hence the result follows from $\sum_{i=1}^{n} v_i q(v|e_i) = 2$, see Lemma 5.7($a$). □

Notice that a vertex is exceptional with respect to a maximal root. Since there might exist several maximal roots, exceptional vertices are in general not inherent to unit forms (but to specific maximal roots), as the following example shows.

*Example 5.10.* Consider the quadratic form $q_B$ associated to the bigraph depicted below.




Then there are two maximal roots (indicated by the numbers at the vertices)




where encircled numbers indicate the exceptional vertex in each case.

The following important result will be used below in Theorem 5.13.

**Corollary 5.11.** *For any critical unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ and any positive $q$-root $v$ there is a vertex $i \in \{1, \ldots, n\}$ with $q(v|e_i) < 0$. In particular $q$ has infinitely many positive roots.*

*Proof.* By Theorem 5.2, if $q$ is critical then either $q$ is the Kronecker form $q_m(x_1, x_2) = x_1^2 + x_2^2 - mx_1x_2$ with $m \geq 3$, or $q$ is nonnegative with radical generated by a positive vector $z$.

Consider first the former case, and take $v = (v_1, v_2)$ a positive root of $q_m$, thus in particular $v_1^2 + v_2^2 = 1 + mv_1v_2$. Then

$$\begin{aligned} q(v|e_1)q(v|e_2) &= (2v_1 - mv_2)(2v_2 - mv_1) = (4 + m^2)v_1v_2 - 2m(v_1^2 + v_2^2) \\ &= (4 + m^2)v_1v_2 - 2m(1 + mv_1v_2) \\ &= (2 + m)(2 - m)v_1v_2 - 2m, \end{aligned}$$

and since $v$ is positive and $m > 2$ we have $q(v|e_1)q(v|e_2) < 0$.

Now, for the second case consider a positive root $v$ with $q(v|e_i) \geq 0$ for all $i$ and take $w := v + z$. Since $z$ is positive and sincere we have $w_i > v_i > 0$ for $i = 1, \ldots, n$. Then

$$q(z) = q(w-v) = q(w)+q(v)-q(w|v) = 2-\sum_{i=1}^{n} w_iq(v|e_i) < 2-\sum_{i=1}^{n} v_iq(v|e_i) = 0,$$

which is a contradiction. We conclude in any case that there is an index $i$ with $q(v|e_i) < 0$. For the last claim, for any positive root $v$ with $q(v|e_i) < 0$ we have that $\sigma(v)$ is a positive root larger than $v$, thus the assertion follows. □

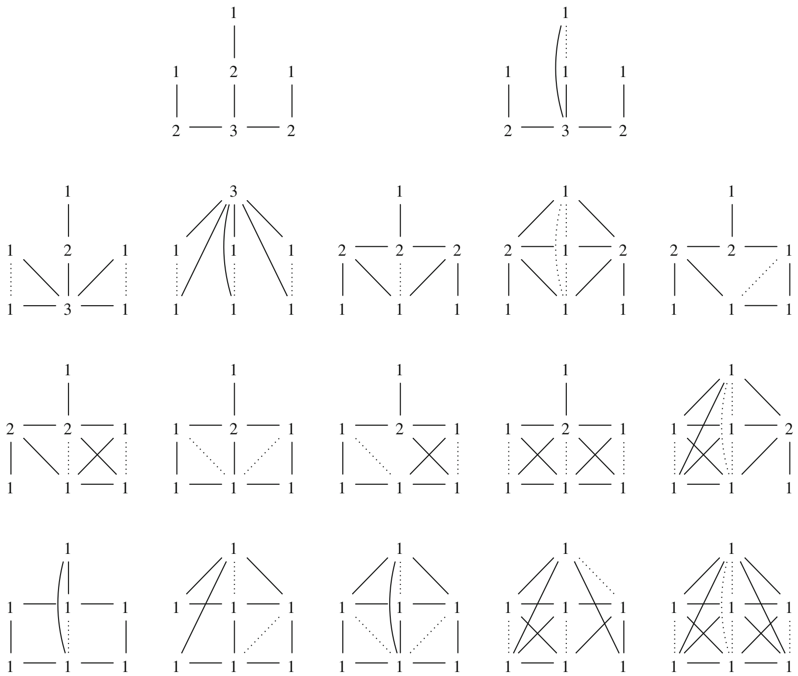
If $q$ is a critical unit form in more than two variables, then $q$ is connected and nonnegative by Theorem 5.2. As defined in Sect. 3.2 the Dynkin type $\mathbf{Dyn}(q)$ of $q$ is a Dynkin graph. For instance, in Table 5.1 we exhibit all critical forms of Dynkin type $\mathbb{E}_6$.

## 5.2 Checking for Weak Positivity

As a first (rather obvious) criterion to verify weak positivity notice that a unit form $q$ is weakly positive if and only if it does not contain as a restriction any critical form. The following nontrivial characterization is due to Drozd and Happel (cf. [30]). We need a preliminary observation.

**Lemma 5.12.** *Let $v^1, v^2, v^3, \ldots$ be an infinite sequence of different positive vectors in $\mathbb{Z}^n$. Then there exist $0 < s < t$ such that $v_s < v_t$ (in other words, the poset of positive vectors in $\mathbb{Z}^n$ has* finite width*).*

**Table 5.1** Critical forms of type $\mathbb{E}_6$



*Proof.* We proceed by induction on $n > 0$ (the case $n = 1$ is evident). Consider the nonnegative integer $m^k := \min(v_1^k, \dots, v_n^k)$ for $k \geq 1$. If the sequence $\{m^k\}_{k \geq 1}$ is unbounded the claim is clear (taking $t$ such that $m^t > \max(v_1^1, \dots, v_n^1)$ then we guarantee that $v^1 < v^t$), hence we may assume that $\{m^k\}_{k\geq 1}$ is bounded.

Taking subsequences if necessary, we may further assume that there is an index $1 \leq i \leq n$ such that the sequence $\{v_i^k\}$ is bounded, thus we may actually assume that the integer $v_i^k$ is fixed for all $k \geq 1$. Consider the vector $\widehat{v}^k = (v_1^k, \dots, v_{i-1}^k, v_{i+1}^k, \dots, v_n^k)$ in $\mathbb{Z}^{n-1}$ for $k \geq 1$, and observe that $\{\widehat{v}^k\}_{k\geq 1}$ is a sequence of different positive vectors in $\mathbb{Z}^{n-1}$. Hence the result follows by induction. □

**Theorem 5.13 (Drozd–Happel).** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form. Then $q$ is weakly positive if and only if $q$ accepts only finitely many positive roots. Moreover, in this case there is an iterated deflation $T$ such that $qT = q_G$ where $G$ is a bigraph with only dotted edges and no loop.*

*Proof.* We start by proving the last statement. Suppose that $R^+(q)$ is a finite set and that $q$ is weakly positive. If $q_{ij} < 0$ for some $i \neq j$ then $q_{ij} = -1$ by Lemma 5.5$(b)$. Consider the deflation $T = T_{ij}^-$ for $q$ and take $q_1 = qT$ (which is a unit form by Proposition 2.17). Then $T : R^+(q_1) \to R^+(q)$ is a proper embedding (Lemma 2.19) thus $R^+(q_1)$ is a finite set. To continue we will look for indices $k \neq \ell$

such that $(q_1)_{k\ell} < 0$. Since this procedure may be iterated only a finite number of times, we get a composition of deflations $T$ taking $q$ to $q_G$ where $G$ has only dotted edges.

Assume first that $q$ is not weakly positive. Then there exists a critical restriction $q^I$ of $q$, and by Corollary 5.11 the forms $q^I$ and $q$ have infinitely many positive roots.

Assume now that $q : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly positive unit form such that $R^+(q)$ is an infinite set. Let $n$ be minimal with this property, so that for each index $i$ the weakly positive unit form $q^{(i)}$ has finitely many positive roots. In particular, $q$ has infinitely many *sincere* positive roots, and by Lemma 5.5($c$), for any such root $v$ we have $q(v|e_i) \in \{-1, 0, 1\}$. Therefore there should be an infinite sequence $\{v^k\}_{k\geq 1}$ of positive $q$-roots with $(q(v^k|e^i))_{i=1}^n$ a fixed vector in $\mathbb{Z}^n$. By Lemma 5.12 we can find two comparable roots $v^s < v^t$, and we have

$$0 < q(v^t - v^s) = \frac{1}{2}q(v^t - v^s|v^t - v^s)$$
$$= \frac{1}{2}\sum_{i=1}^{n}(v_i^t - v_i^s)[q(v^t|e_i) - q(v^s|e_i)] = 0,$$

which is impossible. Therefore $R^+(q)$ is a finite set. □

The iterated simple reflections of a unit form $q$ may also be used to check for weak positivity of $q$ (cf. Proposition 2.5 and Remark 2.6).

**Proposition 5.14.** *A unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is weakly positive if and only if there exists $N > 0$ such that for every sequence of $q$-roots with the shape*

$$e_i < \sigma_{\ell_1}(e_i) < \sigma_{\ell_2}\sigma_{\ell_1}(e_i) < \ldots < \sigma_{\ell_r}\cdots\sigma_{\ell_1}(e_i),$$

*we have $r < N$.*

*Proof.* For $q$ weakly positive the condition is necessary since $R^+(q)$ is a finite set (Theorem 5.13).

If $q$ is not weakly positive then there is a critical restriction $q'$ of $q$. By Corollary 5.11, for any positive $q'$-root $v$ there is a vertex $\ell$ such that $q'(v|e_\ell) = q(v|e_\ell) < 0$. In particular, if $v = \sigma_{\ell_r}\cdots\sigma_{\ell_1}(e_i)$ is a positive root, then $v < \sigma_\ell(v)$ which completes the result. □

Ovsienko's Theorem (see Theorem 5.25 below) claims that if $v \in \mathbb{Z}^n$ is a positive root of a weakly positive unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, then $v_i \leq 6$ for $i = 1, \ldots, n$. This establishes a priori the bound $N = 6^n$ in the algorithm of Proposition 5.14.

Recall from Proposition 2.5 and Remark 2.6 that we may construct all positive roots (inductively using reflections) for a unit form $q$ known to be weakly positive. However, we usually do not know beforehand that $q$ is weakly positive. Still, we could start to construct $q$-roots inductively using reflections, and find a way to stop the process using the following simple criterion.

**Proposition 5.15.** *If $q : \mathbb{Z}^n \to \mathbb{Z}$ is a nonweakly positive unit form with $q_{ij} \geq -2$ for $1 \leq i, j \leq n$, then there exists a positive $q$-root $w$ and a vertex $i$ such that $q(w|e_i) \leq -2$.*

*Proof.* Since $q$ is not weakly positive there exists a critical restriction $q^I$ of $q$ which, by the hypothesis $q_{ij} \geq -2$ and Theorem 5.2, has a positive sincere radical vector $z$. For an index $i$ in $I$ and identifying $z \in \mathbb{Z}^I$ with its inclusion in $\mathbb{Z}^n$, we have $q(z|e_i) = q^I(z|e_i) = 0$. Take $w = z - e_i \in \mathbb{Z}^n$ which is a positive root of $q$, and calculate

$$q(w|e_i) = q(z|e_i) - q(e_i|e_i) = 0 - 2. \qquad \square$$

**Algorithm 5.16.** *By iteratively calculating positive $q$-roots using reflections one of the following two situations appear after a finite number of steps: either one finds a positive root $w$ and a vertex $i$ such that $q(w|e_i) \leq -2$ and conclude that the form was not weakly positive, or we end up with a finite number of positive roots unable to produce any new positive roots using reflections, hence concluding that the form is weakly positive (and we have reached all positive roots).*

The last result of this section will be heavily used in the rest of this chapter.

**Lemma 5.17.** *Let $q$ be a sincere weakly positive unit form and consider its associated bigraph $B_q$. Then the subgraph of $B_q$ determined by all solid edges is connected.*

*Proof.* Suppose that the opposite holds, namely, that the set of vertices may be divided into two disjoint subsets $I$ and $J$ such that $q_{ij} \geq 0$ whenever $i \in I$ and $j \in J$. Consider a positive sincere root $v$ and write $v = v^I + v^J$ where $\mathbf{supp}(v^I) = I$ and $\mathbf{supp}(v^J) = J$. Then each summand on the right side of the following equation is nonnegative,

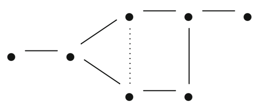
$$1 = q(v^I + v^J) = q(v^I) + q(v^J) + \sum_{i \in I,\ j \in J} v_i^I v_j^J q_{ij},$$

hence we must have $q(v^I) = 0$ or $q(v^J) = 0$, that is, either $I = \emptyset$ or $J = \emptyset$. $\square$

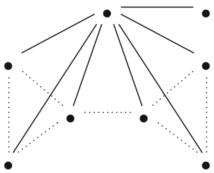
**Exercises 5.18.**

1. Find the exceptional vertices of the maximal positive root of the quadratic form associated to each Dynkin diagram.
2. Calculate the root-picture for $q_B$ (that is, the Hasse diagram of the poset of positive $q_B$-roots) where $B$ is the following bigraph,

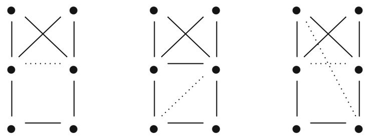
$B$:



3. Find all the maximal roots of $q_B$ and their exceptional vertices, where




4. Determine which of the following bigraphs correspond to a weakly positive unit form.



Which of these forms is sincere?
5. Give an iterated deflation $T$ such that the bigraph associated to the form $qT$ has no dotted edges, where $q$ is the following weakly positive unit form,

$$q(x) = x_1^2 + \ldots + x_7^2 - x_2(x_1 + x_3 + x_4) - x_3(x_5 + x_6) + x_4(x_1 + x_6 - x_7) - x_6 x_7.$$

## 5.3 Edge Reduction

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form and take different indices $i$ and $j$ such that $q_{ij} < 0$. Define a new unit form $q' : \mathbb{Z}^{n+1} \to \mathbb{Z}$ by the formula

$$q'(x) = q(\rho(x)) + x_i x_j, \quad \text{where} \quad \rho(e_k) = \begin{cases} e_k, & \text{if } 1 \leq k \leq n; \\ e_i + e_j, & \text{if } k = n+1. \end{cases}$$

We say that $q'$ is obtained from $q$ by *edge reduction* with respect to indices $i$ and $j$ (see [53]). The quadratic form $q$ can be recovered from $q'$ using the nonlinear map $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ defined as

$$\pi(x)_k = x_k, \quad \text{for } k \notin \{i, j, n+1\} \text{ and}$$

$$(\pi(x)_i, \pi(x)_j, \pi(x)_{n+1}) = \begin{cases} (0, x_j - x_i, x_i), & \text{if } x_i \leq x_j; \\ (x_i - x_j, 0, x_j), & \text{if } x_i > x_j. \end{cases}$$

Indeed, we have $\rho \circ \pi = \mathbf{Id}_{\mathbb{Z}^n}$ and $q(x) = q(\rho(\pi(x))) = q'(\pi(x)) - \pi(x)_i \pi(x)_j = q'(\pi(x))$ for every $x \in \mathbb{Z}^n$.

**Lemma 5.19.** *If $q$ is a unit form and $q'$ is an edge reduction of $q$ with respect to $i$ and $j$, then $q'$ is again a unit form if and only if $q_{ij} = -1$.*

*Proof.* The claim follows from the observations $q'(e_k) = q(\rho(e_k)) = q(e_k) = 1$ for $1 \leq k \leq n$, and

$$q'(e_{n+1}) = q(\rho(e_{n+1})) = q(e_i + e_j) = 2 - q_{ij}. \qquad \square$$

**Proposition 5.20.** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ and $q' : \mathbb{Z}^{n+1} \to \mathbb{Z}$ be unit forms such that $q'$ is obtained from $q$ by edge reduction with respect to vertices $i$ and $j$. The following hold:*

*a) The function $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ induces an injection $\pi : R^+(q) \to R^+(q')$.*
*b) The form $q$ is weakly positive if and only if $q'$ is weakly positive. In this case $\pi : R^+(q) \to R^+(q')$ is a bijection.*

*Proof.*

($a$) If $x$ is a positive $q$-root then

$$q'(\pi(x)) = q(\rho(\pi(x))) + \pi(x)_i \pi(x)_j = q(x) = 1,$$

since by definition either $\pi(x)_i = 0$ or $\pi(x)_j = 0$. Clearly $\pi$ is an injection.

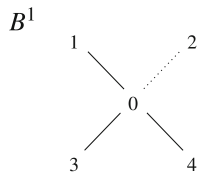
($b$) Assume $q$ is weakly positive and take $y$ to be a positive vector in $\mathbb{Z}^{n+1}$. Then clearly $\rho(y)$ is a positive vector in $\mathbb{Z}^n$ and $q'(y) = q(\rho(y)) + y_i y_j$, where the first summand is strictly positive and the second nonnegative. Hence $q'(y) > 0$.

For the converse, assume that $q'$ is weakly positive and take $x$ a positive vector in $\mathbb{Z}^n$. By construction $\pi(x)$ is a positive vector in $\mathbb{Z}^n$ and $q(x) = q'(\pi(x)) > 0$.

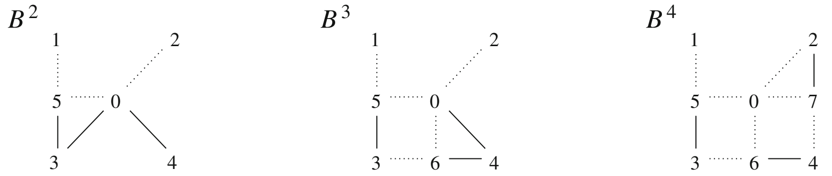
Finally suppose $q$ is weakly positive and take a positive root $y \in R^+(q')$. Then $1 = q'(y) = q(\rho(y)) + y_i y_j > y_i y_j \geq 0$, which means that $y_i y_j = 0$ and $\rho(y) \in R^+(q)$ with $\pi(\rho(y)) = y$, that is, $\pi : R^+(q) \to R^+(q')$ is a bijection $\square$

*Examples 5.21.* Next we illustrate graphically the edge reduction procedure.

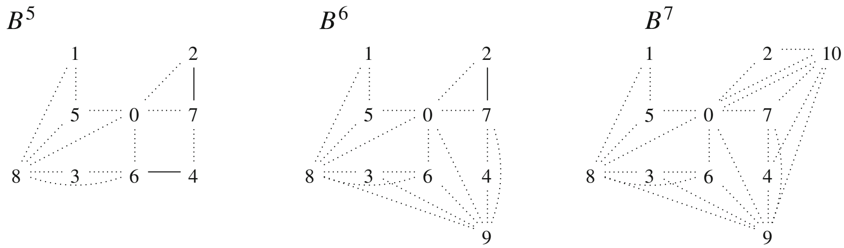
a) Consider $q^1 = q_{B^1}$, where $B^1$ is the bigraph



Reducing $q^1$ with respect to 0 and 1 yields $q^2 = q_{B^2}$, where $B^2$ is the bigraph below (and the added vertex is labeled 5).

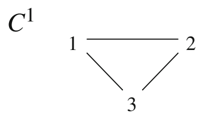



Reduce $q^2$ with respect to 0 and 3 to get $q^3 = q_{B^3}$, after reducing bigraph $B^3$ to avoid both types of edges between two vertices (regularization). Continue with edge 0 and 4 to get $B^4$, and similarly as indicated below:

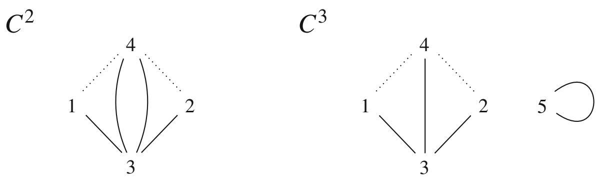



At the end we get a bigraph $B^7$ containing only dotted edges, hence we cannot continue to perform reductions. According to Proposition 5.20, all quadratic forms $q^7, q^6, \dots, q^1$ are weakly positive. Observe also that $B^7$ has a double dotted edge between a pair of vertices (0 and 10).

b) As an illustration of Lemma 5.19, consider the unit form $q = q_{C^1}$ where $C^1$ is the bigraph




After applying edge reduction with respect to 1 and 2 we get $q^2 = q_{C^2}$, where $C^2$ is the bigraph below



Now we get the following interesting situation: Reduction with respect to vertices 3 and 4, which are joined by a couple of solid edges. This reduction yields quadratic form $q^3 = q_{C^3}$, where $C^3$ is (the regularization of) the bigraph above, which has an isolated loop. Hence $q^3$ is not a unit form, and therefore none of the forms $q$, $q^2$ and $q^3$ is weakly positive.

The following *reduction procedure* for weakly positive unit forms is presented by von Höhne in [53], and it forms the basis of the algorithm described below.

**Theorem 5.22.** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly positive unit form and consider a sequence of forms $q = q^n, q^{n+1}, \ldots, q^s$ such that $q^i$ is obtained from $q^{i-1}$ by edge reduction (hence $q^i : \mathbb{Z}^i \to \mathbb{Z}$) for $i > n$. The following hold:*

*a) Each $q^i$ is a weakly positive unit form for $i \geq n$.*
*b) We have $s \leq |R^+(q)|$ and if $s = |R^+(q)|$ then $q^s$ has coefficients $q^s_{ij} \geq 0$ for every pair of indices $1 \leq i, j \leq s$.*

*Proof.* By Proposition 5.20 each $q^i$ is a weakly positive unit form, and the mapping $\pi^i : \mathbb{Z}^{n+1} \to \mathbb{Z}^{i+1}$ induces a bijection $\pi^i : R^+(q^i - 1) \to R^+(q^i)$ for every $i > n$. Since each canonical vector $e^j$ is a positive $q^i$-root, we have

$$i \leq |R^+(q^i)| = |R^+(q)|.$$

If $s = |R^+(q)|$ and if for some pair of indices $i < j$ we have $q^s_{ij} < 0$, then $q^s_{ij} = -1$ by Lemma 5.5, hence $q^s(e_i + e_j) = 1$, which is impossible since $R^+(q^s) = \{e^i \mid 1 \leq i \leq n + s\}$. □

As consequence of Theorem 5.22, if $q$ is a weakly positive unit form there is a bound for the length of any possible iterated edge reduction for $q$, namely $|R^+(q)| - n$ where $n$ is the number of variables of $q$. The converse is false. For instance, the (classical) Kronecker unit form $q_2$ admits iterated edge reductions of length at most two, although $q_2$ is not weakly positive. Now we describe an algorithm to verify weak positivity for a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, constructing on the way all positive $q$-roots.

**Algorithm 5.23.** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form.*

*Step 1. Construct a sequence of quadratic forms $q^n, q^{n+1}, \ldots, q^N$, where $q^n = q$ and $q^{k+1}$ is obtained from $q^k$ by edge reduction with respect to vertices $i_k$ and $j_k$ (in particular, $q^{k+1} : \mathbb{Z}^{k+1} \to \mathbb{Z}$ is a quadratic form), for $n \leq k < N$.*
*Step 2. Define the sequence of vectors $z^1, \ldots, z^N$ in $\mathbb{Z}^n$ as follows: For $k = 1, \ldots, n$ take $z^k = e_k$ the canonical vector, and for $k \geq n$ define*

$$z^{k+1} = z^{i_k} + z^{j_k}.$$

*Step 3.* *For each $N \geq n$ verify the following stopping conditions:*

*a)* $q_{ij}^N \geq 0$ *for all* $1 \leq i, j \leq N$.
*b)* $q_{ij}^N \leq -2$ *for some* $i \neq j$.
*c)* $N > 6^n$.

*Then the algorithm must stop after finitely many steps, and the unit form $q$ is weakly positive if condition $(a)$ in Step 3 is satisfied at some point.*

*Proof.* First, if case $(a)$ arises for some $N \geq n$, then $q^N$ is weakly positive with $R^+(q^N) = \{e_1, \ldots, e_N\}$. By Theorem 5.22 we conclude that $q$ itself is a weakly positive unit form and $|R^+(q)| = N$. Moreover, it can be shown that

$$R^+(q) = \{z^1, \ldots, z^N\}.$$

If one of the cases $(b)$ or $(c)$ holds, the form $q^N$ is not weakly positive (respectively by Lemma 5.5$(b)$ and Ovsienko's Theorem 5.25 below). In any case, $q$ is not weakly positive. □

In practice it is never necessary to go so far as the bound $6^n$ in the algorithm above, and in the next chapter we will review this algorithm and see how to improve it to make it one of the fastest of all.

**Theorem 5.24.** *A unit form $q$ is weakly positive if and only if any iterated edge reduction $q'$ of $q$ is unitary.*

*Proof.* The necessity is a consequence of Theorem 5.22$(a)$. Let us assume that the quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ is unitary, but not weakly positive.

Assume first that there are vertices $i \neq j$ such that $q_{ij} \leq -3$. If $q'$ is edge reduction of $q$ with respect to $i$ and $j$, then clearly $q'(e_{n+1}) = 2 - m \leq -1$, that is, $q$ is a nonunitary form.

Assume now that $q_{ij} \geq -2$ for all indices $i$ and $j$, and take a critical restriction $q^I$ of $q$. By Theorem 5.2 the restriction $q^I$ is nonnegative and has a critical vector $z$ in $\mathbb{Z}^I$ (which will be identified with its inclusion in $\mathbb{Z}^n$). Since $q$ is unitary the weight $|z| = \sum_{i=1}^n z_i$ of $z$ is larger than 1. Consider the following evident facts:

i) If $|v| > 1$ for a positive vector with $q(v) = 0$, then there are vertices $i \neq j$ in the support $\mathbf{supp}(v)$ of $v$ such that $q_{ij} < 0$.
ii) If moreover $q'$ is the edge reduction of $q$ with respect to vertices $i$ and $j$, and $v' = \pi(v) \in \mathbb{Z}^{n+1}$, then $v'$ is a positive vector with $q'(v') = 0$ and $|v'| < |v|$.

Starting with the critical vector $z$, the result follows by induction using points $(i)$ and $(ii)$ above. □

It follows from the proof of Theorem 5.24 that in the reduction process we may find quadratic forms $q$ with $q_{ii} \leq 1$ for some vertex $i$. These are called *pre-unit* forms, and will be considered again in next chapter when addressing the weakly nonnegative setting.

## 5.4 Ovsienko's Theorem

As shown in Proposition 2.22, the absolute values of the entries of any root $v$ of a positive unit form are bounded by 6. This is now extended to positive roots of weakly positive unit forms, the celebrated Ovsienko's Theorem. The proof given closely follows Ringel in [46] (see also Gabriel and Roiter [26]).

**Theorem 5.25 (Ovsienko).** *For any vertex $i \in \{1, \dots, n\}$ and any positive root $v \in \mathbb{Z}^n$ of a weakly positive unit form $q : Z^n \to \mathbb{Z}$ we have $v_i \leq 6$.*

*Proof.* The proof is combinatorial and done in several steps. We have already seen a positive (hence weakly positive) unit form with a root having an entry 6, namely $q_{\mathbb{E}_8}$ (see Table 2.2).

Let $s \geq 6$ be an integer. Among those weakly positive unit forms $q$ with

$$s = M(q) := \max\{v_i \mid i \text{ is an index of } q \text{ and } v \text{ is a positive root of } q\},$$

we choose one, say $q$, having minimal number of positive roots. Fix a maximal positive $q$-root $v$ such that $v_k = s$ for some index $k$. By minimality, $v$ is a sincere root.

Step 1. *We show that $q_{ij} \geq 0$ for all $i, j \neq k$.*
Suppose $v_i \leq v_j$. We see from $0 < q(e_i + e_j) = 2 + q_{ij}$ that $q_{ij} \geq -1$. If $q_{ij} = -1$ then we know from Lemma 2.19 that $q^- = qT_{ij}^-$ has fewer positive roots than $q$. Take $v' = (T_{ij}^-)^{-1}v = v - v_i e_j$, which is a positive root of $q^-$ satisfying $v'_k = v_k = s$. This contradicts the assumed minimality of $q$, since $M(q^-) = M(q)$.

Step 2. *We show that $q_{ij} \leq 1$ for all $i, j \neq k$.*
It follows from Lemma 5.5$(c)$ that $q_{ij} \leq 2$. If $q_{ij} = 2$, assuming that $q(v|e_i) \leq q(v|e_j)$ and taking the positive vector $w = v - v_j e_j + v_j e_i$, we obtain

$$1 \leq q(w) = q(v) + 2v_j^2 - v_j[q(v|e_j) - q(v|e_i)] - v_j^2 q_{ij} \leq 1.$$

Hence $w$ is a positive root of the restriction $q^{(j)}$ with $w_k = s$. But $q^{(j)}$ certainly has fewer positive roots than $q$, again in contradiction to minimality.

Step 3. *We have $q_{ki} = -1$ for every vertex $i \neq k$.* This is a direct consequence of Steps 1 and 2, and Lemma 5.17.

Step 4. *The root $v$ has exactly one exceptional vertex $\ell$ and $v_\ell = 2$.*
Otherwise Lemma 5.9 implies that there are precisely two exceptional vertices $\ell$ and $\ell'$ with $q(v|e_\ell) = q(v|e_{\ell'}) = 1$ and $v_\ell = v_{\ell'} = 1$. But in that case, $\sigma_\ell(v) = v - e_\ell$ is a sincere positive root of $q^{(\ell)}$ with $\sigma_\ell(v)_k = s$, in contradiction to the assumed minimality of $q$.

Step 5. *Define the sets of vertices $I = \{i \neq \ell \mid q_{i\ell} = 1\}$ and $J = \{i \neq \ell \mid q_{i\ell} = 0\}$, where $\ell$ is the exceptional vertex for $v$. Then we have $q_{ij} = 1$ for all $i, j$ in $I$.*

Indeed, consider the positive vector $w = v + e_k - e_\ell + e_i + e_j$. Since $\ell$ is the unique exceptional vertex for $v$ we have

$$q(v|e_k) = q(v|e_i) = q(v|e_j) = 0, \quad \text{and} \quad q(v|e_\ell) = 1,$$

thus we deduce from $w_k = v_k + 1 > s$ that

$$2 \leq q(w) = 5 - q(v|e_\ell) - q_{k\ell} + q_{ki} + q_{kj} - q_{\ell i} - q_{\ell j} + q_{ij} = 1 + q_{ij}.$$

Hence $q_{ij} = 1$ by Step 2.

Step 6. *We have* $v_k = 3 + \sum_{i \in I} v_i = -1 + \sum_{i \in J} v_i$.

Indeed, from $1 = q(v|e_\ell) = 2v_\ell - v_k + \sum_{i \in I} v_i$ we get $v_k = s = 3 + \sum_{i \in I} v_i$, while from

$$\begin{aligned} 0 = q(v|e_k) &= 2s - \sum_{i \neq k} v_i \\ &= s + \left( v_k - \sum_{i \in I} v_i \right) - \sum_{i \in J} v_i - v_\ell = s + 3 - \sum_{i \in J} v_j - 2, \end{aligned}$$

we obtain $s = -1 + \sum_{i \in J} v_i$.

Step 7. *For all* $i \in I$ *and* $j \in J$ *we have* $v_i = 1$ *and* $q_{ij} = 0$.

Indeed, we calculate

$$\begin{aligned} 0 &= q(v|e_i) \\ &= 2v_i + \sum_{m \neq i} q_{im} v_m \\ &= v_i + \sum_{m \in I} v_m + \sum_{j \in J} q_{ij} v_j + v_\ell - v_k \\ &= v_i + (v_k - 3) + \sum_{j \in J} q_{ij} v_j + 2 - v_k \\ &= -1 + v_i + \sum_{j \in J} q_{ij} v_j. \end{aligned}$$

Since $q_{ij} \geq 0$ for all $j \in J$ we must have $v_i = 1$ and $q_{ij} = 0$.

Step 8. *Let* $z \in J$ *be such that* $v_z \geq v_j$ *for all* $j \in J$. *Then there exist two vertices* $j_1 \neq j_2$ *in* $J$ *with* $q_{zj_1} = q_{zj_2} = 0$.

By Step 7 we have

$$0 = q(v|e_z) = 2v_z + \sum_{j \in J,\, j \neq z} q_{zj} v_j - v_k = v_z + \sum_{j \in J} q_{zj} v_j - \sum_{j \in J} v_j + 1.$$

Thus we infer that

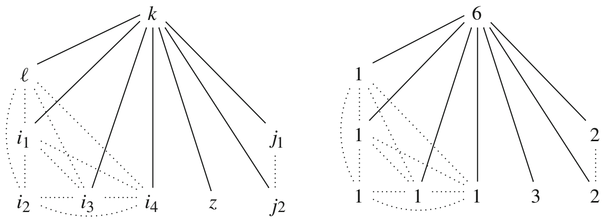
$$v_z < \sum_{j \in J}(1 - q_{zj})v_j \leq \sum_{j \in J}(1 - q_{zj})v_z,$$

hence $2 \leq \sum_{j \in J}(1 - q_{zj})$, which implies the claim.

Step 9. *For vertices $j_1$ and $j_2$ as in Step 8 we have $q_{j_1 j_2} = 1$.*

Otherwise the restriction of $q$ to vertices $\{k, \ell, z, j_1, j_2\}$ equals $q_{\mathbb{D}_4}$ in contradiction to the weak positivity of $q$.

We have now collected enough information to conclude the proof. Assume $s \geq 7$. Then by Steps 6 and 7 the set $I$ has at least four vertices $i_1, i_2, i_3, i_4$. Hence the restriction of $q$ to the set $\{k, \ell, i_1, i_2, i_3, i_4, z, j_1, j_2\}$ has exactly the following associated bigraph (left)




But $q$ evaluates to zero at the positive vector indicated by the number on the vertices in the figure above (right), a contradiction (the bigraph above corresponds to a critical unit form, see figure $\mathscr{C}(6)$ in Table 5.3). □

We now present a suitable generalization of Proposition 1.32 to the weakly positive case due to Zeldych [55] and based on unpublished notes by S. Brenner, where the assumption of $q$ being unitary is dropped. Recall that the *adjugate* $\mathbf{ad}(B)$ of a square matrix $B$ is the transpose of the matrix of cofactors of $B$.

**Theorem 5.26 (Zeldych).** *Let $A$ be the associated symmetric matrix of an integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ (that is, $q(x) = x^t Ax$ for any vector $x$ in $\mathbb{Z}^n$). Then the following conditions are equivalent:*

*a) The form $q$ is weakly positive.*
*b) For each principal submatrix $B$ of $A$ we have either* $\det(B) > 0$*, or* $\mathbf{ad}(B)$ *is not positive (that is,* $\mathbf{ad}(B)$ *has a nonpositive entry).*

*Proof.* Assume $(a)$ holds, let $B$ be a principal submatrix of $A$ and suppose that $\mathbf{ad}(B)$ is a positive matrix. By the Perron–Frobenius Theorem 1.36 there exists a positive eigenvector $v \in \mathbb{R}^n$ of $\mathbf{ad}(B)$ with eigenvalue $\rho > 0$. Considering $q$ as a real function $q_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}$ it is clear that $q_{\mathbb{R}}(x) \geq 0$ for any positive vector $x$ in $\mathbb{R}^n$. That actually $q_{\mathbb{R}}(x) > 0$ can be argued as in the proof of Proposition 2.3. Then the inequality $\det(B) > 0$ is deduced from

$$0 < q_{\mathbb{R}}(v) = v^t B v = \frac{1}{\rho} v^t B(\mathbf{ad}(B)v) = \frac{1}{\rho} \det(B)\|v\|^2,$$

since we have $(B)\mathbf{ad}(B) = \det(B)\mathbf{Id}$.

For the converse we assume that $q$ satisfies $(b)$ but is not weakly positive. Take such a form minimal in the number of variables. Since taking principal submatrices corresponds to restrictions, we infer from minimality that $q$ is critical. Hence each proper restriction of $q$ is positive (see Corollary 5.3), and thus by Proposition 1.32 we have $\det(B) > 0$ for each proper principal submatrix $B$ of $A$.

Thus $\det(A) \leq 0$ since otherwise $q$ would be positive (again by Proposition 1.32). Take $\mathbf{ad}(A) = (v_{ij})_{i,j=1}^n$, thus by hypothesis there must exist $i, j$ with $v_{ij} \leq 0$. Let $v$ be the $j$-th column of $\mathbf{ad}(A)$, so that $Av = \det(A)e_j$ and $q(v) = \det(A)v_{jj}$. Further, let $w > 0$ be a sincere positive vector with $q(w) \leq 0$. For $\lambda = -\frac{v_{ij}}{w_i} \geq 0$ we have $(v + \lambda w)_i = 0$ and (since the restriction $q^{(i)}$ is a positive form)

$$\begin{aligned} 0 &< q(v + \lambda w) \\ &= q(v) + 2\lambda w^t A v + \lambda^2 q(w) \\ &\leq \det(A)[v_{jj} + 2\lambda w_j] \\ &= \frac{\det(A)}{w_i}[v_{jj} w_i - 2v_{ij} w_j]. \end{aligned}$$

If $v_{jj} < 0$ (thus we may take $i = j$) then

$$0 < q(v + \lambda w) \leq \det(A)(-v_{jj}) \leq 0,$$

a contradiction. If $v_{jj} \geq 0$ then $v_{jj} w_i - 2v_{ij} w_j \geq 0$ and the following equation yields another contradiction

$$0 < q(v + \lambda w) \leq \frac{\det(A)}{w_i}[v_{jj} w_i - 2v_{ij} w_j] \leq 0,$$

which completes the proof. □

For convenience in what follows we collect the different **Criteria for Weak Positivity** shown in this chapter.

**Theorem 5.27.** *For a quadratic unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ the following claims are equivalent:*

*a) The form $q$ is weakly positive.*
*b) The form $q$ admits only finitely many positive roots.*
*c) For any positive root $v$ and any vertex $i$ we have $v_i \leq 6$.*

d) *For any positive nonsimple root $v$ and any vertex $i$ we have $q(v|e_i) \geq -1$.*
e) *For each principal submatrix $B$ of $A_q$ we have* $\det(B) > 0$ *or* $\mathbf{ad}(B)$ *is positive.*
f) *For all vertices $i \neq j$ we have $q_{ij} \geq -2$ and $q^{-1}(0) \cap \mathbb{N}_0^n = \{0\}$.*
g) *For all vertices $i \neq j$ we have $q_{ij} \geq -2$ and for all subset of vertices $I$ we have* $\mathbf{rad}(q^I) \cap \mathbb{N}_0^n = \{0\}$.

*Proof.* The equivalence of $(a)$ and $(b)$ was shown in Theorem 5.13, that of $(a)$ and $(e)$ in Proposition 5.26, that $(a)$ implies $(c)$ is Ovsienko's Theorem 5.25 and that $(c)$ implies $(b)$ is obvious. That $(a)$ implies $(d)$ is shown in Lemma 5.5$(c)$ and that $(d)$ implies $(a)$ is Proposition 5.15. This already show the equivalence of $(a - e)$.

Now, $(f)$ and $(g)$ are reformulations of the fact that no critical form can be contained in a weakly positive unit form: Suppose $q$ is not weakly positive. Then there exists a restriction $p = q^I$ which is critical, that is, $p$ is either an $m$-Kronecker form $p(x_i, x_j) = x_i^2 - mx_ix_j + x_j^2$ for some $-m = p_{ij} < -2$, or $p$ is nonnegative with a positive sincere radical vector. Therefore $(f)$ and $(g)$ imply $(a)$. Conversely, if $(f)$ or $(g)$ do not hold then $q$ admits a critical restriction, which completes the proof. □

**Exercises 5.28.**

1. Consider a sequence of quadratic forms $q^n, q^{n+1}, \ldots, q^N$, where $q^n = q$ and $q^{k+1}$ is obtained from $q^k$ by edge reduction with respect to vertices $i_k$ and $j_k$, as in Algorithm 5.23. Also take vectors $z^{(k)} = e_k$ for $k = 1, \ldots, n$ and $z^{(k+1)} = z^{(i_k)} + z^{(j_k)}$ for $k \geq n$. For $k > n$ define recursively transformations $\rho^{k-n} : \mathbb{Z}^k \to \mathbb{Z}^n$ as

$$\rho^1 = \rho_{i_n, j_n} \qquad \text{and} \qquad \rho^{k+1-n} = \rho^{k-n} \circ \rho_{i_{n+k}, j_{n+k}},$$

   where $\rho_{ij}$ is the transformation associated to the edge reduction with respect to vertices $i$ and $j$.

   a) Show that $z^{(k)} = \rho^{N-n}(e_k)$ for $k = 1, \ldots, N$.
   b) Conclude that if $q_{ij}^N \geq 0$ for all $1 \leq i, j \leq N$, then

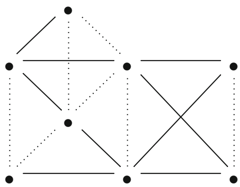
$$R^+(q) = \{z^{(1)}, \ldots, z^{(N)}\}.$$

2. Give an example of a weakly positive unit form with corank two.
3. Find an iterated edge reduction $\sigma$ for the following forms $q$ such that the bigraph associated to $q\sigma$ has no solid edge.

   i) $q(x) = x_1^2 + \ldots + x_4^2 - x_1(x_2 + x_3 + x_4)$.
   ii) $q(x) = x_1^2 + \ldots + x_5^2 - x_1(x_2 + x_3 + x_4 + x_5) + x_2x_3$.

4. Give a weakly positive unit form $q$ such that $M(q) = 5$ (see proof of Theorem 5.25).
5. Provide an example of a weakly positive unit form that fails to be nonnegative.

6. Consider the quadratic form $q$ associated to the following bigraph, and show that $q$ is weakly positive.

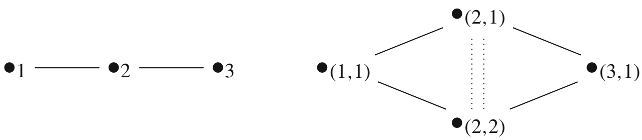


## 5.5 Explosions and Centered Forms

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form. We say that a unitary form $\overline{q}$ is a *(radical) explosion* of $q$ if $q$ is a particular type of restriction of $\overline{q}$, namely: There is a vector $s = (s_1, \dots, s_n)$ in $\mathbb{N}^n$ such that the set

$$I_s = \{(i,k) \mid 1 \leq i \leq n \text{ and } 1 \leq k \leq s_i\},$$

is an index set for $\overline{q}$ satisfying $e_{i,k} - e_{i,1} \in \mathbf{rad}(\overline{q})$ for $1 < k \leq s_i$ (where $\{e_{i,k}\}_{(i,k)\in I_s}$ denotes the canonical basis of $\mathbb{Z}^{I_s}$) and $q$ is the restriction of $\overline{q}$ to the indices $(1,1), \dots, (n,1)$. If $s_i > 1$ for some index $i$ we will say that the vertex $i$ is *exploded* $s_i - 1$ *times*. If $s_\omega = 1$ we say that $\overline{q}$ is an *explosion of* $q$ *with respect to* $\omega$, for $\omega \in \{1, \dots, n\}$. If $s_j = 1$ for $j \neq i$ and $s_i = 2$, then we say that $\overline{q}$ is obtained from $q$ by *doubling* vertex $i$ (cf. Exercise 3.32.4). Below we show a small example, doubling vertex 2 in the Dynkin graph $\mathbb{A}_3$.




The following result collects some elementary properties of explosions of weakly positive unit forms. For instance, it shows that the new quadratic form in the example above has no sincere root.

**Proposition 5.29.** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly positive unit form and $\overline{q}$ an explosion of $q$ with index set $I_s$ for $s = (s_1, \dots, s_n)$. The following hold:*

*a) The form $\overline{q}$ is weakly positive.*
*b) If $q$ has a maximal sincere root, then $\overline{q}$ has a maximal sincere root $\overline{z}$ if and only if $s \leq v$ for a sincere maximal positive root $v$ of $q$. Moreover:*

  *i) If $s = v$ then $\overline{z} = \sum_{(i,k)\in I_s} e_{i,k}$.*

*ii) If $s_\omega = 1$ for some $\omega$ in $\{1, \dots, n\}$ and $s_i = v_i$ for $i \neq \omega$, then*

$$\overline{z} = v_\omega e_{\omega,1} + \sum_{\substack{(i,k)\in I_s \\ (i,k)\neq(\omega,1)}} e_{i,k}.$$

*In both cases $\overline{z}$ is uniquely determined.*

*In situation $(i)$ we say that $\overline{q}$ is a* full explosion of *$q$ (with respect to the maximal root $v$). In situation $(ii)$ we say that $\overline{q}$ is a* full explosion of q with respect to vertex *$\omega$ (and the maximal root $v$).*

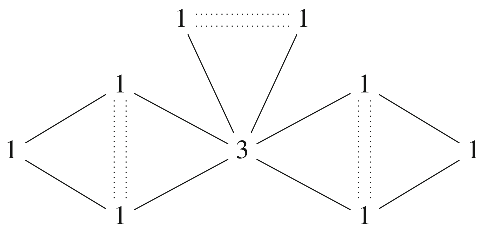
*Proof.* Consider $r_{i,k} = e_{i,k} - e_{i,1} \in \mathbf{rad}(\overline{q})$ for $k = 1, \dots, s_i$ (notice that $r_{i,1} = 0$ for $i = 1, \dots, n$), and the function $\Phi : \mathbb{Z}^{I_s} \to \mathbb{Z}^n$ given by

$$\overline{z} \mapsto \Phi(\overline{z}) = \overline{z} - \sum_{(i,k)\in I_s} \overline{z}_{i,k} r_{r,k}.$$

Considering $\mathbb{Z}^n$ as a subgroup of $\mathbb{Z}^{I_s}$ by means of the inclusion $e_i \mapsto e_{i,1}$, we observe that $\Phi$ is a projection of $\mathbb{Z}^{I_s}$ onto $\mathbb{Z}^n$ satisfying $\overline{q}(\overline{z}) = q(\Phi(\overline{z}))$, and that $\overline{z} > 0$ implies $\Phi(\overline{z}) > 0$. Therefore $\overline{q}$ is weakly positive if so is $q$.

Assume now that $z \in \mathbb{Z}^n$ is a sincere positive vector. Clearly there is a sincere positive vector $\overline{z} \in \mathbb{Z}^{I_s}$ such that $\Phi(\overline{z}) = z$ if and only if $s \leq v$. In this case $\overline{z}$ is a maximal sincere root of $\overline{q}$ if and only if $z$ is a maximal sincere root of $q$, which shows $(b)$. The description of $\overline{z}$ can be easily verified. □

For instance, the full explosion of $q_{\mathbb{E}_6}$ with respect to the star center is given by




where the numbers at the vertices indicate the maximal positive root.

A unit form $q$ is said to be *centered at vertex $c$* if $q_{ci} = -1$ for all $i \neq c$ and $q_{ij} \geq 0$ for all $i, j \neq c$. The importance of centered forms (already used in the proof of Theorem 5.25) relies on the following result. Recall that

$$M(q) := \max\{v_i \mid i \text{ is an index of } q \text{ and } v \text{ is a positive root of } q\}.$$

**Proposition 5.30.** *For each $S \in \{2, \dots, 6\}$ let $q_S$ be a weakly positive unit form with $M(q_S) = S$ such that*

$$|R^+(q_S)| = \min\{|R^+(q)| \text{ such that } q \text{ is weakly positive with } M(q) = s\}.$$

*Then $q_S$ is a centered form, with a maximal sincere positive root having a unique exceptional vertex.*

*Proof.* Arguing as in Steps 1 and 2 of Ovsienko's Theorem 5.25, we see that there exists a vertex $c$ such that $0 \le q_{ij} \le 1$ for all $i, j \ne c$. Let $v$ be a root with $v_c = M(q)$.

Since for each $i \notin \mathbf{supp}(v)$ the restriction $q^{(i)}$ has fewer roots than $q$, but still $M(q^{(i)}) = M(q)$, we deduce from the minimality in the number of positive roots of $q$ that $v$ is sincere. As a consequence of Lemma 5.17,we obtain $q_{ci} = -1$ for all $i \ne c$.

If $v$ has two exceptional vertices $i \ne j$ then $v_i = 1$ and $q(v|e_i) = 1$. Hence $\sigma_i(v) = v - e^i$ is a sincere root of $q^{(i)}$ and again $q^{(i)}$ has fewer roots than $q$, but still $M(q^{(i)}) = M(q)$, contradicting minimality. Thus the result. □

It is important to observe that the maximal value $M(q)$ may not be attained at a sincere root of $q$. To see this, define

$$M_{sin}(q) := \max\{v_i \mid i \text{ is an index of } q \text{ and } v \text{ is a positive sincere root of } q\},$$

and observe that $M_{sin}(q) \le M(q)$. Let us consider some examples where the inequality is strict. For each bigraph $B$ in Table 5.2 observe that there is a unique sincere root $v$ of $q_B$, the one displayed by the integers at the vertices. However, there exists another positive root $w$ satisfying

$$\max\{w_i \mid i \in \mathbf{supp}(w)\} > \max\{v_i \mid i \in \mathbf{supp}(v)\}.$$

Indeed, the bigraph on top fully contains the Dynkin graph $\mathbb{D}_4$, those in the middle fully contain Dynkin graphs $\mathbb{E}_6$ and $\mathbb{E}_7$, and both in the bottom fully contain $\mathbb{E}_8$.

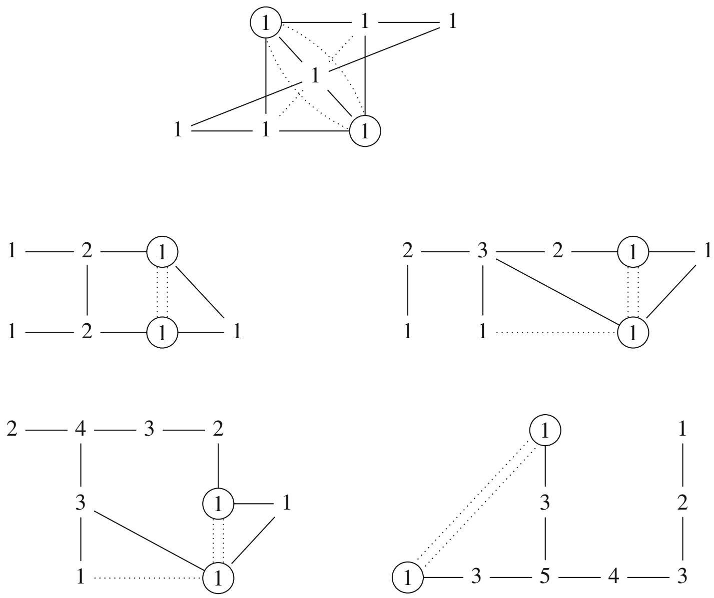
The unit forms $q$ in Table 5.2 are examples of the situation $M_{sin}(q) < M(q)$ for $M_{sin}(q) = 1, \dots, 5$. By Ovsienko's Theorem we cannot expect to find a similar example for $M_{sin}(q) = 6$.

We will now determine those centered forms which are critical (nonweakly positive). Since critical Kronecker forms are not centered, by Theorem 5.2 any critical centered form $q$ is nonnegative of corank one with a sincere positive radical vector. We can say even more:

**Proposition 5.31.** *If $q$ is a critical centered unit form then $q = p[w]$ where $p$ is a positive centered unit form and $w$ is a sincere positive root of $p$.*

*Proof.* Denote by $c$ the center of $q$, and let $v$ be a sincere positive radical vector of $q$ with mutually coprime entries. Then there exists an index $i$ with $v_i = 1$ (an omissible vertex, see Proposition 3.20).

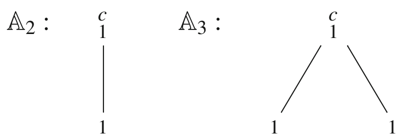
**Table 5.2** Some examples of weakly positive unit forms $q$ with $M_{sin}(q) \in \{1, \dots, 5\}$ and satisfying $M_{sin}(q) < M(q)$



Encircled numbers correspond to exceptional vertices of the displayed maximal root

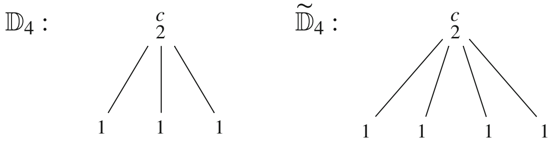
For $j \neq c$ we have $0 = q(v|e_j) = 2v_j + \sum_{\ell \neq j,c} q_{j\ell} v_\ell - v_c$, that is, $v_c = 2v_j + \sum_{\ell \neq j,c} q_{j\ell} v_\ell > 1$, therefore $i \neq c$. Hence $q^{(i)}$ is a positive connected centered unit form with $\mathbf{Dyn}(q^{(i)}) = \mathbf{Dyn}(q)$ (again by Proposition 3.20) and $v' = v - e^i$ may be seen as a positive sincere $q^{(i)}$-root. From Lemma 3.26 we have $q = q^{(i)}[v']$, thus the result. □

Since any root of a positive connected unit form of Dynkin type $\mathbb{A}_n$ has as support a line (see Proposition 2.39), there are only two positive centered unit forms $p$ of Dynkin type $\mathbb{A}_n$ which admit a sincere positive root $v$, namely $q_{\mathbb{A}_2}$ and $q_{\mathbb{A}_3}$,




In any case, however, the form $p[v]$ is not centered.

In order to ensure that $p[v]$ is centered again we need the condition $p(v|e_c) = 1$ and $p(v|e_i) \leq 0$ for all $i \neq c$. From Lemma 5.9, the only possibility for a centered positive form of type $\mathbb{D}_m$ is $\mathbb{D}_4$, with centered critical extension $\widetilde{\mathbb{D}}_4$,

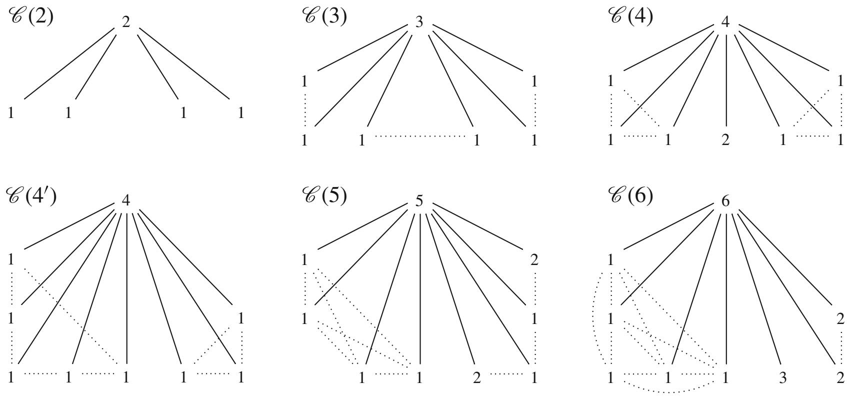



In a similar way, we calculate all cases for $\mathbb{E}_p$ and obtain the list in Table 5.3.

Since the approach in this book is based on algorithms, we do not present a 'paper proof' of the fact that Table 5.4 contains all weakly positive centered forms $q$ admitting a sincere positive root and satisfying $q_{ij} \leq 1$ for all vertices $i \neq j$ (*graphical forms*). By induction any sincere weakly positive centered form admits a restriction to a sincere weakly positive centered form in one less variable. Hence a paper proof could show that no form $q$ in the list admits an extension to a centered form $\overline{q}$ by a vertex $k$ with $\overline{q}(w|e_k) = -1$ for any sincere $q$-root $w$ not containing any of the critical centered forms above.

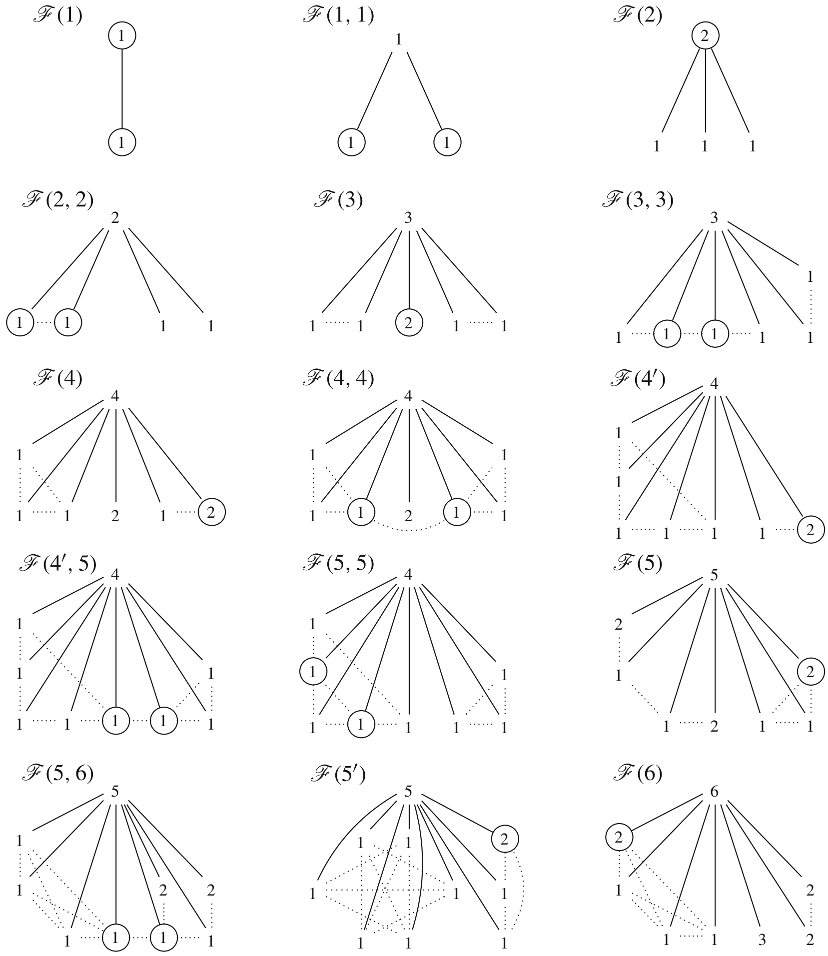
Our list is not entirely complete, since we removed from it all forms which can be obtained by explosions of noncentered vertices. For a weakly positive unit form $q$ with associated bigraph belonging to Table 5.3, and vector $v$ with entries as indicated in the vertices, the maximal number of times a noncentered point may be exploded is $v_i - 1$. This is due to the fact that this number is the corresponding entry of the (unique) maximal sincere positive root of (any) restricted centered form $q^{(k)}$ with $v_k = 1$, cf. Proposition 5.29($b$).

**Table 5.3** Critical centered forms




The minimal positive radical vector is indicated by the values at the vertices

**Table 5.4** Sincere weakly positive centered forms without multiple edges (graphical forms)




## 5.6 Roots with an Entry 6

By direct inspection of the list of sincere centered weakly positive unit forms (Table 5.4), we observe that some of these forms are indefinite. However, there need not exist an indefinite weakly positive form $q$ with $M(q) = s$ for all possible values $s = 1, \dots, 6$. In fact, in the following we will prove that if $q$ is a weakly positive unit form having a sincere positive root $v$ with $v_\omega = 6$ for some vertex $\omega$, then $q$ is a nonnegative unit form (Theorem 5.38 due to Ostermann and Pott [42]).

A brief description of the proof is in order. The starting point is Ringel's Lemma 5.32 below, where centered weakly positive unit forms having a positive root $v$ with an entry $v_i = 6$ for some vertex $i$ (plus certain additional properties) are described. One of these properties, indicating that all other entries $v_j$ for $j \neq i$ are equal either to 1 or 0, is the main technical condition of so-called *regular pairs*. This definition is meant to keep track of forms having positive roots with this particular shape. In Lemmas 5.33, 5.34 and Proposition 5.35 it is shown how iterated deflations can be used to reduce our problem to *centered forms*. With the help of Lemma 5.36 we prove the main technical result in [42] (Theorem 5.37 below), ensuring the existence of radical vectors that somehow control vertices outside the support of the maximal root in a centered regular pair. This result is used to sketch the proof of Ostermann and Pott's Theorem 5.38.

Let $q$ be a weakly positive unit form and $v$ a maximal sincere positive $q$-root with $v_\omega = 6$ for some vertex $\omega$. Denote by $\tilde{q}$ the unit form obtained from $q$ by exploding each vertex $i \neq \omega$ exactly $v_i$ times (that is, a full explosion with respect to vertex $\omega$ as in Proposition 5.29) and let $\widetilde{v}$ be the maximal root of $\widetilde{q}$ given in Proposition 5.29$(b)(ii)$. Notice that $\tilde{v}_\omega = 6$ and $\tilde{v}_x = 1$ for any other vertex $x$. Since $q$ is nonnegative if and only if so is $\tilde{q}$, we can restrict our attention to the case where $v_i = 1$ for any $i \neq \omega$. Explosion was our first reduction step. Our second step will be reduction to centered forms by means of deflations for full edges $i - j$ with $i \neq \omega \neq j$. After each such deflation $T_{ij}^-$, the corresponding vector $(T_{ij}^-)^{-1}v = T_{ij}^+ v$ will have smaller support than $v$, so we have to keep track of the points running out of the support of $v$. This motivated the definition of regular pairs as given in [42]. For simplicity, for the rest of this chapter we consider pairs $(q, v)$ where $q$ is a unit form and $v$ is a root of $q$, and referred to them simply as *(unit) pairs*. The following terminology will be useful for the technical results below.

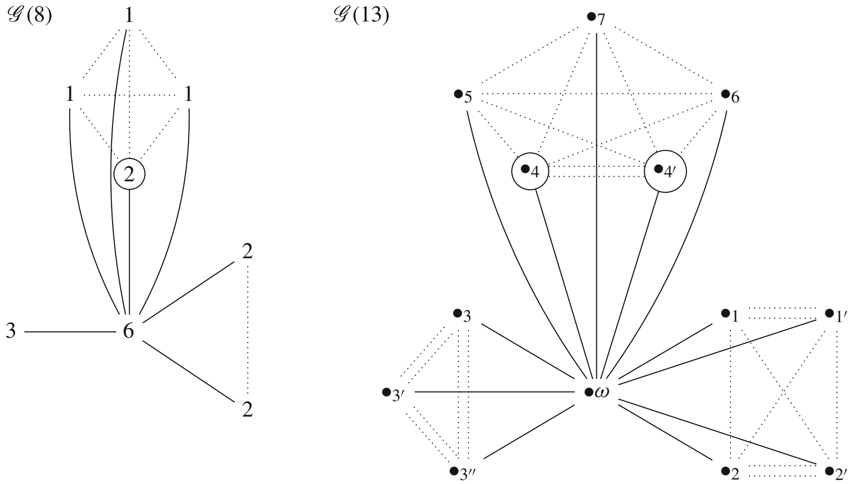
a) A pair $(q, v)$ is *weakly positive* if $q$ is a weakly positive form and $v$ is a positive root.
b) A pair $(q, v)$ is *sincere* if $v$ is a sincere root.
c) A weakly positive pair $(q, v)$ is *centered* if $q$ is a centered form.
d) A weakly positive pair $(q, v)$ is *regular* if

    i) $v$ is a maximal $q$-root.
    ii) $v_\omega = 6$ and $0 \leq v_i \leq 1$ for all $i \neq \omega$.
    iii) $q_{ij} \leq 2$ for all $i \neq j$.
    iv) $q_{\omega i} = -1$ and $q(v|e_i) = 0$ for all $i \notin \mathbf{supp}(v)$.

Notice that a pair $(q, v)$ is regular and sincere if and only if $v$ is a maximal root of $q$ with $v_\omega = 6$ and $v_i = 1$ for $i \neq \omega$ (cf. Lemma 5.5$(c)$). In view of Lemma 5.7 and Proposition 5.8, for a positive $q$-root $v$ condition $(i)$ is equivalent to having $0 \leq q(v|e_i) \leq 1$ for all $i \in \mathbf{supp}(v)$. By an *exceptional vertex* of a regular pair $(q, v)$ we mean an exceptional vertex of the maximal $q$-root $v$, that is, a vertex $i \in \mathbf{supp}(v)$ such that $q(v|e_i) = 1$ (cf. Lemma 5.9).

Proposition 5.29$(b)$ may be reinterpreted as follows: To any sincere positive maximal $q$-root $z$ with $z_\omega = 6$, where $q$ is a weakly positive unit form $q$, we can assign a regular sincere pair $(\overline{q}, \overline{z})$ where $\overline{q}$ is a full explosion of $q$ with respect to vertex $\omega$.

**Table 5.5** Weakly positive centered forms $g_{(8)} = q_{\mathscr{G}(8)}$ and $g_{(13)} = q_{\mathscr{G}(13)}$ having a maximal sincere positive root $z_{(8)}$ and $z_{(13)}$ with an entry 6




On the left we have $|(g_{(8)})_{ij}| \leq 1$ for all $1 \leq i, j \leq 8$ (numbers on the vertices indicate vector $z_{(8)}$). On the right the pair $(g_{(13)}, z_{(13)})$ is regular. Encircled points indicate exceptional vertices

The following lemma, whose proof we skip (Part $(a)$ is shown by Ovsienko in [43] whereas Part $(b)$ is Lemma 4.2 in [42]), is a fundamental part of (and perhaps the inspiration behind) Ostermann and Pott's results concerning weakly positive unit forms having a positive root with entry 6.

**Lemma 5.32.** *Let $(q, v)$ be a sincere maximal centered pair with $v_\omega = 6$ for $\omega$ the center vertex of $q$.*

*a) If $|q_{ij}| \leq 1$ for all $i, j$ then $(q, v)$ is, up to a permutation of vertices, the pair $(g_{(8)}, z_{(8)})$ given in Table 5.5.*
*b) If $(q, v)$ is a regular pair then $(q, v)$ is, up to a permutation of vertices, the pair $(g_{(13)}, z_{(13)})$ given in Table 5.5.*

Next we prove the basic results for our second reduction step. Notice that Lemma 5.32 plays a key role in the proof of Lemma 5.34. If $(q, v)$ is a unit form and $T$ is a flation for $q$ such that $qT$ is a unit form, then we denote by $(q, v)T$ the unit pair $(qT, T^{-1}v)$.

**Lemma 5.33.** *Let $(q, v)$ be a regular pair and $i, j \in$ **supp**$(v) - \{\omega\}$ two different vertices with $q_{\omega j} = q_{ij} = -1$.*

*a) Then the restriction of $qT_{ij}^-$ and $T_{ij}^+ v$ to* **supp**$(T_{ij}^+ v)$ *is a sincere regular pair.*
*b) If moreover $q(v|e_j) = 0$, then $(q, v)T_{ij}^-$ is a regular pair.*

*Proof.* Let $q' = qT_{ij}^-$ and $v' = T_{ij}^+ v = v - v_i e_j$. If $v'$ is not maximal then there exists a root $w > v'$ and hence $T_{ij}^- w = w + w_i e_j > v' + v_i e_j = v$, in contradiction to the maximality of $v$. This shows point $(i)$ in the definition of a regular pair for both $(a)$ and $(b)$, whereas $(ii)$ is obvious, since $v'_\ell = v_\ell$ for all $\ell \neq j$ and $0 = v'_j = v_j - v_i$. Hence $(a)$ holds by the discussion after the definition of a sincere pair.

Let us assume now that $q(v|e_j) = 0$ to show $(b)$. For $(iii)$, observe that $q'_{\ell k} = q_{\ell k}$ for all $\ell, k \neq i$. Now, for $\ell \notin \mathbf{supp}(v')$ we have

$$\begin{aligned} 1 &\leq q'(v' + e_\ell - e_i) = 3 - q'(v'|e_i) - q'_{i\ell} \\ &\leq 3 - q'_{i\ell}, \end{aligned}$$

where the last inequality is due to the maximality of $v'$. Therefore $q'_{i\ell} \leq 2$, and for $\ell \in \mathbf{supp}(v')$ the same inequality holds by Lemma 5.5$(c)$.

Finally, for $(iv)$ observe that $\mathbf{supp}(v') = \mathbf{supp}(v) - \{j\}$. So, if $\ell \notin \mathbf{supp}(v')$ then $\ell \neq i$ and we have $q'_{\omega\ell} = q_{\omega\ell}$ and $q'(v'|e_\ell) = q(v|e_\ell)$. For $\ell \neq j$, we use that $(q, v)$ is regular whereas for $\ell = j$, it follows directly from the hypothesis that $q'_{\omega\ell} = -1$ and $q'(v'|e_\ell) = 0$. □

The previous result gives an inductive tool as long as we can find different vertices $i, j \in \mathbf{supp}(v) - \{\omega\}$ with $q_{\omega j} = q_{ij} = -1$ and $q(v|e_j) = 0$. Now, if $q$ is not centered, then it follows from Lemma 5.17 that there exist different vertices $i, j \in \mathbf{supp}(v) - \{\omega\}$ with $q_{\omega j} = q_{ij} = -1$. So the question is whether we can always find such vertices for which, in addition, $q(v|e_j) = 0$. This is affirmatively shown in the following lemma.

**Lemma 5.34.** *Let $(q, v)$ be a regular, noncentered pair. Then there exist $i, j \in \mathbf{supp}(v) - \{\omega\}$ with $q_{\omega j} = -1 = q_{ij}$ such that $q(v|e_j) = 0$.*

*Proof.* Assume that $v$ is a sincere $q$-root. Since $v$ is a maximal positive root, recall from Lemma 5.9 that $v$ has exactly two exceptional vertices, say $k$ and $k'$. Assume on the contrary that $(q, v)$ satisfies the following:

*[$*$] The pair $(q, v)$ is a sincere regular noncentered pair such that for any $i, j \neq \omega$ with $q_{\omega j} = -1 = q_{ij}$ we have $q(v|e_j) = 1$.*

Consider the set $\mathscr{A}_{(q,v)} = \{\ell \in \mathbf{supp}(v) - \{\omega\} \mid q_{\ell\omega} \geq -1\}$, which by hypothesis is nonempty. Since the bigraph of $q$ is connected by solid walks (cf. Lemma 5.17), there are $\ell \in \mathscr{A}_{(q,v)}$ and $k'' \in \mathbf{supp}(v) - \{\omega\}$ with $q_{k''\ell} = -1 = q_{\omega k''}$. By hypothesis $q(v|e_{k''}) = 1$, therefore $k'' \in \{k, k'\}$. Let us say that $\ell, \ell' \in \mathscr{A}_{(q,v)}$ are such that $q_{k\ell} = -1$ and $q_{k'\ell'} = -1$ (possibly $\ell = \ell'$).

Take $\widetilde{q} = (qT^-_{\ell k})|_{\mathbf{supp}(T^+_{\ell k}v)}$ and $\widetilde{v} = (T^+_{\ell k}v)|_{\mathbf{supp}(T^+_{\ell k}v)}$, and notice by Lemma 5.33$(a)$ that $(\widetilde{q}, \widetilde{v})$ is a sincere regular pair.

Step 1. *The sincere regular pair $(\widetilde{q}, \widetilde{v})$ satisfies condition [$*$] above.*

Take $p, r \in \mathbf{supp}(\widetilde{v}) - \{\omega\}$ with $\widetilde{q}_{\omega p} = -1 = \widetilde{q}_{pr}$.

If $p = \ell$ then

$$\widetilde{q}(\widetilde{v}|e_\ell) = q(v|T^-_{\ell k}e_\ell) = q(v|e_\ell) + q(v|e_k) = q(v|e_k) = 1.$$

If $p \neq \ell$, assume first that $r \neq \ell$. Then $q_{\omega p} = \widetilde{q}_{\omega p} = -1 = \widetilde{q}_{pr} = q_{pr}$, and by hypothesis [$*$] we have $p = k'$. Calculate

$$\widetilde{q}(\widetilde{v}|e_p) = q(v|T^{-}_{\ell k}e_p) = q(v|_{k'}) = 1.$$

Assume finally that $p \neq \ell$ and $r = \ell$. Since $-1 = \widetilde{q}_{p\ell} = q_{p\ell} + q_{pk}$, and $q_{\omega p} = q_{\omega k} = -1$, we must have $q_{p\ell} = -1$. Hence $p = k'$ and

$$\widetilde{q}(\widetilde{v}|e_p) = 1.$$

Step 2. *The vertex $\ell$ is exceptional for $(\widetilde{q}, \widetilde{v})$. In particular $\widetilde{q}_{\omega\ell} = 1$, thus $q_{\omega\ell} = 0$.*
We calculate

$$\widetilde{q}(\widetilde{v}|e_\ell) = (qT^{-}_{\ell k})(T^{+}_{\ell k}v|e_\ell) = q(v|T^{-}_{\ell k}e_\ell) = q(v|e_\ell) + q(v|e_k) = 1.$$

Consider now $k''$, the second exceptional vertex of $\widetilde{v}$, and take $w = \sigma_{k''}(\widetilde{v}) = \widetilde{v} - e_{k''}$. By connectedness with solid walks (Lemma 5.17), and the fact that [$*$] holds for $(\widetilde{q}, \widetilde{v})$, we notice that if $q_{\omega\ell} \geq 0$ then there is a solid walk from $\ell$ to $\omega$ that does not pass through the exceptional vertex $k''$. Hence [$*$] implies that there must be a third exceptional vertex, a contradiction. Then $\widetilde{q}_{\omega\ell} = -1$, and therefore $q_{\omega\ell} = 0$.

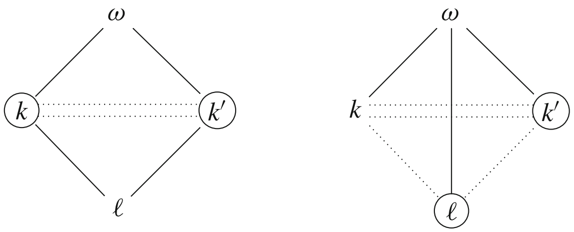
Step 3. *We have $|\mathscr{A}_{(\widetilde{q},\widetilde{v})}| = |\mathscr{A}_{(q,v)}| - 1$.*
This follows from Step 2 considering that after applying a flation $T^{\epsilon}_{ij}$ to a quadratic form $q$, all modified edges in the bigraph $B_{qT^{\epsilon}_{ij}}$ have as end-point vertex $j$.

Using Steps 1–3 as many times as necessary we may assume that $(q, v)$ is a sincere regular and centered pair satisfying [$*$] with $\mathscr{A}_{(q,v)} = \{\ell\}$. We next observe that $q_{kk'} = 2$, and deduce from $q_{\omega k} = q_{\omega k'} = -1 = q_{k\ell} = q_{k'\ell}$ and $q_{\omega\ell} = 0$ (by Step 2) that $0 < q_{kk'} \leq 2$. Assume that $q_{kk'} = 1$, and notice that $\sigma_\omega(\sigma_k(v)) = v - e_k - e_\omega$ (for $q(\sigma_k(v)|e_\omega) = q(v - e_k|e_\omega) = -q_{\omega k} = 1$). Moreover, we have

$$q(\sigma_\omega\sigma_k(v)|e_{k'}) = q(v - e_k - e_\omega|e_{k'}) = q(v|e_{k'}) - q_{kk'} - q_{\omega k'} = 1,$$

and therefore $w := \sigma_{k'}\sigma_\omega\sigma_k(v) = v - e_k - e_\omega - e_{k'}$. Since $k, k' \notin \mathbf{supp}(w)$, there must exist a vertex $k'' \in \mathbf{supp}(w) - \{\omega\}$ connecting $\ell$ with $\omega$, that is, $q_{\omega k''} = -1$. However, by [$*$] the vertex $k''$ is exceptional for $(q, v)$, a contradiction.

So far we have shown that we may assume that the restriction of $q$ to the set $\{\omega, k, k', \ell\}$ has the following associated bigraph (left):




Apply once more deflation $T^-_{\ell k}$ to the pair $(q, v)$ and restrict to the support of $T^+_{\ell k} v$ to obtain a sincere regular pair $(\widetilde{q}, \widetilde{v})$ as before (bigraph on the right above), which is centered by Step 3. The same step shows that $k'$ and $\ell$ are the exceptional vertices of $(\widetilde{q}, \widetilde{v})$. But notice that in this case we have $\widetilde{q}_{k\ell} = 1$ (since $q_{k'k} = 2$ and $q_{k'\ell} = -1$).

On the other hand, by Lemma 5.32($b$) the pair $(\widetilde{q}, \widetilde{v})$ coincides with the pair $(g_{(13)}, z_{(13)})$, where $(g_{(13)})_{k'\ell} = 2$ (the exceptional vertices of the maximal $g_{(13)}$-root $z_{(13)}$ are joined by a double dotted edge, see Table 5.5). This is a contradiction, which completes the proof. □

**Proposition 5.35.** *Let $q$ be a weakly positive unit form and $v$ a maximal positive $q$-root such that $v_\omega = 6$ and $v_i = 1$ for $i \neq \omega$. Then there is an iterated deflation $T$ for $q$ such that $(q, v)T := (qT, T^{-1}v)$ is a regular centered pair.*

*Proof.* Since $v$ is a sincere vector, by assumption $(q, v)$ is a regular pair. If $(q, v)$ is a noncentered pair, use Lemmas 5.33($b$) and 5.34 to find a deflation $T^-_{ij}$ such that $(qT^-_{ij}, T^+_{ij} v)$ is a regular pair. This process has to stop, since

$$|v| = \sum_i v_i > \sum_i v_i - 1 = |T^+_{ij} v|.$$

Hence the result. □

We need a final preliminary result.

**Lemma 5.36.** *Let $(q, v)$ be a regular centered pair, and take $j \in$ **supp**$(v)$ and $k \notin$ **supp**$(v)$ such that $q_{jk} = 2$. Then*

*a) Vertex $j$ is nonexceptional for $v$.*
*b) Vector $e_j - e_k$ is radical for the form $q|_{\mathbf{supp}(v) \cup \{k\}}$.*
*c) For $\ell \notin$ **supp**$(v) \cup \{k\}$ we have $q_{j\ell} \leq q_{k\ell}$.*

*Proof.* For ($a$) we have

$$0 < q(v - e_j + e_k) = 3 - q(v|e_j) + q(v|e_k) - q_{jk} = 1 - q(v|e_j),$$

therefore $q(v|e_j) = 0$.

Notice now that $y = v - e_j + e_k$ is a positive $q$-root. Let $a$ and $a'$ be the exceptional vertices of $v$, and observe that they are also the exceptional vertices for $y$ (indeed, by Lemma 5.32($b$) the restriction of $q$ to the support of $y$ is $g_{(13)}$, and in this form the exceptional vertices are characterized as the unique pair of vertices with $q_{aa'} = 2$ in the component of $\mathscr{G}(13)^{(\omega)}$ with five vertices, cf. Table 5.5).

Thus if $\ell \in \mathbf{supp}(y) - \{a, a'\}$ then

$$0 = q(y|e_\ell) = q(z|e_\ell) - q_{j\ell} + q_{k\ell} = q_{k\ell} - q_{j\ell},$$

and if $\ell \in \{a, a'\}$ then

$$1 = q(y|e_\ell) = q(z|e_\ell) - q_{j\ell} + q_{k\ell} = 1 + q_{k\ell} - q_{j\ell}.$$

In any case, if $\ell \in \mathbf{supp}(y) = (\mathbf{supp}(v) \cup \{k\}) - \{j\}$, we have $q_{j\ell} = q_{k\ell}$, and the same equality holds for $\ell = j$ by hypothesis. This shows ($b$).

Take now $\ell \notin \mathbf{supp}(v) \cup \{k\}$ and observe by Ovsienko's Theorem 5.25 that $y + e_\ell$ is not a root of $q$, since otherwise

$$(\sigma_\omega(y + e_\ell))_\omega = y_\omega - q(y + e_\ell|e_\omega) = 6 - q_{\ell\omega} = 7.$$

Hence

$$2 \leq q(y + e_\ell) = 2 + q(y|e_\ell) = 2 + q(v) - q_{j\ell} + q_{k\ell},$$

which shows ($c$). □

**Theorem 5.37.** *Let* $(q, v)$ *be a regular centered pair, and consider vertices* $i \in \mathbf{supp}(v)$ *and* $k \notin \mathbf{supp}(v)$ *such that* $q_{ik} = 1$ *and such that* $qT^+_{ik}$ *is a weakly positive unit form. Then there exists a* $j \in \mathbf{supp}(v)$ *such that* $e_j - e_k \in \mathbf{rad}(q)$.

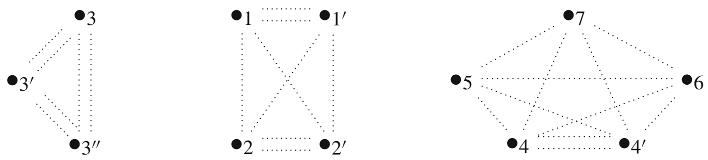
*Proof.* Take $q^+ = qT^+_{ik}$ and $v^+ = T^-_{ik}(v) = v + e_k$.

Step 1. *There is a* $j \in \mathbf{supp}(v)$ *with* $q_{jk} = 2$.

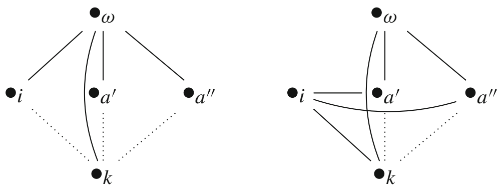
Assume on the contrary that $q_{jk} < 2$ for all $j \in \mathbf{supp}(v)$. It can be shown (Exercise 7 below) that in this case the set

$$\mathscr{N}(k) = \{j \in \mathbf{supp}(v) \mid q_{jk} = 1\}$$

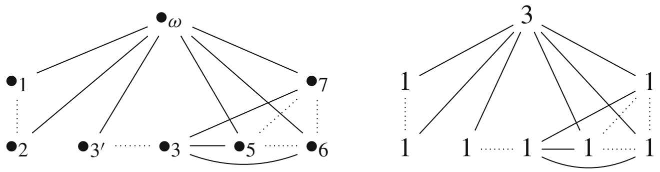
coincides, up to symmetries of vertices, with one of the sets $\{1, 1', 3, 3', 3'', 5\}$. $\{3, 3', 3'', 5, 6, 7\}$ or $\{1, 1', 2, 2', 5, 6\}$ in the following figure (the connected components of the restriction $\mathscr{G}(13)^{(\omega)}$ of bigraph $\mathscr{G}(13)$ in Table 5.5).



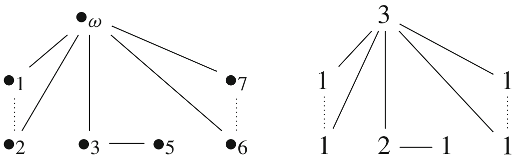

Assume $\mathscr{N}(k) = \{1, 1', 3, 3', 3'', 5\}$ and take $a', a'' \in \mathscr{N}(k)$ such that $i$, $a'$ and $a''$ belong to different components $\mathscr{K}$, $\mathscr{K}'$ and $\mathscr{K}''$ of $\mathscr{G}(13)^{(\omega)}$. Then the restriction of $q$ and $q^+$ to the set of vertices $\{\omega, i, a', a'', k\}$ have the following forms




hence the restriction of $q^+$ to the set $\{\omega, i, a', a''\}$ has the form $\widetilde{\mathbb{A}}_3$, contradicting the weak positivity of $q^+$.
Assume $\mathscr{N}(k) = \{3, 3', 3'', 5, 6, 7\}$ and $i = 3$. Then the restriction of $q^+$ to the set $\{1, 2, 3, 3', 5, 6, 7\}$ has the following shape,

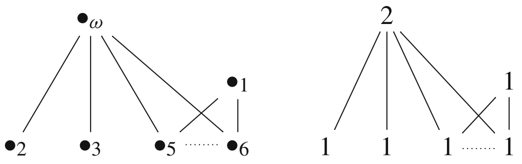



where the positive vector $z$ indicated on the right satisfies $q^+(z) = 0$, a contradiction. Up to symmetry the remaining case is $i = 5$, in which case the restriction of $q^+$ to the set of vertices $\{\omega, 1, 2, 3, 5, 6, 7\}$ has the shape




where again the positive vector on the left is radical.
Assume $\mathscr{N}(k) = \{1, 1', 2, 2', 5, 6\}$. Then vertex $i$ is (up to symmetry of vertices) one of vertices $i = 1$ or $i = 5$. In any case the restriction of $q^+$ to the set

$\{\omega, 1, 1', 2, 2', 5, 6\}$ has the shape




where again on the left we exhibit a positive radical vector. In any case we reach a contradiction, completing the proof of Step 1.

Step 2. *For every $\ell$ we have $q(e_j - e_k|e_\ell) = 0$.*

Take $z = e_j - e_k$ and let us assume that $q(z|e_\ell) \neq 0$ for some $\ell$. By Lemma 5.36(*b*) and (*c*) we have $\ell \notin \mathbf{supp}(v) \cup \{k\}$ and $q(z|e_\ell) = q_{j\ell} - q_{k\ell} < 0$. Consider the following facts:

i) $\mathbf{supp}(v^+) = \mathbf{supp}(v) \cup \{k\}$.
ii) $q^+_{jk} = 2$.
iii) $q^+(v^+|e_\ell) = q(v|e_\ell) = 0$.
iv) $q^+(v^+|e_k) = q(v|e_k) = 0$.

Take now $y = v^+ - e_k + e_j$ and observe that $y$ is a positive root of $q^+$. Indeed, since $q(v|e_j) = 0$ by Lemma 5.36(*a*), we have

$$q^+(y) = q^+(v^+ - e_k + e_j) = 3 - q^+(v^+|e_k) + q^+(v^+|e_j) - q^+_{kj}$$
$$= 1 + q^+(v^+|e_j) = 1 + q(v|e_j) = 1.$$

Moreover, $q^+(y|e_\ell) = q^+_{j\ell} - q^+_{k\ell} < 0$. Therefore $\sigma_\ell(y)$ is a positive $q^+$-root with $\ell \in \mathbf{supp}(\sigma_\ell(y))$, and also

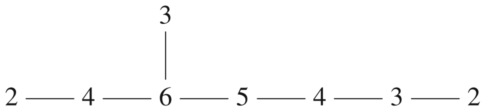
$$q^+(\sigma_\ell(y)|e_\omega) = q^+(y - q^+(y|e_\ell)e_\ell|e_\omega) = q^+(y|e_\omega) - q^+(y|e_\ell)q^+_{\ell\omega}$$
$$= q^+(v^+|e_\ell) - q^+_{k\ell} + q^+_{j\ell} = q_{j\ell} - q_{k\ell} < 0,$$

since $q^+(y|e_\omega) = 0$. Hence $\sigma_\omega(\sigma_\ell(y))$ is a positive $q^+$-root with $\sigma_\omega(\sigma_\ell(y))_\omega = v_\omega - (q_{j\ell} - q_{k\ell}) > 6$, contradicting Ovsienko's Theorem 5.25.

This completes the proof. □

Before we can prove the main result of this section we have to analyze another extreme situation. Let $q : \mathbb{Z}^8 \to \mathbb{Z}$ be a connected positive unit form of Dynkin type $\mathbb{E}_8$ having a maximal positive root $v$ with $v_\omega = 6$. By Theorem 2.20 there exists an iterated inflation $T$ such that $qT = q_{\mathbb{E}_8}$ and $T^{-1}v$ is the maximal root $v_8$ of $q_{\mathbb{E}_8}$.

Hence $|v| = \sum_{i=1}^{8} \leq \sum_{i=1}^{8}(v_8)_i = |v_8| = 29.$




Therefore, if $q^* : \mathbb{Z}^n \to \mathbb{Z}$ and $q^*_{\mathbb{E}_8} : \mathbb{Z}^m \to \mathbb{Z}$ are respectively the full explosion of $q$ and $q_{\mathbb{E}_8}$ with respect of vertex $\omega$, then $n \leq m$ and $m = 8 + 16 = 24$. The bigraph associated to $q^*_{\mathbb{E}_8}$ is shown in Fig. 5.1.

**Theorem 5.38.** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly positive unit form having a sincere positive root $v$ and a vertex $\omega \in \{1, \dots, n\}$ with $v_\omega = 6$. Then $q$ is a nonnegative unit form with Dynkin type* $\mathbf{Dyn}(q) = \mathbb{E}_8$ *and corank $n - 8$. In particular*

$$8 \leq n \leq 24 \qquad \textit{and} \qquad 113 \leq |\mathscr{R}^+(q)| \leq 418\,923\,665 = 5 \cdot 83\,784\,733$$

*where the last equality is a prime factorization.*

*Sketch of Proof.* Assume that $v$ is a maximal sincere $q$-root and take the full explosion $q' : \mathbb{Z}^m \to \mathbb{Z}$ of $q$ with respect to vertex $\omega$ (and maximal root $v$). By Proposition 5.29, the pair $(q', v')$ is sincere and regular, where $v'$ is the root given in Proposition 5.29$(b(ii))$.

We proceed by induction on $m$. If $m \in \mathbb{N}$ is minimal such that there is a sincere regular pair $(q', v')$, then $(q', v')$ is a centered pair (for otherwise by Proposition 5.35 there is a deflation $T$ such that the restriction of $(q'T, T^{-1}v')$ to the support of $T^{-1}v$ contradicts the minimality of $m$). Hence by Lemma 5.32$(b)$ we have $q' \cong g_{(13)}$, which is nonnegative of Dynkin type $\mathbb{E}_8$.

Now, for nonminimal $m$ we have, by Proposition 5.35, an iterated deflation $T$ such that $(q'T, T^{-1}v') = (q'', v'')$ is a centered regular pair. Then $T$ is nontrivial, thus there exist $i \in \mathbf{supp}(v'')$ and $k \notin \mathbf{supp}(v'')$ such that $q''_{ik} = 1$ and $qT^+_{ik}$ is weakly positive. By Theorem 5.37 there is a $j \in \mathbf{supp}(v'')$ with $e_j - e_k \in \mathbf{rad}(q'')$. Consequently $q''$ is an explosion of the restriction $(q'')^{(k)}$, which by induction is nonnegative of Dynkin type $\mathbb{E}_8$. Then by Proposition 5.29 $q''$ is nonnegative of Dynkin type $\mathbb{E}_8$, and so are $q'$ (since $q' \cong q''$) and $q$ (cf. Theorem 3.28). In particular, $\mathbf{Dyn}(q) = \mathbb{E}_8$ and $\mathbf{cork}(q) = n - 8$.

For the last claim it is clear that $8 \leq n$. The proof of $n \leq 24$ is briefly sketched: Take $n$ maximal such that a weakly positive unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ has a maximal sincere positive root $v$ with $v_\omega = 6$. By maximality of $n$ the sincere pair $(q, v)$ is regular. By the above, $q$ is a full explosion of a positive unit for $\widetilde{q}$ of Dynkin type $\mathbb{E}_8$ with respect of $\omega$ and a maximal $\widetilde{q}$-root $\widetilde{v}$. But a positive unit form with a sincere maximal positive root $\widetilde{v}$ that maximizes the weight $|\widetilde{v}| = \sum_{i=1}^{n} \widetilde{v}_i$ must be precisely $\widetilde{q} = q_{\mathbb{E}_8}$. Therefore $q = q^*_{\mathbb{E}_8}$, the full explosion of $q_{\mathbb{E}_8}$ with respect of the star center

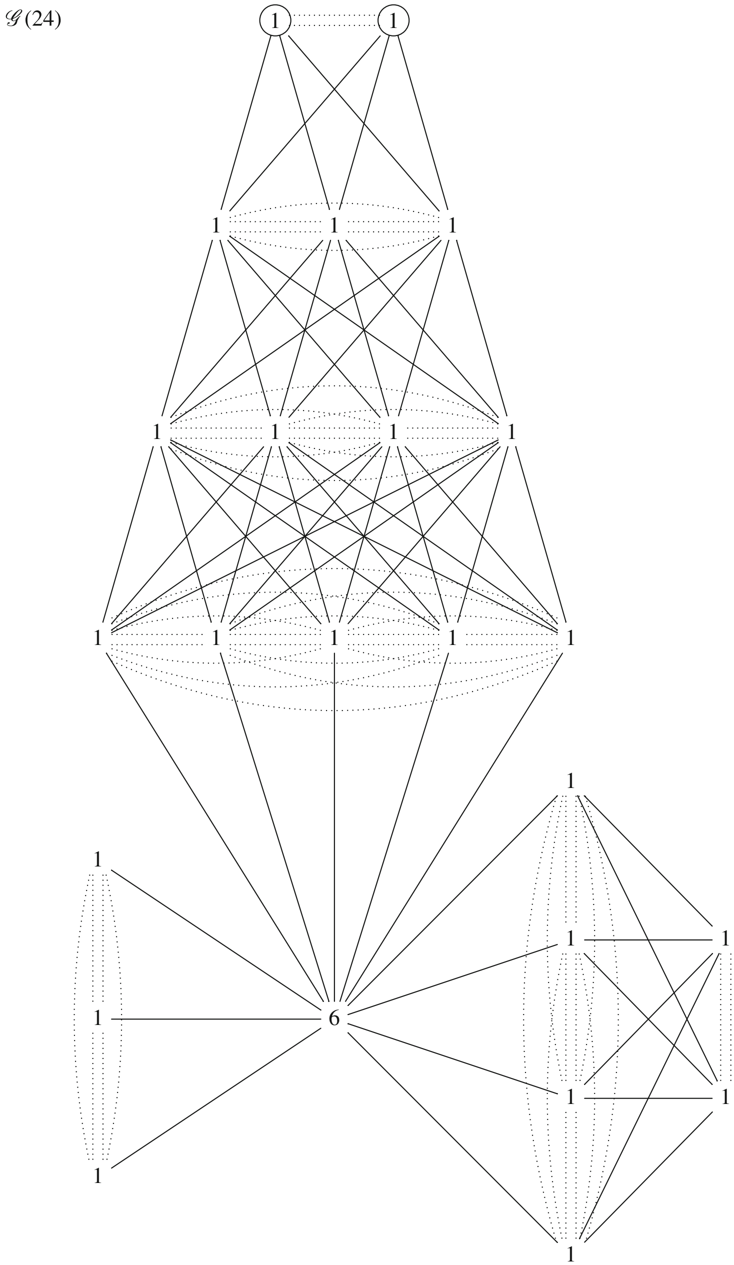



**Fig. 5.1** Full explosion $q_{\mathscr{G}(24)} = q^*_{\mathbb{E}_8}$ of $q_{\mathbb{E}_8}$ with respect to the star center. Encircled vertices correspond to exceptional vertices of the indicated (maximal) positive root

(see Fig. 5.1). This shows that $n \leq 24$. The bound for the number of positive roots of $q$ is computed by Ostermann and Pott in [42]. □

**Exercises 5.39.**

1. Show that if $q$ is a positive centered form with a positive root $w$, then $q[w]$ is a critical centered form.
2. Determine which of the bigraphs in Table 5.4 correspond to nonnegative forms.
3. How many centered regular pairs $(q, v)$ are there (up to permutation of vertices) with associated bigraph $G_q$ having exactly one double dotted edge?
4. Show that the encircled vertices in the bigraphs of Table 5.5 are in fact exceptional vertices of the corresponding quadratic forms.
5. With the notation of Table 5.5, show that $g_{(13)}$ is a full explosion of $g_{(8)}$ with respect to vertex $\omega$.
6. Prove that if $q : \mathbb{Z}^{24} \to \mathbb{Z}$ is a weakly positive unit form having a sincere root $v$ with $v_\omega = 6$ for some $1 \leq \omega \leq 24$, then $q = q^*_{\mathbb{E}_8}$ as in Fig. 5.1.
7. Let $(q, v)$ be a regular centered pair. Show that if $k \notin \mathbf{supp}(v)$ and $q_{jk} \leq 1$ for all $i \in \mathbf{supp}(v)$, then the set $\{i \in \mathbf{supp}(v) \mid q_{ik} = 1\}$ is (up to symmetry of $\mathbf{supp}(v)$) one of the following subsets of vertices of $g_{(13)}$ (cf. Table 5.5):

   i) $\{1, 1', 3, 3', 3'', 5\}$;
   ii) $\{3, 3', 3'', 5, 6, 7\}$;
   iii) $\{1, 1', 2, 2', 5, 6\}$.

   [Hint: Show that otherwise one of the critical centered forms $\mathscr{C}(2) - \mathscr{C}(6)$ in Table 5.3 is a restriction of $q$.]

## 5.7 Thin Forms

In this section we further reduce weakly positive unit forms, following Dräxler, Drozd, Golovachtchuk, Ovsienko and Zeldych [22], to get a so-called *good thin weakly positive unit form*. Since this reduction process is reversible, a classification of such forms determines, in principle, all weakly positive forms. This classification (partially achieved computationally) is presented in [22], cf. Theorem 5.46 and Tables 5.6, 5.7 and 5.8.

A unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is called *thin* if $q((1, \dots, 1)) = 1$, that is, if the sincere vector $\tau^{(n)}$ with $\tau^{(n)}_i = 1$ for $i = 1, \dots, n$ (called the *thin vector* of $\mathbb{Z}^n$) is a $q$-root. In particular, weakly positive thin forms are sincere. In the following we write $\tau$ instead of $\tau^{(n)}$ if no confusion arises.

**Proposition 5.40.** *For any weakly positive sincere unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ there is an iterated deflation $T$ such that $qT$ is a thin weakly positive unit form having thin vector $\tau^{(n)}$ as unique (thus maximal) sincere root.*

*Proof.* Let $v$ be a maximal sincere $q$-root and $\tau = \tau^{(n)}$ the thin vector. We prove the result by induction on $|v| = \sum_{i=1}^{n} v_i$. If $|v| = n$ then $v = \tau$ and we have nothing to do, so assume $v > \tau$. Since $v$ is a root, $v$ cannot be a multiple of $\tau$, and thus using Lemma 5.17 we may find vertices $1 \leq i \neq j \leq n$ with $q_{ij} < 0$ and $v_i < v_j$. By Lemma 5.5($b$) we have $q_{ij} = -1$.

Take $q' = qT_{ij}^-$ and $v' = T_{ij}^+ v = v - v_i e_j > 0$. Then $q'$ is a weakly positive unit form (for if $x > 0$ then $T_{ij}^- x > 0$) and has a maximal sincere root $v'$ with $|v'| < |v|$. By the induction hypothesis there is an iterated deflation $T'$ such that $q'T'$ is a weakly positive thin unit form having the thin vector as unique (maximal) sincere root. Take $T = T_{ij}^- T'$ to complete the proof. □

We now restrict our attention to deflations that preserve the thin property. If $q : \mathbb{Z}^n \to \mathbb{Z}$ is a thin weakly positive unit form with $\tau^{(n)}$ a nonmaximal root, then there is a vertex $j \in \{1, \dots, n\}$ such that $q(\tau^{(n)}|e_j) = -1$ (see Lemma 5.7 and Proposition 5.8). In this case, a deflation $T_{ij}^-$ for $q$ is called a *$\tau$-deflation*. Notice that if $y := \sigma_j(\tau^{(n)}) = \tau^{(n)} + e_j$, then $T_{ij}^+(y) = y - y_i e_j = \tau^{(n)}$. Therefore $qT_{ij}^-$ is again a thin form. An iterated deflation consisting of corresponding $\tau$-deflations will be referred to as an *iterated $\tau$-deflation*. For a $\tau$-deflation $T_{ij}^-$ for $q$, taking $q^- = qT_{ij}^-$, the inflation $T_{ij}^+$ for $q^-$ is called a *$\tau$-inflation*, and *iterated $\tau$-inflations* are defined similarly. The following result is evident from the discussion above.

**Lemma 5.41.** *Let $q$ be a thin weakly positive unit form. Then there is an iterated $\tau$-deflation $T$ such that the thin vector $\tau^{(n)}$ is maximal for the thin weakly positive unit form $qT$.*

In order to have at hand an effective inductive tool to construct weakly positive unit forms, we define following [22] a new type of extension on weakly positive pairs $(q, v)$. We call a weakly positive pair $(q', v')$ a *reflection-extension* of $(q, v)$ if there exists a vertex $i$ of $q$ (the *extension vertex*) such that $(q')^{(i)} = q$ and $q'(v'|e_i) = v'_i$, and if $\sigma'_i$ denotes the reflection with respect to the unit form $q'$ and $v$ is identified with its inclusion in $\mathbb{Z}^n$, then $\sigma'_i(v') = v$. If furthermore $v'$ is a maximal $q'$-root with two exceptional vertices (cf. Lemma 5.9), we say that $(q', v')$ is a *main reflection-extension* of $(q, v)$.

A sincere pair $(q, v)$ is called *bad* if there is a radical vector $\mu \in \mathbf{rad}(q)$ such that both $v + \mu$ and $v - \mu$ are positive $q$-roots. Otherwise $(q, v)$ is called a *good pair*. Recall that, for a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ and a $q$-root $z$, the one point extension $q[z]$ is defined as the root-induced form $q_{e(z)}$ where $e(z) = (e_1, \dots, e_n, -z)$ (cf. Sect. 3.5), that is

$$q[z](y_1, \dots, y_n, y_{n+1}) = q(y_1 e_1 + \dots + y_n e_n - y_{n+1} z).$$

**Proposition 5.42.** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ and $q' : \mathbb{Z}^{n+1} \to \mathbb{Z}$ be weakly positive unit forms, and assume that $q$ is a thin form.*

*a) The pair* $(q', \tau^{(n+1)})$ *is a main reflection-extension of* $(q, \tau^{(n)})$ *if and only if there is an* $i \in \{1, \dots, n\}$ *such that* $q'(x) = q[\tau^{(n)}](x) + x_i x_{n+1}$.
*b) If* $(q, \tau^{(n)})$ *is a bad pair and* $(q', \tau^{(n+1)})$ *is a reflection-extension of* $(q, \tau^{(n)})$, *then* $(q', \tau^{(n+1)})$ *is a bad pair.*
*c) If* $(q, \tau^{(n)})$ *is a good pair, then* $(q', \tau^{(n+1)})$ *is reflection-extension of* $(q, \tau^{(n)})$ *and is a bad pair if and only if there is a* $q$*-root with* $|z_i| \leq 1$ *for* $i = 1, \dots, n$ *such that* $q(z|\tau^{(n)}) = -1$ *and* $q' = q[-z]$.

*Proof.*

($a$) By definition of reflection-extension we have $q'(\tau^{(n+1)}|e_{n+1}) = 1$. By maximality of $\tau^{(n+1)}$ there is exactly one other exceptional vertex for $\tau^{(n+1)}$, say $i \in \{1, \dots, n\}$, that is, $q'(\tau^{(n+1)}|e_j) = \delta_{ij}$ for $j \in \{1, \dots, n\}$. Therefore

$$q'(e_{n+1}|e_j) = q'(\tau^{(n+1)} - \tau^{(n)}|e_j) = q(-\tau^{(n)}|e_j) + \delta_{ij},$$

that is, $q'(x) = q[\tau^{(n)}](x) + x_i x_{n+1}$. Conversely, since $(q')^{(n)} = q$ notice that

$$q'(\tau^{(n+1)}|e_{n+1}) = q'(\tau^{(n+1)}) + q'(e_{n+1}) - q'(\tau^{(n+1)} - e_{n-1}) = 2 - 1 = 1.$$

Now, for $j \in \{1, \dots, n\}$ and $j \neq i$ we have

$$q'(\tau^{(n+1)}|e_j) = q'(\tau^{(n)}|e_j) + q'(e_{n+1}|e_j) = q(\tau^{(n)}|e_j) - q(\tau^{(n)}|e_j) = 0,$$

whereas $q'(\tau^{(n+1)}|e_i) = q(\tau^{(n)}|e_i) - q(\tau^{(n)}|e_i) + 1 = 1$. Hence $\tau^{(n+1)}$ is a maximal $q'$-root and $(q', \tau^{(n+1)})$ is a reflection-extension of the pair $(q, \tau^{(n)})$.

($b$) Take $\mu \in \mathbf{rad}(q)$ with $\mu_i \in \{1, 0, -1\}$ for $i = 1, \dots, n$ and define $\mu' \in \mathbb{Z}^{n+1}$ with $\mu'_i = \mu_i$ for $i = 1, \dots, n$ and $\mu'_{n+1} = 0$. We show that $\mu' \in \mathbf{rad}(q')$.

Since $q'(\mu', e_i) = q(\mu, e_i) = 0$ for $i = 1, \dots, n$, let us assume that $q'(\mu'|e_{n+1}) > 0$ (multiplying $\mu'$ by $-1$ if necessary). Then

$$q'(\mu'|\tau^{(n+1)}) = q'(\mu'|e_{n+1}) \geq 1,$$

and therefore for the positive vector $\tau^{(n+1)} - \mu'$ in $\mathbb{Z}^{n+1}$ we have

$$q'(\tau^{(n+1)} - \mu') = q'(\tau^{(n+1)}) + q'(\mu') - q'(\mu'|\tau^{(n+1)}) = 1 - q'(\mu'|\tau^{(n+1)}) \leq 0,$$

a contradiction.

($c$) Assume first that $(q', \tau^{(n+1)})$ is a bad extension of $(q, \tau^{(n)})$, and take $\mu \in \mathbf{rad}(q')$ with $|\mu_i| \leq 1$. If $\mu_{n+1} = 0$ then $(q, \tau^{(n)})$ is itself a bad pair, therefore we may also assume that $\mu_{n+1} = 1$. Then $z := e_{n+1} - \mu$ is a $q$-root with entries $z_i \in \{1, 0, -1\}$ such that

$$q(z|\tau^{(n)}) = q'(e_{n+1} - \mu|\tau^{(n)}) = q'(e_{n+1}|\tau^{(n+1)}) - q'(e_{n+1}|e_{n+1}) = 1 - 2 = -1.$$

By definition of $z$ we have $q' = q[-z]$, since for $x_1, \ldots, x_n, x_{n+1} \in \mathbb{Z}$, taking $x = \sum_{i=1}^{n} x_i e_i$, we have

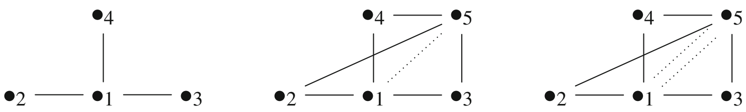
$$\begin{aligned} q'(x + x_{n+1}e_{n+1}) &= q'(x) + x_{n+1}^2 + x_{n+1}q'(x|e_{n+1}) \\ &= q'(x) + x_{n+1}^2 + x_{n+1}q'(x|z+\mu) \\ &= q(x) + q(x_{n+1}z) + q(x|z) \\ &= q(x + x_{n+1}z). \end{aligned}$$

Conversely, since $q' = q[-z]$, the restriction of $q'$ to the first $n$ variables is $q$. Moreover,

$$\begin{aligned} q'(\tau^{(n+1)}|e_{n+1}) &= q'(e_{n+1}|e_{n+1}) + q'(\tau^{(n)}|e_{n+1}) \\ &= 2 + q(\tau^{(n)}|z) = 1. \end{aligned}$$

Hence $(q', \tau^{(n+1)})$ is reflection-extension of $(q, \tau^{(n)})$. □

A small example is in order. Consider the thin unit form $q = q_{\mathbb{D}_4}$, which is positive, hence weakly positive. The thin vector $\tau^{(4)}$ is nonmaximal (we have $q(\tau^{(4)}|e_1) = -1$, see the figure on the left below). The bigraph associated to the one-point extension $q[\tau^{(4)}]$ has the following shape (center):




The figure on the right corresponds to a reflection-extension $(q', \tau^{(5)})$ of $(q, \tau^{(4)})$ satisfying both point $(a)$ and $(b)$ of Proposition 5.42. That is, the pair $(q', \tau^{(5)})$ is a bad main reflection-extension of $(q, \tau^{(4)})$, with both $q$ and $q'$ weakly positive unit forms.

**Lemma 5.43.** *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a thin weakly positive unit form. Then there is a sequence of thin weakly positive unit forms $q_i : \mathbb{Z}^i \to \mathbb{Z}$ for $i = 1, \ldots, n$ such that $q_n = q$ and $(q_{i+1}, \tau^{(i+1)})$ is a reflection-extension of $(q_i, \tau^{(i)})$ for $1 \leq i < n$.*

*Proof.* We proceed by induction on $n \geq 1$. For $n = 1$ there is nothing to show. For $n > 1$ consider the (nonsimple) thin vector $\tau^{(n)} \in \mathbb{Z}^n$ and apply Lemma 5.7$(c)$ to get a vertex $i \in \{1, \ldots, n\}$ with $q(\tau^{(n)}|e_i) = 1$. Then $\sigma_i(\tau^{(n)}) = \tau^{(n)} - e_i$, which is the thin vector $\tau$ for the restriction $q^{(i)}$. Then $(q, \tau^{(n)})$ is a reflection-extension of $(q^{(n)}, \tau)$, and the result follows by induction. □

**Theorem 5.44.** *Let $(q', \tau^{(n+1)})$ be a reflection-extension of $(q, \tau^{(n)})$ with both $q$ and $q'$ weakly positive unit forms. Then there exist an iterated $\tau$-deflation $T$ for $q$*

*and an iterated $\tau$-deflation $T'$ for $q'$ such that $(q'T', \tau^{(n+1)})$ is a main reflection-extension of $(qT, \tau^{(n)})$.*

*Proof.* We proceed by induction on the number $|R^+(q')|$ of positive roots of $q'$. If $\tau^{(n+1)}$ is a maximal $q'$-root there is nothing to show (in particular if $|R^+(q')| = 1$). Assume now that $\tau^{(n+1)}$ is a nonmaximal $q'$-root. By Lemma 5.7 and Proposition 5.8 there is a vertex $j \in \{1, \dots, n\}$ such that $q'(\tau^{(n+1)}|e_j) = -1$. Since $q'$ is weakly positive we get

$$0 < q'(e_j + \tau^{(n)}) = 2 + q'(\tau^{(n+1)} - e_{n+1}|e_j) = 1 - q'_{j,n+1},$$

therefore $q'_{j,n+1} \leq 0$.

If $q'_{j,n+1} = 0$, then by Lemma 5.17 there is a vertex $i \in \{1, \dots n\}$ such that $q'_{ij} = -1$ (hence $i \neq j$). Since $q(\tau^{(n)}|e_j) = q'(\tau^{(n+1)}|e_j) - q'_{j,n+1} = -1$, the deflation $T^-_{ij}$ is a $\tau$-deflation for both $q'$ and $q$. Observe also that the restriction of $q'T^-_{ij}$ to $\mathbb{Z}^n$ coincides with $qT^-_{ij}$. Moreover, $(q'T^-_{ij}, \tau^{(n+1)})$ is a reflection-extension of $(qT^-_{ij}, \tau^{(n)})$, since

$$\begin{aligned}(q'T^-_{ij})(\tau^{(n+1)}|e_{n+1}) &= q'(T^-_{ij}(\tau^{(n+1)})|T^-_{ij}(e_{n+1}))\\ &= q'(\tau^{(n+1)} + e_j|e_{n+1})\\ &= q'(\tau^{(n+1)}|e_{n+1}) + q'_{j,n+1} = 1.\end{aligned}$$

If $q'_{j,n+1} < 0$ then $q'_{j,n+1} = -1$ (by Lemma 5.5(*b*)). Then $T^-_{n+1,j}$ is a $\tau$-deflation for $q'$ and the restriction of $q'T$ to $\mathbb{Z}^n$ is $q$. Again we have

$$\begin{aligned}(q'T^-_{n+1,j})(\tau^{(n+1)}|e_{n+1}) &= q'(T^-_{n+1,j}(\tau^{(n+1)})|T^-_{n+1,j}(e_{n+1}))\\ &= q'(\tau^{(n+1)} + e_j|e_{n+1} + e_j)\\ &= q'(\tau^{(n+1)}|e_{n+1}) + q'(\tau^{(n+1)}|e_j) + q'_{j,n+1} + q'(e_j|e_j)\\ &= 1 - 1 - 1 + 2 = 1,\end{aligned}$$

therefore $(qT^-_{n+1,j}, \tau^{(n+1)})$ is reflection-extension of $(q, \tau^{(n)})$.

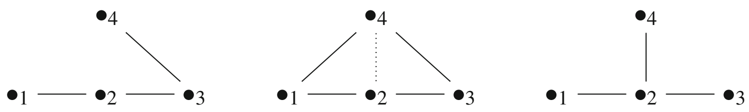
To complete the proof we use induction observing that in both cases the number of positive $q'$-roots decreases (see Lemma 2.19). □

**Algorithm 5.45.** *Theorem 5.44 is used in [22] to sketch a four step algorithm to produce all good thin weakly positive unit forms in $n + 1$ variables starting from those forms in n variables.*

*Step 1. Apply all possible iterated $\tau$-deflations to the good thin weakly positive unit forms in $n$-variables.*

*Step 2.* *Construct all main reflection-extensions (using Proposition 5.42(a)) of the obtained forms.*
*Step 3.* *Apply all possible iterated $\tau$-inflations to the list obtained in Step 2.*
*Step 4.* *Filter the final list to sort out any bad thin forms.*

From Theorem 5.44 it is clear that every good thin weakly positive unit form in $n+1$ variables belongs to the list obtained in Step 4 of Algorithm 5.45. For instance, in $n = 1, 2, 3$ variables there is exactly one good thin weakly positive unit form, namely $q_{\mathbb{A}_1}$, $q_{\mathbb{A}_2}$ and $q_{\mathbb{A}_3}$ respectively. For $n = 4$ apply Step 2 to $q_{\mathbb{A}_3}$ to get the two forms on the left below




The third form on the right (for which the thin vector $\tau^{(4)}$ is nonmaximal) is obtained after applying Step 3. Case $n = 5$ is sketched in Exercise 4 below.

Before we can state the main classification result of this section we consider yet another construction of unit forms. We say that a point $i$ in a bigraph $B$ is a *linking vertex* if it has exactly two neighbors and is joint to them by simple solid edges. A linking vertex of a unit form $q$ is a linking vertex of its associated bigraph. By a *chain* in a bigraph (or unit form) we mean a sequence of vertices $a_{-1}, a_0 \dots, a_k, a_{k+1}$ where $a_i$ is a linking vertex for $i = 0, \dots, k$ joined precisely to $a_{i-1}$ and $a_{i+1}$. The number $k+1$ will be referred to as the *length of the chain*.

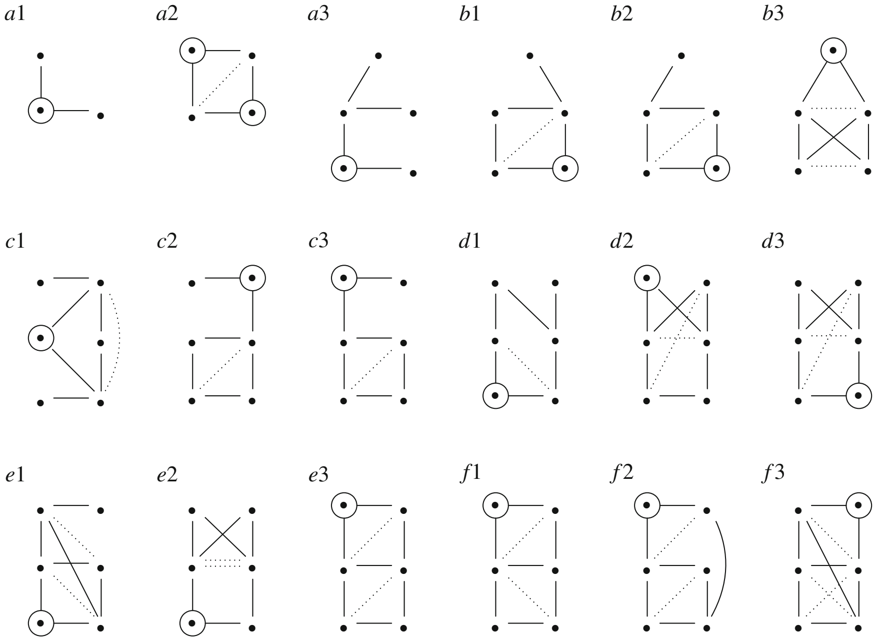
For $u \geq 1$, the *u-blow up* $q^{(a\star u)}$ of a unit form $q$ with respect to a linking vertex $a$ is the form with bigraph $B^\star$ obtained by replacing vertex $a$ by a chain of length $u$. To be precise, if $a$ is joined to vertices $a_{-1}$ and $a_{u+1}$ in $B_q$, we get $B^\star$ from the restriction $B_q^{(a)}$, by adding vertices $a_0, \dots, a_u$ such that $a_i$ is joined by solid simple edges only to $a_{i-1}$ and $a_{i+1}$, for $i = 1, \dots, u$. Now, if $\Lambda$ is a set of linking vertices of $q$ and $\overline{u} = (u_\lambda)_{\lambda\in\Lambda}$ is a vector of natural numbers, then the *blow up* of $q$ with respect to $(\Lambda, \overline{u})$ is the unit form $q^{(\Lambda\star\overline{u})}$ defined recursively as

$$q^{(\Lambda\star\overline{u})} = (q^{(\Lambda-\{\lambda\}\star\overline{u}-\{u_\lambda\})})^{(\lambda\star u_\lambda)},$$

for some $\lambda \in \Lambda$. This procedure yields, for a unit form $q$ with a set of linking vertices $\Lambda$, a series of unit forms $\{q^{(\Lambda\star\overline{u})}\}$ indexed by $\overline{u} \in \mathbb{N}^\Lambda$. Whether the forms in the series associated to $q$ and $\Lambda$ are (good and thin) weakly positive, assuming that $q$ is (good and thin) weakly positive, is the subject of investigation in [22, Section 5]. Their outcome leads to the following classification result.

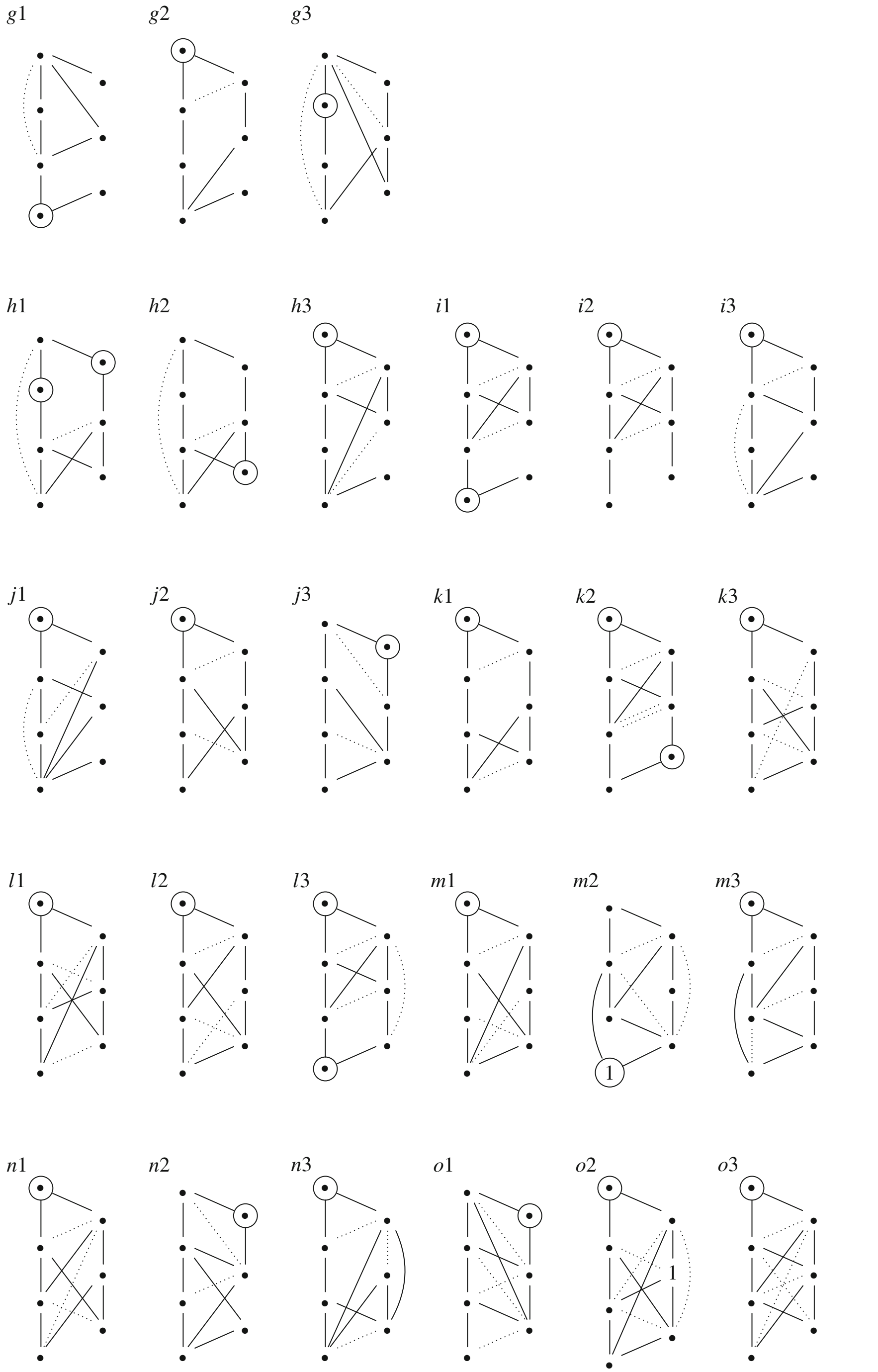
**Theorem 5.46.** *Every good thin weakly positive unit form in $n \geq 15$ variables is a blow up of one of the* 63 *unit forms in Tables 5.6, 5.7 and 5.8, referred to as* basic *good thin weakly positive unit forms.*

**Table 5.6** Basic good thin weakly positive unit forms in $n$ variables for $n = 3, 4, 5, 6$



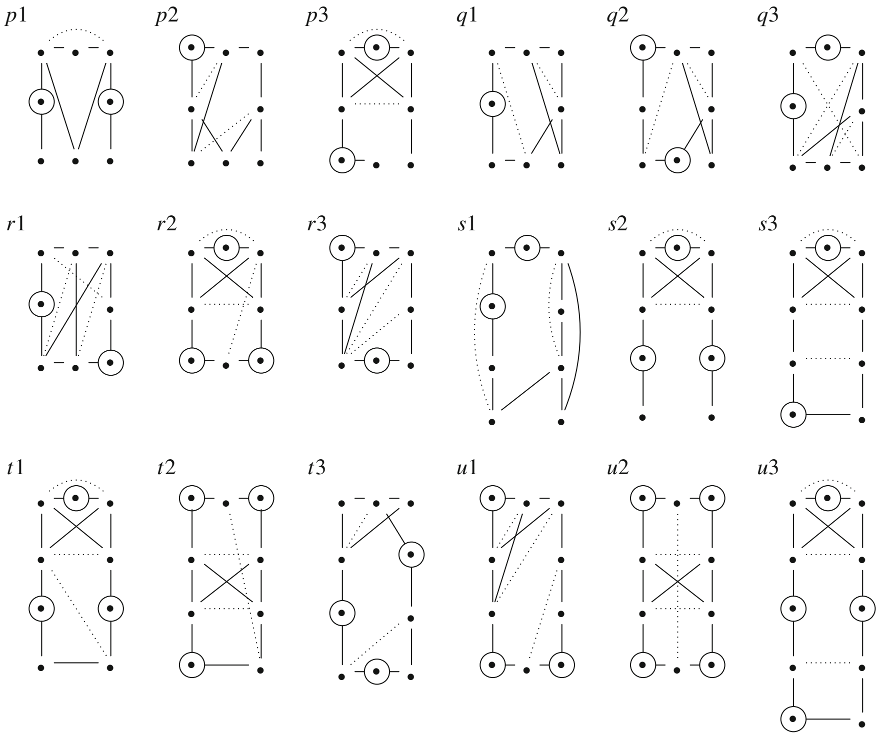

The distinguished set of linking points $\Lambda$ is denoted by encircled vertices

**Table 5.7** Basic good thin weakly positive unit forms in $n = 7$ variables




The distinguished set of linking points $\Lambda$ is denoted by encircled vertices

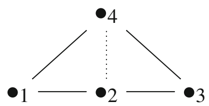
**Table 5.8** Basic good thin weakly positive unit forms in $n$ variables for $n = 8, 9, 10, 11$




The distinguished set of linking points $\Lambda$ is denoted by encircled vertices

**Exercises 5.47.**

1. Show that if $(q, v)$ is a bad sincere pair and $(q', v')$ is a reflection-extension of $(q, v)$, then $(q', v')$ itself is a bad pair.
2. Find a thin weakly positive unit form $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$ with $\tau^{(n+1)}$ a maximal root such that $(q, \tau^{(n+1)})$ is a reflection-extension of $(q^{(n)}, \tau^{(n)})$ and $\tau^{(n)}$ is a nonmaximal $q^{(n)}$-root.
3. Give an example of a good thin weakly positive unit form $q$ and a $\tau$-deflation $T_{ij}^{-}$ for $q$ such that $qT_{ij}^{-}$ is thin weakly positive but not good.
4. Consider the three good thin weakly positive unit forms in 4 variables $q_{\mathbb{A}_4}$, $q_{\mathbb{D}_4}$ and $q'$ with associated bigraph as below.




   i) Determine all five main reflection-extensions of the forms $q_{\mathbb{A}_4}$, $q_{\mathbb{D}_4}$ and $q'$.
   ii) Using $\tau$-inflations determine two remaining good thin weakly positive unit forms in five variables.
   iii) From the seven obtained forms, how many are bad?

5. Use Algorithm 5.45 to produce the complete list of good thin weakly positive unit forms in 6 variables. [Hint: There are exactly 26 such forms.]
6. From the lists obtained in Exercises 4 and 5, how many good thin forms are blow ups of one of the 63 unit forms in Tables 5.6, 5.7 and 5.8 (cf. Theorem 5.46).
7. Show that the quadratic form associated to the following bigraph is a good thin weakly positive unit form which is not blow up of one of the 63 forms in Tables 5.6, 5.7 and 5.8.