Chapter 3
Nonnegative Quadratic Forms

In this chapter we deal with *semi-unit forms* that are *nonnegative*, that is, integral quadratic forms $q(x_1, \dots, x_n) = \sum_{1 \le i \le j \le n} q_{ij} x_i x_j$ with diagonal coefficients q_{ii} in the set $\{0, 1\}$ for $i = 1, \dots, n$ such that $q(x) \ge 0$ for any vector $x = (x_1, \dots, x_n)$ in $\mathbb{Z}^n$. As before it will be convenient to set $q_{ji} = q_{ij}$ for $i \ne j$. We begin by describing nonnegative forms related to (solid) graphs (those forms q satisfying $q_{ij} \le 0$ for $i \ne j$).

3.1 Extended Dynkin Graphs

Recall that *extended Dynkin diagrams* (also known as *Euclidean graphs*) are obtained from Dynkin graphs Δ by adding a vertex ω and edges joining ω with certain *exceptional vertices* in Δ (cf. Tables 2.1 and 2.2 in Chap. 2 and Lemma 5.9 in Chap. 5). It was shown in Proposition 2.2 that the quadratic form q_G associated to a (solid) connected graph G is positive if and only if G is a Dynkin diagram. This result is generalized below to the nonnegative setting, by means of extended Dynkin diagrams (cf. [46]). Recall that for a set of indices G_0 the *support* of a vector $x \in \mathbb{Z}^{G_0}$ is given by $\mathbf{supp}(x) = \{i \in G_0 \mid x_i \ne 0\}$, and that the vector x is *positive* if $x \ne 0$ and $x_i \ge 0$ for all $i \in G_0$.

Proposition 3.1. *Let G be a connected (solid) graph. Then the associated quadratic form q_G is semi-unitary nonpositive and nonnegative if and only if G is a loop or an extended Dynkin diagram $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_m$ or $\widetilde{\mathbb{E}}_p$ for $n \ge 1$, $m \ge 4$ or $p = 6, 7, 8$ (see Table 2.2).*

Proof. Consider an extended Dynkin diagram $\widetilde{\Delta}$ and its associated quadratic form $q = q_{\widetilde{\Delta}}$. Observe that any proper subgraph of $\widetilde{\Delta}$ is union of Dynkin diagrams. Hence by Lemma 2.1 any proper restriction of q is positive. Now, it can be directly verified that the vector $p_{\widetilde{\Delta}}$ displayed as vertices in Table 2.2 is an isotropic vector

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for q (see Exercise 2.10.4). Therefore q is a critical nonpositive unit form, and by Theorem 2.12 the form q is nonnegative (again since q has an isotropic vector and Kronecker forms q_m with $|m| > 2$ are anisotropic, see Proposition 1.20). If $\widetilde{\Delta}$ is a single loop, then $q_{\widetilde{\Delta}}$ is clearly a nonnegative semi-unit form (the zero form in one variable ξ).

Let now G be a connected graph with n vertices such that $q := q_G$ is nonpositive and nonnegative and take the canonical vectors $\{e_i\}_{i \in G_0}$ of $\mathbb{Z}^{G_0}$.

First notice that if $I \subset G_0$ is a subset of vertices such that q^I has a positive radical vector w, then $I = G_0$. Indeed, if $i \in G_0 - I$ we may complete w by zeros to a vector v in $\mathbb{Z}^{G_0}$, which is a (positive) radical vector of q by Lemma 2.11. Since $v_i = 0$, as shown in Lemma 1.1 we have

$$q(v|e_i) = 2q_{ii}v_i + \sum_{j \neq i} q_{ij}v_j = \sum_{j \neq i} q_{ij}v_j < 0,$$

where the last inequality is due to the connectedness of G, since $q_{ij} \leq 0$ for all $i \neq j$ and v is a positive vector. This is impossible since v is a radical vector of q.

Assume that G has a loop, say in vertex i (that is, $q_{ii} \leq 0$). By nonnegativity we have $0 \leq q(e_i) = q_{ii}$, that is, the vertex i has exactly one loop on it. Then e_i is a positive radical vector for the one-variable restriction $q^{\{i\}}$, and by the above we have $n = 1$ and G is a single loop (that is, q is the zero form ξ in one variable). Assume now that G has no loop, but has multiple edges (say $q_{ij} < -1$ for vertices $i \neq j$). Then $0 \leq q(e_i + e_j) = 2 + q_{ij}$, that is $q_{ij} = -2$, and in particular $e_i + e_j$ is a positive radical vector of the restriction $q^{\{i,j\}}$. Therefore $n = 2$ and q is the Kronecker form $q_2(x_1, x_2) = (x_1 - x_2)^2$.

Hence we may assume that G is a simple graph (with no loops nor multiple edges). By Proposition 2.2 the graph G is not a Dynkin diagram, for q is nonpositive. Recall that for any connected simple graph G that is not a Dynkin graph, there is a subset $E_0 \subset G_0$ such that the full subgraph E of G determined by E_0 is an extended Dynkin diagram (cf. Table 2.2 and Exercise 2.10.5). For any such diagram E, the restriction q^{E_0} of q has a positive radical vector p_E, the one exhibited in Table 2.2. Using again the above argument we have $E_0 = G_0$, that is, G is an extended Dynkin diagram which completes the proof. $\square$

We give a useful result for nonnegative semi-unit forms which is analogous to Lemma 2.14, compare also with Lemma 2.11. We say that an integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ is *pre-unitary* or a *pre-unit* form if $q(e_i) \leq 1$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the canonical basis for $\mathbb{Z}^n$. Recall that a vector z in $\mathbb{Z}^n$ is an *isotropic vector for* q if $q(z) = 0$.

Lemma 3.2. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a nonnegative pre-unit form. Then q is semi-unitary and the following hold.*

a) Any isotropic vector for q is a radical vector.
b) We have $|q_{ij}| \leq 2$ for all indices $i, j \in \{1, \ldots, n\}$.
c) If $q(e_i) = 0$ for some $i \in \{1, \ldots, n\}$, then $q_{ij} = 0$ for all $j \in \{1, \ldots, n\}$.

Proof. That q is semi-unitary is evident. For (a) consider an isotropic vector $x \in \mathbb{Z}^n$, an arbitrary integer m and an index $i \in \{1, \ldots, n\}$. Then we have

$$0 \leq q(mx + e_i) = m^2 q(x) + q(e_i) + mq(x|e_i) \leq 1 + mq(x|e_i).$$

Since m is arbitrary the equality $q(x|e_i) = 0$ must hold, and since this is true for any index i, the vector x is radical for q.

Take now indices $i \neq j$ and observe that by nonnegativity, since $q_{ij} = q(e_i|e_j)$, we have

$$0 \leq q(e_i \pm e_j) = q(e_i) + q(e_j) \pm q_{ij} \leq 2 \pm q_{ij},$$

which shows (b). Assume finally that $q(e_i) = 0$ for some $i \in \{1, \ldots, n\}$. By (a) the canonical vector e_i is radical for q, that is, for any $j \neq i$ we have

$$0 = q(e_i|e_j) = q_{ij},$$

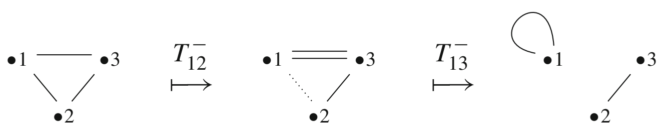
thus (c) holds. □

Let q be an integral quadratic form. Recall form Sect. 2.4 that a *flation for* q is a linear transformation $T_{ij}^{\epsilon} : \mathbb{Z}^n \to \mathbb{Z}^n$ given by

$$T_{ij}^{\epsilon} : v \mapsto v - \epsilon v_i e_j,$$

where $\epsilon \in \{+, -\}$ is a sign such that $\epsilon q_{ij} = |q_{ij}|$. When $q_{ij} > 0$ we say that T_{ij}^{+} is an *inflation* for q, and when $q_{ij} < 0$ the transformation T_{ij}^{-} is called a *deflation*. A finite composition of flations is called an *iterated flation*.

In contrast to the positive case, nonnegative unit forms are not preserved under flations, as the following example shows. Let ξ denote the zero quadratic form in one variable and take $q_{\widetilde{\mathbb{A}}_2}$ to be the form associated to the extended Dynkin diagram $\widetilde{\mathbb{A}}_2$. Then $q := q_{\widetilde{\mathbb{A}}_2} T = \xi \oplus q_{\mathbb{A}_2}$ is not unitary, where the iterated flation T is the composition $T_{12}^{-} T_{13}^{-}$.



Notice that if q' denotes the quadratic form associated to the bigraph in the middle, then the vector $e_1 + e_3$ generates the radical of q', and $T_{12}^{-}(e_1 + e_3) = e_1 + e_2 + e_3$ is a generator of the radical of $q_{\widetilde{\mathbb{A}}_2}$. Notice also that T_{13}^{-} is not a Gabrielov transformation, and that its inverse T_{13}^{+} is neither an inflation nor a deflation for q (see Proposition 2.17). However, we show next that *semi-unitary forms* are actually preserved under flations.

Lemma 3.3. *For $n \geq 1$, the set of nonnegative semi-unit forms with n variables (denoted $\mathfrak{SU}^{\geq 0}(n)$) is invariant under inflations and deflations.*

Proof. Since flations are equivalences, we only need to show that they preserve semi-unitary forms in the nonnegative case.

Let us assume that T_{ij}^{ϵ} is a flation for a nonnegative semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ with $q_{ij} \neq 0$. By Lemma 3.2 we have $|q_{ij}| \in \{1, 2\}$ and $q(e_i) = 1 = q(e_j)$. Notice that

$$q(e_i - \epsilon e_j) = q(e_i) + q(e_j) - \epsilon q_{ij} = 2 - |q_{ij}| \in \{0, 1\}.$$

Take $q^{\epsilon} = qT_{ij}^{\epsilon}$. Since $q^{\epsilon}(e_i) = q(e_i - \epsilon e_j)$ and $q^{\epsilon}(e_k) = q(e_k)$ for $k \neq i$, by the above we conclude that q^{ϵ} is semi-unitary. □

The fact that flations do not necessarily preserve connectedness of nonnegative semi-unit forms (as exhibited in the example above) was used in [7] to classify those forms.

Lemma 3.4. *If q is a nonzero nonnegative connected semi-unit form, then q is unitary. Moreover, if T_{ij}^{ϵ} is a flation for q such that $q^{\epsilon} = qT_{ij}^{\epsilon}$ is not connected, then $|q_{ij}| = 2$ and there is a nonnegative connected unit form q' such that*

$$q^{\epsilon} = q' \oplus \xi,$$

where ξ is the zero form in one variable.

Proof. Assume first that q is connected but nonunitary, say $q(e_1) = 0$. If $n > 1$ then for any other index $1 < i \leq n$ we have $q_{1i} = 0$ by Lemma 3.2(c), which is impossible since q is connected. Then $n = 1$ and q is the zero form.

For the second claim let us assume that $q^{\epsilon} = qT_{ij}^{\epsilon}$ is not connected. Observe that we must have $|q_{ij}| = 2$ (for if $|q_{ij}| = 1$ then T_{ij}^{ϵ} is a Gabrielov transformation, hence q^{ϵ} is connected by Proposition 2.17). By Lemma 3.2(c) we have $q(e_i) = 1 = q(e_j)$, and

$$q^{\epsilon}(e_i) = q(e_i - \epsilon e_j) = 2 - |q_{ij}| = 0.$$

Again by Lemma 3.2(c) the bigraph B^{ϵ} associated to q^{ϵ} has an isolated loop at vertex i. The result will follow by showing that B^{ϵ} has exactly two connected components, that is, we will show that for any vertex $k \neq i$, if $k \neq j$ then k and j belong to the same connected component of B^{ϵ}. Let B be the bigraph associated to q. Since for $k \neq i$ we have $q_{k,i}^{\epsilon} = 0 = q^{\epsilon}(e_i)$, and considering that q is unitary, then

$$q^{\epsilon}(e_k + e_i) = q^{\epsilon}(e_k) + q^{\epsilon}(e_i) + q_{ki}^{\epsilon} = q(e_k) = 1,$$

and therefore, if moreover $k \neq j$,

$$\begin{aligned} 1 = q^{\epsilon}(e_k + e_i) &= q(e_k + e_i - \epsilon e_j) \\ &= q(e_k) + q(e_i) + q(e_j) - |q_{ij}| + q_{ki} - \epsilon q_{kj} \\ &= 1 + q_{ki} - \epsilon q_{kj}, \end{aligned}$$

that is, $q_{ki} = \epsilon q_{kj}$. Hence, since B is connected, for every $k \neq i, j$ there exists a walk w in B joining k and j and not containing vertex i. The same is true in B^{ϵ} since the bigraph B^{ϵ} differs from B only on edges containing vertex i (for clearly $q^{(i)} = (q^{\epsilon})^{(i)}$). □

Recall that a vector x in $\mathbb{Z}^n$ is called *sincere* if $\mathbf{supp}(x) = \{1, \dots, n\}$. The proof of the following result, analogous to Theorem 2.20 and due originally to Ovsienko [43], is based on an argument given by von Höhne in [52].

Theorem 3.5. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected nonnegative unit form with $\mathbf{rad}(q) = \mathbb{Z}v$ for a sincere positive vector v. Then there exists an iterated inflation T and an extended Dynkin graph $\widetilde{\Delta}$ such that $qT = q_{\widetilde{\Delta}}$.*

Proof. Let B^0 be the bigraph associated to $q = q_0$. If B^0 has no dotted edges, by Proposition 3.1 the connected graph B^0 is an extended Dynkin graph and we are done. Assume otherwise that $(q_0)_{ij} > 0$ for some $i \neq j$ and consider the inflation $T_0 = T_{ij}^{+}$ and $q_1 = q_0 T_0$ with associated bigraph B^1. Notice that if $(q_0)_{ij} > 1$, then $q_0(e_i - e_j) = 2 - (q_0)_{ij} \leq 0$. By nonnegativity and Lemma 3.2(a), the vector $e_i - e_j$ is radical for q_0, contradicting the hypothesis on the generator of $\mathbf{rad}(q_0)$. Therefore $(q_0)_{ij} = 1$ and by Lemma 3.4 the unit form q_1 is connected.

Observe that the radical of q_1 is generated by the positive sincere vector $v^1 := (T_{ij}^{+})^{-1} v = v + v_i e_j$. Iterating this process we find a sequence of connected nonnegative forms $q_0, q_1, q_2, \dots$ with associated connected bigraphs $B^0, B^1, B^2, \dots$ and an inflation T_r for q_r such that $q_{r+1} = q_r T_r$. Moreover, the radical of each form q_r is generated by a sincere positive vector v^r. We show that this process is finite, arriving in this way at a quadratic form q_r with B^r having no dotted edge, therefore B^r is an extended Dynkin graph again by Proposition 3.1.

For $r \geq 0$ consider the set

$$C_{q_r} = \{x \in \mathbb{Z}^n \mid q_r(x) = 1 \text{ and there are indices } i, j \text{ with } x_i > 0 \text{ and } x_j < 0\}.$$

We divide the proof into two steps.

Step 1. *The set C_{q_r} is finite for each $r \geq 0$.*

Let us assume that C_{q_r} is an infinite set. Notice that for any of its elements x and an arbitrary index $i \in \{1, \dots, n\}$ we have

$$0 \leq q(x \pm e_i) = q(x) + q(e_i) \pm q(x|e_i) = 2 \pm q(x|e_i),$$

therefore $|q(x|e_i)| \leq 2$. Consequently we may find a sequence $\{a^0, a^1, a^2, \dots, \}$ of different vectors in C_{q_r} such that for any $i \in \{1, \dots, n\}$ and any $k \geq 0$ we have $q(a^0|e_i) = q(a^k|e_i)$. By construction, for any $k \neq \ell$ the difference $a^k - a^\ell$ is a radical vector for q_r, therefore a nonzero integral multiple of v^r. In particular, $a^k - a^\ell$ is a sincere vector, hence $a_i^k \neq a_i^\ell$ for any index i. This implies that for any integer $m \geq 1$ there is an integer $M > 0$ such that for any $k \geq M$ none of the entries a_i^k of the vector a^k belongs to the interval $[-m, m]$. Therefore we may find $k < \ell$ such that

$$\min_{i=1,\dots,n}(a_i^\ell) < \min_{i=1,\dots,n}(a_i^k) < 0 < \max_{i=1,\dots,n}(a_i^k) < \max_{i=1,\dots,n}(a_i^\ell).$$

Then the difference $a^k - a^\ell$ is a radical vector for q_r with a negative entry as well as a positive entry. This is impossible since the radical of q_r is generated by a positive vector v^r. Thus C_{q_r} is a finite set for $r \geq 0$.

Step 2. *For $r \geq 0$ the inflation T_r determines a proper inclusion $C_{q_{r+1}} \to C_{q_r}$.* First assume that for a q_{r+1}-root x the vector $T_r(x)$ does not belong to C_{q_r}, and assume that $T_r = T_{ij}^+$ for indices $i \neq j$. Multiplying by (-1) if necessary, we may assume that $T_r(x) = x - x_i e_j$ is a positive vector, that is, that $x_k \geq 0$ for $k \neq j$ and $x_j - x_i \geq 0$. Since $x_i \geq 0$ we must have $x_j \geq 0$, that is, the vector x itself is positive. This shows that $T_r(x) \in C_{q_r}$ for any vector x in $C_{q_{r+1}}$. Thus $T_r : C_{q_{r+1}} \to C_{q_r}$ is an inclusion (for T_r is $\mathbb{Z}$-invertible) which is proper since $T_r(e_i) = e_i - e_j \in C_{q_r} - T_r(C_{q_{r+1}})$.

Using Steps 1 and 2 we get a sequence of proper inclusions between finite sets

$$C_{q_r} \xrightarrow{T_{r-1}} C_{q_{r-1}} \xrightarrow{T_{r-2}} \dots \longrightarrow C_{q_2} \xrightarrow{T_1} C_{q_1} \xrightarrow{T_0} C_{q_0},$$

hence the iterative process must stop, which completes the proof. □

In the last result of this section we reformulate Vinberg's characterization of extended Dynkin diagrams (presented originally in the context of Cartan matrices, see [51] and [32]) to the setting of integral quadratic forms (adapting the short presentation given in [3]). For a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ denote by $\mathbf{rad}^+(q)$ the subset of $\mathbf{rad}(q)$ consisting of positive vectors.

Theorem 3.6 (Vinberg). *Let G be a connected (solid) graph without loops. The following are equivalent:*

a) The graph G is an extended Dynkin diagram (see Table 2.2).
b) The associated unit form q_G satisfies $\mathbf{rad}^+(q_G) \neq \emptyset$.

Proof. That (a) implies (b) is clear, since for an extended Dynkin graph $\widetilde{\Delta}$, the vector $p_{\widetilde{\Delta}}$ given in Table 2.2 belongs to $\mathbf{rad}^+(q_{\widetilde{\Delta}})$.

For the converse consider the following classical terminology (cf. [3] or [32]). For a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, a vector x in $\mathbb{Z}^n$ is said to be *subadditive* if x is positive and $q(x|e_i) \geq 0$ for $i = 1, \dots, n$. If moreover $q(x|e_i) = 0$ for all i then x

is said to be an *additive* vector for q (observe that x is additive for q if and only if $x \in \mathbf{rad}^+(q)$). Next we divide the proof into several steps.

Step 1. *The Kronecker form $q_m(x_1, x_2) = x_1^2 + x_2^2 - mx_1x_2$ admits no subadditive vector for $m > 2$.* A direct calculation shows that for $x = (x_1, x_2)$ we have

$$q_m(x|e_1) = 2x_1 - mx_2, \quad \text{and} \quad q_m(x|e_2) = 2x_2 - mx_1.$$

This shows that for $m > 2$ and x a positive vector we have either $q_m(x|e_1) < 0$ or $q_m(x|e_2) < 0$.

Step 2. *If G is an extended Dynkin diagram then any subadditive vector for q_G is additive.* Let x be a subadditive vector for q_G and take $y = p_G$ where p_G is the positive vector given in Table 2.2. Since y is a radical vector for q_G, then

$$0 = q_G(x|y) = \sum_{i=1}^{n} y_i q_G(x|e_i).$$

This implies that $q_G(x|e_i) = 0$ since $q_G(x|e_i) \geq 0$ and $y_i > 0$ for $i = 1, \ldots, n$, that is, x is an additive vector for q_G.

Step 3. *Let G be a connected (solid) graph without loops. If q_G admits a subadditive vector x, then for any proper restriction q_G^I of q_G, the restriction x' of x to the coordinates of I is a subadditive vector for q_G^I which is not additive.* First notice that x must be a sincere vector (otherwise, by connectedness there are vertices $i \in \mathbf{supp}(x)$ and $j \notin \mathbf{supp}(x)$ with $(q_G)_{ij} < 0$, and therefore $q_G(x|e_j) < 0$). In particular the restriction x' is also a positive vector. Then for $i \in I$, using that x is a positive vector and that $(q_G)_{ij} \leq 0$ for $j \neq i$, we have

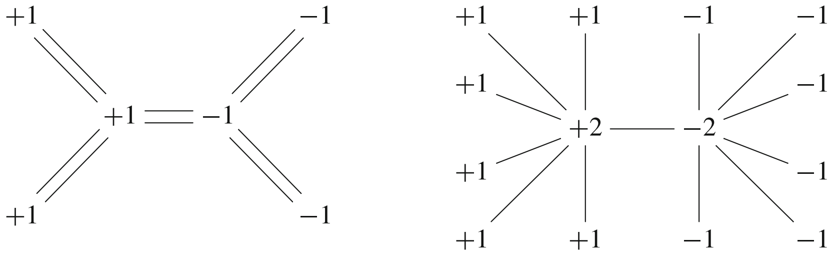
$$0 \leq q_G(x|e_i) = 2x_i + \sum_{j \neq i} (q_G)_{ij} x_j \leq 2x_i + \sum_{j \in I,\ j \neq i} (q_G)_{ij} x_j = q_G^I(x'|e_i),$$

which shows that x' is a subadditive vector for q_G^I. To show that x' is not additive observe that, since G is connected and I is a proper subset of vertices, we may find vertices $i \in I$ and $j \notin I$ such that $(q_G)_{ij} < 0$. Since x is sincere, this shows that the second inequality in the expression above is strict, therefore $q_G^I(x'|e_i) > 0$ for such $i \in I$.

We are able now to complete the proof. Take a graph G as in the hypothesis, and assume that $x \in \mathbf{rad}^+(q_G)$, that is, x is an additive vector for q_G. Steps 3 and 1 imply that q_G has no Kronecker restriction of the shape q_m for $m > 2$ (that is, G has at most double edges).

Now, if G has as full subgraph an extended Dynkin graph G', then Step 3 implies that the restriction x' is a subadditive vector for $q_{G'}$, which is not additive. This contradicts Step 2, therefore G admits no extended Dynkin diagram as proper full subgraph. Since G is not a Dynkin diagram (for $\mathbf{rad}(q_G) \neq 0$), then G is an extended Dynkin diagram (see Exercise 2.10.5). $\square$

The following examples show that, in the Theorem above, condition $\mathbf{rad}^+(q_G) \neq \emptyset$ cannot be replaced by $\mathbf{rad}(q_G) \neq \emptyset$.



Indeed, the depicted connected graphs G are not extended Dynkin diagrams, but the vector with entries as displayed in the figures is a radical vector of q_G. Subadditive roots of q, also called *locally maximal roots*, will be studied later in Sect. 6.2.

3.2 Dynkin Type and Corank

An integral quadratic form q is said to be *balanced* if $q^{-1}(0) = \mathbf{rad}(q)$, that is, if the linear form $q(x|-)$ vanishes for every $x \in \mathbb{Z}^n$ with $q(x) = 0$.

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a nonnegative quadratic semi-unit form. It was shown in Lemma 3.2(a) and (b) that q is a balanced form and that $|q_{ij}| \leq 2$ for all $1 \leq i < j \leq n$. We show next that these conditions characterize all nonnegative forms. This **Nonnegativity Criterion**, given in [7], (see also [8]), will be useful in subsequent chapters. Observe that m-Kronecker forms $q_m(x_1, x_2) = x_1^2 - mx_1x_2 + x_2^2$ are balanced (cf. Proposition 1.20), but for $|m| \geq 3$ they fail to be nonnegative forms.

Theorem 3.7. *A semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ is nonnegative if and only if the following conditions hold:*

N1) For $1 \leq i < j \leq n$ we have $|q_{ij}| \leq 2$.
N2) The form q is balanced, that is, every isotropic vector for q is a radical vector.

Proof. The necessity was shown in Lemma 3.2(a) and (b).

Assume now that q satisfies conditions ($N1$) and ($N2$), and let us also assume that n is minimal such that there is a vector $v \in \mathbb{Z}^n$ with $q(v) < 0$. Notice in particular that $n > 2$, since Kronecker forms satisfying condition ($N1$) are nonnegative. For any vertex $i \in \{1, \ldots, n\}$ the restriction $q^{(i)}$ satisfies condition ($N1$), which is condition ($P1$) in Theorem 2.15. If $q^{(i)}$ is anisotropic then it is positive by Theorem 2.15. Hence there exists an index i and an isotropic vector z for $q^{(i)}$, for otherwise q would be critical nonpositive, thus nonnegative by Theorem 2.12. Viewing z as a vector in $\mathbb{Z}^n$ by setting $z_i = 0$, notice that there are nonzero integers a and b such that $av + bz$ is nonsincere. But since q is balanced and $q(bz) = 0$, the vector bz is radical for q, therefore

$$q(av + bz) = a^2 q(v) < 0.$$

This is impossible by minimality of n, for clearly any restriction of q satisfies conditions $(N1)$ and $(N2)$ (see Exercise 1 below). □

The following reduction theorem, given in [7], is the main tool for the classification of nonnegative semi-unit forms in terms of Dynkin diagrams. For $\varepsilon \in \{+, -\}$ and a vector x in $\mathbb{Z}^n$ define the vectors x^ε by taking $x_i^\varepsilon = \max(\varepsilon x_i, 0)$ for $i = 1, \dots, n$, so we have $x = x^+ - x^-$ (recall that x is positive if $x \neq 0$ and $x = x^+$). Consider also the *weight* of a vector x in $\mathbb{Z}^n$ given by $|x| = \sum_{i=1}^n |x_i|$. Recall that the *corank* of a semi-unit form q is the rank of its radical.

Theorem 3.8. *Let q be a connected nonnegative semi-unit form with corank c. Then there exists an iterated flation T such that $qT = p \oplus \xi^c$, where ξ^c is the zero quadratic form in c variables and p is a connected positive unit form.*

Proof. Notice first that by connectedness and Lemma 3.2(c), we may assume that q is a unitary form. We proceed by induction on the corank c of q. If $c = 0$ then q is positive and there is nothing to show. For $c > 0$ the proof is divided into two steps:

Step 1. *There is an iterated inflation T such that qT has a positive radical vector.* For a nonzero radical vector v assume that there are vertices $i \in \mathbf{supp}(v^+)$ and $j \in \mathbf{supp}(v^-)$ with $q_{ij} > 0$ and $|v_i| \leq |v_j|$ (exchange the roles of i and j otherwise). Define $q' = qT_{ij}^+$ and $v' = (T_{ij}^+)^{-1}v = v + v_i e_j$, and observe that since v_i and v_j have opposite sign we have $|v_i + v_j| < |v_j|$. Since $|v'| < |v|$ this process must stop, getting an iterated inflation T, a quadratic semi-unit form $\widehat{q} = qT$ and a vector $\widehat{v} = T^{-1}v$ satisfying

$$0 = \widehat{q}(\widehat{v}) = \widehat{q}(\widehat{v}^+ - \widehat{v}^-) = \widehat{q}(\widehat{v}^+) + \widehat{q}(\widehat{v}^-) + \sum_{(i,j)} \widehat{q}_{ij}\widehat{v}_i\widehat{v}_j,$$

where the sum runs over the set $\mathbf{supp}(\widehat{v}^+) \times \mathbf{supp}(\widehat{v}^-)$. Since every summand on the right side of the equation is nonnegative, all of them are equal to zero (for $(\widehat{q})_{ij} \leq 0$ if $(i, j) \in \mathbf{supp}(\widehat{v}^+) \times \mathbf{supp}(\widehat{v}^-)$). By Lemma 3.2$(a)$ all three vectors $\widehat{v}^+$, $\widehat{v}^-$ and $\widehat{v}^+ + \widehat{v}^-$ are positive radical vectors of $\widehat{q}$. Notice that by Lemma 3.4, if the form $\widehat{q}$ is not connected then there is a connected unit form $\widehat{q}'$ and an integer c' with $\widehat{q} = \widehat{q}' \oplus \xi^{c'}$, for $0 \leq c' \leq c$. Thus by induction we may assume that $\widehat{q}$ is connected.

Step 2. *If q has a positive radical vector, there exists an iterated deflation T' such that qT' is the direct sum of a zero form in k variables (for $1 \leq k \leq c$) and a connected nonnegative unit form with corank $c - k$.*
Assume that v is a positive radical vector of q and that there exist $i, j \in \mathbf{supp}(v)$ with $q_{ij} < 0$ and $v_i \leq v_j$.

Take $q' = qT_{ij}^{-}$ and $v' = (T_{ij}^{-})^{-1}v = v - v_i e_j$, and observe that v' is a positive radical vector for q' with $|v'| < |v|$. Repeating this procedure as long as possible we end up with a quadratic form $\widetilde{q}$ and a positive radical vector $\widetilde{v}$ such that

$$0 = \widetilde{q}(\widetilde{v}) = \sum_{i=1}^{n} \widetilde{q}_{ii}\widetilde{v}_i^2 + \sum_{1 \leq i < j \leq n} \widetilde{q}_{ij}\widetilde{v}_i\widetilde{v}_j.$$

Again, both summands on the right side are nonnegative, hence zero. Then $\widetilde{q}_{ii} = 0$ for any i in the support of $\widetilde{v}$, and the claim follows from Lemma 3.4.

We conclude the proof of the theorem by induction, using Steps 1 and 2 above, and Lemma 3.4 for connectivity. □

Considering Theorems 3.8 and 2.20, for a nonnegative semi-unit form q there is an iterated flation T such that $qT = q_\Delta \oplus \xi^c$, where Δ is a disjoint union of Dynkin diagrams, q_Δ is its associated (positive) unit form, and c is the corank of q. Notice that if there are iterated inflations T and T' for q such that $qT = p \oplus \xi^c$ and $qT' = p' \oplus \xi^c$, then p and p' are equivalent positive unit forms, therefore by Theorem 2.20 the disjoint union of Dynkin graphs Δ related to q is unique up to a permutation of its components. This disjoint union Δ is referred to as the *Dynkin type* of q, written $\mathbf{Dyn}(q) = \Delta$. We now show that the Dynkin type of a nonnegative semi-unit form, together with its corank, determine the equivalence class of such forms. Here and in what follows, the zero quadratic form in $c \geq 1$ variables will be denoted by ξ^c.

Corollary 3.9. *Let q and q' be nonnegative semi-unit forms. Then q and q' are equivalent forms if and only if they have the same Dynkin type and the same corank.*

Proof. Assume first that q and q' are equivalent forms. Then $\mathbf{cork}(q) = \mathbf{cork}(q') =: c$. Take iterated flations T and T' such that $qT = p \oplus \xi^c$ and $q'T' = p' \oplus \xi^c$, where p and p' are positive unit forms. By transitivity observe that p and p' are equivalent forms, hence using Corollary 2.21 we have

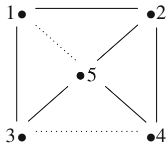
$$\mathbf{Dyn}(q) = \mathbf{Dyn}(p) = \mathbf{Dyn}(p') = \mathbf{Dyn}(q').$$

For the converse assume there is a disjoint union of Dynkin graphs Δ with $\mathbf{Dyn}(q) = \Delta = \mathbf{Dyn}(q')$, and an integer c with $\mathbf{cork}(q) = c = \mathbf{cork}(q')$. Then there are iterated flations T and T' with

$$qT = q_\Delta \oplus \xi^c = q'T'.$$

In particular, we have $q' = qT(T')^{-1}$, that is, q and q' are equivalent forms. □

Example 3.10. The quadratic form q associated to the following bigraph is nonnegative with Dynkin type $\mathbf{Dyn}(q) = \mathbb{D}_4$ and corank one.



Its radical is generated by the vector $e_2+e_4+e_5$. Moreover, the restricted forms $q^{(2)}$, $q^{(4)}$ and $q^{(5)}$ are positive with Dynkin type $\mathbb{D}_4$, while $\mathbf{Dyn}(q^{(1)}) = \mathbf{Dyn}(q^{(3)}) = \mathbb{A}_3$.

Exercises 3.11. 1. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a balanced semi-unit form, and take a subset of indices $I \subset \{1, \dots, n\}$. Show that the restricted form q^I is also balanced.

2. Show that if $q : \mathbb{Z}^n \to \mathbb{Z}$ is a nonnegative unit form of Dynkin type $\mathbb{A}_n$ with radical generated by a single positive sincere vector then $q = q_{\widetilde{\mathbb{A}}_n}$ for the extended Dynkin diagram $\widetilde{\mathbb{A}}_n$.
3. Find a connected bigraph B with at least three dotted edges such that q_B is a nonnegative unit form with Dynkin type $\mathbb{E}_6$ and radical generated by a sincere positive vector.
4. Which of the following unit forms is nonnegative?

   a) $q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 - x_3(x_1 - x_2 + x_4)$.
   b) $q(x) = x_1^2 + \ldots + x_5^2 - x_1(x_2 + x_3 + x_4 + x_5) + x_4x_5$.
   c) $q(x) = x_1^2 + \ldots + x_5^2 - x_1(x_2 + x_3 + x_4) + x_5(x_2 - x_3 + x_4) - x_3(x_2 - x_4)$.

5. Show that for any integer $c \geq 0$ and any Dynkin graph G there is a connected nonnegative unit form q with $\mathbf{Dyn}(q) = G$ and corank c.
6. Prove that the following unit forms are nonnegative and determine their Dynkin type and corank.

   a) $x_1^2 + \ldots + x_5^2 - x_2(x_1 - x_5) + x_3(x_1 - x_2 + x_4 - x_5) + x_4(x_1 + x_5)$.
   b) $x_1^2+\ldots+x_6^2+x_1(x_2-x_3-x_5+x_6)-x_4(x_2-x_3+x_5-x_6)+(x_2+x_3)(x_5-x_6)$.
   c) $x_1^2+\ldots+x_7^2-x_1(x_2+x_3+x_4)+x_2x_3+x_4(x_2+2x_3)+x_5(x_6-x_7)+x_6x_7$.

3.3 Radicals and Their Extensions

Recall that a quadratic form $q : \mathbb{Z}^I \to \mathbb{Z}$ is said to be *regular* if $\mathbf{rad}(q) = 0$. For a subset of indices $J \subset \{1, \dots, n\}$ consider the inclusion $\sigma : \mathbb{Z}^J \to \mathbb{Z}^n$ determined by $e_j \mapsto e_j$. The restriction $q^J : \mathbb{Z}^J \to Z$ of q is given by $q^J(x) = q(\sigma(x))$ for $x \in \mathbb{Z}^J$. In that situation we say $\mathbf{rad}(q^J) \subseteq \mathbf{rad}(q)$ if the restriction of σ to the radical of q^J determines an injective map $\sigma : \mathbf{rad}(q^J) \to \mathbf{rad}(q)$. As mentioned in Lemma 2.11 and its following example, it is not always true that $\mathbf{rad}(q') \subset \mathbf{rad}(q)$ for a unit form q. Our purpose here is to show that this property characterizes nonnegativity.

Instead of using the somehow clumsy term "critical not nonnegative form", we say that a quadratic form q is *hypercritical nonnegative* if any proper restriction q'

of q is nonnegative, but q itself is not. Notice, for instance, that a Kronecker form q_m is hypercritical nonnegative if and only if $|m| \geq 3$. The following graph $\overset{\approx}{\mathbb{E}}_8$ with 10 vertices,

$$\begin{array}{ccccccccc} & & 3 & & & & & & \\ & & | & & & & & & \\ 2 \text{ --- } & 4 \text{ --- } & 6 & \text{ --- } 5 & \text{ --- } 4 & \text{ --- } 3 & \text{ --- } 2 & \text{ --- } 1 & \text{ --- } \bullet \end{array}$$

has hypercritical nonnegative associated form $q = q_{\overset{\approx}{\mathbb{E}}_8}$, where the bullet $\bullet$ is the unique vertex in $\overset{\approx}{\mathbb{E}}_8$ satisfying that the restriction $q^{(\bullet)}$ is nonpositive (thus critical nonpositive.). The vector in $\mathbb{Z}^9$ indicated by the numbers at the vertices is the generator of the radical of $q^{(\bullet)}$. It is convenient to point out that if q is simultaneously a critical nonpositive and hypercritical nonnegative form then q is a Kronecker form q_m with $|m| \geq 3$. We say that a nonzero vector z in $\mathbb{Z}^n$ is called a *critical vector* for a critical nonpositive form $q : \mathbb{Z}^n \to \mathbb{Z}$ if z generates the radical of q (cf. Theorem 2.12).

Proposition 3.12. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form with $n \geq 3$. Then q is hypercritical nonnegative if and only if q is not nonnegative and for any critical nonpositive restriction q^I of q, there exists an index i such that $I = \{1, \dots, n\} - \{i\}$ and a critical vector z' of q^I such that $q(z|e_i) < 0$ where z is the vector in $\mathbb{Z}^n$ obtained by extending z' by zeros.*

Proof. First let q be a hypercritical nonnegative unit form, and take $v \in \mathbb{Z}^n$ with $q(v) < 0$. Since any proper restriction of q is nonnegative, v is a sincere vector. Assume q^I is a critical nonpositive form. Since $n \geq 3$, the form q^I is a proper restriction of q. Further, q^I is not the Kronecker form q_m with $|m| \geq 3$, for q is hypercritical nonnegative. Therefore q^I has a critical vector z' (see Theorem 2.12). Complete z' with zeros to a vector z in $\mathbb{Z}^n$.

Take integers m, k and a vertex $j \in I$ such that $(kv + mz)_j = 0$. Since

$$0 \leq q^{(j)}(kv + mz) = k^2 q(v) + m^2 q(z) + kmq(z|v) < km \sum_{i=1}^{n} v_i q(z|e_i),$$

there must exist a vertex $i \in \{1, \dots, n\}$ satisfying $q(z|e_i) \neq 0$ (hence $i \notin I$). Multiplying z by (-1) if necessary we may assume that $q(z|e_i) < 0$. Moreover,

$$q(2z + e_i) = 4q(z) + 1 + 2q(z|e_i) = 1 + 2q(z|e_i) < 0,$$

therefore q hypercritical implies that $2z + e_i$ is a sincere vector, that is, $I = \{1, \dots, n\} - \{i\}$ (and z' is a critical vector for $q^{(i)}$).

For the converse we need to show that $q^{(i)}$ is nonnegative for any $i = 1, \dots, n$. If $q^{(i)}$ is not nonnegative for some $i \in \{1, \dots, n\}$ then there is a critical nonpositive restriction q^I of $q^{(i)}$ (that is, $I \subset \{1, \dots, \widehat{i}, \dots, n\}$). By hypothesis, $I = \{1, \dots, \widehat{i}, \dots, n\}$ and $q^{(i)}$ is the m-Kronecker form for some m with $|m| \geq 3$

(since $q^{(i)}$ is not nonnegative, cf. Theorem 2.12). This contradicts the existence of a critical vector for q^I (cf. Proposition 1.20). □

Corollary 3.13. *Every hypercritical nonnegative unit form q is regular (that is, $\mathbf{rad}(q) = 0$).*

Proof. If q is a binary form, then q is the Kronecker form q_m with $|m| \geq 3$, and by Proposition 1.20 the form q is anisotropic, in particular regular.

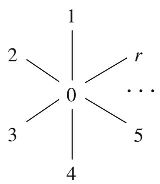
Let v be a radical vector of a hypercritical nonnegative unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ with $n > 2$. Consider a vertex i such that q^I is a critical nonpositive restriction of q for $I = \{1, \ldots, n\} - \{i\}$, with critical vector z' whose extension by zeros z to $\mathbb{Z}^n$ satisfies $q(z|e_i) < 0$. If $v_i = 0$ then v is an integral multiple of z (for the restriction of v to a vector in $\mathbb{Z}^I$ is a radical vector for q^I), which is impossible since $z \notin \mathbf{rad}(q)$). Suppose now that $v_i \neq 0$ and consider the vector v' in $\mathbb{Z}^I$ such that $v = v' + v_i e_i$. Then

$$0 = q(v|z) = q(v'|z) + v_i q(e_i|z) = q^I(v'|z) + v_i q(z|e_i) = v_i q(z|e_i) \neq 0,$$

again a contradiction. □

As an illustration consider the (solid) *r-pointed star graph* $\mathbb{S}_r$ with $r+1$ vertices and r edges

$\mathbb{S}_r =$



for $r \geq 1$. Observe that $q_{\mathbb{S}_r}$ is nonnegative if and only if $r \leq 4$, and is regular if and only if $r \neq 4$. The first assertion is consequence of $q_{\mathbb{S}_5}$ being hypercritical nonnegative. For the second claim, take $q = q_{\mathbb{S}_r}$ and $x = (x_0, x_1, \ldots, x_r)$ in $\mathbb{Z}^{r+1}$ such that $q(x|e_i) = 0$ for $i = 0, \ldots, r$. These equations can be written as

$$\begin{aligned} 2x_0 &= x_1 + \ldots + x_r, \\ 2x_1 &= x_0, \\ \cdots & \quad \cdots \\ 2x_r &= x_0, \end{aligned}$$

and in particular $4x_0 = rx_0$. Therefore there exists such nonzero x if and only if $r = 4$.

Theorem 3.14. *For a semi-unit form q the following are equivalent:*

a) The form q is nonnegative.
b) For any restriction q' of q we have $\mathbf{rad}(q') \subseteq \mathbf{rad}(q)$.

Proof. That (a) implies (b) was shown in Lemma 3.2 (see also Lemma 2.11). Assume that q is not nonnegative and take a hypercritical nonnegative restriction q^I of q (with $I \subset \{1, \dots, n\}$) and a vertex $i \in I$ such that $q' = (q^I)^{(i)}$ is a critical nonpositive restriction of q^I (see Proposition 3.12). Then there is a critical vector $z \in \mathbf{rad}(q')$ but its extension by zeros $\sigma(z) \in \mathbb{Z}^I$ is not radical for q^I by Corollary 3.13, in particular not a radical vector for q. □

We say that a semi-unit form $q' : \mathbb{Z}^n \to \mathbb{Z}$ is a *radical extension* of a semi-unit form $q : \mathbb{Z}^m \to \mathbb{Z}$ (with $m \leq n$) if there is a subgroup U of $\mathbb{Z}^n$ and a subgroup U' of $\mathbf{rad}(q')$ such that $\mathbb{Z}^n = U \oplus U'$ and $q = q'|_U$. In other words, q' is radical extension of q if there is a $\mathbb{Z}$-invertible transformation $S : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

$$q'S = q \oplus \xi^{n-m}.$$

In particular the columns of S consists of roots or isotropic vectors of q'. Throughout the text we will find many instances of radical extensions: Theorem 3.8 implies that every nonnegative semi-unit form with Dynkin type Δ is radical extension of the positive unit form q_Δ (see details below in Theorem 3.15). In Sect. 3.5 we will consider *one-point extensions*, one of the main tools in the construction of unitary forms. *Radical explosions* are defined in Sect. 5.5, with a particular case known as *doubling of vertices* used in Sect. 6.5 for the construction of graphical forms.

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a nonnegative semi-unit form. Observe that $\mathbf{rad}(q)$ is a *pure subgroup* of $\mathbb{Z}^n$ (that is, if $0 \neq n \in \mathbb{Z}$ with $nv \in \mathbf{rad}(q)$, then $v \in \mathbf{rad}(q)$), hence there is an isomorphism $\mathbb{Z}^n/\mathbf{rad}(q) \to \mathbb{Z}^{n-c}$ where $c = \mathbf{cork}(q)$ is the corank of q. Recall that for $v \in \mathbf{rad}(q)$ we have $q(w+v) = q(w)$ for any $w \in \mathbb{Z}^n$, thus we may consider a well-defined induced mapping

$$\begin{aligned} \overline{q}: \ \mathbb{Z}^n/\mathbf{rad}(q) &\longrightarrow \mathbb{Z}, \\ w + \mathbf{rad}(q) &\longmapsto q(w). \end{aligned}$$

We show that there is a basis in $\mathbb{Z}^n/\mathbf{rad}(q)$ which makes $\overline{q}$ a positive unit form.

Theorem 3.15. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a nonnegative semi-unit form. Then the induced mapping $\overline{q} : \mathbb{Z}^n/$*$\mathbf{rad}(q)$ *$\to \mathbb{Z}$ is $\mathbb{Z}$-equivalent to q_Δ, where Δ is the Dynkin type of q. In particular, q is radical extension of q_Δ.*

Proof. Applying Theorems 3.8 and 2.20 to each connected component of q, there is a $\mathbb{Z}$-invertible linear transformation $T : \mathbb{Z}^n \to \mathbb{Z}^c \oplus \mathbb{Z}^{n-c}$ such that

$$qT^{-1} = \xi^c \oplus q_\Delta : \mathbb{Z}^c \oplus \mathbb{Z}^{n-c} \to \mathbb{Z},$$

where Δ is the Dynkin type of q and $c = \mathbf{cork}(q)$. Notice that $\mathbb{Z}^c = T(\mathbf{rad}(q)) = \mathbf{rad}(qT^{-1})$, thus we have an induced isomorphism $\overline{T} : \mathbb{Z}^n/\mathbf{rad}(q) \to \mathbb{Z}^{n-c}$ which makes the following diagram commutative,

$$\begin{tikzcd}
 & \mathbb{Z}^n/\mathbf{rad}(q) \arrow[dd, "\overline{T}"] \arrow[dr, "\overline{q}"] & \\
\mathbb{Z}^n \arrow[ur, two heads] \arrow[rr, "q"'] \arrow[dd, "T"'] & & \mathbb{Z} \arrow[dd, equal] \\
 & \mathbb{Z}^{n-c} \arrow[dr, "p_\Delta"] & \\
\mathbb{Z}^c \oplus \mathbb{Z}^{n-c} \arrow[ur, two heads] \arrow[rr, "\xi^c \oplus p_\Delta"'] & & \mathbb{Z}
\end{tikzcd}$$

where $\twoheadrightarrow$ denotes canonical projections. Hence $\overline{T}$ is the desired equivalence, since we have $\overline{q} = p_\Delta \overline{T}$. Taking $U = T^{-1}(Z^{n-c})$ and $U' = T^{-1}(\mathbb{Z}^c)$ as in the definition of radical extension above, it is clear that q is a radical extension of q_Δ. □

Exercises 3.16.

1. Show that the quadratic forms associated to the following bigraphs are nonnegative, and find their radicals and Dynkin type.

2. Prove that a semi-unit form q is positive if and only if $\mathbf{rad}(q') = 0$ for any restriction q' of q.
3. Give an example of a nonregular unit form q which fails to be nonnegative.
4. Determine all hypercritical nonnegative unit forms in 5 variables.

3.4 Omissible Variables

In this section we analyze, following [7] and [9], how Dynkin type and corank change under restrictions of nonnegative semi-unit forms.

Lemma 3.17. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a nonnegative semi-unit form. For any vertex $i \in \{1, \dots, n\}$ we have*

$$0 \leq \mathbf{cork}(q) - \mathbf{cork}(q^{(i)}) \leq 1.$$

Proof. Take $I = \{1, \dots, n\} - \{i\}$ and consider the canonical inclusion $\sigma : \mathbb{Z}^I \to \mathbb{Z}^n$. By Theorem 3.14 we have $\sigma(\mathbf{rad}(q^{(i)})) \to \mathbf{rad}(q)$, which shows the first inequality. Observe now that $\mathbf{rad}(q)$ is a pure subgroup of $\mathbb{Z}^n$, and that $\sigma(\mathbf{rad}(q^{(i)}))$ is a pure subgroup of $\mathbf{rad}(q)$ (cf. Remark 4.2). Hence, if $v^1, \ldots, v^r$ is a $\mathbb{Z}$-basis of $\mathbf{rad}(q^{(i)})$, then their image $w^i := \sigma(v^i)$ may be completed to a basis $w^1, \ldots, w^r, w^{r+1}, \ldots, w^c$ of $\mathbf{rad}(q)$ (where $c = \mathbf{cork}(q)$ and $r = \mathbf{cork}(q^{(i)})$, see Proposition 4.1). If $r < c - 1$ then there are non-zero integers a and b with $(aw^{c-1} + bw^c)_i = 0$, which means that $aw^{c-1} + bw^c$ is a radical vector of q belonging to $\sigma(\mathbf{rad}(q^{(i)}))$. This is impossible since $w^1, \ldots, w^r, w^{r+1}, \ldots, w^c$ is linearly independent and $w^1, \ldots, w^r$ generate $\sigma(\mathbf{rad}(q^{(i)}))$. Therefore $c - r \leq 1$, which completes the proof. $\square$

We now generalize Proposition 2.25 to the nonnegative setting. The following partial ordering of Dynkin graphs was introduced in Sect. 2.4.

$$\mathbb{A}_m \leq \mathbb{A}_n, \quad \text{for } m \leq n;$$
$$\mathbb{A}_n < \mathbb{D}_n \leq \mathbb{D}_p, \quad \text{for } 4 \leq n \leq p;$$
$$\mathbb{D}_p < \mathbb{E}_p \leq \mathbb{E}_q, \quad \text{for } 6 \leq p \leq q \leq 8.$$

As before we take $r_{\mathbb{A}_n} = 1$, $r_{\mathbb{D}_m} = 2$, $r_{\mathbb{E}_6} = 3$, $r_{\mathbb{E}_7} = 4$ and $r_{\mathbb{E}_7} = 6$ to be the maximal value the entries of a maximal positive root of q_Δ may attain, where Δ is a Dynkin graph (cf. Table 2.1 and Remark 2.24).

Proposition 3.18. *Let q be a connected nonnegative unit form. Then for any connected restriction q' of q we have* $\mathbf{Dyn}(q') \leq \mathbf{Dyn}(q)$.

Proof. We will show that $\mathbf{Dyn}(q^{(i)}) \leq \mathbf{Dyn}(q)$ for any $1 \leq i \leq n$ such that $q^{(i)}$ is still connected (see Exercise 6 below). For simplicity we take $i = n$.

Suppose first that $\mathbf{cork}(q^{(n)}) = c = \mathbf{cork}(q)$. Using Theorems 3.8 and 2.20 there is an iterated flation $T : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}$ such that $q^{(n)}T = \xi^c \oplus q_{\Delta'}$ where $\Delta' = \mathbf{Dyn}(q^{(n)})$. Consider the linear transformation $\widetilde{T} = T \oplus [1] : \mathbb{Z}^n \to \mathbb{Z}^n$, and observe that $q\widetilde{T} = \xi^c \oplus \widetilde{q}$ with $\widetilde{q}$ a positive unit form (for $\mathbf{cork}(q\widetilde{T}) = \mathbf{cork}(q) = c$). Let $v = p_{\Delta'}$ be the maximal positive root of $q_{\Delta'}$ (see Table 2.1) and take $v_n = 0$, so that we may view v as a positive root of $\widetilde{q}$. If $\mathbf{Dyn}(q) = \Delta$, then $\mathbf{Dyn}(\widetilde{q}) = \Delta$ and we have

$$r_{\Delta'} = \max_{i=1,\ldots,n} (|v_i|) \leq r_\Delta,$$

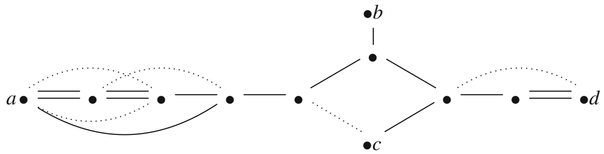
where the last inequality follows from Proposition 2.22. Since the number of vertices of Δ' is $n - 1 - c$, and that of Δ is $n - c$, we get $|\Delta'_0| < |\Delta_0|$ and $r_{\Delta'} \leq r_\Delta$. Thus by Remark 2.24 we have $\Delta' \leq \Delta$.

Suppose now that $\mathbf{cork}(q^{(n)}) \neq \mathbf{cork}(q)$ hence by Lemma 3.17 we have $\mathbf{cork}(q^{(n)}) = \mathbf{cork}(q) - 1$. Taking $\mathbf{Dyn}(q^{(n)}) = \Delta'$ and $\mathbf{Dyn}(q) = \Delta$, we notice as above that $|\Delta'_0| = |\Delta_0|$. As in the proof of Theorem 3.15, the inclusion $\sigma : \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ induces an injection $\overline{\sigma} : \mathbb{Z}^{n-1}/\mathbf{rad}(q^{(n)}) \to \mathbb{Z}^n/\mathbf{rad}(q)$. If $\overline{q}$ and $\overline{q^{(n)}}$ are the induced positive unit forms of Theorem 3.15, then $\overline{\sigma}$ determines an inclusion

$R(\overline{q^{(n)}}) \to R(\overline{q})$. Observe finally that if $|\Delta'_0| = |\Delta_0|$ and $|R(q_{\Delta'})| \leq |R(q_\Delta)|$, then $\Delta' \leq \Delta$, which completes the proof. □

In what follows we give conditions on an index $1 \leq i \leq n$ and a nonnegative semi-unit form q ensuring that the restriction q^i and q have same Dynkin type. We say that an index $i \in \{1, \dots, n\}$ is an *omissible point* (or an *omissible variable*) for a nonnegative semi-unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ if $q(e_i) = 1$ and there is a radical vector v of q with $v_i = 1$. In Example 3.10, for instance, indices 2, 4 and 5 are omissible points.

Example 3.19. Let q be the quadratic form associated to the following bigraph:



Then q is nonnegative, $\mathbf{Dyn}(q) = \mathbb{E}_8$ and $\mathbf{cork}(q) = 3$. Moreover, vertices a and d are omissible points, $\mathbf{Dyn}(q^{(b)}) = \mathbb{D}_7$ and $\mathbf{Dyn}(q^{(c)}) = \mathbb{E}_7$ with $\mathbf{cork}(q^{(b)}) = \mathbf{cork}(q^{(c)}) = 3$.

Proposition 3.20. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected nonnegative semi-unit form.*

a) For any omissible variable i for q, the restriction $q^{(i)}$ is connected and satisfies $\mathbf{Dyn}(q^{(i)}) = \mathbf{Dyn}(q)$.
b) If q is unitary and $\mathbf{cork}(q) > 0$*, then q admits an omissible variable.*

Proof. Let i be an omissible point of q and $v \in \mathbf{rad}(q)$ with $v_i = 1$. Consider $\overline{x} = \{\overline{x}^1, \dots, \overline{x}^\ell\}$ a $\mathbb{Z}$-basis of $\mathbb{Z}^n/\mathbf{rad}(q)$ (recall that $\mathbf{rad}(q)$ is a pure subgroup of $\mathbb{Z}^n$) and take a representative $x^j \in \mathbb{Z}^n$ of $\overline{x}^j$ with $x^j_i = 0$ for $j = 1, \dots, \ell$ (which is possible since $v_i = 1$). Denote by $\sigma : \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ the canonical inclusion with $q^{(i)} = q\sigma$, and take $y^j \in \mathbb{Z}^{n-1}$ with $x^j = \sigma(y^j)$. Since $\mathbf{rad}(q^{(i)}) \subset \mathbf{rad}(q)$ (Theorem 3.14), the set $\overline{y} = \{\pi(y^1), \dots, \pi(y^\ell)\}$ is linearly independent, where $\pi : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}/\mathbf{rad}(q^{(i)})$ is the canonical projection. Since $\mathbf{cork}(q^{(i)}) = \mathbf{cork}(q) - 1$ (Lemma 3.17), the rank of $\mathbb{Z}^{n-1}/\mathbf{rad}(q^{(i)})$ is ℓ, thus $\overline{y}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{n-1}/\mathbf{rad}(q^{(i)})$. If T denotes the change of basis transformation between $\overline{x}$ and $\overline{y}$, we have $\overline{q} = \overline{q^{(i)}}T$.

Since $\overline{q}$ is connected, this implies first that $\overline{q^{(i)}}$ is connected, thus $q^{(i)}$ is also connected. Moreover, we have $\mathbf{Dyn}(q) = \mathbf{Dyn}(q^{(i)})$, which shows (a).

Assume now that $\mathbf{cork}(q) > 0$ and that q is unitary. By Lemma 3.17, and restricting q to a subset of vertices if necessary, we may assume that $\mathbf{cork}(q) = 1$ and that $\mathbf{rad}(q)$ is generated by a sincere vector v. Moreover, composing with a point inversion S we get a nonnegative quadratic form $q' = qS$ with radical generated by a positive sincere vector $v' = Sv$. By Theorem 3.5, there is an iterated flation T such that $q'T$ is the quadratic form associated to an extended Dynkin graph. All these forms have an omissible point (cf. Table 2.2), hence the same is true for q'. Since $|v_i| = |v'_i|$ for $i = 1, \dots, n$, the form q admits an omissible variable. □

Remark 3.21. As a consequence of Proposition 3.20, for any nonnegative semi-unit quadratic form q and any $c \leq \mathbf{cork}(q)$, there exists a restriction q' of q such that $\mathbf{cork}(q') = c$ and $\mathbf{Dyn}(q') = \mathbf{Dyn}(q)$.

In particular, taking $c = 0$ in the last remark, there is a positive restriction q' of q with $\mathbf{Dyn}(q') = \mathbf{Dyn}(q)$, called a *core* of q. The form q in Example 3.10 has exactly three cores, namely $q^{(2)}$, $q^{(4)}$ and $q^{(5)}$.

Exercises 3.22.

1. Give an example of a semi-unit form q and a flation T for q such that qT is no longer semi-unitary.
2. Determine which of the following quadratic forms are nonnegative:

   i) $q_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 + x_1x_3 - x_1x_4 + x_2x_3 + x_2x_4 - 2x_3x_4$.
   ii) $q_2 = x_1^2 + \ldots + x_6^2 - (x_1 + x_4)(x_2 + x_3) - x_3x_5 - x_4x_6 + x_2x_3 + x_4x_5$.
   iii) $q_3 = x_1^2 + \ldots + x_4^2 - x_1(x_2 + x_3 + x_4) - x_2x_4 + x_3x_4$.

3. Prove that if q is a nonnegative unit form then q is connected if and only if the induced quadratic form $\overline{q}$ given in Theorem 3.15 is connected.
4. Show that if the quadratic form q associated to a complete bigraph with at least four vertices is nonnegative, then q is a positive form.
5. Let q be a nonnegative unit form such that $\mathbf{Dyn}(q) = \mathbf{Dyn}(q^{(i)})$. Is i necessarily is an omissible variable for q?
6. Show that if q is a connected nonnegative unit form and q' is a connected restriction of q, then there is a sequence of indices $i_1, \dots, i_r$ with $q' = q^{(i_1)\cdots(i_r)}$ and such that $q^{(i_1)\cdots(i_s)}$ is connected for any $s = 1, \dots, r$.

3.5 Root Induction and One-Point Extensions

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form. Following [9], for a finite collection of q-roots $r = (r^j)_{j \in J}$ define the *quadratic form induced by* r, denoted by $q_r : \mathbb{Z}^J \to \mathbb{Z}$, to be the form

$$q_r(y) = q\left(\sum_{j \in J} y_j r^j\right).$$

Notice that $q_r(e_j) = q(r^j) = 1$ for any $j \in J$, that is, q_r is a unit form. Moreover, if q is nonnegative then q_r is again nonnegative. Observe also that if $I \subset \{1, \dots, n\}$ and $r = (e_i)_{i \in I}$, then the root induction q_r is precisely the restriction q^I. We say that two unit forms q and q' are *root equivalent* if q is root induced from q', and q'

is root induced from q. First we show that root equivalence is indeed an equivalence relation in the set of unit quadratic forms. Clearly we only need to prove transitivity.

Lemma 3.23. *Let $r = (r^i)_{i \in I}$ and $s = (s^j)_{j \in J}$ be finite collections of q-roots and q_r-roots respectively, where q is a unit form. Then there exists a collection t of q-roots such that $(q_r)_s = q_t$.*

Proof. Take $t^j = \sum_{i \in I} s_i^j r^i$ for $j \in J$. Observe that $q(t^j) = q\left(\sum_{i \in I} s_i^j r^i\right) = q_r(s^j) = 1$, thus $t = (t^j)_{j \in J}$ is a collection of q-roots. Then we have

$$\begin{aligned}(q_r)_s(x) &= q_r\left(\sum_{j \in J} x_j s^j\right) = q\left(\sum_{i \in I}\left(\sum_{j \in J} x_j s^j\right)_i r^i\right) \\ &= q\left(\sum_{j \in J} x_j\left(\sum_{i \in I} s_i^j r^i\right)\right) = q\left(\sum_{j \in J} x_j t^j\right) = q_t(x).\end{aligned}$$ □

We now show how root induction behaves with respect to connectivity. For convenience, for an empty collection of q-roots r we denote by q_r the trivial quadratic form in zero variables. Let us first analyze the positive case.

Remark 3.24. Consider a positive unit form q that decomposes as $q = q^1 \oplus q^2$, and take a finite collection $r = (r^j)_{j \in J}$ of q-roots. There is a partition $J = J^1 \cup J^2$ such that $q^1(\overline{r}^j) = 1$ for $j \in J^1$ and $q^2(\overline{r}^j) = 1$ for $j \in J^2$, where r^j is obtained from $\overline{r}^j$ by extending by zeros. Take collections $r' = (\overline{r}^j)_{j \in J^1}$ and $r'' = (\overline{r}^j)_{j \in J^2}$. Then

$$q_r(x) = q\left(\sum_{j \in J} x_j r^j\right) = q^1\left(\sum_{j \in J^1} x_j \overline{r}^j\right) + q^2\left(\sum_{j \in J^2} x_j \overline{r}^j\right) = q^1_{r'}(x') \oplus q^2_{r''}(x''),$$

where x' and x'' are the restrictions of x to the entries indexed by J^1 and J^2 respectively.

Lemma 3.25. *Let $p : \mathbb{Z}^J \to \mathbb{Z}$ and $q : \mathbb{Z}^I \to \mathbb{Z}$ be root equivalent positive unit forms. If $p = p^1 \oplus \cdots \oplus p^m$ and $q = q^1 \oplus \cdots \oplus q^n$ are decompositions with p^a and q^b connected for $a = 1, \ldots, m$ and $b = 1, \ldots, n$, then $m = n$ and there is a permutation π such that p^k is root equivalent to $q^{\pi(k)}$ for $k = 1, \ldots, n$.*

Proof. Let $r = (r^j)_{j \in J}$ and $s = (s^i)_{i \in I}$ be finite collections of q-roots and p-roots respectively with $p = q_r$ and $q = p_s$. Using Remark 3.24, we have a partition $I = \bigcup_{a=1}^m I_a$ such that $\overline{s}^i$ is a p^a-root for $i \in I_a$, where s^i is obtained from $\overline{s}^i$ by extending by zeros. Take $s^{(a)} = (\overline{s}^i)_{i \in I_a}$ for $a = 1, \ldots, m$.

Observe first that $|I| = |J|$. Indeed, if there is an integral linear dependence in the collection s, say $\sum x_j s^j = 0$, then taking $x = (x_j)$ we have $q(x) = p_s(x) =$

$p(\sum_{i\in I} x_j s^j) = p(0) = 0$, contradicting the positivity of q. Then s is a linearly independent set, and $|I| \leq |J|$. Exchanging positions of p and q we get $|J| \leq |I|$.

Now we have

$$q = p_s = p^1_{s(1)} \oplus \ldots \oplus p^a_{s(a)} \oplus \ldots \oplus p^m_{s(m)},$$

and from $|I| = |J|$ it follows that $s^{(a)}$ is a $\mathbb{Q}$-basis in the domain of p^a. In particular, $m \leq n$. Exchanging the roles of p and q we get $n \leq m$. Hence $m = n$ and there is a permutation π of the set $\{1, \ldots, n\}$ such that p^a is root equivalent to $q^{\pi(a)}$ using the Remark above. □

Assume that $q : \mathbb{Z}^n \to \mathbb{Z}$ is a nonnegative semi-unit form for which the last variable n is omissible, and take a radical vector $\overline{v}$ of q with $\overline{v}_n = 1$. As shown before, the restriction $q^{(n)}$ has the same Dynkin type as q. We want to recover q from its restriction $q^{(n)}$. With that purpose define the *one-point extension* $p[v] : \mathbb{Z}^n \to \mathbb{Z}$ of a semi-unit form $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ with respect to a p-root v as

$$p[v] = p_{e(v)}, \qquad \text{where } e(v) = (e_1, \ldots, e_{n-1}, -v).$$

Lemma 3.26. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a nonnegative semi-unit form such that n is an omissible point for q, and take $p = q^{(n)}$. Then there exists a p-root $v \in \mathbb{Z}^{n-1}$ such that $q = p[v]$ and such that $v + e_n$ is a radical vector for q.*

Proof. Take $\overline{v}$ a radical vector for q such that $\overline{v}_n = 1$, and let $v \in \mathbb{Z}^{n-1}$ with $\overline{v} = v + e_n$. Then

$$p(v) = q(\overline{v} - e_n) = q(\overline{v}) + q(e_n) + q(\overline{v}|e_n) = 1,$$

since $q(e_n) = 1$. Observe that the coefficients of $p[v]$ are given as follows,

$$\begin{aligned} p[v](e_i|e_n) &= p[v](e_i + e_n) - p[v](e_i) - p[v](e_n) \\ &= p(e_i - v) - p(e_i) - p(-v) \\ &= -p(e_i|v). \end{aligned}$$

Notice now that $q(e_i|e_n) = q^{(n)}[v](e_i|e_n)$ for $i = 1, \ldots, n-1$. Since $\overline{v}$ is a radical vector for q we have

$$0 = q(e_i|\overline{v}) = q(e_i|v + e_n) = q^{(n)}(e_i|v) + q(e_i|e_n) = q(e_i|e_n) - q^{(n)}[v](e_i|e_n),$$

which completes the proof. □

Proposition 3.27. *Let q be a nonnegative quadratic unit form, and consider a core p of q. Then q and p are root equivalent forms.*

Proof. Being a restriction of q, the core p is the root induced from q. For the converse we proceed by induction on $c = \mathbf{cork}(q)$. If $c = 0$ then $p = q$ and there is nothing to prove. Assume that $c > 0$ and take an omissible variable i (using Proposition 3.20(*b*)) such that p is a restriction of $q^{(i)}$ (written $p \subset q^{(i)}$). By Proposition 3.20 we have $\mathbf{Dyn}(q^{(i)}) = \mathbf{Dyn}(p)$, thus p is a core of $q^{(i)}$. By induction there is a collection r of p-roots such that $q^{(i)} = p_r$. By Lemma 3.26 there is a $q^{(i)}$-root v such that $q = q^{(i)}[v] = (p_r)_{e(v)}$, and by the Transitivity Lemma 3.23, q is root induced from p. □

We proceed now to prove the main result of this section, as provided in [9].

Theorem 3.28 (Barot-de la Peña). *Two nonnegative unit forms have the same Dynkin type if and only if they are root equivalent forms.*

Proof. Assume first that $p : \mathbb{Z}^I \to \mathbb{Z}$ and $q : \mathbb{Z}^J \to \mathbb{Z}$ are root equivalent forms. By Proposition 3.27, p and any of its cores are root equivalent, as well as q and any of its cores. Thus we may assume that both p and q are positive unit forms. In this case we have shown that root induction preserves connected components (Lemma 3.25), therefore we may also assume that p and q are connected.

Take a collection of q-roots $r = (r^i)_{i \in I}$ with $p = q_r$ and a collection of p-roots $s = (s^j)_{j \in J}$ with $q = p_s$. Consider the linear maps

$$\begin{array}{ccc} \mathbb{Z}^I \xrightarrow{\varphi} \mathbb{Z}^J & \text{and} & \mathbb{Z}^J \xrightarrow{\psi} \mathbb{Z}^I \\ x \longmapsto \sum_{i \in I} x_i r^i & & y \longmapsto \sum_{j \in J} y_j s^j . \end{array}$$

Since $p(x) = q_r(x) = q(\varphi(x))$ and $q(y) = p_s(y) = p(\psi(y))$ and p, q are positive unit forms, both φ and ψ are injective maps, which implies $|I| = |J|$. Moreover, φ and ψ induce respectively injective functions $p^{-1}(1) \to q^{-1}(1)$ and $q^{-1}(1) \to p^{-1}(1)$, and by Proposition 2.3 both sets $p^{-1}(1)$ and $q^{-1}(1)$ are finite. Hence p and q are connected positive unit forms in the same number of variables and with the same number of roots. This implies that p and q have the same Dynkin type (cf. Table 2.1).

Assume now that $\mathbf{Dyn}(p) = \mathbf{Dyn}(q)$, and take cores p' and q' of p and q respectively. By Proposition 3.20 we have $\mathbf{Dyn}(p') = \mathbf{Dyn}(q')$. Since p' and q' are positive unit forms, they are equivalent by Corollary 2.21. Take a matrix T with columns $r^1, \ldots, r^m$ such that $p' = q'T$. Then r^i is a q'-root for $i = 1, \ldots, m$ (for p' is unitary) and the collection $r = (r^1, \ldots, r^m)$ of q'-roots clearly satisfies $p' = q'_r$, that is, p' and q' are root equivalent unit forms. By Proposition 3.27, p is root equivalent to p' and q is root equivalent to q', hence by transitivity we conclude that p and q are root equivalent forms. □

As an interesting consequence of the result above, we show that the number of nonnegative unit forms q without double edges (that is, such that $|q_{ij}| < 2$ for all $i < j$) of fixed Dynkin type is bounded.

Proposition 3.29. *Let q be a nonnegative unit form of Dynkin type Δ, and take $p = q_\Delta$. Then q has no double edge if and only if there exists a collection of p-roots r such that $r \cap -r = \emptyset$ and $q = p_r$.*

Proof. From Theorem 3.28 we know that there is a finite collection r of p-roots such that $q = p_r$. Assume first that there are $r^i = \varepsilon r^j$ in the collection r with $i \neq j$ and $\varepsilon = \pm 1$. Then

$$\varepsilon q_{ij} = p_r(e_i|\varepsilon e_j) = p_r(e_i + \varepsilon e_j) - p_r(e_i) - p_r(e_j) = p(r^i + \varepsilon r^j) - 2 = 2,$$

that is, q has a double edge. On the other hand, if $r^i \neq r^j$ and $r^i \neq -r^j$ for any $i \neq j$, then

$$0 < p(r^i \pm r^j) = p_r(e_i \pm e_j) = 2 \pm q(e_i|e_j) = 2 \pm q_{ij},$$

that is, q has no double edge. □

Proposition 3.29 yields the following immediate consequence.

Corollary 3.30. *There are only finitely many nonnegative unit forms without double edges of a given Dynkin type.*

We end this section with a result necessary for Chaps. 5 and 6.

Corollary 3.31. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a nonnegative unit form with radical generated by a vector $v \in \mathbb{Z}^n$. Then $|v_i| \leq 6$ for $i = 1, \ldots, n$.*

Proof. Consider a core $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ of q and take a p-root w such that $q = p[w]$. By Proposition 2.22 we have $|w_i| \leq 6$ for $i = 1, \ldots, n-1$, and the result follows since $v = \pm(w + e_n)$, see Lemma 3.26. □

Exercises 3.32.

1. Find all nonnegative unit forms of Dynkin type $\mathbb{A}_3$ without double edges.
2. Find a bound for the number of connected nonnegative unit forms of a given Dynkin type Δ without double edges.
3. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form with a root u, and consider a flation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ for q. Show that $T^{-1}u$ is a qT-root, and that $q[u]\overline{T} = (qT)[T^{-1}u]$, where $\overline{T} = T \oplus [1]$.
4. **Doubling vertices.** Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form. For an index $i \in \{1, \ldots, n\}$ the one-point extension $q[i] = q[-e_i] = q_{v_i}$, where $v_i = (e_1, \ldots, e_n, e_i)$, is called the *doubling of the vertex* i. Consider the morphism $\pi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ given by $\pi(e_k) = e_k$ if $k \leq n$ and $\pi(e_{n+1}) = e_i$. Show that:

   a) The mapping π is order preserving.
   b) For all $x, y \in \mathbb{Z}^{n+1}$ we have $q[i](x, y) = q(\pi(x), \pi(y))$.
   c) $\mathbf{rad}(q[i]) = \mathbf{rad}(q) \oplus \mathbb{Z}(e_{n+1} - e_i)$.
   d) A vector $x \in \mathbb{N}^{n+1}$ is a maximal positive root of $q[i]$ if and only if $\pi(x)$ is a maximal positive root of q.

5. Considering Exercise 4 above, show that for any pair of vertices $i \neq j$ in I and $q : \mathbb{Z}^I \to \mathbb{Z}$ a unit form we have $q[i][j] = q[j][i]$.
6. Show that if $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is a semi-unit form such that $e_{n+1} - e_i \in \mathbf{rad}(q)$, then $q = q^{(n+1)}[i]$.

3.6 Order of Dynkin Types

With the above analysis of root induction we may generalize the partial order within Dynkin diagrams studied in Proposition 3.18. As defined in [9], for two Dynkin types Γ and Δ we set $\Gamma \leq \Delta$ if there is a nonnegative unit form q such that $\mathbf{Dyn}(q) = \Delta$ and a q-root induced form p with $\mathbf{Dyn}(p) = \Gamma$. In what follows an empty graph will be considered as a Dynkin type, corresponding to the form q_r for an empty set of q-roots r.

Lemma 3.33. *Let Γ and Δ be Dynkin types such that Γ is an immediate predecessor of Δ. Then either $\Delta = \Gamma \sqcup \mathbb{A}_1$ or there is a Dynkin type Θ with $\Gamma = \Theta \sqcup \Gamma'$ and $\Delta = \Theta \sqcup \Delta'$, where Δ' is connected and Γ' is an immediate predecessor of Δ'.*

Proof. Notice first that for Dynkin types Δ_1 and Δ_2, any predecessor of $\Delta_1 \sqcup \Delta_2$ has the shape $\Gamma_1 \sqcup \Gamma_2$ where $\Gamma_i \leq \Delta_i$ for $i = 1, 2$. Indeed, suppose that q is a nonnegative unit form with Dynkin type $\Delta := \Delta_1 \sqcup \Delta_2$. By Theorem 3.28 there is a collection of q_Δ-roots s such that $q = (q_\Delta)_s$. Thus for any q-root induced form q_r (where r is a collection of q-roots), using Lemma 3.23 we have q_Δ-roots t with $q_r = ((q_\Delta)_s)_r = (q_\Delta)_t$. Since $q_\Delta = q_{\Delta_1} \oplus q_{\Delta_2}$ and q_Δ is a positive unit form, by Remark 3.24 we have

$$q_r = (q_{\Delta_1} \oplus q_{\Delta_2})_t = (q_{\Delta_1})_{t'} \oplus (q_{\Delta_2})_{t''},$$

for appropriate collections of q_{Δ_i}-roots t' and t''. This shows that any predecessor of $\Delta_1 \sqcup \Delta_2$ has the shape $\mathbf{Dyn}(q_r) = \Gamma_1 \sqcup \Gamma_2$ with $\Gamma_i \leq \Delta_i$ for $i = 1, 2$.

Now, if Δ is connected, taking $\Theta = \emptyset$ there is nothing to prove. Otherwise there is a Dynkin type Θ such that $\Delta = \Theta \sqcup \Delta'$ with Δ' connected. By the above we have $\Gamma = \Theta' \sqcup \Gamma'$ with $\Theta' \leq \Theta$ and $\Gamma' \leq \Delta'$, with exactly one strict inequality since Γ is an immediate predecessor of Δ (see Exercise 1 below). If $\Gamma' = \Delta'$ then we apply the result to Θ (using induction on the number of connected components) and rearrange components. If $\Theta' = \Theta$ then Γ' is an immediate predecessor of Δ', which completes the proof (observe that Γ' is empty if and only if $\Delta' = \mathbb{A}_1$). $\square$

In order to understand the partial relation in Dynkin types determined by root induction, using Lemma 3.33 it is sufficient to determine the immediate predecessors of all (connected) Dynkin diagrams. This is done in the following result, given in [9], which is used below in Table 3.1 to compute immediate predecessors of Dynkin graphs.

Table 3.1 Immediate predecessors of a Dynkin diagram Δ

Dynkin diagram Δ	Immediate predecessors Γ of Δ with $\lvert\Gamma\rvert = \lvert\Delta\rvert$	Immediate predecessors Γ of Δ with $\lvert\Gamma\rvert < \lvert\Delta\rvert$
$\mathbb{A}_n$ $(n \geq 1)$		$\mathbb{A}_{n-1}$
		$\mathbb{A}_i \sqcup \mathbb{A}_{n-i-1}$ $(i = 1, \dots, n-2)$
$\mathbb{D}_4$	$\mathbb{A}_1^4$	$\mathbb{A}_3$
$\mathbb{D}_5$	$\mathbb{A}_1^2 \sqcup \mathbb{A}_3$	$\mathbb{A}_4$
		$\mathbb{D}_4$
$\mathbb{D}_6$	$\mathbb{A}_1^2 \sqcup \mathbb{D}_4$	$\mathbb{A}_5$
	$\mathbb{A}_2^3$	$\mathbb{D}_5$
$\mathbb{D}_m$ $(m > 6)$	$\mathbb{A}_1^2 \sqcup \mathbb{D}_{m-2}$	$\mathbb{A}_{m-1}$
	$\mathbb{A}_3 \sqcup \mathbb{D}_{m-3}$	$\mathbb{D}_{m-1}$
	$\mathbb{D}_i \sqcup \mathbb{D}_{m-i}$ $(i = 4, \dots, m-4)$	
$\mathbb{E}_6$	$\mathbb{A}_1 \sqcup \mathbb{A}_5$	$\mathbb{D}_5$
	$\mathbb{A}_2^3$	
$\mathbb{E}_7$	$\mathbb{A}_7$	$\mathbb{D}_6$
	$\mathbb{A}_1 \sqcup \mathbb{D}_6$	
	$\mathbb{A}_2 \sqcup \mathbb{A}_5$	
$\mathbb{E}_8$	$\mathbb{A}_8$	
	$\mathbb{D}_8$	
	$\mathbb{A}_1 \sqcup \mathbb{E}_7$	
	$\mathbb{A}_2 \sqcup \mathbb{E}_6$	
	$\mathbb{A}_3 \sqcup \mathbb{D}_5$	
	$\mathbb{A}_4^2$	

The notation Σ^m indicates the disjoint union of m copies of a Dynkin type Σ

Theorem 3.34. *Let Γ be an immediate predecessor of a Dynkin diagram Δ. Then Γ is a restriction (by either one or two points) of the extended Dynkin diagram $\widetilde{\Delta}$.*

Proof. Take a nonnegative unit form $q : \mathbb{Z}^J \to \mathbb{Z}$ with $\mathbf{Dyn}(q) = \Delta$, and consider a collection $r = (r^i)_{i \in I}$ of q-roots such that $p = q_r$ has Dynkin type $\mathbf{Dyn}(p) = \Gamma$. The *multi-point* extension $q[r]$, defined by the root induction $q_{e(r)}$ where $e(r) = (e_j)_{j \in J} \sqcup (-r^i)_{i \in I}$ (see Exercise 4 below), also satisfies $\mathbf{Dyn}(q) = \Delta$, and clearly p is equal to the restriction $q[r]^I$. Thus substituting q by $q[r]$ we may assume that $p = q^I$ for some subset of indices $I \subset J$.

Take $j_1, \dots, j_t$ such that $J = I \sqcup \{j_1, \dots, j_t\}$, and for $0 \leq a \leq t$ define $I_a = I \sqcup \{j_1, \dots, j_a\}$. Then we have

$$\Gamma = \mathbf{Dyn}(q^{I_0}) \leq \mathbf{Dyn}(q^{I_1}) \leq \ldots \leq \mathbf{Dyn}(q^{I_{t-1}}) \leq \mathbf{Dyn}(q^{I_t}) = \Delta,$$

and since Γ is an immediate predecessor of Δ, there is exactly one a for which $\mathbf{Dyn}(q^{I_{a-1}}) \neq \mathbf{Dyn}(q^{I_a})$. Of course we may substitute q by q^{I_a} and p by $q^{I_{a-1}}$, so that there is a vertex $i \in J$ such that $p = q^{(i)}$. Observe that Lemma 3.2(a) implies that any omissible vertex j for p is also omissible for q, therefore if the restriction $p^{I'}$ is a core of p, then the restriction $q^{I'}$ has Dynkin type Δ (by Proposition 3.20(a)) and $q^{I'} = (q^{I'})^{(i)}$. Hence we may assume from the beginning that p is a positive unit form with $\mathbf{Dyn}(p) = \Gamma$ such that $p = q^{(i)}$ for a nonnegative unit form q with $\mathbf{Dyn}(q) = \Delta$. Take for simplicity $i = n$. By Theorem 2.20 there is an iterated inflation T for p such that $pT = q_\Gamma$, and clearly $q_\Delta = (q(T \oplus [1]))^{(n)}$. Replacing q by $q(T \oplus [1])$, altogether we get a nonnegative unit form q with $\mathbf{Dyn}(q) = \Delta$ such that $q^{(n)} = q_\Gamma$. In particular by Lemma 3.17 we have $\mathbf{cork}(q) = 0$ or $\mathbf{cork}(q) = 1$.

By construction we have $q_{ij} \leq 0$ for all $i \neq j$ in $J - \{n\}$. If $q_{ni} > 0$ for some $i \neq n$ then the inflation T^+_{ni} does not modify the restriction $q^{(n)}$ and takes q into a root equivalent unit form $q' = qT^+_{ni}$. Iterating this process we consider two cases, when q is positive and when q has corank one. In the first case take a q-root w with $w_n > 0$ and observe that $w' = T^-_{ni}(w) = w + w_n e_i$ is a q'-root with $w'_n > 0$. Since the entries of all roots of a positive unit form are bounded in absolute value by 6 (Proposition 2.22), the process must stop after finitely many steps. Similarly, if q has corank one, take v to be the radical vector with $\mathbf{rad}(q) = \mathbb{Z}v$ and $v_n > 0$. Then $v' = T^-_{ni}v = v + v_n e_i$ is a generator of $\mathbf{rad}(q')$ with $v'_n > 0$. Now by Corollary 3.31, all entries of these generators are bounded in absolute value by 6, thus the process must be finite again.

Then we may assume that the associated bigraph G of q has no dotted edges, and by Propositions 2.2 and 3.1 we conclude that G is either Δ or the corresponding extended Dynkin diagram $\widetilde{\Delta}$, hence the result. □

Exercises 3.35.

1. Show that if Γ is an immediate predecessor of a Dynkin type Δ, then for any Dynkin type Σ we have that $\Sigma \sqcup \Gamma$ is an immediate predecessor of $\Sigma \sqcup \Delta$.
2. Let q be a nonnegative unit form with finite collections of roots $r = (r^i)_{i \in I}$ and $s = (s^i)_{i \in I}$. Prove that $q_r = q_s$ if and only if $r - s$ is a collection of radical vectors for q.
3. Show that if $\Gamma \sqcup \Sigma \leq \Delta \sqcup \Sigma$ then $\Gamma \leq \Delta$.
4. For a collection of q-roots $r = (r^1, \dots, r^m)$ where $q : \mathbb{Z}^n \to \mathbb{Z}$ is a nonnegative unit form, define $q[r]$ as $q_{e(r)}$ where $e(r) = (e_1, \dots, e_n, -r^1, \dots, -r^m)$, called the *multi-point extension* of q by the collection r. Show that the iterated one-point extension $q[r^1][r^2] \cdots [r^m]$ is a multi-point extension.
5. Prove or give a counterexample: A quadratic unit form q with $|q_{ij}| \leq 2$ for $i < j$ is nonnegative if and only if $q^{-1}(0)$ is an abelian group.