9

Singular Integrals with Respect to the Gaussian Measure

Singular integrals are among the most important operators in classical harmonic analysis. They first appear naturally in the proof of the $L^p(\mathbb{T})$ convergence of Fourier series, $1 < p < \infty$, where the notion of the *conjugated function* is needed[1]

$$\tilde{f}(x) = p.v. \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x-y)}{2\tan\frac{y}{2}} dy = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\pi > |y| > \varepsilon} \frac{f(x-y)}{2\tan\frac{y}{2}} dy.$$

This notion was extended to the non-periodic case with the definition of the *Hilbert transform,*

$$Hf(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-y)}{y} dy = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(x-y)}{y} dy,$$

and then to $\mathbb{R}^d$, with the notion of the *Riesz transform* (see E. Stein [252, Chap III, §1]),

$$\begin{aligned} R_j f(x) &= p.v.\, C_d \int_{\mathbb{R}^d} \frac{y_i}{|y|^{d+1}} f(x-y) dy \\ &= \lim_{\varepsilon \to 0} C_d \int_{|y| > \varepsilon} \frac{y_j}{|y|^{d+1}} f(x-y) dy, \end{aligned} \tag{9.1}$$

for $j = 1, \cdots, d$, $f \in L^p(\mathbb{R}^d)$ with $C_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}}$. Taking Fourier transform, we get

$$\widehat{(R_j f)}(\zeta) = C_d \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{y_j}{|y|^{d+1}} f(x-y) dy \right] e^{-i<\xi, x>} dx$$

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[1] For a detailed study of this problem see, for instance, R. Weeden & A. Zygmund [294, Chapter 12], E. Stein [252, Chapter II, III], J. Duoandikoetxea [72, Chapter 4, 5], L. Grafakos [118, Chapter 4] or A. Torchinski [275, Chapter XI].

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$$= C_d \int_{\mathbb{R}^d} \frac{y_j}{|y|^{d+1}} \Big[\int_{\mathbb{R}^d} f(x-y) e^{-i<\xi,x>} dx \Big] dy$$
$$= C_d \int_{\mathbb{R}^d} \frac{y_j}{|y|^{d+1}} e^{-i<\xi,y>} \hat{f}(\xi) dy = C_d \, i \frac{\xi_j}{|\xi|} \hat{f}(\xi).$$

Hence,

$$\widehat{(R_j f)}(\zeta) = i \frac{\zeta_j}{|\zeta|} \hat{f}(\zeta);$$

thus, $R_j f$ is a classical multiplier operator, with multiplier $m(y) = C_d \, i \frac{y_j}{|y|}$, and hence

$$R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2} \tag{9.2}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator and $(-\Delta)^{-1/2}$ is the (classical) *Riesz potential* of order $1/2$. For more details on this, see E. Stein [252, Chap V].

Moreover, we have seen (see 2.2), $e^{i<\cdot,y>}$, $|y|^2 = -\lambda$, for $\lambda < 0$ are the eigenfunctions of the Laplacian, then,

$$R_j(e^{<\cdot,y>})(x) = -\frac{1}{|y|} \frac{\partial}{\partial x_j} e^{i<x,y>} = -i \frac{y_j}{|y|} e^{i<x,y>} = -i \frac{y_j}{\sqrt{\lambda}} e^{i<x,y>}. \tag{9.3}$$

In their seminal paper [43], A. P. Calderón and A. Zygmund considered a general class of singular operators in $\mathbb{R}^d$, which is nowadays called the Calderón–Zygmund theory.

In this chapter, we consider singular integrals with respect to the Gaussian measure. Singular integrals have been, without any doubt, one of the topics in Gaussian harmonic analysis that have been more extensively researched over the last 40 years. We begin with the study of the Gaussian Riesz transform, then the higher-order Gaussian Riesz transforms, and finally, we consider a fairly general class of Gaussian singular integrals initially studied by W. Urbina in [278] and later extended by S. Pérez in [220]. For completeness, and to facilitate comprehension of the topic, we give full proof of the boundedness properties in each case, even though the Gaussian Riesz transform and higher-order Gaussian Riesz transforms are particular cases of the general class of Gaussian singular integrals that we are going to study in Section 9.4. Additionally, in Section 9.3, we study an alternative class of Riesz transforms introduced by H. Aimar, L. Forzani, and R. Scotto in [5].

9.1 Definition and Boundedness Properties of the Gaussian Riesz Transforms

In analogy with the classical case (9.2), the Gaussian Riesz transforms in $\mathbb{R}^d$ are defined in terms of the Gaussian derivatives and Riesz potentials.

Definition 9.1. *The Gaussian j-th Riesz transform in* $\mathbb{R}^d$ *is defined spectrally, for* $1 \leq i \leq d$, *as*

$$\mathscr{R}_j = \partial_\gamma^j I_{1/2} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_j} (-L)^{-1/2}, \tag{9.4}$$

where $L = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle$ *is the Ornstein–Uhlenbeck operator,* $I_{1/2}$ *the Gaussian Riesz potential of order* $1/2$, *and* $\partial_i^\gamma = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i}$ *is the Gaussian partial derivative with respect to the variable* x_i. *The meaning of this is that for any multi-index* ν *such that* $|\nu| > 0$, *its action on the Hermite polynomial* $\mathbf{H}_\nu$ *is*

$$\mathscr{R}_j \mathbf{H}_\nu = \sqrt{\frac{2}{|\nu|}} \nu_j \mathbf{H}_{\nu - \mathbf{e}_j} \tag{9.5}$$

where $\mathbf{e_j}$ *is the unitary vector with zeros in all coordinates except for the j-th coordinate, which is one.*

Observe that (9.5) is the Gaussian analogous to (9.3). Moreover, for the normalized Hermite polynomials $\mathbf{h}_\nu$, we have

$$\mathscr{R}_j \mathbf{h}_\nu = \mathscr{R}_j \Big(\frac{\mathbf{H}_\nu}{(2^{|\nu|} \nu!)^{1/2}} \Big) = \frac{1}{(2^{|\nu|} \nu!)^{1/2}} \sqrt{\frac{2}{|\nu|}} \nu_j \mathbf{H}_{\nu - \mathbf{e_j}} = \mathbf{h}_{\nu - \mathbf{e_j}}. \tag{9.6}$$

From the integral representation of the Riesz potential (8.8), obtained in Theorem 8.3, using the kernel (8.9), we immediately get the kernel of $\mathscr{R}_j$,

$$\begin{aligned} \mathscr{K}_j(x,y) &= \frac{\partial}{\partial x_i} N_{1/2}(x,y) \\ &= \frac{1}{\pi^{d/2}\Gamma(1/2)} \int_0^1 \left(\frac{1-r^2}{-\log r} \right)^{1/2} \frac{y_j - r x_j}{(1-r^2)^{\frac{(d+3)}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}} dr; \end{aligned} \tag{9.7}$$

therefore, we get the integral representation of $\mathscr{R}_i$,

$$\begin{aligned} \mathscr{R}_j f(x) &= p.v. \int_{\mathbb{R}^d} \mathscr{K}_j(x,y) f(y) dy \qquad (9.8) \\ &= p.v. \frac{1}{\pi^{d/2}\Gamma(1/2)} \int_{\mathbb{R}^d} \Big(\int_0^1 \left(\frac{1-r^2}{-\log r} \right)^{1/2} \frac{y_j - r x_j}{(1-r^2)^{\frac{(d+3)}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}} \Big) dr f(y) dy. \end{aligned}$$

In particular, for $d = 1$, the *Gaussian Hilbert transform* is defined spectrally as

$$\mathscr{H} = \partial^\gamma I_{1/2} = \frac{1}{\sqrt{2}} \frac{d}{dx} (-L)^{-1/2}. \tag{9.9}$$

meaning that

$$\mathscr{H} H_n(x) = \frac{1}{\sqrt{2}} \frac{d}{dx} ((-L)^{-1/2} H_n(x)) = \frac{1}{\sqrt{2n}} \frac{d}{dx} H_n(x) = \sqrt{2n} H_{n-1}(x).$$

As a particular case of (9.8), we get the following integral representation of $\mathscr{H}$

$$\begin{aligned}\mathscr{H}f(x) &= p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\Big(\int_0^1 (\frac{1-r^2}{-\log r})^{1/2}\frac{y-rx}{(1-r^2)^2} \\ &\qquad\qquad \times \exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})\,dr f(y)\Big)\,\gamma_1(dy) \\ &= p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\Big(\int_0^1 (\frac{1-r^2}{-\log r})^{1/2}\frac{y-rx}{(1-r^2)^2}e^{-\frac{-|y-rx|^2}{1-r^2}}\,dr\Big) f(y)\,dy.\end{aligned}$$

Theorem 9.2. *The Gaussian Riesz transforms $\mathscr{R}_j,\ j=1,\cdots,d$ are $L^p(\gamma_d)$ bounded for $1<p<\infty$, that is to say, there exists $C>0$, depending on p, β and dimension d such that*

$$\|\mathscr{R}_j f\|_{p,\gamma} \le \|f\|_{p,\gamma}, \tag{9.10}$$

for any $f \in L^p(\gamma_d)$.

In 1969, B. Muckenhoupt considered the one-dimensional case of the Gaussian Hilbert transform $\mathscr{H}$, using real analysis methods, based on Natanson's lemma (see Lemma 10.27). Then, in 1984, P. A. Meyer [189] established the $L^p(\gamma_d)$-boundedness of the Gaussian Riesz transforms $\mathscr{R}_j$ with respect to the Gaussian measure $\gamma_d(dx)$ in $\mathbb{R}^d$, for $1<p<\infty$, using probabilistic methods, by considering the Brownian motion and the famous Burkholder–Gundy inequality (see also [82] for a simpler proof of P. A. Meyer's theorem). After these two landmark papers, several other proofs of the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_j$ were obtained. In 1986, R. Gundy [121] got one, also by using the Brownian motion and the notion of background radiation as a stochastic process, and G. Pisier [227] got one by using the method of rotations and transference methods introduced in [57] by R. Coifman and G. Weiss. In both proofs, the estimates are independent of dimension. In 1988, W. Urbina [278], in his doctoral dissertation, got the first proof using real analysis methods in $\mathbb{R}^d$, $d>1$ by studying the kernel directly, extending B. Muckenhoupt's proof to the higher dimensional case, but the constants are strongly dependent on dimension. Then, in 1994, C. Gutiérrez [122] got an alternative proof, using the Littlewood–Paley–Stein theory, with constants independent of dimension. Finally, in 1996, S. Pérez, S. & F. Soria [223] (see also [220]), got an alternative real analysis proof using refined estimates of the kernel, with constants dependent on dimension, by using analog estimates of those they obtained for the maximal function of the Ornstein–Uhlenbeck semigroup.

On the other hand, the weak type $(1,1)$ with respect to γ_d of $\mathscr{R}_j$ was proved by B. Muckenhoupt, in the case $d=1$, in his 1969 paper [194], and R. Scotto proved it for the case $d>1$ in his doctoral dissertation [244] (see also [77]), by using the method developed by P. Sjögren in [247] to prove the weak type $(1,1)$ of T^*, the maximal function of the Ornstein–Uhlenbeck semigroup already discussed in Chapter 4. Also, S. Pérez has an alternative proof of this result (see [220, 221]).

Theorem 9.3. *(Scotto) There exists a constant C such that*

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : \mathscr{R}_j f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda}||f||_{1,\gamma_d}. \tag{9.11}$$

for all $f \in L^1(\gamma_d)$.

Observe that, in general, if T is a linear operator associated with a given kernel $K(x,y)$, its adjoint with respect to the Gaussian measure has kernel $K\ (x,y) = K(y,x)e^{|x|^2-|y|^2}$. Then, as $\mathscr{K}_j(y,x)e^{|x|^2-|y|^2} = \mathscr{K}_j(x,y)$, it follows that the adjoint of $\mathscr{R}_j$ is also of weak type $(1,1)$ with respect to γ_d.

To prove Theorem 9.2 and Theorem 9.3, we essentially follow the proof given by S. Pérez, S. and F. Soria in [223]. As was done for the Ornstein–Uhlenbeck maximal function T^*, we split the operator $\mathscr{R}_j$ into a local part and a global part. Given $x \in \mathbb{R}^d$, the *local part* of the operator $\mathscr{R}_j$ is its restriction to the admissible ball

$$B_h(x) = B(x, d\,m(x)) = \{y \in \mathbb{R}^d : |y-x| < d\,m(x)\},$$

and we have seen that the Gaussian density is essentially constant on admissible balls (see 4.102). The *global part* of the operator $\mathscr{R}_j$ is its restriction to the complement of $B_h(x)$. Thus,

$$\begin{aligned}\mathscr{R}_j f(x) &= C_d \int\limits_{|x-y|<d\,m(x)} \mathscr{K}_j(x,y)f(y)dy + C_d \int\limits_{|x-y|\geq d\,m(x)} \mathscr{K}_j(x,y)|f(y)|dy \\ &= \mathscr{R}_{j,L}f(x) + \mathscr{R}_{j,G}f(x),\end{aligned}$$

where $\mathscr{R}_{j,L}f(x) = \mathscr{R}_j(f\chi_{B_h(x)})(x)$ is the *local part* and $\mathscr{R}_{j,G}f(x) = \mathscr{R}_j(f\chi_{B_h^c(x)})(x)$ is the *global part* of $\mathscr{R}_j$.

To study the local part of the Gaussian Riesz transform $\mathscr{R}_j$, we use Theorem 4.30, to see that the local part $\mathscr{R}_{j,L}$ corresponds essentially to a classical Calderón–Zygmund singular integral. First, we need to verify the size and smooth conditions (4.29),

$$\begin{aligned}|\nabla_y \mathscr{K}_j(x,y)| &= \left|\nabla_y\left(\int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} (y_j - rx_j)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}}dr\right)\right| \\ &= \left(\sum_{j=1}^d \left|\int_0^1 \left(\frac{1-r^2}{-\log r}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}}\left(\delta_{i,j} - \frac{2(y_j-rx_j)(y_i-rx_i)}{1-r^2}\right)dr\right|^2\right)^{\frac{1}{2}} \\ &\leq C_d\int_0^1 \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}}dr + C_d\int_0^1 \frac{|y-rx|^2}{1-r^2}\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}}dr,\end{aligned}$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

Let us recall the notation introduced in Proposition 4.23, given $x,y \in \mathbb{R}^d$ and $t > 0$. Writing $a = |x|^2 + |y|^2, b = 2\langle x,y\rangle$, $u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |x|^2$. Therefore, taking the change of variables, $t = 1 - r^2$

$$|\nabla_y \mathscr{K}_j(x,y)| \leq C_d \int_0^1 \frac{e^{-u(t)}}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}} + C_d \int_0^1 u(t) \frac{e^{-u(t)}}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}}.$$

Also, it is easy to see that for the kernel $\mathscr{K}_j$ we have

$$|\mathscr{K}_j(x,y)| \leq C_{|\beta|} \int_0^1 (u(t))^{1/2} \frac{e^{-u(t)}}{t^{(d+2)/2}} dt.$$

Therefore, using Lemma 4.35, with exponent $d-1$ instead of d, we get

$$|\mathscr{K}_\beta(x,y)| \leq \frac{C}{|x-y|^d},$$

and then, we can apply Theorem 4.30 to the kernel $\mathscr{K}_j$ and the operator determined by it.

The global part $\mathscr{R}_{j,G}$ can be bounded using the following result.

Theorem 9.4. *(Pérez) If $|x-y| \geq C_d \left(1 \wedge \frac{1}{|x|}\right) = C_d m(x)$, then, for $1 \leq j \leq d$,*

$$|\mathscr{K}_j(x,y)| \leq C_d \overline{\mathscr{K}}(x,y), \tag{9.12}$$

where $\overline{\mathscr{K}}$ is the Gaussian maximal kernel defined in (4.40).

Proof. Let $\mathscr{K}(x,y)$ be the kernel defined as

$$\mathscr{K}(x,y) = \int_0^1 \frac{|y-rx|}{(1-r^2)^{(d+3)/2}} e^{-\frac{|y-rx|^2}{1-r^2}} dr. \tag{9.13}$$

Given that $(\frac{1-r^2}{-\log r})^2$ is a bounded function for $0 \leq r \leq 1$, then,

$$|\mathscr{K}_j(x,y)| \leq C_d \mathscr{K}(x,y).$$

Thus, it is enough to prove that

$$\mathscr{K}(x,y) \leq \overline{\mathscr{K}}(x,y),$$

when $|x-y| \geq C_d m(x)$. Making the change of variables $t = 1-r^2$, we get

$$\begin{aligned}
\mathscr{K}(x,y) &= \frac{1}{2} \int_0^1 \frac{|y - \sqrt{1-t}x|}{t^{1/2}} \frac{1}{t^{d/2}} e^{-\frac{|y-\sqrt{1-t}x|^2}{t}} \frac{dt}{t\sqrt{1-t}} \\
&= \frac{1}{2} \int_0^1 u^{1/2}(t) e^{-u(t)} \frac{dt}{t^{\frac{d}{2}+1}\sqrt{1-t}}.
\end{aligned}$$

Then, using Lemma 4.38, we immediately get

$$|\mathscr{K}_j(x,y)| \leq C_d \mathscr{K}(x,y) \leq C_d \overline{\mathscr{K}}(x,y). \qquad \square$$

From the inequality obtained in Theorem 9.4 and using Theorem 4.24, we immediately get that $\mathscr{R}_{j,G}$ is of weak type $(1,1)$ with respect to the Gaussian measure and with that we conclude the proof of Theorem 9.3. Moreover, observe that in general, if T is the linear operator associated with a kernel $K(x,y)$, its adjoint with respect to the Gaussian measure has kernel $K^*(x,y) = K(y,x)e^{|x|^2-|y|^2}$. As $\overline{\mathscr{K}}(y,x)e^{|x|^2-|y|^2} = \overline{\mathscr{K}}(x,y)$, it follows easily that the adjoint of $\mathscr{R}_j$ is also of weak type $(1,1)$ with respect to the Gaussian measure.

In [37], T. Bruno gives an alternative and simpler proof of Theorem 9.3 (see [37, Theorem 1.1]), proving that $\mathscr{K}_j$, the kernel of the j-th Riesz transform is also bounded by its kernel $\tilde{K}$, (4.59), in the global region, [37, Proposition 3.8], and then apply [37, Lemma 3.5].

As we have mentioned earlier, the main goal of C. Gutiérrez's article [122] is, following Stein's scheme in [253, Chapter IV], to prove Theorem 9.2, using the Littlewood–Paley theory. Let us see the basics of his arguments. First, he gets the following identity:

$$\frac{\partial P_t^{(1)}}{\partial t}(\mathscr{R}_j f)(x) = -\frac{1}{\sqrt{2}}\frac{P_t f}{\partial x_j}(x), \; j = 1, \cdots, d. \tag{9.14}$$

To prove this identity, it is enough to check it for the Hermite polynomials $\{\mathbf{H}_\nu\}$. From (9.5),

$$\begin{aligned}\frac{\partial P_t^{(1)}}{\partial t}(\mathscr{R}_j \mathbf{H}_\nu)(x) &= \left(\sqrt{\frac{2}{|\nu|}}\nu_j\right)\frac{\partial P_t^{(1)}}{\partial t}(\mathbf{H}_{\nu-\mathbf{e_j}})(x)\\ &= \left(\sqrt{\frac{2}{|\nu|}}\nu_j\right)\frac{\partial}{\partial t}(e^{-\sqrt{|\nu-\mathbf{e_j}|+1}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x))\\ &= \left(\sqrt{\frac{2}{|\nu|}}\nu_j\right)\frac{\partial}{\partial t}(e^{-\sqrt{|\nu|}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x)) = -\sqrt{2}\nu_j e^{-\sqrt{|\nu|}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x),\end{aligned}$$

and by (1.60)

$$\frac{\partial P_t \mathbf{H}_\nu}{\partial x_j}(x) = e^{-\sqrt{|\nu|}}\frac{\partial \mathbf{H}_\nu}{\partial x_j}(x) = 2\nu_j e^{-\sqrt{|\nu|}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x).$$

Thus,

$$\frac{\partial P_t^{(1)}}{\partial t}(\mathscr{R}_j \mathbf{H}_\nu)(x) = -\frac{1}{\sqrt{2}}\frac{P_t \mathbf{H}_\nu}{\partial x_j}(x),$$

and the formula can be extended immediately to polynomial functions, which are dense in $L^p(\gamma_d)$. Therefore,

$$g^{(1)}_{t,\gamma}(\mathscr{R}_i f)(x) = \Big(\int_0^\infty \Big|t\frac{\partial P^{(1)}_t}{\partial t}(\mathscr{R}_i f)(x)\Big|^2 \frac{dt}{t}\Big)^{1/2} = \frac{1}{\sqrt{2}}\Big(\int_0^\infty \Big|t\frac{P_t f}{\partial x_j}(x)\Big|^2 \frac{dt}{t}\Big)^{1/2}$$
$$\leq \frac{1}{\sqrt{2}} g_\gamma(f)(x).$$

Then, using Theorem 5.2 and Theorem 5.8, we get

$$1/C'_p \|\mathscr{R}_i f\|_{p,\gamma} \leq \|g^{(1)}_{t,\gamma}(\mathscr{R}_i f)\|_{p,\gamma} \leq \frac{1}{\sqrt{2}}\|g_\gamma(f)\|_{p,\gamma} \leq C_p\|f\|_{p,\gamma}.$$

An important advantage of this proof is that the constants C_p, C'_p are independent of dimension.

Finally, the atomic definition of the Gaussian Hardy spaces, given by G. Mauceri and S. Meda in [174], does not provide a fully satisfying theory. Unfortunately, that may not relate to the Ornstein–Uhlenbeck operator as well as classical Hardy spaces relate to the usual Laplacian (see [79]). In particular, G. Mauceri and S. Meda in [174], proved that the imaginary powers of L, $(-L)^{i\alpha}$ and the adjoint of the Riesz transforms $\mathscr{R}^*_j$ are bounded from $H^{1,r}_{at}(\gamma_d)$ to $L^1(\gamma_d)$, but later in [176, Theorem 3.1] G. Mauceri, S. Meda, and P. Sjögren proved that the Riesz transforms $\mathscr{R}_j$ are bounded from L^∞ to the dual of $H^{1,r}_{at}(\gamma_d) = BMO(\gamma_d)$ in any dimension, but they are not bounded from $H^1_{at}(\gamma_d)$ to $L^1(\gamma_d)$ in a dimension greater than one. Thus, their definition does not contain all the machinery that makes Fefferman–Stein [79] so outstanding, and has proven useful in a range of applications, specially in the study of partial differential equations. This was the main reason why J. Maas, J. van Neerven, and P. Portal developed a program to find an alternative definition of the Hardy spaces. In [231, Theorem 6.1], P. Portal proved that the Riesz transforms $\mathscr{R}_j$ are bounded from $H^1_{max}(\gamma_d) = H^1_{quad}(\gamma_d)$ to $L^1(\gamma_d)$, with a similar approach to that in the proof of Theorem 7.16, using an appropriated Calderón reproducing formula (see [231, Lemma 6.2]). More recently, T. Bruno proved that the Riesz transforms are bounded from the atomic Gaussian Hardy space $X^1(\gamma_d)$ to $L^1(\gamma_d)$ (see [37, Theorem 1.2]).

9.2 Definition and Boundedness Properties of the Higher-Order Gaussian Riesz Transforms

In the Gaussian case, the *higher-order Gaussian Riesz transforms* are defined directly.

Definition 9.5. *For $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \mathbb{N}^d_0$, the higher order Riesz transforms are defined spectrally as*

$$\mathscr{R}_\beta = \partial^\beta_\gamma (-L)^{-|\beta|/2}, \tag{9.15}$$

where $|\beta| = \sum_{j=1}^d \beta_j$ and $\partial^\gamma_\beta = \frac{1}{2^{|\beta|/2}}\partial^{\beta_1}_{x_1}\cdots\partial^{\beta_d}_{x_d}$. The meaning of this is that for any multi-index ν such that $|\nu| > 0$, its action on the Hermite polynomial $\mathbf{H}_\nu$ is

$$\mathscr{R}_\beta \mathbf{H}_\nu = \Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta} \tag{9.16}$$

if $\beta_i \leq \nu_i$ *for all* $i = 1,2,\cdots,d$, *and zero otherwise.*

Observe that (9.16) follows directly from the definition of $\mathscr{R}_\beta$, because $\mathbf{H}_\nu$ is the eigenfunction of the Ornstein–Uhlenbeck operator $-L$, with eigenvalue $|\nu|$; therefore,

$$(-L)^{-|\beta|/2}\mathbf{H}_\nu = \frac{1}{|\nu|^{|\beta|/2}}\mathbf{H}_\nu.$$

Hence, using (1.57) and (1.36), we get

$$\begin{aligned}
\mathscr{R}_\beta \mathbf{H}_\nu(x) &= \partial_\beta^\gamma(-L)^{-|\beta|/2}\mathbf{H}_\nu(x) = \partial_\beta^\gamma\Big(\frac{1}{|\nu|^{|\beta|/2}}\mathbf{H}_\nu(x)\Big)\\
&= \frac{1}{2^{|\beta|/2}|\nu|^{|\beta|/2}}\partial_1^{\beta_1}\cdots\partial_d^{\beta_d}\Big(\prod_{i=1}^{d}H_{\nu_i}(x_i)\Big) = \frac{1}{2^{|\beta|/2}|\nu|^{|\beta|/2}}\prod_{i=1}^{d}(\partial_i^{\beta_i}H_{\nu_i}(x_i))\\
&= \frac{1}{2^{|\beta|/2}|\nu|^{|\beta|/2}}\prod_{i=1}^{d}(2^{\beta_i}[\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]H_{\nu_i-\beta_i}(x_i))\\
&= \frac{2^{|\beta|/2}}{|\nu|^{|\beta|/2}}\prod_{i=1}^{d}([\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]H_{\nu_i-\beta_i}(x_i))\\
&= \Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta}(x).
\end{aligned}$$

Observe that this implies that

$$\mathscr{R}_\beta \mathbf{h}_\nu(x) = \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]^{1/2}\mathbf{h}_{\nu-\beta}(x), \tag{9.17}$$

because

$$\begin{aligned}
\mathscr{R}_\beta \mathbf{h}_\nu(x) &= \mathscr{R}_\beta\Big(\frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}\Big) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}}\Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\\
&\qquad\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta}(x)\\
&= \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\frac{\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)}{(\nu_i!)^{1/2}}\Big]\frac{\mathbf{H}_{\nu-\beta}(x)}{(2^{|\nu|-|\beta|})^{1/2}}\\
&= \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\frac{[\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]^{1/2}}{(\nu_i-\beta_i!)^{1/2}}\Big]\frac{\mathbf{H}_{\nu-\beta}(x)}{(2^{|\nu-\beta|})^{1/2}}\\
&= \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\prod_{i=1}^{d}[\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]^{1/2}\frac{\mathbf{H}_{\nu-\beta}(x)}{(2^{|\nu-\beta|}(\nu-\beta)!)^{1/2}}
\end{aligned}$$

$$= \left(\frac{1}{|\nu|}\right)^{|\beta|/2} \Big[\prod_{i=1}^{d} \nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]^{1/2} \mathbf{h}_{\nu-\beta}(x).$$

The higher-order Gaussian Riesz transforms have a kernel given by

$$\mathscr{K}_\beta(x,y) = \partial_\gamma^\beta N_{|\beta|/2}(x,y) \tag{9.18}$$
$$= \frac{1}{\pi^{d/2}\Gamma(|\beta|/2)} \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|} \mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r}.$$

Therefore,

$$\mathscr{R}_\beta f(x) = p.v. \int_{\mathbb{R}^d} \mathscr{K}_\beta(x,y) f(y)dy \tag{9.19}$$
$$= p.v. \frac{1}{\pi^{d/2}\Gamma(|\beta|/2)} \int_{\mathbb{R}^d} \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|} \mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)$$
$$\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r} f(y)dy.$$

Let us study the $L^p(\gamma_d)$ boundedness of these operators, for $1 < p < \infty$,

Theorem 9.6. *The higher-order Gaussian Riesz transforms $\mathscr{R}_\beta$, $|\beta| > 1$ are $L^p(\gamma_d)$ bounded for $1 < p < \infty$, that is, there exists $C > 0$, dependent only on p and dimension such that*

$$\|\mathscr{R}_\beta f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}, \tag{9.20}$$

for any $f \in L^p(\gamma_d)$.

There are several analytic proofs of this result. The first analytic proof was given by W. Urbina in [278] with constants dependent on dimension. A clever proof was given by G. Pisier [227], which combines probabilistic and analytic techniques (method of rotations and transference methods), with constants independent of dimension, but valid only for the case $|\beta|$ odd. In [124], C. Gutiérrez, C. Segovia, and J. L. Torrea obtained a proof, with constants independent of dimension, following the work of C. Gutiérrez in [122], by using the Littlewood–Paley theory, with higher-order Gaussian Littlewood–Paley functions, which were discussed in Chapter 6. In [223], S. Pérez and F. Soria provide an analytic proof, with constants dependent on dimension, with a similar technique to that developed to study the Ornstein–Uhlenbeck maximal function T^* already discussed in Chapter 4. We study their proof in detail. Finally, L. Forzani, R. Scotto, and W. Urbina in [88] have a very simple proof, with constants independent of dimension, based on Meyer's multiplier theorem (Theorem 6.2; see Corollary 9.12).

Proof. As in the case of the Gaussian Riesz transforms, we follow the proof of S. Pérez and F. Soria ([223]). Again, we split these operators into a local part and a global part,

$$\begin{aligned}\mathscr{R}_\beta f(x) &= C_d \int\limits_{|x-y|<d\,m(x)} \mathscr{K}_\beta(x,y) f(y) dy + C_d \int\limits_{|x-y|\geq d\,m(x)} \mathscr{K}_\beta(x,y)|f(y)|dy \\ &= \mathscr{R}_{\beta,L} f(x) + \mathscr{R}_{\beta,G} f(x),\end{aligned}$$

where $\mathscr{R}_{\beta,L} f(x) = \mathscr{R}_\beta(f\chi_{B_h(\cdot)})(x)$ is the *local part* , $\mathscr{R}_{\beta,G} f(x) = \mathscr{R}_\beta(f\chi_{B_h^c(\cdot)})(x)$ is the *global part* of $\mathscr{R}_\beta$, and $B_h = B(x, C_d\, m(x)) = \{y \in \mathbb{R}^d : |y-x| < C_d\, m(x)\}$, is an admissible ball.

I) It has been clear, since W. Urbina's work in [278], that the local part, as in the case of the Gaussian Riesz transforms, corresponds to a classical Calderón–Zygmund singular integral.
Now, we see that the kernel $\mathscr{K}_\beta$ satisfies the decay conditions (4.29) in the local region. Observe that $r^{|\beta|-2}\left(\frac{-\log r}{1-r^2}\right)^{|\beta|-2}$ is bounded for every $r \in (0,1)$ and any $\beta, \geq 2$. We also use the fact that, $|\mathbf{H}_\beta(x)| \leq C|x|^{|\beta|}$. Then,

$$\begin{aligned}&\left|\nabla_y\left(e^{-\frac{|y-rx|^2}{1-r^2}}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\right)\right| \\ &= \Bigg(\sum_{i=1}^d e^{-\frac{|y-rx|^2}{1-r^2}}\Bigg| -\frac{2(y_i-rx_i)}{1-r^2}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right) \\ &\quad -\frac{2\beta_i}{\sqrt{1-r^2}}H_{\beta_1}\left(\frac{y_1-rx_1}{\sqrt{1-r^2}}\right)\dots H_{\beta_{i-1}}\left(\frac{y_i-rx_i}{\sqrt{1-r^2}}\right)\dots H_{\beta_d}\left(\frac{y_d-rx_d}{\sqrt{1-r^2}}\right)\Bigg|^2\Bigg)^{\frac{1}{2}} \\ &\leq C_\beta\left(\frac{|y-rx|^{|\alpha|+1}}{(1-r^2)^{\frac{|\beta|+1}{2}}} + \frac{|y-rx|^{|\beta|-1}}{(1-r^2)^{\frac{|\beta|-1}{2}}}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{1}{2}}}.\end{aligned}$$

Again, using the notation of Proposition 4.23, given $x,y \in \mathbb{R}^d$ and $t > 0$, we write $a = |x|^2+|y|^2, b = 2\langle x,y\rangle$ and $u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |x|^2$. We can conclude that the above expression is bounded by

$$\int_0^1 \left(u^{\frac{|\alpha|-1}{2}}(t) + u^{\frac{|\alpha|+1}{2}}(t)\right)\frac{e^{-u(t)}}{t^{\frac{d+3}{2}}}dt;$$

therefore, using Lemma 4.35, we have, in the local region,

$$\left|\nabla_y \mathscr{K}_\beta(x,y)\right| \leq \frac{C}{|x-y|^{d+1}}.$$

Also, it is easy to see that for the kernel $\mathscr{K}_\beta$ we have

$$|\mathscr{K}_\beta(x,y)| \le C_{|\beta|} \int_0^1 (u(t))^{|\beta|/2} \frac{e^{-u(t)}}{t^{(d+2)/2}} dt.$$

Therefore, again using Lemma 4.35, with exponent $d-1$ instead of d, we get

$$|\mathscr{K}_\beta(x,y)| \le \frac{C}{|x-y|^d}.$$

Therefore, we can apply Theorem 4.30 to $\mathscr{K}_\beta$ and the operator determined by it.

II) For the global part of $\mathscr{R}_\alpha$, we use a generalization of Theorem 9.4.

First, let us consider the following kernel:

Definition 9.7. *For each $m \ge 2$ the m-*modified maximal Gaussian kernel is defined as

$$\overline{\mathscr{K}}_m(x,y) = \begin{cases} (|x+y||x-y|)^{\frac{m-2}{2}} \overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle \le 0 \\ (|x+y||x-y|)^{\frac{m-2}{2}} \left(|x+y||x-y|)^{\frac{1}{2}} \frac{|x|\,|y|}{|x|^2+|y|^2} + 1 \right) \overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle \ge 0 \end{cases} \tag{9.21}$$

where $\overline{\mathscr{K}}$ is the Gaussian maximal kernel defined in (4.40), and define the m-modified maximal operator

$$\overline{T}_m f(x) = \int_{\mathbb{R}^d} \overline{\mathscr{K}}_m(x,y) f(y) dy. \tag{9.22}$$

Theorem 9.8. *(Pérez–Soria) For the kernel $\mathscr{K}_\beta$ of the Gaussian Riesz transform of order β, $|\beta| \ge 2$. Then, we have*

$$|\mathscr{K}_\beta(x,y)| \le C\overline{\mathscr{K}}_{|\beta|}(x,y), \tag{9.23}$$

on the region $|x-y| > C_d(1 \wedge 1/|x|)$.

Proof. Observe that the function $r^{|\beta|-2}\left(\frac{-\log r}{1-r^2}\right)^{(|\beta|-2)/2}$ is bounded for any $r \in (0,1)$ and any $\beta \ge 2$. Again, using the fact that $|\mathbf{H}_\beta(x)| \le C|x|^{|\beta|}$, and making the change of variables $t = 1-r^2$, we get

$$|\mathscr{K}_\beta(x,y)| \le C \int_0^1 \left| \mathbf{H}_\beta\left(\frac{y-\sqrt{1-t}x}{\sqrt{t}}\right)\right| \frac{e^{-\frac{|y-\sqrt{1-t}x|^2}{t}}}{t^{\frac{d+2}{2}}} dt \le C \int_0^1 \frac{u^{|\beta|/2}(t) e^{-u(t)}}{t^{\frac{d+2}{2}}} dt.$$

Thus, it is enough to prove that the last integral is bounded by $\mathscr{K}^*_{|\beta|}(x,y)$. We need to analyze two cases:

- Case #1: $b = 2\langle x,y\rangle \le 0$. In this case, we see that

$$\int_0^1 \frac{u^{|\beta|/2}(t)e^{-u(t)}}{t^{\frac{d+2}{2}}} dt \le Ca^{\frac{|\beta|-2}{2}} e^{-|y|^2},$$

Using the inequality (4.76):

$$\frac{a}{t} - |x|^2 \leq u(t) \leq \frac{2a}{t},$$

from Proposition 4.23, the change of variables $a\left(\frac{1}{t}-1\right) = s$, and the fact that, in the global region, $a > 1/2$, we obtain,

$$\int_0^1 \frac{u^{|\beta|/2}(t)e^{-u(t)}}{t^{\frac{d+2}{2}}}dt \leq e^{-|y|^2}\int_0^1 \exp\left(-\frac{a}{t}+a\right)\left(\frac{2a}{t}\right)^{|\alpha|/2}\frac{dt}{t^{\frac{d}{2}+1}}$$

$$\leq C_\beta a^{\frac{|\beta|-2}{2}} e^{-|y|^2}\int_0^\infty e^{-s}(2s+1)^{\frac{d+|\beta|-2}{2}}ds \leq Ca^{\frac{|\beta|-2}{2}}e^{-|y|^2}.$$

- Case #2: $b = 2\langle x,y\rangle > 0$.
  Using the same argument as in Theorem 9.4, we have that for $d \geq 2$ (4.78) holds,

$$\frac{e^{-\frac{d-2}{d}u(t)}}{t^{\frac{d-2}{2}}} \leq C\frac{e^{-\frac{d-2}{d}u_0}}{t_0^{\frac{d-2}{2}}}.$$

  Then, using Lemma 4.37 for $\nu = 2/d$, we get

$$\int_0^1 u^{|\alpha|/2}(t)e^{-\frac{2u(t)}{d}}\frac{dt}{t^2} \leq \frac{C_d e^{-\frac{2u_0}{2}}}{t_0}\left(u_0^{\frac{|\alpha|-1}{2}}\frac{b}{a}u(t_0)^{\frac{|\alpha|-2}{2}}+1\right),$$

  because $u_0 \leq |x+y||x-y|, b/a \leq 2|x||y|/(|x|^2+|y|^2)$ and $d \leq |x+y||x-y|$ if $\langle x,y\rangle \geq 0$ and $|x-y| > C_d(1\wedge 1/|x|)$. □

Similar to the case of the Riesz transforms, the symmetry of the non-exponential factor of the kernel $\overline{\mathscr{K}}_{|\beta|}(x,y)$ allows us to obtain that the adjoint operator to the higher-order Riesz transforms are also of weak type $(1,1)$ with respect to the Gaussian measure, as

$$\overline{\mathscr{K}_{|\alpha|}}^*(x,y) = \overline{\mathscr{K}}_{|\alpha|}(y,x)e^{|y|^2-|x|^2}.$$

As mentioned already, the main goal of C. Gutiérrez, C. Segovia, and J. L. Torrea's article [124, Chapter 4] is, also following Stein's scheme in [253], to prove Theorems 9.2 using higher-order Littlewood–Paley functions. To do so, they first get the following identity: given a multi-index $\beta \in \Gamma_k$ of order k, i.e., $|\beta| = k$ using translated Poisson semigroups $\{P_t^{(k)}\}_{t\geq 0}$ (see 3.56),

$$\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathscr{R}_\beta f)(x) = (-\frac{1}{\sqrt{2}})^k \partial^\beta P_t f(x). \tag{9.24}$$

To prove this identity, it is enough to check it for the Hermite polynomials $\{\mathbf{H}_\nu\}$. From (9.16) and (1.60),

$$\begin{aligned}
\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathscr{R}_\beta \mathbf{H}_\nu(x)) &= \Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathbf{H}_{\nu-\beta})(x)\\
&= \Big(\frac{2}{|\nu|}\Big)^{k/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\\
&\qquad\times\frac{\partial^k}{\partial t^k}\Big(e^{-\sqrt{|\nu-\beta|+k}\,t}\big)\mathbf{h}_{\nu-\beta}(x)\\
&= \Big(\frac{2}{|\nu|}\Big)^{k/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\Big)\frac{\partial^k}{\partial t^k}(e^{-\sqrt{|\nu|}\,t})\mathbf{H}_{\nu-\beta}(x)\\
&= (-\frac{1}{\sqrt{2}})^k e^{-\sqrt{|\nu|}\,t}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta}(x)\\
&= (-\frac{1}{\sqrt{2}})^k e^{-\sqrt{|\nu|}\,t}(\partial^\beta \mathbf{H}_\nu)(x) = (-\frac{1}{\sqrt{2}})^k(\partial^\beta P_t \mathbf{H}_\nu)(x).
\end{aligned}$$

Then, let $\mathscr{R}_k f = (\mathscr{R}_\beta f)_{\beta\in\Lambda_k}$

$$\begin{aligned}
\mathbf{g}^k_{t,\gamma}(\mathscr{R}_k f)(x) &= \Big(\int_0^\infty \sum_{\beta\in\Lambda_k}\Big|t^k\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathscr{R}_\beta f)(x)\Big|^2\frac{dt}{t}\Big)^{1/2}\\
&= C\left(\int_0^{+\infty}\Big|t^k(\partial^\beta P_t f)(x)\Big|^2\frac{dt}{t}\right)^{\frac{1}{2}} = C g^k_{x,\gamma} f(x).
\end{aligned}$$

Therefore, using Theorem 5.13, we get

$$\|\,|\mathscr{R}_k f|\,\|_{p,\gamma} \le C_p\|\mathbf{g}^k_{t,\gamma}(\mathscr{R}_k f)\|_{p,\gamma} = C_p\|g^k_{x,\gamma}(f)\|_{p,\gamma} \le C_p\|f\|_{p,\gamma}.$$

Thus, we get the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_\beta$, for any β, $|\beta|>1$ with constants independent of dimension.

The Riesz transforms of order 2 are of weak type $(1,1)$ with respect to the Gaussian measure, that is, they map $L^1(\gamma_d)$ into $L^{1,\infty}(\gamma_d)$. This result has been shown, by L. Forzani and R. Scotto for the case $d=1$ in [86], and for general $d>1$ by J. García-Cuerva, G. Mauceri, P. Sjögren and J. L. Torrea in [102], but their proof contains a gap. Additionally, S. Pérez and F. Soria [223] have an alternative proof using the fact that the 2-modified maximal Gaussian kernel $\overline{\mathscr{K}}_2$ bounds the kernels of the Gaussian Riesz transforms of order 2, based on the following result (see [223, Theorem 4.4]):

Theorem 9.9. *The operator $\overline{T}_2$ is of weak type $(1,1)$ with respect to the Gaussian measure.*

The proof of this theorem involves heavily all the arguments used to prove Theorem 4.24, with some slight modifications. In particular, it is important to recall some of the notation and facts:

- Let $\alpha := \alpha(x,y) = \sin \angle(x,y)$, where $\angle(x,y) \in [0,\pi]$ denotes the shortest angle between the vectors x and y if $\langle x,y\rangle > 0$, we have $\angle(x,y) \in [0,\pi/2]$).
- Define for $k = 1,2$ and $l \in \mathbb{N}$

$$\Gamma_l^k(x) = \left\{ y : \langle x,y\rangle > 0, |x| \leq |y|, \alpha(x,y) \leq l/|x|^k \right\},$$

(see (4.51)).
- Then, for fixed values of k and l, the average operator, defined by

$$\mathtt{T}_l^k f(x) = \frac{1}{\gamma_d(\Gamma_l^k(x))} \int_{\Gamma_l^k(x)} |f(y)| e^{-|y|^2} dy$$

(see (4.52)), is of weak type $(1,1)$ with respect to γ_d.

The arguments follow closely the proof of Lemmas 4.25, 4.26 and 4.27 (see also [185, Lemma 2.6, 2.7 and 2.8]).

Proof. Without loss of generality, we may assume that $f \geq 0$. As the operator $\overline{T}$, defined in (4.46), is of weak type $(1,1)$ with respect to the Gaussian measure (see Theorem 4.24), and $\overline{\mathscr{K}}_2(x,y)$ is dominated by $\overline{\mathscr{K}}(x,y)$ if $\langle x,y\rangle > 0$ and $|x| \leq 10$, or on the local region, the operator $\overline{T}_2$ is also of weak type $(1,1)$ with respect to the Gaussian measure on those regions. Thus, it remains to consider the case when we are outside of those regions.

When $|x| > |y|$, as $|x+y||x-y| > d$, the kernel $\overline{\mathscr{K}}_2(x,y)$ satisfies

$$\begin{aligned}
\overline{\mathscr{K}}_2(x,y) &\leq \frac{|x+y|^d}{(|x+y||x-y|)^{(d-1)/2}} \exp\Big(- \frac{|y|^2 - |x|^2}{2} - \frac{|x+y||x-y|}{2} \Big) \\
&\leq C|x|^d \exp\Big(- \frac{|x||x-y|}{2} \Big) e^{|x|^2 - |y|^2}.
\end{aligned}$$

It is easy to check that $\overline{\mathscr{K}}_2(x,y)e^{|y|^2-|x|^2} \in L^1(\gamma_d)$, uniformly in y; thus, the operator is of strong type (p,p), $1 < p < \infty$ with respect to the Gaussian measure in the global region.

Next, we consider for $\langle x,y\rangle > 0$ and $|x| > 10$ two operators defined $\tilde{\mathtt{T}}_1$ and $\tilde{\mathtt{T}}_2$ defined by the restriction of $\overline{\mathscr{K}}_2(x,y)$ to the regions,

$$\begin{aligned}
\mathtt{B}_1 = \Big\{ &(x,y) \in \mathbb{R}^{2d} : y \notin B_h(x), \langle x,y\rangle > 0, |x| \leq |y| \text{ and } \alpha(x,y) > 1/|x| \\
&\qquad \text{or } |x| \leq 2|y|, \alpha(x,y) \leq 1/|x| \Big\}, \\
\mathtt{B}_2 = \Big\{ &(x,y) \in \mathbb{R}^{2d} : y \notin B_h(x) : \langle x,y\rangle > 0, |y|/2 \leq |x| < |y|, \alpha(x,y) \leq 1/|x| \Big\},
\end{aligned}$$

respectively.

On $\mathtt{B}_1$, we have $\overline{\mathscr{K}}_2(x,y) \leq C|x|\overline{\mathscr{K}}(x,y)$; therefore $\tilde{\mathtt{T}}_1 f(x) \leq \overline{\mathtt{T}}_1 f(x)$, where $\overline{\mathtt{T}}_1$ corresponds to the operator associated with the restriction of $\overline{\mathscr{K}}(x,y)$ on $\mathtt{B}_1$. Now, from the estimate (4.54) we obtain,

$$\tilde{\mathrm{T}}_1 f(x) \leq C\mathrm{T}_1^1 f(x) + C\sum_{m\geq 1} e^{-m^2/4}\mathrm{T}_{m+1}^1 f(x),$$

similar to the proof of Lemma 4.25.

To estimate $\tilde{\mathrm{T}}_2$, we follow the same arguments and notation as in the proof of Lemma 4.26. We have that

$$\overline{\mathscr{K}}_2(x,y) \leq CA^{1/2}\overline{\mathscr{K}}(x,y), \text{ if } y \in \Lambda(x),$$

as $A \geq c\alpha(x,y)|x|^2$, and

$$\overline{\mathscr{K}}_2(x,y) \leq C\alpha(x,y)^{1/2}\overline{\mathscr{K}}(x,y), \text{ if } y \in \overline{\Gamma_1^1(x)} \setminus \Lambda(x),$$

as $A \leq C\alpha(x,y)|x|^2$.

Consider now the average operator,

$$\tilde{\mathscr{A}_2} f(x) = \frac{1}{\gamma_d(\Lambda(x))}\int_{\Lambda(x)} G_2(x,y)f(y)e^{-|y|^2}dy,$$

with $G_2(x,y) = A^{-d/2}G(x,y)$ and $G(x,y) = A^{-d/2}e^{\alpha^2|y|^4/16A}$. Then, we conclude with the same arguments as in the proof of Lemma 4.26, that

$$\tilde{\mathrm{T}}_2 f(x) \leq C\sum_{l\geq 2}^{\infty} e^{-\delta l}\mathrm{T}_l^2 f(x) + C\tilde{\mathscr{A}} f(x).$$

The value of $\delta > 0$ can be chosen as before.

It remains only to show that $\tilde{\mathscr{A}_2}$ is of weak type $(1,1)$ and the proof of that is similar to the proof of Lemma 4.27, replacing $G(x,y)$ by $G_2(x,y)$. □

Moreover, the Gaussian higher-order Riesz transforms $\mathscr{R}_\beta$ are of weak type $(1,1)$ with respect to the Gaussian measure if and only if $|\beta| \leq 2$; equivalently, it can be proved that the result breaks down for $|\beta| > 2$. This is a surprising result, compared with the classical case, and it was initially proved by R. Scotto and L. Forzani in the one-dimensional case in [86]. The case for higher dimensions $d > 1$ was considered by J. L. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea in [102], even though there are certain technical issues in their proof, and by S. Pérez and F. Soria [223]. This fact implies then that the theory of Gaussian singular integrals is different from the classical Calderón–Zygmund and, in particular, it cannot be developed using interpolation results.

Now, let us discuss the counterexample that Riesz transforms of at least order three are not of weak type $(1,1)$ with respect to the Gaussian measure. This is taken from [102]. The idea of the counterexample is to consider a function $f \in L^1(\gamma_d)$ which is "equivalent" to a point mass at $y \in \mathbb{R}^d$ properly normalized in $L^1(\gamma_d)$, that is to say, $f \sim e^{|y|^2}\delta_y$, for $|y|$ large.

Theorem 9.10. *Let $|\beta| \geq 3$. Then, the Riesz transform $\mathscr{R}_\beta$ is not of weak type $(1,1)$ with respect to the Gaussian measure.*

Proof. Let $y \in \mathbb{R}^d$ with $|y| = \eta$ large and $y_i \geq C\eta,\ i = 1,\cdots,d$. Write $x \in \mathbb{R}^d$ as $x = \xi\frac{y}{\eta} + v$ with $\xi \in \mathbb{R}$ and $v \perp y$. Consider the tubular region

$$\mathbf{J} = \{x \in \mathbb{R}^d : x = \xi\frac{y}{\eta} + v \quad \text{with} \quad \eta/2 < \xi < 3\eta/4, v \perp y,\ |v| < 1\}.$$

It follows that for $x \in \mathbf{J}$, there is a $C > 0$ so that

$$\frac{y_i - rx_i}{\sqrt{1-r^2}} \geq \frac{C\eta}{\sqrt{1-r^2}} \geq C\eta,\ i = 1,\cdots,d. \tag{9.25}$$

Hence,

$$\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right) > C|y|^\beta.$$

In particular, the integrand in (9.18) is positive for $0 < r < 1$, and observe that

$$e^{-\frac{|y-rx|^2}{1-r^2}} = e^{\xi^2-\eta^2}e^{-\frac{|\xi-r\eta|^2+r^2|v|^2}{1-r^2}}, \tag{9.26}$$

so that for $1/4 < r < 3/4$ and $x \in \mathbf{J}$

$$e^{-\frac{|y-rx|^2}{1-r^2}} \geq e^{\xi^2-\eta^2}e^{-C|\xi-r\eta|^2}. \tag{9.27}$$

These estimates imply that

$$|\mathscr{K}_\beta(x,y)| \geq C_d\eta^\beta e^{\xi^2-\eta^2}\int_{1/4}^{3/4} e^{-C|\xi-r\eta|^2}dr \geq C_d\eta^{|\beta|-1}e^{\xi^2-\eta^2}.$$

for $x \in \mathbf{J}$.

Now, let $f \in L^1(\gamma_d)$, $f \geq 0$ be a close approximation of a point mass at y, with norm $\|f\|_{1,\gamma} = 1$. Then, $\mathscr{R}_\beta f(x)$ is close to $e^{\eta^2}\mathscr{K}_\beta(x,y)$ when $x \in \mathbf{J}$. We conclude that

$$\mathscr{R}_\beta f(x) \geq C\eta^{\beta-1}e^{\xi^2} \geq C\eta^{\beta-1}e^{(\eta/2)^2},$$

for $x \in \mathbf{J}$.

On the other hand, because $\gamma_d(\mathbf{J}) \geq \frac{C}{\eta}e^{-(\eta/2)^2}$, and

$$\gamma_d(\mathbf{J}) \leq \gamma_d\left(\left\{x \in \mathbb{R}^d : \mathscr{R}_\beta f(x) > C\eta^{\beta-1}e^{(\eta/2)^2}\right\}\right) \leq C\frac{e^{-(\eta/2)^2}}{\eta^{\beta-1}}.$$

Then,

$$||\mathscr{R}_\beta f||_{1,\infty,\gamma} \geq C\eta^{|\beta|-2} \to \infty,$$

if $\eta \to \infty$, for $|\beta| \geq 3$. □

In [85], L. Forzani, E. Harboure, and R. Scotto give a different and simpler proof of this result for a more general class of Gaussian singular integrals that includes the Gaussian higher-order Riesz transforms, which are discussed later (see Theorem 9.18 and Theorem 9.19).

Additionally, S. Pérez and F. Soria [223, Theorem 4.5] obtained the following result on the boundedness of Gaussian higher-order Riesz transforms of order greater than or equal to 3 on Orlicz spaces, "near" $L^1(\gamma_d)$, using the estimates of the size of the kernel of $\mathscr{R}_\beta$.

Theorem 9.11. *The higher-order Gaussian Riesz transform $\mathscr{R}_\beta$, $|\beta| \geq 3$ is of weak type in the Orlicz space $L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)$. In other words, there exists a constant C such that*

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : |\mathscr{R}_{\beta,G} f(x)| \geq \lambda\right\}\right) \leq \frac{C}{\lambda}(\|f\|_{L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)} + 1), \tag{9.28}$$

where, as before $\mathscr{R}_{\beta,G} f(x) = \mathscr{R}_\beta(f\chi_{B_h^c(\cdot)})(x)$, is the global part of the Riesz transform $\mathscr{R}_\beta$ and $\|\cdot\|_{L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)}$ denotes the functional associated with the space $L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)$. Thus, $\mathscr{R}_{\beta,G}$ sends the space $L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)$ continuously into $L^{1,\infty}(\gamma_d)$.

Proof. From Theorem 9.8, it is enough to work with the m-modified maximal operator $\overline{T}_m$, as it controls $\mathscr{R}_{\beta,G}$, with $m = |\beta|$. Thus, we will prove that $\overline{T}_m$ satisfies (9.28) for $m \geq 3$. When we restrict ourselves to the region $|x| > |y|$, the usual arguments, which show that $\overline{T}$ or $\overline{T}_2$ are of strong type 1 (see [185, Theorem 2.3] or [223, Theorem 4.4]), tell us that $\overline{T}_m$, is also of strong type 1 in this region. This is easy to see, for $\langle x,y\rangle \leq 0$ and $|x| > |y|$then

$$\overline{\mathscr{K}}_m(x,y) \leq C|x|^m e^{|y|^2},$$

whereas $\langle x,y\rangle > 0$ and $|x| > |y|$ then

$$\begin{aligned}\overline{\mathscr{K}}_m(x,y) &\leq C(|x+y||x-y|)^{\frac{m-1}{2}}|x|^m e^{-\frac{|y|^2-|x|^2}{2}} e^{-\frac{|x-y||x+y|}{2}} e^{|x|^2-|y|^2}\\ &\leq C|x|^m \exp\left(-\frac{|x|\,|x-y|}{3}\right) e^{|x|^2-|y|^2}\end{aligned}$$

In both cases, the integral in the variable x is uniformly bounded in y and the strong type $(1,1)$ follows.

For $|x| < |y|$, we use the crude estimate

$$\overline{\mathscr{K}}_m(x,y) \leq C(|x+y||x-y|)^{\frac{m-2}{2}}\overline{\mathscr{K}}_2(x,y).$$

Hence,

$$\int_{|x|<|y|} \overline{\mathscr{K}}_m(x,y)|f(y)|dy \le C\int_{\mathbb{R}^d} \overline{\mathscr{K}}_2(x,y)|f(y)||y|^{m-2}dy.$$

We use a particular case of Young's inequality: given positive u and v, we have $u\cdot v \le u(1+\log^+ u)+e^v$, which implies with more generality that

$$u\cdot v \le \delta^k\Big(u^{1/k}\frac{v^{1/k}}{\delta}\Big)^k \le C_k\delta^k(u(1+\log^+ u)^k + e^{(k/\delta)v^{1/k}}).$$

Taking $u=|f(y)|$, $v=|y|^{m-2}$ and $k/\delta = 1/2$, we obtain

$$\begin{aligned}\int_{|x|<|y|} \overline{\mathscr{K}}_m(x,y)|f(y)|dy &\le C\int_{\mathbb{R}^d} \overline{\mathscr{K}}_2(x,y)|f(y)|(1+\log^+|f(y)|)^{m-2}\,dy \\ &\quad + \int_{\mathbb{R}^d} \overline{\mathscr{K}}_2(x,y)e^{|y|^2/2}dy \\ &= \overline{T}_2(|f|(1+\log^+|f|)^{m-2}+e^{|\cdot|^2/2})(x).\end{aligned}$$

Because $\overline{T}_2$ is of weak type $(1,1)$ with respect to the Gaussian measure, as we have seen, we conclude that

$$\begin{aligned}\gamma_d\Big(\Big\{x\in\mathbb{R}^d : \overline{T}_2(|f|(1+\log^+|f|)^{m-2}+e^{|\cdot|^2/2})(x)\ge\lambda\Big\}\Big) \\ \le \frac{C}{\lambda}\int_{\mathbb{R}^d}\Big[|f(y)|(1+\log^+|f(y)|)^{m-2}+e^{|y|^2/2}\Big]\gamma_d(dy) \\ \le \frac{C}{\lambda}(\|f\|_{L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)}+1). \qquad \square\end{aligned}$$

Finally, as was mentioned before, the $L^p(\gamma_d)$-boundedness, $1<p<\infty$, of the higher-order Riesz transforms, with constants independent of dimension, can be obtained as a consequence of Meyer's multiplier theorem (Theorem 6.2; see [88]).

Corollary 9.12. *The higher-order Gaussian Riesz transforms $\mathscr{R}_\beta$, $|\beta|>1$, are $L^p(\gamma_d)$ bounded for $1<p<\infty$, that is to say, there exists $C>0$, dependent only on p and β, but not on dimension, such that*

$$\|\mathscr{R}_\beta f\|_{p,\gamma} \le C\|f\|_{p,\gamma}, \tag{9.29}$$

for any $f\in L^p(\gamma_d)$.

Proof. Given the multi-index $\beta=(\beta_1,\dots,\beta_d)$, from (9.17), we know that the action of $\mathscr{R}_\beta$ over the normalized Hermite polynomial $\mathbf{h}_\nu$ is given by

$$\mathscr{R}_\beta\mathbf{h}_\nu(x) = \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^d \nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]^{1/2}\mathbf{h}_{\nu-\beta}(x),$$

with $\beta_i\le\nu_i$ for all $i=1,\cdots,d$, otherwise $\mathscr{R}_\beta\mathbf{h}_\nu(x)=0$.

Now, for the same multi-index β, let us consider the operator

$$\mathscr{R}_1^{\beta_1}\mathscr{R}_2^{\beta_2}\ldots\mathscr{R}_d^{\beta_d},$$

the composition of powers of the Riesz transforms, $\mathscr{R}_1^{\beta_1},\mathscr{R}_2^{\beta_2},\ldots,\mathscr{R}_d^{\beta_d}$. Then,

$$\mathscr{R}_1^{\beta_1}\mathscr{R}_2^{\beta_2}\ldots\mathscr{R}_d^{\beta_d}\mathbf{h}_\nu(x) = \Big[\prod_{i=1}^{d}\frac{\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)}{|\nu|(|\nu|-1)\ldots(|\nu|-\beta_i+1)}\Big]^{1/2}\mathbf{h}_{\nu-\beta}(x).$$

Now, define the multiplier operator T_β as

$$\begin{aligned}
T_\beta\mathbf{h}_\nu(x) &= \Big[\frac{\prod_{i=1}^{d}|\nu|(|\nu|-1)\ldots(|\nu|-\beta_i+1)}{|\nu|^{|\beta|}}\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\frac{\prod_{i=1}^{d}(|\nu|-1)\ldots(|\nu|-(\beta_i-1))}{|\nu|^{|\beta|-d}}\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{i=1}^{d}\frac{(|\nu|-1)\ldots(|\nu|-(\beta_i-1))}{|\nu|^{\beta_i-1}}\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{i=1}^{d}(1-\frac{1}{|\nu|})\ldots(1-\frac{(\beta_i-1)}{|\nu|})\Big]^{1/2}\mathbf{h}_\nu(x).
\end{aligned}$$

Then, T_β is a Meyer's multiplier (6.4), with multiplier ϕ defined using the function,

$$h(x) = [\prod_{i=1}^{d}(1-x)\ldots(1-(\beta_i-1)x)]^{1/2}.$$

By construction, T_β satisfies,

$$\mathscr{R}_\beta = \mathscr{R}_1^{\beta_1}\mathscr{R}_2^{\beta_2}\ldots\mathscr{R}_d^{\beta_d}\circ T_\beta \tag{9.30}$$

Therefore, the $L^p(\gamma_d)$ boundedness of $\mathscr{R}_\beta$ is obtained immediately from the $L^p(\gamma_d)$ boundedness of the Riesz transforms $\mathscr{R}_j$ using Meyer's multiplier theorem (Theorem 6.2), where the constant is dependent on p and β, but independent of the dimension d, as long as we have proof of the L^p-boundedness of the (first-order) Riesz transforms with constants independent of dimension[2]. □

9.3 Alternative Gaussian Riesz Transforms

We have mentioned before in Chapter 2, that the Gaussian partial derivatives in $\mathbb{R}^d$, ∂_γ^i are not self-adjoint in $L^2(\gamma_d)$, and its adjoint is given by

[2]It has been mentioned before that there are several proofs of this fact (see, for instance, G. Pisier [227] or C. Gutiérrez [122])

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_i} + \sqrt{2}x_i I_d$$

(see 2.12). The Ornstein–Uhlenbeck operator can be written as

$$(-L) = \sum_{i=1}^d (\partial_\gamma^i)^* \partial_\gamma^i.$$

Therefore, there is another "natural" differential operator, the alternative Ornstein–Uhlenbeck operator, (2.14), which is given by

$$(-\overline{L}) = \sum_{i=1}^d \partial_\gamma^i (\partial_\gamma^i)^* = (-L) + dI = -\frac{1}{2}\Delta + \langle x, \nabla_x \rangle + dI.$$

H. Aimar, L. Forzani, and R. Scotto in [5] considered the following *alternative Riesz transforms*, by taking the derivatives $(\partial_\gamma^i)^*$ and the operator $(-\overline{L})$,

$$\overline{\mathscr{R}}_j = (\partial_j^\gamma)^* (-\overline{L})^{-1/2}, \tag{9.31}$$

Moreover, we can also consider alternative higher-order Gaussian Riesz transforms, that is, for a multi-index β, $|\beta| \geq 1$ we use the gradient

$$(\partial_\gamma^\beta)^* = \frac{(-1)^{|\beta|}}{2^{|\beta|/2}} e^{|x|^2} (\partial^\beta e^{-|x|^2} I)$$

and the Riesz potentials associated with $\overline{L}$. Then, these new singular integral operators are defined as follows:

Definition 9.13. *The alternative Gaussian Riesz transform $\overline{\mathscr{R}}_\beta$ for $|\beta| \geq 1$ is defined spectrally as*

$$\overline{\mathscr{R}}_\beta f(x) = (\partial_\gamma^\beta)^* (-\overline{L})^{-|\beta|/2} f(x).$$

Thus, the action of $\overline{\mathscr{R}}_\beta$ over the Hermite polynomial $\mathbf{H}_\nu$ is given by

$$\overline{\mathscr{R}}_\beta \mathbf{H}_\nu = \frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}} \mathbf{H}_{\nu+\beta}, \tag{9.32}$$

because, using the fact that the Hermite polynomials $\{\mathbf{H}_\nu\}$ are eigenfunctions of $\overline{L}$,

$$(-\overline{L})^{-|\beta|/2} H_\nu = \frac{1}{(|\nu|+d)^{|\beta|/2}} \mathbf{H}_\nu,$$

and using Rodrigues' formula (1.28), we get

$$\begin{aligned}\overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x) &= (\partial_\gamma^\beta)^* (-\overline{L})^{-|\beta|/2} \mathbf{H}_\nu(x) = \frac{(-1)^{|\beta|}}{(|\nu|+d)^{|\beta|/2}} e^{|x|^2} \partial^\beta (e^{-|x|^2} \mathbf{H}_\nu(x)) \\ &= \frac{(-1)^{|\beta+\nu|}}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}} e^{|x|^2} \partial^{\beta+\nu} (e^{-|x|^2}) = \frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}} \mathbf{H}_{\nu+\beta}(x);\end{aligned}$$

therefore,

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+d)^{|\beta|/2}}\Big[\prod_{i=1}^{d}(\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2}\mathbf{h}_{\nu+\beta}(x), \quad (9.33)$$

because,

$$\begin{aligned}\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) &= \overline{\mathscr{R}}_\beta\Big(\frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}\Big) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}}\overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x)\\ &= \frac{1}{(2^{|\nu|}\nu!)^{1/2}}\frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}}\mathbf{H}_{\nu+\beta}(x) = \frac{1}{(\nu!)^{1/2}(|\nu|+d)^{|\beta|/2}}\frac{\mathbf{H}_{\nu+\beta}(x)}{2^{|\nu|/2+|\beta|/2}}\\ &= \frac{1}{(|\nu|+d)^{|\beta|/2}}\Big(\frac{(\nu+\beta)!}{\nu!}\Big)^{1/2}\frac{\mathbf{H}_{\nu+\beta}(x)}{(2^{|\nu+\beta|}(\nu+\beta)!)^{1/2}}\\ &= \frac{1}{(|\nu|+d)^{|\beta|/2}}\Big[\prod_{i=1}^{d}(\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2}\mathbf{h}_{\nu+\beta}(x).\end{aligned}$$

With an argument analogous to Lemma 8.3, we can get that the alternative higher-order Gaussian Riesz transforms then have the following integral representation

$$\overline{\mathscr{R}}_\beta f(x) = \text{p.v. } e^{|x|^2}\int_{\mathbb{R}^d}\overline{\mathscr{K}}_\beta(x,y)f(y)\,\gamma_d(dy)$$

where

$$\overline{\mathscr{K}}_\beta(x,y) = C_\beta\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\,dr.$$

Formally, $\overline{\mathscr{K}}_\beta$ is obtained by differentiating with respect to the adjoint of ∂^γ the kernel corresponding to the Riesz potentials associated with $\overline{L}$, (8.62),

$$\begin{aligned}(-\overline{L})^{-|\beta|/2}f(x) &= \frac{1}{\Gamma(|\beta|/2)}\int_0^\infty t^{\frac{|\beta|-2}{2}}T_t^{(d)}f(x)\,dt\\ &= C_\beta e^{|x|^2}\int_{\mathbb{R}^d}\Big(\int_0^1(-\log r)^{\frac{|\beta|-2}{2}}r^d\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\frac{dr}{r}\Big)f(y)\,\gamma_d(dy).\\ &= C_\beta\int_{\mathbb{R}^d}\Big(\int_0^1(-\log r)^{\frac{|\beta|-2}{2}}r^d\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\frac{dr}{r}\Big)f(y)\,dy.\end{aligned}$$

Similar to Corollary 9.12, the $L^p(\gamma_d)$ boundedness of $\overline{\mathscr{R}}_\beta$, $1<p<\infty$ can be obtained from P. A. Meyer's multiplier theorem (Theorem 6.2).

Corollary 9.14. *The alternative Gaussian Riesz transforms $\overline{\mathscr{R}}_\beta$ are $L^p(\gamma_d)$ bounded for $1<p<\infty$, that is to say, there exists $C>0$, dependent only on p and β, but not on dimension, such that*

$$\|\overline{\mathscr{R}}_\beta f\|_{p,\gamma} \le C\|f\|_{p,\gamma}, \quad (9.34)$$

for any $f\in L^p(\gamma_d)$.

Proof. Given the multi-index $\beta = (\beta_1, \ldots, \beta_d)$, from (9.33), we know that the action of $\mathscr{R}_\beta$ over the normalized Hermite polynomial $\mathbf{h}_\nu$ is given by

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+d)^{|\beta|/2}} \Big[\prod_{j=1}^{d} (\nu_j+\beta_j)\cdots(\nu_j+1)\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x).$$

Now, for the same multi-index β, let us consider the operator

$$\overline{\mathscr{R}}_1^{\beta_1} \overline{\mathscr{R}}_2^{\beta_2} \ldots \overline{\mathscr{R}}_d^{\beta_d},$$

the composition of powers of the Riesz transforms, $\overline{\mathscr{R}}_1^{\beta_1}, \overline{\mathscr{R}}_2^{\beta_2}, \ldots \overline{\mathscr{R}}_d^{\beta_d}$. Then,

$$\begin{aligned}\overline{\mathscr{R}}_1^{\beta_1} \overline{\mathscr{R}}_2^{\beta_2} \ldots \overline{\mathscr{R}}_d^{\beta_d} \mathbf{h}_\nu(x) &= \prod_{j=1}^{d} \Big(\prod_{i=1}^{\beta_j} \Big(\frac{\nu_j+i}{|\nu|+d+(i-1)}\Big)\Big)^{1/2} \mathbf{h}_{\nu+\beta}(x) \\ &= \Big[\prod_{j=1}^{d} \frac{(\nu_j+\beta_j)\cdots(\nu_j+1)}{(|\nu|+d+\beta_j-1)\cdots(|\nu|+d)}\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x)\end{aligned}$$

Consider the multiplier T_β defined as

$$\begin{aligned}T_\beta \mathbf{h}_\nu(x) &= \Big[\frac{\prod_{j=1}^{d}(|\nu|+d+\beta_j-1)\cdots(|\nu|+d)}{(|\nu|+d)^{|\beta|}}\Big]^{1/2} \mathbf{h}_\nu(x) \\ &= \Big[\frac{\prod_{j=1}^{d}(|\nu|+d+\beta_j-1)\cdots(|\nu|+2)}{(|\nu|+d)^{|\beta|-d}}\Big]^{1/2} \mathbf{h}_\nu(x) \\ &= \Big[\prod_{j=1}^{d} \Big(\frac{|\nu|+d+\beta_j-1)\cdots(|\nu|+d+1)}{(|\nu|+d)^{\beta_j-1}}\Big)\Big]^{1/2} \mathbf{h}_\nu(x) \\ &= \Big[\prod_{j=1}^{d} \Big(\frac{(|\nu|+d)+(\beta_j-1)}{|\nu|+d}\Big)\cdots\frac{(|\nu|+d)+1}{|\nu|+d}\Big)\Big]^{1/2} \mathbf{h}_\nu(x) \\ &= \Big[\prod_{j=1}^{d} \Big(1+\frac{(\beta_j-1)}{|\nu|+d}\Big)\cdots\Big(1+\frac{1}{|\nu|+d}\Big)\Big]^{1/2} \mathbf{h}_\nu(x)\end{aligned}$$

By construction, T_β satisfies,

$$\overline{\mathscr{R}}_\beta = \overline{\mathscr{R}}_1^{\beta_1} \overline{\mathscr{R}}_2^{\beta_2} \ldots \overline{\mathscr{R}}_d^{\beta_d} \circ T_\beta \tag{9.35}$$

As in the case of the Gaussian Bessel potentials, T_β is the composition of two Meyer's multipliers 6.4), one of the multipliers defined using the function,

$$h(x) = \Big[\prod_{j=1}^{d} (1+x(\beta_j-1))\cdots(1+x)\Big]^{1/2}.$$

Therefore, the $L^p(\gamma_d)$ boundedness of $\overline{\mathscr{R}}_\beta$ is obtained immediately from the $L^p(\gamma_d)$ boundedness of the Riesz transforms $\overline{\mathscr{R}}_j$ using Meyer's multiplier theorem (Theorem 6.2), where the constant is dependent on p and β, but independent of the dimension d, as long as we have proof of the L^p-boundedness of the (first-order) Riesz transforms with constants independent of dimension. □

In [5], H. Aimar, L. Forzani, and R. Scotto obtained a surprising result: the alternative Riesz transforms $\overline{\mathscr{R}}_\beta$ are of weak type $(1,1)$ for all multi-index β, i.e., independently of their orders, which is a contrasting fact with respect to the anomalous behavior of the higher-order Riesz transforms $\mathscr{R}_\beta$.

Theorem 9.15. *For any multi-index β, there exists a constant C dependent only on d and β such that for all $\lambda > 0$, $f \in L^1(\gamma_d)$, we have*

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : \overline{\mathscr{R}}_\beta f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|\gamma_d(dy),$$

i.e., $\overline{\mathscr{R}}_\beta f$ is of γ_d-weak type $(1,1)$.

Proof. The main feature, to prove this theorem, is to apply Theorem 4.18 with an special Φ. For each $x \in \mathbb{R}^d$, as usual, we write this operator as the sum of two operators that are obtained by splitting $\mathbb{R}^d$ into a local region, $B_h(x) = \{y \in \mathbb{R}^d : |y-x| < C_d m(x)\}$, an admissible ball and its complement $B_h^c(x)$ called the global region. Thus,

$$\overline{\mathscr{R}}_\beta f(x) = \overline{\mathscr{R}}_{\beta,L} f(x) + \overline{\mathscr{R}}_{\beta,G} f(x)$$

where $\overline{\mathscr{R}}_{\beta,L} f(x) = \overline{\mathscr{R}}_\beta(f\chi_{B_h(\cdot)})(x)$ is the local part of $\mathscr{R}_\beta$ and $\overline{\mathscr{R}}_{\beta,G} f(x) = \overline{\mathscr{R}}_\beta(f(\chi^c_{B_h(\cdot)})(x)$ is the global part of $\mathscr{R}_\beta$.

We prove that these two operators are γ_d-weak type $(1,1)$; thus, also $\overline{\mathscr{R}}_\beta$ is weak type $(1,1)$. To prove that $\overline{\mathscr{R}}_{\beta,L}$ is of γ-weak type $(1,1)$, we apply Theorem 4.30. In our case,

$$Tf(x) = p.v.\int_{\mathbb{R}^d} \mathscr{K}(x,y) f(y)dy$$

with

$$\begin{aligned}\mathscr{K}(x,y) &= e^{|x|^2}\overline{\mathscr{K}}_\beta(x,y)e^{-|y|^2}\\ &= C_\beta\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\,dr.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial\mathscr{K}}{\partial y_j}(x,y) = 2C_\beta\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}&\left[\frac{-r\beta_j}{\sqrt{1-r^2}}\mathbf{H}_{\beta-\mathbf{e_j}}\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right.\\ &\left.+\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{(rx_j-y_j)}{1-r^2}\right]\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\,dr.\end{aligned}$$

Now, we show that the hypotheses of Theorem 4.30 are fulfilled for this operator. Thus, we prove that, in the local region $B_h(x)$, we have,

$$|\mathscr{K}(x,y)| \leq \frac{C}{|x-y|^d}$$

and

$$|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)| \leq \frac{C}{|x-y|^{d+1}}.$$

There exists a constant $C>0$ such that for every $y \in B_h$ $C^{-1} \leq e^{|y|^2-|x|^2} \leq C$, then

$$|\mathscr{K}(x,y)| \leq C|e^{-|x|^2+|y|^2}\mathscr{K}(x,y)| = C|\overline{\mathscr{K}}_\beta(x,y)|$$

and

$$\left|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)\right| \leq C\left|e^{-|x|^2+|y|^2}\frac{\partial \overline{\mathscr{K}}}{\partial y_j}(x,y)\right|.$$

On the other hand, on B_h, for any $c>0$,

$$e^{-c\frac{|x-ry|^2}{1-r^2}} = e^{-c\frac{|x-y|^2}{1-r^2}}e^{-c\frac{1-r}{1+r}|y|^2}e^{-c\frac{(x-y)\cdot y}{1-r}} \leq Ce^{-c\frac{|x-y|^2}{1-r}};$$

thus, with thisinequality and taking into account that $t^m e^{-ct^2} \leq C_m$, for all $t \geq 0$, we get

$$\left|\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right|e^{-\frac{|x-ry|^2}{1-r^2}} \leq C\sum_{m=0}^{|\beta|}\left|\frac{x-ry}{\sqrt{1-r^2}}\right|^m e^{-\frac{|x-ry|^2}{2(1-r^2)}}e^{-\frac{|x-ry|^2}{2(1-r^2)}} \leq Ce^{-c\frac{|x-y|^2}{1-r}}.$$

Therefore, by combining all the above remarks, on B_h we have,

$$\begin{aligned}|\mathscr{K}(x,y)| &\leq C\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}}\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d}{2}+1}}\,dr\\ &\leq C\left[\int_0^{\frac{1}{2}}(-\log r)^{\frac{|\beta|-2}{2}}\,dr+\int_{\frac{1}{2}}^1\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d}{2}+1}}\,dr\right] \leq C\left(1+\frac{1}{|x-y|^d}\right) \leq \frac{C}{|x-y|^d}\end{aligned}$$

and

$$\begin{aligned}|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)| &\leq C\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}}\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d+3}{2}}}\,dr\\ &\leq C\left[\int_0^{\frac{1}{2}}(-\log r)^{\frac{|\beta|-2}{2}}\,dr+\int_{\frac{1}{2}}^1\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d+3}{2}}}\,dr\right]\\ &\leq C\left(1+\frac{1}{|x-y|^{d+1}}\right) \leq \frac{C}{|x-y|^{d+1}}.\end{aligned}$$

Let us prove that the operator $\overline{\mathscr{R}}_\beta$ is bounded on $L^2(\gamma_d)$. Given $f \in L^2(\gamma_d)$, with Hermite expansion $f = \sum_\nu \widehat{f}_\gamma(\nu)\mathbf{h}_\nu = \sum_\nu \langle f, \mathbf{h}_\nu\rangle_\gamma \mathbf{h}_\nu$. Then, because the action of $\overline{\mathscr{R}}_\beta$ over the normalized Hermite polynomial $\mathbf{h}_\nu$ is given by (9.33),

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{2^{|\beta|/2}} \frac{\prod_{j=1}^d [(\nu_j+d)\cdots(\nu_j+\beta_j)]^{\frac{1}{2}}}{(|\nu|+d)^{|\beta|/2}} h_{\nu+\beta}(x).$$

Therefore,

$$\begin{aligned}\|\overline{\mathscr{R}}_\beta f\|^2_{L^2(d\gamma)} &= \sum_\nu \frac{\prod_{j=1}^d [(\nu_j+1)\cdots(\nu_j+\beta_j)]}{2^{|\beta|}(|\nu|+d)^{|\beta|}} |\widehat{f}_\gamma(\nu)|^2 \\ &\leq \sum_\nu \prod_{j=1}^d (\beta_j+1)^{\beta_j} |\widehat{f}_\gamma(\nu)|^2 \leq (|\beta|+1)^{|\beta|} \sum_\nu |\widehat{f}_\gamma(\nu)|^2 \leq C\|f\|^2_{L^2(\gamma_d)}.\end{aligned}$$

Therefore, using Theorem 4.30, the γ_d-weak type $(1,1)$ of $\overline{\mathscr{R}}_{\beta,L}$ follows.

To prove that $\overline{\mathscr{R}}_{\beta,G}$ is also γ_d-weak type $(1,1)$, we prove on $\mathbb{R}^d \setminus B_h$,

$$|\overline{\mathscr{R}}_{\beta,G} f(x)| \leq C\mathscr{M}_\Phi f(x), \tag{9.36}$$

with $\Phi(t) = e^{-ct^2}$. Then, using Theorem 4.18, we get the weak type $(1,1)$ inequality for $\overline{\mathscr{R}}_{\beta,G}$.

$$\begin{aligned}|\overline{\mathscr{K}}_\beta(x,y)| &= \left| \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\,dr \right| \\ &\leq C\int_0^{\frac{3}{4}} (-\log r)^{\frac{|\beta|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}}\,dr \\ &\quad + C\int_{\frac{3}{4}}^{1-\zeta/|x|^2} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d-1}{2}}} \left(|x| \vee (1-r^2)^{-\frac{1}{2}}\right) \frac{dr}{|x|(1-r^2)^{3/2}} \\ &\quad + C\int_{1-\zeta/|x|^2}^1 \frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d-1}{2}}} \left(|x| \vee (1-r^2)^{-\frac{1}{2}}\right) \frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\,dr.\end{aligned}$$

Hence,

$$|\overline{\mathscr{K}}_\beta(x,y)| \leq C\left(\overline{\mathscr{K}}^1_\beta(x,y) + \overline{\mathscr{K}}^2_\beta(x,y) + \overline{\mathscr{K}}^3_\beta(x,y)\right),$$

where the inequality is obtained by annihilating the Hermite polynomial with part of the exponential, then splitting the unit interval of the integral into three subintervals $[0,3/4]$, $[3/4, 1-\zeta/|x|^2]$, and $[1-\zeta/|x|^2, 1]$ and taking into account that on the second one $|x| \vee (1-r^2)^{-1/2} \geq |x|$, on the third one $|x| \vee (1-r^2)^{-1/2} \geq (1-r^2)^{-1/2}$

and $|x-ry| \geq \bar{c}|x-y|$, and on the last two intervals, the function $-\log r/(1-r^2)$ is bounded by a constant.

Thus, by using the definition of kernels $\overline{\mathscr{K}}_\beta^j$ with $j=1,2,3$, interchanging the order of integration on each operator $\overline{\mathscr{R}}_{\beta,G}{}^j$ with $j=1,2,3$, using Lemma 1.23 and taking $\Phi(t)=e^{-ct^2}$, we get, using Fubini's theorem

$$\overline{\mathscr{R}}_{\beta,G}{}^1 f(x) = e^{|x|^2}\int_{\mathbb{R}^d}\int_0^{\frac{3}{4}}(-\log r)^{\frac{|\beta|-2}{2}}\frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d}{2}}}\,dr\,|f(y)|\,\gamma_d(dy)$$

$$= \int_0^{\frac{3}{4}}(-\log r)^{\frac{|\beta|-2}{2}}e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d}{2}}}|f(y)|\gamma_d(dy)\,dr$$

$$\leq C\int_0^{\frac{3}{4}}(-\log r)^{\frac{|\beta|-2}{2}}dr\quad \mathscr{M}_\Phi f(x) \leq C\,\mathscr{M}_\Phi f(x),$$

$$\overline{\mathscr{R}}_{\beta,G}{}^2 f(x) = e^{|x|^2}\int_{\mathbb{R}^d}\int_{\frac{3}{4}}^{1-\zeta/|x|^2}\frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d-1}{2}}}\,(|x|\vee(1-r^2)^{-\frac{1}{2}})\times\frac{dr}{|x|(1-r^2)^{3/2}}|f(y)|\,\gamma_d(dy)$$

$$= \int_{3/4}^{1-\zeta/|x|^2}e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(d-1)/2}}(|x|\vee(1-r^2)^{-1/2})|f(y)|\gamma_d(dy)\times\frac{dr}{|x|(1-r^2)^{3/2}}$$

$$\leq C\frac{1}{|x|}\int_{3/4}^{1-\zeta/|x|^2}\frac{dr}{(1-r)^{3/2}}\;\;\mathscr{M}_\Phi f(x) \leq C\mathscr{M}_\Phi f(x),$$

and, finally,

$$\overline{\mathscr{R}}_{\beta,G}{}^3 f(x) = e^{|x|^2}\int_{\mathbb{R}^d}\int_{1-\zeta/|x|^2}^1\frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d-1}{2}}}\,(|x|\vee(1-r^2)^{-\frac{1}{2}})\frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\,dr\,|f(y)|\,\gamma_d(dy)$$

$$= \int_{1-\zeta/|x|^2}^1 e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(d-1)/2}}(|x|\vee(1-r^2)^{-1/2})\frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\times|f(y)|\,\gamma_d(dy)\,dr$$

$$\leq \int_{1-\zeta/|x|^2}^1 e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(d-1)/2}}(|x|\vee(1-r^2)^{-1/2})\times\frac{1}{|x-y|^2}\,|f(y)|\gamma_d(dy)\,dr$$

$$\leq C|x|^2 \int_{1-\zeta/|x|^2}^{1} dr \;\; \mathscr{M}_\Phi f(x) \leq C\mathscr{M}_\Phi f(x).$$

Thus, because $|\overline{\mathscr{R}}_{\beta,G} f(x)| \leq C_\beta \sum_{j=1}^{3} \overline{\mathscr{R}}^{j}_{\beta,G} f(x)$, then (9.36) follows. □

9.4 Definition and Boundedness Properties of General Gaussian Singular Integrals

Finally, we define *general Gaussian singular integrals*, generalizing the Gaussian higher-order Riesz transforms. We follow, essentially, the outline developed for them. The first formulation of general Gaussian singular integrals was given by W. Urbina in [278]. Later, S. Pérez [221] extended it. We consider S. Pérez's class, as it is a much larger class.

Definition 9.16. *Given a C^1-function F, satisfying the orthogonality condition*

$$\int_{\mathbb{R}^d} F(x)\gamma_d(dx) = 0, \tag{9.37}$$

and such that for every $\varepsilon > 0$, there exist constants, C_ε y C'_ε such that

$$|F(x)| \leq C_\varepsilon e^{\varepsilon|x|^2} \quad \text{and} \quad |\nabla F(x)| \leq C'_\varepsilon e^{\varepsilon|x|^2}. \tag{9.38}$$

Then, for each $m \in \mathbb{N}$ the generalized Gaussian singular integral is defined as

$$T_{F,m} f(x) = \int_{\mathbb{R}^d} \int_0^1 \Big(\frac{-\log r}{1-r^2}\Big)^{\frac{m-2}{2}} r^m F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r} f(y)dy. \tag{9.39}$$

$T_{F,m}$ can be written as

$$T_{F,m} f(x) = \int_{\mathbb{R}^d} \mathscr{K}_{F,m}(x,y) f(y)dy,$$

denoting,

$$\begin{aligned}
\mathscr{K}_{F,m}(x,y) &= \int_0^1 \Big(\frac{-\log r}{1-r^2}\Big)^{\frac{m-2}{2}} r^{m-1} F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} dr \\
&= \int_0^1 \varphi_m(r) F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} dr \\
&= \int_0^1 \psi_m(t) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt,
\end{aligned} \tag{9.40}$$

with $\varphi_m(r) = \big(\frac{-\log r}{1-r^2}\big)^{\frac{m-2}{2}} r^{m-1}$; and taking the change of variables $t = 1-r^2$, with $\psi_m(t) = \varphi_m(\sqrt{1-t})/\sqrt{1-t}$, and $u(t) = \frac{|\sqrt{1-t}x-y|^2}{t}$.

In [278], instead of condition (9.38), it was asked that F and ∇F would have at most polynomial growth, which of course is a particular case of (9.38). On the other hand, the higher-order Riesz transforms $\mathscr{R}_\beta$ are clearly particular cases of the operators $T_{F,m}$ by simply taking $F = H_\beta$, and $m = |\beta|$.

We will prove that the operator $T_{F,m}$ is a bounded operator in $L^p(\gamma_d)$, $1 < p < \infty$.

Theorem 9.17. *The operators $T_{F,m}$ are $L^p(\gamma_d)$-bounded for $1 < p < \infty$; that is to say, there exists $C > 0$, dependent only on p and on dimension such that*

$$\|T_{F,m}f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}, \tag{9.41}$$

for any $f \in L^p(\gamma_d)$.

Proof. As usual, we split $T_{F,m}$ into its local part and its global part,

$$T_{F,m}f(x) = T_{F,m}(f\chi_{B_h(\cdot)})(x) + T_{F,m}(f\chi_{B_h^c(\cdot)})(x) = T_{F,m,L}f(x) + T_{F,m,G}f(x).$$

I) For the local part $T_{F,m,L}$, we prove that it is always of weak type $(1,1)$. The estimates needed follow from an idea that appeared initially in W. Urbina's article [278], that the local part differs from a Calderón–Zygmund singular integral by an operator that is $L^1(\gamma_d)$-bounded; in other words, the operator is defined by the difference of $T_{F,m}$ and an appropriate approximation of it (which is an operator defined as the convolution with a Calderón–Zygmund kernel) is $L^1(\mathbb{R}^d)$-bounded.

- First, observe that if F satisfies the orthogonality condition (9.37) and (9.38), setting
$$K(x) = \int_0^\infty F\Big(-\frac{x}{t^{1/2}}\Big)e^{-|x|^2/t}\frac{dt}{t^{d/2+1}},$$
then, K is a Calderón–Zygmund kernel of convolution type (see (4.67)), as the integral is absolutely convergent when $x \neq 0$. Taking the change of variables $s = |x|/t^{1/2}$ we get
$$K(x) := \frac{2\int_0^\infty F\Big(-\frac{x}{|x|}s\Big)e^{-s^2}s^{d-1}ds}{|x|^d} = \frac{\Omega(x)}{|x|^d},$$
with Ω homogeneous of degree zero; therefore, K is homogeneous of degree $-d$. Moreover, Ω is C^1 with mean zero on S^{d-1}, because
$$\begin{aligned}\int_{S^{d-1}}\Omega(x')d\sigma(x') &= 2\int_0^\infty\int_{S^{d-1}}F(-x's)d\sigma(x')e^{-s^2}s^{d-1}ds\\ &= 2\int_{\mathbb{R}^d}F(-y)e^{-|y|^2}dy = 0.\end{aligned}$$
Therefore, according to the classical Calderón–Zygmund theory, the convolution operator defined using convolution with the kernel K is continuous in $L^p(\mathbb{R}^d)$, $1 < p < \infty$ and weak type $(1,1)$, with respect to the Lebesgue measure. Therefore, using Theorem 4.32, its local part S_L is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and of weak type $(1,1)$ with respect to γ_d.

- Second, we need to get rid of the function ψ_m. Taking a limit from the right, we can define $\psi(0) := \psi_m(0^+) = 2^{-(m-2)/2}$, then ψ_m is continuous on $[0,1)$. Moreover,

$$|\psi_m(t) - \psi_m(0)| \leq C\frac{t}{\sqrt{1-t}}.$$

Thus, from (9.40), we can write,

$$\begin{aligned}\mathscr{K}_{F,m}(x,y) &= \psi_m(0)\int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt\\ &\quad + \int_0^1 (\psi_m(t)-\psi_m(0))F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt.\end{aligned}$$

Set

$$K_1(x,y) := \int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt,$$

Now, over the local part we know that $u(t) \geq |y-x|^2/t - 2d$, then, using condition (9.38), we get

$$\begin{aligned}&\int_0^1 |\psi_m(t)-\psi_m(0)|\Big|F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\Big|\frac{e^{-u(t)}}{t^{d/2+1}}dt\\ &\qquad \leq \int_0^1 |\psi_m(t)-\psi_m(0)|\Big|F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\Big|\frac{e^{-u(t)}}{t^{d/2+1}}dt\\ &\qquad \leq C\int_0^1 \frac{e^{-(1-\varepsilon)u(t)}}{t^{d/2}}\frac{dt}{\sqrt{1-t}} \leq C\int_0^1 \frac{e^{-\frac{\delta|x-y|^2}{t}}}{t^{d/2}}\frac{dt}{\sqrt{1-t}}.\end{aligned}$$

Set

$$K_2(x) := \frac{e^{-\frac{\delta|x|^2}{t}}}{t^{d/2}}.$$

- Third, we need to control the difference between K_1 and the Calderón–Zygmund kernel K.
<u>Claim</u>

$$|K_1(x,y) - K(x-y)| \leq C\frac{1+|x|^{1/2}}{|x-y|^{d-1/2}} \quad \chi_{\{|x-y|<d(1\wedge 1/|x|)\}}(x,y)$$

Proof of the claim We need to estimate,

$$\begin{aligned}|K_1(x,y) - K(x-y)| &= \Big|\int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt\\ &\quad - \int_0^\infty F\Big(\frac{y-x}{t^{1/2}}\Big)e^{-|x-y|^2/t}\frac{dt}{t^{d/2+1}}\Big|.\end{aligned}$$

Using again the notation of Proposition 4.23, consider t_0 defined in (4.45),

$$t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}.$$

Now, if $t \geq t_0$, because

$$t_0 \sim \frac{\sqrt{a^2-b^2}}{a} \sim \frac{\sqrt{a-b}}{\sqrt{a+b}} = \frac{|x-y|}{|x+y|} \wedge 1,$$

and again using that on the local part $u(t) \geq |y-x|^2/t - 2d$, there is a $\delta > 0$ such that,

$$\begin{aligned}
&\Big| \int_{t_0}^{1} F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt - \int_{t_0}^{\infty} F\Big(\frac{y-x}{\sqrt{t}}\Big)e^{-|x-y|^2/t}\frac{dt}{t^{d/2+1}}\Big| \\
&\quad \leq \int_{t_0}^{1} \Big|F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\Big|\frac{e^{-u(t)}}{t^{d/2+1}}dt + \int_{t_0}^{\infty} \Big|F\Big(\frac{y-x}{\sqrt{t}}\Big)\Big|e^{-|x-y|^2/t}\frac{dt}{t^{d/2+1}} \\
&\quad \leq C\int_{t_0}^{1} \frac{e^{-\delta|x-y|^2/t}}{t^{(d-1)/2}}\frac{dt}{t^{3/2}} \leq C\frac{1}{|x-y|^{d-1}}\frac{1}{t_0^{1/2}} \leq C\frac{1+|x|^{1/2}}{|x-y|^{d-1/2}}.
\end{aligned}$$

For $t \leq t_0$ setting $v(s) = y - \sqrt{1-s}\,x$, we then have

$$\begin{aligned}
&\Big|F\Big(\frac{v(t)}{t^{1/2}}\Big)e^{-\frac{|v(t)|^2}{t}} - F\Big(\frac{v(0)}{t^{1/2}}\Big)e^{-\frac{|v(0)|^2}{t}}\Big| = \Big|\int_0^t \frac{\partial}{\partial s}\Big(F\Big(\frac{v(s)}{t^{1/2}}\Big)e^{-\frac{|v(s)|^2}{t}}\Big)ds\Big| \\
&\quad = \Big|\int_0^t \Big(\Big\langle \frac{v'(s)}{t^{1/2}}, (\nabla F)\Big(\frac{v(s)}{t^{1/2}}\Big)\Big\rangle e^{-\frac{|v(s)|^2}{t}} \\
&\qquad\qquad -2\Big\langle v'(s), \frac{v(s)}{t}\Big\rangle F\Big(\frac{v(s)}{t^{1/2}}\Big)e^{-\frac{|v(s)|^2}{t}}\Big)ds\Big| \\
&\quad \leq \int_0^t \Big|\frac{v'(s)}{t^{1/2}}\Big|\Big|(\nabla F)\Big(\frac{v(s)}{t^{1/2}}\Big)\Big|e^{-\frac{|v(s)|^2}{t}}ds \\
&\qquad\qquad +2\int_0^t \frac{|v'(s)|}{t^{1/2}}\frac{|v(s)|}{t^{1/2}}\Big|F\Big(\frac{v(s)}{t^{1/2}}\Big)\Big|e^{-\frac{|v(s)|^2}{t}}ds.
\end{aligned}$$

Using the hypothesis (9.38) and the fact that in the local part

$$\frac{-|v(s)|^2}{t} \leq \frac{-|x-y|^2}{t} + 2d\frac{s}{t},$$

we get, for some $\delta > 0$,

$$\begin{aligned}
&\int_0^{t_0} \Big|F\Big(\frac{v(t)}{t^{1/2}}\Big)e^{-\frac{|v(t)|^2}{t}} - F\Big(\frac{v(0)}{t^{1/2}}\Big)e^{-\frac{|v(0)|^2}{t}}\Big|\frac{dt}{t^{d/2+1}} \\
&\qquad \leq \int_0^{t_0}\int_0^t \frac{|v'(s)|}{t^{1/2}}e^{-\delta\frac{|v(s)|^2}{t}}ds\,\frac{dt}{t^{d/2+1}} \\
&\qquad \leq C|x|\int_0^{t_0}\int_0^t \frac{1}{\sqrt{1-s}}ds\,\frac{1}{t^{1/2}}e^{-\delta\frac{|x-y|^2}{t}}\frac{dt}{t^{d/2+1}}
\end{aligned}$$

$$\leq C|x| \int_0^{t_0} \frac{1}{t^{1/2}} \frac{e^{-\delta \frac{|x-y|^2}{t}}}{t^{d/2}} dt$$
$$\leq C \frac{|x|}{|x-y|^d} \int_0^{t_0} \frac{1}{t^{1/2}} dt \leq C \frac{|x| t_0^{1/2}}{|x-y|^d} \leq C \frac{1+|x|^{1/2}}{|x-y|^{d-1/2}}.$$

Set

$$K_3(x,y) := \frac{1+|x|^{1/2}}{|x-y|^{d-1/2}}.$$

Observe that $K_3(x,y)$ defines a function in the variable x, which is $L^1(\mathbb{R}^d)$, uniformly in the variable y.

Hence, writing $\mathscr{K}_{F,m}(x,y)$ as

$$\begin{aligned}
\mathscr{K}_{F,m}(x,y) &= \int_0^1 \psi_m(t) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt, \\
&= \psi_m(0) \int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt \\
&\qquad + \int_0^1 (\psi_m(t)-\psi_m(0)) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt \\
&= \psi_m(0) \int_0^1 \Big[F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} - F\Big(\frac{y-x}{\sqrt{t}}\Big) \frac{e^{-|x-y|^2/t}}{t^{d/2+1}}\Big] dt \\
&\quad + \psi_m(0) \int_0^1 F\Big(\frac{y-x}{\sqrt{t}}\Big) \frac{e^{-|x-y|^2/t}}{t^{d/2+1}} dt \\
&\qquad + \int_0^1 (\psi_m(t)-\psi_m(0)) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt.
\end{aligned}$$

Using the estimates above, we conclude that the local part $T_{F,m,L}$ can be bounded as

$$\begin{aligned}
|T_{F,m,L} f(x)| &= |T_{F,m} f(\chi_{B_h(x)})(x)| = \Big| \int_{B_h(x)} \mathscr{K}_{F,m}(x,y) f(y)\, dy \Big| \\
&\leq C \int_{B_h(x)} K_3(x,y)|f(y)|\, dy + C \Big| p.v. \int_{B_h(x)} K(x,y) f(y)\, dy \Big| \\
&\qquad + \int_{B_h(x)} K_2(x-y)|f(y)|\, dy \\
&= (I) + (II) + (III).
\end{aligned}$$

Using Theorem 4.32, (II) is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and is of weak type $(1,1)$ with respect to γ_d. Thus, it remains to prove that (I) and (III) are also bounded. To do so, we use Lemma 4.3, taking a countable family of admissible balls. $\mathscr{F}$

Now, given $B \in \mathscr{F}$, if $x \in B$ then $B_h(x) \subset \hat{B}$; therefore,

$$(I) = (1+|x|^{1/2}) \sum_{k=0}^{\infty} \int_{2^{-(k+1)}C_d m(x)<|x-y|<2^{-k}C_d m(x)} \frac{|f(y)|\chi_{\hat{B}}}{|x-y|^{d-1/2}} dy$$
$$\leq C_d 2^d \mathscr{M}(f\chi_{\hat{B}})(x)(1+|x|^2)m(x)^{1/2} \sum_{k=0}^{\infty} 2^{-(k+1)/2} \leq LC\mathscr{M}(f\chi_{\hat{B}})(x)(\chi_{B_h(\cdot)})(x).$$

On the other hand, let us consider $\varphi(y) = C_\delta e^{-\delta|y|^2}$, where C_δ is a constant such that $\int_{\mathbb{R}^d} \varphi(y)dy = 1$. φ is a non-increasing radial function, and given $t > 0$, we rescale this function as $\varphi_{\sqrt{t}}(y) = t^{-d/2}\phi(y/\sqrt{t})$, and, because $0 \leq \varphi \in L^1(\mathbb{R}^d)$, $\{\varphi_{\sqrt{t}}\}_{t>0}$ is an approximation of the identity (see the Appendix). Then, because $\int_0^1 (1/\sqrt{1-t})dt < \infty$,

$$(III) = \int_{B_h(x)} K_2(x-y)|f(y)|\, dy = \int_{B_h(x)} \Big(\int_0^1 \varphi_{\sqrt{t}}(x-y) \frac{dt}{\sqrt{1-t}} \Big) |f(y)|dy$$
$$\leq \int_{B_h(x)} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y) \Big) \Big(\int_0^1 \frac{dt}{\sqrt{1-t}} \Big) |f(y)|dy$$
$$\leq C \int_{B_h(x)} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y) \Big) |f(y)|dy.$$

Again, using the family $\mathscr{F}$, if $x \in B$, $B_h(x) \subset \hat{B}$, and then, using a similar argument to previously,

$$(III) = \int_{B_h(x)} K_2(x-y)|f(y)|\, dy \leq C \int_{\mathbb{R}^d} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y) \Big) |f(y)|\chi_{\hat{B}}(y)dy$$

which yields, using Theorem 4 in Stein's book [252, Chapter II §4.],

$$(III) = \int_{B_h(x)} K_2(x-y)|f(y)|\, dy \leq \sum_{B\in\mathscr{F}} \sup_{t>0} \Big| (\varphi_{\sqrt{t}} * |f\chi_{\hat{B}}|)(x) \Big| \chi_B(x)$$
$$\leq \sum_{B\in\mathscr{F}} \mathscr{M}(f\chi_{\hat{B}})(x)\chi_B(x).$$

Therefore, the local part $T_{F,m,L}$ is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and is of weak type $(1,1)$ with respect to γ_d,

II) Now, for the global part $T_{F,m,G}$, we prove that it is $L^p(\gamma_d)$-bounded for all $1 < p < \infty$ using similar techniques to those used for the Gaussian Riesz transform, to estimate the kernel $\mathscr{K}_{F,m}$. The idea is to exploit the size of the kernel and treat T as a positive operator.
Observe that the function $\varphi_m(r) = (\frac{-\log r}{1-r^2})^{\frac{m-2}{2}} r^{m-1}$ is bounded in $(0,1)$ for any $m \in \mathbb{N}$. Hence, using (9.38), we get

$$|\mathscr{K}_{F,m}(x,y)| \leq \int_0^1 \Big| F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}} \Big) \Big| \frac{e^{-u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} \leq C_\varepsilon \int_0^1 e^{\varepsilon u(t)} \frac{e^{-u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}},$$

for some $\varepsilon > 0$ to be determined. As before, we consider two cases:

- Case #1: $b = 2\langle x,y\rangle \le 0$. In this case we use again the inequality (4.76):

$$\frac{a}{t} - |x|^2 \le u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t} b - |x|^2 \le \frac{2a}{t};$$

thus, the change of variables $s = a(\frac{1}{t} - 1)$ gives

$$\int_0^1 \frac{e^{-(1-\varepsilon)u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} \le e^{-(1-\varepsilon)|y|^2} \frac{1}{a^{d/2}} \int_0^\infty e^{-(1-\varepsilon)s} (s+a)^{(d-1)/2} \frac{ds}{\sqrt{s}}$$
$$\le Ce^{-(1-\varepsilon)|y|^2},$$

as $a > 1/2$ over the global region. Therefore, using Hölder's inequality

$$\|T_{F,m,G} f\|_{p,\gamma}^p \le \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} |\mathcal{K}_{F,m}(x,y)||f(y)|dy \Big)^p e^{-|x|^2} dx$$
$$\le C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} e^{-(1-\varepsilon)|y|^2} e^{|y|^2/p} e^{-|y|^2/p} |f(y)|dy \Big)^p e^{-|x|^2} dx$$
$$\le C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} e^{-(1-\varepsilon)q|y|^2} e^{q|y|^2/p} dy \Big)^{p/q} e^{-|x|^2} dx \, \|f\|_{p,\gamma}^p,$$

where $q = \frac{p}{p-1}$. Now, we select an appropriate $\varepsilon > 0$ so that the above integral is finite. We can see that any $\varepsilon > 0$, with $\varepsilon < 1 - 1/p$, suffices.

- Case #2: $b = 2\langle x,y\rangle > 0$.
We have,

$$|\mathcal{K}_{F,m}(x,y)| \le \int_0^1 \Big| F\Big(\frac{y - \sqrt{1-t}\,x}{\sqrt{t}} \Big) \Big| \frac{e^{-u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} \le C_\varepsilon \frac{e^{-(1-\varepsilon)u(t_0)}}{t_0^{d/2}}.$$

When $d = 1$, this inequality follows directly from Lemma 4.36, by taking $\eta = 0$, and $\nu = 1 - \varepsilon$ for $0 < \varepsilon < 1$.
For $d \ge 2$, we use (4.77) and the boundedness of F for ε smaller than $1/d$. Thus, using Lemma 4.36, we have

$$|\mathcal{K}_{F,m}(x,y)| \le C_\varepsilon \frac{e^{\frac{-d-1}{d} u(t_0)}}{t_0^{(d-1)/2}} \int_0^1 e^{\varepsilon u(t)} \frac{e^{-u(t)/d}}{t^{3/2}} \frac{dt}{\sqrt{1-t}} \le C_\varepsilon \frac{e^{-(1-\varepsilon)u(t_0)}}{t_0^{d/2}},$$

with $\varepsilon > 0$ to be determined. Then,

$$\|T_{F,m,G} f\|_{p,\gamma}^p \le \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} |\mathcal{K}_{F,m}(x,y)||f(y)|dy \Big)^p e^{-|x|^2} dx$$
$$\le C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} \frac{e^{-(1-\varepsilon)u(t_0)}}{t_0^{d/2}} |f(y)|dy \Big)^p e^{-|x|^2} dx$$
$$= C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} e^{\frac{|y|^2 - |x|^2}{p}} e^{\varepsilon u(t_0)} \frac{e^{-u(t_0)}}{t_0^{d/2}} |f(y)| e^{-\frac{|y|^2}{p}} dy \Big)^p dx.$$

Therefore, it is enough to check that the operator defined using the kernel,

$$\tilde{K}(x,y) = e^{\frac{|y|^2-|x|^2}{p}} e^{\varepsilon u(t_0)} \frac{e^{-u(t_0)}}{t_0^{d/2}} \chi_{B_h^c(x)}(y),$$

is of strong type p with respect to the Lebesgue measure. Using the inequality $||y|^2 - |x|^2| \leq |x+y||x-y|$, and that, as $b > 0$, on the global region, $|x+y||x-y| \geq d$, we have

$$\begin{aligned} e^{\frac{|y|^2-|x|^2}{p}} e^{\varepsilon u(t_0)} \frac{e^{-u(t_0)}}{t_0^{d/2}} &= \frac{1}{t_0^{d/2}} e^{(\frac{1}{p}-\frac{1-\varepsilon}{2})(|y|^2-|x|^2)} e^{-\frac{1-\varepsilon}{2}|x+y||x-y|} \\ &\leq C|x+y|^d e^{-\alpha_p|x+y||x-y|}, \end{aligned}$$

where $\alpha_p = \frac{1-\varepsilon}{2} - |\frac{1}{p} - \frac{1-\varepsilon}{2}|$. Since $p > 1$, we can choose $\varepsilon > 0$ so that $\alpha_p > 0$. Observe that the last expression is symmetric in x and y; therefore, it suffices to prove its integrability with respect to one of the

$$\begin{aligned} \int_{\mathbb{R}^d} |x+y|^d e^{-\alpha_p|x+y||x-y|} dy &\leq C + C\int_{|x-y|<1} |x|^d e^{-\alpha_p|x||x-y|} dy \\ &\quad + C\int_{|x-y|<1} |x+y|^d e^{-\alpha_p|x+y|} dy \\ &\leq C\int_{\mathbb{R}^d} e^{\alpha_p|v|} dv + C_d \int_0^\infty r^{2d-1} e^{-\alpha_p r} dr \leq C. \end{aligned}$$

Observe that, once $p > 1$ is chosen, then the operator defined using the kernel $\tilde{K}(x,y)$ is in fact $L^q(\mathbb{R}^d)$-bounded for $1 \leq q \leq \infty$, but for the proof of the theorem, the case $p = q$ is enough. □

Now, we discuss the results corresponding to the weak type $(1,1)$ for the operators $T_{F,m}$. First of all, observe that condition (9.38) provides a function Φ satisfying the property

iii) $|F(x)| \leq \Phi(|x|)$ for some continuous function $\Phi : [0,\infty) \to [0,\infty)$ for which there exists a $\delta > 0$ with $1 - 2/d < \delta < 1$, such that $\Phi(t)e^{-(1-\delta)t^2}$ is a non-increasing function for all $t \geq 0$.

Indeed, for $0 < \varepsilon < 2/d$ we set $\Phi(t) = C_\varepsilon e^{\varepsilon t^2}$ and $\delta 1 - \varepsilon$. In what follows, we denote by Φ any function satisfying the property *iii)*. We see that the smaller the function Φ is taken, the better the result that can be obtained. The goal of the following two theorems is to answer the question: what are the precise conditions needed on F and on m to guarantee the weak type $(1,1)$ with respect to the Gaussian measure of $T_{F,m}$? The answer is given in the following two theorems. First, let us consider the *negative result*, which roughly says that if the function $\Phi(t)$ increases at infinity more than t^2, then the operator $T_{F,m}$ fails to be of weak type $(1,1)$ and it is a generalization of what is already known about the behavior on $L^1(\gamma_d)$ of the Gaussian higher-order Riesz transforms.

Theorem 9.18. *Let* $\Omega_t = \left\{z \in \mathbb{R}^d : \min_{1\le i\le d} |z_i| \ge t\right\}$ *and* $\Theta(t) = \frac{\inf_{\Omega_t} F(z)}{t^2}$, *if* $\limsup_{t\to\infty} \Theta(t) = \infty$, *then the operator* $T_{F,m}$ *is not of weak type* $(1,1)$ *with respect to the Gaussian measure.*

Proof. We follow the proof of Theorem 9.10. Again, let $y \in \mathbb{R}^d$ with $|y| = \eta$ large and $y_i \ge C\eta$, $i = 1,\cdots,d$. Write $x \in \mathbb{R}^d$ as $x = \xi\frac{y}{\eta} + v$ with $\xi \in \mathbb{R}$ and $v \perp y$. Consider the tubular region

$$\mathbf{J} = \{x \in \mathbb{R}^d : x = \xi\frac{y}{\eta} + v \quad \text{with} \quad \eta/2 < \xi < 3\eta/4, v \perp y,\ |v| < 1\}.$$

It follows that for $x \in \mathbf{J}$ (9.25) holds; therefore,

$$F\Big(\frac{y - rx}{\sqrt{1-r^2}}\Big) \ge C\eta^2\Theta(c\eta).$$

Thus, for $x \in \mathbf{J}$ using this estimate and (9.26) and (9.27) we get

$$\begin{aligned}\mathscr{K}_{F,m}(x,y) &\ge C_d\eta^2\Theta(c\eta)\int_{1/4}^{3/4} \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}dr\\ &\ge C_d\eta^2\Theta(c\eta)e^{\xi^2-\eta^2}\int_{1/4}^{3/4} e^{-C|\xi-r\eta|^2}dr \ge C_d\eta\Theta(c\eta)e^{\xi^2-\eta^2}\end{aligned}$$

Now, let $f \in L^1(\gamma_d)$, $f \ge 0$ be a close approximation of a point mass at y, with norm $\|f\|_{1,\gamma} = 1$. Then, for $x \in \mathbf{J}$

$$T_{F,m}f(x) \ge C\eta\Theta(c\eta)e^{\xi^2} \ge C\eta\Theta(c\eta)e^{(\eta/2)^2}.$$

Let us assume that $T_{F,m}$ is of weak type $(1,1)$ with respect to the Gaussian measure. Then,

$$\gamma_d(\mathbf{J}) \le \gamma_d\Big(\Big\{x \in \mathbb{R}^d : T_{F,m}f(x) > C\eta\Theta(c\eta)e^{(\eta/2)^2}\Big\}\Big) \le C\frac{e^{-(\eta/2)^2}}{\eta\Theta(c\eta)},$$

but $\gamma_d(\mathbf{J}) \ge \frac{C}{\eta}e^{-(\eta/2)^2}$; therefore, $\Theta(\eta)$ is bounded for η large, which is a contradiction with the assumption on Θ. □

The positive result is contained in the following theorem. To get sufficient conditions on F for the weak type $(1,1)$ of $T_{F,m}$, because the weak type is not true, the natural question is: what weights can be put to get a weak type inequality? From the proof of Theorem 9.11, it is clear that for $|\beta| \ge 3$, the weight should be of the form $w(y) = 1 + |y|^{|\beta|-2}$. Moreover, for every $0 < \varepsilon < |\beta| - 2$, there exists a function $F \in L^1((1+|\cdot|^{\varepsilon})\gamma_d)$ such that $\mathscr{R}_\beta f \notin L^{1,\infty}(\gamma_d)$ (see [86]). The weights w that are considered, to ensure that $T_{F,m}$ is bounded from $L^1(w\gamma_d)$ into $L^{1,\infty}(\gamma_d)$, depend on the function Φ.

Theorem 9.19. *The operator $T_{F,m}$ maps continuously $L^1(w\gamma_d)$ into $L^{1,\infty}(\gamma_d)$ with $w(y) = 1 \vee \max_{1\leq t\leq |y|} \eta(t)$ and*

$$\eta(t) = \begin{cases} \Phi(t)/t \text{ if} & 1 \leq m < 2, \\ \Phi(t)/t^2 \text{ if} & m \geq 2, \end{cases}$$

The proof is long and technical; it is based on the refinement of several inequalities used by S. Pérez in [220], and the application of a technique developed by García-Cuerva et al. in gmst4. For details, see [85, Theorem 2].

As an immediate consequence we get:

Corollary 9.20. *If for t large either $\Phi(t) \leq Ct$ when $1 \leq m < 2$ or $\Phi(t) \leq Ct^2$ when $m \geq 2$, then the operator $T_{F,m}$ is of weak type $(1,1)$ with respect to the Gaussian measure.*

9.5 Notes and Further Results

1. What is known as *Meyer's inequality* in Malliavin calculus is given in the following terms. Given L a self-adjoint, second-order differential operator on $L^2(\mathbb{R}^d, d\mu)$, for some probability measure μ, and suppose that L is the infinitesimal generator of a Markov semigroup, then there exist constants c_p, C_p such that
$$c_p \|L^{1/2} f\|_{p,\mu} \leq \|\nabla f\|_{p,\mu} \leq C_p \|L^{1/2} f\|_{p,\mu},$$
holds for all p, $1 < p < \infty$. Observe that that statement is equivalent to the $L^p(\mu)$-boundedness of the corresponding Riesz transforms.

2. In [194] B. Muckenhoupt introduces, for $d = 1$, the Gaussian Hilbert transform in a different way. He follows the classical definition of the conjugated function as the limit of the conjugated Fourier series, using the Cauchy–Riemann equations. In more detail, he considers the conjugated Poisson–Hermite semigroup $P_t^c f(x)$, based on the Gaussian Cauchy–Riemann equations (see section 3.4). As we know from Chapter 3, the Poisson–Hermite operator P_t on f is defined as
$$P_t f(x) = u(x,t) = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} p(t,x,y) f(y)\, dy,$$
$t > 0$, where
$$p(t,x,y) = \int_0^1 \frac{t \exp(\frac{t^2}{2\log r})}{r(-\log r)^{3/2}} \frac{\exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})}{(1-r^2)^{1/2}}\, dr,$$
then, $u(x,t)$ satisfies:
$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} - 2x \frac{\partial u(x,t)}{\partial x} = 0,$$

which is equivalent to

$$\frac{\partial^2 u(x,t)}{\partial t^2} + e^{x^2} \frac{\partial}{\partial x}(e^{-x^2} \frac{\partial u(x,t)}{\partial x}) = 0.$$

Then, considering a L-harmonic conjugated v given by the Gaussian Cauchy–Riemann equations (3.44),

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} &= -\frac{\partial v(x,t)}{\partial t}\\ \frac{\partial u(x,t)}{\partial t} &= e^{x^2} \frac{\partial}{\partial x}(e^{-x^2} \frac{\partial v(x,t)}{\partial x}),\end{aligned}$$

it is easy to see that v can be written as

$$P_t^c f(x) = v(x,t) = \int_{-\infty}^{\infty} Q(t,x,y) f(y)\, dy,$$

$t > 0$, where

$$Q(t,x,y) = \frac{\sqrt{2}}{\pi} \int_0^1 \frac{(y-rx)\exp(\frac{t^2}{2\log r})}{(-\log r)^{1/2}} \frac{\exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})}{(1-r^2)^{3/2}}\, dr.$$

B. Muckenhoupt [194] proved that v is $L^p(\gamma_1)$-bounded for $p > 1$, and for $f \in L^p(\gamma_1)$, $p > 1$, $v(x,t)$ tends to Gaussian Hilbert transform $\mathscr{H}$, as $t \to 0^+$; therefore,

$$\mathscr{H} f(x) = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} \int_0^1 \frac{(y-rx)}{(-\log r)^{1/2}} \frac{\exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})}{(1-r^2)^{3/2}}\, dr f(y)\, \gamma_1(dy).$$

The convergence is in $L^p(\gamma_1)$-norm sense, $1 < p < \infty$ and also almost everywhere (a.e.). He also proved the $L^p(\gamma_1)$-boundedness and the weak type $(1,1)$ with respect to the Gaussian measure γ_1 using analytic methods based on Natanson's lemma, see (10.27).

3. In his doctoral dissertation, [244][3] R. Scotto got the extension of this approach to the higher dimensions $d > 1$, by considering the Gaussian Cauchy–Riemann equations in $\mathbb{R}^d$ (3.50),

$$\begin{aligned}\frac{\partial u}{\partial x_j}(x,t) &= -\frac{\partial v_j}{\partial t}(x,t), j = 1,\ldots,d\\ \frac{\partial v_i}{\partial x_j}(x,t) &= \frac{\partial v_j}{\partial x_i}(x,t), i,j = 1,\ldots,d\\ \frac{\partial u}{\partial t}(x,t) &= \frac{1}{2}\sum_{j=1}^{d} e^{|x|^2} \frac{\partial}{\partial x_j}(e^{-|x|^2} v_j(x,t)).\end{aligned}$$

[3]See also [77].

Then, he defined a system of conjugates

$$(u(x,t), v_1(x,t), v_2(x,t), \ldots, v_d(x,t)),$$

in a similar way to the one-dimensional case, and then, taking $t \to 0^+$, he proved that the system of conjugates converges to the vector where the first coordinate is the function f and the other coordinates are the Gaussian Riesz transforms of f,

$$(f(x), \mathscr{R}_1 f(x), \cdots, \mathscr{R}_d f(x)).$$

The convergence is in $L^p(\gamma_1)$-norm sense, $1 < p < \infty$ and also a.e. For more details, see [244, Chapter] (see also section 3.4).

4. The definition used by B. Muckenhoupt and R. Scotto for the Gaussian Riesz transforms using Cauchy–Riemann equations is, of course, equivalent (up to a constant) to the one given in this chapter. To see this, observe that according to the general semigroup theory, we have

$$-\int_0^\infty P_s ds = (-L)^{-1/2},$$

because, as $(-L)^{1/2}$ is the infinitesimal generator of the Poisson–Hermite semigroup $\{P_t\}$, then, at least formally

$$\begin{aligned} -(-L)^{1/2}(\int_0^\infty P_s ds) &= -\lim_{t\to 0^+} \frac{1}{t}[P_t(\int_0^\infty P_s ds) - \int_0^\infty P_s ds] \\ &= -\lim_{t\to 0^+} \frac{1}{t}[(\int_0^\infty P_{(t+s)} ds) - \int_0^\infty P_s ds] \\ &= -\lim_{t\to 0^+} \frac{1}{t}[(\int_t^\infty P_s ds) - \int_0^\infty P_s ds] \\ &= \lim_{t\to 0^+} \frac{1}{t}[(\int_0^t P_s ds)] = P_0 = I. \end{aligned}$$

Therefore, from (3.45) we know that the conjugated Poisson–Hermite integral can be written as

$$v(x,t) = \int_{-\infty}^{\infty} Q(t,x,y) f(y)\, dy,$$

and from (3.47), $Q(t,x,y)$ can be written as

$$Q(t,x,y) = -\int_t^\infty \frac{\partial p(s,x,y)}{\partial x} ds = -\frac{\partial}{\partial x}\int_t^\infty p(s,x,y) ds$$

then,

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} Q(t,x,y) f(y)\, dy = -\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_t^\infty p(s,x,y) ds f(y)\, dy \\ &= -\frac{\partial}{\partial x}\int_t^\infty \int_{-\infty}^{\infty} p(s,x,y) f(y)\, dy ds = -\frac{\partial}{\partial x}\int_t^\infty P_s f(x) ds, \end{aligned}$$

formally we get, taking $t \to 0^+$,

$$v(x,t) \to \frac{\partial}{\partial x}(-L)^{1/2} f(x) = \sqrt{2}\mathscr{H} f(x).$$

The argument for higher dimensions is analogous, using (3.51) and (3.52).

5. In [127], E. Harboure, R. A. Macías, M. T. Menárguez, and J. L. Torrea studied the rate of convergence for the family of truncations of the Gaussian Riesz transforms and Hermite–Poisson semigroup through the oscillation and variation operators. More precisely, they search for their $L^p(\gamma_d)$-boundedness properties, by looking at the oscillation and variation operators from a vector-valued point of view.

6. We know that the Gaussian Hilbert transform is defined spectrally as

$$\mathscr{H} = \frac{1}{\sqrt{2}}\frac{d}{dx}(-L)^{-1/2},$$

then,

$$\mathscr{H} H_n(x) = (-L)^{-1/2} H_n(x) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n}}\frac{d}{dx} H_n(x) = \sqrt{2n} H_{n-1}(x),$$

which of course is a particular version of (9.5). Therefore,

$$\mathscr{H} h_n = \mathscr{H}\left(\frac{H_n}{(2^n n!)^{1/2}}\right) = \sqrt{2n}\frac{H_{n-1}}{(2^n n!)^{1/2}} = h_{n-1}. \tag{9.42}$$

Hence, given $f \in L^2(\gamma_1)$ with Hermite expansion $f = \sum_{n=0}^{\infty} \langle f, H_n \rangle_\gamma H_n$, then its Gaussian Hilbert transform is the *conjugated series*

$$\mathscr{H} f = \sum_{n=1}^{\infty} \sqrt{2n} \langle f, H_n \rangle H_{n-1}, \tag{9.43}$$

This fact motivates the study of the Gaussian Hilbert transform $\mathscr{H}$ from the operator theory point of view. These results are contained in M. D. Morán and W. Urbina's article [191]. Let $\mathbb{D}$ be the open unit disk, $\mathbb{T}$ the circumference, consider the square integrable functions in $\mathbb{T}$

$$L^2(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : \int_{-\pi}^{\pi} |f(e^{it})|^2 dt < \infty\}.$$

Let $\{e_n\}_{n\in\mathbb{Z}}$ be the trigonometric system, $e_n(\xi) = \xi^n, \xi \in \mathbb{T}$, which is a complete orthonormal system in $L^2(\mathbb{T})$, and finally let $\mathscr{S} : setL^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the shift operator given by

$$(\mathscr{S} f)(\xi) = \xi f(\xi),$$

for all $\xi \in \mathbb{T}$. For more details on the shift operator, we refer the reader to Nikol'skii [206].

If we consider

$$H^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty\},$$

then we can identify $H^2(\mathbb{D})$ with the subspace of $L^2(\mathbb{T})$ consisting of functions such that

$$\langle f, e_n \rangle = 0, \forall n < 0.$$

The restriction of the bilateral forward shift operator to $H^2(\mathbb{D})$, which, abusing the notation, we also call $\mathscr{S}$, is the *unilateral forward shift*, which leaves the space $H^2(\mathbb{D})$ invariant and

$$\mathscr{S}(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{n=1}^{\infty} a_{n-1} z^n.$$

The main result in this direction is that the Gaussian Hilbert transform is unitary equivalent to the adjoint of the unilateral shift operator acting on $H^2(\mathbb{D})$; thus, we are able to completely characterize the invariant subspaces and the commutant of the Gaussian Hilbert transform. The main results are as follows:

Theorem 9.21. *The Gaussian Hilbert transform $\mathscr{H}$ as an operator on $L^2(\gamma_1)$ is unitarily equivalent to the adjoint of the restriction of the shift operator on $H^2(\mathbb{D})$.*

Proof. Let us consider $\Omega : L^2(\gamma_1) \to H^2(\mathbb{D})$ defined by

$$\Omega(\sum_{n=0}^{\infty} \langle f, h_n \rangle h_n) = \sum_{n=0}^{\infty} \langle f, h_n \rangle e_n.$$

It is easy to see, by Parseval's identity, that Ω is a well-defined operator and unitary, also Ω intertwining $\mathscr{H}$ and $\mathscr{S}^*$, that is, $\Omega\mathscr{H} = \mathscr{S}^*\Omega$:

$$\begin{aligned}
\Omega\mathscr{H}(\sum_{n=0}^{\infty} \langle f, h_n \rangle h_n) &= \Omega\mathscr{H}(\sum_{n=0}^{\infty} (\sqrt{2^n n!})^{-2} \langle f, H_n \rangle H_n) \\
&= \Omega(\sum_{n=1}^{\infty} (\sqrt{2^n n!})^{-2} \sqrt{2n} \langle f, H_n \rangle H_{n-1}) \\
&= \Omega(\sum_{n=1}^{\infty} \langle f, h_n \rangle h_{n-1}) = \sum_{n=1}^{\infty} \langle f, h_n \rangle e_{n-1} \\
&= \mathscr{S}^*(\sum_{n=0}^{\infty} \langle f, h_n \rangle e_n) = \mathscr{S}^*\Omega(\sum_{n=0}^{\infty} \langle f, h_n \rangle h_n). \qquad \square
\end{aligned}$$

There are several consequences of this result. The first completely characterizes the invariant subspaces of the Gaussian Hilbert transform.

Theorem 9.22. *Given the Gaussian Hilbert transform* $\mathscr{H}$ *and A a proper and closed subspace of* $L^2(\gamma_1)$*, then,* $\mathscr{H}(A) \subset A$ *if and only if there exists a sequence of complex numbers* $\{a_n\}$ *such that* $|\sum_{n=0}^{\infty} a_n z^n| = 1$ *almost everywhere in* $\mathbb{T}$ *and*

$$A = \{f = \sum_{n=0}^{\infty} \langle f, h_n \rangle h_n \in L^2(\gamma_1) : \sum_{n \geq k}^{\infty} \langle f, h_n \rangle \overline{a_{n-k}} = 0, \forall k \geq 0\}$$

Proof. We shall prove first that the condition is necessary. Let Ω intertwining $\mathscr{H}$ and $\mathscr{S}^*$ as in the previous theorem. It is clear that $\mathscr{H}(A) \subset A$ if and only if $\mathscr{S}^*(\Omega A) \subset \Omega A$, and this is equivalent to

$$S(H^2(\mathbb{D}) \ominus \Omega A) \subset H^2(\mathbb{D}) \ominus \Omega A.$$

Now $H^2(\mathbb{D}) \ominus \Omega A = \Omega(L^2(\gamma_1) \ominus A) \neq 0$ according to the hypothesis, then $H^2(\mathbb{D}) \ominus \Omega A$ is a non-trivial, closed subspace of $H^2(\mathbb{D})$, then (see [31] or [132]), there exists $\theta \in H^2(\mathbb{D})$ with $|\theta(\xi)| = 1$ for almost all $\xi \in \mathbb{T}$ such that

$$H^2(\mathbb{D}) \ominus \Omega A = \theta H^2(\mathbb{D}),$$

or equivalently

$$A = \Omega^{-1}[H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})] = L^2(\gamma_1) \ominus \Omega^{-1} \theta H^2(\mathbb{D}).$$

Let $\theta(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f = \sum_{n=0}^{\infty} \langle f, h_n \rangle h_n \in A$ if and only if

$$\langle f, \Omega^{-1} \theta u \rangle = 0, \text{for all } u \in H^2(\mathbb{D}).$$

Given that $k \geq 0$, let us take $u = e_k$, then

$$0 = \langle f, \Omega^{-1} \theta e_k \rangle = \langle \Omega f, \theta e_k \rangle = \sum_{n=0}^{\infty} \langle f, h_n \rangle \langle e_n, \theta e_k \rangle,$$

but we have that

$$\langle e_n, \theta e_k \rangle = \begin{cases} \{\overline{a_{n-k}}, & \text{if } k \geq n, \\ 0 & \text{if } k < n, \end{cases}$$

then for all $k \geq 0$, $\sum_{n=0}^{\infty} \langle f, h_n \rangle \overline{a_{n-k}} = 0$.

We shall now prove the sufficiency. Let $\theta(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, given $f \in L^2(\gamma_1)$,

$$f \in A \text{ if and only if for all } k \geq 0 \ \langle \Omega f, \theta e_k \rangle = 0.$$

Thus, if $u \in H^2(\mathbb{D})$, then we have

$$|\langle \Omega f, \theta u \rangle| \leq |\langle \Omega f, \theta(u - \sum_{k=0}^{n} \langle u, e_k \rangle e_k) \rangle| \leq ||\Omega f||_{H^2} ||\theta(u - \sum_{k=0}^{n} \langle u, e_k \rangle e_k)||_{H^2}$$

$$\leq ||f||_{L^2(\gamma_1)} ||u - \sum_{k=0}^{n} \langle u, e_k \rangle e_k||_{H^2},$$

but

$$\lim_{n\to\infty} ||u - \sum_{k=0}^{n} \langle u, e_k \rangle e_k||_{H^2} = 0.$$

Therefore, $\langle \Omega f, \theta u \rangle = 0$ for all $f \in A, u \in H^2(\mathbb{D})$; thus, $\Omega A = H^2 \ominus \theta H^2(\mathbb{D})$, and then

$$\begin{aligned} \mathscr{H}A &= \Omega^{-1}\Omega\mathscr{H}A = \Omega^{-1}S^*\Omega A \\ &= \Omega^{-1}\mathscr{S}^*(H^2 \ominus \theta H^2(\mathbb{D})) \subset \Omega^{-1}(H^2 \ominus \theta H^2(\mathbb{D})) = A. \qquad \square \end{aligned}$$

The next result characterizes the commutant of the Gaussian Hilbert transform.

Theorem 9.23. *Let $\mathscr{F}$ be a linear operator on $L^2(\gamma_1)$. If $\mathscr{F}\mathscr{H} = \mathscr{H}\mathscr{F}$ then there exists $g \in H^\infty(\mathbb{D})$ such that*

$$\begin{aligned} \mathscr{F}f &= \mathscr{F}(\sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} h_n) \\ &= \sum_{k=0}^{\infty} \sum_{n\geq k}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} \langle g, e_{n-k} \rangle_{H^2(\mathbb{D})} h_k. \end{aligned}$$

Conversely, if this relation holds and $P_0 f = \langle f, h_0 \rangle_{L^2(\gamma_1)} h_0$, then

$$\mathscr{F}\mathscr{H} = \mathscr{H}\mathscr{F}(I - P_0).$$

Proof. Let $G = \Omega\mathscr{F}\Omega^{-1}$. It is clear that $G \in L(H^2(\mathbb{D}))$ and

$$\mathscr{S}^*G = \mathscr{S}^*\Omega\mathscr{F}\Omega^{-1} = \Omega\mathscr{H}\mathscr{F}\Omega^{-1} = \Omega\mathscr{F}\mathscr{H}\Omega^{-1} = \Omega\mathscr{F}\Omega^{-1}\mathscr{S}^* = G\mathscr{S}^*;$$

thus,

$$\mathscr{S}^*G = G\mathscr{S}^*, \text{ and also } G^*\mathscr{S} = \mathscr{S}G^*.$$

Let $g = G^*e_0$. Then, it is easy to check that $g \in H^\infty(\mathbb{D})$, $G^*u = gu$ for all $u \in \mathbb{D}$ and

$$\begin{aligned} \mathscr{F}f &= \mathscr{F}(\sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} h_n) = \sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} \mathscr{F}h_n \\ &= \sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} \Omega^{-1}\mathscr{F}\Omega h_n = \sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} \Omega^{-1}(\sum_{k=0}^{\infty} \langle Ge_n, e_k \rangle e_k) \\ &= \sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} \Omega^{-1}(\sum_{k=0}^{\infty} \langle e_n, ge_k \rangle e_k) = \sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} \Omega^{-1}(\sum_{k=0}^{n} \overline{\langle g, e_{n-k} \rangle} e_k) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \langle f, h_n \rangle_{L^2(\gamma_1)} \overline{\langle g, e_{n-k} \rangle} h_k = \sum_{k=0}^{\infty} (\sum_{k\geq n} \langle f, h_n \rangle_{L^2(\gamma_1)} \overline{\langle g, e_{n-k} \rangle} h_k). \end{aligned}$$

Conversely,

$$\mathscr{F}\mathscr{H}f = \mathscr{F}\mathscr{H}(\sum_{n=0}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} h_n) = \mathscr{F}(\sum_{n=1}^{\infty} \langle f, h_n \rangle_{L^2(\gamma_1)} h_{n-1})$$

$$= \mathscr{F}(\sum_{n=0}^{\infty}\langle f, h_{n+1}\rangle_{L^2(\gamma_1)} h_n) = \sum_{k=0}^{\infty}\sum_{n\geq k}\langle f, h_{n+1}\rangle_{L^2(\gamma_1)}\overline{\langle g, e_{n-k}\rangle} h_k$$

and

$$\begin{aligned}\mathscr{H}\mathscr{F}(I-P_0)f &= \mathscr{H}\mathscr{F}(\sum_{n=1}^{\infty}\langle f, h_n\rangle_{L^2(\gamma_1)} h_n) = \mathscr{H}(\sum_{k=0}^{\infty}\sum_{n\geq k}\langle f, h_{n+1}\rangle_{L^2(\gamma_1)} \\ &\qquad \times\overline{\langle g, e_{n-k}\rangle} h_k) \\ &= \mathscr{H}(\sum_{k=0}^{\infty}\sum_{n\geq k}\langle f, h_n\rangle_{L^2(\gamma_1)}\overline{\langle g, e_{n-k}\rangle} h_k) = \sum_{k=1}^{\infty}\sum_{n\geq k}\langle f, h_n\rangle_{L^2(\gamma_1)} \\ &\qquad \times\overline{\langle g, e_{n-k}\rangle} h_{k-1} \\ &= \sum_{k=0}^{\infty}\sum_{n\geq k+1}\langle f, h_n\rangle_{L^2(\gamma_1)}\overline{\langle g, e_{n-(k+1)}\rangle} h_k = \sum_{k=0}^{\infty}\sum_{n\geq k}\langle f, h_{n+1}\rangle_{L^2(\gamma_1)} \\ &\qquad \times\overline{\langle g, e_{n-k}\rangle} h_k;\end{aligned}$$

thus, $\mathscr{F}\mathscr{H} = \mathscr{H}\mathscr{F}(I-\mathbf{J}_0)$. □

Finally,

Theorem 9.24. *Let A be a (closed) subspace of $L^2(\gamma_1)$ that is invariant for $\mathscr{H}$ and P_A is the orthogonal projection of $L^2(\gamma_1)$ onto A. If $\mathscr{F}$ is a linear operator on A such that*

$$\mathscr{F}(P_A\mathscr{H}^*) = (P_A\mathscr{H}^*)\mathscr{F},$$

then there exists $\mathscr{F}_1$ a linear operator acting on $L^2(\gamma_1)$ such that $\mathscr{F}_1\mathscr{H} = \mathscr{H}\mathscr{F}_1$ and

$$\mathscr{F} = P_A\mathscr{F}_1{}^*|_A.$$

Proof. Recall that $\mathscr{H}(A) \subset A$ implies that $\mathscr{H}(A^{\perp}) \subset A^{\perp}$. Set $\mathbf{B} = \Omega A$, then

$$\mathbf{S}(H^2(\mathbb{D}) \ominus B) \subset H^2(\mathbb{D}) \ominus B.$$

Let T be a linear operator in B given by $T = \Omega\mathscr{F}\Omega^{-1}$, then

$$\begin{aligned}TP_B\mathscr{S}|_B &= \Omega\mathscr{F}\Omega^{-1}P_B\mathscr{S}|_B = \Omega\mathscr{F}P_A\Omega^{-1}\mathscr{S}|_B = \Omega\mathscr{F}P_A\mathscr{H}^*\Omega^{-1}|_B \\ &= \Omega P_A\mathscr{H}^*\mathscr{F}\Omega^{-1}|_B = P_B\Omega\mathscr{H}^*\Omega^{-1}T|_B = P_B\mathscr{S}T|_B.\end{aligned}$$

Thus, using the Sarason generalized interpolation theorem [241], there exists $g \in \mathscr{H}^{\infty}(\mathbb{D})$ such that $T = P_BM_g|_B$ (i.e., $Tf = P_B(fg)\forall f \in B$), and if $\mathscr{F}_1 = \Omega M_g\Omega^{-1}$, then

$$\begin{aligned}P_A\mathscr{F}_1|_A &= P_A\Omega M_g\Omega^{-1}|_A = \Omega^{-1}P_B\Omega\Omega^{-1}M_g\Omega|_A \\ &= \Omega^{-1}P_BM_g\Omega|_A = \Omega^{-1}T\Omega|_A = \mathscr{F},\end{aligned}$$

and

$$\begin{aligned}\mathscr{F}_1\mathscr{H}^* &= \Omega^{-1}M_g\Omega\mathscr{H}^* = \Omega^{-1}M_gS\Omega = \Omega^{-1}SM_g\Omega \\ &= \Omega^{-1}S\Omega\Omega^{-1}M_g\Omega = \Omega^{-1}\Omega\mathscr{H}^*\mathscr{F}_1 = \mathscr{H}^*\mathscr{F}_1.\end{aligned}$$ □

The generalization of these results to higher dimensions is an open problem.

7. G. Pisier's proof of the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_j$, $1 < p < \infty$ is analytic in the sense that it does not use the Brownian motion, but instead uses a variation of the Calderón's method of rotations and methods of transference developed by R. R. Coifman and G. Weiss in [57]. It turns out that the inequalities needed, also include the classical case, are consequences of the one-dimensional results. Using the same method, he is also able to prove the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_\beta$, if $|\beta|$ is odd (see [227]).

8. In [69], O. Dragicevic and A. Volberg get the $L^p(\gamma_d)$-boundedness of the vector of Gaussian Riesz transforms $(\mathscr{R}_1, \cdots, \mathscr{R}_d)$

$$\|\Big(\sum_{j=1}^{d} |\mathscr{R}_j f|^2\Big)^{1/2}\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma},$$

obtained from a dimensionless bilinear estimate of the Littlewood–Paley type, using the Bellman function technique (see [203]). This technique also works in the classical case.

9. The boundedness of the Riesz transforms can be used to obtain the Littlewood–Paley estimates for the spatial gradient; that is to say, the opposite direction of Stein's scheme is also possible.

10. In [174], G. Mauceri and S. Meda also proved that imaginary powers of the Ornstein–Uhlenbeck operator $(-L)^{i\alpha}$ and Riesz transforms $\mathscr{R}_\beta$, of any order $|\beta| > 0$, are bounded from L^∞ to $BMO(\gamma_d)$ (with a bound dependent on the dimension). They also proved that imaginary powers are bounded from $H^1_{at}(\gamma_d)$ to $L^1(\gamma_d)$.

11. In [63], E. Dalmaso and R. Scotto have studied the boundedness of general Gaussian singular integrals in variable $L^{p(\cdot)}$ Gaussian spaces following S. Pérez's approach in [221].

12. The *Jacobi–Riesz transform* can be defined spectrally as

$$R^{\alpha,\beta} = \sqrt{1-x^2}\,\frac{d}{dx}(\mathscr{L}^{\alpha,\beta})^{-1/2}, \tag{9.44}$$

where $(\mathscr{L}^{\alpha,\beta})^{-\nu/2}$ is the Jacobi–Riesz potential of order $\nu/2$. $(\mathscr{L}^{\alpha,\beta})^{-\nu/2}$ can be represented as

$$(\mathscr{L}^{\alpha,\beta})^{-\nu/2} f = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} P_t^{(\alpha,\beta)} f\,dt;$$

moreover, it is easy to see that for $f \in L^2\left([-1,1], \mu_{(\alpha,\beta)}\right)$, with Laguerre expansion

$$f = \sum_{k=0}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{h_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)},$$

where

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

then, $(\mathscr{L}^{\alpha,\beta})^{-\nu/2} f$ will have a Jacobi expansion

$$(\mathscr{L}^{\alpha,\beta})^{-\nu/2} f = \sum_{k=0}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{\hat{h}_k^{(\alpha,\beta)}} \lambda_k^{-\nu/2} P_k^{(\alpha,\beta)} \tag{9.45}$$

and since

$$\frac{d}{dx}\left\{P_k^{(\alpha,\beta)}(x)\right\} = \frac{(k+\alpha+\beta+1)}{2} P_{k-1}^{(\alpha+1,\beta+1)}(x), \tag{9.46}$$

then its Jacobi–Riesz transform have an expansion

$$R^{\alpha,\beta} f = \sum_{k=1}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{\hat{h}_k^{(\alpha,\beta)}} \lambda_k^{-1/2} \frac{(k+\alpha+\beta+1)}{2} \sqrt{1-x^2} P_{k-1}^{(\alpha+1,\beta+1)}. \tag{9.47}$$

where $\lambda_k = k(k+\alpha+\beta+1)$.

Observe that (9.47) is not a proper Jacobi expansion given the presence of the factor $\sqrt{1-x^2}$. This is different than the Hermite case, and complicates the arguments. The L^p-boundedness of the Riesz–Jacobi transform $R^{\alpha,\beta}$, was proved by Z. Li [157], in the case $d=1$, and by A. Nowak and P. Sjögren, [213, Theorem 5.1] in the case $d \geq 1$,

Theorem 9.25. *Assume that $1 < p < \infty$ and $\alpha, \beta \in [-1/2, \infty)^d$. There exists a constant c_p such that*

$$\|R_i^{\alpha,\beta} f\|_{p,(\alpha,\beta)} \leq c_p \|f\|_{p,(\alpha,\beta)}. \tag{9.48}$$

for all $i = 1, \cdots, d$.

For the particular case of the Gegenbauer polynomials, this result was obtained in the one-dimensional case by B. Muckenhoupt and E. Stein in their seminal article of 1965 [199].

13. The *Laguerre–Riesz transform* can be defined spectrally, as

$$R^{\alpha} = \sqrt{x} \frac{d}{dx} (\mathscr{L}^{\alpha})^{-1/2}; \tag{9.49}$$

therefore for $f \in L^2((0,\infty),\mu_\alpha)$ with Laguerre expansion

$$f = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+1)} \langle f, L_k^\alpha \rangle L_k^\alpha$$

its Laguerre–Riesz transform has expansion

$$R^\alpha f(x) = -\sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+1)} (\sqrt{k})^{-1} \langle f, L_k^\alpha \rangle \sqrt{x} L_{k-1}^{\alpha+1}(x). \tag{9.50}$$

Observe that (9.50) is not a Laguerre expansion given the presence of the factor $\sqrt{x}$. This is different from the Hermite case, and complicates the arguments; thus, the proofs in the Laguerre setting are more involved than that of the Hermite case. The L^p boundedness of the Laguerre–Riesz transform was proved, for the case $d = 1$ by B. Muckenhoupt [196], and the case $d \geq 1$ was proved by A. Nowak [209] using Littlewood–Paley's theory and also following Stein's scheme in [253].

Theorem 9.26. *Assume that $1 < p < \infty$ and $\alpha \in [-1/2,\infty)^d$. There exists a constant C_p such that*

$$\|R_i^\alpha f\|_{p,\alpha} \leq C_p \|f\|_{p,\alpha}. \tag{9.51}$$

for all $i = 1,\cdots,d$,.

14. In [201], E. Navas and W. Urbina develop a transference method to obtain the L^p-boundedness, $1 < p < \infty$ of the Gaussian Riesz transforms $\mathscr{R}_i$, and the L^p-boundedness of the Laguerre–Riesz transform R_i^α from the L^p-boundedness of the Jacobi–Riesz transform $R_i^{\alpha,\beta}$ for the one-dimensional case by using the well-known asymptotic relations between Jacobi polynomials and other classical orthogonal polynomials (10.64) and (10.67) (see also [262, (5.3.4),(5.6.3)]). The transference for the higher dimensional case is open.

15. In [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, and W. Urbina proved the $L^p(\mu_\alpha)$-boundedness of higher-order Laguerre–Riesz transforms and the weak type $(1,1)$ for the Riesz–Laguerre transform of order 2. However, the methods they used impose substantial restrictions on the admissible values of type multi-index α, because the result is obtained by means of transference from the Hermite setting using the classical relations formulas that relate to the Hermite polynomials and Laguerre polynomials (10.36). This method has been used by many authors, for instance [152, 19, 68, 123], to study different properties of Laguerre semigroups of half-integer type, which are related to Hermite semigroups. They provide a considerable extension of this technique, and show how to transfer higher-order Riesz type operators and certain differential operators. Although the corresponding formulas are rather complex, because their combinatorial component, they shed some light on an interplay between Hermite and Laguerre expansions.

16. In [237], E. Sasso proved that the first-order Riesz–Laguerre transforms associated with the Laguerre semigroup are of weak type $(1,1)$ with respect to the Gamma measure, analogous to the Gaussian case. She also presents a counterexample showing that for the Riesz transforms of order three or higher, the weak type $(1,1)$ estimate fails.

17. In 2006, A. Nowak and K. Stempak in [210] proposed a fairly general and unified approach to the theory of Riesz transforms and conjugacy in the setting of multi-dimensional orthogonal expansions (polynomials and functions), proving their L^2-boundedness under certain conditions. Additionally, they attempt to offer a unified conjugacy scheme that includes definitions of Riesz transforms and conjugate Poisson integrals for a broad class of expansions. The postulated definitions were supported by a good L^2-theory, the existence of Cauchy–Riemann-type equations, and numerous examples in the literature that are covered by the scheme. There is, however, a shortcoming of this unified conjugacy scheme manifested in a lack of symmetry in the decomposition (2.13). Asymmetry of the decomposition of L has, in fact, a deep impact on the whole conjugacy scheme postulated in [210]. Then, in [211], they proposed a symmetrization procedure and consider the resulting symmetrized situation. The construction is motivated to some extent by the setting of the Dunkl harmonic oscillator with the underlying reflection group isomorphic to $\mathbb{Z}_2^d = \{0,1\}^d$, and gives a different notion of conjugacy (for more details see [211]).

18. In 2015, L. Forzani, E. Sasso, and R. Scotto [93] extended Nowak and Stempak's approach to the general case of multi-dimensional orthogonal polynomial expansions, proving the L^p boundedness, $1 < p < \infty$ of those Riesz transforms, with constants independent of dimension.

19. In [296], B. Wróbel derives a scheme to deduce the L^p boundedness of certain d-dimensional Riesz transforms from the L^p boundedness of appropriate one-dimensional Riesz transforms, by using an H^∞ joint functional calculus for strongly commuting operators. Moreover, the L^p bounds obtained are independent of the dimension. The scheme is applied to Riesz transforms connected with orthogonal expansions and discrete Riesz transforms on products of groups with polynomial growth, which of course include the Gaussian case. For the vector case, an explicit Bellman function is used to prove a bilinear embedding theorem for operators associated with general multi-dimensional orthogonal expansions on product spaces and as a consequence the L^p boundedness of the vector of Riesz transforms is obtained (see [297]).