8

Gaussian Fractional Integrals and Fractional Derivatives, and Their Boundedness on Gaussian Function Spaces

In this chapter, we study several important operators in Gaussian harmonic analysis. First, we consider Riesz and Bessel potentials with respect to the Ornstein–Uhlenbeck operator L, and then, Riesz and Bessel fractional derivatives. We study their regularity on Gaussian Lipschitz spaces, on Gaussian Besov–Lipschitz spaces, and on Gaussian Triebel–Lizorkin spaces. The results obtained are essentially similar to the classical results, as mentioned before, the methods of proofs are completely different. The boundedness results for Gaussian Besov–Lipschitz and Triebel–Lizorkin spaces were obtained by A. E. Gatto, E. Pineda, and W. Urbina, and appeared initially in [110] and [111]. These results can be extended to the case of Laguerre and Jacobi expansions by analogous arguments.

8.1 Riesz and Bessel Potentials with Respect to the Gaussian Measure

Gaussian Riesz Potentials

In the classical case, the Riesz potential of order $\beta > 0$ is defined as the negative fractional powers of $-\Delta$,

$$(-\Delta)^{-\beta/2},$$

which means, using Fourier transform, that

$$((-\Delta)^{-\beta/2} f)\widehat{\ }(\xi) = (2\pi|\xi|)^{-\beta} \hat{f}(\xi). \tag{8.1}$$

For more details, see [252, 118].

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The Gaussian fractional integrals or Gaussian Riesz potentials can also be defined as negative fractional powers of $(-L)$. However, because the Ornstein–Uhlenbeck operator has eigenvalue 0, the negative powers are not defined on all of $L^2(\gamma_d)$; thus, we need to be more careful with the definition. Let us consider $\Pi_0 f = f - \int_{\mathbb{R}^d} f(y)\gamma_d(dy)$ the $L^2(\gamma_d)$ for $f \in L^2(\gamma_d)$, the orthogonal projection on the orthogonal complement of the eigenspace corresponding to the eigenvalue 0.

Definition 8.1. *The Gaussian fractional integral or Riesz potential of order $\beta > 0$, I_β is defined spectrally as*

$$I_\beta = (-L)^{-\beta/2}\Pi_0, \tag{8.2}$$

which means that for any multi-index ν, $|\nu| > 0$ its action on the Hermite polynomial $\mathbf{H}_\nu$ is given by

$$I_\beta \mathbf{H}_\nu(x) = \frac{1}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x), \tag{8.3}$$

and for $\nu = 0 = (0,\ldots,0)$, $I_\beta(\mathbf{H}_0) = 0$.

By linearity, using the fact that the Hermite polynomials are an algebraic basis of $\mathscr{P}(\mathbb{R}^d)$, I_β can be defined for any polynomial function $f(x) = \sum_\nu \widehat{f}_\gamma(\nu)\mathbf{H}_\nu$ as

$$I_\beta f(x) = \sum_\nu \frac{\widehat{f}_\gamma(\nu)}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x) = \sum_{k\geq 1}\frac{1}{k^{\beta/2}}\mathbf{J}_k f(x). \tag{8.4}$$

and similarly for $f \in L^2(\gamma_d)$.

From (8.4), it is clear that the Gaussian Riesz potentials I_β are the simplest Meyer's multipliers, because in this case

$$m(k) = \frac{1}{k^\beta} = h(\frac{1}{k^\beta}), \tag{8.5}$$

with $h(x) = x$ the identity function.

Proposition 8.2. *The Gaussian Riesz potential I_β, $\beta > 0$, has the following integral representations, for $f \in (\mathbb{R}^d)$ is a polynomial or $f \in C_b^2(\mathbb{R}^d)$,*

$$I_\beta f(x) = \frac{1}{\Gamma(\beta/2)}\int_0^\infty t^{\beta/2-1}T_t(I-\mathbf{J}_0)f(x)\,dt, \tag{8.6}$$

with respect to the Ornstein–Uhlenbeck semigroup, and

$$I_\beta f(x) = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}P_t(I-\mathbf{J}_0)f(x)\,dt, \tag{8.7}$$

with respect to the Poisson–Hermite semigroup, *Proof.* It is enough to prove that (8.6) holds for the Hermite polynomials. By the change of variables $u = |\nu|t$

$$\begin{aligned}\frac{1}{\Gamma(\beta/2)}\int_0^\infty t^{\beta/2-1}(T_t(I-\mathbf{J}_0)\mathbf{H}_\nu)(x)\,dt &= \frac{1}{\Gamma(\beta/2)}\int_0^\infty t^{\beta/2-1}e^{-t|\nu|}\,dt\,\mathbf{H}_\nu(x)\\ &= \frac{1}{\Gamma(\beta/2)}\int_0^\infty \frac{u^{\beta/2-1}}{|\nu|^{\beta/2-1}}e^{-u}\frac{du}{|\nu|}\,\mathbf{H}_\nu(x)\\ &= \frac{1}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x).\end{aligned}$$

Then, again as the Hermite polynomials are an algebraic base of the set of polynomials $\mathscr{P}(\mathbb{R}^d)$, the formula holds for any polynomial. It can be proved that (8.6) also holds for $f \in C_b^2(\mathbb{R}^d)$.

Observe that the integral representation (8.7) only means a change of scale, as $I_\beta = [(-L)^{1/2}]^{-\beta}$. Taking the change of variables $u = t\sqrt{|\nu|}$,

$$\begin{aligned}\frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}(P_t(I-\mathbf{J}_0)\mathbf{H}_\nu)(x)\,dt &= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-t\sqrt{|\nu|}}\,dt\,\mathbf{H}_\nu(x)\\ &= \frac{1}{\Gamma(\beta)}\int_0^\infty \frac{u^{\beta-1}}{|\nu|^{(\beta-1)/2}}e^{-u}\frac{du}{\sqrt{|\nu|}}\,\mathbf{H}_\nu(x)\\ &= \frac{1}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x),\end{aligned}$$

again using that the Hermite polynomials are an algebraic base of the set of polynomials $\mathscr{P}(\mathbb{R}^d)$, □

Following the classical case, in general, we prefer to use the representation of I_β (8.7), using the Poisson–Hermite semigroup. This representation will be crucial later to get several boundedness results to operators associated with L.

On the other hand, let us recall that in the classical case (see [252, Chapter V §1]), Riesz potentials have the following integral representation:

$$(-\Delta)^{-\beta/2}f(x) = C_\beta\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-\beta}}dy.$$

In the Gaussian case, we can also get an integral representation, as follows:

Theorem 8.3. *The Gaussian Riesz potential I_β, $\beta > 0$, has an integral representation,*

$$I_\beta f(x) = \int_{\mathbb{R}^d} N_{\beta/2}(x,y)f(y)dy, \tag{8.8}$$

where the kernel $N_{\beta/2}(x,y)$ is defined as

$$N_{\beta/2}(x,y) = \frac{1}{\pi^{d/2}\Gamma(\beta/2)}\int_0^1(-\log r)^{\beta/2-1}\Big(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2}\Big)\frac{dr}{r}. \tag{8.9}$$

Proof. To find the integral representation of I_β, because the negative powers of L do not exist in all of $L^2(\gamma_d)$, we add a small multiple of the identity. Hence, let us consider the operator $(\varepsilon I_d - L)$, where I_d is the identity in $\mathbb{R}^d$ and $\varepsilon > 0$, and let us take its negative powers. The advantage of this trick is that it can be represented as a Laplace transform and this allows us to use the expression for Mehler's kernel $M_t(x,y)$. More precisely, for $\varepsilon > 0$ and $\beta > 0$,

$$(\varepsilon I - L)^{-\beta/2} = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-(\varepsilon I - L)t} dt; \tag{8.10}$$

therefore, the kernel of $(\varepsilon I - L)^{-\beta/2}$ is

$$\begin{aligned} N_{\beta/2,\varepsilon}(x,y) &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} M_t(x,y) dt \\ &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} dt, \end{aligned}$$

because, if $f \in L^1(\gamma_d)$,

$$\begin{aligned} (\varepsilon I - L)^{-\beta/2} f(x) &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-(\varepsilon I - L)t} f(x) dt \\ &= \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^d} \Big(\int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} M_t(x,y) dt \Big) f(x) dy. \end{aligned}$$

As Π_0 is the orthogonal projection of the orthogonal complement of the eigenspace corresponding to the eigenvalue 0, then $\mathbf{J}_0 = I - \Pi_0$, where $\mathbf{J}_0$ is the orthogonal projection on the subspace generated by $\mathbf{H}_0 \equiv 1$ (that is, the constants), and then we have

$$(\varepsilon I - L)^{-\beta/2} \Pi_0 = (\varepsilon I - L)^{-\beta/2} - \varepsilon^{-\beta/2} \mathbf{J}_0.$$

The kernel of $\mathbf{J}_0$ is clearly $\pi^{-d/2} e^{-|y|^2}$ and trivially $\varepsilon^{-\beta} = \int_0^\infty t^{\beta-1} e^{-\varepsilon t} dt$, then the kernel of $(\varepsilon I - L)^{-\beta/2} \Pi_0$ is

$$\frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} \Big(M_t(x,y) - \pi^{-d/2} e^{-|y|^2} \Big) dt.$$

We can take $\varepsilon \to 0$ in the integral above without problems, then

$$I_\beta = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} T_t (I - \mathbf{J}_0) dt.$$

Therefore, the kernel of I_β is given by

$$\begin{aligned} N_{\beta/2}(x,y) &= \frac{1}{\pi^{d/2}\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} \Big(\frac{e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}}{(1-e^{-2t})^{d/2}} - e^{-|y|^2} \Big) dt \\ &= \frac{1}{\pi^{d/2}\Gamma(\beta/2)} \int_0^1 (-\log r)^{\beta/2-1} \Big(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \Big) \frac{dr}{r}. \end{aligned} \tag{8.11}$$

taking $r = e^{-t}$. Thus

$$I_\beta f(x) = \int_{\mathbb{R}^d} N_{\beta/2}(x,y) f(y) dy.$$

□

In [102], it is proven that these operators are not of weak type $(1,1)$ with respect to γ. On the other hand, the strong type (p,p), for $1 < p < \infty$,

$$||I_\beta||_{p,\gamma_d} \leq C_p ||f||_{p,\gamma_d}, \tag{8.12}$$

follows either directly, from the hypercontractivity property of the Ornstein–Uhlenbeck semigroup, or by applying P. A. Meyer's multiplier theorem, Theorem 6.2.

The classical Riesz potentials are homogeneous (see E. Stein [252, Chapter V (10)]), but it is easy to see that this is not the case for the Gaussian Riesz potentials I_β.

Moreover, it is well-known that the classical Riesz potentials are of strong type (p,q) with $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$, that is to say, the classical Riesz potentials "improve" in the sense that $I_\beta : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ continuously, with $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$. The Gaussian Riesz potentials, however, do not improve integrability. More formally, for any $\beta > 0$ for the Gaussian Riesz potential I_β, there is no $q > p$ such that it sends $L^p(\gamma_d) \to L^q(\gamma_d)$ continuously. This can be proved using the following counterexample, due to L. Forzani and W. Urbina, [87]. For every $a > 0$, let us split I_β as,

$$I_\beta f(x) = I_1 f(x) + I_2 f(x) = \int_{\mathbb{R}^d} N^1_\beta(x,y) f(y) dy + \int_{\mathbb{R}^d} N^2_\beta(x,y) f(y) dy,$$

where the kernel (8.11) is split into the sum of two parts,

$$N^1_\beta(x,y) = C_{\beta,d} \int_0^{e^{-a}} (-\log r)^{\beta-1} \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \right) \frac{dr}{r}$$

$$N^2_\beta(x,y) = C_{\beta,d} \int_{e^{-a}}^1 (-\log r)^{\beta-1} \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \right) \frac{dr}{r}.$$

The operator

$$I_1 f(x) = \int_{\mathbb{R}^d} N^1_\beta(x,y) f(y) dy$$

can be written as

$$I_1 f(x) = \frac{1}{\Gamma(\beta)} \int_a^{+\infty} t^{\beta-1} T^t \Pi_0 f(x) dt,$$

where $T_t = e^{Lt}$ is the Ornstein–Uhlenbeck semigroup (see Chapter 2). Taking into account that T^t is a hypercontractive semigroup, I_1 turns out to be of strong type (p,q), with $q = 1 + (p-1)e^{4t}$.

Additionally, I_2 is an operator defined for every function on $L^p(\gamma_d)$. To prove that it does not improve integrability, it would be enough to show that for every $q > p$ there is a function $f \in L^p(d\gamma)$ such that $I_2 f \notin L^q(\gamma_d)$. Let us take $\frac{1}{q} < c < \frac{1}{p}$ and $f(y) = e^{c|y|^2}\chi_{|y|\geq 1} \in L^p(d\gamma)$.[1] It can be proved (see [86]), that the kernel $N^2_\beta(x,y) \geq Ce^{|x|^2}\frac{e^{|y|^2}}{|y|}$ in the region $\{(x,y): \quad |x| \geq 1, \quad \frac{1}{4}|y|^2 + 1 < |x|^2 < \frac{3}{4}|y|^2\}$. Hence, $I_2(e^{c|x|^2}) \geq \frac{e^{c|x|^2}}{|x|^2}$ for $|x| \geq 1$; therefore, $I_2(e^{c|x|^2}) \notin L^q(\gamma_d)$.

The reason why Gaussian Riesz potentials do not improve integrability is the fact that L satisfies a logarithmic Sobolev inequality and not a Sobolev inequality. Nevertheless, a $L^p \log L(\gamma_d)$ inequality can still be pulled out. Following E. Fabes' suggestion, applying certain techniques used by L. Gross in [119], to prove that hypercontractivity implies a Sobolev logarithmic inequality, we can prove the following result:

Proposition 8.4. *For any $\beta > 0$ the Gaussian Riesz potential I_β maps $L^p(\gamma_d)$ into $L^p \log L(\gamma_d)$ continuously; in other words, the following inequality holds*

$$\int_{\mathbb{R}^d} |I_\beta f(x)|^p \log |I_\beta f(x)| \, \gamma(dx) \leq C\left(\int_{\mathbb{R}^d} |f(x)|^p \, d\gamma + ||f||^p_{p,\gamma} \log ||f||_{p,\gamma}\right), \quad (8.13)$$

for each $f \in L^p(\gamma_d)$.

Proof. Indeed, for $\beta > 0$, consider the generalized Poisson–Hermite semigroup $P^\beta_t = e^{-(-L)^\beta t}$, defined in (3.38). Let f be a polynomial, such that $\int_{\mathbb{R}^d} f \, d\gamma = 0$, $I_\beta f \neq 0$, and set $F(t) = P^\beta_t(I_\beta f)$, then for every $t > 0$,

$$\frac{||F(t)||_{1+(p-1)e^{4t},\gamma} - ||F(0)||_{p,\gamma}}{t} \leq \frac{1-1}{t} ||I_\beta f||_{p,\gamma} = 0 \quad (8.14)$$

where the above inequality is a consequence of the hypercontractivity of P^β_t. In (8.14) we let $t \to 0^+$ to get

$$\frac{d}{dt} ||F(t)||_{1+(p-1)e^{4t},\gamma}\Big|_{t=0} \leq 0 \quad (8.15)$$

Using a lemma proved in [119],

$$\frac{d}{dt} ||F(t)||_{1+(p-1)e^{4t},\gamma}\Big|_{t=0} = ||I_\beta f||^{1-p}_{p,\gamma}[p^{-1}4(p-1)(\int_{\mathbb{R}^d} |I_\beta|^p \log |I_\beta| \, d\gamma \quad (8.16)$$
$$- ||I_\beta f||_{p,\gamma} \log ||I_\beta f||_{p,\gamma}) + Re\langle F'(0), \operatorname{sgn}(I_\beta f)|I_\beta f|^{p-1}\rangle_\gamma].$$

[1] For $d = 1$ the function f, defined above, is the same as that used by H. Pollard in his famous counterexample in [230].

But $F'(0) = (-L)^{\beta} I_{\beta} f = f$. Now, combining (8.15) and (8.16) we get

$$\int_{\mathbb{R}^d} |I_\beta f(x)|^p \log |I_\beta f(x)| \, d\gamma \leq C(||I_\beta f||_{p,\gamma}^p \log ||I_\beta f||_{p,\gamma} + \langle |f|, |I_\beta f|^{p-1} \rangle_\gamma).$$

By applying Hölder's inequality to the second term of the sum appearing on the right-hand side of the above inequality, and then the $L^p(d\gamma)$ continuity of I_β, we get inequality (8.13). □

Thus, although I_β do not improve in the $L^p(\gamma_d)$ "scale," they do improve in the "logarithmic scale" $L^p(\gamma_d) \log L(\gamma_d)$.

Gaussian Bessel Potentials

Definition 8.5. *The Gaussian Bessel potential of order $\beta > 0$, $\mathscr{J}_\beta$, is defined spectrally as*

$$\mathscr{J}_\beta = (I + \sqrt{-L})^{-\beta}, \tag{8.17}$$

meaning that for the Hermite polynomials we have,

$$\mathscr{J}_\beta \mathbf{H}_\nu(x) = \frac{1}{(1+\sqrt{|\nu|})^\beta} \mathbf{H}_\nu(x). \tag{8.18}$$

Again, by linearity, $\mathscr{J}_\beta$ can be extended to any polynomial; thus, if $f = \sum_k \mathbf{J}_k f$, then

$$\mathscr{J}_\beta = \sum_k \frac{1}{(1+\sqrt{|k|})^\beta} \mathbf{J}_k f.$$

From (8.18), it is clear that the Gaussian Bessel potentials $\mathscr{J}_\beta$ are not Meyer's multipliers, but a composition of two Meyer's multipliers, because in this case

$$\frac{1}{(1+\sqrt{k})^\beta} = \left(\frac{1}{\sqrt{k}} + 1\right)^{-\beta} \frac{1}{k^{\beta/2}} = m_1(L)(m_2(L)(k)), \tag{8.19}$$

with $h_1(x) = (1+x)^{-\beta}$ and $h_2(x) = x$.

Using a similar argument to that above (8.7), the Bessel potentials can be represented as

$$\mathscr{J}_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^\beta e^{-t} P_t f(x) \frac{dt}{t} = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^{\beta-1} e^{-t} P_t f(x) dt \tag{8.20}$$

P. A. Meyer's multiplier theorem, Theorem 6.2, shows that $\mathscr{J}_\beta$ is a bounded operator on $L^p(\gamma_d)$, $1 < p < \infty$, and again (8.20) can be extended to $L^p(\gamma_d)$, using the density of the polynomials there.

On the other hand, again using P. A. Meyer's multiplier theorem, Theorem 6.2, we get that the operators

$$\frac{I_\beta}{\mathscr{I}_\beta}, \text{ and } \frac{\mathscr{I}_\beta}{I_\beta}$$

are bounded on every $L^p(\gamma_d), 1 < p < \infty$; because, for instance, for any multi-index ν, $|\nu| > 0$

$$\left(\frac{I_\beta}{\mathscr{I}_\beta}\right)\mathbf{H}_\nu(x) = \left(\frac{(1+\sqrt{|\nu|})^\beta}{|\nu|^{\beta/2}}\right)\mathbf{H}_\nu(x) = \left(\frac{1}{\sqrt{|\nu|}}+1\right)^\beta \mathbf{H}_\nu(x) = h(\frac{1}{|\nu|^{1/2}})\mathbf{H}_\nu(x),$$

with $h(x) = (x+1)^\beta$. These give the relation between the Riesz and Bessel potentials, similar to those in the classical case (see [252, Chapter V. Lemma. 2]).

It is easy to see, from the fact that $\mathscr{I}_\beta$ is a multiplier, that it is also a bijection over the set of polynomials $\mathscr{P}$. Additionally, the Gaussian Sobolev spaces can be characterized in terms of Gaussian Bessel potentials,

Proposition 8.6. *For* $\beta \geq 0$ *and* $1 \leq p < \infty$

$$L^p_\beta(\gamma_d) = \{\mathscr{I}_\beta f : f \in L^p(\gamma_d)\} \tag{8.21}$$

Proof. First of all, observe that $\mathscr{I}_\beta$ maps the family of polynomials $\mathscr{P}(\mathbb{R}^d)$ into itself injectively. Then, as we already know $\mathscr{I}_\beta$ is continuous in $L^p(\gamma_d)$, then we conclude $\mathscr{I}_\beta : L^p(\gamma_d) \to L^p_\alpha(\gamma_d)$ is bijective. □

Moreover, considering the family $\{\mathscr{I}_\beta\}_\beta$ it is easy to see that it is a strongly continuous semigroup on $L^p(\gamma)$, $1 \leq p < \infty$, having as infinitesimal generator $\frac{1}{2}\log(I - L)$.

8.2 Fractional Derivatives with Respect to the Gaussian Measure

Gaussian Riesz Fractional Derivate

In the classical case, fractional derivates for the Laplacian operator are defined as,

$$(-\triangle)^{\beta/2} f(x) = c_\beta \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{f(x+y) - f(x)}{|y|^{d+\beta}} dy$$

for $0 < \beta < 2$, $c_\beta = \frac{2^\beta \Gamma(d+\beta/2)}{\pi^{d/2}\Gamma(-\beta/2)}$, see [255].

For the case of doubling measures, and more recently for s-dimensional non-doubling measures, this has been generalized by A. E. Gatto, C. Segovia, and S. Vàgi in [108].

On the other hand, observe that

$$\int_{\mathbb{R}^d} \frac{f(x+y)-f(x)}{|y|^{d+\beta}}dy = C_{\beta,d}\int_0^\infty t^{-\beta-1}\left(P_t f(x)-f(x)\right)dt, \tag{8.22}$$

where P_t is the classical Poisson semigroup. Then, following the classical case:

Definition 8.7. *The Gaussian Riesz fractional derivative of order $\beta > 0$, D^β is defined spectrally as*

$$D^\beta = (-L)^{\beta/2}, \tag{8.23}$$

meaning that for the Hermite polynomials, we have

$$D^\beta \mathbf{H}_\nu(x) = |\nu|^{\beta/2}\mathbf{H}_\nu(x). \tag{8.24}$$

Thus, by linearity, D^β can be extended to any polynomial (see [164] and [224]).

Now, if f is a polynomial, by the linearity of the operators I_β and D^β, (8.3) and (8.24), we get

$$I_\beta(D^\beta f) = D^\beta(I_\beta f) = \Pi_0 f. \tag{8.25}$$

In the case of $0<\beta<1$ we have the following integral representation for f a polynomial,

$$D^\beta f(x) = \frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}(I-P_t)f(x)dt, \tag{8.26}$$

where $c_\beta = \int_0^\infty u^{-\beta-1}(1-e^{-u})du$: because for the Hermite polynomials we have, by the change of variables $u=\sqrt{|\nu|}t$,

$$\begin{aligned}\frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}(I-P_t)\mathbf{H}_\nu(x)(x)dt &= \Big(\frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}\left(e^{-t\sqrt{|\nu|}}-1\right)dt\Big)\mathbf{H}_\nu(x)\\ &= |\nu|^{\beta/2}\Big(\frac{1}{c_\beta}\int_0^\infty u^{-\beta-1}\left(e^{-u}-1\right)du\Big)\mathbf{H}_\nu(x)\\ &= |\nu|^{\beta/2}\mathbf{H}_\nu(x) = D^\beta\mathbf{H}_\nu(x).\end{aligned}$$

The identity (8.26) is very important in the development of a version of A. P. Calderón's reproduction formula (see Theorem 8.31 below).

Now, if $\beta \geq 1$, let k be the smallest integer greater than β i.e. $k-1\leq\beta<k$, then the fractional derivative D^β can be represented as

$$D^\beta f = \frac{1}{c_\beta^k}\int_0^\infty t^{-\beta-1}(I-P_t)^k f\,dt, \tag{8.27}$$

where $c_\beta^k = \int_0^\infty u^{-\beta-1}(1-e^{-u})^k\,du$ and f a polynomial function (see [239]).

As was mentioned earlier, fractional derivatives D^β can be used to characterize the Gaussian Sobolev spaces $L^p_\beta(\gamma_d)$. First, we need to extend the fractional derivative operator D^{β} to all the Gaussian Sobolev spaces $L^p_\beta(\gamma_d)$, $1 < p < \infty$. The union of these spaces

$$L_\beta(\gamma_d) := \bigcup_{p>1} L^p_\beta(\gamma_d)$$

is a natural domain of D^{β}. Observe that the definition of D^{β} in all the spaces $L^p_\beta(\gamma_d)$, $1 < p < \infty$, is based on an application of Meyer's multiplier theorem, Theorem 6.2.

Theorem 8.8. *Let $\beta > 0$ and $1 < p < \infty$.*

i) If $\{P_n\}_n$ is a sequence of polynomials such that $\lim_{n\to\infty} P_n = f$ in $L^p_\beta(\gamma_d)$, then $\lim_n D^\beta P_n$ exists in $L^p_\beta(\gamma_d)$ and does not depend on the choice of a sequence $\{P_n\}_n$. If $f \in L^p_\beta(\gamma_d) \cap L^r_\beta(\gamma_d)$, then the limit does not depend on the choice of p or r. Thus, the fractional derivative is well defined by

$$D^\beta f = \lim_{n\to\infty} D^\beta P_n \text{ in } L^p_\beta(\gamma_d), \quad as \quad \lim_{n\to\infty} P_n = f \text{ in } L^p_\beta(\gamma_d),$$

$f \in L_\beta(\gamma_d)$, is well defined.

ii) $f \in L^p_\beta(\gamma_d)$ if and only if $D^\beta f \in L^p(\gamma_d)$. Moreover,

$$B_{p,\beta} \|f\|_{p,\beta} \le \left\|D^\beta f\right\|_{p,\gamma_d} \le A_{p,\beta} \|f\|_{p,\beta} . \tag{8.28}$$

Proof. *ii*) Let f be a polynomial. Then

$$D^\beta f = \sum_{n\ge 0} \left(\frac{n}{1+n}\right)^{\beta/2} \mathbf{J}_n g,$$

where $g = (1-L)^{-\beta/2} f$. Note that g is also a polynomial. Observe that by construction,

$$\|f\|_{p,\beta} = \|g\|_{p,\gamma}.$$

Using Meyer's multiplier theorem, Theorem 6.2, with the holomorphic function $h(z) = (1+z)^{-\beta/2}$, we get

$$\left\|D^\beta f\right\|_{p,\gamma} \le C_1 \|g\|_{p,\gamma}.$$

To prove the converse inequality, observe that the polynomial g can be rewritten as

$$g = \sum_{n\ge 0} \left(\frac{n}{1+n}\right)^{\beta/2} \mathbf{J}_n(D^\beta f),$$

and using Meyer's multiplier theorem again we obtain,

$$\|h\|_{p,\gamma} \le C_2 \left\|D^\beta f\right\|_{p,\gamma}.$$

Thus, we get (8.28) for polynomials.

i) The completeness of $L^p_\beta(\gamma_d)$ can be proved using (8.28), and the fact that for $r \geq p$ the embedding $L^r_\beta(\gamma_d) \subset L^p_\beta(\gamma_d)$ is continuous. Finally, from there, we can obtain (8.28) for any $f \in L^p_\beta(\gamma_d)$. □

From the previous result and Proposition 7.3, we can immediately obtain a characterization of the Gaussian Sobolev spaces.

Corollary 8.9. *Assume that* $1 < p < \infty$ *and* $\beta > 0$*. Then*

$$L^p_\beta(\gamma_d) = \Big\{ f \in L_\beta(\gamma_d) : D^\beta f \in L^p(\gamma_d) \Big\}. \tag{8.29}$$

If $\beta = k \in \mathbb{N}$*, then*

$$L^p_k(\gamma_d) = \Big\{ f \in L_k(\gamma_d) : D^j f \in L^p(\gamma_d), j \leq k \Big\}. \tag{8.30}$$

This characterization of Sobolev spaces is the most common one in the classical case.

Gaussian Bessel Fractional Derivates

We can also define the Gaussian Bessel fractional derivatives, $\mathscr{D}^\beta$.

Definition 8.10. *The Gaussian Bessel fractional derivatives of order* β*,* $\mathscr{D}^\beta$*, are defined spectrally as*

$$\mathscr{D}^\beta = (I + \sqrt{-L})^\beta, \tag{8.31}$$

which means that for the Hermite polynomials, we have

$$\mathscr{D}^\beta \mathbf{H}_\nu(x) = (1 + \sqrt{|\nu|})^\beta \mathbf{H}_\nu(x); \tag{8.32}$$

thus, by linearity, it can be extended to any polynomial (see [224]).

In the case of $0 < \beta < 1$, we have the following integral representation,

$$\mathscr{D}^\beta f = \frac{1}{c_\beta} \int_0^\infty t^{-\beta-1}(I - e^{-t}P_t) f dt, \tag{8.33}$$

where, as before, $c_\beta = \int_0^\infty u^{-\beta-1}(1-e^{-u})du$ and f is a polynomial.

Moreover, if $\beta \geq 1$, let k be the smallest integer greater than β, i.e.. $k-1 \leq \beta < k$, then we have the following representation of $\mathscr{D}^\beta f$

$$\mathscr{D}^\beta f = \frac{1}{c^k_\beta} \int_0^\infty t^{-\beta-1}(I - e^{-t}P_t)^k f\, dt, \tag{8.34}$$

where $c^k_\beta = \int_0^\infty u^{-\beta-1}(1-e^{-u})^k du$ and f is a polynomial (see [239]).

8.3 Boundedness of Fractional Integrals and Fractional Derivatives on Gaussian Lipschitz Spaces

The boundedness results in the case of Gaussian Lipschitz spaces initially appeared in A. E. Gatto and W. Urbina's article [109]. First, observe that the Gaussian Riesz potentials are not bounded operators on $L^\infty(\gamma_d)$ and, therefore, not on $Lip_\alpha(\gamma)$ either. Then, to make sense of Riesz potentials on L^∞, we consider, for $\beta > 0$, the *truncated Gaussian Riesz potentials,*

$$I_\beta^T f(x) = \int_0^1 t^{\beta-1} P_t f(x) dt.$$

We want to study the truncated Gaussian Riesz potentials I_β^T on the Gaussian Lipschitz spaces $Lip_\alpha(\gamma_d)$,

Theorem 8.11. *For $0 < \beta < 1$ and $\alpha > 0$, the Riesz potential of order β, I_β^T : $Lip_\alpha(\gamma_d) \to Lip_{\alpha+\beta}(\gamma_d)$ is bounded.*

Proof. Let $f \in Lip_\alpha(\gamma_d)$, i.e., $f \in L^\infty$ such that $\left\| \frac{\partial P_t f}{\partial t} \right\|_{\infty,\gamma_d} \leq A t^{-1+\alpha}$. First, observe that

$$|P_t f(x)| \leq \int_{\mathbb{R}^d} p(t,x,y)|f(y)|dy \leq \|f\|_{\infty,\gamma},$$

that is, $P_t f \in L^\infty$ and then

$$|I_\beta^T f(x)| \leq \int_0^1 t^{\beta-1}|P_t f(x)|dt \leq \int_0^1 t^{\beta-1}\|f\|_{\infty,\gamma} dt = \frac{1}{\beta}\|f\|_{\infty,\gamma}.$$

Therefore, $I_\beta^T f \in L^\infty$. Now, using the semigroup property and Fubini's theorem,

$$P_s I_\beta^T f(x) = \int_{\mathbb{R}^d} p(s,x,y) I_\beta^T f(y) dy = \int_0^1 t^{\beta-1} P_{s+t} f(y) dt = v(x,s).$$

If $\alpha + \beta < 1$, then for $0 \leq s \leq 1$

$$\begin{aligned} \frac{\partial v}{\partial s}(x,s) &= \int_0^1 t^{\beta-1} \frac{\partial}{\partial s} P_{s+t} f(x) dt = \int_0^1 t^{\beta-1} \frac{\partial}{\partial t} P_{s+t} f(x) dt \\ &= \int_0^s t^{\beta-1} \frac{\partial}{\partial t} P_{s+t} f(x) dt + \int_s^1 t^{\beta-1} \frac{\partial}{\partial t} P_{s+t} f(x) dt \\ &= (I) + (II). \end{aligned}$$

Now, for (I), because $t < s$

$$\begin{aligned} |(I)| &\leq \int_0^s t^{\beta-1} \left|\frac{\partial}{\partial t} P_{s+t} f(x)\right| dt \leq C \int_0^s t^{\beta-1}(t+s)^{\alpha-1} dt \\ &\leq C s^{\alpha-1} \int_0^s t^{\beta-1} dt = C s^{(\alpha+\beta)-1}, \end{aligned}$$

and, for (II), as $t > s$

$$|(II)| \leq \int_s^1 t^{\beta-1} |\frac{\partial}{\partial t} P_{s+t} f(x)| dt \leq C \int_s^\infty t^{\beta-1}(t+s)^{\alpha-1} dt$$
$$\leq C \int_s^\infty t^{\beta-1} t^{\alpha-1} dt = C s^{(\alpha+\beta)-1}.$$

Thus,

$$\left\| \frac{\partial}{\partial s} I_\beta f \right\|_{\infty,\gamma_d} < C s^{(\alpha+\beta)-1},$$

which implies $I_\beta f \in Lip_{\alpha+\beta}(\gamma_d)$. The general case follows in a similar manner. □

Now, we study the action of the Bessel potentials on the Gaussian Lipschitz spaces $Lip_\alpha(\gamma)$, which is much better than the case of the Riesz potentials:

Theorem 8.12. *Let $\alpha, \beta > 0$ then $\mathscr{J}_\beta$ is bounded from $Lip_\alpha(\gamma)$ to $Lip_{\alpha+\beta}(\gamma)$.*

Proof. Let $f \in Lip_\alpha(\gamma)$ and consider a fixed integer $n > \alpha + \beta$, then

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_\infty \leq A_\beta(f) t^{-n+\alpha}, \quad t > 0.$$

Using (8.20), the fact that $f \in L^\infty$, and consequently $P_{t+s} f \in L^\infty$, we obtain

$$P_t(\mathscr{J}^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} P_{t+s} f(x)\, ds; \tag{8.35}$$

therefore,

$$\|P_t(\mathscr{J}_\beta f)\|_\infty \leq \|f\|_\infty,$$

i.e. $P_t(\mathscr{J}_\beta f) \in L^\infty$.

Now, we want to verify the Lipschitz condition. Differentiating (8.35), we get

$$\frac{\partial^n P_t(\mathscr{J}_\beta f)(x)}{\partial t^n} = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} \frac{\partial^n P_{t+s} f(x)}{\partial t^n} ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} \frac{\partial^n P_{t+s} f(x)}{\partial (t+s)^n} ds,$$

and this implies

$$\left\| \frac{\partial^n P_t(\mathscr{J}_\beta f)}{\partial t^n} \right\|_\infty \leq \frac{1}{\Gamma(\beta)} \int_0^t s^{\beta-1} e^{-s} \left\| \frac{\partial^n P_{t+s} f}{\partial (t+s)^n} \right\|_\infty ds$$
$$+ \frac{1}{\Gamma(\beta)} \int_t^{+\infty} s^{\beta-1} e^{-s} \left\| \frac{\partial^n P_{t+s} f}{\partial (t+s)^n} \right\|_\infty ds$$
$$= (I) + (II).$$

Because $\beta > 0$ as $t+s > t$,

$$\begin{aligned}(I) &\leq \frac{A_\beta(f)}{\Gamma(\beta)}\int_0^t s^{\beta-1}(t+s)^{-n+\alpha}e^{-s}\,ds\\ &\leq \frac{A_\beta(f)}{\Gamma(\beta)}t^{-n+\alpha}\int_0^t s^{\beta-1}ds(\gamma) \leq Ct^{-n+\alpha+\beta}\,\|f\|_{Lip_\beta(\gamma)}.\end{aligned}$$

On the other hand, because $n > \alpha+\beta$, as $t+s > s$

$$\begin{aligned}(II) &\leq \frac{A_\beta(f)}{\Gamma(\beta)}\int_t^\infty s^{\beta-1}e^{-s}(t+s)^{-n+\alpha}\,ds \leq \frac{A_\beta(f)}{\Gamma(\beta)}\int_t^\infty s^{\beta-1}e^{-s}s^{-n+\alpha}\,ds\\ &\leq \frac{A_\beta(f)}{\Gamma(\beta)}\int_t^\infty s^{-n+\alpha+\beta-1}ds = CA_\beta(f)t^{-n+\alpha+\beta}.\end{aligned}$$

Therefore,

$$\left\|\frac{\partial^n P_t(\mathscr{I}_\beta f)}{\partial t^n}\right\|_\infty \leq CA_\beta(f)t^{-n+\alpha+\beta}, \quad t > 0.$$

Thus, $\mathscr{I}_\beta f \in Lip_{\alpha+\beta}(\gamma)$, and moreover

$$\begin{aligned}\|\mathscr{I}_\beta f\|_{Lip_{\alpha+\beta}(\gamma)} &= \|\mathscr{I}_\beta f\|_{\infty,\gamma} + A_\beta(\mathscr{I}_\beta f)\\ &\leq \|f\|_{\infty,\gamma} + CA_\beta(f) \leq C\|f\|_{Lip_\beta(\gamma)}.\end{aligned}$$

□

Finally, let us study the action of the fractional derivative D^β on the Gaussian Lipschitz spaces.

Theorem 8.13. *For $0 < \beta < \alpha < 1$, the fractional derivate of order β, D^β : $Lip_\alpha(\gamma_d) \to Lip_{\alpha-\beta}(\gamma_d)$ is bounded.*

Proof. Let $f \in Lip_\alpha(\gamma_d)$, i.e., $f \in L^\infty$ such that $\left\|\frac{\partial P_t f}{\partial t}\right\|_{\infty,\gamma} \leq At^{-1+\alpha}$. Observe that using (7.44) and Proposition 7.23, we get

$$\begin{aligned}|D^\beta f(x)| &\leq \frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}|P_t f(x) - f(x)|dt\\ &= \frac{1}{c_\beta}\int_0^1 t^{-\beta-1}|P_t f(x) - f(x)|dt + \frac{1}{c_\beta}\int_1^\infty t^{-\beta-1}|P_t f(x) - f(x)|dt\\ &\leq \frac{1}{c_\beta}\int_0^1 t^{-\beta-1}\|P_t f(x) - f(x)\|_{\infty,\gamma}dt + \frac{2\|f\|_{\infty,\gamma}}{c_\beta}\int_1^\infty t^{-\beta-1}dt\\ &\leq \frac{A_1(f)}{c_\beta}\int_0^1 t^{\alpha-\beta-1}dt + \frac{2\|f\|_{\infty,\gamma}}{c_\beta}\int_1^\infty t^{-\beta-1}dt\\ &= \frac{A_1(f)}{c_\beta(\alpha-\beta)} + \frac{2\|f\|_{\infty,\gamma}}{\beta c_\beta} \leq C_{\beta,\beta}\|f\|_{Lip_\beta(\gamma)}.\end{aligned}$$

Thus, $D^{\beta} f \in L^{\infty}(\gamma_d)$. Now, using (8.26), and fixing s, we have

$$\begin{aligned}
\frac{\partial}{\partial s}(P_s D^{\beta} f(x)) &= \frac{1}{c_{\beta}} \frac{\partial}{\partial s} \int_0^{\infty} t^{-\beta-1} [P_{s+t} f(x) - P_s f(x)] dt \\
&= \frac{1}{c_{\beta}} \int_0^{\infty} t^{-\beta-1} [\frac{\partial}{\partial s}(P_{s+t} f(x)) - \frac{\partial}{\partial s} P_s f(x)] dt \\
&= \frac{1}{c_{\beta}} \int_0^{s} t^{-\beta-1} [\frac{\partial}{\partial s}(P_{s+t} f(x)) - \frac{\partial}{\partial s} P_s f(x)] dt \\
&\quad + \frac{1}{c_{\beta}} \int_s^{\infty} t^{-\beta-1} [\frac{\partial}{\partial s}(P_{s+t} f(x)) - \frac{\partial}{\partial s} P_s f(x)] dt \\
&= (I) + (II).
\end{aligned}$$

Using Proposition 7.27, we have

$$\Big\| \frac{\partial^2}{\partial u^2} P_u f \Big\|_{\infty,\gamma_d} \leq A u^{\alpha-2}, \tag{8.36}$$

and then, using the fundamental theorem of calculus, we get

$$\begin{aligned}
|\frac{\partial}{\partial s}(P_{s+t} f(x)) - \frac{\partial}{\partial s} P_s f(x)| &\leq \int_s^{s+t} |\frac{\partial^2}{\partial u^2} P_u f(x)| du \leq A \int_s^{s+t} u^{\alpha-2} du \\
&\leq A \int_s^{\infty} u^{\alpha-2} du \leq \frac{A}{1-\alpha} s^{\alpha-1}.
\end{aligned}$$

Then, as $t < s$,

$$\begin{aligned}
|(I)| &\leq \frac{1}{c_{\beta}} \int_0^{s} t^{-\beta-1} |\frac{\partial}{\partial s}(P_{s+t} f(x)) - \frac{\partial}{\partial s} P_s f(x)| dt \\
&\leq A \frac{s^{-1}}{c_{\beta}} \int_0^{s} t^{-\beta-1} s^{\alpha} dt \leq C_{\alpha,\beta} s^{-1} \int_0^{s} t^{\alpha-\beta-1} dt = C_{\alpha,\beta}\, s^{\alpha-\beta-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|(II)| &\leq \frac{1}{c_{\beta}} \int_s^{\infty} t^{-\beta-1} |\frac{\partial}{\partial s}(P_{s+t} f(x)) - \frac{\partial}{\partial s} P_s f(x)| dt \\
&\leq \frac{A s^{\alpha-1}}{(\beta-1) c_{\beta}} \int_s^{\infty} t^{-\beta-1} dt = C_{\alpha,\beta}\, s^{\alpha-\beta-1}.
\end{aligned}$$

Therefore,

$$\Big\| \frac{\partial}{\partial s}(P_s D^{\beta} f) \Big\|_{\infty,\gamma_d} \leq C s^{\alpha-\beta-1},$$

which implies $D^{\beta} f \in Lip_{\alpha-\beta}(\gamma_d)$. □

8.4 Boundedness of Fractional Integrals and Fractional Derivatives on Gaussian Besov–Lipschitz Spaces

As we discussed in the previous section, in the case of the Lipschitz spaces only a truncated version of the Riesz potentials is bounded from $Lip_\alpha(\gamma_d)$ to $Lip_{\alpha+\beta}(\gamma_d)$. Now, we study the boundedness properties of the Riesz potentials on Besov–Lipschitz spaces, and we see that in this case, the results are better.

Theorem 8.14. *Let* $\alpha \geq 0, \beta > 0$, $1 < p < \infty, 1 \leq q \leq \infty$ *then* I_β *is bounded from* $B^{\alpha}_{p,q}(\gamma_d)$ *into* $B^{\alpha+\beta}_{p,q}(\gamma_d)$.

Proof. Let $k > \alpha+\beta$ a fixed integer, $f \in B^{\alpha}_{p,q}(\gamma_d)$, using the integral representation of Riesz potentials (8.7), the semigroup property of $\{P_t\}_{t\geq 0}$ and the fact that $P_\infty f(x)$ is a constant and the semigroup is conservative, we get

$$\begin{aligned} P_t I_\beta f(x) &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta-1} P_t(P_s f(x) - P_\infty f(x))ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta-1}(P_{t+s} f(x) - P_\infty f(x))ds. \end{aligned} \tag{8.37}$$

Using the fact that $P_\infty f(x)$ is a constant again, and the chain rule,

$$\begin{aligned} \frac{\partial^k}{\partial t^k}(P_t I_\beta f)(x) &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta-1}\frac{\partial^k}{\partial t^k}(P_{t+s} f(x) - P_\infty f(x))\,ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta-1} u^{(k)}(x,t+s)\,ds. \end{aligned} \tag{8.38}$$

Then, using Minkowski's integral inequality

$$\left\|\frac{\partial^k}{\partial t^k} P_t I_\beta f\right\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta-1}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds. \tag{8.39}$$

Hence, if $1 \leq q < \infty$,

$$\begin{aligned} &\left(\int_0^{+\infty}\left(t^{k-(\alpha+\beta)}\left\|\frac{\partial^k}{\partial t^k}(P_t I_\beta f)\right\|_{p,\gamma}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\quad\leq \frac{1}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_0^{+\infty} s^{\beta-1}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\quad\leq C_\beta\left(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_0^{t} s^{\beta-1}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\qquad + C_\beta\left(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_t^{+\infty} s^{\beta-1}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} = (I)+(II). \end{aligned}$$

Now, as $\beta > 0$ using Lemma 3.5, and as $t+s > t$,

$$(I) \leq C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big(\int_0^t s^{\beta-1} \|u^{(k)}(\cdot,t)\|_{p,\gamma} ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$= C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma}^q \Big(\frac{t^\beta}{\beta} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$= C'_\beta \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} < +\infty,$$

because $f \in B_p^{\alpha,q}(\gamma_d)$.

On the other hand, as $k > \alpha+\beta$ using Lemma 3.5 again, because $t+s > s$, and Hardy's inequality (10.101), we obtain

$$(II) \leq C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big(\int_t^{+\infty} s^\beta \|u^{(k)}(\cdot,s)\|_{p,\gamma} \frac{ds}{s} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq \frac{C_\beta}{k-(\alpha+\beta)} \int_0^{+\infty} \Big(s^{k-\alpha} \Big\| \frac{\partial^k P_s f}{\partial s^k} \Big\|_{p,\gamma} \Big)^q \frac{ds}{s} \Big)^{\frac{1}{q}} < +\infty,$$

because $f \in B_{p,q}^\alpha(\gamma_d)$. Therefore, $I_\beta f \in B_{p,q}^{\alpha+\beta}(\gamma_d)$ and, moreover,

$$\|I_\beta f\|_{B_{p,q}^{\alpha+\beta}} = \|I_\beta f\|_{p,\gamma} + \Big(\int_0^{+\infty} \Big(t^{k-(\alpha+\beta)} \Big\| \frac{\partial^k}{\partial t^k}(P_t I_\beta f) \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq C\|f\|_{p,\gamma} + C_{\alpha,\beta} \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \leq C\|f\|_{B_{p,q}^\beta}.$$

Now, if $q = \infty$, (8.39) can be written as

$$\Big\| \frac{\partial^k}{\partial t^k} P_t I_\beta f \Big\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^t s^{\beta-1} \|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds$$
$$+ \frac{1}{\Gamma(\beta)} \int_t^\infty s^{\beta-1} \|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds$$
$$= (I) + (II).$$

Using that $\beta > 0$, Lemma 3.5, as $t+s > t$, and because $f \in B_{p,\infty}^\alpha(\gamma_d)$,

$$(I) \leq \frac{1}{\Gamma(\beta)} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \int_0^t s^{\beta-1} ds \leq \frac{1}{\Gamma(\beta)} \frac{t^\beta}{\beta} A_k(f) t^{-k+\alpha} = C_\beta A_k(f) t^{-k+\alpha+\beta}.$$

Now, because $k > \alpha+\beta$, using Lemma 3.5, as $t+s > s$, and because $f \in B^{\alpha}_{p,\infty}(\gamma_d)$, we get

$$(II) \leq \frac{1}{\Gamma(\beta)} \int_t^{\infty} s^{\beta-1} \Big\| \frac{\partial^k P_s f}{\partial s^k} \Big\|_{p,\gamma} ds \leq \frac{A_k(f)}{\Gamma(\beta)} \int_t^{\infty} s^{-k+\alpha+\beta-1} ds$$
$$= \frac{A_k(f)}{\Gamma(\beta)} \frac{t^{-k+\alpha+\beta}}{k-(\alpha+\beta)} = C_{k,\alpha,\beta}\, t^{-k+\alpha+\beta}.$$

Therefore,

$$\Big\| \frac{\partial^k}{\partial t^k} P_t I_\beta f \Big\|_{p,\gamma} \leq C A_k(f) t^{-k+\alpha+\beta}, \quad t > 0,$$

and this implies that $I_\beta f \in B^{\alpha+\beta}_{p,\infty}(\gamma_d)$ and $A_k(I_\beta f) \leq C A_k(f)$.

Moreover, as I_β is a bounded operator on $L^p(\gamma_d), 1 < p < \infty$,

$$\|I_\beta f\|_{B^{\alpha+\beta}_{p,\infty}} = \|I_\beta f\|_{p,\gamma} + A_k(I_\beta f) \leq \|f\|_{p,\gamma} + C A_k(f) \leq C \|f\|_{B^{\alpha}_{p,\infty}}. \qquad \square$$

Now, we are going to study the boundedness properties of the Bessel potentials on Besov–Lipschitz spaces.

Theorem 8.15. *Let $\alpha \geq 0$, $1 \leq p,q < \infty$, then for $\beta > 0$,*

i) $\mathscr{J}_\beta$ is bounded on $B^{\alpha}_{p,q}(\gamma_d)$.

ii) Moreover, $\mathscr{J}_\beta$ is bounded from $B^{\alpha}_{p,q}(\gamma_d)$ to $B^{\alpha+\beta}_{p,q}(\gamma_d)$.

iii) Finally, for $q = \infty$, $\mathscr{J}_\beta$ is bounded from $B^{\alpha}_{p,\infty}(\gamma_d)$ into $B^{\alpha+\beta}_{p,\infty}(\gamma_d)$.

Proof.

i) Let us see that $\mathscr{J}_\beta$ is bounded on $B^{\alpha}_{p,q}(\gamma_d)$. Using Lebesgue's dominated convergence theorem, Minkowski's integral inequality, and Jensen's inequality, we have

$$\Big\| \frac{\partial^k P_t}{\partial t^k}\Big(\mathscr{J}_\beta f\Big)\Big\|^q_{p,\gamma_d} = \Big(\int_{\mathbb{R}^d} \Big|\frac{\partial^k P_t}{\partial t^k}\Big(\frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s} P_s f(x) \frac{ds}{s}\Big)\Big|^p \gamma_d(dx)\Big)^{\frac{q}{p}}$$
$$\leq \Big(\frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s} \Big(\int_{\mathbb{R}^d} \Big|\frac{\partial^k P_t P_s f(x)}{\partial t^k}\Big|^p \gamma_d(dx)\Big)^{\frac{1}{p}} \frac{ds}{s}\Big)^q$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s} \Big\|\frac{\partial^k P_t P_s f}{\partial t^k}\Big\|^q_{p,\gamma_d} \frac{ds}{s},$$

and then, using Tonelli's theorem,

$$\int_0^{+\infty} \Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t}{\partial t^k}\Big(\mathscr{J}_\beta f\Big)\Big\|_{p,\gamma_d}\Big)^q \frac{dt}{t}$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\alpha e^{-s}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t\Big(P_s f\Big)}{\partial t^k}\Big\|_{p,\gamma_d}\Big)^q \frac{dt}{t}\Big)\frac{ds}{s}$$

$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\alpha}e^{-s}(\int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^q\frac{dt}{t})\frac{ds}{s}$$
$$=\int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^q\frac{dt}{t}.$$

Therefore,

$$\|\mathscr{I}_\beta f\|_{B^{\alpha}_{p,q}} = \|\mathscr{I}_\beta f\|_{p,\gamma_d} + \int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t}{\partial t^k}\left(\mathscr{I}_\beta f\right)\right\|_{p,\gamma_d}\right)^q\frac{dt}{t}$$
$$\leq \|f\|_{p,\gamma_d} + \int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^q\frac{dt}{t} = \|f\|_{B^{\alpha}_{p,q}}.$$

ii) We use the notation $u(x,t) = P_t f(x)$ and $U(x,t) = P_t\mathscr{I}_\beta f(x)$, using the representation (3.8) of P_t we have,

$$U(x,t) = \int_0^{+\infty} T_s(\mathscr{I}_\beta f)(x)\mu_t^{(1/2)}(ds).$$

Therefore,

$$U(x,t_1+t_2) = P_{t_1}(P_{t_2}(\mathscr{I}_\beta f))(x) = \int_0^{+\infty} T_s(P_{t_2}(\mathscr{I}_\beta f))(x)\mu_{t_1}^{(1/2)}(ds).$$

Now, let k,l be integers greater than α,β respectively, by differentiating k times with respect to t_2 and l times with respect to t_1,

$$\frac{\partial^{k+l}U(x,t_1+t_2)}{\partial(t_1+t_2)^{k+l}} = \int_0^{+\infty} T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f))(x)\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds).$$

Thus,

$$\frac{\partial^{k+l}U(x,t)}{\partial t^{k+l}} = \int_0^{+\infty} T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f))(x)\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds),$$

if $t = t_1+t_2$ and therefore, using the L^p continuity of T_s and (3.21)

$$\left\|\frac{\partial^{k+l}U(\cdot,t)}{\partial t^{k+l}}\right\|_{p,\gamma} \leq \int_0^{+\infty}\|T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f))\Big\|_{p,\gamma}\left|\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds)\right|$$
$$\leq \int_0^{+\infty}\|\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f)\Big\|_{p,\gamma}\left|\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds)\right|$$
$$= \left\|\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f)\right\|_{p,\gamma}\int_0^{+\infty}\left|\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds)\right|$$
$$\leq C t_1^{-l}\left\|\frac{\partial^k}{\partial t_2^k}P_{t_2}\mathscr{I}_\beta f\right\|_{p,\gamma}. \tag{8.40}$$

On the other hand, using the representation of Bessel potential (8.20), we have

$$P_t(\mathscr{I}_\beta f)(x) = \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s} P_{t+s}f(x)\frac{ds}{s}$$

then

$$\begin{aligned}\frac{\partial^k P_t}{\partial t^k}(\mathscr{I}_\beta f)(x) &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\frac{\partial^k P_{t+s}f(x)}{\partial t^k}\frac{ds}{s}\\ &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}\frac{ds}{s},\end{aligned}$$

and this implies that

$$\Big\|\frac{\partial^k P_t}{\partial t^k}(\mathscr{I}_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\Big\|\frac{\partial^k P_{t+s}f}{\partial (t+s)^k}\Big\|_{p,\gamma}\frac{ds}{s} < \infty,$$

because $f \in B^\alpha_{p,q}(\gamma_d)$. Now, because the definition of $B^\alpha_{p,q}(\gamma_d)$ is independent on the integer $k > \alpha$ that we can choose, let us take $k > \alpha+\beta$ and $l > \beta$, then $k+l > \alpha+2\beta > \alpha+\beta$; thus, $k+l$ is an integer greater than $\alpha+\beta$. Let us now see that

$$\Big(\int_0^{+\infty}\Big(t^{k+l-(\alpha+\beta)}\Big\|\frac{\partial^{k+l}U(\cdot,t)}{\partial t^{k+l}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} < +\infty.$$

In fact, taking $t_1 = t_2 = t/2$ in (8.40), we get

$$\begin{aligned}&\Big(\int_0^{+\infty}\Big(t^{k+l-(\alpha+\beta)}\Big\|\frac{\partial^{k+l}U(\cdot,t)}{\partial t^{k+l}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq C\Big(\int_0^{+\infty}\Big(t^{k+l-(\alpha+\beta)}\Big\|\frac{\partial^k P_{\frac{t}{2}}}{\partial(\frac{t}{2})^k}(\mathscr{I}_\beta f)\Big\|_{p,\gamma}(\frac{t}{2})^{-l}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}\Big(\int_0^{+\infty} s^\beta e^{-s}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq \frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\\ &\qquad\qquad + \Big(\int_t^{+\infty} s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}.\end{aligned}$$

Again using that $(a+b)^q \leq C_q(a^q+b^q)$ if $a,b \geq 0, q \geq 1$, but because $(a+b)^{1/q} \leq a^{1/q}+b^{1/q}$ if $a,b \geq 0, q \geq 1$,

$$\begin{aligned}&\frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\\ &\qquad + \Big(\int_t^{+\infty} s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{1/q}$$
$$+\frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_t^{+\infty} s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}$$
$$=(I)+(II).$$

Now, using Lemma 3.5 and because $\beta>0$

$$(I)=\frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^{\beta}\Big\|\frac{\partial^k P_{\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}$$
$$=\frac{C}{\beta\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_{\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$=C_{\alpha,\beta}\Big(\int_0^{+\infty}\Big(u^{k-\alpha}\Big\|\frac{\partial^k P_u f}{\partial u^k}\Big\|_{p,\gamma}\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}}<+\infty,$$

because $f\in B_p^{\alpha,q}(\gamma_d)$.

On the other hand, using Hardy inequality, because $k>\alpha+\beta$ and Lemma 3.5, we get

$$(II)=\frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_t^{+\infty} s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_t^{+\infty} s^{\beta}\Big\|\frac{\partial^k P_s f}{\partial s^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}\frac{1}{k-(\alpha+\beta)}\int_0^{+\infty}\Big(s^{k-\alpha}\Big\|\frac{\partial^k}{\partial s^k}P_s f\Big\|_{p,\gamma}\Big)^q\frac{ds}{s}\Big)^{\frac{1}{q}}<+\infty$$

because $f\in B_{p,q}^{\alpha}(\gamma_d)$. Thus, $\mathscr{I}_\beta f\in B_{p,q}^{\alpha+\beta}(\gamma_d)$ and, moreover,

$$\|\mathscr{I}_\beta f\|_{B_{p,q}^{\alpha+\beta}}\leq C_{\alpha,\beta}\|f\|_{B_{p,q}^{\beta}}.$$

iii) Let $k>\alpha+\beta$ a fixed integer, $f\in B_{p,\infty}^{\alpha}(\gamma_d)$, by using the representation of Bessel potential (8.20), we get

$$P_t(\mathscr{I}_\beta f)(x)=\frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta}e^{-s}P_{t+s}f(x)\frac{ds}{s};$$

thus, using the chain rule, we obtain

$$\frac{\partial^k}{\partial t^k}P_t(\mathscr{I}_\beta f)(x)=\frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta}e^{-s}u^{(k)}(x,t+s)\frac{ds}{s},$$

which implies, using Minkowski's integral inequality,

$$\Big\|\frac{\partial^k}{\partial t^k}P_t(\mathscr{I}_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma}\frac{ds}{s}$$
$$= \frac{1}{\Gamma(\beta)}\int_0^{t} s^\beta e^{-s}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma}\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\beta)}\int_t^{\infty} s^\beta e^{-s}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma}\frac{ds}{s} = (I)+(II).$$

Now, as $\beta > 0$, using Lemma 3.5, as $t+s>t$, and because $f\in B^{\beta}_{p,\infty}(\gamma_d)$,

$$(I) \leq \frac{1}{\Gamma(\beta)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\int_0^t s^\beta e^{-s}\frac{ds}{s} \leq \frac{1}{\Gamma(\beta)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\int_0^t s^{\beta-1}ds$$
$$\leq \frac{1}{\Gamma(\beta)}\frac{t^\beta}{\beta}A_k(f)t^{-k+\alpha} = C_\beta A_k(f)t^{-k+\alpha+\beta}.$$

On the other hand, as $k>\alpha+\beta$ using Lemma 3.5, as $t+s>s$, and because $f\in B^{\alpha}_{p,\infty}(\gamma_d)$

$$(II) \leq \frac{1}{\Gamma(\beta)}\int_t^{\infty} s^\beta e^{-s}\Big\|\frac{\partial^k P_s f}{\partial s^k}\Big\|_{p,\gamma}\frac{ds}{s} \leq \frac{A_k(f)}{\Gamma(\beta)}\int_t^{\infty} s^\beta e^{-s}s^{-k+\alpha}\frac{ds}{s}$$
$$\leq \frac{A_k(f)}{\Gamma(\beta)}\int_t^{\infty} s^{-k+\alpha+\beta-1}ds = \frac{A_k(f)}{\Gamma(\beta)}\frac{t^{-k+\alpha+\beta}}{k-(\alpha+\beta)} = C_{k,\alpha,\beta}A_k(f)t^{-k+\alpha+\beta}.$$

Therefore,

$$\Big\|\frac{\partial^k}{\partial t^k}P_t(\mathscr{I}_\beta f)\Big\|_{p,\gamma} \leq CA_k(f)t^{-k+\alpha+\beta},$$

then $\mathscr{I}_\beta f \in B^{\alpha+\beta}_{p,\infty}(\gamma_d)$ and $A_k(\mathscr{I}_\beta f)\leq CA_k(f)$. Thus,

$$\Big\|\mathscr{I}_\beta f\Big\|_{B^{\alpha+\beta}_{p,\infty}} = \Big\|\mathscr{I}_\beta f\Big\|_{p,\gamma} + A_k(\mathscr{I}_\beta f) \leq \|f\|_{p,\gamma}+CA_k(f)\leq C\|f\|_{B^{\alpha}_{p,\infty}}.$$

□

Now, we study the boundedness of the Riesz fractional derivatives and of the Bessel fractional derivatives on Besov–Lipschitz spaces. We use the representation (8.24) of the fractional derivative and Hardy's inequalities. Because they require different techniques, we consider two cases:

- The bounded case, $0<\beta<\alpha<1$.
- The unbounded case $0<\beta<\alpha$.

Let us start with the bounded case for the Riesz derivative:

Theorem 8.16. *Let* $0<\beta<\alpha<1$*,* $1\leq p<\infty$ *and* $1\leq q\leq\infty$ *then* D^β *is bounded from* $B^{\alpha}_{p,q}(\gamma_d)$ *into* $B^{\alpha-\beta}_{p,q}(\gamma_d)$*.*

Proof. Let $f \in B^{\alpha}_{p,q}(\gamma_d)$, using Hardy's inequality (10.100), with $p=1$, and the fundamental theorem of calculus,

$$|D^{\beta} f(x)| \leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} |P_s f(x) - f(x)| ds \leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} \int_0^s |\frac{\partial}{\partial r} P_r f(x)| dr\, ds$$
$$\leq \frac{1}{c_{\beta}\beta} \int_0^{+\infty} r^{1-\beta} |\frac{\partial}{\partial r} P_r f(x)| \frac{dr}{r}. \tag{8.41}$$

Thus, using Minkowski's integral inequality

$$\left\|D^{\beta} f\right\|_{p,\gamma} \leq C_{\beta} \int_0^{+\infty} r^{1-\beta} \left\|\frac{\partial}{\partial r} P_r f\right\|_{p,\gamma} \frac{dr}{r} < \infty, \tag{8.42}$$

because $f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\beta}_{p,1}(\gamma_d)$, $1 \leq q \leq \infty$ as $\alpha > \beta$, i.e., $D_{\beta} f \in L^p(\gamma_d)$.

Now, by analogous argument

$$\frac{\partial}{\partial t} P_t(D^{\beta} f)(x) = \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} [\frac{\partial}{\partial t} P_{t+s} f(x) - \frac{\partial}{\partial t} P_t f(x)] ds$$
$$= \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} u^{(2)}(x,r)\, dr\, ds$$

and again, using Minkowski's integral inequality

$$\left\|\frac{\partial}{\partial t} P_t(D^{\beta} f)\right\|_{p,\gamma} \leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr\, ds \tag{8.43}$$

Then, if $1 \leq q < \infty$, by (8.43)

$$\int_0^{\infty} \left(t^{1-(\alpha-\beta)} \left\|\frac{\partial}{\partial t} P_t(D_{\beta} f)\right\|_{p,\gamma}\right)^q \frac{dt}{t}$$
$$\leq C_{\beta} \int_0^{\infty} \left(t^{1-(\alpha-\beta)} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr\, ds\right)^q \frac{dt}{t}$$
$$= C_{\beta} \int_0^{\infty} \left(t^{1-(\alpha-\beta)} \int_0^{t} s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr\, ds\right)^q \frac{dt}{t}$$
$$+ C_{\beta} \int_0^{\infty} \left(t^{1-(\alpha-\beta)} \int_t^{+\infty} s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr\, ds\right)^q \frac{dt}{t}$$
$$= (I) + (II).$$

Now, because $r > t$ using Lemma 3.5 and the fact that $0 < \beta < 1$,

$$(I) \leq C_{\beta} \int_0^{\infty} \left(t^{1-(\alpha-\beta)} \int_0^{t} s^{-\beta} ds\, \|u^{(2)}(\cdot,r)\|_{p,\gamma}\right)^q \frac{dt}{t}$$
$$= C_{\beta,q} \int_0^{\infty} \left(t^{2-\alpha} \left\|\frac{\partial^2}{\partial r^2} P_r f\right\|_{p,\gamma}\right)^q \frac{dt}{t}.$$

On the other hand, as $r > t$ using Hardy's inequality (10.101), because $(1-\alpha)q > 0$, we get

$$\begin{aligned}
(II) &\leq C_\beta \int_0^\infty t^{(1-(\alpha-\beta))q} \Big(\int_t^{+\infty} s^{-\beta-1} ds \int_t^\infty \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr \Big)^q \frac{dt}{t} \\
&= C'_\beta \int_0^\infty t^{(1-\alpha)q} \Big(\int_t^\infty \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr \Big)^q \frac{dt}{t} \\
&\leq \frac{C'_\beta}{(1-\alpha)} \int_0^\infty \Big(r^{2-\alpha} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} \Big)^q \frac{dr}{r}.
\end{aligned}$$

Thus,

$$\Big(\int_0^\infty \Big(t^{1-\alpha+\beta} \Big\| \frac{\partial}{\partial t} P_t D_\beta f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} \leq C \Big(\int_0^\infty \Big(t^{2-\alpha} \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} < \infty,$$

as $f \in B^{\alpha}_{p,q}(\gamma_d)$. Then, $D_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$ and

$$\begin{aligned}
\|D_\beta f\|_{B^{\alpha-\beta}_{p,q}} &= \|D_\beta f\|_{p,\gamma} + \Big(\int_0^\infty \Big(t^{1-\alpha+\beta} \Big\| \frac{\partial}{\partial t} P_t D_\beta f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} \\
&\leq C_1 \|f\|_{B^{\alpha}_{p,q}} + C_2 \Big(\int_0^\infty \Big(t^{2-\alpha} \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} \leq C \|f\|_{B^{\alpha}_{p,q}}.
\end{aligned}$$

Therefore, $D_\beta f : B^{\alpha}_{p,q} \to B^{\alpha-\beta}_{p,q}$ is bounded.

Now if $q = \infty$, inequality (8.43) can be written as

$$\begin{aligned}
\Big\| \frac{\partial}{\partial t} P_t (D_\beta f) \Big\|_{p,\gamma} &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} dr\, ds \\
&= \frac{1}{c_\beta} \int_0^t s^{-\beta-1} \int_t^{t+s} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} dr\, ds \\
&\quad + \frac{1}{c_\beta} \int_t^{+\infty} s^{-\beta-1} \int_t^{t+s} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} dr\, ds = (I) + (II).
\end{aligned}$$

Now, using Lemma 3.5, because $r > t$,

$$\begin{aligned}
(I) &\leq \frac{1}{c_\beta} \int_0^t s^{-\beta} \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} ds = C_\beta \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} t^{1-\beta} \\
&\leq C_\beta A(f) t^{-2+\alpha} t^{1-\beta} = C_\beta A(f) t^{-1+\alpha-\beta},
\end{aligned}$$

and by Lemma 3.5, because $r > t$, and the fact that $f \in B^{\alpha}_{p,\infty}$,

$$\begin{aligned}(II) &\leq \frac{1}{c_\beta}\int_t^{+\infty} s^{-\beta-1}\int_t^{\infty}\left\|\frac{\partial^2}{\partial r^2}P_r f\right\|_{p,\gamma} dr\,ds\\ &\leq C_\beta t^{-\beta}\int_t^{\infty}\left\|\frac{\partial^2}{\partial r^2}P_r f\right\|_{p,\gamma} dr \leq C_\beta A(f)t^{-\beta}\int_t^{\infty} r^{-2+\alpha}dr\\ &= C_{\alpha,\beta}A(f)t^{-1+\alpha-\beta}.\end{aligned}$$

Thus,

$$\left\|\frac{\partial}{\partial t}P_t(D_\beta f)\right\|_{p,\gamma} \leq CA(f)t^{-1+\alpha-\beta}, \quad t>0.$$

Hence, $D_\beta f \in B^{\alpha-\beta}_{p,\infty}(\gamma_d)$ then $A(D_\beta f) \leq CA(f)$, and

$$\|D_\beta f\|_{B^{\alpha-\beta}_{p,\infty}} = \|D_\beta f\|_{p,\gamma} + A(D_\beta f) \leq C_1\|g\|_{B^{\alpha}_{p,\infty}} + C_2 A(f) \leq C\|f\|_{B^{\alpha}_{p,\infty}}.$$

Therefore, $D_\beta : B^{\alpha}_{p,\infty} \to B^{\alpha-\beta}_{p,\infty}$ is bounded. □

Next, we study the boundedness of the Bessel fractional derivative on Besov–Lipschitz spaces for the bounded case $0 < \beta < \alpha < 1$:

Theorem 8.17. *Let* $0 < \beta < \alpha < 1$, $1 \leq p < \infty$ *and* $1 \leq q \leq \infty$, *then* $\mathscr{D}_\beta$ *is bounded from* $B^{\alpha}_{p,q}(\gamma_d)$ *into* $B^{\alpha-\beta}_{p,q}(\gamma_d)$.

Proof. Let $f \in L^p(\gamma_d)$, using the fundamental theorem of calculus we can write,

$$\begin{aligned}|\mathscr{D}_\beta f(x)| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|e^{-s}P_s f(x) - f(x)|ds\\ &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}e^{-s}|P_s f(x) - f(x)|ds + \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|e^{-s}-1||f(x)|ds\\ &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\left|\int_0^s \frac{\partial}{\partial r}P_r f(x)dr\right|ds + \frac{|f(x)|}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\left|-\int_0^s e^{-r}dr\right|ds\\ &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s\left|\frac{\partial}{\partial r}P_r f(x)\right|dr\,ds + \frac{|f(x)|}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s e^{-r}dr\,ds.\end{aligned}$$

Now, using Hardy's inequality (10.100), with $p = 1$ in both integrals, we have

$$|\mathscr{D}_\beta f(x)| \leq \frac{1}{\beta c_\beta}\int_0^{+\infty} r^{1-\beta}\left|\frac{\partial}{\partial r}P_r f(x)\right|\frac{dr}{r} + \frac{\Gamma(1-\beta)}{\beta c_\beta}|f(x)|.$$

Therefore, according to Minkowski's integral inequality

$$\|\mathscr{D}_\beta f\|_p \leq \frac{1}{\beta c_\beta}\int_0^{+\infty} r^{1-\beta}\Big\|\frac{\partial}{\partial r}P_r f\Big\|_p \frac{dr}{r} + \frac{\Gamma(1-\beta)}{\beta c_\beta}\|f\|_p < C_1\|f\|_{B^\alpha_{p,q}} < \infty,$$

because $f \in B^\alpha_{p,q}(\gamma_d) \subset B^\beta_{p,1}(\gamma_d)$, $1 \leq q \leq \infty$ as $\alpha > \beta$, i.e. $D_\beta f \in L^p(\gamma_d)$.

On the other hand, using the fundamental theorem of calculus and, Hardy's inequality (10.100) again, with $p = 1$ in the second integral, we have

$$\begin{aligned}
|\frac{\partial}{\partial t}P_t(\mathscr{D}_\beta f)(x)| &\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}|e^{-s}\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)|ds\\
&\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}e^{-s}|\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)|ds\\
&\qquad + \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}|e^{-s}-1||\frac{\partial}{\partial t}P_t f(x)|ds\\
&\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}\int_t^{t+s}|\frac{\partial^2}{\partial r^2}P_r f(x)|dr\,ds\\
&\qquad + \frac{1}{c_\beta}|\frac{\partial}{\partial t}P_t f(x)|\int_0^\infty s^{-\beta-1}\int_0^s e^{-r}dr\,ds,\\
&\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}\int_t^{t+s}|\frac{\partial^2}{\partial r^2}P_r f(x)|dr\,ds + \frac{\Gamma(1-\beta)}{\beta c_\beta}|\frac{\partial}{\partial t}P_t f(x)|.
\end{aligned}$$

Thus, using Minkowski's integral inequality,

$$\Big\|\frac{\partial}{\partial t}P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}\int_t^{t+s}\Big\|\frac{\partial^2}{\partial r^2}P_r f\Big\|_{p,\gamma}dr\,ds + \frac{\Gamma(1-\beta)}{\beta c_\beta}\Big\|\frac{\partial}{\partial t}P_t f\Big\|_{p,\gamma}. \tag{8.44}$$

Then, if $1 \leq q < \infty$, using (8.44) and Minkowski's integral inequality, we get

$$\begin{aligned}
&\Big(\int_0^\infty \Big(t^{1-(\alpha-\beta)}\Big\|\frac{\partial}{\partial t}P_t\mathscr{D}_\beta f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\quad \leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{1-(\alpha-\beta)}\int_0^\infty s^{-\beta-1}\int_t^{t+s}\Big\|\frac{\partial^2}{\partial r^2}P_r f\Big\|_{p,\gamma}dr\,ds\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\qquad + \frac{\Gamma(1-\beta)}{\beta c_\beta}\Big(\int_0^\infty \Big(t^{1-(\alpha-\beta)}\Big\|\frac{\partial}{\partial t}P_t f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} = (I)+(II).
\end{aligned}$$

For the first term, the argument is the same as that considered in the second part of the proof of Theorem 8.16; thus,

$$(I) \leq C_\beta\Big(\int_0^\infty \Big(t^{2-\beta}\Big\|\frac{\partial^2}{\partial t^2}P_t f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \|f\|_{B^\alpha_{p,q}} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$, and for the second term trivially

$$(II) \leq C\|f\|_{B^{\alpha-\beta}_{p,q}} \leq C\|f\|_{B^{\alpha}_{p,q}}$$

because $\alpha > \alpha - \beta$ and the inclusion relation given in Proposition 7.36.

Hence, if $1 \leq q < \infty$,

$$\Big(\int_0^\infty \Big(t^{1-(\alpha-\beta)}\Big\|\frac{\partial}{\partial t}P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} \leq C_2\|f|_{B^{\alpha}_{p,q}},$$

so $\mathscr{D}_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$ and, moreover,

$$\begin{aligned}\|\mathscr{D}_\beta f\|_{B^{\alpha-\beta}_{p,q}} &= \|\mathscr{D}_\beta f\|_{p,\gamma} + \Big(\int_0^\infty \Big(t^{1-\alpha+\beta}\Big\|\frac{\partial}{\partial t}P_t\mathscr{D}_\beta f\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q}\\ &\leq C_1\|f\|_{B^{\alpha}_{p,q}} + C_2\Big(\int_0^\infty \Big(t^{2-\alpha}\Big\|\frac{\partial^2}{\partial t^2}P_t f\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} \leq C\|f\|_{B^{\alpha}_{p,q}}.\end{aligned}$$

If $q = \infty$, using the same argument as in Theorem 8.16, inequality (8.44) can be written as

$$\begin{aligned}\Big\|\frac{\partial}{\partial t}P_t\mathscr{D}_\beta f\Big\|_{p,\gamma} &\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}\int_t^{t+s}\Big\|\frac{\partial^2}{\partial r^2}P_r f\Big\|_{p,\gamma} dr\,ds + \frac{\Gamma(1-\beta)}{\beta c_\beta}\Big\|\frac{\partial}{\partial t}P_t f\Big\|_{p,\gamma}\\ &\leq C_{\alpha,\beta}A(f)t^{-1+\alpha-\beta} + \frac{\Gamma(1-\beta)}{\beta c_\beta}A(f)t^{-1+\alpha-\beta} \leq C_{\alpha,\beta}A(f)t^{-1+\alpha-\beta},\end{aligned}$$

for $t > 0$, then, $\mathscr{D}_\beta f \in B^{\alpha-\beta}_{p,\infty}(\gamma_d)$ and $A(\mathscr{D}_\beta f) \leq C_{\alpha,\beta}A(f)$; thus,

$$\|\mathscr{D}_\beta f\|_{B^{\alpha-\beta}_{p,\infty}} = \|\mathscr{D}_\beta f\|_{p,\gamma} + A(\mathscr{D}_\beta f) \leq C_1\|f\|_{B^{\alpha}_{p,\infty}} + C_2 A(f) \leq C\|f\|_{B^{\alpha}_{p,\infty}}.$$

□

We consider now the unbounded case for fractional derivatives (removing the condition that the indexes must be less than 1). To do this, we need to consider forward differences. Remember that for a given function f, the k-th order forward difference of f starting at t with increment s is defined as

$$\Delta_s^k(f,t) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j f(t+(k-j)s).$$

The forward differences have the following properties (see Appendix Lemma 10.30), which will be needed in what follows. For any positive integer k

i) $\Delta_s^k(f,t) = \Delta_s^{k-1}(\Delta_s(f,\cdot),t) = \Delta_s(\Delta_s^{k-1}(f,\cdot),t)$.
ii) $\Delta_s^k(f,t) = \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-2}}^{v_{k-2}+s}\int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)dv_k dv_{k-1}\ldots dv_2 dv_1$.

iii) For any positive integer k

$$\frac{\partial}{\partial s}(\Delta_s^k(f,t)) = k\Delta_s^{k-1}(f',t+s), \tag{8.45}$$

and for any integer $j > 0$,

$$\frac{\partial^j}{\partial t^j}(\Delta_s^k(f,t)) = \Delta_s^k(f^{(j)},t). \tag{8.46}$$

Observe that using the binomial theorem and the semigroup property of $\{P_t\}$, we have

$$\begin{aligned}(P_t - I)^k f(x) &= \sum_{j=0}^{k}\binom{k}{j}P_t^{k-j}(-I)^j f(x) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j P_t^{k-j} f(x)\\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j P_{(k-j)t} f(x) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j u(x,(k-j)t)\\ &= \Delta_t^k(u(x,\cdot),0),\end{aligned} \tag{8.47}$$

where as usual, $u(x,t) = P_t f(x)$. Additionally, we need the following result:

Lemma 8.18. *Let $f \in L^p(\gamma_d)$, $1 \le p < \infty$ and $k,n \in \mathbb{N}$ then*

$$\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma_d} \le s^k \|u^{(k+n)}(\cdot,t)\|_{p,\gamma_d}$$

Proof. From property *ii*) of forward differences (see Lemma 10.30), we have

$$\Delta_s^k(u^{(n)}(x,\cdot),t) = \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-2}}^{v_{k-2}+s}\int_{v_{k-1}}^{v_{k-1}+s} u^{(k+n)}(x,v_k)dv_k dv_{k-1}\dots dv_2 dv_1,$$

then, using Minkowski's integral inequality k-times and Lemma 3.5,

$$\begin{aligned}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma_d} &\le \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-2}}^{v_{k-2}+s}\int_{v_{k-1}}^{v_{k-1}+s}\|u^{(k+n)}(\cdot,v_k)\|_{p,\gamma_d} dv_k dv_{k-1}\dots dv_2 dv_1\\ &\le s^k\|u^{(k+n)}(\cdot,t)\|_{p,\gamma_d} = s^k\left\|\frac{\partial^{k+n}}{\partial t^{k+n}}u(\cdot,t)\right\|_{p,\gamma_d}.\end{aligned}$$

□

Let us start studying the boundedness of the Riesz fractional derivative in $B^{\alpha}_{p,q}(\gamma_d)$

Theorem 8.19. *Let $0 < \beta < \alpha$, $1 \le p < \infty$ and $1 \le q \le \infty$ then D^β is bounded from $B^{\alpha}_{p,q}(\gamma_d)$ into $B^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in B^{\alpha}_{p,q}(\gamma_d)$, using (8.47), Hardy's inequality (10.100), $p = 1$, the fundamental theorem of calculus, and property iii) of forward differences (see Lemma 10.30), we get

$$|D^{\beta}f(x)| \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(u(x,\cdot),0)|ds \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s |\frac{\partial}{\partial r}\Delta_r^k(u(x,\cdot),0)|dr\,ds$$
$$\leq \frac{1}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}|\frac{\partial}{\partial r}\Delta_r^k(u(x,\cdot),0)|dr = \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}|\Delta_r^{k-1}(u'(x,\cdot),r)|dr.$$

Now, using Minkowski's integral inequality and Lemma 8.18

$$\|D_\beta f\|_{p,\gamma} \leq \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}\|\Delta_r^{k-1}(u',r)\Big\|_{p,\gamma}dr$$
$$\leq \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{k-\beta}\Big\|\frac{\partial^k}{\partial r^k}P_r f\Big\|_{p,\gamma}\frac{dr}{r} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\beta}_{p,1}(\gamma_d)$, as $\alpha > \beta$. Therefore, $D_\beta f \in L^p(\gamma_d)$.

On the other hand,

$$P_t[(P_s - I)^k f(x)] = P_t(\Delta_s^k(u(x,\cdot),0)) = P_t(\sum_{j=0}^{k}\binom{k}{j}(-1)^j P_{(k-j)s}f(x))$$
$$= \sum_{j=0}^{k}\binom{k}{j}(-1)^j P_{t+(k-j)s}f(x) = \Delta_s^k(u(x,\cdot),t). \tag{8.48}$$

Thus, if n is the smaller integer greater than α, i.e., $n-1 \leq \alpha < n$, then according to Lemma 10.30 *iv*),

$$\frac{\partial^n}{\partial t^n}P_t(D_\beta f)(x) = \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\frac{\partial^n}{\partial t^n}(\Delta_s^k(u(x,\cdot),t)$$
$$= \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Delta_s^k(u^{(n)}(x,\cdot),t)ds;$$

therefore, using Minkowski's integral inequality

$$\Big\|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds. \tag{8.49}$$

Now, if $1 \leq q < \infty$, by (8.49),

$$\Big(\int_0^{\infty}\Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{c_\beta}\Big(\int_0^{\infty}\Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{c_\beta}\Big(\int_0^{\infty}\Big(t^{n-(\alpha-\beta)}\int_0^{t} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$+\frac{1}{c_\beta}\Big(\int_0^{\infty}\Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$= (I) + (II).$$

Then, using Lemma 8.18,

$$(I) \leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^{n+k}}{\partial t^{n+k}}P_t f\Big\|_{p,\gamma}\int_0^t s^{k-\beta-1}ds\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$= \frac{1}{c_\beta(k-\beta)}\Big(\int_0^\infty \Big(t^{n+k-\alpha}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$, and using Lemma 3.5

$$(II) \leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t+(k-j)s)\|_{p,\gamma}\Big)ds\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t)\|_{p,\gamma}\Big)ds\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$= \frac{2^k}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^{n}}{\partial t^{n}}P_t f\Big\|_{p,\gamma}\int_t^{+\infty} s^{-\beta-1}ds\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$= \frac{2^k}{c_\beta \beta}\Big(\int_0^\infty \Big(t^{n-\alpha}\Big\|\frac{\partial^{n}}{\partial t^{n}}P_t f\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$. Therefore, if $1 \leq q < \infty$, $D_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$; moreover,

$$\|D_\beta f\Big\|_{B^{\alpha-\beta}_{p,q}} = \|D_\beta f\|_{p,\gamma} + \Big(\int_0^\infty \Big(t^{n-\alpha+\beta}\Big\|\frac{\partial^{n}}{\partial t^{n}}P_t(D_\beta f)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$\leq C_1\|f\|_{B^{\alpha}_{p,q}} + C_2\|f\|_{B^{\alpha}_{p,q}} \leq C\|f\|_{B^{\alpha}_{p,q}}$$

Thus, $D_\beta f : B^{\alpha}_{p,q} \to B^{\alpha-\beta}_{p,q}$ is bounded.

If $q = \infty$, inequality (8.49) can be written as

$$\|\frac{\partial^{n}}{\partial t^{n}}P_t(D_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{c_\beta}\int_0^t s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds$$
$$+\frac{1}{c_\beta}\int_t^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds$$
$$= (I) + (II)$$

and then as $f \in B^{\beta}_{p,\infty}$, by Lemma 8.18,

$$(I) \leq \frac{1}{c_\beta}\int_0^t s^{-\beta-1}s^k\|u^{(n+k)}\|_{p,\gamma}ds = C_\beta\Big\|\frac{\partial^{n+k}}{\partial t^{n+k}}P_t f\Big\|_{p,\gamma} t^{k-\beta}$$
$$\leq C_\beta A(f)t^{-n-k+\alpha}t^{k-\beta} = C_\beta A(f)t^{-n+\alpha-\beta},$$

and as above, using Lemma 3.5,

$$
\begin{aligned}
(II) &\le \frac{1}{c_\beta}\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t+(k-j)s)\|_{p,\gamma}\Big)ds \\
&\le C_\beta\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t)\|_{p,\gamma}\Big)ds = C_\beta t^{-\beta}\Big\|\frac{\partial^n}{\partial t^n}P_t f\Big\|_{p,\gamma} \\
&\le C_\beta A(f)t^{-n+\alpha}t^{-\beta} = C_\beta A(f)t^{-n+\alpha-\beta}.
\end{aligned}
$$

□

There is an alternative proof of the fact that $D_\beta f \in L^p(\gamma_d)$ without using Hardy's inequality following the same scheme as in the proof of *i*) [109, Theorem 3.5], and using the inclusion $B^{\alpha}_{p,q} \subset B^{\beta+\varepsilon}_{p,\infty}$ with $\beta+\varepsilon < k$.

Now, let us consider the case of the Bessel derivative.

Theorem 8.20. *Let* $0<\beta<\alpha$, $1\le p<\infty$ *and* $1\le q\le\infty$ *then* $\mathscr{D}_\beta$ *is bounded from* $B^{\alpha}_{p,q}(\gamma_d)$ *into* $B^{\alpha-\beta}_{p,q}(\gamma_d)$.

Proof. Let $f \in B^{\alpha}_{p,q}(\gamma_d)$, and set $v(x,t) = e^{-t}u(x,t)$. Then, using Hardy's inequality (10.100), the fundamental theorem of calculus, and property *iii*) of forward differences (see Lemma 10.30),

$$
\begin{aligned}
|\mathscr{D}_\beta f(x)| &\le \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(v(x,\cdot),0)|ds \\
&\le \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s \Big|\frac{\partial}{\partial r}\Delta_r^k(v(x,\cdot),0)\Big|dr\,ds \\
&\le \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}|\Delta_r^{k-1}(v'(x,\cdot),r)|dr
\end{aligned}
$$

and this implies, using Minkowski's integral inequality,

$$
\|\mathscr{D}_\beta f\|_{p,\gamma_d} \le \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}\|\Delta_r^{k-1}(v',r)\|_{p,\gamma}dr.
$$

Now, using property *ii*) of forward differences (see Lemma 10.30),

$$
\|\Delta_r^{k-1}(v',r)\|_{p,\gamma} \le \int_r^{2r}\int_{v_1}^{v_1+r}\cdots\int_{v_{k-2}}^{v_{k-2}+r}\|v^{(k)}(\cdot,v_{k-1})\|_{p,\gamma}dv_{k-1}dv_{k-2}\ldots dv_2dv_1
$$

and using Leibniz's differentiation rule for the product

$$
\begin{aligned}
\|v^{(k)}(\cdot,v_{k-1})\Big\|_{p,\gamma} &= \Big\|\sum_{j=0}^{k}\binom{k}{j}(e^{-v_{k-1}})^{(j)}u^{(k-j)}(\cdot,v_{k-1})\Big\|_{p,\gamma} \\
&\le \sum_{j=0}^{k}\binom{k}{j}e^{-v_{k-1}}\|u^{(k-j)}(\cdot,v_{k-1})\|_{p,\gamma}.
\end{aligned}
$$

Then

$$\begin{aligned}
&\|\Delta_r^{k-1}(v',r)\|_{p,\gamma}\\
&\quad\le \sum_{j=0}^{k}\binom{k}{j}\int_r^{2r}\int_{v_1}^{v_1+r}\dots\int_{v_{k-2}}^{v_{k-2}+r} e^{-v_{k-1}}\|u^{(k-j)}(\cdot,v_{k-1})\|_{p,\gamma}dv_{k-1}dv_{k-2}\dots dv_2dv_1\\
&\quad\le \sum_{j=0}^{k}\binom{k}{j} r^{k-1}e^{-r}\|u^{(k-j)}(\cdot,r)\|_{p,\gamma}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathscr{D}_\beta f\|_{p,\gamma} &\le \frac{k}{\beta c_\beta}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k-j)}(\cdot,r)\|_{p,\gamma}dr\\
&= \frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{(k-j)-(\beta-j)-1}e^{-r}\left\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_rf\right\|_{p,\gamma}dr\\
&\qquad+\frac{k}{\beta c_\beta}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|P_rf\|_{p,\gamma}dr\\
&\le \frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{(k-j)-(\beta-j)-1}\left\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_rf\right\|_{p,\gamma}dr\\
&\qquad+\frac{k}{\beta c_\beta}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|f\|_{p,\gamma}dr
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathscr{D}_\beta f\|_{p,\gamma} \le \frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{k-j-(\beta-j)}\left\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_rf\right\|_{p,\gamma}\frac{dr}{r}&\\
+\frac{k\Gamma(k-\beta)}{\beta c_\beta}\|f\|_{p,\gamma}<\infty,&
\end{aligned}$$

because $f\in B^{\alpha}_{p,q}(\gamma_d)\subset B^{\beta-j}_{p,1}(\gamma_d)$ as $\alpha>\beta>\beta-j\ge 0$, for $j\in\{0,\dots,k-1\}$, then $\mathscr{D}_\beta f\in L^p(\gamma_d)$.

On the other hand,

$$P_t(e^{-s}P_s-I)^k f(x)=\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-s(k-j)}u(x,t+(k-j)s).$$

Let n be the smaller integer greater than β, i.e., $n-1\le\beta<n$, we have

$$\begin{aligned}\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)(x) &= \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-s(k-j)}u^{(n)}(x,t+(k-j)s)ds\\ &= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-(t+s(k-j))}u^{(n)}(x,t+(k-j)s)ds\\ &= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Delta_s^k(w(x,\cdot),t)ds,\end{aligned}$$

where $w(x,t)=e^{-t}u^{(n)}(x,t)$. Now, using the fundamental theorem of calculus,

$$\begin{aligned}\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)(x) &= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Delta_s^k(w(x,\cdot),t)ds\\ &= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s \frac{\partial}{\partial r}\Delta_r^k(w(x,\cdot),t)dr\,ds.\end{aligned}$$

Then, using Hardy's inequality (10.100) and property *iii*) of forward differences (see Lemma 10.30),

$$\begin{aligned}\left|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)(x)\right| &\le \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s \left|\frac{\partial}{\partial r}\Delta_r^k(w(x,\cdot),t)\right|dr\,ds\\ &\le \frac{e^t}{c_\beta\beta}\int_0^{+\infty} r\left|\frac{\partial}{\partial r}\Delta_r^k(w(x,\cdot),t)\right|r^{-\beta-1}dr\\ &= \frac{ke^t}{c_\beta\beta}\int_0^{+\infty} r^{-\beta}|\Delta_r^{k-1}(w'(x,\cdot),t+r)|dr\end{aligned}$$

and according to Minkowski's integral inequality, we get

$$\left\|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)\right\|_{p,\gamma} \le \frac{ke^t}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}\|\Delta_r^{k-1}(w',t+r)\|_{p,\gamma}dr.$$

Now, using an analogous argument to that above, Lemma 10.30 and Leibniz's product rule give us

$$\|\Delta_r^{k-1}(w',t+r)\|_{p,\gamma} \le \sum_{j=0}^{k}\binom{k}{j}r^{k-1}e^{-(t+r)}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma},$$

and this implies that

$$\begin{aligned}\left\|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)\right\|_{p,\gamma} &\le e^t\frac{k}{c_\beta\beta}\int_0^{+\infty} r^{-\beta}\Big(\sum_{j=0}^{k}\binom{k}{j}r^{k-1}e^{-(t+r)}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}\Big)dr\\ &= \frac{k}{c_\beta\beta}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}dr.\end{aligned}$$

Thus,

$$\Big\|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma} \leq \frac{k}{c_\beta \beta}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}\,dr.$$

Now, if $1 \leq q < \infty$, using (8.50) we have,

$$\begin{aligned}
&\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\quad\leq \frac{k}{c_\beta\beta}\sum_{j=0}^{k}\binom{k}{j}\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}.
\end{aligned}$$

For each $1 \leq j \leq k$, $0 < \alpha-\beta+k-j \leq \beta$ and using Lemma 3.5

$$\begin{aligned}
&(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_0^\infty r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\qquad\leq (\int_0^\infty\Big(t^{n-(\alpha-\beta)}\|u^{(n+k-j)}(\cdot,t)\|_{p,\gamma}\int_0^{+\infty} r^{k-\beta-1}e^{-r}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\qquad= \Gamma(k-\beta)(\int_0^\infty\Big(t^{n+(k-j)-(\alpha-\beta+k-j)}\|u^{(n+k-j)}(\cdot,t)\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \infty,
\end{aligned}$$

as $f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\alpha-\beta+(k-j)}_{p,q}(\gamma_d)$ for any $0 \leq j \leq k$.

Now, for the case $j = 0$,

$$\begin{aligned}
&\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(n+k)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\qquad\leq \Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{t} r^{k-\beta-1}e^{-r}\|u^{(n+k)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\qquad\qquad+\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(n+k)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\qquad= (I)+(II).
\end{aligned}$$

Using Lemma 3.5, and $k > \beta$,

$$\begin{aligned}
(I) &\leq \Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{t} r^{k-\beta-1}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&= \Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}\int_0^{t} r^{k-\beta-1}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&= \frac{1}{k-\beta}\Big(\int_0^\infty\Big(t^{n+k-\alpha}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \infty,
\end{aligned}$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$ and $n+k > \alpha$. For the second term, using Lemma 3.5 and Hardy's inequality (10.101)

$$(II) \leq \left(\int_0^\infty \left(t^{n-(\alpha-\beta)} \int_t^{+\infty} r^{k-\beta-1} \|u^{(n+k)}(\cdot,r)\Big\|_{p,\gamma} dr\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\leq \frac{1}{n-(\alpha-\beta)} \left(\int_0^\infty \left(r^{n+k-\alpha} \|u^{(n+k)}(\cdot,r)\Big\|_{p,\gamma}\right)^q \frac{dr}{r}\right)^{1/q} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$.

Therefore, $\mathscr{D}_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$. Moreover,

$$\|\mathscr{D}_\beta f\|_{B^{\alpha-\beta}_{p,q}} = \|\mathscr{D}_\beta f\|_{p,\gamma} + \left(\int_0^\infty \left(t^{n-(\alpha-\beta)} \Big\| \frac{\partial^n}{\partial t^n} P_t \mathscr{D}_\beta f\Big\|_{p,\gamma}\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\leq C_1 \|f\|_{p,\gamma} + \frac{k}{c_\beta \beta} \sum_{j=0}^{k} \binom{k}{j} C_2 \left(\int_0^\infty \left(r^{n-\alpha} \Big\| \frac{\partial^n}{\partial r^n} P_r f\Big\|_{p,\gamma}\right)^q \frac{dr}{r}\right)^{1/q}$$
$$\leq C\|f\|_{B^{\alpha}_{p,q}}$$

Finally, if $q = \infty$, from the inequality (8.50)

$$\Big\| \frac{\partial^n}{\partial t^n} P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma} \leq \frac{k}{c_\beta \beta} \sum_{j=0}^{k} \binom{k}{j} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma} dr,$$

and then, the argument is essentially similar to the previous case, as in the last part of the proof of Theorem 8.19. □

8.5 Boundedness of Fractional Integrals and Fractional Derivatives on Gaussian Triebel–Lizorkin Spaces

First, we study the boundedness of the Riesz potentials I_β on Gaussian Triebel–Lizorkin spaces.

Theorem 8.21. *Let* $\alpha \geq 0, \beta > 0$*,* $1 < p < \infty, 1 \leq q < \infty$ *then* I_β *is bounded from* $F^{\alpha}_{p,q}(\gamma_d)$ *into* $F^{\alpha+\beta}_{p,q}(\gamma_d)$*.*

Proof. Let $k > \alpha+\beta+1$ be an integer fixed and $f \in F^{\alpha}_{p,q}(\gamma_d)$. Using the integral representation of Riesz potentials (8.7), the semigroup property of $\{P_t\}_{t\geq 0}$, and the fact that $P_\infty f(x)$ is a constant, we get

$$P_t(I_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} P_t(P_s f(x) - P_\infty f(x)) ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} (P_{t+s} f(x) - P_\infty f(x)) ds. \qquad (8.50)$$

Then, again using that $P_\infty f(x)$ is a constant and the chain rule,

$$\frac{\partial^k}{\partial t^k} P_t(I_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \frac{\partial^k}{\partial t^k}(P_{t+s}f(x) - P_\infty f(x))ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} u^{(k)}(x,t+s)ds. \tag{8.51}$$

i) Case $\beta \geq 1$: Using (8.51), the change of variables $r = t+s$, $dr = ds$, and Hardy's inequality (10.101), we have

$$\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_t^{+\infty} (r-t)^{\beta-1}|u^{(k)}(x,r)|dr\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_t^{+\infty} r^{\beta-1}|u^{(k)}(x,r)|dr\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\frac{1}{(k-(\alpha+\beta))^{1/q}}\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|u^{(k)}(x,r)|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}},$$

and, therefore,

$$\|\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t(I_\beta f)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}$$
$$\leq C_{k,\alpha,\beta}\|\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|\frac{\partial^k P_r f}{\partial r^k}|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} < \infty,$$

because $f \in F^{\alpha}_{p,q}$. By (8.12) and the previous estimate,

$$\|I_\beta f\Big\|_{F^{\alpha+\beta}_{p,q}} \leq C\|f\Big\|_{F^{\alpha}_{p,q}}.$$

ii) Case $0 < \beta < 1$: again using (8.51),

$$\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty} s^{\beta}|u^{(k)}(x,t+s)|\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_0^{t} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}}$$
$$+\frac{C}{\Gamma(\beta)}(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_t^{+\infty} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}}$$
$$= (I) + (II).$$

Writing $s^{\beta-1} = s^{\frac{\beta-1}{q}+\frac{\beta-1}{q'}}$, $\frac{1}{q}+\frac{1}{q'}=1$, and using Hölder's inequality in the internal integral,

$$(I) \le \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} t^{q(k-\alpha)-\beta-1}\int_0^t s^{\beta-1}|u^{(k)}(x,t+s)|^q ds\, dt\Big)^{1/q}.$$

Using the Fubini–Tonelli theorem and using that $q(k-\beta)-\beta-1>0$, as $k>\beta+\beta+1$, we get

$$\begin{aligned}(I) &\le \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_s^{+\infty} t^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^q dt\, ds\Big)^{1/q}\\ &\le \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_s^{+\infty} (t+s)^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^q dt\, ds\Big)^{1/q}.\end{aligned}$$

Then, by the change of variables $r=t+s$ and using Hardy's inequality (10.101),

$$\begin{aligned}(I) &\le \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_{2s}^{+\infty} r^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^q dr\, ds\Big)^{1/q}\\ &\le \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_{s}^{+\infty} r^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^q dr\, ds\Big)^{1/q}\\ &\le \frac{C_\beta}{\beta}\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|u^{(k)}(x,r)|\Big)^q\frac{dr}{r}\Big)^{1/q}.\end{aligned}$$

On the other hand, because $\beta<1$, then $t<s$ implies that $s^{\beta-1}<t^{\beta-1}$, and by the change of variables $r=t+s$ and according to Hardy's inequality (10.101), as $k>\alpha+\beta+1>\alpha+1$, we obtain

$$\begin{aligned}(II) &\le C_\beta\Big(\int_0^{+\infty} t^{q(k-\alpha-1)-1}\Big(\int_t^{+\infty}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}}\\ &\le C_\beta\Big(\int_0^{+\infty} t^{q(k-\alpha-1)-1}\Big(\int_{2t}^{+\infty}|u^{(k)}(x,r)|dr\Big)^q dt\Big)^{\frac{1}{q}}\\ &\le C_\beta\frac{1}{(k-\alpha-1)^{1/q}}\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|\frac{\partial^k P_r f(x)}{\partial r^k}|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big\|\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}\\ &\qquad\le C_{k,\alpha,\beta}\Big\|\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|\frac{\partial^k P_r f}{\partial r^k}|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}<\infty.\end{aligned}$$

as $f\in F^\alpha_{p,q}$. Then, using (8.12) and the previous estimate, we get

$$\|I_\beta f\|_{F^{\alpha+\beta}_{p,q}} \le C\|f\|_{F^\alpha_{p,q}}.$$

□

Next, we study the boundedness properties of the Bessel potentials on Triebel–Lizorkin spaces.

Theorem 8.22. *Let $\alpha \geq 0$, $1 \leq p,q < \infty$ then for every $\beta > 0$,*

i) $\mathscr{I}_\beta$ *is bounded on* $F^{\alpha}_{p,q}(\gamma_d)$.

ii) Moreover, $\mathscr{I}_\beta$ *is bounded from* $F^{\alpha}_{p,q}(\gamma_d)$ *to* $F^{\alpha+\beta}_{p,q}(\gamma_d)$.

Proof.

i) Let us prove that $\mathscr{I}_\beta$ is bounded on $F^{\alpha}_{p,q}(\gamma_d)$. Using Lebesgue's dominated convergence theorem, Minkowski's integral inequality, and *iii*), we have

$$\left(\int_0^\infty (s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathscr{I}_\beta g\right)(x)\right|)^q \frac{ds}{s}\right)^{1/q}$$
$$= \left(\int_0^\infty (s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t} P_t g(x)\frac{dt}{t}\right)\right|)^q \frac{ds}{s}\right)^{1/q}$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left(\int_0^\infty (s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}(x)\right|)^q \frac{ds}{s}\right)^{1/q}\frac{dt}{t},$$

then, again using Minkowski's integral inequality, and *iii*)

$$\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathscr{I}_\beta g\right)\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}$$
$$\leq \left\|\frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\frac{dt}{t}\right\|_{p,\gamma}$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}\frac{dt}{t}$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}\frac{dt}{t}$$
$$= \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}.$$

Thus,

$$\|\mathscr{I}_\beta g\|_{F^{\beta}_{p,q}} = \|\mathscr{I}_\beta g\|_{p,\gamma} + \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathscr{I}_\beta g\right)\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}$$
$$\leq \|g\|_{p,\gamma} + \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma} = \|g\|_{F^{\alpha}_{p,q}}.$$

ii) Let $k > \alpha+\beta+1$ be a fixed integer, let $f \in F^{\alpha}_{p,q}(\gamma_d)$, and let $h = \mathscr{I}_\beta f$. We consider two cases:

ii-1) If $\beta \geq 1$. Taking the change of variables $u = t+s$ and using Hardy's inequality, we get

$$
\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t h(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\qquad\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty}t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty}s^\beta e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}|\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\qquad\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty}t^{q(k-(\alpha+\beta))}\Big(\int_t^{+\infty}(u-t)^{\beta-1}|\frac{\partial^k P_u f(x)}{\partial u^k}|du\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\qquad\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(\int_t^{+\infty}u^{\beta-1}|\frac{\partial^k P_u f(x)}{\partial u^k}|du\Big)^q t^{q(k-(\alpha+\beta))-1}dt\Big)^{\frac{1}{q}}\\
&\qquad\leq \frac{1}{\Gamma(\beta)}\frac{1}{k-(\alpha+\beta)}\Big(\int_0^{+\infty}\Big(u^{k-\alpha}|\frac{\partial^k P_u f(x)}{\partial u^k}|\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big\|\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t h}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}\\
&\qquad\leq \frac{1}{\Gamma(\beta)(k-(\alpha+\beta))}\Big\|\Big(\int_0^{+\infty}\Big(u^{k-\alpha}|\frac{\partial^k P_u f}{\partial u^k}|\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}<\infty,
\end{aligned}
$$

because $f\in F^{\alpha}_{p,q}(\gamma_d)$. Thus $\mathscr{I}_\beta f\in F^{\alpha+\beta}_{p,q}(\gamma_d)$.

ii-2) If $0<\beta<1$.

$$
\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t h(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\qquad\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty}t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty}s^\beta e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}|\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\qquad\leq \frac{C}{\Gamma(\beta)}(\int_0^{+\infty}t^{q(k-(\alpha+\beta))-1}\Big(\int_0^{t}s^{\alpha-1}e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}|ds\Big)^q dt\Big)^{\frac{1}{q}}\\
&\qquad+\frac{C}{\Gamma(\beta)}(\int_0^{+\infty}t^{q(k-(\alpha+\beta))-1}\Big(\int_t^{+\infty}s^{\alpha-1}e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}|ds\Big)^q dt\Big)^{\frac{1}{q}}\\
&\qquad=I+II.
\end{aligned}
$$

Now, $e^{-s}<1$ and as $\beta<1$, then $s^{\beta-1}<t^{\beta-1}$ for $t<s$.
Hence, again by the change of variables $u=t+s$ and using Hardy's inequality, we get

$$
\begin{aligned}
II&\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty}t^{q(k-\beta-1)-1}\Big(\int_t^{+\infty}|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}|ds\Big)^q dt\Big)^{\frac{1}{q}}\\
&\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty}t^{q(k-\beta-1)-1}\Big(\int_t^{+\infty}|\frac{\partial^k P_u f(x)}{\partial u^k}|du\Big)^q dt\Big)^{\frac{1}{q}}\\
&\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(u^{k-\alpha}|\frac{\partial^k P_u f(x)}{\partial u^k}|\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}}
\end{aligned}
$$

On the other hand, using $e^{-s} < 1$ again,

$$I^q \leq \frac{C}{\Gamma(\beta)} \int_0^{+\infty} t^{q(k-(\alpha+\beta))-1} \Big(\int_0^t s^{\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}| ds \Big)^q dt$$
$$= \frac{C}{\Gamma(\beta)\beta^q} \int_0^{+\infty} t^{q(k-\alpha)-1} \Big(\frac{\beta}{t^\beta} \int_0^t s^{\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}| ds \Big)^q dt$$

Now, as $\beta > 0$, $\int_0^t s^{\beta-1} ds = \frac{t^\beta}{\beta}$, then using Jensen's inequality for the probability measure $\frac{\beta}{t^\beta} s^{\beta-1} ds$ and Fubini's theorem, we get

$$I^q \leq \frac{C}{\Gamma(\beta)\beta^q} \int_0^{+\infty} t^{q(k-\alpha)-1} \Big(\frac{\beta}{t^\beta} \int_0^t s^{\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}|^q ds \Big) dt$$
$$= \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_s^{+\infty} t^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}|^q dt \Big) ds$$
$$\leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_s^{+\infty} (t+s)^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}|^q dt \Big) ds$$

as $q(k-\alpha)-\beta-1 > 0$, because $0 < \beta < 1$. Finally, again taking the change of variables $u = t+s$ and using Hardy's inequality, we get

$$I^q \leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_{2s}^{+\infty} u^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_u f(x)}{\partial u^k}|^q du \Big) ds$$
$$\leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_s^{+\infty} u^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_u f(x)}{\partial u^k}|^q du \Big) ds$$
$$\leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} \Big(u^{k-\beta} |\frac{\partial^k P_u f(x)}{\partial u^k}| \Big)^q \frac{du}{u}.$$

Hence,

$$\Big\| \Big(\int_0^{+\infty} \Big(t^{k-(\alpha+\beta)} |\frac{\partial^k P_t h}{\partial t^k}| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq C_{k,\alpha,\beta} \Big\| \Big(\int_0^{+\infty} \Big(u^{k-\alpha} |\frac{\partial^k P_u f}{\partial u^k}| \Big)^q \frac{du}{u} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma} < \infty.$$

Thus, $\mathscr{I}_\beta f \in F^{\alpha+\beta}_{p,q}(\gamma_d)$, for $0 < \beta < 1$.

Therefore, in both cases we have,

$$\|\mathscr{I}_\beta f\|_{F^{\alpha+\beta}_{p,q}} = \|\mathscr{I}_\beta f\|_{p,\gamma} + \| \Big(\int_0^{+\infty} \Big(t^{k-(\alpha+\beta)} |\frac{\partial^k P_t \mathscr{I}_\beta f}{\partial t^k}| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq C_\beta \|f\|_{p,\gamma} + C_{k,\alpha,\beta} \| \Big(\int_0^{+\infty} \Big(u^{k-\alpha} |\frac{\partial^k P_u}{\partial u^k}| \Big)^q \frac{du}{u} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq C_{k,\alpha,\beta} \|f\|_{F^\alpha_{p,q}}.$$

□

Now, we study the boundedness of the Riesz fractional derivatives and of the Bessel fractional derivatives on Triebel–Lizorkin spaces. Again, because they require different techniques, we consider two cases:

- The bounded case, $0 < \beta < \alpha < 1$.
- The unbounded case $0 < \beta < \alpha$.

Let us start with the bounded case for the Riesz derivative.

Theorem 8.23. *Let* $1 \leq p,q < \infty$, *for* $0 < \beta < \alpha < 1$, D^{β} *is bounded from* $F^{\alpha}_{p,q}(\gamma_d)$ *into* $F^{\alpha-\beta}_{p,q}(\gamma_d)$.

Proof. Let $f \in F^{\alpha}_{p,q}(\gamma_d)$, using the fundamental theorem of calculus, and Hardy's inequality (10.100) with $p = 1$,

$$\begin{aligned}|D^{\beta} f(x)| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Big|P_s f(x) - f(x)\Big| ds\\ &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s \Big|\frac{\partial}{\partial r}P_r f(x)\Big| dr\, ds \leq \frac{1}{c_\beta \beta}\int_0^{+\infty} r^{1-\beta}\Big|\frac{\partial}{\partial r}P_r f(x)\Big|\frac{dr}{r}.\end{aligned}$$

Thus,

$$\|D^{\beta} f\|_{p,\gamma} \leq C_\beta \Big\| \int_0^{+\infty} r^{1-\beta}\Big|\frac{\partial}{\partial r}P_r f\Big|\frac{dr}{r}\Big\|_{p,\gamma} \leq C_\beta \|f\|_{F^{\alpha}_{p,q}} < \infty, \qquad (8.52)$$

because $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$ ($\alpha > \beta$ and $q \geq 1$). Now, using an analogous argument using Hardy's inequality (10.100) with $p = 1$,

$$\begin{aligned}\Big|\frac{\partial}{\partial t}P_t(D^{\beta} f)(x)\Big| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Big|\frac{\partial}{\partial t}P_{t+s} f(x) - \frac{\partial}{\partial t}P_t f(x)\Big| ds\\ &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s |u^{(2)}(x,t+r)| dr\, ds \leq \frac{1}{c_\beta \beta}\int_0^{+\infty} r^{-\beta}|u^{(2)}(x,t+r)| dr.\end{aligned}$$

This implies that

$$\begin{aligned}\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\Big|\frac{\partial}{\partial t}P_t(D_\beta f)(x)\Big|\Big)^q \frac{dt}{t} &\leq \frac{1}{c_\beta\beta}\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_0^{+\infty} r^{-\beta}|u^{(2)}(x,t+r)| dr\Big)^q \frac{dt}{t}\\ &\leq C_\beta \int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_0^{t} r^{-\beta}|u^{(2)}(x,t+r)| dr\Big)^q \frac{dt}{t} \qquad (8.53)\\ &\quad + C_\beta \int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_t^{+\infty} r^{-\beta}|u^{(2)}(x,t+r)| dr\Big)^q \frac{dt}{t}\\ &= (I) + (II).\end{aligned}$$

Writing $r^{-\beta} = r^{\frac{-\beta}{q}+\frac{-\beta}{q'}}$, $\frac{1}{q}+\frac{1}{q'} = 1$, and using Hölder's inequality in the internal integral, we have

$$(I) \leq C_\beta (1-\beta)^{1-q} \int_0^\infty t^{(2-\alpha)q-2+\beta} \int_0^t r^{-\beta} |u^{(2)}(x,t+r)|^q dr\, dt.$$

Then, according to the Fubini–Tonelli theorem, we get

$$(I) \leq C_\beta (1-\beta)^{1-q} \int_0^\infty r^{-\beta} \int_r^\infty t^{(2-\alpha)q+\beta-2} |u^{(2)}(x,t+r)|^q dt\, dr.$$

It is easy to prove that $(2-\alpha)q+\beta-2 > -1$. We need to study two cases:

Case #1 – if $(2-\alpha)q+\beta-2 < 0$: as $r < t$ and taking the change of variables $w = t+r$, we have

$$\begin{aligned}(I) &\leq C_\beta (1-\beta)^{1-q} \int_0^\infty r^{(2-\alpha)q-2} \int_r^\infty |u^{(2)}(x,t+r)|^q dt\, dr \\ &\leq C_\beta (1-\beta)^{1-q} \int_0^\infty r^{[(2-\alpha)q-1]-1} \int_{2r}^\infty |u^{(2)}(x,w)|^q dw\, dr \\ &\leq C_\beta (1-\beta)^{1-q} \int_0^\infty r^{[(2-\alpha)q-1]-1} \int_r^\infty |u^{(2)}(x,w)|^q dw\, dr.\end{aligned}$$

Then using Hardy's inequality (10.101) as $(2-\alpha)q-1 > 0$

$$(I) \leq C_\beta (1-\beta)^{1-q} \frac{1}{(2-\beta)q-1} \int_0^\infty \left(w^{2-\beta} |u^{(2)}(x,w)|\right)^q \frac{dw}{w}.$$

Case #2 – if $(2-\beta)q+\beta-2 \geq 0$: taking the change of variables $w = t+r$, we get

$$\begin{aligned}(I) &\leq C_\beta (1-\beta)^{1-q} \int_0^\infty r^{-\beta} \int_r^\infty (t+r)^{(2-\alpha)q+\beta-2} |u^{(2)}(x,t+r)|^q dt\, dr \\ &= C_\beta (1-\beta)^{1-q} \int_0^\infty r^{-\beta} \int_{2r}^\infty w^{(2-\alpha)q+\beta-2} |u^{(2)}(x,w)|^q dw\, dr \\ &\leq C_\beta (1-\beta)^{1-q} \int_0^\infty r^{-\beta} \int_r^\infty w^{(2-\alpha)q+\beta-2} |u^{(2)}(x,w)|^q dw\, dr,\end{aligned}$$

and using Hardy's inequality (10.101),

$$(I) \leq \frac{C_\beta}{(1-\beta)^q} \int_0^\infty \left(w^{2-\alpha} |u^{(2)}(x,w)|\right)^q \frac{dw}{w}.$$

Therefore, in both cases we have

$$(I) \leq C_\beta \int_0^\infty \left(w^{2-\alpha} |u^{(2)}(x,w)|\right)^q \frac{dw}{w}.$$

To estimate (II), observe that $r^{-\beta} < t^{-\beta}$, for $r > t$ and $\beta > 0$, then using the same argument as before to estimate (I) case #1, taking the change of variables $w = t + r$, and using Hardy's inequality (10.101), so that

$$(II) \leq \frac{C_\beta}{1-\alpha} \int_0^\infty \left(w^{2-\alpha} |u^{(2)}(x,w)| \right)^q \frac{dw}{w}.$$

Finally,

$$\left\| \left(\int_0^\infty \left(t^{1-(\alpha-\beta)} |\frac{\partial}{\partial t} P_t(D^\beta f)| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma}$$

$$\begin{aligned} &\leq C \left\| (\int_0^\infty \left(t^{1-(\alpha-\beta)} \int_0^{+\infty} r^{-\beta} |u^{(2)}(\cdot, t+r)| dr \right)^q \frac{dt}{t})^{1/q} \right\|_{p,\gamma} \\ &\leq C \left\| \left(\int_0^\infty \left(w^{2-\beta} |u^{(2)}(\cdot, w)| \right)^q \frac{dw}{w} \right)^{1/q} \right\|_{p,\gamma} < \infty, \end{aligned} \tag{8.54}$$

as $f \in F^\beta_{p,q}(\gamma_d)$. Using the previous estimate and (8.52)

$$\|D^\beta f\|_{F^{\alpha-\beta}_{p,q}} \leq C \|f\|_{F^\beta_{p,q}}. \qquad \square$$

In the following theorem, we study the boundedness of the Bessel fractional derivative on Triebel–Lizorkin spaces for the bounded case $0 < \beta < \alpha < 1$.

Theorem 8.24. *Let $0 < \beta < \alpha < 1$, $1 \leq p, q < \infty$ then $\mathscr{D}^\beta$ is bounded from $F^\alpha_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in L^p(\gamma_d)$, using the fundamental theorem of calculus, we can write

$$\begin{aligned} |\mathscr{D}^\beta f(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} P_s f(x) - f(x)| ds \\ &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} |P_s f(x) - f(x)| ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} - 1| |f(x)| ds \\ &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \left| \frac{\partial}{\partial r} P_r f(x) \right| dr\, ds + \frac{1}{c_\beta} |f(x)| \int_0^{+\infty} s^{-\beta-1} \int_0^s e^{-r} dr\, ds. \end{aligned}$$

Now, using Hardy's inequality (10.100) with $p = 1$ in both integrals, we have

$$|\mathscr{D}^\beta f(x)| \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} \left| \frac{\partial}{\partial r} P_r f(x) \right| \frac{dr}{r} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) |f(x)|.$$

Thus,

$$|\mathscr{D}^\beta f(x)| \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} |u^{(1)}(x,r)| \frac{dr}{r} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) |f(x)|.$$

Therefore, if $f \in F^{\alpha}_{p,q}(\gamma_d)$, we get

$$\begin{aligned}\|\mathscr{D}^{\beta} f\|_{p,\gamma} &\leq \frac{1}{\beta c_{\beta}}\Big\| \int_0^{+\infty} r^{1-\beta}|u^{(1)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma} + \frac{1}{\beta c_{\beta}}\Gamma(1-\beta)\|f\|_{p,\gamma} \\ &\leq C_{\beta}\|f\|_{F^{\beta}_{p,1}} \leq C'_{\beta}\|f\|_{F^{\alpha}_{p,q}}, \end{aligned} \tag{8.55}$$

because $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$, as $\alpha > \beta$, and $q \geq 1$.

Using a similar argument to that above, the fundamental theorem of calculus and Hardy's inequality (10.100) with $p = 1$, we get

$$\begin{aligned}\Big|\frac{\partial}{\partial t}P_t(\mathscr{D}^{\beta} f)(x)\Big| &\leq \frac{1}{c_{\beta}}\int_0^{\infty} s^{-\beta-1}\Big|e^{-s}\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)\Big|ds \\ &\leq \frac{1}{c_{\beta}}\int_0^{\infty} s^{-\beta-1}e^{-s}\Big|\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)\Big|ds \\ &\qquad + \frac{1}{c_{\beta}}\int_0^{\infty} s^{-\beta-1}|e^{-s}-1|\Big|\frac{\partial}{\partial t}P_t f(x)\Big|ds \\ &\leq \frac{1}{c_{\beta}}\int_0^{\infty} s^{-\beta-1}\int_0^{s}|u^{(2)}(x,t+r)|dr\,ds \\ &\qquad + \frac{1}{c_{\beta}}|u^{(1)}(x,t)|\int_0^{\infty} s^{-\beta-1}\int_0^{s} e^{-r}dr\,ds, \\ &\leq \frac{1}{\beta c_{\beta}}\int_0^{\infty} r^{-\beta}|u^{(2)}(x,t+r)|dr + \frac{\Gamma(1-\beta)}{\beta c_{\beta}}|u^{(1)}(x,t)|. \end{aligned}$$

Therefore,

$$\Big|\frac{\partial}{\partial t}P_t(\mathscr{D}^{\beta} f(x))\Big| \leq \frac{1}{\beta c_{\beta}}\int_0^{\infty} r^{-\beta}|u^{(2)}(x,t+r)|dr + \frac{\Gamma(1-\beta)}{\beta c_{\beta}}|u^{(1)}(x,t)|.$$

Then, we have

$$\begin{aligned}&\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\Big|\frac{\partial}{\partial t}P_t(\mathscr{D}^{\beta} f)\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} \\ &\leq \frac{C}{\beta c_{\beta}}\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_0^{\infty} r^{-\beta}|u^{(2)}(\cdot,t+r)|dr\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} \\ &\qquad + \frac{C}{\beta c_{\beta}}\Gamma(1-\beta)\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}|u^{(1)}(\cdot,t)|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}. \end{aligned}$$

Now, the first term can be estimated as in the proof of Theorem 3, estimates (8.53) and (8.54), so that

$$\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_0^{\infty} r^{-\beta}|u^{(2)}(\cdot,t+r)|dr\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} \leq C\Big\|\int_0^{\infty}\Big(t^{2-\alpha}\Big|\frac{\partial^2}{\partial t^2}P_t f\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}$$

which is finite as $f \in F^{\alpha}_{p,q}(\gamma_d)$. For the second term, we have

$$\Big\|\Big(\int_0^\infty \Big(t^{1-(\alpha-\beta)}|u^{(1)}(x,t)|\Big)^q \frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} \leq C\|f\|_{F^{\alpha-\beta}_{p,q}} \leq C\|f\|_{F^{\alpha}_{p,q}}$$

as $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\alpha-\beta}_{p,q}(\gamma_d)$; thus,

$$\Big\|\Big(\int_0^\infty \Big(t^{1-(\alpha-\beta)}|\frac{\partial}{\partial t}P_t(\mathscr{D}^\beta f)|\Big)^q \frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} \leq C\|f\|_{F^{\alpha}_{p,q}}.$$

Therefore, $\mathscr{D}^\beta f \in F^{\alpha-\beta}_{p,q}(\gamma_d)$ and moreover, using the previous estimate and (8.55)

$$\|\mathscr{D}^\beta f\|_{F^{\alpha-\beta}_{p,q}} \leq C\|f\|_{F^{\alpha}_{p,q}}.$$

□

To study the general case for fractional derivatives (removing the condition that the indexes must be less than 1), we need to consider forward differences again. Also, we need the generalized version of Hardy's inequality (see Theorem 10.26 in the Appendix, and also the following technical results):

Lemma 8.25. *For any positive integer k,*

$$\begin{aligned}\Delta_s^k(f,t) &= \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)dv_k\ldots dv_2dv_1\\ &= \int_0^s\cdots\int_0^s f^{(k)}(t+v_1+\ldots+v_k)dv_k\ldots dv_1\end{aligned}$$

For the proof of this result, see Lemma 10.30 in the Appendix, or [109] Lemma 3.1, *ii*).

Lemma 8.26. *Let $t \geq 0, \beta > 0$ and let k be the smallest integer greater than β, and let f differentiable up to order k, then*

$$\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds \leq C_{\beta,k}\int_0^{+\infty} w^{k-\beta-1}|f^{(k)}(t+w)|dw$$

where $C_{\beta,k} = \int_0^1\cdots\int_0^1 (v_1+\ldots+v_k)^{\beta-k}dv_1\ldots dv_k$

Proof. Using Lemma10.26, with $p=1$, and Lemma 8.25 we have,

$$\begin{aligned}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds &\leq \int_0^{+\infty} s^{-\beta-1}\int_0^s\cdots\int_0^s |f^{(k)}(t+v_1+\ldots+v_k)|dv_1\ldots dv_k ds\\ &\leq \int_0^1\cdots\int_0^1\Big(\int_0^{+\infty}(s^k|f^{(k)}(t+s(v_1+\ldots+v_k))|)s^{-\beta-1}ds\Big)dv_1\ldots dv_k\\ &= \int_0^1\cdots\int_0^1\Big(\int_0^{+\infty}(s^{k-\beta-1}|f^{(k)}(t+s(v_1+\ldots+v_k))|)ds\Big)dv_1\ldots dv_k\end{aligned}$$

taking $r = s(v_1+\ldots+v_k)$ then $dr = (v_1+\ldots+v_k)ds$,

$$\begin{aligned}\int_0^{+\infty} s^{k-\beta-1}|f^{(k)}(t+s(v_1+\ldots+v_k)|ds &= \int_0^{+\infty} r^{k-\beta}|f^{(k)}(t+r)|\frac{dr}{r}(v_1+\ldots+v_k)^{\beta-k}\\ &= \int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr(v_1+\ldots+v_k)^{\beta-k}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds\\ &\leq \int_0^1\cdots\int_0^1\Big(\int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr(v_1+\ldots+v_k)^{\beta-k}\Big)dv_1\ldots dv_k\\ &= \Big(\int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr\Big)\int_0^1\cdots\int_0^1(v_1+\ldots+v_k)^{\beta-k}dv_1\ldots dv_k\\ &= C_{\beta,k}\Big(\int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr,\end{aligned}$$

where $C_{\beta,k} = \int_0^1\cdots\int_0^1(v_1+\ldots+v_k)^{\beta-k}dv_1\ldots dv_k < \infty.$ □

We need to use (8.47)

$$(P_s-I)^k f(x) = \Delta_s^k(u(x,\cdot),0),$$

and (8.48)

$$P_t(P_s-I)^k f(x) = \Delta_s^k(u(x,\cdot),t).$$

Let us consider the unbounded case $0<\beta<\alpha$ for the Riesz derivative,

Theorem 8.27. *Let $0<\beta<\alpha$, $1\leq p,q<\infty$, then D^β is bounded from $F^\alpha_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f\in F^\alpha_{p,q}(\gamma_d)$, using (8.47), (8.48) and Lemma 8.26,

$$\begin{aligned}|D_\beta f(x)| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|(P_s-I)^k f(x)|ds\\ &= \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(u(x,\cdot),0)|ds\\ &\leq C_{\beta,k}\int_0^{+\infty} r^{k-\beta-1}|u^{(k)}(x,r)|dr.\end{aligned}$$

Then

$$\|D_\beta f\|_{p,\gamma} \leq C_{\beta,k}\Big\|\int_0^{+\infty} r^{k-\beta}|u^{(k)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma} < \infty,$$

because $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$, ($\alpha > \beta$ and $1 \leq q < \infty$).

Let $n \in \mathbb{N}, n > \alpha$; using Lemma 10.30(8.46) and Lemma 8.26, we get

$$\Big|\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x)\Big| \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(u^{(n)}(x,\cdot),t)|ds$$
$$\leq \frac{1}{c_\beta} C_{\beta,k}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr.$$

Therefore,

$$\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big|\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x)\Big|\Big)^q \frac{dt}{t}$$
$$\leq C_{\beta,k}\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr\Big)^q \frac{dt}{t}$$

which is inequality (8.53) for $n = k = 1$. The rest of the proof follows the argument used in Theorem 8.23, so that

$$\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr\Big)^q \frac{dt}{t}\Big)^{1/q} \tag{8.56}$$
$$\leq C\Big(\int_0^\infty \Big(s^{n+k-\alpha}|u^{(n+k)}(x,s)|\Big)^q \frac{ds}{s}\Big)^{1/q},$$

taking $L^p(\gamma)$-norm both sides of the inequality, we get the result. □

Finally, the following result extends Theorem 8.24 to the general case $0 < \beta < \alpha$:

Theorem 8.28. *Let $0 < \beta < \alpha$, $1 < p < \infty$ and $1 \leq q < \infty$, then $\mathscr{D}^\beta$ is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in F^{\alpha}_{p,q}(\gamma_d)$, k be an integer such that $k-1 \leq \beta < k$ and $v(x,r) = e^{-r}u(x,r)$, using Lemma 8.26 and Leibniz's differentiation rule for the product

$$|\mathscr{D}^\beta f(x)| \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|(e^{-s}P_s - I)^k f(x)|ds = \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(v(x,\cdot),0)|ds$$
$$\leq C_{\beta,k}\int_0^{+\infty} r^{k-\beta}|v^{(k)}(x,r)|\frac{dr}{r} \leq C_{\beta,k}\Big(\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta}e^{-r}|u^{(k-j)}(x,r)|\frac{dr}{r}\Big)$$
$$= C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{k-\beta}e^{-r}|u^{(k-j)}(x,r)|\frac{dr}{r}\Big) + C_{\beta,k}\int_0^{+\infty} r^{k-\beta}e^{-r}|u(x,r)|\frac{dr}{r},$$

then

$$\begin{aligned}
\|\mathscr{D}^\beta f\|_{p,\gamma} &\leq C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\Big\|\int_0^{+\infty} r^{k-\beta}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\Big) + C_{\beta,k}\Big\|\int_0^{+\infty} r^{k-\beta}e^{-r}|u(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\\
&\leq C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\Big\|\int_0^{+\infty} r^{k-\beta}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\Big) + C_{\beta,k}\int_0^{+\infty} r^{k-\beta}e^{-r}\|u(\cdot,r)\|_{p,\gamma}\frac{dr}{r}\\
&\leq C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\Big\|\int_0^{+\infty} r^{k-j-(\beta-j)}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\Big) + C_{\beta,k}\|f\|_{p,\gamma}\Gamma(k-\beta)\\
&\leq C\|f\|_{F^\alpha_{p,q}},
\end{aligned}$$

because $F^\alpha_{p,q}(\gamma_d) \subset F^{\beta-j}_{p,1}(\gamma_d)$, as $\alpha > \beta \geq \beta - j \geq 0$, for $j = 0,\dots,k-1$ and $q \geq 1$.

Now, let $n \in \mathbb{N}, n > \alpha$ and $w(x,t) = e^{-t}u^{(n)}(x,t)$, using Lemma 8.26, we get

$$\begin{aligned}
\Big|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^\beta f)(x)\Big| &\leq \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(w(x,\cdot),t)|ds\\
&\leq e^t C_{\beta,k}\int_0^{+\infty} s^{k-\beta-1}|w^{(k)}(x,t+s)|ds.
\end{aligned}$$

Now, using Leibniz's rule, $w^{(k)}(x,r) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-r}u^{(k+n-j)}(x,r)$ and then

$$|w^{(k)}(x,r)| \leq \sum_{j=0}^{k}\binom{k}{j}e^{-r}|u^{(k+n-j)}(x,r)|,$$

for all $r > 0$. Thus,

$$\Big|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^\beta f)(x)\Big| \leq C_{\beta,k}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} s^{k-\beta-1}e^{-s}|u^{(k+n-j)}(x,t+s)|ds.$$

Therefore,

$$\begin{aligned}
&\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^\beta f)(x)\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\leq C_{\beta,k}\sum_{j=0}^{k}\binom{k}{j}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{k-j-(\beta-j)-1}e^{-s}|u^{(k-j+n)}(x,t+s)|ds\Big)^q\frac{dt}{t}\Big)^{1/q}
\end{aligned}$$

For $0 \leq j \leq k-1$, we have $\beta - j \geq \beta - (k-1) \geq 0$, and taking into account that each term of the above sum is bounded by the left side of the inequality (8.56), with k replaced by $k-j$ and β replaced by $\beta - j$, we get that

$$\Big\|\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{k-j-(\beta-j)-1}e^{-s}|u^{(k+n-j)}(x\cdot,t+s)|ds\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} < C\|f\|_{F^\alpha_{p,q}}$$

for $0 \le j \le k-1$. Unfortunately, the remaining case $j = k$ requires a special argument that uses the following known inequality for the Poisson–Hermite semigroup:

$$\Big|\frac{\partial^n}{\partial t^n} P_t f(x)\Big| \le C T^* f(x) t^{-n} \tag{8.57}$$

(see Lemma 3.4; see also [226, Lemma 1], or [224]). Then

$$\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)} \int_0^{+\infty} s^{k-\beta-1} e^{-s} |u^{(n)}(\cdot, t+s)| ds\Big)^q \frac{dt}{t}\Big)^{1/q}$$

$$\begin{aligned}
&\le C\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)} \int_0^t s^{k-\beta-1} e^{-s} |u^{(n)}(x, t+s)| ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&\quad + C\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)} \int_t^{+\infty} s^{k-\beta-1} e^{-s} |u^{(n)}(x, t+s)| ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&= (I) + (II).
\end{aligned}$$

We first consider the case $k \le \beta$. The term (I) is estimated as term (I) in the proof of Theorem 8.23.

$$(I) \le C\Big(\int_0^\infty \Big(v^{n-(\alpha-k)} |u^{(n)}(x, v)|\Big)^q \frac{dv}{v}\Big)^{1/q}.$$

Because $\beta \ge k-1$, taking the change of variables $v = t+s$, we get

$$\begin{aligned}
(II) &\le C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_t^{+\infty} |u^{(n)}(x, t+s)| ds\Big)^q dt\Big)^{1/q}\\
&= C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_{2t}^{+\infty} |u^{(n)}(x, r)| dr\Big)^q dt\Big)^{1/q}\\
&\le C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_t^{+\infty} |u^{(n)}(x, r)| dr\Big)^q dt\Big)^{1/q}.
\end{aligned}$$

Therefore, using Hardy's inequality (10.101),

$$(II) \le \frac{C}{(n+k-\alpha-1)^{1/q}} \Big(\int_0^\infty \Big(r^{n-(\alpha-k)} |u^{(n)}(x, r)|\Big)^q \frac{dr}{r}\Big)^{1/q},$$

Next, consider the case $k > \alpha$. In this case, using inequality (8.57) and Hardy's inequality (10.100), we have

$$\begin{aligned}
(I) &\le C_n |T^* f(x)| \Big(\int_0^\infty t^{-(\alpha-\beta)q-1} \Big(\int_0^t s^{k-\beta-1} e^{-s} ds\Big)^q dt\Big)^{1/q}\\
&\le C_n |T^* f(x)| \frac{1}{(\alpha-\beta)^{1/q}} \Big(\int_0^\infty s^{(k-\alpha)q-1} e^{-sq} ds\Big)^{1/q}\\
&= C_n |T^* f(x)| \frac{1}{(\alpha-\beta)^{1/q} q^{k-\alpha}} \Big(\Gamma((k-\alpha)q)^{1/q}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(II) &\le \Big(\int_0^1 t^{(n+k-\alpha-1)q-1}\Big(\int_t^{+\infty} e^{-s}|u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&\quad +\Big(\int_1^{\infty} t^{(n+k-\alpha-1)q-1}\Big(\int_t^{+\infty} e^{-s}|u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&= (III)+(IV).
\end{aligned}
$$

Using the usual argument the change of variables $v = t+s$ and Hardy's inequality (10.101), we get

$$
\begin{aligned}
(III) &\le \Big(\int_0^1 t^{(n-1)q-1}\Big(\int_t^{+\infty} |u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&\le \Big(\int_0^{\infty} t^{(n-1)q-1}\Big(\int_t^{+\infty} |u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&= \Big(\int_0^{\infty} t^{(n-1)q-1}\Big(\int_{2t}^{+\infty} |u^{(n)}(x,r)|dr\Big)^q dt\Big)^{1/q}\\
&\le \Big(\int_0^{\infty} t^{(n-1)q-1}\Big(\int_t^{+\infty} |u^{(n)}(x,r)|dr\Big)^q dt\Big)^{1/q}\\
&\le \frac{1}{n-1}\Big(\int_0^{\infty} \Big(r^n|u^{(n)}(x,r)|\Big)^q \frac{dr}{r}\Big)^{1/q}.
\end{aligned}
$$

Finally, using inequality (8.57) again, we get

$$
\begin{aligned}
(IV) &\le \Big(\int_1^{\infty} t^{(n+k-\alpha-1)q-1}\Big(\int_t^{+\infty} e^{-s}C_n|T^*f(x)|t^{-n}ds\Big)^q dt\Big)^{1/q}\\
&= C_n|T^*f(x)|\Big(\int_1^{\infty} t^{(k-\alpha-1)q-1}e^{-tq}dt\Big)^{1/q}\\
&\le C_n|T^*f(x)|\Big(\int_1^{\infty} t^{(k-\alpha-1)q-1}dt\Big)^{1/q} = C_n|T^*f(x)|\Big(\frac{1}{(\alpha+1-k)q}\Big)^{1/q}.
\end{aligned}
$$

Hence, in both cases, we get that

$$
\Big\|\Big(\int_0^{\infty}\Big(t^{n-(\alpha-\beta)}|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^\beta f)|\Big)^q \frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} < \infty,
$$

as $f \in F^{\alpha}_{p,q}(\gamma_d)$. Therefore, $\mathscr{D}^\beta f \in F^{\alpha-\beta}_{p,q}(\gamma_d)$ and moreover,

$$
\|\mathscr{D}^\beta f\|_{F^{\alpha-\beta}_{p,q}} \le C\|f\|_{F^{\alpha}_{p,q}}. \qquad \square
$$

8.6 Notes and Further Results

1. Observe that the arguments given in the proofs of theorems in this chapter are still valid in the classical case taking the Poisson integral; therefore, they are alternative proofs to those given in E. Stein's book [252].

2. Moreover, if instead of considering the *Ornstein–Uhlenbeck operator* and the *Poisson–Hermite semigroup*, we consider the *Laguerre differential operator* in $\mathbb{R}^d_+$, for $\alpha = (\alpha_1, \cdots, \alpha_d)$ a multi-index,

$$\mathscr{L}^\alpha = \sum_{i=1}^{d} \left[x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right] \tag{8.58}$$

and the corresponding *Poisson–Laguerre semigroup*, or if we consider the *Jacobi differential operator* in $(-1,1)^d$,

$$\mathscr{L}^{\alpha,\beta} = -\sum_{i=1}^{d} \left[(1-x_i^2) \frac{\partial^2}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial}{\partial x_i} \right], \tag{8.59}$$

and the corresponding *Poisson–Jacobi semigroup* (for more details, we refer the reader to [279]), the arguments are completely analogous. To see this, it is more convenient to use the representation of P_t in terms of the one-sided stable measure $\mu_t^{(1/2)}(ds)$ and to write Lemma 3.3 in those terms (see [225]). In other words, we can define in an analogous manner *Laguerre–Lipschitz spaces* and *Jacobi–Lipschitz spaces,* and prove the corresponding notions of fractional integrals and fractional derivatives (see [117, 25]).

3. Following similar arguments to those given in Chapter 7, we can define in an analogous manner *Laguerre–Besov–Lipschitz spaces* and *Jacobi–Besov–Lipschitz spaces*, in addition to *Laguerre–Triebel–Lizorkin spaces* and *Jacobi–Triebel–Lizorkin spaces,* and then prove that the corresponding notions of fractional integrals and fractional derivatives of corresponding operators $\mathscr{L}^{\alpha,\beta}$ and $\mathscr{L}^\alpha$ behave similarly.

4. In [146], G. E. Karadzhov & M. Milman show that the Gaussian Riesz potentials I_β maps $L^p(\log L)_a$ continuously into $L^p(\log L)_{a+\beta}$, for $1 < p < \infty$ and $a \in \mathbb{R}$. The proof is using extrapolation in an abstract setting. Moreover, their proof is in fact valid for any hypercontractive semigroup (see [146, Theorem 5.7]).

5. We can also consider *alternative Riesz potentials, alternative Bessel potential, alternative Riesz and alternative Bessel fractional derivatives* using the same formulas as before, but with respect to $\overline{L}$, the alternative Ornstein–Uhlenbeck operator (2.14). This case is actually simpler, as 0 is not a eigenvalue of $\overline{L}$. For instance, for $\beta > 0$ the alternative Riesz potential $\overline{I}_\beta$ can be defined as

$$\overline{I}_\beta = (-\overline{L})^{-\beta/2}, \tag{8.60}$$

meaning that any multi-index ν such that $|\nu| > 0$ its action on the Hermite polynomial $\mathbf{H}_\nu$ is

$$\overline{I}_\beta \mathbf{H}_\nu(x) = \frac{1}{(|\nu| + d)^{\beta/2}} \mathbf{H}_\nu(x). \tag{8.61}$$

$\overline{I}_\beta$ has the following integral representation, using the fact that $\overline{L}$ is the infinitesimal generator of the semigroup $\{T_t^{(d)}\}_t = \{e^{-td}T_t\}_t$, the d-translated Ornstein–Uhlenbeck semigroup (2.78),

$$\begin{aligned}\overline{I}_\beta f(x) = (-\overline{L})^{-\beta/2} f(x) &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta-2}{2}} T_t^{(d)} f(x)\, dt \qquad (8.62)\\ &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta-2}{2}} e^{-dt} T_t f(x)\, dt\\ &= C_\beta e^{|x|^2} \int_{\mathbb{R}^d} \Big(\int_0^1 (-\log r)^{\frac{\beta-2}{2}} r^d \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \Big) f(y)\; \gamma_d(dy).\\ &= C_\beta \int_{\mathbb{R}^d} \Big(\int_0^1 (-\log r)^{\frac{\beta-2}{2}} r^d \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \Big) f(y)(dy).\end{aligned}$$

The integral representation (8.62) is crucial to getting the $L^p(\gamma_d)$-boundedness results of some of the Gaussian singular integrals considered in Chapter 9.

Similar representations can be found for Bessel potentials and the fractional derivatives associated with $\overline{L}$.

6. In [164], alternate representations of I_β and D_β are obtained.

Proposition 8.29. *Suppose $f \in C_B^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, then*

$$D^\beta f = \frac{1}{\beta c_\beta} \int_0^\infty t^{-\beta} \frac{\partial}{\partial t} P_t f dt, \quad 0 < \beta < 1, \qquad (8.63)$$

$$I_\beta f = -\frac{1}{\beta \Gamma(\beta)} \int_0^\infty t^\beta \frac{\partial}{\partial t} P_t f dt, \quad \beta > 0. \qquad (8.64)$$

Proof. Let us start proving (8.63). Integrating by parts in (8.26) ,we get

$$\begin{aligned}D_\beta f(x) &= \frac{1}{c_\beta} \lim_{\substack{a\to 0^+\\ b\to\infty}} \int_a^b t^{-\beta-1} \left(P_t f(x) - f(x)\right) dt\\ &= \frac{1}{c_\beta} \lim_{\substack{a\to 0^+\\ b\to\infty}} \left\{ \frac{t^{-\beta}}{-\beta} \left(P_t f(x) - f(x)\right)\Big|_a^b + \frac{1}{\beta} \int_a^b t^{-\beta} \frac{\partial}{\partial t} P_t f(x) dt \right\}\\ &= \frac{1}{\beta c_\beta} \int_0^\infty t^{-\beta} \frac{\partial}{\partial t} P_t f(x) dt\end{aligned}$$

because, using (3.28) and (3.29), we have

$$\lim_{b\to\infty} \left(\frac{P_b f(x) - f(x)}{b^\beta} \right) = 0$$

and

$$\lim_{a\to 0^+}\left|\frac{P_af(x)-f(x)}{a^\beta}\right| \le \lim_{a\to 0^+}\frac{1}{a^\beta}\int_0^a\left|\frac{\partial}{\partial s}P_sf(x)\right|ds$$
$$\le C_{d,f}(d+|x|)\lim_{a\to 0^+}\frac{1-e^{-a}}{a^\beta}=0.$$

Let us prove now (8.64). Again, by integrating by parts, we have

$$\begin{aligned}
I_\beta f(x) &= \frac{1}{\Gamma(\beta)}\lim_{\substack{a\to 0^+\\ b\to\infty}}\int_a^b t^{\beta-1}P_tf(x)dt\\
&= \frac{1}{\Gamma(\beta)}\lim_{\substack{a\to 0^+\\ b\to\infty}}\left\{\frac{t^\beta}{\beta}P_tf(x)\Big|_a^b-\frac{1}{\beta}\int_a^b t^\beta\frac{\partial}{\partial t}P_tf(x)dt\right\}\\
&= -\frac{1}{\beta\Gamma(\beta)}\int_0^\infty t^\beta\frac{\partial}{\partial t}P_tf(x)dt,
\end{aligned}$$

because, using the previous result

$$\lim_{b\to\infty}\left|P_bf(x)b^\beta\right| \le C_{d,f}(d+|x|)\lim_{b\to\infty}b^\beta e^{-b}=0$$

and

$$\lim_{a\to 0^+}\left|P_af(x)a^\beta\right|=0. \qquad \square$$

Observe that because the previous proposition holds for $f=\mathbf{H}_\beta$, the Hermite polynomial of order $\beta, |\beta|>0$, then it holds for any non-constant polynomial f such that $\int_{\mathbb{R}^d}f(y)\gamma_d(dy)=0$.

By using (3.3) and (8.63), D_β can be expressed explicitly as

$$D_\beta f(x)=\int_{\mathbb{R}^d}K_\beta(x,y)f(y)dy,$$

where,

$$K_\beta(x,y)=C_d\int_0^\infty\int_0^1 t^{-\beta}e^{t^2/4logr}(-logr)^{1/2}\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}}$$
$$\times\left(\frac{2r^2\left|y-rx\right|^2-2r(1-r^2)\langle y-rx,x\rangle-dr^2(1-r^2)}{(1-r^2)^2}\right)\frac{dr}{r}dt.$$

Now, let us write

$$q_t(x,y)=-t\frac{\partial}{\partial t}\left(\int_0^1 t\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}\right), \tag{8.65}$$

and define the operator Q_t as

$$Q_t f(x) = -t\frac{\partial}{\partial t}P_t f(x) = \int_{\mathbb{R}^d} q_t(x,y)f(y)dy. \tag{8.66}$$

Following [108] we immediately get from (8.63) and (8.64) the following formulas:

Corollary 8.30. *Suppose $f \in C_B^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$. Then, we have*

$$-\beta D_\beta f = \frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}Q_t f dt, 0 < \beta < 1, \tag{8.67}$$

$$\beta I_\beta = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}Q_t f dt, \beta > 0. \tag{8.68}$$

7. An interesting use of the family $\{Q_t\}$ is that it allows us to give a version of *A. P. Calderón's reproducing formula* for the Gaussian measure; see [164].

Theorem 8.31.
i) Suppose $f \in L^1(\gamma_d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, then we have

$$f = \int_0^\infty Q_t f \frac{dt}{t}. \tag{8.69}$$

ii) Suppose f a polynomial such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, then we have

$$f = C_\beta \int_0^\infty \int_0^\infty t^{-\beta}s^\beta Q_t\left(Q_s f\right)\frac{ds}{s}\frac{dt}{t} \quad 0 < \beta < 1. \tag{8.70}$$

Also,

$$\int_0^\infty \int_0^\infty t^{-\beta}s^\beta Q_t\left(Q_s f\right)\frac{ds}{s}\frac{dt}{t} = d_\beta \int_0^\infty u\frac{\partial^2}{\partial u^2}P_u f du. \tag{8.71}$$

Formula (8.70) is the aforementioned version of Calderón's reproducing formula for the Gaussian measure.

Proof.
i) Using (3.28) and (3.29) we have,

$$\int_0^\infty Q_t f \frac{dt}{t} = \lim_{\substack{a\to 0^+\\ b\to\infty}}\left(-\int_a^b \frac{\partial}{\partial t}P_t f dt\right) = \lim_{\substack{a\to 0^+\\ b\to\infty}}\left(-P_t f\right)\Big|_a^b = f.$$

ii) Let us prove (8.70), given f, a polynomial such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, by Corollary 8.30, we have

$$D_\beta\left(I_\beta f\right) = \frac{1}{\beta c_\beta}\int_0^\infty t^{-\beta-1}Q_t\left(I_\beta f\right)dt.$$

Now, using the definition of Q_t and Fubini's theorem, we have

$$Q_t\left(I_\beta f\right) = \frac{1}{\beta\Gamma(\beta)}\int_{\mathbb{R}^d}\int_0^\infty s^{\beta-1}Q_s(f)(y)dsdy.$$

Again, using the definition of Q_s, we obtain

$$f = D_\beta\left(I_\beta f\right) = d_\beta\int_0^\infty\int_0^\infty t^{-\beta-1}s^{\beta-1}Q_t\left(Q_s f\right)dsdt.$$

To show (8.71), we see that from (8.66)

$$Q_t\left(Q_s f\right)(x) = ts\frac{\partial}{\partial t}\frac{\partial}{\partial s}P_{t+s}f(x).$$

But

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}P_{t+s}f(x) = \frac{\partial^2}{\partial u^2}P_u f(x)\Big|_{u=t+s},$$

then

$$\begin{aligned}\int_0^\infty\int_0^\infty t^{-\beta-1}s^{\beta-1}Q_t\left(Q_s f\right)dsdt &= \int_0^\infty\int_0^\infty t^{-\beta}s^\beta\frac{\partial^2}{\partial u^2}P_u f\Big|_{u=t+s}dsdt\\ &= d_\beta\int_0^\infty u\frac{\partial^2}{\partial u^2}P_u f du,\end{aligned}$$

where $d_\beta = \frac{B(-\beta+1,\beta+1)}{a_\beta c_\beta}$, $B(-\beta+1,\beta+1)$ being the beta function of parameter $(-\beta+1,\beta+1)$. □

8. Also, in [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, and W. Urbina obtained an analog of A. P. Calderón's reproducing formula for the Laguerre case.

9. Using more abstract approaches to Besov and Triebel–Lizorkin spaces associated with a general differential operator, as in [154], many of the results contained in this chapter would follow from the functional calculus for the Ornstein–Uhlenbeck operator.