7

Function Spaces with Respect to the Gaussian Measure

One of the main goals of functional spaces is to interpret and quantify the smoothness of functions. In this chapter, we discuss the analogs of classical functional spaces with respect to the Gaussian measure. We see that almost all classical spaces with respect to the Lebesgue measure have an analog for the Gaussian measure; nevertheless, we see that in some cases, for instance, Hardy spaces, the analogs to classical spaces are still incomplete and/or imperfect. On the other hand, most of the time, even if the spaces look similar, most of the proofs are different, mainly because the Gaussian measure is not invariant by translation, which implies the need for completely different techniques.

7.1 Gaussian Lebesgue Spaces $L^p(\gamma_d)$

The Gaussian Lebesgue spaces have been used implicitly in previous chapters for the study of continuity properties of the Ornstein–Uhlenbeck semigroup, the Poisson–Hermite semigroup, and maximal functions. For completeness, we are including them in this chapter.

Definition 7.1. *For $1 \leq p < \infty$, the Gaussian Lebesgue space $L^p(\gamma_d)$ is defined as*

$$L^p(\gamma_d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f \textit{ is a measurable function and } \int_{\mathbb{R}^d} |f(x)|^p \gamma_d(dx) < \infty \right\} \tag{7.1}$$

and the L^p-norm is given by

$$\|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^d} |f(x)|^p \gamma_d(dx) \right)^{1/p}. \tag{7.2}$$

Using analogous arguments, as in the classical case, it can be proved that the normed space $(L^p(\gamma_d), \|\cdot\|_{p,\gamma})$ is a Banach space for $1 \leq p < \infty$, that is, $L^p(\gamma_d)$ is a complete space (see for instance [263, Theorem 7.3]).

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As the Gaussian measure is a probability measure, using Hölder's inequality, we have for $1 \leq p < q$,

$$L^q(\gamma_d) \subset L^p(\gamma_d). \tag{7.3}$$

Additionally, from Theorem 10.7, we know that the family of polynomials with real coefficients is not only contained in $L^p(\gamma_d)$, $1 \leq p < \infty$, but is also dense there.

Thus, the Gaussian Lebesgue spaces $L^p(\gamma_d)$ are very different from the classical Lebesgue space $L^p(\mathbb{R}^d)$ theory with respect to the Lebesgue measure, because if $f \in L^p(\mathbb{R}^d)$, then $f(x) \to 0$ as $|x| \to \infty$, but for $f \in L^p(\gamma_d)$, we may have $f(x) \to \infty$ as $|x| \to \infty$, as long as it grows no faster than $e^{\delta|x|^2/p}$ with $\delta < 1$.

Observe that for any $1 \leq p < \infty$, the space $L^p(\gamma_d)$ is not closed under translations. For instance, in dimension one and $p = 1$, taking the function $f(x) = e^{|x|^2-|x|}$, then it is clear that $f \in L^1(\gamma_1)$, but it is easy to see that

$$\tau_1 f(x) = f(x+1) = e^{|x+1|^2-|x+1|} \notin L^1(\gamma_1).$$

Finally, because the Gaussian measure is trivially absolutely continuous with respect to the Lebesgue measure, with the Radon–Nikodym derivative the Gaussian weight, $\frac{d\gamma_d}{dx} = e^{-|x|^2}$, then

$$L^\infty(\gamma_d) = L^\infty(\mathbb{R}^d).$$

7.2 Gaussian Sobolev Spaces $L^p_\beta(\gamma_d)$

Sobolev spaces in the classical case are used to measure the regularity of solutions of partial differential equations (PDEs). Gaussian Sobolev spaces were introduced in the context of Malliavin calculus (see for instance P. Malliavin [172], D. Nualart [218] or S. Watanabe [288]). They play a fundamental role in it because they are used as a scale to measure the regularity of solutions of stochastic differential equations (see [218]). Moreover, similar to the classical case, Gaussian Sobolev spaces are particular cases of Gaussian Besov spaces; therefore, Besov spaces are a "better scale" to measure the regularity of functions.

Definition 7.2. *Given $\beta \geq 0$ and $1 \leq p < \infty$, the Gaussian Sobolev space of order β, $L^p_\beta(\gamma_d)$, is defined as the completion of the set of polynomials $\mathscr{P}(\mathbb{R}^d)$ with respect to the norm*

$$\|f\|_{p,\beta} := \left\|(I-L)^{\beta/2} f\right\|_{p,\gamma}. \tag{7.4}$$

Therefore, the set of polynomials in $\mathbb{R}^d$, $\mathscr{P}(\mathbb{R}^d)$ is trivially a dense set in these spaces. The spaces $L^p_\beta(\gamma_d)$, are also called *potential spaces* (see [145]).

In the classical case, Sobolev spaces appear naturally in partial differential equations to measure the integrability of partial derivatives of a given function, A. P. Calderón proved that Sobolev spaces can be characterized using the integrability of the derivatives. We are going to see that the same holds in the Gaussian case, i.e., fractional derivatives D_β can be used to characterize $L^p_\beta(\gamma_d)$ (see Theorem 8.8). A probabilistic proof of this fact was given by Sugita in [261].

Moreover, from the definition given of the Gaussian Sobolev spaces, $L^p_\alpha(\gamma_d)$, we see they can be characterized as the image of the Gaussian Lebesgue spaces under Gaussian Bessel potentials (see 8.21) Proposition 8.6. They can also be characterized using Riesz fractional derivatives (see Theorem 8.8). Additionally, as an application of the Littlewood–Paley functions $g^k_{x,\gamma}$ and $g^k_{t,\gamma}$, a characterization of Gaussian Sobolev spaces, $L^p_\beta(\gamma_d)$ for $1 < p < \infty$ can also be provided (see Section 9.5 in Chapter 9.

Finally, we have the following *Gaussian Sobolev embeddings*,

Proposition 7.3. *Gaussian Sobolev spaces satisfy*

i) If $p < q$ *then* $L^q_\beta(\gamma_d) \subset L^p_\beta(\gamma_d)$ *for each* $\beta \geq 0$.
ii) If $0 \leq \alpha_1 < \beta_2$ *then* $L^p_{\beta_2}(\gamma_d) \subset L^p_{\beta_1}(\gamma_d)$ *for each* $1 < p < \infty$.

Moreover, the embeddings in i) and ii) are continuous

Proof. Claim *i)* is an immediate consequence of Hölder's inequality.
For claim *ii)*, let f be a polynomial and let us consider $g = (1-L)^{-\beta_2/2} f$, then

$$(1-L)^{(\beta_1-\beta_2)/2} g = (1-L)^{\beta_1/2} f.$$

Using Meyer's multiplier theorem, Theorem 6.2, we can conclude that there exists $C > 0$, such that

$$\|f\|_{p,\beta_1} \leq C \|f\|_{p,\beta_2}.$$

□

7.3 Gaussian Tent Spaces $T^{1,q}(\gamma_d)$

In 1985, R. Coifman, Y. Meyer, and E. M. Stein [55], introduced the *tent spaces* T^p_q with respect to the Lebesgue measure, as the space of functions $F : \mathbb{R}^{d+1}_+ :\to \mathbb{R}$ such that,

$$J_q(f)(x) = \left(\int_{\Gamma(x)} |F(y,t)|^q dy \frac{dt}{t^{d+1}} \right)^{1/q} \in L^p(\mathbb{R}^d),$$

where $\Gamma(x) = \left\{ (y,t) \in \mathbb{R}^{d+1}_+ : |x-y| < t \right\}$, $1 < q < \infty$, and

$$\|F\|_{q,p} = \| J_q(f)\|_p.$$

In 2012, J. Mass, J. Van Neerven, and P. Portal [169] introduced *Gaussian tent spaces* as follows. Let

$$D := \{(x,t) \in \mathbb{R}^d \times (0,\infty) : t < m(x)\},$$

where as usual, $m(x) = 1 \wedge \frac{1}{|x|}$, is the admissibility function. Note that a point $(x,t) \in \mathbb{R}^d \times (0,\infty)$ belongs to $\overline{D}$ if and only if $B(x,t) \in \mathscr{B}_1$.

Definition 7.4. *The Gaussian tent space $T^{1,q}(\gamma_d)$ is the completion of $C_0(D)$ with respect to the norm,*

$$\|F\|_{T^{1,q}(\gamma)} := \|JF\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(D,\gamma_d\times\frac{dt}{t}))}, \tag{7.5}$$

where

$$(JF(x))(y,t) := \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))^{1/q}}F(y,t), \quad F \in C_0(D), \tag{7.6}$$

that is,

$$\|F\|_{T^{1,q}(\gamma)} = \int_{\mathbb{R}^d}\left(\int\int_{\Gamma_x^1(\gamma_d)} \frac{1}{\gamma_d(B(y,t))}|(JF(x))(y,t)|^q\gamma_d(dy)\frac{dt}{t}\right)^{1/q}\gamma_d(dx),$$

where, $\Gamma_x^1(\gamma_d) = \left\{(y,t) \in \mathbb{R}^d \times (0,\infty) : |y-x| < t < m(x)\right\}$ *is a Gaussian cone with* $a = 1$, *see (4.83).*

In [169], J. Mass, J. Van Neerven, and P. Portal obtained an atomic decomposition for $T^{1,q}(\gamma_d)$. As in the Euclidean case, this atomic decomposition turns out to be very useful, because using an atomic decomposition, we only have to check results for atoms and then the rest follows easily. First, let us see what a Gaussian *tent* is:

Definition 7.5. *For a measurable set $E \subset \mathbb{R}^d$ and a real number $a > 0$, we define the tent with aperture α over E by*

$$\mathsf{T}_\alpha(E) = \{(y,t) \in \mathbb{R}^{d+1}_+ : d(y,E^c) \geq \alpha t\}. \tag{7.7}$$

Now, let us define a *Gaussian atom.*

Definition 7.6. *Given $\alpha > 0$ a function $A : D \to \mathbb{C}$ is called a $T^{1,q}(\gamma_d)$ α-atom if there exists a ball B in $\mathscr{B}_\alpha$ such that*

i) A is supported in $\mathsf{T}_1(B) \cap D$, i.e.,

$$supp(A) \subset \{(y,t) \in D : \ t \leq d(y,B^c)\}.$$

ii) $\displaystyle\int\int_D |A(y,t)|^q\,\gamma_d(dy)\frac{dt}{t} \leq \frac{1}{\gamma_d(B)^{q/q'}}$, *where* $\frac{1}{q}+\frac{1}{q'} = 1.$

Lemma 7.7. *If A is a $T^{1,q}(\gamma_d)$ α-atom, then $A \in T^{1,q}(\gamma_d)$ and $\|A\|_{T^{1,q}(\gamma)} \leq 1$.*

Proof. Let A be a $T^{1,q}(\gamma_d)$ α-atom supported in $\mathsf{T}_1(B)\cap D$, for some $B \in \mathscr{B}_\alpha$. If $(y,t) \in \mathsf{T}_1(B)\cap D$ and $x \in B(y,t)$, then $x \in B$. Then, using this fact, Hölder's inequality, and Fubini's theorem, we obtain,

$$\begin{aligned}
\int_{\mathbb{R}^d} \Big(\int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} |A(y,t)|^q \gamma_d(dy) \frac{dt}{t} \Big)^{1/q} \gamma_d(dx) \\
&= \int_{\mathbb{R}^d} \Big(\int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} |A(y,t)|^q \gamma_d(dy) \frac{dt}{t} \Big)^{1/q} \chi_B(x)\gamma_d(dx) \\
&\leq \Big(\int_{\mathbb{R}^d} \int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} |A(y,t)|^q \gamma_d(dy) \frac{dt}{t} \gamma_d(dx) \Big)^{1/q} \gamma_d(B)^{1/q'} \\
&= \Big(\int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} |A(y,t)|^q \gamma_d(dy) \frac{dt}{t} \Big)^{1/q} \gamma_d(B)^{1/q'} \leq 1.
\end{aligned}$$

The set D admits a locally finite cover with tents $\mathsf{T}_1(B)$ based at balls $B \in \mathscr{B}_\alpha$ if and only if $\alpha > 1$; this explains the condition $\alpha > 1$ in the next theorem, which establishes an *atomic decomposition* of $T^{1,q}(\gamma_d)$.

Theorem 7.8. *(Mass, Van Neerven, and Portal) For all $F \in T^{1,q}(\gamma_d)$ and $\alpha > 1$, there exists a sequence $(\lambda_n)_{n\geq 1} \in \ell^1$ and a sequence of $T^{1,q}(\gamma)$ α-atoms $\{A_n\}_{n\geq 1}$ such that*

i) $F = \sum_{n=1}^{\infty} \lambda_n A_n$.
ii) $\sum_{n=1}^{\infty} |\lambda_n| \leq C\|f\|_{T^{1,q}(\gamma)}$, for some constant independent of f.

The proof of this result follows the lines of the classic counterpart in [55]; however, we can only use the doubling property of γ_d for admissible balls. That is why we need the Gaussian Whitney covering (see Theorem 4.10). Before we start with the proof, we need some notations and auxiliary results. Given a measurable set $E \subseteq \mathbb{R}^d$ and a real number $\alpha > 0$, we define

$$R_\alpha(E) = \{(y,t) \in \mathbb{R}^d \times (0,\infty) : \ d(y,E) < \alpha t\} = T^c_\alpha(E^c).$$

We also put, for any measurable set $E \subseteq \mathbb{R}^d$ and real number $\beta > 0$,

$$E^{[\beta]} = \left\{ x \in \mathbb{R}^d : \frac{\gamma_d(E\cap B)}{\gamma_d(B)} \geq \beta \text{ for all } B \in \mathscr{B}_{\frac{3}{2}} \text{ with center } x \right\}.$$

We call $E^{[\beta]}$ the set of points of admissible β-density of E. Note that $E^{[\beta]}$ is a closed subset of $\mathbb{R}^d$ contained in $\overline{E}$.

Lemma 7.9. *For all $\eta \in (\frac{1}{2},1)$ there exists an $\overline{\eta} \in (0,1)$ such that, for all measurable sets $E \subseteq \mathbb{R}^d$ and all non-negative measurable functions F on D, there exists a constant $C > 0$ such that*

$$\iint_{R_{1-\eta}(E^{[\overline{\eta}]})\cap D} F(y,t)\,\gamma_d(dy)\,\frac{dt}{t} \leq C \int_E \left(\iint_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} F(y,t)\,\gamma_d(dy)\,\frac{dt}{t} \right) \gamma_d(dx).$$

Proof. First, let $\overline{\eta} \in (0,1)$ be arbitrary and fixed. Let $(y,t) \in R_{1-\eta}(E^{[\overline{\eta}]}) \cap D$. Note that $(y,t) \in D$ implies $B(y,t) \in \mathscr{B}_1$. There exists $x \in E^{[\overline{\eta}]}$ such that $|y-x| < (1-\eta)t$. Notice first that, because $t \leq m(x)$, we have $|x| \leq (1-\eta)t + \frac{1}{t} \leq \frac{1}{2} + \frac{1}{t} \leq \frac{3}{2}\frac{1}{t}$. Thus, we have that $t \in (0, \frac{3}{2}m(x))$. Moreover, $B(x,\eta t) \subseteq B(y,t) \subseteq B(x,\frac{3}{2}t)$, and thus $B(y,t) \in \mathscr{B}_1$, $B(x,t) \in \mathscr{B}_{\frac{3}{2}}$, and $\gamma_d(B(x,t)) \sim \gamma_d(B(y,t))$ by repeated application of Theorem 1.6 *ii*), the doubling property on admissible balls. Therefore, we have

$$\begin{aligned}\gamma_d(E \cap B(y,t)) &\geq \gamma_d(E \cap B(x,t)) - \gamma_d(B(x,t) \cap B(y,t)^c)\\ &\geq \overline{\eta}\gamma_d(B(x,t)) - \gamma_d(B(x,t)) + \gamma_d(B(x,t) \cap B(y,t))\\ &\geq (\overline{\eta}-1)\gamma_d(B(x,t)) + \gamma_d(B(x,\eta t)).\end{aligned}$$

Now, picking $\overline{\eta}$ close enough to 1 and using the doubling property, we obtain a constant $c = c(\eta,n) \in (0,1)$ such that

$$\gamma_d(E \cap B(y,t)) \geq c\gamma_d(B(x,t)).$$

Therefore, there exists a constant $c' = c'(\eta,n) > 0$ such that

$$\gamma_d(E \cap B(y,t)) \geq c'\gamma_d(B(y,t)),$$

for all $(y,t) \in R_{1-\eta}(E^{[\overline{\eta}]}) \cap D$. Finally,

$$\begin{aligned}\int_E \left(\iint_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} F(y,t)\,\gamma_d(dy)\frac{dt}{t}\right)\gamma_d(dx) &= \iint_D \frac{\gamma_d(E \cap B(y,t))}{\gamma_d(B(y,t))} F(y,t)\,\gamma_d(dy)\,\frac{dt}{t}\\ &\geq c' \iint_{R_{1-\eta}(E^{[\overline{\eta}]})\cap D} F(y,t)\,\gamma_d(dy)\,\frac{dt}{t}. \qquad \square\end{aligned}$$

Lemma 7.10. *If a function $F \in T^{1,q}(\gamma_d)$ admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for some $\alpha > 1$, then it admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for all $\alpha > 1$.*

Proof. Suppose that $F \in T^{1,q}(\gamma_d)$ admits a decomposition in terms of $T^{1,q}(\gamma_d)$ β-atoms for some $\beta > 1$. We will show that f admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for any $\alpha > 1$. This is immediate if $\alpha \geq \beta$, because in this case any $T^{1,q}(\gamma_d)$ β-atom is a $T^{1,q}(\gamma_d)$ α-atom as well.

Therefore, let us assume that $1 < \alpha < \beta$. We claim that it suffices to show that there exists an integer N, depending only upon α, β, and the dimension d, such that if $B \in \mathscr{B}_\beta$, then $\mathsf{T}_1(B) \cap D$ can be covered by at most N tents of the form $\mathsf{T}_1(B')$ with $B' = B(c',r') \in \mathscr{B}_\alpha$ satisfying $r' = \alpha m(c')$.

To prove the claim, it clearly suffices to consider the case that F is a $T^{1,q}(\gamma_d)$ β-atom having support in $\mathsf{T}_1(B) \cap D$ for some ball $B \in \mathscr{B}_\beta$, with center c and radius $r = \beta m(c)$. Let $\{\mathsf{T}_1(B_1'), \cdots, \mathsf{T}_1(B_N')\}$ be a covering of $\mathsf{T}_1(B)$, where each B_j', $j = 1, \cdots N$, is a ball in $\mathscr{B}_\alpha$ with center c_j, radius $r_j = \alpha m(c_j)$, and intersecting B. For $x \in \mathsf{T}_1(B) \cap D$ we set

$$n(x) := \#\{1 \leq j \leq N : x \in \mathsf{T}_1(B_1')\}, \text{ and } F_j(x) = \frac{F(x)}{n(x)}\chi_{\mathsf{T}_1(B_1')}(x).$$

Then it follows that $F = \sum_{j=1}^{N} F_j$. Moreover, each F_j is a $T^{1,q}(\gamma_d)$ α-atom, because F_j is supported in $\mathsf{T}_1(B_j) \cap D$ and

$$\|F_j\|_{L^q(D,\gamma_d dt/t)} \le \|F\|_{L^q(D,\gamma_d dt/t)} \le \gamma_d(B)^{-1/q'} \le C\gamma_d(B'_j)^{-1/q'}.$$

To obtain the latter estimate, we pick an arbitrary $b \in B'_j \cup B$ and use Lemma 1.5 *ii*) to conclude that

$$m(c_j) \le (1+\alpha)m(b) \le 2(1+\alpha)(1+\beta)m(c),$$

and then we estimate,

$$r_j = \alpha m(c_j) \le 2\alpha(1+\alpha)(1+\beta)m(c) = 2\frac{\alpha}{\beta}(1+\alpha)(1+\beta)r.$$

Using the doubling property, Theorem 1.6, we conclude $\gamma_d(B_j) \le C\gamma_d(B)$. It follows that $F = \sum_{j=1}^{N} F_j$ is a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms, which proves the claim.

Fix $R \ge 1+\beta$ large enough such that $\alpha(R-\beta)/(R-\beta+\alpha) > 1$. The set $\{(y,t) \in D : |y| \le R+1\}$ can be covered with finitely many sets – their number depending only upon R, d and α – of the form $\mathsf{T}_1(B')$ with $B' = B(c',r') \in \mathscr{B}_\alpha$ and $r' = \alpha m(c')$.

Take a ball $B = B(c,r) \in \mathscr{B}_\beta$ with $|c| \ge R$ and choose $\delta \in (0,1)$ small enough such that $(1-\delta)\alpha(R-\beta)/(R-\beta+\alpha) > 1$. Observe that if $x \in B$, then $|x| \ge R-\beta \ge 1$, and therefore $m(x) = \frac{1}{|x|}$. Let us define

$$C_B := \{(x,t) \in B \times (0,\infty)\}.$$

Noting that $\mathsf{T}_1(B) \cap D \subset C_B$, it remains to cover C_B with N tents $\mathsf{T}_1(B')$ based on balls $B' \in \mathscr{B}_\alpha$ where the number N depends on α, β, and d only. To do so, let us start picking $c' \in B$, and let $r' = \alpha m(c') = \frac{\alpha}{|c'|}$ and $B' = B(c',r')$. If $(x,t) \in C_B$ is such that $|x-c'| \le \delta r'$, then

$$\begin{aligned} d(x,(B')^c) &= d(c',(B')^c) - |x-c'| \ge (1-\delta)r' = (1-\delta)\frac{\alpha}{|c'|} \\ &\ge (1-\delta)\frac{\alpha}{|x|+|x-c'|} \ge m(x)(1-\delta)\left(\frac{\alpha|x|}{|x|+\alpha}\right) \\ &\ge m(x)(1-\delta)\frac{\alpha(R-\beta)}{R-\beta+\alpha} \ge m(x) \ge t. \end{aligned}$$

Here, we have used the monotonicity of the function $t \to t/(t+\alpha)$.

We have proved that the point $(x,t) \in C_B$ belongs to $\mathsf{T}_1(B')$ whenever $|x-c'| \le \delta r'$. Using that $(|c|+\beta)r \le (|c|+\beta)\frac{\beta}{|c|} \le \beta+\beta^2$, we have

$$r' = \alpha m(c') \ge \frac{\alpha}{|c|+\beta} \ge \frac{\alpha}{\beta+\beta^2}r.$$

This implies that B can be covered with N balls $B' = B(c',\delta r')$ as above, with N dependent only on α, β, and d. The union of the N sets $\mathsf{T}_1(B') \cap D$ then covers C_B, thus completing the proof of the lemma. □

We are ready to prove Theorem 7.8.

Proof. Using Lemma 7.10, it suffices to prove that each $F \in T^{1,q}(\gamma_d)$ admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for some $\alpha > 0$.

Recall that the disjoint sets $A^{(i)}_{p,\kappa}$ have been introduced in Definition 4.8. We shall apply Theorem 4.10 for $p = 4$ and $\kappa = 8$ (the reason for this choice is the constant $16 = 2^4$ produced in the argument below). As

$$\Big(\bigcup_{0\le l\le 4} L_l\Big) \cup \Big(\bigcup_{i\in\{1,\dots,8\}^d} A^{(i)}_{4,8}\Big) = \mathbb{R}^d,$$

we may write

$$f = f\chi_{\{\|Jf\|_2>0\}} = \sum_{0\le l\le 4}\sum_{Q\in\Delta^{\gamma}_{0,l}} f\chi_{Q\cap\{\|Jf\|_2>0\}} + \sum_{i\in\{1,\dots,8\}^d} f\chi_{A^{(i)}_{4,8}\cap\{\|Jf\|_2>0\}}, \tag{7.8}$$

where $f\chi_{\{\|Jf\|_2>0\}}(x,t) := f(x,t)\chi_{\{\|Jf\|_2>0\}}(x)$ and

$$\{\|Jf\|_2 > 0\} := \{x\in\mathbb{R}^d : \|Jf(x)\|_{L^2(D,d\gamma_d\frac{dt}{t})} > 0\}.$$

The first equality in (7.8) is justified as follows. For all $x_0 \in V := \{\|Jf\|_2 = 0\}$ we have $\chi_{B(y,t)}(x_0)f(y,t) = 0$ for almost all $(y,t)\in D$; therefore, using Fubini's theorem, for almost all $y\in\mathbb{R}^d$, we have

$$\chi_{B(y,t)}(x_0)f(y,t) = 0 \ \text{ for almost all } t > 0.$$

Fix $\delta > 0$ arbitrary. Then, for almost all $y \in B(x_0,\delta)$ we have $f(y,t) = 0$ for almost all $t \ge \delta$. Applying again Fubini's theorem, this implies that $f(y,t) = 0$ for almost all $(y,t) \in (B(x_0,\delta)\times[\delta,\infty))\cap D$. Taking the union over all rational $\delta > 0$, it follows that $f \equiv 0$ almost everywhere on $\Gamma_x := \{(y,t)\in D : \ |x-y| < t\}$ the "admissible cone" over x. If K is any compact set contained in V, then by taking the union over a countable dense set of points $x\in K$, it follows that $f(y,t) = 0$ almost everywhere on the "admissible cone" over K. Finally, using the inner regularity of the Lebesgue measure on $\mathbb{R}^d$, it follows that $f(y,t) = 0$ almost everywhere on the "admissible cone" over V. In particular, this proves the first identity in (7.8).

To prove the theorem it suffices to prove that each of the summands on the right-hand side of (7.8) has an atomic decomposition. In view of Theorem 4.10 for $p = 4$ and $\kappa = 8$ it suffices to prove that

$$g := f\chi_{W\cap\{\|Jf\|_2>0\}}$$

has an atomic decomposition for every measurable set W in $\mathbb{R}^d$ such that $W + \mathscr{C}_{16}$ is admissible $2^9\sqrt{d}$-Whitney.

Given $k\in\mathbb{Z}$, let us define

$$O_k := \{\|Jf\|_2 > 2^k\}$$

and $F_k := O_k^c$. Fix an arbitrary $\eta \in (\frac{1}{2},1)$ and let $\overline{\eta}$ be as in Lemma 7.9. With abuse of notation we let $O_k^{[\overline{\eta}]} := (F_k^{[\overline{\eta}]})^c$, where $F_k^{[\overline{\eta}]}$ denotes the set of points of admissible $\overline{\eta}$-density of F_k. We claim that $O_k^{[\bar{\eta}]}$ is contained in $W + \mathscr{C}_{16}$ (see (4.8)).

To prove the claim, we fix $x \in O_k^{[\bar\eta]}$, and check that $x \in W + \mathscr{C}_2$. Indeed, as $Jg(x)$ does not vanish almost everywhere on D, we can find a set $D' \subset D$ of positive measure such that for almost all $(y,t) \in D'$

$$\chi_{B(y,t)}(x)g(y,t) = \chi_{B(y,t)}(x)f(y,t)\chi_{W\cap\{\|Jf\|_2>0\}}(y) \neq 0.$$

For those points, we have $y \in W$, $|x-y| < t$ and $t < m(y)$, so $t < 2m(x)$, using Lemma 1.5 *i*). Thus, $B(x,t)$ belongs to $\mathscr{B}_2$ and intersects W; thus, $x \in W + \mathscr{C}_2$.

As x is not a point of admissible $\overline{\eta}$-density of F_k, there is a ball $B \in \mathscr{B}_{\frac{3}{2}}$ with center x such that $\gamma_d(F_k \cap B) < \bar\eta\gamma_d(B)$. This is only possible if B intersects $O_k = F_k^c$. As O_k is contained in $W + \mathscr{C}_2$, this means that B intersects $W + \mathscr{C}_2$. Fix an arbitrary $x' \in B \cap (W + \mathscr{C}_2)$ and let $B' \in \mathscr{C}_2$ be any admissible ball centered at x' and intersecting W. From $x' \in B$ and $B \in \mathscr{B}_{\frac{3}{2}}$, it follows that $|x-x'| < \frac{3}{2}m(x)$. Also, because $B' \in \mathscr{B}_2$ and intersects W, $d(x',W) < 2m(x)$, it follows that $d(x,W) < \frac{3}{2}m(x) + 2m(x')$. Using Lemma 1.5 *ii*), we have $m(x') < 5m(x)$, and therefore $\operatorname{dist}(x,W) \leq 16m(x)$. This proves the claim.

For each $N \geq 1$ define $g_N(y,t) := \chi_{\{|y|\leq N\}}\chi_{\{|g|\leq N\}}\chi_{(\frac{1}{N},\infty)}(t)g(y,t)$. Clearly, $g_N \in T^{q,q}(\gamma_d)$ and, by dominated convergence, $\lim_{N\to\infty} g_N = g$ in $T^{1,q}(\gamma_d)$. Defining the sets $F_{k,N}$, $O_{k,N}$, $F_{k,N}^{[\overline{\eta}]}$, $O_{k,N}^{[\overline{\eta}]}$ in the same way as above, Lemma 7.9 gives that

$$\begin{aligned}\iint_{R_{1-\eta}(F_{k,N}^{[\overline{\eta}]})\cap D} &|g_N(y,t)|^q\gamma_d(dy)\,\frac{dt}{t}\\ &\leq C\int_{F_{k,N}}\left(\iint_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|g_N(y,t)|^q\gamma_d(dy)\,\frac{dt}{t}\right)\gamma_d(dx) \leq C\|g_N\|^q_{T^{q,q}(\gamma_d)}.\end{aligned}$$

As $k \to -\infty$, the middle term tends to 0; therefore, the support of fg_N is contained in the union $\bigcup_{k\in\mathbb{Z}} \mathsf{T}_{1-\eta}(O_{k,N}^{[\overline{\eta}]}) \cap D$. Clearly, $O_{k,N} \subseteq O_k$ implies $\mathsf{T}_{1-\eta}(O_{k,N}^{[\overline{\eta}]}) \subseteq \mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]})$; therefore, a limiting argument shows that the support of g is contained in the union $\bigcup_{k\in\mathbb{Z}} \mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]}) \cap D$.

Choose cubes $(Q_k^m)_{m\in\mathbb{N}}$ and functions $(\phi_k^m)_{m\in\mathbb{N}}$ as in Lemma 4.12, applied to the open sets $O_k^{[\overline{\eta}]}$ which are contained in $W + \mathscr{C}_8$. Define for $(y,t) \in D$,

$$\begin{aligned} b_k^m(y,t) &:= (\chi_{\mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]})}(y,t) - \chi_{\mathsf{T}_{1-\eta}(O_{k+1}^{[\overline{\eta}]})}(y,t))\phi_k^m(y)f(y,t),\\ \mu_k^m &:= \iint_D |b_k^m(y,t)|^q\,\gamma_d(dy)\,\frac{dt}{t}, \end{aligned}$$

and put

$$\lambda_k^m := (\gamma_d(Q_k^m))^{\frac{1}{q'}}(\mu_k^m)^{\frac{1}{q}}, \quad a_k^m(y,t) := \frac{b_k^m(y,t)}{\lambda_k^m}.$$

Then,

$$g = \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}} \lambda_k^m a_k^m.$$

Let C be a constant to be determined later and denote by $(Q_k^m)^{**}$ the cube that has the same center as Q_k^m, but side length multiplied by C. Let us further denote by δ_k^m the length of the diagonal of Q_k^m and by c_k^m its center. We now show that $\mathrm{supp}(a_k^m) \subseteq \mathsf{T}_1((Q_k^m)^{**})$. We have

$$\mathrm{supp}(a_k^m) \subseteq \mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]}) \cap \{(y,t) \in D:\ y \in (Q_k^m)^*\},$$

where $(Q_k^m)^*$ is as in Lemma 4.12. Therefore, for $(y,t) \in \mathrm{supp}(a_k^m)$, we have $d(y, F_k^{[\overline{\eta}]}) \geq (1-\eta)t$ and $y \in (Q_k^m)^*$. For $z \notin (Q_k^m)^{**}$ this gives

$$d(y,z) \geq d(z,c_k^m) - d(y,c_k^m) \geq \Big(\frac{C}{2\sqrt{n}} - \frac{\rho}{2}\Big)\delta_k^m, \tag{7.9}$$

where $\rho = \rho_{2^{10}\sqrt{d},d}$ is the constant from Lemma 4.12. Moreover, using property *ii*) in Lemma 4.12,

$$d(c_k^m, F_k^{[\overline{\eta}]}) \leq (\rho + \frac{1}{2})\delta_k^m.$$

For $u \in F_k^{[\overline{\eta}]}$ such that $d(c_k^m, u) \leq (\rho + \frac{1}{2})\delta_k^m + \varepsilon$, this gives

$$(1-\eta)t \leq d(y, F_k^{[\overline{\eta}]}) \leq d(y,u) \leq d(y,c_k^m) + d(c_k^m,u) \leq \frac{3\rho+1}{2}\delta_k^m + \varepsilon. \tag{7.10}$$

Upon taking $C = 2\sqrt{n}(\frac{\rho}{2} + \frac{3\rho+1}{2(1-\eta)})$, from (7.9) and (7.10) and letting $\varepsilon \downarrow 0$, we infer that

$$d(y,z) \geq \frac{3\rho+1}{2(1-\eta)}\delta_k^m \geq t.$$

This means that $(y,t) \in \mathsf{T}_1((Q_k^m)^{**})$, thus proving the claim: $\mathrm{supp}(a_k^m) \subseteq \mathsf{T}_1((Q_k^m)^{**})$.

Using the definitions of λ_k^m and a_k^m together with the doubling property for admissible balls, we also get that

$$\iint_D |a_k^m(y,t)|^q \gamma_d(dy)\frac{dt}{t} \leq \frac{1}{\gamma_d(Q_k^m)^{\frac{q}{q'}}} \leq C\frac{1}{\gamma_d((Q_k^m)^{**})^{\frac{q}{q'}}}.$$

Up to a multiplicative constant, the a_k^m are thus $T^{1,q}(\gamma_d)$ α-atoms for some $\alpha = \alpha(C,n) > 0$. To get the norm estimates, we first use Lemma 7.9. Noting that $(y,t) \in \mathsf{T}_1((Q_k^m)^{**})$ and $x \in B(y,t)$ imply $x \in (Q_k^m)^{**}$, we obtain

$$\begin{aligned}
\mu_k^m &\leq \iint_{R_{1-\eta}(F_{k+1}^{[\overline{\eta}]})\cap D} \chi_{\mathsf{T}_1((Q_k^m)^{**})}(y,t)|f(y,t)|^q \gamma_d(dy)\frac{dt}{t} \\
&\leq C\int_{F_{k+1}} \left(\iint_D \frac{\chi_{B(y,t)}(x)\chi_{\mathsf{T}_1((Q_k^m)^{**})}(y,t)}{\gamma(B(y,t))} |f(y,t)|^q \gamma_d(dy)\frac{dt}{t} \right) \gamma_d(dx) \\
&\leq C\int_{F_{k+1}\cap (Q_k^m)^{**}} \|Jf(x)\|^q_{L^q(D,\gamma_d\frac{dt}{t}))} \gamma_d(dx) \\
&\leq C2^{2(k+1)}\gamma_d((Q_k^m)^{**}) \leq C2^{2k}\gamma_d(Q_k^m).
\end{aligned}$$

This then gives

$$\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\lambda_k^m = \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\sqrt{\mu_k^m\gamma_d(Q_k^m)} \leq C\sum_{k\in\mathbb{Z}}2^k\gamma_d(O_k^{[\overline{\eta}]}).$$

Because $x\in O_k^{[\overline{\eta}]}$ implies $\mathscr{M}_\gamma^{\frac{3}{2}}(\chi_{O_k})(x) > 1-\overline{\eta}$ i.e.,

$$O_k^{[\overline{\eta}]} \subset \{x\in W:\ \mathscr{M}_\gamma^{\frac{3}{2}}(\chi_{O_k})(x) > 1-\overline{\eta}\},$$

the weak type $(1,1)$ of the truncated centered Gaussian Hardy–Littlewood maximal function $\mathscr{M}_\gamma^{\frac{3}{2}}$ defined by using only $\mathscr{B}_{\frac{3}{2}}$-balls (see (4.101), gives that

$$(1-\overline{\eta})\gamma(O_k^{[\overline{\eta}]}) \leq C\gamma(O_k)$$

and thus

$$(1-\overline{\eta})\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\lambda_k^m \lesssim \sum_{k\in\mathbb{Z}}2^k\gamma_d(O_k) \lesssim \int_0^\infty \gamma_d(x\in\mathbb{R}^d:\ \|Jf(x)\|_q > s)\,ds = \|f\|_{T^{1,q}(\gamma_d)}.$$

□

As an application of the atomic decomposition, we prove a result on change of aperture of the cones. The proof is different from the classical one (see [55]), because the result is derived directly from the atomic decomposition.

Definition 7.11. *For $\alpha > 0$, the Gaussian tent space $T_\alpha^{1,q}(\gamma_d)$ with aperture α is the completion of $C_0(D)$ with respect to the norm,*

$$\|f\|_{T_\alpha^{1,q}(\gamma_d)} = \|J_\alpha f\|_{L^1(\mathbb{R}^d,\gamma_d);L^q(D,\gamma_d\frac{dt}{t}))}, \tag{7.11}$$

where

$$(J_\alpha f(x))(y,t) := \frac{\chi_{B(y,\alpha t)}(x)}{\gamma_d(B(y,t))^{1/q}}f(y,t), \quad f\in C_0(D). \tag{7.12}$$

Theorem 7.12. *(Change of aperture) For all $1<\alpha_0<\alpha$, we have $T_\alpha^{1,q}(\gamma_d) = T_{\alpha_0}^{1,q}(\gamma_d)$ with equivalent norms.*

Proof. It is clear that $T_\alpha^{1,q}(\gamma_d)\subset T_{\alpha_0}^{1,q}(\gamma_d)$; thus, it suffices to show that $T_{\alpha_0}^{1,q}(\gamma_d)\subset T_\alpha^{1,q}(\gamma_d)$. To get that, it is enough to show that

$$J_\alpha \in L^1(\mathbb{R}^d,\gamma_d;L^q(D,\gamma_d\times\frac{dt}{t})),$$

whenever $f\in T_{\alpha_0}^{1,q}(\gamma_d)$. Observe that $(y,t)\in D$ implies $B(y,t)\in\mathscr{B}_1$; therefore, using the doubling property

$$\begin{aligned}
\|J_\alpha f\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(D,\gamma_d\times\frac{dt}{t}))} &= \int_{\mathbb{R}^d}\Big(\iint_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|f(y,t)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\
&= \int_{\mathbb{R}^d}\Big(\iint_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t/\alpha))}|f(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\
&\le \int_{\mathbb{R}^d}\Big(\iint_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|f(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\
&= \|J\tilde{f}\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))},
\end{aligned}$$

where $\tilde{D} := \{(x,t)\in\mathbb{R}^d\times(0,\infty) : (x,t/\alpha)\in D\}$, and $\tilde{f}(y,t) := f(y,t/\alpha)$. To prove the result, it is enough to show that

$$\|J\tilde{f}\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))} \le C\|J_{\alpha_0} f\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))}, \tag{7.13}$$

for $f\in T^{1,q}_\alpha(\gamma_d)$.

Suppose a is a $T^{1,q}_\alpha(\gamma_d)$ α_0-atom. Then, a is supported in $\mathsf{T}_1(B)\cap D$ for some ball $B = B(c,r)\in\mathscr{B}_{\alpha_0}$. Then $\tilde{a}(y,t) = a(y,y/\alpha)$ is supported in $\tilde{\mathsf{T}}_1(B)\cap\tilde{D}$ where $\tilde{\mathsf{T}}_1(B) := \{(y,t)\in\mathbb{R}^d\times(0,\infty) : (y,t/\alpha)\in\mathsf{T}_1(B)\}$. Using that $(y,t)\in\tilde{\mathsf{T}}_1(B)$ and $x\in B(y,t)$ imply $x\in B(c,\alpha r)$, the doubling property for admissible balls gives,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\
&\quad\le \int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\chi_{B(c,\alpha r)}(x)\gamma_d(dx)\\
&\quad\le \int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\chi_{B(c,\alpha r)}(x)\gamma_d(dx)\\
&\quad\le \Big(\int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(B(c,\alpha r))^{1/q'}\\
&\quad\le C\Big(\int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} |a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(B(c,r))^{1/q'}\\
&\quad\le C\Big(\int_{\mathbb{R}^d}\Big(\int\int_{D} |a(y,t)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(B(c,r))^{1/q'} \le C.
\end{aligned}$$

This shows that $J\tilde{a}\in L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))$. Then, using the atomic decomposition, Theorem 7.8, we can conclude that $J\tilde{f}\in L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))$, for all $f\in T^{1,q}_\alpha(\gamma_d)$. The estimate (7.13) then follows from the closed graph theorem. □

7.4 Gaussian Hardy Spaces $H^1(\gamma_d)$

The real variable theory of Hardy spaces originates from the work of C. Fefferman and E. Stein [79]. There are several equivalent definitions for the Hardy spaces on

$\mathbb{R}^d$, with respect to the Lebesgue measure. We are going to discuss briefly the most important ones, at least for us, here. First, there is the atomic Hardy space $H^1_{\mathrm{at}}(\mathbb{R}^d)$. Here, an atom is a complex-valued function a defined on $\mathbb{R}^d$, which is supported on a cube Q and is such that

$$\int_Q a(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \|a\|_\infty \leq \frac{1}{|Q|}.$$

The atomic space $H^1_{\mathrm{at}}(\mathbb{R}^d)$ is defined by

$$H^1_{\mathrm{at}}(\mathbb{R}^d) := \Big\{ \sum_j \lambda_j a_j \colon a_j \,\text{atoms}, \lambda_j \in \mathbb{C},\ \sum_j |\lambda_j| < \infty \Big\},$$

with norm

$$\|f\|_{H^1_{\mathrm{at}}(\mathbb{R}^d)} := \inf \Big\{ \sum_j |\lambda_j| \colon f = \sum_j \lambda_j a_j\ \sum_j |\lambda_j| < \infty \Big\}.$$

The other relevant characterizations of the classical Hardy space are given using the non-tangential maximal function $\mathscr{T}^*_{NT}$ of the heat semigroup

$$\mathscr{T}^*_{NT} f(x) := \sup_{(y,t)\in\Gamma_x} |\mathscr{T}_{t^2} f(y)|, \tag{7.14}$$

and the conical square function of the heat semigroup

$$\mathscr{S}_{NT} f(x) := \frac{1}{|B(y,t)|} \Big(\int_{\Gamma_x} \Big| t \mathscr{T}_{t^2} f(y) \Big|^2 \mathrm{d}y \frac{dt}{t} \Big)^{\frac{1}{2}}, \tag{7.15}$$

where $\Gamma_x := \big\{ (y,t) \in \mathbb{R}^d \times (0,\infty) \colon |y-x| < t \big\}$ are the usual cones in $\mathbb{R}^{d+1}$ with a vertex at $x \in \mathbb{R}^d$.

The Hardy spaces can then be defined as the completion of the space of compactly supported functions $C_0(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{H^1_{max}} := \|f\|_1 + \|\mathscr{T}^*_{NT} f\|_1,$$

or with respect to the norm

$$\|f\|_{H^1_{quad}} := \|f\|_1 + \|\mathscr{S}_{NT} f\|_1.$$

It can be proved that these norms are equivalent norms.

The Calderón–Zygmund operators are not bounded on $L^1(\mathbb{R}^d)$, but are bounded on weak-L^1, which is not a Banach space. Another characterization of $H^1(\mathbb{R}^d)$ is precisely the subspace of functions $f \in L^1(\mathbb{R}^d)$ such that their Riesz transforms $R_j f$ are also in $L^1(\mathbb{R}^d)$, i.e.,

$$H^1(\mathbb{R}^d) = \Big\{ f \in L^1(\mathbb{R}^d) : R_j f \in L^1(\mathbb{R}^d), j = 1, 2, \cdots, d \Big\}.$$

Finally, in 1971, it was proved by C. Fefferman in [78] (see also [79]), that the dual of $H^1(\mathbb{R}^d)$ is $BMO(\mathbb{R}^d)$, the space of functions with bounded mean oscillations introduced by F. John and L. Nirenberg in [144].

In recent years, the theory of Hardy spaces has been extended to a variety of new settings. These developments involve replacing the (Euclidean) Laplacian with a different semigroup generator L, and the space $\mathbb{R}^d$ endowed with the Borel σ-algebra and the Lebesgue measure with a different metric measure space (M,d,μ). Important references include S. Hofmann and S. Mayboroda's work on the Euclidean space, with the Laplacian replaced by a more general divergence form second-order elliptic differential operator with bounded measurable coefficients (see [136] and the Auscher–McIntosh–Russ Hardy spaces of differential forms associated with the Hodge Laplacian on a Riemannian manifold [13]. These results rely heavily on two assumptions: that the measure μ is a doubling measure (see Appendix), and that the semigroup generated by L, $\{e^{tL}\}$, has some appropriate L^2 off-diagonal decay: for $f \in L^2(\mathbb{R}^d)$, there exists a constant C independent of E,F,t and f such that

$$\left\|\chi_E e^{tL}(\chi_F f)\right\|_2 \leq c(1+\frac{d(E,F)}{t})^{-k}\|\chi_f f\|_2,$$

where E,F, are Borel sets in $\mathbb{R}^d$.

Given the success of Hardy space techniques in deterministic partial differential equations, we can expect that a Gaussian analog would similarly have applications to non-linear stochastic partial differential equations and stochastic boundary value problems.

There have been several attempts to define Gaussian Hardy spaces, but the main difficulty has been the fact that the Gaussian measure is not a doubling measure and the Ornstein–Uhlenbeck semigroup does not satisfy the kernel bounds required to apply the non-doubling theory of Tolsa [274]. The first result was obtained in 2007 by G. Mauceri and S. Meda in their seminal paper [174]. Their work is striking precisely because the Gaussian measure is not doubling, but the key to their success relies on the fact that they discovered that the Gaussian measure is a doubling measure when restricted to the class of admissible balls (see Proposition 1.6). The Mauceri–Meda Hardy spaces $H^1_{at}(\gamma_d)$ are defined via an atomic decomposition. An atom is either the constant function 1 or a function supported in an admissible ball $B \in \mathscr{B}_1$ with vanishing integral and satisfying an appropriate size condition. More precisely,

Definition 7.13. *Let $1 < r < \infty$, a $(1,r)$-atom is either the constant function 1, or a function a in $L^1(\gamma_d)$ supported in a ball $B \in \mathscr{B}_1$ with the following properties:*

$$\int_B a(y)\,\gamma_d(dy) = 0, \tag{7.16}$$

and

$$\Big(\frac{1}{\gamma_d(B)}\int_B |a(y)|^r\,\gamma_d(dy)\Big)^{1/r} \leq \frac{1}{\gamma_d(B)}, \tag{7.17}$$

or equivalently,

$$\|a\|_{r,\gamma} \leq \gamma_d(B)^{1/r-1}. \tag{7.18}$$

Then, we have

Definition 7.14. *The atomic Gaussian Hardy space $H^{1,r}_{at}(\gamma_d)$ is the space of all functions f in $L^1(\gamma_d)$ that admit an atomic decomposition of the form*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \tag{7.19}$$

where a_k is a $(1,r)$-atom and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$, with norm

$$\|f\|_{H^{1,r}_{at}(\gamma)} = \inf\left\{ \sum_{k=1}^{\infty} |\lambda_k| : f = \sum_{k=1}^{\infty} \lambda_k a_k,\ a_k\ (1,r)-\textit{atom and} \sum_{k=1}^{\infty} |\lambda_k| < \infty \right\}. \tag{7.20}$$

By duality with the $BMO(\gamma_d)$ spaces, it can be proved that all Gaussian Hardy spaces $H^{1,r}_{at}(\gamma_d)$ coincide for all $r \in (1,\infty)$ with equivalent norms. Moreover, in [177, Theorem 2.2], G. Mauceri, S. Meda, and P. Sjögren prove that this can be extended to the case $r = \infty$. Thus, we can denote any of them simply by $H^1_{at}(\gamma_d)$ and use any of the equivalent norms. Additionally, the Mauceri–Meda space $H^1_{at}(\gamma_d)$ provides a good endpoint to the L^p scale from the interpolation point of view.

J. Maas, J. van Neerven, and P. Portal in [168] and [169] developed an alternative approach to the theory of Hardy spaces for the Gaussian case. This involved considering adequate dyadic cubes, Whitney-type covering lemmas (which were discussed in Section 4.1), related tent spaces and their atomic decomposition (which were discussed in Section 7.3), and techniques to estimate non-tangential maximal functions and conical square functions (see Section 4.6).

In 2012, P. Portal in [231] gave another characterization of Gaussian Hardy spaces, introducing two new spaces:

Definition 7.15. *i) The (maximal) Gaussian Hardy space, or non-tangential maximal function Hardy space, $H^1_{max,a}(\gamma_d)$ is the completion of the L^2 range of L, $R(L)$,*[1] *with respect to the norm*

$$\|f\|_{H^1_{max,a}(\gamma)} := \|\mathscr{T}^*_\gamma(1,a)f\|_{1,\gamma}, \tag{7.21}$$

where $\mathscr{T}^*_\gamma(1,a)$ is the non-tangential maximal function associated with the Ornstein–Uhlenbeck semigroup (4.84).

[1] In [231], the spaces are defined as completions of $C_0^\infty(\mathbb{R}^d)$. This unfortunate mistake was pointed out in [232]. These spaces, just like other Hardy spaces associated with an operator L, can only be defined on the range of L (where the reproducing formula holds in a L^1 sense). In other situations, this is only a minor technical hindrance. For the Ornstein–Uhlenbeck operator, however, this is critical because of the change of spectrum at $p = 1$.

ii) The (quadratic) Gaussian Hardy space $H^1_{quad,a}(\gamma_d)$ is the completion of the L^2 range of L, R(L), with respect to the norm,

$$\|f\|_{H^1_{quad,a}(\gamma)} := \|f\|_{1,\gamma_d} + \|\mathscr{S}_{a,\gamma} f\|_{1,\gamma}, \tag{7.22}$$

where $\mathscr{S}_{a,\gamma}$ is the "averaged version" of the non-tangential Ornstein–Uhlenbeck maximal function, (4.91).

Then, we have the following crucial result:

Theorem 7.16. *Given $a > 0$, there exists $a' > 0$ such that the norms $\|\cdot\|_{H^1_{quad,a}(\gamma)}$ and $\|\cdot\|_{H^1_{max,a'}(\gamma)}$ are equivalent; therefore,*

$$H^1_{quad,a}(\gamma_d) = H^1_{max,a}(\gamma_d). \tag{7.23}$$

The proof of this result is technically very difficult and long. We give some of the main elements (for full details, see [231, Theorem 1.1]). The proof is based on the Gaussian version of A. P. Calderón's reproducing formula (2.59).

First of all, observe that from Theorem 7.12, we can immediately obtain one of the required inequalities, because

$$\|\mathscr{S}_{a,\gamma} f\|_{1,\gamma} \leq C \|\mathscr{T}^*_\gamma(1,a') f\|_{1,\gamma},$$

for some $C, a' > 0$.[2]

Therefore, to prove Theorem 7.16, we need to prove the reverse inequality.

The (local) part

$$J_1 f(x) := \int_0^{m(x)} (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} f(x) \, \frac{dt}{t}, \tag{7.24}$$

is treated, via atomic decomposition of the tent space $T^{1,q}(\gamma_d)$, leading to the estimate,

$$\|J_1 f\|_{H^1_{max,a'}(\gamma)} \leq C' (\|f\|_{1,\gamma} + \|f\|_{H^1_{quad,a}(\gamma)}). \tag{7.25}$$

The (global) term,

$$J_\infty f(x) := \int_{m(x)}^{\infty} (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} f(x) \, \frac{dt}{t}, \tag{7.26}$$

is very problematic, as the boundedness of the square function norm $\|\mathscr{S}_{a,\gamma}\|_{1,\gamma_d}$ does not give any information about it. Nevertheless, estimates of the Ornstein–Uhlenbeck semigroup give the estimate,

$$\|J_1 f\|_{H^1_{max,a'}(\gamma)} \leq C'' \|f\|_{1,\gamma}. \tag{7.27}$$

[2] Actually, Theorem 4.43 gives a slightly stronger inequality involving $\Upsilon^*_\gamma(1,a')$, the "average" non-tangential maximal function.

Let us look at the main argument of the proof in more detail. Using the Gaussian version of A. P. Calderón's reproducing formula (2.59)

$$f(x) = \int_{\mathbb{R}^d} f(x)\gamma_d(dx) + C\int_0^\infty (t^2L)^{N+1} T_{(1+a)t^2/\alpha} f(x)\,\frac{dt}{t},$$

for $f \in L^2(\gamma_d)$, in L^2-sense, and the atomic decomposition, we can prove the following corollary of Theorem 7.8, for $q = 2$. This corollary is the actual underlying identity for proving Theorem 7.16.

Corollary 7.17. *For all $N \in \mathbb{N}, a > 1, b \geq \frac{1}{2}$ and $\alpha > a^2$ there exists $C_1, C_2, C_3, C_4 > 0$, and d sequences of α-atoms $\{A_{n,j}\}_{n\geq 1}$ and numbers $(\lambda_{n,j})_{n\geq 1} \in \ell^1$, such that for all $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} f(x) = \int_{\mathbb{R}^d} f(y)\gamma_d(dy) \; &- C_1 \sum_{j=1}^d \sum_{n=1}^\infty \lambda_{n,j} \int_0^2 (t^2L)^N T_{t^2/\alpha}\left(t(\partial_\gamma^j)^* A_{n,j}(x,t)\right)\,\frac{dt}{t} \\ &+ C_2 \sum_{j=1}^d \sum_{n=1}^\infty \lambda_{n,j} \int_0^2 \chi_{\left[\frac{m(x)}{b},2\right]}(t)(t^2L)^N T_{t^2/\alpha}\left(t(\partial_\gamma^j)^* A_{n,j}(x,t)\right)\frac{dt}{t} \\ &- C_3 \sum_{j=1}^d \int_0^{\frac{m(x)}{b}} (t^2L)^N T_{t^2/\alpha}\left(\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha} f(x)\right)\,\frac{dt}{t} \\ &+ C_4 \int_{\frac{m(x)}{b}}^\infty (t^2L)^{N+1} T_{(1+a^2)t^2/\alpha} f(x)\,\frac{dt}{t}, \end{aligned} \tag{7.28}$$

and

$$\sum_{j=1}^d \sum_{n=1}^\infty |\lambda_{n,j}| \leq C\|f\|_{H^1_{quad,a}(\gamma)},$$

where $(\partial_\gamma^j)^* = \sqrt{2}x_j I_d - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_j}$, the formal $L^2(\gamma_d)$-adjoint of ∂_γ^j, see (2.12).

Proof. Let us recall that $L = -\sum_{j=1}^d (\partial_\gamma^j)^*\partial_\gamma^j$, see (2.13). Hence, as L and $T_t, t \geq 0$ commute,

$$\begin{aligned} (t^2L)^{N+1} T_{(1+a^2)t^2/\alpha} f(x) &= -\sum_{j=1}^d (t^2L)^N t^2 (\partial_\gamma^j)^* \partial_\gamma^j T_{t^2/\alpha} T_{a^2t^2/\alpha} f(x) \\ &= -\sum_{j=1}^d (t^2L)^N T_{t^2/\alpha} t(\partial_\gamma^j)^* [\chi_D(x,t) + \chi_{D^c}(x,t)] t\partial_\gamma^j T_{a^2t^2/\alpha} f(x). \end{aligned}$$

Set $F_j(x,t) := \chi_D(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha} f(x)$, for $j = 1,\cdots,d$. We need to check that $F_j \in T^{1,2}(\gamma_d)$, i.e., that they have an atomic decomposition.

Using Theorem 1.6 *ii*) we have, taking the change of variables $t = \sqrt{\alpha}s$

$$\begin{aligned}
\|F_j\|_{T^{1,2}_\alpha(\gamma_d)} &\leq C\int_{\mathbb{R}^d}\Big(\iint_{\Gamma^1_x(\gamma_d)}\frac{\chi_D(y,t)}{\gamma_d(B(y,t))}|t\partial^j_\gamma T_{a^2t^2/\alpha}f(y)|^2\gamma_d(dy)\frac{dt}{t}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,t)}\frac{\chi_D(y,t)}{\gamma_d(B(y,t))}|t\partial^j_\gamma T_{a^2t^2/\alpha}f(y)|^2\gamma_d(dy)\frac{du}{u}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,\sqrt{\alpha}s)}\frac{\chi_D(y,\sqrt{\alpha}s)}{\gamma_d(B(y,\sqrt{\alpha}s))}|s\partial^j_\gamma T_{a^2s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,\sqrt{\alpha}s)}\frac{\chi_D(y,\sqrt{\alpha}s)}{\gamma_d(B(y,\sqrt{\alpha}s))}|s\nabla T_{a^2s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,\sqrt{\alpha}s)}\frac{\chi_D(y,\sqrt{\alpha}s)}{\gamma_d(B(y,s))}|s\nabla T_{a^2s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx),
\end{aligned}$$

as $\gamma_d(B(y,\sqrt{\alpha}s)) \geq \gamma_d(B(y,s))$. Then, by the change of aperture formula, Theorem 7.12, and the change of variables $at = s$, we get

$$\begin{aligned}
\|F_j\|_{T^{1,2}_\alpha(\gamma_d)} &\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,a^2t)}\frac{\chi_D(y,a^2t)}{\gamma_d(B(y,t))}|t\nabla T_{a^2t^2}f(y)|^2\gamma_d(dy)\frac{dt}{t}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,as)}\frac{\chi_D(y,as)}{\gamma_d(B(y,t))}|s\nabla T_{s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx) \leq \|f\|_{H^1_{quad,a}}.
\end{aligned}$$

Then, using Theorem 7.8, we conclude that

$$F_j(x,t) = \sum_{n=1}^\infty \lambda_{n,j}A_{n,j}(x,t),$$

with $\sum_{n=1}^\infty |\lambda_{n,j}| < \infty$, for $j = 1,\cdots,d$. Hence, using the Gaussian version of A. P. Calderón's reproducing formula

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}^d} f(y)\gamma_d(dy) + C\int_0^\infty (t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&= \int_{\mathbb{R}^d} f(y)\gamma_d(dy) + C\int_{\frac{m(x)}{b}}^\infty (t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&\quad - C\sum_{j=1}^d\int_0^{\frac{m(x)}{b}} d(t^2L)^N T_{t^2/\alpha}t(\partial^j_\gamma)^*[\chi_D(x,t)+\chi_{D^c}(x,t)]t\partial^j_\gamma T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
&= \int_{\mathbb{R}^d} f(y)\gamma_d(dy) + C\int_{\frac{m(x)}{b}}^\infty (t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&\quad - C\sum_{j=1}^d\int_0^{\frac{m(x)}{b}} (t^2L)^N T_{t^2/\alpha}t(\partial^j_\gamma)^*[\chi_D(x,t)+\chi_{D^c}(x,t)]t\partial^j_\gamma T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
&= \int_{\mathbb{R}^d} f(y)\gamma_d(dy) + C\int_{\frac{m(x)}{b}}^\infty (t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}
\end{aligned}$$

$$
\begin{aligned}
&-C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*\chi_D(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
&-C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
=&\int_{\mathbb{R}^d}f(y)\gamma_d(dy)+C\int_{\frac{m(x)}{b}}^{\infty}(t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&-C\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}\\
&-C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}f(x)\frac{dt}{t}
\end{aligned}
$$

It is easy to check that the interchange of the (Bochner) integral with the sum is allowed. Finally, using that $m(x)/b\le 2$, we get

$$
\begin{aligned}
&\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}\\
&\quad=\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{2}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}\\
&\quad\quad-\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{2}\chi_{\left[\frac{m(x)}{b},2\right]}(t)(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}.
\end{aligned}
$$

This gives (7.28). Thus, we have shown that $\|F_j\|_{T_\alpha^{1,2}(\gamma_d)}\le C\|f\|_{H^1_{quad,a}(\gamma)}$, so

$$
\sum_{j=1}^{d}\sum_{n=1}^{\infty}|\lambda_{n,j}|\le C\|f\|_{H^1_{quad,a}(\gamma)}. \qquad \square
$$

The proof of Theorem 7.16 uses (7.28) obtained in Corollary 7.17.

For $a>0$, Theorem 7.12 gives that there exists $a'>0$ such that $H^1_{max,a'}(\gamma_d)\subset H^1_{quad,a}(\gamma_d)$. Let us fix a' and pick

$$
\alpha>\max\left\{2^{38},32e^4,4\sqrt{a}e^{2a^2}\right\},\ b\ge\max\left\{2e,\sqrt{\frac{32e^4}{(\alpha-32e^4)(1-e^{-2a^2/\alpha})}}\right\},
$$

and $N>d/4$. Let $f\in C_c^\infty(\mathbb{R}^d)$ and apply Corollary 7.17. We have

$$\begin{aligned}\|f\|_{H^1_{max,a'}(\gamma)} \leq C = &\left\| \mathscr{T}_\gamma^*(1,a)\left(\int_{\mathbb{R}^d} f(y)\gamma_d(dy)\right)\right\|_{1,\gamma} \\ &+ C\sum_{j=1}^{d}\sum_{n=1}^{\infty}|\lambda_{n,j}|\left\|\int_0^2 (t^2L)^N T_{t^2/\alpha}\left(t(\partial_\gamma^j)^* A_{n,j}(\cdot,t)\right)\,\frac{dt}{t}\right\|_{H^1_{max,a'}} \\ &+ C\sum_{j=1}^{d}\sum_{n=1}^{\infty}|\lambda_{n,j}|\left\|\int_0^2 \chi_{\left[\frac{m(\cdot)}{b},2\right]}(t)(t^2L)^N T_{t^2/\alpha}\left(t(\partial_\gamma^j)^* A_{n,j}(\cdot,t)\right)\,\frac{dt}{t}\right\|_{H^1_{max,a'}} \\ &+ C\sum_{j=1}^{d}\left\|\int_0^{\frac{m(\cdot)}{b}} (t^2L)^N T_{t^2/\alpha}\left(\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}\, f(\cdot)\right)\,\frac{dt}{t}\right\|_{H^1_{max,a'}} \\ &+ C\left\|\int_{\frac{m(\cdot)}{b}}^{\infty} (t^2L)^{N+1} T_{(1+a^2)t^2/\alpha} f(\cdot)\,\frac{dt}{t})\right\|_{H^1_{max,a'}} + \|f\|_{1,\gamma}.\end{aligned}$$

As the Ornstein–Uhlenbeck semigroup is conservative, i.e., $T_t 1 =, t \geq 0$ then

$$\left\| \mathscr{T}_\gamma^*(1,a)\left(\int_{\mathbb{R}^d} f(y)\gamma_d(dy)\right)\right\|_{1,\gamma} \leq \|f\|_{1,\gamma} \leq \|f\|_{H^1_{quad,a'}(\gamma)}.$$

To bound the rest of the terms above, several estimates of Mehler's kernel (off-diagonal estimates) are needed, in addition to the introduction of the notion of molecules (see Sections 3 and 4 of [231]). Once that is done, we can then bound the remaining terms. Using [231, Proposition 5.5], we get

$$\left\|\int_{\frac{m(\cdot)}{b}}^{\infty} (t^2L)^{N+1} T_{(1+a^2)t^2/\alpha} f(\cdot)\,\frac{dt}{t})\right\|_{H^1_{max,a'}(\gamma)} \leq C \leq \|f\|_{1,\gamma} \leq C\|f\|_{H^1_{quad,a'}(\gamma)}.$$

Now, for $j = 1,\cdots,d$, using [231, Proposition 5.4], we obtain

$$\left\|\int_0^{\frac{m(\cdot)}{b}} (t^2L)^N T_{t^2/\alpha}\left(\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}\, f(\cdot)\right)\,\frac{dt}{t}\right\|_{H^1_{max,a'}(\gamma)} \leq C \leq \|f\|_{1,\gamma} \leq C\|f\|_{H^1_{quad,a'}(\gamma)}.$$

Applying [231, Proposition 5.3] gives that

$$\left\|\int_0^2 \chi_{\left[\frac{m(\cdot)}{b},2\right]}(t)(t^2L)^N T_{t^2/\alpha}\left(t(\partial_\gamma^j)^* A_{n,j}(\cdot,t)\right)\,\frac{dt}{t}\right\|_{H^1_{max,a'}(\gamma)} \leq C,$$

whereas Proposition 4.2 combined with Theorem 4.3 of [231] gives, for $j = 1,\cdots,d$,

$$\left\| \mathscr{T}_\gamma^*(1,a)\left(\int_{\mathbb{R}^d} f(y)\gamma_d(dy)\right)\right\|_{1,\gamma} \leq C.$$

Therefore,

$$\|f\|_{H^1_{max,a'}(\gamma)} \leq C\|f\|_{H^1_{quad,a'}(\gamma)} + C\sum_{j=1}^{d}\sum_{n=1}^{\infty}|\lambda_{n,j}| \leq C\|f\|_{H^1_{quad,a'}(\gamma)}. \qquad \square$$

G. Mauceri and S. Meda proved that the topological dual of $H^{1,r}_{at}(\gamma_d)$ is isomorphic to $BMO(\gamma_d)$. They also proved that the imaginary power of the Ornstein–Uhlenbeck operator, $(-L)^{i\alpha}$ and the adjoint of the Riesz transforms $\mathscr{R}_j^*$ are bounded from $H^{1,r}_{at}(\gamma_d)$ to $L^1(\gamma_d)$. Unfortunately, it was proved by G. Mauceri, S. Meda, and P. Sjögren in [176, Theorem 3.1] that the Riesz transforms $\mathscr{R}_j$ are not bounded from $H^1_{at}(\gamma_d)$ to $L^1(\gamma_d)$ in a dimension greater than one. On the other hand, P. Portal proved, in [231, Theorem 6.1], that the Riesz transforms $\mathscr{R}_j$ are bounded from $H^1_{max}(\gamma_d)$ to $L^1(\gamma_d)$, but it is not known if the imaginary powers of $(-L)$ are bounded there. Also, nothing is known about duality and interpolation for $H^1_{max}(\gamma_d)$. Thus, these spaces are different.

As we have seen, Portal's proof is based on the theory of Gaussian tent spaces $T^{1,2}_{\alpha}(\gamma_d)$. Although these tent spaces are defined using an atomic decomposition, and the equivalence of $H^1_{max}(\gamma_d)$ and $H^1_{quad}(\gamma_d)$ uses the atomic decomposition of $T^{1,2}_{\alpha}(\gamma_d)$ via the Gaussian version of Calderón's reproducing formula, their explicit characterization is not provided in [231]. In [37], T. Bruno introduces a new atomic Gaussian Hardy space $X^1(\gamma_d)$, which is strictly contained in the space $H^1_{at}(\gamma_d)$.

First, we need the following notation,

Definition 7.18. *Let E be a bounded open set and K be a compact set in $\mathbb{R}^d$.*

i) We denote by $q^2(E)$ the space of all functions $f \in L^2(E)$ such that Lf is constant on E, and by $q^2(K)$ the space of all functions on K, which are restriction to K of a function in $q^2(E')$ for some bounded open set, such that $K \subset E'$.
ii) We denote by $h^2(E)$ the space of all functions $f \in L^2(E)$ such that $Lf = 0$ on E, and by $h^2(K)$ the space of all functions on K that are restriction to K of a function in $h^2(E')$ for some bounded open set, such that $K \subset E'$.

The spaces $h^2(E)^{\perp}$ and $q^2(E)^{\perp}$ are the orthogonal complements of $h^2(E)$ and $q^2(E)$ in $L^2(E,\gamma_d)$ respectively. The spaces $h^2(K)^{\perp}$ and $q^2(K)^{\perp}$ are the orthogonal complements of $h^2(K)$ and $q^2(K)$ in $L^2(K,\gamma_d)$ respectively.

Now, following G. Mauceri, S. Meda, and P. Vallarino in [178], we defined the atomic Gaussian Hardy space $X^1(\gamma_d)$.

Definition 7.19. *An X^1-atom is a function $a \in L^2(\gamma_d)$, supported in a ball $B \in \mathscr{B}_1$, with the following properties:*

i) $a \in q^2(\overline{B})^{\perp}$.
ii) $\|a\|_{2,\gamma} \leq \gamma_d(B)^{1/2}$.

Definition 7.20. *The atomic Gaussian Hardy space $X^1(\gamma_d)$ is the space of all functions f in $L^1(\gamma_d)$ that admit an atomic decomposition of the form*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \tag{7.29}$$

where a_k is a X^1-atom and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$, with norm

$$\|f\|_{X^1(\gamma)} = \inf\Big\{ \sum_{k=1}^{\infty} |\lambda_k| : f = \sum_{k=1}^{\infty} \lambda_k a_k,\ a_k \ X^1\text{-atom and } \sum_{k=1}^{\infty} |\lambda_k| < \infty \Big\}. \tag{7.30}$$

If $B \in \mathscr{B}_1$, the functions in $q^2(\overline{B})$ are referred to as *Gaussian quasi-harmonic functions* in B.

Observe that the space $X^1(\gamma_d)$ is strictly contained in the atomic Gaussian space $H^1_{at}(\gamma_d)$ of Mauceri and Meda. Indeed, the atoms defining $H^1_{at}(\gamma_d)$ are supported on admissible balls of $\mathscr{B}_1$, but have only zero integral, a much weaker condition than being in $q^2(\overline{B})^{\perp}$. The great advantage of the space $X^1(\gamma_d)$ is that T. Bruno proved that the Riesz transforms are bounded from $X^1(\gamma_d)$ to $L^1(\gamma_d)$. However, the understanding of the space $X^1(\gamma_d)$ is far from complete; for instance, it seems that $X^1(\gamma_d)$ is also a subspace of $H^1_{max}(\gamma_d)$.

7.5 Gaussian $BMO(\gamma_d)$ Spaces

In 1961, F. John and L. Nirenberg [144] introduced the space of functions of bounded mean oscillations (BMO) with respect to the Lebesgue measure, as the space of all locally integrable functions on $\mathbb{R}^d$ such that

$$\sup_{Q \in \mathscr{Q}} \frac{1}{|Q|} \int_{\mathbb{R}^d} |f(y) - f_Q| dy < \infty, \tag{7.31}$$

where $\mathscr{Q}$ is the family of all open cubes in $\mathbb{R}^d$ with sides parallel to the coordinate axes, and $f_Q = \frac{1}{|Q|} \int_{\mathbb{R}^d} |f(y)| dy$, the average of f over Q with respect to the Lebesgue measure. It is easy to see that by replacing the family $\mathscr{Q}$ with the family of balls $\mathscr{B}$ in the formula above, we obtain an equivalent norm on BMO.

Extensions of the space of functions of bounded mean oscillations have been considered in the literature. In particular, a theory of functions of bounded mean oscillations that parallels the Euclidean theory has been developed on spaces of homogeneous type by R. Coifman and G. Weiss [56] (see also [170]). As mentioned before, $(\mathbb{R}^d, |\cdot|, \gamma_d)$ is not a space of homogeneous type and the theory of BMO spaces developed in [56] and [170] does not apply to this setting.

More recently, spaces of functions of bounded mean oscillations have been introduced on measured metric spaces not of homogeneous type, specifically on

$(\mathbb{R}^d, |\cdot|, \mu)$, where μ is a (possibly non-doubling) non-negative Radon measure. In particular, X. Tolsa [274] has defined a regular BMO space, $RBMO(\mu)$, whenever μ is a non-negative Radon measure on $\mathbb{R}^d$, which is n-dimensional, i.e., there exists a constant $C > 0$ such that for any ball $B(x,r) \subset \mathbb{R}^d$

$$\mu(B(x,r)) \leq Cr^n,$$

for some $n \in [1,d]$. Tolsa's space enjoys many good properties of BMO of spaces of homogeneous type. In particular, Calderón–Zygmund singular integrals are bounded from $L^\infty(\mu)$ to $RBMO(\mu)$.

As mentioned before, γ_d is trivially a d-dimensional measure. However, $RBMO(\gamma_d)$ is not the appropriate space to study the boundedness on $L^\infty(\gamma_d)$ of Gaussian singular integrals, because the kernel of these operators does not satisfy the standard estimates uniformly in the whole complement of the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$. As we discuss in detail in Chapter 9, the local part of Gaussian singular integrals satisfies the usual estimates of a Calderón–Zygmund operator. In 2007, G. Mauceri and S. Meda in [174] also introduced *Gaussian BMO spaces, $BMO(\gamma_d)$*, as follows:

Definition 7.21. *The Gaussian space of functions of bounded mean oscillations $BMO(\gamma_d)$, is the space of functions $f \in L^1(\gamma_d)$ that satisfy*

$$\sup_{B\in\mathscr{B}_1} \frac{1}{\gamma_d(B)} \int_B |f(x) - f_B^\gamma| \gamma(dx) < \infty, \tag{7.32}$$

where

$$f_B^\gamma = \frac{1}{\gamma_d(B)} \int_B f(x)\gamma_d(dx),$$

the average of f over B. We define

$$\|f\|_*^{\mathscr{B}_1} = \sup_{B\in\mathscr{B}_1} \frac{1}{\gamma_d(B)} \int_B |f(x) - f_B^\gamma| \gamma(dx), \tag{7.33}$$

and the norm in $BMO(\gamma_d)$ is then defined as

$$\|f\|_{BMO(\gamma)} = \|f\|_{1,\gamma} + \|f\|_*^{\mathscr{B}_1}.$$

Observe that by definition $BMO(\gamma_d) \subset L^1(\gamma_d)$. Moreover, it can be proved that $BMO(\gamma_d)$ is a Banach space, and also that if we replace the family $\mathscr{B}_1$ with any other family $\mathscr{B}_a$ in the definition of $BMO(\gamma_d)$, we obtain the same space with an equivalent norm (see [174, Proposition 2.4]),[3]

[3] Also, we obtain the same space with an equivalent norm if instead of $\mathscr{B}_a$, we consider $\mathscr{Q}_a$ the *admissible cubes of parameter a*, i.e., the cubes Q with sides parallel to the axes , with a center at c_q and a side length $l_q \leq am(cQ)$.

We define the (local) sharp function $f^\sharp$ as follows:

Definition 7.22. *Given $f \in L^1(\gamma_d)$, the (local) sharp function $f^\sharp$ is defined as*

$$f^\sharp(x) = \sup_{B\in\mathscr{B}_1, x\in B} \frac{1}{\gamma_d(B)} \int_B |f(y) - f_B^\gamma| \gamma_d(dy). \tag{7.34}$$

Clearly, $f \in BMO(\gamma_d)$ if and only if $f^\sharp \in L^\infty(\gamma_d)$, and $\|f\|_*^{\mathscr{B}_1} = \|f^\sharp\|_{\infty,\gamma}$. Moreover, it is straightforward to prove that $f^\sharp \leq 2\mathscr{M}_\gamma^a 1 f(x)$, for any $x \in \mathbb{R}^d$.

Additionally, G. Mauceri and S. Meda in [174] prove that an inequality of John–Nirenberg type for admissible balls holds for functions in $BMO(\gamma_d)$ (see [174, Proposition 4.1]) and that the topological dual of $H^1_{at}(\gamma_d)$ is isomorphic to $BMO(\gamma_d)$. The proof of this result is modeled over the classical result of Fefferman, although there are several additional difficulties to overcome to adapt the original proof to the Gaussian setting (see [174, Theorem 5.2]).

7.6 Gaussian Lipschitz Spaces $Lip_\alpha(\gamma)$

The standard Euclidean Lipschitz space $Lip_\alpha(\mathbb{R}^n)$ consists of all bounded functions f such that for some $C > 0$

$$|f(y) - f(x)| \leq C|x-y|^\alpha, \quad x, y \in \mathbb{R}^n. \tag{7.35}$$

This characterization is based on the regularity of the functions. It is known that the space $Lip_\alpha(\mathbb{R}^n)$ can also be characterized by convolution with the standard Poisson kernel,

$$q_t(x) = c_n \frac{t}{(t^2 + |x-y|^2)^{(d+1)/2}},$$

see E. Stein [252, Section V. 4. 2], as $f \in Lip_\alpha(\mathbb{R}^n)$ if and only if

$$\left\| \frac{\partial \mathscr{P}_t}{\partial t}(x,y) f \right\|_{L^\infty} \leq C t^{\alpha-1}, \tag{7.36}$$

for all $t > 0$.

We would like to define Lipschitz spaces associated with the Gaussian measure. Observe that, as mentioned above, the spaces $L^p(\gamma_d)$ are not closed under the action of the classical translation operator; thus, it would not be a good idea to try to define them following the classical definition (7.35). Therefore, we use the Poisson–Hermite semigroup to define Gaussian Lipschitz spaces.

In what follows, we need the technical result about the L^1-norm of the derivatives discussed in Lemma 3.16. From there, we then get the following key result,

Proposition 7.23. *Suppose $f \in L^\infty(\gamma)$ and $\alpha > 0$. Let k and l be two integers both greater than α. The two conditions*

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{\infty,\gamma} \leq A_{\alpha,k} t^{-k+\alpha} \tag{7.37}$$

and

$$\left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{\infty,\gamma} \leq A_{\alpha,l} t^{-l+\alpha}, \tag{7.38}$$

are equivalent. Moreover, the smallest $A_{\alpha,k}$ and $A_{\alpha,l}$ holding in the above inequalities, are comparable.

Proof. It suffices to prove that if $k > \alpha$,

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{\infty,\gamma} \leq A_{\alpha,k} t^{-k+\alpha} \tag{7.39}$$

and

$$\left\| \frac{\partial^{k+1} P_t f}{\partial t^{k+1}} \right\|_{\infty,\gamma} \leq A_{\alpha,k+1} t^{-(k+1)+\alpha}, \tag{7.40}$$

are equivalent.

Let us assume (7.39). Applying the semigroup property, if $t = t_1 + t_2$, $P_t f = P_{t_1}(P_{t_2} f)$, then using the hypothesis and Lemma 3.3,

$$\begin{aligned} \left\| \frac{\partial^{k+1} P_t f}{\partial t^{k+1}} \right\|_{\infty,\gamma} = \left\| \frac{\partial P_{t_1}}{\partial t_1} \Big(\frac{\partial^k P_{t_2} f}{\partial t_2^k} \Big) \right\|_{\infty,\gamma} &\leq \left\| \frac{\partial^k P_{t_2} f}{\partial t_2^k} \right\|_{\infty,\gamma} \int_{\mathbb{R}^d} \left| \frac{\partial p(t_1, \cdot, y)}{\partial t_1} \right| dy \\ &\leq A_{\alpha,k} t_2^{-k+\alpha} C t_1^{-1}. \end{aligned}$$

For $t_1 = t_2 = t/2$ we get (7.40).

Now, assume (7.40). Observe that, again by Lemma 3.3,

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{\infty,\gamma} \leq \|f\|_\infty \int_{\mathbb{R}^d} \left| \frac{\partial^k p(t,x,y)}{\partial t^k} \right| dy \leq \frac{C}{t^k} \|f\|_\infty;$$

thus, $\frac{\partial^k P_t f}{\partial t^k} \to 0$ as $t \to \infty$, and then using hypothesis

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{\infty,\gamma} \leq \int_t^\infty \left\| \frac{\partial^{k+1} P_s f}{\partial s^{k+1}} \right\|_{\infty,\gamma} ds \leq A_{\alpha,k+1} \frac{t^{-k+\alpha}}{-k+\alpha} = C t^{-k+\alpha}. \qquad \square$$

Now, we can define the Gaussian Lipschitz spaces as follows:

Definition 7.24. *For $\alpha > 0$ let n be the smallest integer greater than α. The Gaussian Lipschitz space $Lip_\alpha(\gamma)$ is defined as the set of L^∞ functions for which there exists a constant A such that*

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_\infty \le A t^{-n+\alpha}. \tag{7.41}$$

The norm of $f \in Lip_\alpha(\gamma)$ is defined as

$$\|f\|_{Lip_\alpha(\gamma)} := \|f\|_{\infty,\gamma} + A_\alpha(f), \tag{7.42}$$

where $A_\alpha(f)$ is the smallest constant A appearing in (7.41).

Observations 7.25. *For the Gaussian Lipschitz spaces, we have*

i) The definition of $Lip_\alpha(\gamma)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent, according to Proposition 7.23.

ii) Condition (7.41) is of interest for t near zero, because the inequality

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_\infty \le A t^{-n}, \tag{7.43}$$

which is stronger away from zero, follows for $f \in L^\infty$ immediately from (3.17),

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{\infty,\gamma} \le \int_{\mathbb{R}^d} \left| \frac{\partial^n p(t,x,y)}{\partial t^n} \right| |f(y)| dy \le \frac{C}{t^n} \|f\|_\infty.$$

iii) For the completeness of the Gaussian Lipschitz spaces see Lemma 7.35.

We also define, for $\alpha > 0$, homogeneous Gaussian Besov spaces $\dot{B}^{\alpha}_{\infty,\infty}(\gamma)$ as follows:

Definition 7.26. *For $\alpha > 0$, let n be the smallest integer greater than α, then the homogeneous Gaussian Besov space type $\dot{B}^{\alpha}_{\infty,\infty}(\gamma)$ is defined as the set of $L^1(\gamma)$ functions such that (7.41) holds for a constant $B_{\alpha,n}$.*

All these spaces can also be obtained using abstract interpolation theory using the Poisson–Hermite semigroup (see [271] 1.6.5.)

Observe that $Lip_\alpha(\gamma) \subset \dot{B}^{\alpha}_{\infty,\infty}(\gamma)$. There are also inclusion relations among the Gaussian Lipschitz spaces,

Proposition 7.27. *If $0 < \alpha_1 < \alpha_2$, then we have the inclusion*

$$Lip_{\alpha_2}(\gamma) \subset Lip_{\alpha_1}(\gamma).$$

Proof. Take $f \in Lip_{\alpha_2}(\gamma)$ and $n \ge \alpha_2$, then

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{\infty,\gamma} \le A_\alpha(f) t^{-n+\alpha_2}.$$

If $0 < t < 1$, then $t^{-n+\alpha_2} \le t^{-n+\alpha_1}$; therefore,

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{\infty,\gamma} \le A_\alpha(f) t^{-n+\alpha_1}.$$

Now, if $t \geq 1$, then we know from (7.43) that

$$\left\|\frac{\partial^n P_t f}{\partial t^n}\right\|_{\infty,\gamma} \leq A_\alpha(f) t^{-n}$$

and as $t^{-n+\alpha_1} > t^{-n}$, we also get in this case

$$\left\|\frac{\partial^n P_t f}{\partial t^n}\right\|_{\infty,\gamma} \leq A_\alpha(f) t^{-n+\alpha_1}$$

because $n > \alpha_1$; then, $f \in Lip_{\alpha_1}(\gamma)$. □

Proposition 7.28. *If $f \in Lip_\alpha(\gamma)$ with $0 < \alpha < 1$, then*

$$||P_t f - f||_{\infty,\gamma} \leq A_\alpha(f) t^\alpha. \tag{7.44}$$

Proof. Applying the fundamental theorem of calculus,

$$\begin{aligned} ||P_t f - f||_{\infty,\gamma} &= \left\| \int_0^t \frac{\partial P_s f}{\partial s} ds \right\|_\infty \leq \int_0^t \left\| \frac{\partial P_s f}{\partial s} \right\|_{\infty,\gamma} ds \\ &\leq A_\alpha(f) \int_0^t s^{-1+\alpha} ds = A_\alpha(f) t^\alpha. \qquad \square \end{aligned}$$

Gaussian Lipschitz spaces were defined by A. E. Gatto and W. Urbina in [109] following E. Stein's approach in [252, Chapter V], using the Poisson–Hermite semigroup. After the given definition of those spaces in this way, it is natural to ask if there is a characterization based on the regularity of the functions involved, as in the classical case. In [159], L. Liu and P. Sjögren have characterized these spaces, for $0 < \alpha < 1$, in terms of a combination of ordinary Lipschitz continuity conditions, giving a positive answer to the question posed. The main result of Liu and Sjögren's paper is the following:

Theorem 7.29. *Let $\alpha \in (0,1)$, an essentially bounded function $f \in Lip_\alpha(\gamma)$ if and only if there exists a constant K such that for all $x, y \in \mathbb{R}^n$,*

$$|f(y) - f(x)| \leq K \min\left\{ |x-y|^\alpha, \left(\frac{|x-y|}{1+|x|+|y|}\right)^{\alpha/2} + ((|x|+|y|)\sin\theta)^\alpha \right\}, \tag{7.45}$$

after a correction of f on a null set. Here, θ denotes the angle between the vectors x and y; if $x = 0$ or $y = 0$, then θ is understood to be 0.

In one dimension, the inequality becomes,

$$|f(y) - f(x)| \leq K \min\left\{ |x-y|^\alpha, \left(\frac{|x-y|}{1+|x|+|y|}\right)^{\alpha/2} \right\}. \tag{7.46}$$

This is a combined Lipschitz condition, with exponent α for a short distance $|x-y|$ and exponent $\alpha/2$ with a different coefficient, for a long distance.

As usual in Gaussian harmonic analysis, the two parts of this estimate correspond to the "local part" (for short distance $|x-y|$), in which the estimate coincides with the Euclidean case, and the "global part" corresponding to the long distance (i.e., $|x-y|$ big), in which the effect of the Gaussian measure makes the estimate a little different.

In higher dimensions, the expression $(|x|+|y|)\sin\theta$ describes the "orthogonal component" of the vector $x-y$, because it is the distance from x to the line in the direction x. To make this clearer, Liu and Sjögren state a non-symmetric inequality equivalent to (7.45). For $x,y \in \mathbb{R}^n$ with $x \neq 0$, we decompose y as $y = y_x + y'_x$, where y_x is parallel to x and y'_x orthogonal to x,

$$|f(y)-f(x)| \leq K' \min\left\{|x-y|^{\alpha}, \left(\frac{|x-y_x|}{1+|x|}\right)^{\alpha/2} + |y'_x|\right\}. \tag{7.47}$$

This inequality means that the combined Lipschitz condition applies in the radial direction, but in the orthogonal direction, the exponent is always α. The equivalence between these two inequalities is valid in any dimension, with a constant $K' > 0$ comparable with K.

The proof of (7.45) relies on very precise pointwise estimates of the Poisson–Hermite kernel $p(t,x,y)$ and its derivatives; for all $t > 0$ and $x,y \in \mathbb{R}^n$,

$$p(t,x,y) \leq C[K_1(t,x,y)+K_2(t,x,y)+K_3(t,x,y)+K_4(t,x,y)], \tag{7.48}$$

where,

$$K_1(t,x,y) = \frac{t}{(t^2+|x-y|^2)^{(n+1)/2}} \exp(-C_1 t(1+|x|)),$$

for some constant C_1,

$$K_2(t,x,y) = \frac{t}{|x|}\left(t^2 + \frac{|x-y_x|}{|x|} + |y'_x|^2\right)^{-(n+2)/2} \times \exp\left(-C_2 \frac{(t^2+|y'_x|^2)|x|}{|x-y_x|}\right)\chi_{\{|x|>1, x\cdot y>0, |x|/2 \leq |y_x| < |x|\}};$$

for some constant C_2,

$$K_3(t,x,y) = \min(1,t)\exp(-C_3|y|^2);$$

for some constant C_3, and

$$K_4(t,x,y) = \frac{t}{|y_x|}\left(\log\frac{|x|}{|y_x|}\right)^{-3/2} \exp\left(-C_4 \frac{t^2}{\log\frac{|x|}{|y_x|}}\right)\exp(-C_5|y'_x|^2)\chi_{\{x\cdot y>0, 1<|y_x|<|x|/2\}};$$

for some constant C_4.

Similar estimates are also possible for the derivatives of $p(t,x,y)$, both $\partial_t p(t,x,y)$ and $\partial_{x_i} p(t,x,y)$. Thus,

$$p(t,x,y) + |t\partial_t p(t,x,y)| + |t\partial_{x_i} p(t,x,y)| \\ \leq C[K_1(t,x,y) + K_2(t,x,y) + K_3(t,x,y) + K_4(t,x,y)]. \quad (7.49)$$

Moreover, Liu and Sjögren prove that these estimates are also sharp. For each of the four kernels $K_i(t,x,y)$ there is a set $\tilde{E}_i$ of points (t,x,y) in which $p(t,x,y)$ is equivalent to $K_i(t,x,y)$, but where the other terms are much smaller; thus, none of the four terms can be suppressed in the estimate. The estimates are product of a very deep understanding of the kernel $p(t,x,y)$ and how it compares with the standard Poisson kernel $q_t(x)$ (for more details, we refer the reader to their paper [159]).

The estimates of the Poisson–Hermite kernel $p(t,x,y)$ and its derivatives are of independent interest, and the proof of the main result is almost straightforward once we have those estimates. It would be interesting to know if alternative characterization of the Gaussian Besov–Lipschitz and the Gaussian Triebel–Lizorkin spaces, which are defined in the next two sections, using higher order derivatives of the Poisson–Hermite kernel, can be obtained using similar estimates.

Another open question would be if the characterization of the Gaussian Lipschitz spaces obtained by Liu and Sjögren is related to the notion of translation operator introduced by C. Markett in [173].

In the Euclidean case, as mentioned above, condition (7.36) characterizes the ordinary Lipchitz space only if the functions considered are bounded. Thus, we obtain the *inhomogeneous Lipschitz space*; without the boundedness assumption, we get the larger *homogeneous Lipschitz space.*

In the Gaussian setting, as no homogeneity is involved, the condition (7.41) without the boundedness assumption defines a space that had been considered by L. Liu and P. Sjögren in [160]. It is called the *global Gaussian Lipschitz space*. Using a result by G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, and B. Viviani [106], Liu and Sjögren consider measurable functions f in $\mathbb{R}^d$ with the condition

$$\int_{\mathbb{R}^d} \frac{e^{-|y|^2}}{\sqrt{\ln(e+|y|)}} |f(y)| dy < \infty, \quad (7.50)$$

which according to Theorem 1.1 of [106] guarantees that the $P_t f$ is well defined. Moreover, the same condition ensures that $P_t f(x) \to f(x)$ as $t \to 0$ a.e. $x \in \mathbb{R}^n$. Therefore,

Definition 7.30. *Let $\alpha \in (0,1)$. A measurable function f defined in $\mathbb{R}^n$ and satisfying (7.50) belongs to the global Gaussian Lipschitz space $GLip_\alpha(\gamma)$ if (7.41) holds. The corresponding norm is*

$$\|f\|_{GLip_\alpha(\gamma)} = \inf\{A > 0 : A \textit{ satisfies } (7.41)\}.$$

This space is actually a space of equivalence classes, as it consists of functions modulo constants.

A natural question is what continuity condition characterizes these spaces? To answer this, Liu and Sjögren introduce the following distance:

$$d(x,y) = \left| \int_x^y \frac{d\xi}{1+|\xi|} \right| = |\ln(1+|x|) - \operatorname{sgn} xy \ln(1+|y|)|, \quad x,y \in \mathbb{R}, \tag{7.51}$$

with the convention $\operatorname{sgn} 0 = 1$. In several dimensions, we use this distance on the line spanned by x, defining

$$d(x,y) = |\ln(1+|x|) - \operatorname{sgn} < x,y > \ln(1+|y_x|)|, \quad x,y \in \mathbb{R}^n,$$

with y_x as before. The main result in [160] is the following:

Theorem 7.31. *Let $\alpha \in (0,1)$ and let f be a measurable function in $\mathbb{R}^n$. The following conditions are equivalent:*

i) f satisfies condition (7.50) and $f \in GLip_\alpha(\gamma)$.
ii) There exists a positive constant K such that for all $x,y \in \mathbb{R}^n$

$$|f(y) - f(x)| \leq K \min\left\{ |x-y|^\alpha, d(x,y_x)^{\alpha/2} + |y_x'|^\alpha \right\}, \quad x,y \in \mathbb{R}^n \tag{7.52}$$

after a correction of f on a null set.

Moreover, the space $GLip_\alpha(\gamma)$ is defined in terms of the distance function d. Indeed, (7.47) implies boundedness, then (7.47) holds if and only if there exists a constant $K" > 0$ such that,

$$|f(y) - f(x)| \leq K \min\left\{ 1, |x-y|^\alpha, d(x,y_x)^{\alpha/2} + |y_x'|^\alpha \right\},$$

for $x,y \in \mathbb{R}^n$. This also tells us that for bounded functions (7.47) and (7.52) are equivalent.

The condition (7.52) implies only

$$f(x) = O(\ln|x|)^{\alpha/2} \quad \text{as } |x| \to \infty.$$

Liu and Sjögren show that this condition is sharp using a counterexample in Section 7.5.

To obtain (7.52), they need to modify the kernel K_3 to decay for large values of x, refining a few of the previous arguments. The estimates (7.48) and (7.49) remain valid if the kernel $K_3(t,x,y)$ is replaced by

$$\widetilde{K}_3(t,x,y) = \min\left\{ 1, \frac{t}{[\ln(e+|x|)]^{1/2}} \right\} \exp(-C_3|y|^2)$$

The introduction in (7.51) of the distance d in the context of Gaussian harmonic analysis is an interesting point that may be used in other problems.

After several technical results, analogous estimates can obtained for $f \in GLip_\alpha(\gamma)$ with norm 1:

- For all $i = 1,2,\cdots n, t > 0$, and $x \in \mathbb{R}^n$,

$$|\partial_{x_i} P_t f(x)| \leq C t^{\alpha-1}.$$

- For all $t > 0$ and $x = (x_1, 0, \cdots, 0) \in \mathbb{R}^n$ with $x_1 \geq 0$,

$$|\partial_{x_i} P_t f(x)| \leq C t^{\alpha-2}(1 + x_1)^{-1}.$$

The proof of the main result, Theorem 1.2, follows almost immediately from all the previous estimates.

7.7 Gaussian Besov–Lipschitz Spaces $B^{\alpha}_{p,q}(\gamma_d)$

In the next two sections, we study the Gaussian Besov–Lipschitz and the Gaussian Triebel–Lizorkin spaces. They were introduced initially by E. Pineda in his doctoral dissertation (see [224] and also [226]).

For any $\alpha \geq 0$, we define Gaussian Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$, following E. Stein [252] to define and study the $B^{\alpha}_{p,q}(\gamma_d)$ spaces, using the Poisson–Hermite semigroup. But because the Poisson–Hermite semigroup is not a convolution semigroup, the proofs of the results are totally different to those given there.

As in the case of Gaussian Lipschitz spaces, Besov–Lipschitz spaces can also be obtained as interpolated spaces using interpolation theory for semigroups defined on a Banach space (see for instance Chapter 3 of [38, 112] or [271]).

We use the representation of the Poisson–Hermite semigroup (3.8) in a crucial way, using the one-sided stable measure

$$\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds = g(t,s) ds,$$

and the estimates (3.19), (3.20) and (3.21).

In Chapter 3, we have obtained an estimate of the $L^p(\gamma_d)$-norms of the derivatives of the Poisson–Hermite semigroup (see Lemma 3.5); additionally, we have

Lemma 7.32. *Given $f \in L^p(\gamma_d), \alpha \geq 0$ and k, l integers greater than α, then*

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma} \leq A_k t^{-k+\alpha} \text{ if and only if } \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{p,\gamma} \leq A_l t^{-l+\alpha}.$$

Moreover, if $A_k(f), A_l(f)$ are the smallest constants appearing in the above inequalities, then there exist constants $A_{k,l,\alpha}$ and $D_{k,l,\alpha}$ such that

$$A_{k,l,\alpha} A_k(f) \leq A_l(f) \leq C D_{k,l,\alpha} A_k(f),$$

for all $f \in L^p(\gamma_d)$.

Proof. Let us suppose, without loss of generality, that $k \geq l$. We prove the direct implication first. For this, we use again the representation of the Poisson–Hermite semigroup (3.8),

$$P_t f(x) = \int_0^{+\infty} T_s f(x) \mu_t^{(1/2)}(ds).$$

Then, differentiating k-times with respect to t,

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds).$$

Using the identity (3.19), it is easy to prove that for all $m \in \mathbb{N}$

$$\lim_{t \to +\infty} \frac{\partial^m P_t f(x)}{\partial t^m} = 0;$$

therefore, given $n \in \mathbb{N}, n > \alpha$

$$\frac{\partial^n P_t f(x)}{\partial t^n} = -\int_t^{+\infty} \frac{\partial^{n+1} P_s f(x)}{\partial s^{n+1}} ds$$

Thus,

$$\begin{aligned} \left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{p,\gamma} &\leq \int_t^{+\infty} \left\| \frac{\partial^{n+1} P_s f}{\partial s^{n+1}} \right\|_{p,\gamma} ds \leq \int_t^{+\infty} A_{n+1}(f) s^{-(n+1)+\alpha} ds \\ &= \frac{A_{n+1}(f)}{n-\alpha} t^{-n+\alpha}. \end{aligned}$$

Then,

$$A_n(f) \leq \frac{A_{n+1}(f)}{n-\alpha},$$

and as $n > \alpha$ is arbitrary, by using the above result $k-l$ times, we get

$$\begin{aligned} A_l(f) &\leq \frac{A_{l+1}(f)}{l-\alpha} \leq \frac{A_{l+2}}{(l-\alpha)(l+1-\alpha)} \leq \cdots \leq \frac{A_k(f)}{(l-\alpha)(l+1-\alpha)\ldots(k-1-\alpha)} \\ &= D_{k,l,\alpha} A_k(f). \end{aligned}$$

To prove the converse implication, using again the representation of the Poisson–Hermite semigroup (3.8),

$$u(x, t_1 + t_2) = P_{t_1}(P_{t_2} f)(x) = \int_0^{+\infty} T_s(P_{t_2} f)(x) \mu_{t_1}^{(1/2)}(ds).$$

Therefore, taking $t = t_1 + t_2$ and differentiating l times with respect to t_2 and $k-l$ times with respect to t_1, we get

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s\Big(\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big) \frac{\partial^{k-l}}{\partial t_1^{k-l}} \mu_{t_1}^{(1/2)}(ds). \tag{7.53}$$

Thus, using the inequality (3.21) and the fact that the Ornstein–Uhlenbeck semigroup is a contraction semigroup, we get

$$\begin{aligned}\Big\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\Big\|_{p,\gamma} &\leq \int_0^{+\infty}\Big\|T_s\Big(\frac{\partial^l P_{t_2} f}{\partial t_2^l}\Big)\Big\|_{p,\gamma}\Big|\frac{\partial^{k-l}\mu_{t_1}^{(1/2)}}{\partial t_1^{k-l}}(ds)\Big|\\ &\leq \Big\|\frac{\partial^l P_{t_2} f}{\partial t_2^l}\Big\|_{p,\gamma}\int_0^{+\infty}\Big|\frac{\partial^{k-l}}{\partial t_1^{k-l}}\mu_{t_1}^{(1/2)}(ds)\Big| \leq C_{k-l}\Big\|\frac{\partial^l}{\partial t_2^l}P_{t_2} f\Big\|_{p,\gamma} t_1^{l-k}\\ &\leq C_{k-l}A_l(f)t_2^{-l+\alpha}t_1^{l-k}.\end{aligned}$$

Therefore, taking $t_1 = t_2 = \frac{t}{2}$,

$$\Big\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\Big\|_{p,\gamma} \leq C_{k-l}A_l(f)(\frac{t}{2})^{-k+\alpha},$$

and then,

$$A_k(f) \leq \frac{C_{k-l}}{2^{-k+\alpha}}A_l(f).$$

□

The following technical result is crucial for defining Gaussian Besov–Lipschitz spaces:

Lemma 7.33. *Given $\alpha \geq 0$ and k,l integers greater than α. Then,*

$$\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} < \infty$$

if and only if

$$\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha}, D_{k,l,\alpha}$ such that

$$\begin{aligned}D_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} &\leq \Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq A_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\end{aligned}$$

Proof. Let us suppose, without loss of generality, that $k \geq l$. We prove the converse implication first; from Lemma 7.32, we have

$$\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma} \leq C_{k-l}\Big\|\frac{\partial^l P_{\frac{t}{2}} f}{\partial(\frac{t}{2})^l}\Big\|_{p,\gamma}(\frac{t}{2})^{l-k}.$$

Thus,

$$\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} \leq \frac{C_{k-l}}{2^{l-k}}\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_{t/2} f}{\partial(\frac{t}{2})^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= A_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(s^{l-\alpha}\Big\|\frac{\partial^l P_s f}{\partial s^l}\Big\|_{p,\gamma}\Big)^q\frac{ds}{s}\Big)^{\frac{1}{q}}$$

with $A_{k,l,\alpha} = C_{k-l}2^{k-\alpha}$.

For the direct implication, given $n \in \mathbb{N}$, $n > \alpha$, using the previous lemma again, we get

$$\Big\|\frac{\partial^n P_t f}{\partial t^n}\Big\|_{p,\gamma} \leq \int_t^{+\infty}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma} ds$$

Therefore, using Hardy's inequality (10.101),

$$\Big(\int_0^{+\infty}\Big(t^{n-\alpha}\Big\|\frac{\partial^n P_t f}{\partial t^n}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \Big(\int_0^{+\infty}\Big(t^{n-\alpha}\int_t^{+\infty}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma} ds\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \Big(\int_0^{+\infty}\Big(\int_t^{+\infty}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma} ds\Big)^q t^{(n-\alpha)q-1} dt\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{n-\alpha}\Big(\int_0^{+\infty}\Big(s^{n+1-\alpha}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma}\Big)^q\frac{ds}{s}\Big)^{\frac{1}{q}}.$$

Now, as $n > \alpha$ is arbitrary, using the above result $k-l$, times

$$\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{l-\alpha}\Big(\int_0^{+\infty}\Big(t^{l+1-\alpha}\Big\|\frac{\partial^{l+1} P_t f}{\partial t^{l+1}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{(l-\alpha).(l+1-\alpha)}\Big(\int_0^{+\infty}\Big(t^{l+2-\alpha}\Big\|\frac{\partial^{l+2} P_t f}{\partial t^{l+2}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\cdots$$
$$\leq D_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$

where $D_{k,l,\alpha} = \dfrac{1}{(l-\alpha).(l+1-\alpha)\dots(k-1-\alpha)}$. □

Following the classical case, we are going to define the Gaussian Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$ or Besov–Lipschitz spaces for Hermite polynomial expansions.

Definition 7.34. *Let $\alpha \geq 0$, k be the smallest integer greater than α, and $1 \leq p,q \leq \infty$. For $1 \leq q < \infty$ the Gaussian Besov–Lipschitz space $B^{\alpha}_{p,q}(\gamma_d)$ is defined as the set of functions $f \in L^p(\gamma_d)$, for which*

$$\left(\int_0^{\infty} (t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma})^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{7.54}$$

The norm of $f \in B^{\alpha}_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{B^{\alpha}_{p,q}} := \|f\|_{p,\gamma} + \left(\int_0^{\infty} (t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma})^q \frac{dt}{t} \right)^{1/q} \tag{7.55}$$

For $q = \infty$, the Gaussian Besov–Lipschitz space $B^{\alpha}_{p,\infty}(\gamma_d)$ is defined as the set of functions $f \in L^p(\gamma_d)$ for which exists a constant A, such that

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma} \leq A t^{-k+\alpha}$$

and then the norm of $f \in B^{\alpha}_{p,\infty}(\gamma_d)$ is defined as

$$\|f\|_{B^{\alpha}_{p,\infty}} := \|f\|_{p,\gamma} + A_k(f), \tag{7.56}$$

where $A_k(f)$ is the smallest constant A appearing in the above inequality.

In particular, the space $B^{\alpha}_{\infty,\infty}(\gamma_d)$ is the Gaussian Lipschitz space $Lip_{\alpha}(\gamma_d)$.

Lemma 7.33 shows us that we could have replaced k with any other integer l greater than α and that the resulting norms are equivalent. Let us prove now that the Gaussian Besov–Lipschitz spaces are complete.

Lemma 7.35. *For any $\alpha \geq 0$, $1 \leq p,\ q \leq \infty$, the Gaussian Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$ are Banach spaces.*

Proof. To prove the completeness, it is enough to see that if $\{f_n\}$ is a sequence in $B^{\alpha}_{p,q}(\gamma_d)$, such that $\sum_{n=1}^{\infty} \|f_n\|_{B^{\alpha}_{p,q}} < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges in $B^{\alpha}_{p,q}(\gamma_d)$. Because

$$\sum_{n=1}^{\infty} \|f_n\|_{B^{\alpha}_{p,q}} = \sum_{n=1}^{\infty} \left(\|f_n\|_{p,\gamma} + \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t f_n \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right) < \infty.$$

In particular, this implies that

$$\sum_{n=1}^{\infty} \|f_n\|_{p,\gamma} < \infty, \text{ and } \sum_{n=1}^{\infty} \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t f_n \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

But as $L^p(\gamma_d)$ is complete, there exists a function $f \in L^p(\gamma_d)$, such that

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{a.e.} x.$$

We need to prove that $\sum_{n=1}^{\infty} f_n = f$ in $B^{\alpha}_{p,q}$, i.e. $\lim_{n\to\infty} \|\sum_{i=1}^{n} f_i - f\|_{B^{\alpha}_{p,q}} = 0$.

Given $t > 0$ and $x \in \mathbb{R}^d$, by linearity, $P_t(\sum_{i=1}^{n} f_i(x)) = \int_{\mathbb{R}^d} p(t,x,y) \sum_{i=1}^{n} f_i(y)dy$ and then

$$\lim_{n\to\infty} p(t,x,y) \sum_{i=1}^{n} f_i(y) = p(t,x,y) \sum_{i=1}^{\infty} f_i(y) = p(t,x,y) f(y) \quad \text{a.e.}\, y$$

and for all $n \in \mathbb{N}$

$$|p(t,x,y) \sum_{i=1}^{n} f_i(y)| \leq p(t,x,y) \sum_{i=1}^{n} |f_i(y)| \leq p(t,x,y) g(y) \quad \text{a.e.}$$

As $\int_{\mathbb{R}^d} p(t,x,y) g(y) dy = P_t g(x) < \infty$, i.e., $p(t,x,y)g(y)$ is integrable, we conclude using Lebesgue's dominated convergence theorem, for any $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} P_t(\sum_{i=1}^{n} f_i(x)) = \lim_{n\to\infty} \int_{\mathbb{R}^d} p(t,x,y) \sum_{i=1}^{n} f_i(y) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy = P_t f(x).$$

Similarly, we have, $\lim_{n\to\infty} T_t(\sum_{i=1}^{n} f_i(x)) = T_t f(x)$, for any $t \geq 0$ and $x \in \mathbb{R}^d$, and again using Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t(f_i(x)) &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t(\sum_{i=1}^{n} f_i(x)) = \lim_{n\to\infty} \int_0^{\infty} T_s(\sum_{i=1}^{n} f_i(x)) \frac{\partial^k}{\partial t^k} \mu_t^{1/2}(ds) \\
&= \int_0^{\infty} \lim_{n\to\infty} T_s(\sum_{i=1}^{n} f_i(x)) \frac{\partial^k}{\partial t^k} \mu_t^{1/2}(ds) \\
&= \int_0^{\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{1/2}(ds) = \frac{\partial^k}{\partial t^k} P_t f(x),
\end{aligned}$$

for any $t \geq 0$ and $x \in \mathbb{R}^d$. Then, for $t > 0$, using Fatou's lemma,

$$\begin{aligned}
\Big\| \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma}^p &= \int_{\mathbb{R}^d} \Big| \frac{\partial^k}{\partial t^k} P_t f \Big|^p \gamma_d(dx) \\
&= \int_{\mathbb{R}^d} \Big| \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) \Big|^p \gamma_d(dx) \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} \Big| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) \Big|^p \gamma_d(dx).
\end{aligned}$$

Thus, for any $t > 0$, by triangle inequality,

$$\Big\| \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma} \leq \liminf_{n\to\infty} \Big\| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} \leq \liminf_{n\to\infty} \sum_{i=1}^{n} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma},$$

and again, by triangle inequality,

$$\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \Big(\int_0^{+\infty}\Big(t^{k-\alpha}\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \liminf_{n\to\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \sum_{n=1}^{\infty}\Big(\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_n\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big) < \infty.$$

Then, $f \in B^{\alpha}_{p,q}$.

Let, for each $t > 0$,

$$h(t) = t^{k-\alpha}\Big(\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big) + \Big\|\frac{\partial^k}{\partial^k}P_t f\Big\|_{p,\gamma}\Big).$$

Then,

$$\int_0^{+\infty}|h(t)|^q\frac{dt}{t}$$
$$\leq \int_0^{+\infty}\Big(t^{k-\alpha}\Big(\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big) + \Big\|\frac{\partial^k}{\partial t^k}P_t f\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}$$
$$\leq \int_0^{+\infty}\Big(t^{k-\alpha}\Big(2\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)\Big)^q\frac{dt}{t}$$
$$\leq 2\liminf_{n\to\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big);$$

hence,

$$\Big(\int_0^{+\infty}|h(t)|^q\frac{dt}{t}\Big)^{1/q} \leq 2\liminf_{n\to\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq 2\liminf_{n\to\infty}\sum_{i=1}^{n}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$= 2\sum_{n=1}^{\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_n\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} < \infty.$$

Thus, $h \in L^q((0,\infty),\frac{dt}{t})$; therefore,

$$h(t) = t^{k-\alpha}\Big(\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big) + \Big\|\frac{\partial^k}{\partial t^k}P_t f\Big\|_{p,\gamma}\Big) < \infty \quad \text{a.e. } t$$

and this immediately implies

$$\sum_{n=1}^{\infty} \Big\| \frac{\partial^k}{\partial t^k} P_t f_n \Big\|_{p,\gamma} + \Big\| \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma} < \infty \quad \text{a.e.}\, t. \tag{7.57}$$

Let $t > 0$ such that $h(t) < \infty$, we know that for all $x \in \mathbb{R}^d$

$$\lim_{n\to\infty} \Big(\sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big) = \lim_{n\to\infty} \frac{\partial^k}{\partial t^k} P_t \Big(\sum_{i=1}^{n} f_i(x) - f(x) \Big) = 0,$$

Set, for each $x \in \mathbb{R}^d$,

$$H(x) := 2 \sum_{n=1}^{\infty} \Big| \frac{\partial^k}{\partial t^k} P_t f_n(x) \Big|.$$

Then, from the above $H \in L^p(\gamma_d)$ and, therefore, as for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}^d$,

$$\Big| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big| \leq 2 \sum_{i=1}^{\infty} \Big| \frac{\partial^k}{\partial t^k} P_t f_i(x) \Big| = H(x).$$

Then, using Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \Big\| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i - \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma} & \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \Big| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big|^p \gamma_d(dx) \\ &= \int_{\mathbb{R}^d} \lim_{n\to\infty} \Big| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big|^p \gamma_d(dx) = 0, \end{aligned}$$

and as $h(t) < \infty \quad a.e.t$, we conclude,

$$\lim_{n\to\infty} \Big\| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i - \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma} = 0, \quad \text{a.e.}\, t.$$

Now, for each $n \in \mathbb{N}$,

$$\begin{aligned} \Big\| \frac{\partial^k}{\partial t^k} P_t \Big(\sum_{i=1}^{n} f_i - f \Big) \Big\|_{p,\gamma} &\leq \sum_{i=1}^{\infty} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} + \liminf_{n\to\infty} \Big(\sum_{i=1}^{n} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} \Big) \\ &= 2 \liminf_{n\to\infty} \Big(\sum_{i=1}^{n} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} \Big). \end{aligned}$$

For each $t > 0$, let $G(t) = \liminf_{n\to\infty} \Big(2t^{k-\alpha} \sum_{i=1}^{n} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} \Big)$. Then, using Fatou's lemma and triangle inequality,

$$\Big(\int_0^\infty |G(t)|^q \frac{dt}{t}\Big)^{1/q} \leq 2\liminf_{n\to\infty}\Big(\int_0^\infty \Big(t^{k-\alpha}\sum_{i=1}^n \Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$\leq 2\sum_{n=1}^\infty \Big(\int_0^\infty \Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_n\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} < \infty.$$

Thus, $G \in L^q((0,\infty),\frac{dt}{t})$, so $\liminf_{n\to\infty}\Big(t^{k-\alpha}\sum_{i=1}^n \Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)^q \frac{1}{t}$ is integrable, and therefore, using Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\Big(\int_0^\infty \Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^n f_i - f\Big)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$= \Big(\int_0^\infty \lim_{n\to\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^n f_i - f\Big)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} = 0.$$

Hence,

$$\lim_{n\to\infty}\Big\|\sum_{i=1}^n f_i - f\Big\|_{B^{\alpha}_{p,q}}$$
$$= \lim_{n\to\infty}\Big(\Big\|\sum_{i=1}^n f_i - f\Big\|_{p,\gamma} + \lim_{n\to\infty}\Big(\int_0^\infty \Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^n f_i - f\Big)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} = 0.$$

□

Finally, we study some inclusions among the Gaussian Besov–Lipschitz spaces:

Proposition 7.36. *The inclusion $B^{\alpha_1}_{p,q_1}(\gamma_d) \subset B^{\alpha_2}_{p,q_2}(\gamma_d)$ holds if either:*

i) $\alpha_1 > \alpha_2 > 0$ (q_1 and q_2 need not be related), or
ii) If $\alpha_1 = \alpha_2$ and $q_1 \leq q_2$.

Proof. To prove $ii)$, we set $A = \Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{1}{q_1}}$
Now, fixing $t_0 > 0$

$$\int_{\frac{t_0}{2}}^{t_0}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_1}\frac{dt}{t} \leq A^{q_1}.$$

Using Lemma 3.5, $\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}$ takes its minimum value at the upper end point ($t = t_0$) of the above integral; thus, we get

$$\Big\|\frac{\partial^k P_{t_0} f}{\partial t^k}\Big\|_{p,\gamma}^{q_1}\int_{\frac{t_0}{2}}^{t_0} t^{(k-\alpha)q_1}\frac{dt}{t} \leq A^{q_1}.$$

That is $\left\|\frac{\partial^k P_{t_0} f}{\partial t^k}\right\|_{p,\gamma} \leq CAt_0^{-k+\alpha}$, but because t_0 is arbitrary, then

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} \leq CAt^{-k+\alpha},$$

for all $t > 0$. In other words, $f \in B^{\alpha}_{p,q_1}$ also implies that $f \in B^{\alpha}_{p,\infty}$. Thus, as $q_2 \geq q_1$

$$\begin{aligned}
\int_0^{+\infty} &\left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_2}\frac{dt}{t} \\
&\leq \int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_2-q_1}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_1}\frac{dt}{t} \\
&\leq (CA)^{q_2-q_1}\int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_1}\frac{dt}{t} \\
&= (CA)^{q_2-q_1}A^{q_1} = CA^{q_2} < +\infty;
\end{aligned}$$

therefore $f \in B^{\alpha}_{p,q_2}$.

Now, to prove part i), using Lemma 3.5, we have

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} \leq Ct^{-k}, t > 0.$$

Then, given $f \in B^{\alpha_1}_{p,q_1}$, taking again

$$A = \left(\int_0^{+\infty}\left(t^{k-\alpha_1}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_1}\frac{dt}{t}\right)^{\frac{1}{q_1}},$$

we get, as in part ii),

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} \leq CAt^{-k+\alpha_1},$$

for all $t > 0$. Thus,

$$\begin{aligned}
\int_0^{+\infty}\left(t^{k-\alpha_2}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_2}\frac{dt}{t} &= \int_0^{1}\left(t^{k-\alpha_2}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_2}\frac{dt}{t} \\
&\quad + \int_1^{+\infty}\left(t^{k-\alpha_2}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}\right)^{q_2}\frac{dt}{t} \\
&= (I) + (II).
\end{aligned}$$

Now,

$$\begin{aligned}
(I) &= \int_0^1 t^{(k-\alpha_2)q_2}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}^{q_2}\frac{dt}{t} \leq \int_0^1 t^{(k-\alpha_2)q_2}(CA)^{q_2}t^{(\alpha_1-k)q_2}\frac{dt}{t} \\
&= (CA)^{q_2}\int_0^1 t^{(\alpha_1-\alpha_2)q_2}\frac{dt}{t} = CA^{q_2},
\end{aligned}$$

and

$$(II) = \int_1^{+\infty} t^{(k-\alpha_2)q_2} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma}^{q_2} \frac{dt}{t} \leq \int_1^{+\infty} t^{(k-\alpha_2)q_2} C^{q_2} t^{-kq_2} \frac{dt}{t}$$
$$= C^{q_2} \int_1^{+\infty} t^{-\alpha_2 q_2} \frac{dt}{t} = C.$$

Hence,

$$\int_0^{+\infty} \Big(t^{k-\alpha_2} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \Big)^{q_2} \frac{dt}{t} < +\infty;$$

thus, $f \in B^{\alpha_2}_{p,q_2}$. □

7.8 Gaussian Triebel–Lizorkin Spaces $F^{\alpha}_{p,q}(\gamma_d)$

Finally, we define Gaussian Triebel–Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$ for any $\alpha \geq 0$. The following technical result is key for their definition:

Lemma 7.37. *Let $\alpha \geq 0$ and k,l integers such that $k \geq l > \alpha$. Then*

$$\Big\| \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big| \frac{\partial^k}{\partial t^k} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma} < \infty$$

if and only if

$$\Big\| \Big(\int_0^{+\infty} \Big(t^{l-\alpha} \Big| \frac{\partial^l}{\partial t^l} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha}, D_{k,l,\alpha}$ such that

$$D_{k,l,\alpha} \Big\| \Big(\int_0^{+\infty} \Big(t^{l-\alpha} \Big| \frac{\partial^l}{\partial t^l} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \|_{p,\gamma} \leq \Big\| \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big| \frac{\partial^k}{\partial t^k} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq A_{k,l,\alpha} \Big\| \Big(\int_0^{+\infty} \Big(t^{l-\alpha} \Big| \frac{\partial^l}{\partial t^l} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}.$$

Proof. Let $n \in \mathbb{N}$ such that $n > \alpha$. It can be proved that

$$\Big| \frac{\partial^n}{\partial t^n} P_t f(x) \Big| \leq \int_t^{+\infty} \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| ds$$

Then, using Hardy's inequality,

$$\Big(\int_0^{+\infty} \Big(t^{n-\alpha} \Big| \frac{\partial^n}{\partial t^n} P_t f(x) \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \leq \Big(\int_0^{+\infty} \Big(t^{n-\alpha} \int_t^{+\infty} \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{n-\alpha} \Big(\int_0^{+\infty} \Big(s \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| \Big)^q s^{(n-\alpha)q-1} ds \Big)^{\frac{1}{q}}$$
$$= \frac{1}{n-\alpha} \Big(\int_0^{+\infty} \Big(s^{n+1-\alpha} \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| \Big)^q \frac{ds}{s} \Big)^{\frac{1}{q}}.$$

Now, as $n > \alpha$ is arbitrary, iterating the previous argument $k-l$ times, we have

$$\begin{aligned}
\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l}{\partial t^l}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}
&\leq \frac{1}{l-\alpha}\Big(\int_0^{+\infty}\Big(t^{l+1-\alpha}\Big|\frac{\partial^{l+1}}{\partial t^{l+1}}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\leq \frac{1}{(l-\alpha)(l+1-\alpha)}\Big(\int_0^{+\infty}\Big(t^{l+2-\alpha}\Big|\frac{\partial^{l+2}}{\partial t^{l+2}}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\cdots\\
&\leq C_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^{k}}{\partial t^{k}}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}
\end{aligned}$$

where $C_{k,l,\alpha} = \dfrac{1}{(l-\alpha)(l+1-\alpha)\dots(k-1-\alpha)}$. Thus,

$$D_{k,l,\alpha}\Big\|\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l}{\partial t^l}P_t f\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} \leq \Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma},$$

where $D_{k,l,\alpha} = 1/C_{k,l,\alpha}$.

The converse inequality is also obtained by an inductive argument from the case $k = l+1$. Let us recall (7.53),

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s\Big(\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big)\frac{\partial^{k-l}}{\partial t_1^{k-l}}\mu_{t_1}^{(1/2)}(ds),$$

and because, from (3.19), $\dfrac{\partial}{\partial t_1}\mu_{t_1}^{(1/2)}(ds) = \Big(t_1^{-1} - \dfrac{t_1}{2s}\Big)\mu_{t_1}^{(1/2)}(ds)$ we get

$$\begin{aligned}
\Big|\frac{\partial^k u(x,t)}{\partial t^k}\Big|
&\leq \int_0^{+\infty} T_s\Big(\Big|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big|\Big)\Big|\Big(t_1^{-1} - \frac{t_1}{2s}\Big)\Big|\mu_{t_1}^{(1/2)}(ds)\\
&\leq t_1^{-1}\int_0^{+\infty} T_s(\Big|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big|)\mu_{t_1}^{(1/2)}(ds) + \frac{t_1}{2}\int_0^{+\infty} T_s(\Big|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big|)\frac{1}{s}\mu_{t_1}^{(1/2)}(ds).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}\Big|\frac{\partial^k u(x,t)}{\partial t^k}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}
&\leq C_q\Big[\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\int_0^{+\infty} T_s\Big(\Big|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big|\Big)\mu_{t_1}^{(1/2)}(ds)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\\
&+\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}\frac{t_1}{2}\int_0^{+\infty} T_s\Big(\Big|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big|\Big)\frac{1}{s}\mu_{t_1}^{(1/2)}(ds)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big]\\
&= (I) + (II)
\end{aligned}$$

Then, using Minkowski's integral inequality twice (because T_s is an integral transformation with a positive kernel) and the fact that $\mu_{t_1}^{(1/2)}(ds)$ is a probability, we get

$$\begin{aligned}(I) &= C_q\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\int_0^{+\infty}T_s\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)\mu_{t_1}^{(1/2)}(ds)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\\ &\leq C_q\int_0^{+\infty}\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(T_s\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\mu_{t_1}^{(1/2)}(ds)\\ &\leq C_q\int_0^{+\infty}T_s\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big)\mu_{t_1}^{(1/2)}(ds)\\ &\leq C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big)\end{aligned}$$

and, using the same argument for (II) and (3.20), we have

$$\begin{aligned}(II) &\leq C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big)\frac{1}{t_1^2}\\ &= C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big).\end{aligned}$$

Taking $t_1 = t_2 = \frac{t}{2}$ and changing the variable, we get

$$(I) \leq C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big)^q\Big(\Big|\frac{\partial^l P_t f(x)}{\partial t^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big)$$

and

$$(II) \leq C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big)^q\Big(\Big|\frac{\partial^l P_t f(x)}{\partial t^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big).$$

Hence, using the L^p boundedness of T^*

$$\begin{aligned}&\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k u(x,t)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}\\ &\qquad\leq C_{q,k,\alpha}\Big\|T^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l P_t f(x)}{\partial u^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big)\Big\|_{p,\gamma}\\ &\qquad\quad+C_q\Big\|T^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l P_t f(x)}{\partial u^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big)\Big\|_{p,\gamma}\Big)\\ &\qquad\leq C_{k,\alpha,q}\Big\|\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l P_t f(x)}{\partial t^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}. \qquad \square\end{aligned}$$

Now, we can introduce the Gaussian Triebel–Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$ following the classical case:

Definition 7.38. *Let $\alpha \geq 0$, k be the smallest integer greater than α, and $1 \leq p, q < \infty$. The Gaussian Triebel–Lizorkin space $F^{\alpha}_{p,q}(\gamma_d)$ is the set of functions $f \in L^p(\gamma_d)$ for which*

$$\left\| \left(\int_0^{\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma} < \infty. \tag{7.58}$$

The norm of $f \in F^{\alpha}_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{F^{\alpha}_{p,q}} := \|f\|_{p,\gamma} + \left\| \left(\int_0^{\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma}. \tag{7.59}$$

Observe that according to Lemma 7.37, the definition of $F^{\alpha,q}_p(\gamma_d)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent.

Let us prove now that the Gaussian Triebel–Lizorkin spaces are complete,

Lemma 7.39. *For any $\alpha \geq 0$, $1 \leq p, q < \infty$, the Gaussian Triebel–Lizorkin space $F^{\alpha}_{p,q}(\gamma_d)$ is a Banach space.*

Proof. To prove the completeness, it is enough to see that if (f_n) is a sequence in $F^{\alpha}_{p,q}(\gamma_d)$ such that $\sum_{n=1}^{\infty} \|f_n\|_{F^{\alpha}_{p,q}} < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges in $F^{\alpha}_{p,q}(\gamma_d)$. Since,

$$\sum_{n=1}^{\infty} \|f_n\|_{F^{\alpha}_{p,q}} = \sum_{n=1}^{\infty} \|f_n\|_{p,\gamma} + \left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f_n}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} < \infty.$$

In particular, this implies that

$$\sum_{n=1}^{\infty} \|f_n\|_{p,\gamma} < \infty, \text{ and } \sum_{n=1}^{\infty} \left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f_n}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} < \infty.$$

But as $L^p(\gamma_d)$ is complete, there exist functions $f, g \in L^p(\gamma_d)$, such that

$$g(x) = \sum_{n=1}^{\infty} |f_n(x)|, \text{ and } \sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{a.e.}x.$$

Moreover, $\sum_{n=1}^{\infty} f_n = f$ in $L^p(\gamma_d)$. Analogously, there exists $h \in L^p(\gamma_d)$, such that

$$\sum_{n=1}^{\infty} \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f_n(x)}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = h(x) \quad \text{a.e.}x,$$

and
$$\sum_{n=1}^{\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f_n}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} = h$$
in $L^p(\gamma_d)$.

We need to prove that $\sum_{n=1}^{\infty} f_n = f$ in $F^{\alpha}_{p,q}$, i.e., $\lim_{n\to\infty}\|\sum_{i=1}^{n} f_i - f\|_{F^{\alpha}_{p,q}} = 0$. Let $h_n(x) = \sum_{i=1}^{n}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f_i(x)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$, then $h(x) = \lim_{n\to\infty} h_n(x)$ a.e.x, and for each x, $\{h_n(x)\}$ is a non-decreasing sequence of real numbers, also $h_n(x) \leq h(x)$ a.e. x.

As in the proof of the completeness of the Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$, we have, using Lebesgue's dominated convergence theorem, for any $t \geq 0$ and $x \in \mathbb{R}^d$,
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x) = \frac{\partial^k}{\partial t^k}P_t f(x).$$

Now, let us prove that $f \in F^{\alpha}_{p,q}$. In fact, using the triangle inequality and Fatou's lemma,
$$\begin{aligned}
\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} &= \int_0^{+\infty}\lim_{n\to\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\leq \liminf_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\leq \liminf_{n\to\infty}\sum_{i=1}^{n}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&= \sum_{n=1}^{\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f_n(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} = h(x) \text{ a.e.} x.
\end{aligned}$$

Therefore,
$$\Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} \leq \|h\|_{p,\gamma} < \infty.$$

Because for any $t \geq 0$ and $x \in \mathbb{R}^d$,
$$\begin{aligned}
\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x) - \frac{\partial^k}{\partial t^k}P_t f(x)\Big| &\leq \sum_{i=1}^{n}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big| + \Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\\
&\leq \sum_{i=1}^{\infty}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big| + \Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|,
\end{aligned}$$

and

$$\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\sum_{i=1}^{\infty}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|+\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \sum_{i=1}^{\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}+\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= h(x)+\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\leq 2h(x)<\infty \quad \text{a.e.}x,$$

thus, $\Big(t^{k-\alpha}\Big(\sum_{i=1}^{\infty}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|+\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)\Big)^q\frac{1}{t}$ is integrable a.e. x, and, therefore, according to Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)-\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)=0 \text{ a.e.}x,$$

and,

$$\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n}f_i(x)-f(x)\Big)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$=\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)-\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq 2h(x),$$

a.e.x, for all $n\in\mathbb{N}$, where $h\in L^p(\gamma_d)$; thus,

$$\lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n}f_i(x)-f(x)\Big)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$=\lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)-\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)=0 \quad \text{a.e.}x.$$

Then, again using Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n}f_i-f\Big)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}$$
$$\Big\|\lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n}f_i-f\Big)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}=0.$$

Finally,

$$\lim_{n\to\infty}\Big\|\sum_{i=1}^{n} f_i - f\Big\|_{F^{\alpha}_{p,q}}$$
$$= \lim_{n\to\infty}\left(\Big\|\sum_{i=1}^{n} f_i - f\Big\|_{p,\gamma} + \Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i - f\Big)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}\right)$$
$$= \lim_{n\to\infty}\Big\|\sum_{i=1}^{n} f_i - f\Big\|_{p,\gamma} + \lim_{n\to\infty}\Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i - f\Big)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} = 0.$$

□

Observe that using the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley $g_{t,\gamma}$-function (5.13),

$$g_{t,\gamma}(f)(x) = \left(\int_0^{\infty} t\left|\frac{\partial P_t f}{\partial t}\right|^2 dt\right)^{1/2},$$

it can be seen, for $1 < p < \infty$, that

$$L^p(\gamma_d) = F^{0}_{p,2}(\gamma_d).$$

Also, by the trivial identification of the L^p spaces with the Hardy spaces, we have

$$H^p(\gamma_d) = F^{0}_{p,2}(\gamma_d).$$

For Gaussian Triebel–Lizorkin spaces, we have the following inclusion result, which is analogous to Proposition 7.36 *i*:

Proposition 7.40. *The inclusion $F^{\alpha_1}_{p,q_1}(\gamma_d) \subset F^{\alpha_2}_{p,q_2}(\gamma_d)$ holds for $\alpha_1 > \alpha_2 > 0$ and $q_1 \geq q_2$.*

Proof. Let us consider $f \in F_p^{\alpha_1,q_1}(\gamma_d)$. Then,

$$\left(\int_0^{+\infty}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}$$
$$= \left(\int_0^{1}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t} + \int_1^{+\infty}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}$$
$$\leq \left(\int_0^{1}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}} + \left(\int_1^{+\infty}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}$$
$$= (I) + (II).$$

Let us observe that for the first term I, the result for the case $q_1 = q_2$ is immediate, because, as $t < 1$, $t^{k-\alpha_2} < t^{k-\alpha_1}$ and then

$$(I)^{q_2} \leq \int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}.$$

Now, in the case $q_1 > q_2$, taking $r = \frac{q_1}{q_2}$, $s = \frac{q_1}{q_1 - q_2}$ then $r, s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, then, using Hölder's inequality

$$(I)^{q_2} = \int_0^1 t^{(\alpha_1-\alpha_2)q_2}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t} \le \Big(\int_0^1 t^{(\alpha_1-\alpha_2)q_2 s}\frac{dt}{t}\Big)^{\frac{1}{s}} \times\Big(\int_0^1 \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2 r}\frac{dt}{t}\Big)^{\frac{1}{r}}$$
$$= \frac{1}{(\alpha_1-\alpha_2)q_2 s}\Big(\int_0^1 \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{q_2}{q_1}} \le C\Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{q_2}{q_1}}.$$

Now, for the second term II, using Lemma 3.4, we have

$$(II) = \Big(\int_1^{+\infty} \Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}} \le CT^*f(x)\Big(\int_1^{+\infty}\Big(t^{k-\alpha_2}t^{-k}\Big)^{q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}}$$
$$= CT^*f(x)\Big(\int_1^{+\infty} t^{-\alpha_2 q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}} = CT^*f(x).$$

Then, using the $L^p(\gamma_d)$ continuity of T^*, we get

$$\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}}\Big\|_{p,\gamma}$$
$$\le C\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{1}{q_1}}\Big\|_{p,\gamma} + C\|T^*f\|_{p,\gamma_d}$$
$$\le C\Big[\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}|\frac{\partial^k P_t f}{\partial t^k}|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{1}{q_1}}\Big\|_{p,\gamma} + \|f\|_{p,\gamma}\Big] < +\infty.$$

Thus, $f \in F_p^{\alpha_2,q_2}(\gamma_d)$. □

Observe that the Gaussian Besov–Lipschitz spaces and the Gaussian Triebel–Lizorkin spaces are, by construction, subspaces of $L^p(\gamma_d)$ and the inclusions are trivially continuous.

Additionally, it is clear that for all $t > 0$ and $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial t^k}P_t h_\beta(x) = (-1)^k|\beta|^{k/2}e^{-t\sqrt{|\beta|}}h_\beta(x);$$

therefore,

$$\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t h_\beta\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} = \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}}\Big(\Gamma((k-\alpha)q)\Big)^{1/q}\|h_\beta\|_{p,\gamma} < \infty.$$

Thus, $h_\beta \in B_{p,q}^\alpha(\gamma_d)$ and

$$\|h_\beta\|_{B_{p,q}^\alpha} = (1 + \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}}\Big(\Gamma((k-\alpha)q)\Big)^{1/q})\|h_\beta\|_{p,\gamma}.$$

Similarly, $h_\beta \in F^{\alpha}_{p,q}(\gamma_d)$ and

$$\|h_\beta\|_{F^{\alpha}_{p,q}} = (1+\frac{|\beta|^{\alpha/2}}{q^{k-\alpha}}\Big(\Gamma((k-\alpha)q)\Big)^{1/q})\|h_\beta\|_{p,\gamma} = \|h_\beta\|_{B^{\alpha}_{p,q}}.$$

Therefore, the set of polynomials $\mathscr{P}$ is included in $B^{\alpha}_{p,q}(\gamma_d)$ and in $F^{\alpha}_{p,q}(\gamma_d)$. An open question is to prove whether or not $\mathscr{P}$ is dense in $B^{\alpha}_{p,q}(\gamma_d)$ or $F^{\alpha}_{p,q}(\gamma_d)$.

Also, we have the following inclusion relations between Gaussian Triebel–Lizorkin spaces and Gaussian Besov–Lipschitz spaces:

Proposition 7.41. *Let $\alpha \geq 0$ and $p,q > 1$*

i) If $p = q$, then $F^{\alpha}_{p,p}(\gamma_d) = B^{\alpha}_{p,p}(\gamma_d)$.
ii) If $q > p$, then $F^{\alpha}_{p,q}(\gamma_d) \subset B^{\alpha}_{p,q}(\gamma_d)$.
iii) If $p > q$, then $B^{\alpha}_{p,q}(\gamma_d) \subset F^{\alpha}_{p,q}(\gamma_d)$.

Proof.

i) Using Tonelli's theorem, we trivially have

$$\begin{aligned}\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\Big\|_{p,\gamma} &= \Big(\int_0^{+\infty} t^{(k-\alpha)p}\int_{\mathbb{R}^d}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|^p\gamma_d(dx)\frac{dt}{t}\Big)^{\frac{1}{p}}\\ &= \Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}.\end{aligned}$$

ii) Suppose $q > p$, by Minkowski's integral inequality we then have,

$$\begin{aligned}\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{p/q} &= \Big(\int_0^{\infty} t^{(k-\alpha)q}\Big(\int_{\mathbb{R}^d}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|^p\gamma_d(dx)\Big)^{q/p}\frac{dt}{t}\Big)^{p/q}\\ &\leq \int_{\mathbb{R}^d}\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{p/q}\gamma_d(dx).\end{aligned}$$

Therefore,

$$\begin{aligned}\|f\|_{B^{\alpha}_{p,q}} &= \|f\|_{p,\gamma} + \Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\leq \|f\|_{p,\gamma} + \Big\|\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} = \|f\|_{F^{\alpha}_{p,q}}.\end{aligned}$$

iii) Finally, if $p > q$, again using Minkowski's integral inequality, we get

$$\begin{aligned}\|f\|_{F^{\alpha}_{p,q}} &= \|f\|_{p,\gamma} + \Big\|\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}\\ &\leq \|f\|_{p,\gamma} + \Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} = \|f\|_{B^{\alpha}_{p,q}}. \qquad \square\end{aligned}$$

Moreover, Gaussian Sobolev spaces $L^p_\alpha(\gamma_d)$ are contained in some Besov–Lipschitz and Triebel–Lizorkin spaces; therefore, these spaces are "finer scales" for measuring the regularity of functions.

Theorem 7.42. *Let us suppose that* $1 < p < +\infty$ *and* $\alpha > 0$. *Then*

i) $L^p_\alpha(\gamma_d) \subset F^{\alpha}_{p,2}(\gamma_d)$ *if* $p > 1$.
ii) $L^p_\alpha(\gamma_d) \subset B^{\alpha}_{p,p}(\gamma_d) = F^{\alpha}_{p,p}(\gamma_d)$ *if* $p \geq 2$.
iii) $L^p_\alpha(\gamma_d) \subset B^{\alpha}_{p,2}(\gamma_d)$ *if* $p \leq 2$.

Proof. For the proof of these inclusions, we need to use a characterization of the Gaussian Sobolev spaces, which will be discussed in the next chapter (see 8.21).

i) We have to consider two cases:

i-1) If $\alpha \geq 1$. Suppose $h \in L^p_\alpha(\gamma_d)$ then $h = \mathscr{J}_\alpha f$, $f \in L^p(\gamma_d)$, by the change of variable $u = t+s$, using the fact of the representation of the Bessel potentials (8.20) and Hardy's inequality to get,

$$\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h(x)}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad= \Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big|\frac{\partial^k P_t \mathscr{J}_\alpha f(x)}{\partial t^k}\Big|^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big(\int_0^{+\infty} s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big(\int_t^{+\infty} (u-t)^{\alpha-1}e^{t-u}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty}\Big(\int_t^{+\infty} u^{\alpha-1}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2 t^{2(k-\alpha)-1}dt\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\frac{1}{k-\alpha}\Big(\int_0^{+\infty}\Big(u^k\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2\frac{du}{u}\Big)^{\frac12}.
\end{aligned}$$

Hence, using the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley $g^k_{t,\gamma}$-function (see Theorem 5.13),

$$\begin{aligned}
\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac12}\Big\|_{p,\gamma} &\leq \frac{1}{\Gamma(\alpha)}\frac{1}{k-\alpha}\Big\|\Big(\int_0^{+\infty}\Big(u^k\Big|\frac{\partial^k P_u f}{\partial u^k}\Big|\Big)^2\frac{du}{u}\Big)^{\frac12}\Big\|_{p,\gamma}\\
&= C_{k,\alpha}\|g_k f\|_{p,\gamma} \leq C_{k,\alpha}\|f\|_{p,\gamma} = C_{k,\alpha}\|h\|_{p,\alpha};
\end{aligned}$$

thus, $h \in F^{\alpha}_{p,2}(\gamma_d)$.

i-2) If $0 \leq \alpha < 1$. Suppose $h \in L^p_\alpha(\gamma_d)$, then $h = \mathscr{J}_\alpha f$, $f \in L^p(\gamma_d)$, again using (8.20),

$$\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h(x)}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big(\int_0^{+\infty} s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)-1}\Big[\Big(\int_0^t s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\\
&\qquad\qquad + \Big(\int_t^{+\infty} s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\Big]dt\Big)^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)-1}\Big(\int_0^t s^{\alpha-1}e^{-s}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\Big)^{\frac{1}{2}}\\
&\qquad + \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)-1}\Big(\int_t^{+\infty} s^{\alpha-1}e^{-s}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\Big)^{\frac{1}{2}}\\
&= (I) + (II).
\end{aligned}
$$

Now, because $e^{-s} < 1$, $s^{\alpha-1} < t^{\alpha-1}$ as $\alpha < 1$, and, using the change of variables $u = t+s$ and Hardy inequality we get,

$$
\begin{aligned}
(II) &\leq \Big(\int_0^{+\infty} t^{2(k-1)-1}\Big(\int_t^{+\infty}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\Big)^{\frac{1}{2}}\\
&= \Big(\int_0^{+\infty} t^{2(k-1)-1}\Big(\int_{2t}^{+\infty}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2 dt\Big)^{\frac{1}{2}}\\
&\leq \Big(\int_0^{+\infty} t^{2(k-1)-1}\Big(\int_t^{+\infty}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2 dt\Big)^{\frac{1}{2}}\\
&\leq \Big(\int_0^{+\infty}\Big(u\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2 u^{2(k-1)-1}du\Big)^{\frac{1}{2}}.\\
&= \Big(\int_0^{+\infty}\Big|u^k\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2\frac{du}{u}\Big)^{\frac{1}{2}} = g^k_{t,\gamma}f(x).
\end{aligned}
$$

In addition, again using that $e^{-s} < 1$, we get

$$
\begin{aligned}
(I)^2 &\leq \int_0^{+\infty} t^{2(k-\alpha)-1}\Big(\int_0^t s^{\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\\
&= \frac{1}{\alpha^2}\int_0^{+\infty} t^{2k-1}\Big(\frac{\alpha}{t^\alpha}\int_0^t s^{\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt
\end{aligned}
$$

Then, as $\alpha > 0$ using Jensen's inequality (for the measure $\frac{\alpha}{t^\alpha}s^{\alpha-1}ds$) and Tonelli's theorem,

$$
\begin{aligned}
(I)^2 &\leq \frac{1}{\alpha^2}\int_0^{+\infty} t^{2k-1}\Big(\frac{\alpha}{t^\alpha}\int_0^t s^{\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|^2 ds\Big)dt\\
&\leq \frac{1}{\alpha}\int_0^{+\infty} s^{\alpha-1}\Big(\int_s^{+\infty}(t+s)^{2k-\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|^2 dt\Big)ds,
\end{aligned}
$$

as $2k-\alpha-1 > 0$. Finally, again using the change of variables $u = t+s$ and the Hardy inequality

$$
\begin{aligned}
(I)^2 &\leq \frac{1}{\alpha}\int_0^{+\infty} s^{\alpha-1}\Big(\int_{2s}^{+\infty} u^{2k-\alpha-1}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2 du\Big)ds\\
&\leq \frac{1}{\alpha}\int_0^{+\infty} s^{\alpha-1}\Big(\int_s^{+\infty} u^{2k-\alpha-1}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2 du\Big)ds\\
&\leq \frac{1}{\alpha}\int_0^{+\infty}\Big|u^k\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2\frac{du}{u} = \frac{1}{\alpha}(g^k_{t,\gamma}f(x))^2.
\end{aligned}
$$

Hence, again using the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley $g^k_{t,\gamma}$-function,

$$\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_{p,\gamma} \leq C_{k,\alpha}\|g_k f\|_{p,\gamma} \leq C_{k,\alpha}\|f\|_{p,\gamma} = C_{k,\alpha}\|h\|_{p,\alpha}.$$

Thus, $h \in F^{\alpha}_{p,2}(\gamma_d)$, for $0 < \alpha < 1$.

ii) Suppose $h \in L^p_\alpha(\gamma_d)$ with $p \geq 2$, then $h = \mathscr{I}_\alpha f$, $f \in L^p(\gamma_d)$. Using the inequality $(a+b)^p \leq C_p(a^p+b^p)$, if $a,b \geq 0, p \geq 1$, we get

$$\begin{aligned}
\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t \mathscr{I}_\alpha f}{\partial t^k}\Big\|_{p,\gamma}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}
&\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\int_0^{+\infty}s^\alpha e^{-s}\Big\|\frac{\partial^k P_{t+s}f}{\partial(t+s)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_0^t s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\\
&\qquad +\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}.
\end{aligned}$$

Using the inequality $(a+b)^{1/p} \leq a^{1/p}+b^{1/p}$ if $a,b \geq 0, p \geq 1$

$$\begin{aligned}
&\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_0^t s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\\
&\qquad +\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{(k-\alpha)p}\Big(\int_0^t s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\qquad +\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{(k-\alpha)p}\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&= (I)+(II).
\end{aligned}$$

Now, again using Hardy's inequality, because $k > \alpha$ and Lemma 3.5

$$\begin{aligned}
(II) &= \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_s f}{\partial s^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq \frac{C}{\Gamma(\alpha)}\frac{1}{k-\alpha}\Big(\int_0^{+\infty}\Big(s^\alpha\Big\|\frac{\partial^k}{\partial s^k}P_s f\Big\|_{p,\gamma}\Big)^p s^{(k-\alpha)p-1}ds\Big)^{\frac{1}{p}}\\
&= C_{k,\alpha}\Big(\int_0^{+\infty}\Big(s^k\Big\|\frac{\partial^k}{\partial s^k}P_s f\Big\|_{p,\gamma}\Big)^p\frac{ds}{s}\Big)^{\frac{1}{p}} = C_{k,\alpha}\|\Big(\int_0^{+\infty}\Big|s^k\frac{\partial^k P_s f}{\partial s^k}\Big|^p\frac{ds}{s}\Big)^{\frac{1}{p}}\|_p,
\end{aligned}$$

using Tonelli's theorem.

Now, because $p \geq 2$ using Lemma 3.4, we have

$$\int_0^{+\infty} \Big|u^k \frac{\partial^k P_u f(x)}{\partial u^k}\Big|^p \frac{du}{u} = \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^{p-2} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}$$

$$\leq C\Big(T^* f(x)\Big)^{p-2} \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}.$$

Therefore,

$$\Big\|\Big(\int_0^{+\infty} |u^k \frac{\partial^k P_u f}{\partial u^k}|^p \frac{du}{u}\Big)^{\frac{1}{p}}\Big\|_p^p$$

$$= \int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big|u^k \frac{\partial^k P_u f(x)}{\partial u^k}\Big|^p \frac{du}{u}\Big) \gamma_d(dx)$$

$$\leq C \int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^{p-2} \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2 \frac{du}{u}) \gamma_d(dx)$$

Using Hölder's inequality, with $\theta = \dfrac{2}{p}$, and the $L^p(\gamma_d)$ continuity of T^* and g_k, we have

$$\Big\|\Big(\int_0^{+\infty} \Big|u^k \frac{\partial^k P_u f}{\partial u^k}\Big|^p \frac{du}{u}\Big)^{\frac{1}{p}}\Big\|_p^p$$

$$\leq C \int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^{p-2} \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2 \frac{du}{u}\Big) \gamma_d(dx)$$

$$\leq C\Big(\int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^{(p-2)\cdot\frac{1}{1-\theta}} \gamma_d(dx)\Big)^{1-\theta}.$$

$$\times \Big(\int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}\Big)^{\frac{1}{\theta}} \gamma_d(dx)\Big)^{\theta}$$

$$= C\Big(\int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^p \gamma_d(dx)\Big)^{\frac{p-2}{p}}.$$

$$\times \Big(\int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}\Big)^{\frac{p}{2}} \gamma_d(dx)\Big)^{\frac{2}{p}}$$

$$= C\|T^* f\|_{p,\gamma}^{p-2} \|g_k f\|_{p,\gamma}^2 \leq C\|f\|_{p,\gamma}^p.$$

Thus,

$$(II) \leq C_{k,\alpha} \|h\|_{p,\alpha}.$$

Now, again using Lemma 3.5 and because $\alpha > 0$

$$
\begin{aligned}
(I) &= \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{p(k-\alpha)}\Big(\int_0^t s^{\alpha}\Big\|\frac{\partial^k}{\partial(s+t)^k}P_{s+t}f\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{p(k-\alpha)}\Big(\int_0^t s^{\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&= \frac{1}{\alpha}\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^k\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}^p\frac{dt}{t}\Big)^{\frac{1}{p}} \leq C_{k,\alpha}\|h\|_{p,\alpha},
\end{aligned}
$$

Thus, $h \in B^{\alpha}_{p,p}(\gamma_d)$, if $p \geq 2$.

iii) This inclusion could be proved using similar arguments as in i) and ii), but it is an immediate consequence of *i*) and of Proposition 7.41 ii). □

In [166], using Theorem 3.2, it is claimed that the Gaussian Sobolev spaces $L^p_\alpha(\gamma_d)$ coincide with the homogeneous Gaussian Triebel–Lizorkin $\dot{F}^{\alpha}_{p,2}$, but the proof of that theorem is wrong because it is assumed that the operator involved is linear; however, it is actually only sublinear.

Now, let us prove some interpolation results for the Gaussian Besov–Lipschitz spaces and for the Gaussian Triebel–Lizorkin Spaces.

Theorem 7.43. *We have the following interpolation results:*

i) For $1 < p_j, q_j < +\infty$ *and* $\alpha_j \geq 0$*, if* $f \in B^{\alpha_j}_{p_j,q_j}(\gamma_d)$*,* $j = 0, 1$*, then* $f \in B^{\alpha}_{p,q}(\gamma_d)$*, where* $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$*, and*

$$\frac{1}{p} = \frac{1}{p_0}(1-\theta) + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1}{q_0}(1-\theta) + \frac{\theta}{q_1},\ 0 < \theta < 1.$$

ii) For $1 < p_j, q_j < +\infty$ *and* $\alpha_j \geq 0$*, if* $f \in F^{\alpha_j}_{p_j,q_j}(\gamma_d)$*,* $j = 0, 1$*, then* $f \in F^{\alpha}_{p,q}(\gamma_d)$*, where* $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$*, and*

$$\frac{1}{p} = \frac{1}{p_0}(1-\theta) + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1}{q_0}(1-\theta) + \frac{\theta}{q_1},\ 0 < \theta < 1.$$

Proof. The proof of both results is based on the following interpolation result for $L^p(\gamma_d)$ spaces (actually true for any measure μ) obtained using Hölder's inequality:

For $1 < r_0, r_1 < \infty$ and $\frac{1}{r} = \frac{1}{r_0}(1-\eta) + \frac{\eta}{r_1}, 0 < \eta < 1$. If $f \in L^{r_j}(\gamma_d)$, $j = 0, 1$ then $f \in L^r(\gamma_d)$ and

$$\|f\|_{r,\gamma} \leq \|f\|_{r_0,\gamma}^{1-\eta}\|f\|_{r_1,\gamma}^{\eta}. \tag{7.60}$$

Let us prove *i*). Let k be any integer greater than α_0 and α_1. By using the above result, we get for $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$,

$$\begin{aligned}
\int_0^{+\infty} &\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\\
&\leq \int_0^{+\infty} \Big(t^{k-(\alpha_0(1-\theta)+\alpha_1\theta)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{1-\theta}_{p_0,\gamma}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{\theta}_{p_1,\gamma}\Big)^q \frac{dt}{t}\\
&= \int_0^{+\infty} \Big(t^{(1-\theta)(k-\alpha_0)+\theta(k-\alpha_1)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{1-\theta}_{p_0,\gamma}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{\theta}_{p_1,\gamma}\Big)^q \frac{dt}{t}\\
&= \int_0^{+\infty} \Big(t^{k-\alpha_0}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_0,\gamma}\Big)^{(1-\theta)q} \Big(t^{k-\alpha_1}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_1,\gamma}\Big)^{\theta q} \frac{dt}{t}.
\end{aligned}$$

Now, if $\lambda = \dfrac{\theta q}{q_1}$ then $0 < \lambda < 1$ and $q = (1-\lambda)q_0 + \lambda q_1$. Therefore, by using Hölder's inequality again,

$$\begin{aligned}
&\int_0^{+\infty} \Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\\
&\leq \Big(\int_0^{+\infty} \Big(t^{k-\alpha_0}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_0,\gamma}\Big)^{q_0} \frac{dt}{t}\Big)^{1-\lambda} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_1,\gamma}\Big)^{q_1} \frac{dt}{t}\Big)^{\lambda} < \infty;
\end{aligned}$$

thus $f \in B^{\alpha}_{p,q}(\gamma_d)$.

ii) Analogously, by taking $\beta = \frac{p\theta}{p_1}$, $\lambda = \frac{q\theta}{q_1}$, we have $0 < \beta, \lambda < 1$ and $p = (1-\beta)p_0 + \beta p_1, q = (1-\lambda)q_0 + \lambda q_1$. Let k be any integer greater than α_0 and α_1, by using Hölder's inequality we get for $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$,

$$\begin{aligned}
\int_0^{+\infty} &\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q \frac{dt}{t}\\
&= \int_0^{+\infty} \Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{(1-\theta)q} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{\theta q} \frac{dt}{t}\\
&= \int_0^{+\infty} \Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{(1-\lambda)q_0} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{\lambda q_1} \frac{dt}{t}\\
&\leq \Big(\int_0^{+\infty} \Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0} \frac{dt}{t}\Big)^{1-\lambda} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1} \frac{dt}{t}\Big)^{\lambda}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Big\|\Big(\int_0^{+\infty} &\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|^p_{p,\gamma} = \int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{p}{q}} \gamma_d(dx)\\
&\leq \int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0} \frac{dt}{t}\Big)^{\frac{(1-\lambda)p}{q}} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{\lambda p}{q}} \gamma_d(dx)\\
&= \int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0} \frac{dt}{t}\Big)^{\frac{(1-\theta)p}{q_0}} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{\theta p}{q_1}} \gamma_d(dx)\\
&= \int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0} \frac{dt}{t}\Big)^{\frac{(1-\beta)p_0}{q_0}} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{\beta p_1}{q_1}} \gamma_d(dx),
\end{aligned}$$

and then again using Hölder's inequality,

$$\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}^p$$
$$\leq \Big(\int_{\mathbb{R}^d}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0}\frac{dt}{t}\Big)^{\frac{p_0}{q_0}}\gamma_d(dx)\Big)^{1-\beta}$$
$$\times\Big(\int_{\mathbb{R}^d}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{p_1}{q_1}}\gamma_d(dx)\Big)^{\beta}$$
$$=\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0}\frac{dt}{t}\Big)^{\frac{1}{q_0}}\Big\|_{p_0,\gamma}^{p_0(1-\beta)}$$
$$\times\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{1}{q_1}}\Big\|_{p_1,\gamma}^{p_1\beta}<+\infty.$$

Hence, $f\in F_{p,q}^{\alpha}(\gamma_d)$. □

Finally, we are going to study the continuity properties of the Ornstein–Uhlenbeck semigroup and the Poisson–Hermite semigroup on the Gaussian Besov–Lipschitz and Triebel–Lizorkin spaces. In the next chapter, we consider the boundedness property of other operators on those spaces.

Theorem 7.44. *For The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ and the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$,*

i) Both are bounded on $B_{p,q}^{\alpha}(\gamma_d)$.
ii) Both are bounded on $F_{p,q}^{\alpha}(\gamma_d)$.

Proof.

i) Let us prove the $B_{p,q}^{\alpha}(\gamma_d)$-continuity of P_t for any $t>0$; the proof for T_t is totally analogous. Using the L^p-continuity of the Poisson–Hermite semigroup, Lebesgue's dominated convergence theorem, and Jensen's inequality, we get

$$\int_{\mathbb{R}^d}\Big|\frac{\partial^k P_t(P_s f)}{\partial t^k}(x)\Big|^p\gamma_d(dx)=\int_{\mathbb{R}^d}\Big|P_s\Big(\frac{\partial^k P_t f}{\partial t^k}\Big)(x)\Big|^p\gamma_d(dx)$$
$$\leq\int_{\mathbb{R}^d}P_s\Big(\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|^p\Big)\gamma_d(dx)$$
$$=\int_{\mathbb{R}^d}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|^p\gamma_d(dx).$$

Thus,
$$\Big\|\frac{\partial^k P_t(P_s f)}{\partial t^k}\Big\|_{p,\gamma}\leq\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma};$$

therefore,

$$\|P_s f\|_{B_{p,q}^{\alpha}}=\|P_s f\|_{p,\gamma}+\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t(P_s f)}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq\|f\|_{p,\gamma}+\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}=\|f\|_{B_{p,q}^{\alpha}}.$$

ii) Let us prove the $F^{\alpha}_{p,q}$-continuity of P_t for any $t > 0$; the proof for T_t is totally analogous. Using Lebesgue's dominated convergence theorem and Minkowski's integral inequality, we have

$$\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_t(P_s g)}{\partial s^k}(x)\right|\right)^q\frac{ds}{s}\right)^{1/q}$$
$$= \left(\int_0^{\infty}\left(s^{k-\alpha}\left|\int_{\mathbb{R}^d} p(t,x,y)\frac{\partial^k P_s g(y)}{\partial s^k}dy\right|\right)^q\frac{ds}{s}\right)^{1/q}$$
$$\leq \int_{\mathbb{R}^d} p(t,x,y)\left(\int_0^{\infty}(s^{k-\alpha}\left|\frac{\partial^k P_s g(y)}{\partial s^k}\right|\right)^q\frac{ds}{s}\right)^{1/q} dy$$
$$= P_t\left(\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q\frac{ds}{s}\right)^{1/q}\right)(x).$$

Therefore, by the L^p continuity of P_t we get

$$\left\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q\frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}$$
$$\leq \left\|P_t\left(\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q\frac{ds}{s}\right)^{1/q}\right)\right\|_{p,\gamma}$$
$$\leq \left\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q\frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}$$

Thus,

$$\|P_t g\|_{F^{\alpha}_{p,q}} = \|P_t g\|_{p,\gamma} + \left\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q\frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}$$
$$\leq \|g\|_{p,\gamma} + \left\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q\frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma} = \|g\|_{F^{\alpha}_{p,q}}. \qquad \square$$

7.9 Notes and Further Results

1. In [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, and W. Urbina define and study Sobolev spaces associated with multi-dimensional Laguerre expansions of type α. The result is obtained by means of transference from a Hermite setting using the relationship between Laguerre and Hermite polynomials (see G. Szegő's book [262, (5.6.1)]).

2. In [177], G. Mauceri, S. Meda, and P. Sjögren found a maximal characterization of $H^1_{at}(\gamma_d)$ that unfortunately is only valid for $d = 1$. In the same paper, they give a description of the non-negative functions in $H^1_{at}(\gamma_d)$ and use it to prove that $L^p(\gamma_d) \subset H^1_{at}(\gamma_d)$, for $1 < p \leq \infty$.

3. In 1995, J. Epperson [75] considered Triebel–Lizorkin spaces with respect to the Hermite function expansions. Those spaces are completely different than the spaces that we are considering here, because the reference measure is the Lebesgue measure; therefore it should not be confused with them, because he was working with the Lebesgue measure.

4. In [161], L. Liu and D. Yang consider Gaussian bounded lower oscillation (BLO) spaces $BLO_a(\gamma_d)$, the space of functions with bounded lower oscillation associated with a given class of admissible balls with parameter a.

5. In [166], I. López defines and briefly studies Besov spaces and Triebel–Lizorkin spaces for Hermite and Laguerre expansions. There are some technical problems in the definitions and some gaps in the proofs.

6. More abstract approaches to Besov and Triebel–Lizorkin spaces associated with a general differential operator can be found, for instance, in [154].

7. Hardy spaces for Jacobi expansions have a curious story. The first construction obtained by L. Cafarelli in his doctoral dissertation in 1971, under the direction of C. P. Calderón, [39]. He defined the conjugation as a smooth differential operator, and from there he was able to give a definition of them. Unfortunately, that memoir, which contains very original and novel ideas, for example, the proof that the Jacobi measure is doubling, well before the notion of doubling measure was formulated, was never published. Then, 25 years later, in 1996, Zhongkai Li [157, 158], formulated another definition of Hardy spaces for Jacobi expansions, closely following the work of B. Muckenhoupt and E. Stein [199] in the ultraspherical case.

8. There is a class of spaces that are an intermediate generalization between the classical Lebesgue spaces and the Orlicz spaces; they are the variable Lebesgue spaces, which have been intensively studied over the last 25 years, extending almost all the boundedness properties of classical harmonic analysis operators with respect to the Lebesgue measure (see, for instance, [61] or [66]). For the study of variable Lebesgue spaces with respect to general Radon measures, see [3]. In particular, some results for variable Lebesgue spaces with respect to the Gaussian measure can be found in [63] and [192].