6

Spectral Multiplier Operators with Respect to the Gaussian Measure

In this chapter, we study spectral multiplier operators for Hermite polynomial expansions. First, we consider Meyer's multiplier theorem, which is one of the most basic and most useful results for Hermite expansions. Then, we consider spectral multipliers of Laplace transform type. In both cases, we prove their boundedness in $L^p(\gamma_d)$, for $1 < p < \infty$. For the case of spectral multipliers of Laplace transform type, we also study the boundedness in the case $p = 1$. Finally, we discuss the fact that the Ornstein–Uhlenbeck operator has a bounded holomorphic functional calculus.

6.1 Gaussian Spectral Multiplier Operators

Definition 6.1. *Given a bounded function $m : \mathbb{N}_0 \to \mathbb{C}$. According to the spectral theorem, we may form the operator $m(L)$[1] defined for any $f \in L^2(\gamma_d)$ [2]*

$$m(L)f = \sum_{k=0}^{\infty} m(k)\mathbf{J}_k f = \sum_{k=0}^{\infty} m(k) \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu. \tag{6.1}$$

Observe that $m(L)$ is trivially bounded in $L^2(\gamma_d)$, as

$$\|m(L)f\|_{2,\gamma} = \sum_{k=0}^{\infty} |m(k)|^2 \|\mathbf{J}_k f\|_{2,\gamma} \leq \|m\|_\infty \sum_{k=0}^{\infty} \|\mathbf{J}_k f\|_{2,\gamma} = \|m\|_\infty \|f\|_{2,\gamma}.$$

We call $m(L)$ the *spectral multiplier operator* associated with the *spectral multiplier m.*

[1] Formally speaking, it should be denoted by $m(-L)$ because of (2.7); for simplicity we just write it as $m(L)$.

[2] Alternatively, we could define $m(L)$ on the set of polynomials in d-variables, $\mathscr{P}(\mathbb{R}^d)$, as they have finite Hermite expansion $f = \sum_{k=0}^{\infty} \mathbf{J}_k f = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu$.

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Moreover, because $m(L)$ is well defined in $\mathscr{P}(\mathbb{R}^d)$, and we know that $\mathscr{P}(\mathbb{R}^d)$ is dense in $L^p(\gamma_d)$ for any $1 \leq p < \infty$, the multiplier operator $m(L)$ is densely defined in $L^2(\gamma_d)$ with domain

$$\mathscr{D}_m = \left\{ f \in L^2(\gamma_d) : \sum_{k=0}^{\infty} |m(k)|^2 \|J_k f\|_{2,\gamma} < \infty \right\}.$$

The basic problem of the multiplier theory is to determine the conditions on the spectral multiplier m such that the spectral multiplier operator $m(L)$, initially defined in $L^2(\gamma_d) \cap L^p(\gamma_d)$, has a bounded extension on $L^p(\gamma_d), 1 < p < \infty$; in other words, when we can find a constant $C_p > 0$ dependent only on p such that

$$\|m(L)f\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma}, \tag{6.2}$$

for all $f \in L^p(\gamma_d)$.
We also want to consider under which conditions $m(L)$ is of weak type $(1,1)$ with respect to the Gaussian measure; in other words, when we can find a constant $C > 0$ dependent only on p such that

$$\gamma_d\left(\left\{ x \in \mathbb{R}^d : m(L)f(x) > \lambda \right\}\right) \leq \frac{C}{\lambda} \|f\|_{1,\gamma} \tag{6.3}$$

for any $f \in L^p(\gamma_1)$.

6.2 Meyer's Multipliers

One of the most basic results in Gaussian multiplier theory was obtained by P. A. Meyer in [189] (see also [288] and [218]), using in a fundamental way the hypercontractivity property of the Ornstein–Uhlenbeck semigroup. Therefore, the multiplier theory for Hermite expansions and the hypercontractivity property of the Ornstein–Uhlenbeck semigroup are closely related.

Theorem 6.2. *(Meyer) Given a function h, holomorphic in a neighborhood of the origin, and let m be a spectral multiplier such that $m(k) = h\left(\frac{1}{k^\alpha}\right)$, for some $\alpha > 0$ and $k \geq n_0$, for some $n_0 \geq 0$, then the spectral multiplier operator $m(L)$ admits an $L^p(\gamma_d)$-bounded extension for any $1 < p < \infty$. Moreover, its $L^p(\gamma_d)$-norm does not depend on the dimension.*

Proof. Using Corollary 2.17, Lemma 2.18 and the inequality (3.42), the proof is almost immediate. Let us decompose $m(L)$ into its finite and infinite parts.

$$m(L)f = \sum_{k=0}^{n_0-1} m(k)\,\mathbf{J}_k f + \sum_{k=n_0}^{\infty} m(k)\,\mathbf{J}_k f = m_1(L)f + m_2(L)f.$$

Using Corollary 2.17, we know that $\mathbf{J}_n$ is $L^p(\gamma_d)$-bounded; therefore $m_1(L)$ is $L^p(\gamma_d)$-bounded,

$$\|m_1(L)f\|_{p,\gamma} \leq \sum_{k=0}^{n_0-1} m(k)\|\mathbf{J}_k f\|_{p,\gamma} \leq C_p\|f\|_{p,\gamma}.$$

Thus, it is enough to prove that m_2 is $L^p(\gamma_d)$-bounded,

$$\|m_2(L)f\|_{p,\gamma} \leq C_p\|f\|_{p,\gamma}.$$

Using the generalized potential operators (3.41) and the inequality (3.42), then, as $h(x) = \sum_{n=0}^{\infty} a_n x^n$,

$$\begin{aligned} m_2(L)f &= \sum_{k=n_0}^{\infty} m(k)\mathbf{J}_k f = \sum_{k=n_0}^{\infty} \Big(\sum_{n=0}^{\infty} a_n \frac{1}{k^{\alpha n}}\Big)\mathbf{J}_k f \\ &= \sum_{n=0}^{\infty} a_n \Big(\sum_{k=n_0}^{\infty} \frac{1}{k^{\alpha n}}\mathbf{J}_k f\Big) = \sum_{n=0}^{\infty} a_n (U_{n_0,\alpha} f)^n. \end{aligned}$$

Using the $L^p(\gamma_d)$-boundedness of $U_{n_0,\alpha}$ n-times, we get

$$\|m_2(L)f\|_{p,\gamma} \leq \sum_{k=1}^{\infty} |a_n|\|(U_{n_0,\alpha} f)^n\|_{p,\gamma} \leq C\Big(\sum_{n=0}^{\infty} |a_n|\frac{1}{n_0^{\alpha n}}\Big)\|f\|_{p,\gamma} = C\|f\|_{p,\gamma}. \qquad \square$$

Definition 6.3. *A spectral multiplier operator $m(L)$ is called Meyer's multiplier if it satisfies the hypothesis of Theorem 6.2, i.e., there exists a function h holomorphic in a neighborhood of the origin such that*

$$m(k) = h\Big(\frac{1}{k^{\alpha}}\Big), \tag{6.4}$$

for some $\alpha > 0$ and $k \geq n_0$, for some $n_0 \geq 0$.

We see in Chapter 8 that the Gaussian Riesz potentials are the simplest Meyer's multipliers possible (see 8.5), and that the Gaussian Bessel potentials are not Meyer's multipliers, but the composition of two Meyer's multipliers (see 8.19). On the other hand, the Ornstein–Uhlenbeck and the Poisson–Hermite operators and their variations are Gaussian multipliers but are not Meyer's multipliers. Finally, as we are going to see in Chapter 9, the Gaussian Riesz transforms are not Gaussian multipliers, different than the Riesz transforms in the classical case.

6.3 Gaussian Laplace Transform Type Multipliers

Following E. Stein [253, Chapter 4], let us consider *Laplace type multipliers*.

Definition 6.4. *A function $m : (0,\infty) \to \mathbb{C}$ is said to be of Laplace transform type if and only if*

$$m(k) = k\int_0^{\infty} \phi(t)e^{-tk}dt, \quad k > 0, \tag{6.5}$$

where $\phi : (0,\infty) \to \mathbb{C}$, is a bounded measurable function.

Observe that taking the change of variables $r = e^{-t}$, we see that m can be rewritten as

$$m(k) = k\int_0^1 \psi(t) r^k \frac{dr}{r}, \quad k > 0, \tag{6.6}$$

where $\psi(r) = \phi(-\log r)$.

Definition 6.5. *A spectral multiplier operator $m(L)$ is said to be a Laplace transform type multiplier, if the spectral multiplier m is a function of Laplace transform type. Then, $m(L)$ can be written as*

$$m(L)f(x) = \sum_{k=0}^{\infty} m(\sqrt{k})\mathbf{J}_k f = \sum_{k=0}^{\infty} m(\sqrt{k}) \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu, \tag{6.7}$$

for a function f with Hermite expansion $f = \sum_{k=0}^{\infty} \mathbf{J}_k f = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu$.

Observe that if we ask the function ϕ to be not only bounded but integrable, then we can get the following easy result:

Proposition 6.6. *If $m : (0,\infty) \to \mathbb{C}$ is a spectral multiplier of Laplace transform type function such that ϕ is bounded and integrable, then the spectral multiplier operator $m(L)$ is a $L^p(\gamma_d)$-bounded operator, for $1 < p < \infty$.*

Proof. Let $f = \sum_{k=0}^{\infty} \mathbf{J}_k f$, then

$$\begin{aligned} m(L)f &= \sum_{k=0}^{\infty} \phi(\sqrt{k})\mathbf{J}_k f = \sum_{k=0}^{\infty} \Big(\int_0^{\infty} \phi(t) e^{-\sqrt{k}t} dt\Big)\mathbf{J}_k f = \int_0^{\infty} \Big[\sum_{k=0}^{\infty} e^{-\sqrt{k}t}\mathbf{J}_k f\Big] \phi(t) dt \\ &= \int_0^{\infty} P_t f \, \phi(t) dt. \end{aligned}$$

Therefore, using Minkowski's integral inequality, and the $L^p(\gamma_d)$-boundedness of the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$,

$$||m(L)(f)||_{p,\gamma} = \Big\| \int_0^{\infty} P_t f \, \phi(t) dt \Big\|_{p,\gamma} \leq \int_0^{\infty} ||P_t f||_{p,\gamma} |\phi(t)| dt \leq C_p ||f||_{p,\gamma}. \qquad \square$$

Now for the general case, using the Littlewood–Paley theory, following E. Stein [253, Chapter II], we get

Theorem 6.7. *Given a Laplace transform type spectral multiplier m, the spectral multiplier operator $m(L)$ has a $L^p(\gamma_d)$-bounded extension, for $1 < p < \infty$.*

Proof. The proof is given here for the case of the Poisson–Hermite semigroup for completeness, but it is still valid in far more general settings, as is clear from E. Stein's monograph [253].[3]

We need to prove the following identity,

$$m(L)(f)(x) = -\int_0^{\infty} \frac{\partial P_t f}{\partial t}(x) \phi(t) dt. \tag{6.8}$$

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[3] In fact, nowadays this theorem is known as *Stein's universal multiplier theorem.*

For it suffices to check the identity for the normalized Hermite polynomials $\{\mathbf{h}_\nu\}$,

$$\begin{aligned}\int_0^\infty \frac{\partial P_t \mathbf{h}_\nu}{\partial t}(x)\phi(t)dt &= \int_0^\infty \frac{d}{dt}(e^{-\sqrt{|\nu|}t})\mathbf{h}_\nu(x)dt\,\phi(t)\\ &= -\sqrt{|\nu|}\int_0^\infty e^{-\sqrt{|\nu|}t}\phi(t)dt\,\mathbf{h}_\nu(x) = -m(\sqrt{|\nu|})\,\mathbf{h}_\nu(x).\end{aligned}$$

Now,

$$P_{t_1}(m(L)f)(x) = -\int_0^\infty P_{t_1}\Big(\frac{\partial P_t f}{\partial t}(x)\Big)\phi(t)dt = -\int_0^\infty \frac{\partial P_{t+t_1}f}{\partial t}(x)\phi(t)dt.$$

Hence,

$$\frac{\partial P_{t_1}(m(L)f)}{\partial t_1}(x) = -\frac{\partial}{\partial t_1}\Big(\int_0^\infty P_{t_1}\Big(\frac{\partial P_t f}{\partial t}(x)\Big)\phi(t)dt\Big) = -\int_0^\infty \frac{\partial^2 P_{t+t_1}f}{\partial t^2}(x)\phi(t)dt,$$

thus, as ϕ is bounded, using the Cauchy–Schwartz inequality

$$\begin{aligned}\Big|\frac{\partial P_{t_1}(m(L)f)}{\partial t_1}(x)\Big| &\leq \int_0^\infty \Big|\frac{\partial^2 P_{t+t_1}f}{\partial t^2}(x)\Big||\phi(t)|dt \leq M\int_0^\infty \Big|\frac{\partial^2 P_{t+t_1}f}{\partial t^2}(x)\Big|dt\\ &= M\int_{t_1}^\infty s\Big|\frac{\partial^2 P_s f}{\partial s^2}(x)\Big|\frac{ds}{s} \leq M\Big(\int_{t_1}^\infty s^2\Big|\frac{\partial^2 P_s f}{\partial s^2}(x)\Big|^2 ds\Big)^{1/2} t_1^{-1/2}.\end{aligned}$$

Therefore, according to the same argument used in the proof of Proposition 5.12, we have using Fubini's theorem

$$\begin{aligned}g_{t,\gamma}((m(L)f)(x) &= \left(\int_0^\infty t_1\left|\frac{\partial P_{t_1}m(L)f}{\partial t_1}(x)\right|^2 dt_1\right)^{1/2}\\ &\leq C\left(\int_0^\infty\left(\int_{t_1}^\infty s^2\left|\frac{\partial^2 P_s f}{\partial s^2}(x)\right|^2 ds\right)dt_1\right)^{1/2}\\ &= C\left(\int_0^\infty s^3\left|\frac{\partial^2 P_s f}{\partial s^2}(x)\right|^2 ds\right)^{1/2} = Cg^2_{t,\gamma}f(x).\end{aligned}$$

Now, using Theorem 5.6 and Definition 5.7, we get

$$C'_p\|m(L)f\|_{p,\gamma} \leq \|g_{t,\gamma}((m(L)f)(x)\| \leq C\|g^2_{t,\gamma}f\|_{p,\gamma} \leq C_p\|f\|_{p,\gamma}. \qquad \Box$$

In particular, the imaginary powers $(-L)^{i\lambda}$ arising from $\phi(t) = \frac{t^{-i\lambda}}{\Gamma(1-i\lambda)}$ admits a $L^p(\gamma_d)$-bounded extension for any $1 < p < \infty$, because

$$\lambda^{-i\alpha} = \frac{\lambda}{\Gamma(1-i\alpha)}\int_0^\infty e^{-\lambda s}s^{-i\alpha}ds.$$

Theorem 6.7, is a weak version of *the Marcinkiewicz multiplier theorem* in the Euclidean case for the d-dimensional torus $\mathbb{T}^d$. The link is that if ϕ is of Laplace type then

$$|x^k \phi(x)| \leq C_k$$

for any $k \geq 0$, which is a particular case of the Marcinkiewicz condition, then $\phi(|x|)$ is a multiplier in $L^p(\mathbb{T}^d)$, $1 < p < \infty$.

6.4 Functional Calculus for the Ornstein–Uhlenbeck Operator

Now, we are going to discuss the fact that the Ornstein–Uhlenbeck operator has a bounded holomorphic functional calculus. In [105] J. García-Cuerva, G. Mauceri, S. Meda, and P. Sjögren, J. L. Torrea proved that for Gaussian multipliers if $p \neq 2$ there is no reasonable non-holomorphic functional calculus in $L^p(\gamma_d)$ for L. In particular, they proved that there is not an analog of the classical Hörmander multiplier theorem. In fact, for each $p \neq 2$, there exists a spectral multiplier m_p such that $m_p(L)$ does not extend to a bounded operator on $L^p(\gamma_d)$, which is a restriction of a holomorphic function in a neighborhood of $\mathbb{R}^d_+$, which satisfies the conditions

$$\sup_{x>0} |x^j \partial_j m_p(x)| < \infty,$$

for all $j \in \mathbb{N}$.

Moreover, in [103], J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea proved that a spectral multiplier operator $m(L)$ of Laplace type is also of weak type $(1,1)$ with respect to the Gaussian measure. Given a spectral multiplier m is of Laplace transform type, then $m(L)$ is a continuous operator from the space of test functions to the space of distributions on $\mathbb{R}^d$; thus, it has a distributional kernel. Let us prove that, off the diagonal, this kernel has a density K_ψ with respect to the measure $\gamma_d(dx) \otimes dy$, which satisfies the standard Calderón–Zygmund estimates in a suitable neighborhood of the diagonal (see [103, Lemma 2.1 and Theorem 2.2]). Consider the operator $r^L, 0 \leq r < 1$, whose integral kernel

$$M_r(x,y) = \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} e^{-\frac{|y-rx|^2}{1-r^2}},$$

may be obtained from Mehler's kernel by the change of variables $t = -\log r$. Thus

$$r^L f(x) = \int_{\mathbb{R}^d} M_r(x,y) f(y) dy,$$

for all test functions f. As Mehler's kernel satisfies the heat equation $\partial_t M_t(x,y) = LM_t(x,y)$ (see (2.53)), the kernel $M_r(x,y)$ satisfies the transformed equation $r\partial_r M_r(x,y) = -LM_r(x,y)$. If $\psi \in L^\infty(\mathbb{R}^d)$, define

$$K_\psi(x,y) = \int_0^1 \psi(r) \partial_r M_r(x,y) dr.$$

For $t > 0$, the local region N_t defined in (4.63) is the neighborhood of the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$.

Lemma 6.8. *If $x \neq y$, the integral defining K_ψ is absolutely convergent. Moreover, for $t > 0$ and each pair of multi-indices $\alpha, \beta \in \mathbb{N}^d$, there exists a constant C such that*

$$|\partial_x^\alpha \partial_y^\beta K_\psi(x,y)| \leq C \frac{\|\psi\|_\infty}{|x-y|^{d+|\alpha|+|\beta|}} \tag{6.9}$$

for all $(x,y) \in N_t$, $x \neq y$.

Proof. Using Rodrigues' formula for the Hermite polynomials (1.28), we have

$$\partial_x^\alpha \partial_y^\beta M_r(x,y) = \frac{(-r)^{|\alpha|}}{\pi^{d/2}(1-r^2)^{d/2+|\alpha|+|\beta|}} H_{\alpha+\beta}\left(\frac{xr-y}{\sqrt{1-r^2}}\right) e^{(1-r^2)^{d/2}}. \tag{6.10}$$

An elementary computation shows that the function $r \mapsto \partial_x^\alpha \partial_y^\beta M_r(x,y)$ is the product of the positive function,

$$\frac{1}{\pi^{d/2}} \frac{1}{(1-r^2)^{d/2+|\alpha|+|\beta|}} e^{-\frac{|y-rx|^2}{1-r^2}},$$

and a polynomial in r of degree at most $2|\alpha| + |\beta| + 3$, whose coefficients depend on x and y. Hence, as a function of r, it changes sign a finite number of times and there exists a constant C such that

$$\int_0^1 |\psi(r)| |\partial_r \partial_x^\alpha \partial_y^\beta M_r(x,y)| dr \leq C \|\psi\|_\infty \max_{0<r<1} |\partial_x^\alpha \partial_y^\beta M_r(x,y)|,$$

for all $x, y \in \mathbb{R}^d$. According to (6.10), we have that

$$\max_{0<r<1} |\partial_x^\alpha \partial_y^\beta M_r(x,y)| \leq \frac{C}{(1-r^2)^{(d+|\alpha|+|\beta|)/2}} \exp\left(-c_0 \frac{|y-rx|^2}{1-r^2}\right)$$

for some positive constant c_0. Because, in the local region N_t,

$$|rx-y|^2 \geq |x-y|^2 - 2(1-r)|x||x-y| \geq |x-y|^2 - 2(1-r)t,$$

the right-hand side of the previous inequality can be estimated by

$$C(t)(1-r^2)^{(d+|\alpha|+|\beta|)/2} \exp\left(-c_0 \frac{|y-rx|^2}{1-r^2}\right) \leq C|x-y|^{(d+|\alpha|+|\beta|)},$$

for all $(x,y) \in N_t$. □

Using this lemma, we can obtain the following representation of $m(L)$ in terms of $K_\psi(x,y)$.

Theorem 6.9. *Given a spectral multiplier m of Laplace transform type given by the formula (6.6), then the spectral multiplier operator has the following integral representation*

$$m(L) = \int_0^1 \psi(r) L r^L \frac{dr}{r} \tag{6.11}$$

where the integral converges on the weak operator topology of $L^2(\gamma_d)$. Moreover, f is a test function,

$$m(L)f(x) = \int_0^1 K_\psi(x,y) f(y) dy, \tag{6.12}$$

for all x in the support of f.

Proof.

$$\begin{aligned}
\langle m(L)f, g\rangle_\gamma &= \sum_{k=1}^{\infty} m(k) \langle \mathbf{J}_k f, g\rangle_\gamma = \sum_{k=1}^{\infty} k \int_0^{\infty} \phi(t) e^{-tk} dt \langle \mathbf{J}_k f, g\rangle_\gamma \\
&= \sum_{k=1}^{\infty} k \int_0^1 \psi(r) r^k \frac{dr}{r} \langle \mathbf{J}_k f, g\rangle_\gamma = \int_0^1 \psi(r) \sum_{k=1}^{\infty} k r^k \langle \mathbf{J}_k f, g\rangle_\gamma \frac{dr}{r} \\
&= \int_0^1 \psi(r) \langle L r^L f, g\rangle_\gamma \frac{dr}{r} = \int_0^1 \langle \int_0^1 \psi(r) L r^L f \frac{dr}{r}, g\rangle_\gamma
\end{aligned}$$

where we have used that $\sum_{k=1}^{\infty} |\langle \mathbf{J}_k f, g\rangle_\gamma| \leq \|f\|_{2,\gamma} \|g\|_{2,\gamma}$; thus, we may interchange the order of summation and integration. Therefore, we have obtained (6.11).

To compute the kernel of the spectral multiplier operator $m(L)$, assume that f and g are test functions on $\mathbb{R}^d$. Then

$$\begin{aligned}
\langle L r^L f, g\rangle_\gamma = \langle r^L f, L g\rangle_\gamma &= \iint M_r(x,y) f(y) dy\, \overline{Lg} \gamma_d(dx) \\
&= \langle M_r \gamma_d(dx) \otimes dy, L(\overline{g} \otimes f)\rangle
\end{aligned}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the pairing between distributions and test functions on $\mathbb{R}^d \times \mathbb{R}^d$ and $M_r \gamma_d(dx) \otimes dy$ is the distribution whose density with respect to the measure $\gamma_d(dx) \otimes dy$ is M_r. As the operator L is symmetric with respect to the Gaussian measure,

$$\begin{aligned}
\langle r^L f, L g\rangle_\gamma &= \langle L(M_r) \gamma_d(dx) \otimes dy, \overline{g} \otimes f\rangle \\
&= \iint r \partial_r M_r(x,y) \overline{g}(x) f(y) dy \gamma_d(dx)
\end{aligned}$$

Thus, using (6.11),

$$\langle m(L)f, g\rangle_\gamma = \int_0^1 \psi(r) \iint \partial_r M_r(x,y) \overline{g}(x) f(y) \gamma_d(dx) dy dr.$$

If f and g have disjoint supports, the triple integral in the identity above is absolutely convergent because of the previous lemma. Thus, using Fubini's theorem

$$\langle m(L)f,g\rangle_\gamma = \iint K_\psi(x,y)f(y)dy\overline{g}(x)\gamma_d(dx).$$

This proves that K_ψ is the restriction to the complement of the diagonal of the kernel of $m(L)$, i.e., we have proved that off the diagonal, $m(L)$ has density K_ψ with respect to the measure $\gamma_d(dx)\otimes dy$ □

Now, it can be proved that a spectral multiplier operator $m(L)$ of Laplace type is also of weak type $(1,1)$ with respect to the Gaussian measure [103, Theorem 3.8]. The proof uses these two previous results. The operator is split, as usual, into a local part and a global part, using in this case the local region R_t(4.63). J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea improved the treatment of the local part by making a smooth truncation and reducing the estimates to the general Calderón–Zygmund theory. Then, the global part is immediately bounded by the maximal Mehler's kernel used by P. Sjögren, in [247] (for more details, we refer the reader to [103, Section 3]).

Additionally, in [103], they also investigate how to define the multiplier operator in terms of its kernel, as a limit of truncated integrals. In particular, we see under what conditions the multiplier is given by a principal value integral. Boundedness is also proved for the maximal multiplier operator, via a vector-valued version of the estimates. The result applies, in particular, to the imaginary powers of $(-L)$, $(-L)^{i\lambda}$. Here, the growth of the operator (quasi-)norm for large imaginary powers is of special interest. As $-L$ has a non-trivial kernel, to define imaginary powers, it is first needed to restrict $-L$ to the orthogonal complement of the kernel. This amounts to considering $L^{i\alpha}\Pi_0$, where, as before, $\Pi_0 = I - \mathbf{J}_0$. The weak type $(1,1)$ constant of $L^{i\alpha}\Pi_0$ increases at most exponentially as $|\alpha|\to\infty$. They proved that this estimate cannot be improved to polynomial growth.

On the other hand, the assumption that a spectral multiplier m of Laplace type implies that m can be extended to a holomorphic function on the half-plane $\{z\in\mathbb{C}: \operatorname{Re} z > 0\}$, which is bounded on every sector $S_\theta = \{z\in\mathbb{C}: |\arg z| < \theta\}, 0<\theta<\pi/2$ (see Figure 2.1). As the spectrum of the Ornstein–Uhlenbeck L on $L^1(\gamma_d)$ is the closed right half-plane (see Theorem 2.7), it is natural to impose a holomorphy condition on the multiplier m if we want the operator $m(L)$ to be defined on $L^1(\gamma_d)$. Nevertheless, because the spectrum of $-L$ on $L^p(\gamma_d)$, for $1<p<\infty$, is the set $\mathbb{N}_0$ of non-negative integers, it seems too restrictive to require holomorphy of the multiplier m to obtain the $L^p(\gamma_d)$-boundedness of $m(L)$. In [182], S. Meda gave a sufficient condition for the existence of a non-holomorphic functional calculus for the generator A of a symmetric contraction semigroup on $L^p(M), 1<p<\infty$, where M is a σ-finite measure space.

If we fix $p\in(1,\infty)$, as we have mentioned before, it is important to determine the minimal regularity conditions of the spectral multiplier m, which imply that the spectral multiplier operator $m(L)$ is bounded in $L^p(\gamma_d)$. These conditions are sometimes best expressed in terms of Banach spaces of holomorphic functions. If $\theta \in (0,\pi/2)$, consider the open sector $S_\theta = \{z \in \mathbb{C} : |\arg z| < \theta\}$, and denote by $H^\infty(S_\theta)$ the space of bounded holomorphic functions on S_θ. A consequence of an abstract result by M. Cowling [59, Theorem 2] is that if $\theta > \pi|\frac{1}{q} - \frac{1}{2}|$, the spectral multiplier m is bounded and there exists $\tilde{m} \in H^\infty(S_\theta)$ such that $m(k) = \tilde{m}(k)$, $k = 1,2,3,\cdots$, then $m(L)$ extends to a bounded operator on $L^q(\gamma_d)$.

Moreover, in [105] it is shown that requiring holomorphy of a spectral multiplier m, in a sector of angle smaller than $\phi_p^* = \arcsin|\frac{2}{p} - 1|$, is not sufficient for the boundedness of $m(L)$ on $L^p(\gamma_d)$. Observe that $\phi_p^* \to \pi/2$ as $p \to 1$ is in line, with the fact, already mentioned, that the spectrum of L on $L^1(\gamma_d)$ is the (closed) right half-plane (see Theorem 2.7). Furthermore, the $L^1(\gamma_d)$-boundedness of dilation-invariant spectral multiplier operators $m(L)$ was characterized in [131, Theorem 3.5].

Finally, let us mention the main result in [105, Theorem 1] by J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. L. Torrea, which is an improvement, in the finite dimensional case of Cowling's result. Using the notation introduced in Chapter 2, the statement of the theorem is roughly as follows: for every $p \in (1,\infty)$, $p \neq 2$, and consider the sector $S_{\phi_p^*} := \{z \in \mathbb{C} : |\arg z| < \phi_p^*\}$. If m is a bounded holomorphic function on $S_{\phi_p^*}$ whose boundary values on $\partial S_{\phi_p^*}$ satisfy suitable *Hörmander-type conditions*, then the spectral multiplier $m(L)$ extends to a bounded operator on $L^p(\gamma_d)$ and hence to $L^q(\gamma_d)$ for all q such that $|\frac{1}{q} - \frac{1}{2}| \leq |\frac{1}{p} - \frac{1}{2}|$.

To establish the theorem, we first need the following notation. Suppose that J is a non-negative integer and that $\theta \in (0,\pi/2)$. Denote by $H^\infty(S_\theta;J)$ the Banach space of all $m \in H^\infty(S_\theta)$ for which a *Hörmander condition* of order J holds: there exists a constant C such that

$$\sup_{R>0} \int_R^{2R} |x^j \partial_j m(e^{\pm i\theta} x)|^2 \frac{dx}{x} \leq C^2, \text{ for } j = 0,1,\cdots,J. \tag{6.13}$$

$H^\infty(S_\theta;J)$ is endowed with the norm

$$\|m\|_{\theta,J} = \inf\left\{C : \sup_{R>0} \int_R^{2R} |x^j \partial_j m(e^{\pm i\theta} x)|^2 \frac{dx}{x} \leq C^2, \text{ for } j = 0,1,\cdots,J\right\}.$$

Note that (6.13) implies that

$$\sup_{z\in S_\theta} |m(z)| \leq 2C,$$

if $J > 0$.

Theorem 6.10. *Let $1 < p < \infty$, $p \neq 2$, let $m : \mathbb{N} \to \mathbb{C}$ be a bounded function, and assume that there exists a bounded holomorphic function $\tilde{M}$ in $S_{\phi_p^*}$, such that*

$$\tilde{M}(k) = m(k), \quad k = 1,2,3,\cdots$$

then,

i) If $\tilde{M} \in H^\infty(S_{\phi_p^*};4)$, *then* $m(L)$ *extends to a bounded operator on* $L^p(\gamma_d)$*; hence, on* $L^q(\gamma_d)$ *for all q such that* $|\frac{1}{q}-\frac{1}{2}| \leq |\frac{1}{p}-\frac{1}{2}|$.

ii) If $\tilde{M} \in H^\infty(S_{\phi_p^*})$, *and* $|\frac{1}{q}-\frac{1}{2}| < |\frac{1}{p}-\frac{1}{2}|$, *then* $m(L)$ *extends to a bounded operator on* $L^q(\gamma_d)$.

A significant feature of Theorem 6.10 is that the number of derivatives in the Hörmander condition in *i*) is independent of the dimension. However, the estimates depend strongly on dimension; thus, they fail to give a result for the infinite dimensional case, but Cowling's result holds in the infinite dimensional case. Also, the theorem may be sharpened using $H^\infty(S_{\phi_p^*};J)$, for J non-integer.

Moreover, the size of the region of holomorphy, measured by the aperture of the cone, cannot be reduced, as is proved in the following result:

Theorem 6.11. *Let* $1 < p < \infty$, $p \neq 2$, *and* $\theta < \phi_p^*$. *Then, there exists a function* m, *which decays exponentially at infinity and belongs to* $H^\infty(S_{\phi_p^*};J)$, *for* J *for every positive integer* J, *such that* $m(L)$ *does not extend to a bounded operator on* $L^p(\gamma_d)$.

For details of the proofs of Theorem 6.10 and 6.11 we refer the reader to [105, Section 3]. They use an abstract multiplier result for generators of holomorphic semigroups, which is a variant of an earlier result by S. Meda (see [182] or [60]).

6.5 Notes and Further Results

1. In [260], D. Stroock also considers the case of spectral multipliers m, being the Laplace transform of a measure μ in $[0,\infty)$ such that, for some integer N

$$\int_0^\infty e^{-Nt}\nu(dt) < \infty,$$

and then m is defined as

$$m(k) = \begin{cases} 0, & \text{if } 0 \leq k < N-1 \\ \lambda \int_0^\infty e^{-\lambda t}\nu(dt), & \text{if } k \geq N. \end{cases}$$

The proof that the spectral multiplier operator $m(L)$ has an extension to $L^p(\gamma_d)$ is completely analogous to that of Theorem 6.7.

2. Some other examples of spectral multipliers, whose spectral multiplier operator $m(L)$ is $L^p(\gamma_d)$-bounded, but that are not Meyer's multipliers, are:
   - Let us consider the *even part projection* multiplier operator

$$m_e(L)f = \sum_{k=0}^\infty \mathbf{J}_{2k}f = \sum_{k=0}^\infty m(1/k)\mathbf{J}_k f$$

where

$$m(x) = \begin{cases} 1, & \text{if } x = \frac{1}{2n}, n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $m_e(L)$ is a $L^p(\gamma_d)$-multiplier because the even part of f, $f_e(x) = \frac{f(x)-f(-x)}{2} = \sum_{k=0}^{\infty} \mathbf{J}_{2k} f = m_e(L)f$. Therefore,

$$\|m_e(L)f\|_{p,\gamma} = \|f_e\|_{p,\gamma} \leq \|f\|_{p,\gamma},$$

but m_e is not a Meyer's multiplier.

- Analogously, we can consider the *odd part projection* multiplier operator,

$$m_o(L) = \sum_{k=0}^{\infty} \mathbf{J}_{2k+1} f.$$

Since $f = m_e(L)f + m_o(L)f$, and we know that $m_e(L)$ is $L^p(\gamma_d)$-bounded, then we conclude that $m_o(L)$ is a $L^p(\gamma_d)$-multiplier, which is not a Meyer's multiplier either.

- Let us consider the spectral multiplier operator

$$m_-(L)f = \sum_{k=0}^{\infty} (-1)^k \mathbf{J}_k f.$$

As

$$-f(-x) = 2f_p(x) - f(x) = \sum_{k=0}^{\infty} \mathbf{J}_{2k} f - \sum_{k=0}^{\infty} \mathbf{J}_2 f = m_-(L)f,$$

then it is clear that

$$||m_-(L)f||_{p,\gamma_d} = ||f||_{p,\gamma_d},$$

but $m_-(L)f$ is not a Meyer's multiplier.

3. Meyer's theorem admits an extension to spectral multiplier operators of the form

$$m(L)f(x) = \sum_{k=0}^{\infty} m(k,x) \mathbf{J}_k f(x), \tag{6.14}$$

where $f = \sum_{k=0}^{\infty} \mathbf{J}_k f$. The same proof carries over, if m admits an expansion of the form

$$m(t,x) = \sum_n a_n(x) t^n,$$

$|a_n(x)| \leq M_n$ and $\sum_{n=0}^{\infty} |M_n| \frac{1}{n_0^{\delta n}} < \infty$. Operators of the form (6.14) are in a sense *pseudo-differential operators* in the Gaussian context, and require further analysis and study.

4. An open question, as far as we know, is what is the boundedness property of Meyer's multiplier operators for the case of $p = 1$?

5. In his seminal article [28], W. Beckner proved, among other things, that the hypercontractivity property for the Ornstein–Uhlenbeck semigroup is a consequence of Young's generalized inequality, which itself is obtained from an inequality for multipliers of Hermite expansions. In fact, Beckner proved the continuity $L^p(\gamma_d) - L^{p'}(\gamma_d)$ of the operators T_t, but with a purely imaginary parameter $t = i\sqrt{p-1}$, something that is closely related to Weissler's representation [292] given in (2.34) and the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$. Moreover, the proof is quite interesting by itself, using in a decisive way the classical central limit theorem (CLT). Beckner makes clear the intimate relationship between classical harmonic analysis and Gaussian harmonic analysis, because, for example, the multiplier result allows him to obtain the best constant in the Haussdorff–Young inequality for the Fourier transform on $\mathbb{R}^d$.

6. In B. Muckenhoupt's monograph [198], he uses transplantation theorems to get spectral multiplier theorems for Jacobi expansions. This idea could be explored for the Hermite expansions, but to do that we would need to work with the whole family of generalized Hermite polynomials $\{H_n^{\mu}\}$ (see note 4 in Chapter 1; see also T. Chihara [54]).

7. In [148], M. Kemppainen studies a method of decomposing a spectral multiplier operators $m(L)$ into three parts according to the notion of admissibility, which quantifies the doubling behavior of the underlying Gaussian measure. He proves that the above-mentioned admissible decomposition is bounded in $L^p(\gamma_d)$ for $1 < p \leq 2$ in a certain sense involving the Gaussian conical square function. The proof relates admissibility to E. Nelson's hypercontractivity theorem in a novel way.

8. In [147], M. Kemppainen studies a class of spectral multiplier operators $m(L)$, defined using spectral multipliers m such that,

$$m(\lambda) = \int_0^{\infty} e^{-\lambda t}(t\lambda)^2 \phi(t) \frac{dt}{t},\ \lambda \geq 0,$$

where $\phi : (0,\infty) \to \mathbb{C}$ is twice continuously differentiable, satisfying

$$\sup_{0<t<\infty} (|\phi(t)| + t|\phi'(t)| + t^2|\phi''(t)|) + \int_1^{\infty} (|\phi'(t)| + t|\phi''(t)|)dt < \infty,$$

and finds a sufficient condition for the integrability of $m(L)$ in terms of the admissible conical square function (5.51) and a maximal function using a decomposition method presented in [231].

9. In [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, & W. Urbina established a version of P. A. Meyer's multiplier theorem for the Laguerre case, because, as we have mentioned in Chapter 2, point 12. of Section 2.5, the Laguerre semigroup is also hypercontractive.

10. In [236] E. Sasso obtains a multiplier theorem for spectral multipliers of Laplace transform type in the Laguerre case, proving that they are of weak type $(1,1)$ for the Gamma measure.

11. In [49], A. Carbonaro and O. Dragicević have an impressive result, using Bellman function techniques. It provides an alternative to the results in [105], but is also valid in infinite dimensions.